ALGEBRAIC DESIGN OF DISCRETE MULTIWAVELET TRANSFORMS

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ABSTRACT

An algebraic approach to the design of different kinds of discrete wavelet transforms (orthogonal and biorthogonal single-/multiwavelet transforms, multiwavelet-like transforms) is taken. The different transforms are analysed with respect to computational efforts, approximation properties and symmetry. The design of the orthogonal and biorthogonal single-/multiwavelets requires the solution of a system of linear and nonlinear equations. Only the biorthogonal case enables symmetric coefficients. The basis matrix of the multiwavelet-like transform is easy to compute, orthogonal and ultimately symmetric. Modifications of this multiwavelet-like transform are given with respect to practical applications.

1. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in many application fields, e.g. signal processing [4],solving differential and integral equations [1, 2]. Different variations of wavelet bases (orthogonal, biorthogonal, multiwavelet, multiwavelet-like) have been presented and the design of the corresponding wavelet and scaling functions has been adressed.

The single-wavelet transform is based on one scaling function and one wavelet [7]. Any function is decomposed in dilated and translated versions of the scaling and wavelet functions. Orthogonal single-wavelets are not symmetric. Releasing the orthogonality condition by using different wavelets and scaling functions for synthesis and analysis yields symmetric coefficients, while perfect reconstruction can be preserved (biorthogonal wavelets) [3]. The methods for designing orthogonal wavelet bases can be extended to orthogonal multiwavelet bases. Multiwavelet transforms are still based on a single scaling function but use several wavelet functions. Therefore, a signal is again decomposed in dilated and translated versions of the scaling function, while the detail information (multiresolution analysis) is now represented by dilated and translated versions of several wavelet functions. Multiwavelet-like bases are based on as many scaling functions as wavelet functions [1, 2].They are called multiwavelet-like, since the wavelets (coefficients) used in the different stages of the wavelet transform are different.

In this paper we will take a purely algebraic point of view of all these wavelet transforms in their discrete form. The corresponding discrete wavelet transformations $t = \mathbf{U}s$ (s is the signal vector, t is the transformed signal vector) are described by their transformation matrices $\mathbf{U}$ as the DFT is described by the DFT-matrix. The algebraic design of the wavelet basis matrix $\mathbf{U}$ requires the solution of a system of linear and nonlinear equations in the case of orthogonal multiwavelets [6](single wavelets are a special case of multiwavelets). For gaining symmetric bases the orthogonality must be released and designing biorthogonal wavelets also leads to a systems of partly nonlinear equations. It is shown that except for approximation order p=1 , orthogonal multiwavelets require overlapping blocks in the basis matrix $\mathbf{U}$. The proof of this property is obtained by considering the QR-decomposition of the moment matrix (the moment matrix reflects the approximation order). This proof directly leads to the design of multiwavelet-like bases. The design of these multiwavelet-like bases only requires linear algebra operations (QR-decomposition, matrix-matrix-multiplication) [1, 2]. In this case the basis matrix can be built by non-overlapping orthogonal blocks. Furthermore, although the stages of the multiwavelet-like transform show no symmetry, the rows of the final basis matrix $\mathbf{U}$ are symmetric. The main drawback compared to multiwavelets is, that the multiwavelet-like bases have different stages (different filter coefficients in each stage). The stages, however, converge rapidly to very small differences in their coefficients. Therefore, it is usually sufficient for practical applications to work with a small set of different stages (2 different stages, i.e. 2 sets of filter coefficients were used in our examples).

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2. THE DISCRETE MULTIWAVELET TRANSFORM

The discrete multiwavelet transform is based on one scaling function and $k-1$ wavelets. For k=2 we have the special case of single-wavelets [2, 7]. The rows of the basis matrix $\mathbf{U}$ represent the dilated and translated scaling and wavelet functions. $\mathbf{U}$ is gained out of a matrix $\mathbf{W}$ of size kxn, that is decomposed in an upper half $\mathbf{W}^U$ (one row), that represents the scaling function and a lower half $\mathbf{W}^L$ (k-1 rows), that represents the multiwavelets. These parts $\mathbf{W}^U$ and $\mathbf{W}^L$ appear in the matrices $\mathbf{U}_j^{'}$ of stage j.

$$\mathbf{U}_j^{'} = \begin{pmatrix} \mathbf{W}^L & & & \\ & \mathbf{W}^L & & \\ & & \ddots & \\ & & & \mathbf{W}^L \\ \mathbf{W}^U & & & \\ & \mathbf{W}^U & & \\ & & \ddots & \\ & & & \mathbf{W}^U \end{pmatrix}$$

For the multiwavelet transform the computational complexity reduces with each stage by the factor k. Therefore, the mxm matrix $\mathbf{U}_j$ of the stage j contains the identity matrix and the matrix $\mathbf{U}_j^{'}$ of size $\frac{m}{k^j}$x$\frac{m}{k^j}$.

$$\mathbf{U}_j = \begin{pmatrix} \mathbf{I}_{m-m/k^{j-1}} & \\ & \mathbf{U}_j^{'} \end{pmatrix} \quad 1 \le j \le l$$

The whole transform matrix $\mathbf{U}$ is the product of the matrices $\mathbf{U}_j$: $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2\ldots\mathbf{U}_l$. The structure of an orthogonal $\mathbf{U}$ for k=3 and n=6 is shown in Fig.1.

The coefficients of $\mathbf{W}$ are computed by solving a nonlinear system of equations, by decomposing the matrix $\mathbf{W}$ of size kxn in n/k matrices $\mathbf{A}_\nu$ of size kxk.

$$\mathbf{W} = [\mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_{n/k}] \qquad n/k \in N$$

$\mathbf{W}$ has to satisfy the property of orthonormality,

$$\mathbf{W}\mathbf{W}^T = \mathbf{I} \tag{1}$$

what leads to $k(k+1)/2$ equations. As the multiwavelet transform matrix has overlapping blocks, we obtain shifted orthogonality conditions.

$$\sum_{i=1}^{j} \mathbf{A}_i\mathbf{A}_{n/k+i-j}^T = 0, \qquad j = 1,2\ldots n/k-1$$

This leads to another $(n/k-1)\,k^2$ equations. The remaining free parameters are reduced by fulfilling as many order equations as possible:

$$\mathbf{W}\mathbf{C} = \begin{pmatrix} x^T \\ \mathbf{0} \end{pmatrix}, \tag{2}$$

where the elements of the matrix of vanishing moments $\mathbf{C}$ are $c_{ij} = i^{j-1}$, $1 \le i \le n$, $1 \le j \le p$ and p is the approximation order. Usually an amount of $p(k-1)$ equations are gained by the approximation conditions.

As wavelets are often implemented with FIR filters, it is advantageous to have symmetric coefficients. With exception of the single-wavelets of order 1, the so-called Haarbasis, the discussed scaling functions and wavelets are not symmetric. Using biorthogonal single-wavelets enables these symmetric coefficients. A disadvantage is, that different filters have to be used for analysis ($\mathbf{W}$) and synthesis ($\tilde{\mathbf{W}}$). The filter coefficients ,that are equal to the elements of the rows of $\mathbf{W}$ and $\tilde{\mathbf{W}}$, have to fulfill the following, partly nonlinear equations.

$$\mathbf{W}\tilde{\mathbf{W}}^T = \mathbf{I}, \qquad \sum \mathbf{A}_i\tilde{\mathbf{A}}_{n/k+i-j}^T = 0$$

$$\mathbf{W}\mathbf{C}_{pn} = \begin{pmatrix} x^T \\ \mathbf{0} \end{pmatrix} \qquad \tilde{\mathbf{W}}\mathbf{C}_{pn} = \begin{pmatrix} \tilde{x}^T \\ \mathbf{0} \end{pmatrix}$$

The computational efforts increase with the size of $\mathbf{W}$ rapidly. The trial to generate nonoverlapping multiwavelets with only one block $\mathbf{W}=\mathbf{A}_1$ in order to get no shifted orthogonality conditions fails for block length bigger than 2.

Theorem: Except for $k=2$ (Haar wavelet) there is no kxk matrix $\mathbf{W} = [\mathbf{A}_1]$, that meets the conditions (1) and (2). There is no nonoverlapping basis matrix $\mathbf{U}$.

Proof: According to (1) and (2) an orthogonal $\mathbf{W}$ is required such that

$$\mathbf{W}\mathbf{C} = \begin{pmatrix} x^T \\ \mathbf{0} \end{pmatrix}$$

Using a QR-decomposition of $\mathbf{C}$ yields

$$\mathbf{Q}^T\mathbf{C} = \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix} \tag{3}$$

where $\mathbf{R}$ is an upper triangular matrix. In order to find an orthogonal $\mathbf{W}$ that fulfills (2), there must be an orthogonal transformation $\mathbf{Q}_W^T$ such that

$$\mathbf{Q}_W^T\mathbf{Q}^T\mathbf{C} = \mathbf{W}\mathbf{C}$$

i.e.

$$\mathbf{Q}_W^T \begin{pmatrix} \mathbf{R} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} x^T \\ \mathbf{0} \end{pmatrix}$$

Since $\mathbf{C}$ is a full rank matrix, there is no such orthogonal matrix $\mathbf{Q}_W^T$ and therefore no kxk matrix $\mathbf{W}$. This implies overlapping blocks in the basis matrix $\mathbf{U}$, since $\mathbf{W}$ must have the size kxn with $n > k$ in order to fulfill (2).

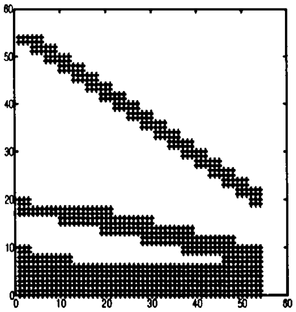

Figure 1: Structure of a multiwavelet transform matrix

3. MULTIWAVELET-LIKE TRANSFORMS

The only way to get a matrix with nonoverlapping blocks is to determine multiwavelet-like bases. This method relies on the QR-decomposition of the matrix of vanishing moments $\mathbf{C}$ of size kxk, what leads to as many scaling functions as wavelets. The $k/2$ orthogonal rows of the lower half of Q^T of (3) fulfill at least $p' = k/2$ approximation conditions. Therefore, they represent orthogonal wavelets of order p'. The upper $k/2$ orthogonal rows represent the corresponding $k/2$ scaling functions, with increasing approximation order ($p' = 0, 1, \ldots, k/2 - 1$).
The QR-decomposition is executed on the kxk matrix $\mathbf{C}'_1$, whose elements are shifted and scaled for numerical reasons:

$$c'_{ij} = \frac{c_{ij} - \mu}{\sigma}, \quad \sigma = \frac{1}{2}(2N+1), \quad \mu = \frac{1}{2}(2N-1)$$

The resulting matrix of stage 1 is decomposed in an upper half $\mathbf{W}_1^U$ of size $\frac{k}{2}$xk, whose rows correspond to the $k/2$ scaling functions, and a lower half $\mathbf{W}_1^L$ of the same size, whose rows are the $k/2$ wavelets.

$$\begin{pmatrix} \mathbf{W}_1^U \\ \mathbf{W}_1^L \end{pmatrix} = Orth(\mathbf{C}'_1)$$

The matrices $\mathbf{W}_1^L$ and $\mathbf{W}_1^U$ appear in the matrix of the first stage $\mathbf{U}_1$, as it is discussed in the previous section.
The discussed transform is only multiwavelet-like, as the different stages differ from each other. For each stage j a new matrix $\mathbf{C}'_j$ is computed. In order to improve the numerical properties, the result is multiplied with the matrices $\mathbf{S}_1$ and $\mathbf{S}_2$ [1, 2].

$$\mathbf{C}'_j = \begin{pmatrix} \mathbf{W}_{j-1}^U \mathbf{C}'_{j-1} \mathbf{S}_1 \\ \mathbf{W}_{j-1}^U \mathbf{C}'_{j-1} \mathbf{S}_2 \end{pmatrix}$$

After the QR-decomposition of $\mathbf{C}'_j$ the matrices $\mathbf{W}_j^U$ and $\mathbf{W}_j^L$ are inserted into a matrix $\mathbf{U}'_j$ as shown in the previous section. The computational complexity of the multiwavelet-like transform reduces with each stage by the factor k=2. Therefore the matrix $\mathbf{U}_j$ of stage j contains the identity matrix and the matrix $\mathbf{U}'_j$ of size $\frac{m}{2^j}$x$\frac{m}{2^j}$. The whole transform matrix $\mathbf{U}$ is again the product of the matrices $\mathbf{U}_j$: $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2\ldots\mathbf{U}_l$.
The multiwavelet-like transform leads automatically to orthogonal, symmetric and nonoverlapping bases of approximation order p'. Furthermore the bases are computed quite easily by linear matrix operations and not by solving a system of nonlinear equations. The structure of $\mathbf{U}$ for p'=3 is shown in Fig.3. In Fig.2 the p' scaling functions and the p' wavelet functions are shown for p'=3 (ploted are the last $2p'$ rows of $\mathbf{U}$).

Remark: $\mathbf{W}^U$ contains in its rows p' scaling functions with the qth scaling function has q-1 vanishing moments. $\mathbf{W}^L$ contains in its rows p' wavelets with the qth wavelet has p'+q-1 vanishing moments.

$$\begin{pmatrix} \mathbf{W}_1^U \\ \mathbf{W}_1^L \end{pmatrix} \mathbf{C} = \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix}$$

It is also possible to perform a QR-decomposition with a reduced matrix $\mathbf{C}'_r$ of size $2p'$xp'. Then the qth scaling function has again q-1 vanishing moments, while all wavelets have p' vanishing moments, what leads also to a satisfying solution.

$$\begin{pmatrix} \mathbf{W}_1^U \\ \mathbf{W}_1^L \end{pmatrix} \mathbf{C}'_r = \begin{pmatrix} x & x \\ 0 & x \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

To form an exact transform matrix you have to compute all matrices $\begin{pmatrix} \mathbf{W}_j^U \\ \mathbf{W}_j^L \end{pmatrix}$, $j = 1\ldots l$. Since the difference in the coefficients of these matrices converge very rapidly for increasing j, it is usually sufficient for practical applications to use only 2 or 3 different matrices $\begin{pmatrix} \mathbf{W}_j^U \\ \mathbf{W}_j^L \end{pmatrix}$.
A computation of multiwavelet-like bases with a reduced matrix $\mathbf{C}'_r$ and only 2 different stages of transform ($\mathbf{W}_j^U = \mathbf{W}_2^U$, $\mathbf{W}_j^L = \mathbf{W}_2^L$, for $j = 2\ldots l$) leads to almost the same basisfunctions as in Fig.2 (the absolute error is smaller than 10^{-4}). In this simplified form the multiwavelet-like transform becomes more amenable to the filter bank implementation usually used for single-

and multiwavelet transforms. For practical applications (e.g. unitary image transforms [4]) using these modified bases works as well as using the exact bases.

4. CONCLUSION

In this paper we have taken an algebraic approach to different kinds of discrete wavelet transforms and have compared them in the algebraic domain. The multiwavelet-like transform offers orthogonality and symmetry. It is easy to compute by linear matrix operations. Also, it is as well suited for a parallel implementation (computation of the bases as well as the transform itself) [4]. Suitable modifications were given with respect to practical applications. Finally, we want to note, that the algebraic design method can be extended to overlapping wavelet transforms using several scaling functions [8], and that the biorthogonal spline wavelet transforms are included in the algebraic design method.

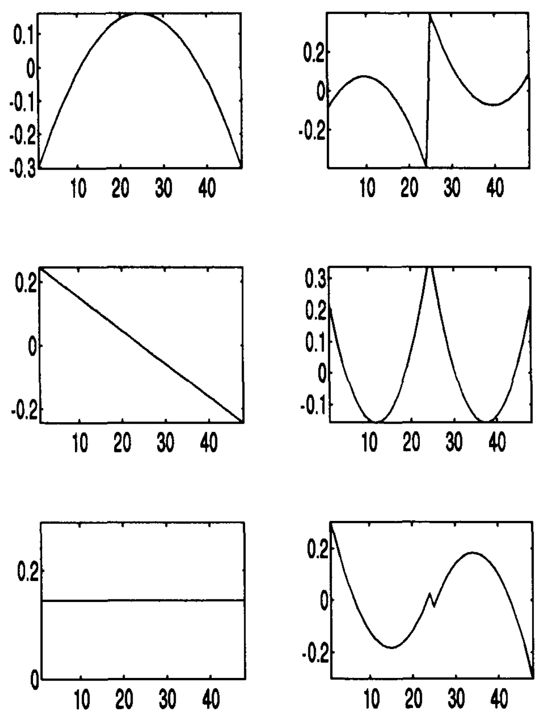

Figure 2: The scaling functions and wavelets for $p' = 3$

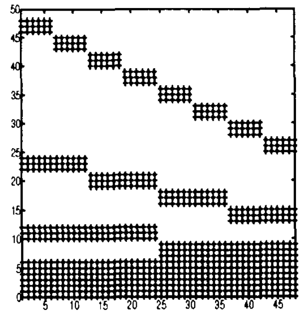

Figure 3: Structure of a multiwavelet-like transform matrix

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