SMOOTH MULTIWAVELETS BASED ON TWO SCALING FUNCTIONS

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ABSTRACT

In this paper new multiwavelets based on several scaling functions are designed. The resulting wavelets exhibit the following properties: compact support, symmetry and orthogonality, arbitrary approximation order as well as good frequency resolution. The new bases are quite similar to the well known splines and also have close relationships to multirate filterbanks (multiwavelets based on one scaling function). The good performance of the new wavelets with respect to the smoothness and the frequency resolution is documented.

1. INTRODUCTION

In recent years wavelet transforms have gained a lot of interest in many application fields, e.g. signal processing [2], solving differential and integral equations [1]. Different variations of wavelet bases (orthogonal, biorthogonal, multiwavelets) have been presented and the design of the corresponding wavelet and scaling functions has been adressed [3, 8, 7, 9]. Most attention was focussed on single wavelet transforms [3], whereby any signal is approximated by dilated and translated versions of the wavelet function $\Psi(t)$. Single wavelet transforms are based on one scaling function $\Phi(t)$ and one wavelet function $\Psi(t)$, which meet the following dilation equations:

$$\Phi(t) = \sum_k g_k \Phi(2t-k); \qquad \Psi(t) = \sum_k h_k \Phi(2t-k).$$

The discrete wavelet transform can be implemented by the filterbank structure of Figure 1a with the complementary filters $G(z) = \sum_k g_k z^{-k}$ (low pass) and $H(z) = \sum h_k z^{-k}$ (high pass).

Because orthogonal, compactly supported single wavelets not being symmetric for arbitrary approximation orders, other types of orthogonal wavelet transforms were designed. Multiwavelet transforms based on one scaling function were constructed in [10]. These are symmetric and show good

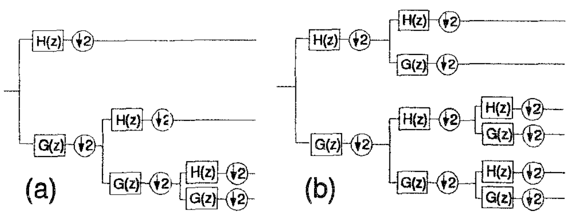

Figure 1: Filterbank implementation of discrete wavelet transforms

frequency behavior, though the approximation order is limited to $p = 1$ affecting the smoothness of the bases [6]. Only the approach of designing multiwavelet transforms based on several scaling functions allows compact support, orthogonality, arbitrary approximation order and symmetry at the same time [11, 4, 5, 9].

This paper presents new multiwavelet transforms that are based on 2 scaling functions. Additionally to the just mentioned properties they exhibit a very good frequency resolution (similar to the multirate filterbanks of [10]). Furthermore, the new bases are quite smooth. The maximally smooth, non-orthogonal splines are approximated quite well, while orthogonality is guaranteed by the presented design method.

This paper is organized as follows: Section 2 reviews the algebraic design method [9] being applied to construct multiwavelets based on several scaling functions. Section 3 presents the new bases and describes their properties.

2. ALGEBRAIC DESIGN OF MULTIWAVELET TRANSFORMS BASED ON SEVERAL SCALING FUNCTIONS

A wavelet transform is always related to dilation equations. These dilation equations describe the wavelet system, its coefficients are used to execute the discrete wavelet transforms and therefore, appear in the basis matrix $\mathbf{W}$. This basis matrix $\mathbf{W}$ can be divided into an upper part $\mathbf{W}^U$ representing the scaling function(s) and a lower part $\mathbf{W}^L$ representing the wavelet(s). Since the number of rows of $\mathbf{W}^U$ is equal to the amount of scaling functions, and the number of rows of $\mathbf{W}^L$ is equal to the amount of wavelets, in the case of single wavelets $\mathbf{W}^U$ consists of the coefficients g_k and $\mathbf{W}^L$ consists of the coefficients h_k.

In this paper only wavelet transforms using 2 wavelets and 2 scaling functions are discussed requiring 4 dilation equations:

$$\Phi_1(t) = \sum_{k=0}^{2h-1} a_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} a_{2k+1}\Phi_2(2t-k);$$

$$\Phi_2(t) = \sum_{k=0}^{2h-1} b_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} b_{2k+1}\Phi_2(2t-k);$$

$$\Psi_1(t) = \sum_{k=0}^{2h-1} c_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} c_{2k+1}\Phi_2(2t-k);$$

$$\Psi_2(t) = \sum_{k=0}^{2h-1} d_{2k}\Phi_1(2t-k) + \sum_{k=0}^{2h-1} d_{2k+1}\Phi_2(2t-k).$$

Therefore, the basis matrix $\boldsymbol{W}$ is of size 4xn. $\boldsymbol{W}$ can be divided into h matrices $\boldsymbol{A}_\nu$ of size 4x4, with n=4h.

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}^U \\ \mathbf{W}^L \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \dots & a_{4h-1} \\ b_0 & b_1 & \dots & b_{4h-1} \\ c_0 & c_1 & \dots & c_{4h-1} \\ d_0 & d_1 & \dots & d_{4h-1} \end{pmatrix}$$

$$\boldsymbol{W} = \begin{pmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \dots & \mathrm{A}_h \end{pmatrix};$$

For the computation of the coefficients a_i, b_i, c_i, d_i, the algebraic design method is used [9]. Initially the properties of the continuous functions $\Phi_1(t), \Phi_2(t), \Psi_1(t), \Psi_2(t)$ are formulated. By inserting the dilation equations, the properties for orthogonality and approximation criteria of the continuous functions are converted to equations for the discrete coefficients a_i, b_i, c_i, d_i. Solving the resulting systems of equations leads to these coefficients that define the multiwavelet system.

Orthogonality: A sufficient condition for orthogonality ($\Phi_{1,2} \perp \Phi_{1,2}, \Phi_{1,2} \perp \Psi_{1,2}, \Psi_{1,2} \perp \Psi_{1,2}$) is, that the matrix $\mathbf{W}$ fullfills the orthogonality and the shifted orthogonality conditions:

$$\mathbf{W}\mathbf{W}^T = \mathbf{I}; \quad \sum_{i=1}^{j} \mathrm{A}_i \mathrm{A}_{h+i-j}^T = \mathbf{0}, \quad j = 1, 2 \ldots h-1;$$

Approximation: The continuous wavelet functions have to fullfill the equations of vanishing moments for a maximum approximation order p.

$$0 = \int_{-\infty}^{+\infty} t^j \Psi_1(t)\,dt; \quad 0 = \int_{-\infty}^{+\infty} t^j \Psi_2(t)\,dt; \quad 0 \le j \le p.$$

The moments of the scaling functions Φ_1 and Φ_2 do not vanish, what leads to I_j (Φ_1) and J_j (Φ_2).

$$I_j = \int_{-\infty}^{+\infty} t^j \Phi_1(t)\,dt; \quad J_j = \int_{-\infty}^{+\infty} t^j \Phi_2(t)\,dt; \quad 0 \le j \le p.$$

The conversion of the approximation conditions of the continuous bases leads to the following equations for the discrete coefficients a_i, b_i, c_i, d_i after inserting the dilation equations [9]:

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k c_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k c_{2k+1} k^{j-r},$$

$$0 = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k d_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k d_{2k+1} k^{j-r},$$

$$2^j I_j = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k a_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k a_{2k+1} k^{j-r},$$

$$2^j J_j = \sum_{r=0}^{j} \binom{j}{r} I_r \sum_k b_{2k} k^{j-r} + \sum_{r=0}^{j} \binom{j}{r} J_r \sum_k b_{2k+1} k^{j-r}.$$

These approximation equations together with the orthogonality conditions have to be solved to design the multiwavelets and the corresponding multiscaling functions.

3. CONSTRUCTION OF SMOOTH MULTIWAVELETS

The simplest multiwavelet transforms based on 2 scaling functions can be constructed out of single wavelet transforms, if not only a wavelet decomposition with the wavelet tree of Figure 1a is executed, but with the extended version of Figure 1b, what leads to a more detailed frequency resolution. Using the Haar basis ($H(z) = \frac{1-z^{-1}}{\sqrt{2}}, G(z) = \frac{1+z^{-1}}{\sqrt{2}}$) leads to the 2 scaling functions Φ_1, Φ_2 and the 2 wavelets Ψ_1, Ψ_2 of Figure 2, which are orthogonal and symmetric, but exhibit only 1 vanishing moment, i.e. only a poor approximation order $p = 1$. The basis matrix $\boldsymbol{W}_H$ has many elements being zero, since Φ_1 and Φ_2 only depend on $\Phi_1(2t-k)$, Ψ_1 and Ψ_2 only depend on $\Phi_2(2t-k)$.

$$\boldsymbol{W}_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Applying the same procedure to a wavelet transform based on quadratic spline wavelets leads to the basis matrix $\boldsymbol{W}_S$ and the corresponding scaling functions and wavelets of Figure 3. These bases show a maximal approximation order with respect to their length, however splinewavelets are only semiorthogonal [12].

$$\boldsymbol{W}_S = \frac{1}{4} \begin{pmatrix} 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 3 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 \end{pmatrix}.$$

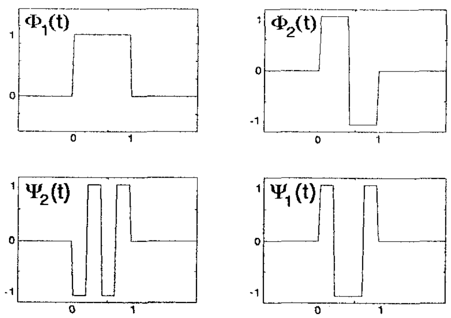

Figure 2: Basis functions of the Haar–based multiwavelet transform ($p = 1$)

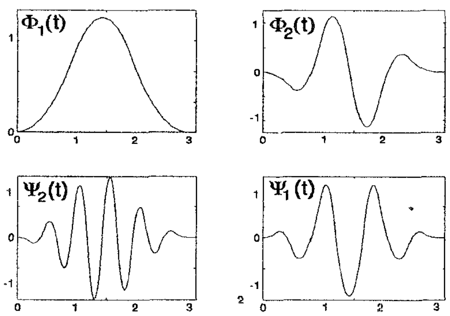

Figure 3: Basis functions of the spline–based multiwavelet transform ($p = 3$)

The task is to modify these matrices of well-known wavelet systems such that they meet the desired properties orthogonality, symmetry and arbitrary approximation order, simultaneously. The numerous zeros in $\boldsymbol{W}_H$ or $\boldsymbol{W}_S$ offer the necessary degrees of freedom.

With the assumption of Φ_1 (1st row of $\boldsymbol{W}$) and Ψ_1 (3rd row of $\boldsymbol{W}$) being symmetric, and Φ_2 (2nd row of $\boldsymbol{W}$) and Ψ_2 (4th row of $\boldsymbol{W}$) being antisymmetric, $\boldsymbol{W}$ has a predetermined structure. This structure also guarantees the orthogonality of spaces ($\Phi_{1,2} \perp \Psi_{1,2}$).

$$\boldsymbol{W} = \begin{pmatrix} a_0 & b_0 & a_1 & b_1 & a_1 & -b_1 & a_0 & -b_0 \\ -a_0 & b_0 & a_1 & -b_1 & -a_1 & -b_1 & a_0 & b_0 \\ b_0 & -a_0 & -b_1 & a_1 & -b_1 & -a_1 & b_0 & a_0 \\ b_0 & a_0 & b_1 & a_1 & -b_1 & a_1 & -b_0 & a_0 \end{pmatrix};$$

The 4 unknowns a_0, a_1, b_0, b_1 can be chosen such that the whole system is orthogonal and fullfills 2 vanishing moments. Setting

$$a_0 = 0.009977, a_1 = 0.697129, b_0 = b_1 = -0.083399$$

results in the bases plotted in Figure 4.

Figure 5 shows the frequency characteristics of the 4 bases. They stress the claim, that the presented multiwavelet system based on 2 scaling functions has similar frequency behavior in comparison to multiwavelet systems based on 1 scaling function of adequate length [10]. Note, that there are also symmetries in the frequency characteristics of the bases (between Φ_1, Ψ_2 respectively between Φ_2, Ψ_1). This is due to the structure of $\boldsymbol{W}$. Giving up these symmetries offers a degree of freedom that can be used to increase the amount of vanishing moments by one. In the case of symmetry, basis matrices $\boldsymbol{W}$ of size 4x4h allow an approximation order p=h. If releasing this symmetry, each wavelet can fullfill p=h+1 vanishing moments. In Figure 6 the bases of a multiwavelet system with approximation order p=4 are plotted, whereby $\boldsymbol{W}$ has dimension 4x12.

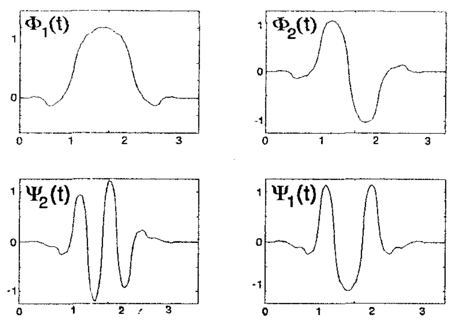

Figure 4: Smooth multiwavelets of $p = 2$ and the corresponding scaling functions

As far as a filterbank implementation is concerned, multiwavelet transforms based on 2 scaling functions require structures different from Figure 1, since $\boldsymbol{W}$ has no zero elements. In Figure 7 a filterbank structure for the presented multiwavelet transform (with only 2 equal stages) is shown. $H_1(z)$, $H_2(z)$ belongs to the rows of $\boldsymbol{W}^L$, $G_1(z)$, $G_2(z)$ belongs to the rows of $\boldsymbol{W}^U$. The downsampling by 2 in each stage (see also the dilation equations) is realized by the multiplexer between the 2 low pass paths, after each is downsampled by 4.

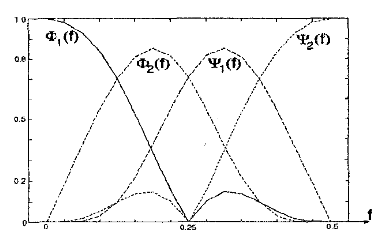

Figure 5: Frequency characteristics of the smooth multiwavelets of p=4 and the corresponding scaling functions

With respect to the discrete wavelet transform there are decisive differences between single- and multiwavelet transforms. While in the case of one scaling function, the co-

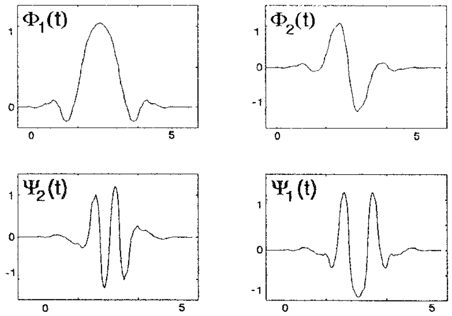

Figure 6: Smooth Multiwavelets of p=4 and the corresponding scaling functions

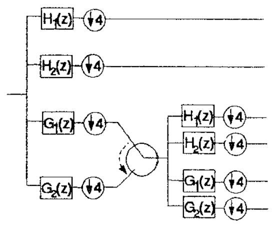

Figure 7: Filterbank implementation of a multiwavelet transform based on 2 scaling functions

efficients (g_k, h_k) approximate the continuous functions Φ, Ψ, in the case of 2 scaling functions the coefficients of $\boldsymbol{W}$ do not approximate Φ_1, Φ_2, Ψ_1, Ψ_2, they only extend the coarse approximation of one stage to the next finer of the following stage (dilation equation). Therefore, in [13] pre- and postfilters were proposed for the discrete transform. Alternatively, it is also possible to work with a wavelet-like approach [1, 9].

4. CONCLUSION

In this paper new multiwavelet transforms based on two scaling functions are presented. They are compactly supported, orthogonal and symmetric, whereby arbitrary approximation orders (vanishing moments) are possible. They are almost as smooth as spline wavelets and show good frequency characteristics. The approach of several scaling functions might lead to interesting novelties in the field of multirate filterbanks.

5. REFERENCES

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