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Efficient Numerical Solution Of The Variance-Optimal Hedging Problem In Geometric Lévy Models

Bernhard Antonin Johannes Vesenmayer

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Vorsitzender:		Univ.-Prof. Dr. Folkmar Bornemann
Prüfer der Dissertation:	1.	Univ.-Prof. Dr. Jan Kallsen, Christian-Albrechts-Universität zu Kiel
	2.	Univ.-Prof. Dr. Christoph Schwab, Eidgenössische Technische Hochschule Zürich/Schweiz, (schriftliche Beurteilung)
	3.	Jun.-Prof. Dr. Brigitte Forster-Heinlein

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Zusammenfassung

Das Ziel dieser Arbeit ist es, ein offenes effizientes numerisches Verfahren zur Berechnung des varianz-optimalen Hedgefehlers einer europäischen Option für exponentielle Lévy Prozesse im Martingalfall mit eingehender Fehleranalyse zu entwickeln. Effizient heißt hierbei, dass das Verhältnis Aufwand zu Konvergenzrate gering gehalten wird. Offen bedeutet, dass es die Möglichkeit einer ebenfalls effizienten Erweiterung für bestimmte pfadabhängige Optionen gibt.

Dies geschieht auf Basis des Verfahrens von [MSW06]. In dieser Arbeit wird sich dazu auf das Problem der Berechnung des Hedgefehlers einer europäischen Option beschränkt. Falls die zugrundeliegende Aktie durch eine geometrische Brownsche Bewegung modelliert wird, führt dies zu einem vollständigen Markt, in welchem alle solche Zufallsvariablen exakt dupliziert werden können. Die Nachteile einer solchen Modellierung in Bezug auf das Risiko bei großen Marktbewegungen führten zu Modellen, in welchen auch Sprünge erlaubt sind. Dadurch entstehen unvollständige Märkte, in welchen die europäische Option im allgemeinen nicht dupliziert werden kann. In diesem Fall wird klassischerweise versucht, den entstehenden folgenden *varianz-optimalen Hedgefehler*

$$J_0 := E((H(S_T) - v + \vartheta \bullet S_T)^2)$$

über allen zulässigen Hedgingstrategien ϑ und allen möglichen Anfangsinvestitionen v zu minimieren. Hierbei sei H die Auszahlungsfunktion der Option, S der diskontierte Aktienpreisprozess, der Malpunkt repräsentiere die stochastische Integration, und T sei der Zeithorizont. In diesem Falle entspricht die Berechnung des Hedgefehlers einer Projektion des Optionspreisprozesses $V_t := E(H(S_T)|\mathcal{F}_t)$ auf einen Raum stochastischer Integrale.

Das Hedging Problem wurde schon eingehend untersucht. Einen Überblick hierüber liefern [Pha00] und [Sch01b]. Im weiteren Verlauf wurden für verschiedene Prozessklassen mehrere mehr oder minder explizite Darstellungen der optimalen Hedgingstrategie und des Hedgefehlers entwickelt. In [BL89] wird hierzu der carré-du-champ Operator verwendet, der Malliavin Kalkül ist die Grundlage für die Darstellung in [BNLk$^+$03], [HPS01] verwendeten für eine Klasse von stetigen Modellen einen partiellen Differentialgleichungsansatz, und in [HKK06] ist die Laplace Transformation maßgebend. In [ČK07] wurden weitere Darstellungen auch für den Nicht-Martingal Fall entwickelt.

Explizit berechnet wurde der Hedgefehler als solcher z.B. in [CTV05], [HPS01] und in [HKK06] mit einer Erweiterung für stochastische Volatilitätsmodelle von [Pau07]. Er-

stere verwenden dazu ein rechenintensives Monte-Carlo Verfahren. Im zweiten Ansatz, der für eine Klasse von stetigen Modellen gezeigt wurde, wird eine partielle Differentialgleichung mit einem Finite Differenzen Verfahren gelöst. Letztere verwenden die Integraltransformationsmethode, um einen Ausdruck für J_0 als komplexes Doppelintegral zu gewinnen. Ein explizites numerisches Verfahren mit Fehleranalyse wurde hierzu jedoch noch nicht vorgestellt, und es besteht hier keine Erweiterungsmöglichkeit zu pfadabhängigen Optionen.

In dieser Arbeit wird nun eine neue numerische Methode basierend auf der Darstellung des Hedgefehlers mit Hilfe des carré-du-champs Operators nach [BL89] entwickelt und mit Fehlerabschätzungen versehen. Wir beschränken uns in dieser Arbeit jedoch auf den eindimensionalen exponentiellen Lévy Prozess als Aktienkursmodell im Martingalfall.

Für glatte Auszahlungsfunktionen H^ϵ wird der entsprechende Hedgefehler J^ϵ dazu auf eine neue Weise repräsentiert, welche auf die Ergebnisse von [ČK07] zurückgreift und eine Entsprechung von [HPS01] für den Lévy-Fall darstellt. Und zwar als Lösung einer parabolischen Integro-Differentialgleichung, welche die Optionspreisfunktion als Datum verwendet. Der Hedgefehler J_0^ϵ ist nämlich gegeben durch $J_0^\epsilon = J^\epsilon(T, S_0)$, wobei $J^\epsilon(t,x)$ die folgende Gleichung löst:

$$\begin{aligned} \frac{\partial}{\partial t} J^\epsilon(t,x) + AJ^\epsilon(t,x) &= \psi(V^\epsilon, V^\epsilon)(t,x), \quad \forall (t,x), \\ J^\epsilon(0,x) &= 0, \quad \forall x. \end{aligned}$$

Hierbei bezeichnet A den Generator von S und

$$\psi(V^\epsilon, V^\epsilon) = \tilde{c}^{V^\epsilon} - (\tilde{c}^{SV^\epsilon})^2 (\tilde{c}^S)^{-1},$$

wobei hier $\tilde{c}$ als Funktion aus der modifizierten differentiellen Semimartingalcharakteristik von (S, V^ϵ) gewonnen werden kann. Die Lösung des usprünglichen Hedging Problems wird dann durch Lösungen des Problems unter Verwendung der approximativen Auszahlungsfunktionen H^ϵ approximiert.

Wegen der Ähnlichkeit zur bekannten Kolmogorowschen Rückwärtsgleichung, welche verwendet wird, um den Optionspreisprozess V zu bestimmen, kann nun die effiziente numerische Behandlung dieser Art von Differentialgleichungen aus [MSW06] angewendet werden. Dies wird so bewerkstelligt, dass die Implementierung lediglich als Zusatz zu derjenigen des Optionspreises realisiert werden kann. Das heißt, es werden hierzu nur Objekte verwendet, welche leicht aus den für die Optionspreisberechnung vorher assemblierten gewonnen werden können.

Ebenso wie in der obigen Referenz wird die Gleichung nun zunächst im Ort lokalisiert und dann in eine variationelle Form unter Verwendung des N-dimensionalen diskreten Raumes X_h gebracht. Dieser besteht aus allen insgesamt stetigen Funktionen, welche eingeschränkt auf ein Teilintervall durch ein Polynom vom Grad p beschrieben werden. Der übliche Finite-Elemente Ansatz führt aber in diesem Falle zu vollbesetzten Matrizen. Daher wird eine Matrixkompressionsmethode eingesetzt, welche die Zahl der nichttrivialen Einträge auf $O(N \log N)$ reduziert. Die Assemblierung der rechten

Seite, d.h. die Berechnung von $(\psi(V^\epsilon, V^\epsilon), v), v \in X_h$, wird als möglicher Zusatz zur Implementierung der Optionspreisberechnung realisiert. Insgesamt beläuft sich der Rechenaufwand der Assemblierung der Gleichung und deren Lösung mit Hilfe des GMRES Verfahrens auf $O(N(\log N)^7)$ Rechenschritte pro Zeitpunkt. Für die fehlende Zeitdiskretisierung wird nun das unstetige Galerkin Schema unter Ausnutzung der Analytizität der Lösung eingesetzt. Zusammen mit der glatten Annäherung durch H^ϵ beläuft sich damit Gesamtaufwand immer noch auf $O(N(\log N)^8 \epsilon^{-(6+\delta)\varrho})$ Rechenschritte, während der Fehler als Potenz der Gitterweite abgeschätzt werden kann. Hierbei bezeichnet $0 < \varrho \leq 2$ die Ordnung von A und ϵ den Parameter der Glättung von H zu H^ϵ.

Letztlich werden dann noch Implementierungsdetails erörtert, während schließlich numerische Experimente vorgestellt werden. Dazu werden die Ergebnisse der Berechnung mit den entsprechenden Funktionen, welche mit Hilfe der Integraltransformationsmethode gewonnen wurden, verglichen. Damit können die Konvergenz und deren Geschwindigkeit, welche vorher theoretisch ermittelt wurden, anhand dieser Ergebnisse nachgewiesen werden.

Abstract

The aim of this thesis is to provide an open and efficient numerical method for the computation of the variance-optimal hedging error of a European option for exponential Lévy models in the martingale case using the method of [MSW06] together with a thorough error analysis. Efficient in this case means that the ratio between complexity and order of convergence is kept small while open means that there is the possibility of generalizing it to certain path-dependent options.

More specifically, in this thesis the problem of computing the hedging error of a European option is considered. If the underlying is modeled via a geometric Brownian motion, this leads to a complete market, where every such claim can be replicated. But the shortcomings of such models in representing the risk related to large market movements have led to models of the underlying which allow for jumps. Those lead to incomplete markets, where the replication of a European option claim is typically impossible. In this setting the classical approach is to minimize the *variance-optimal hedging error*

$$J_0 := E((H(S_T) - v + \vartheta \bullet S_T)^2)$$

over all reasonable hedging strategies ϑ and possibly all endowments v. Here, H represents the payoff function, S the discounted price process of the underlying, the dot refers to stochastic integration, and T is the time horizon. In this case the computation of the hedging error J_0 amounts to computing the projection of the option price process $V_t := E(H(S_T)|\mathcal{F}_t)$ onto a space of stochastic integrals.

The hedging problem has already been extensively studied. An overview over the literature is given in [Pha00] and [Sch01b]. In due course several more or less explicit representations of the hedging strategy and the error have been developed. [BL89] have provided an expression using the carré-du-champ operator, the Malliavin derivative is used in [BNLk$^+$03], the approach in [HPS01] is based upon PDE representation, and Laplace transforms are used in [HKK06]. In [ČK07] several representations in the general semimartingale setting are given, where S does not have to be a martingale.

Explicit computation of the hedging error was done for instance in [CTV05], [HPS01] or in [HKK06] with a generalization by [Pau07] for stochastic volatility models. The first uses an expensive Monte-Carlo simulation to get the results. The second approach is based upon a PDE representation and solved by applying a finite difference scheme. The last approach uses an integral transformation method, thus developing an expression for J, which can be solved by computing a complex double integral. However, for this method an explicit numerical scheme with error analysis has up to now not been

presented. Furthermore, there is no way of extending the approach to path-dependent options.

In this thesis a new method shall be presented that allows for an efficient numerical treatment. It is based upon the representation of the hedging error of [BL89]. The thesis, however, will be restricted to European options having as underlying an exponential Lévy process and it will be studied under the martingale measure in one dimension.

For smooth payoff functions H^ϵ the corresponding hedging error J_0^ϵ is expressed in a new way, which is a kind of adaptation of [HPS01] to the Lévy case, using the results of [ČK07]. This is done in terms of a parabolic integro-differential equation, which uses the option price V^ϵ as data. More specifically, the hedging error is given by $J^\epsilon = J^\epsilon(T, S_0)$, where $J^\epsilon(t,x)$ solves the following initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} J^\epsilon(t,x) + AJ^\epsilon(t,x) &= \psi(V^\epsilon, V^\epsilon)(t,x), \quad \forall (t,x), \\ J^\epsilon(0,x) &= 0, \quad \forall x. \end{aligned}$$

Here, A denotes the generator of S and

$$\psi(V^\epsilon, V^\epsilon) = \tilde{c}^{V^\epsilon} - (\tilde{c}^{SV^\epsilon})^2 (\tilde{c}^S)^{-1},$$

where $\tilde{c}$ as function can be derived from the modified differential semimartingale characteristics of (S, V^ϵ). The solution to the original hedging problem is then obtained by approximation with the solutions to the problems corresponding to smooth payoff functions H^ϵ.

Due to the strong resemblance to the well-known Kolmogorov backward equation used to obtain the option price V, the efficient numerical treatment developed in [MSW06] is adapted. This is done in such a way that the implementation can be realized as add-on to the option price implementation. That means, only objects are used which can easily be assembled with the already implemented ones for the option price backward equation.

Along the lines of [MSW06] the equation is first spatially localized and then cast into a variational setting with the finite element space X_h of order p and of dimension N. That means on each interval of the discretization acts a polynomial of degree p. However, the usual finite element approach in space discretization results in equation systems with densely-populated matrices. A wavelet compression technique deals with this problem and reduces the number of non-trivial entries of the matrix to $O(N \log N)$. The assembly of the right hand side, i.e. the computation of $(\psi(V^\epsilon, V^\epsilon), v), v \in X_h$, is realized as possible add-on to the implementation of the option price computation. The overall assembly and solution of the semi-discrete problem (i.e. only discretized in space) via GMRES amounts to a complexity of $O(N(\log N)^7)$. Time discretization is done via the discontinuous Galerkin scheme. Including the smooth approximation via H^ϵ this results in an overall complexity of $O(N(\log N)^8 \epsilon^{-(6+\delta)\varrho})$, while still maintaining an error estimate which is a power of the spatial mesh width. Here, $0 < \varrho \leq 2$ is the order of the operator A and ϵ is the parameter corresponding to the smooth approximation from H to H^ϵ.

Implementation issues are presented and finally numerical experiments are given. Here, the option price function, the trading strategy as well as the hedging error function are visualized. They are compared with the corresponding functions that are computed with the integral transform method of [HKK06] thus showing the claimed convergence and its order.

Contents

Notational Remarks

Throughout the thesis we choose a sufficiently small constant $\delta > 0$ which may only depend upon global parameters like $\sigma^2, \eta, \varrho, \nu$ and the kernel k. Additionally, let $C = O(1)$ with respect to the relevant parameters h, ϵ, t, d or the variables that are involved in the estimations like x, y, z or the corresponding functions like f, g, u, v. That means C denotes a constant that is independent of these, but can stand for different numbers within one computation. C may, however, depend upon constant global parameters like $\sigma^2, \eta, \varrho, \nu, \delta$, the kernel k and local ones like $\omega, \omega_f, \omega_g$. The derivative of $f : \mathbb{R} \to \mathbb{R}$ is denoted by f' or by Df. For $f : \mathbb{R}^d \to \mathbb{R}$ we denote by $D_i f$ the derivative with respect to the i-th component. For higher derivatives we also use $f^{(k)}$. Sometimes the time derivative is denoted by $\dot{f}$. The subscript on L^2_x denotes the dimension in which the norm is to be applied. If not stated otherwise the domains of the functions of the spaces L^k, H^s and C^s are the whole real line $\mathbb{R}$. For vectors $v \in \mathbb{R}^d$ the following usual norms are used:

$$
\begin{aligned}
\|v\|_1 &:= \sum_{i=1}^{d} |v_i|, \\
\|v\|_2 &:= \left(\sum_{i=1}^{d} |v_i|^2 \right)^{1/2}, \\
\|v\|_\infty &:= \max_{i=1,\dots,d} |v_i|.
\end{aligned}
$$

The induced operator norms for matrices $A \in \mathbb{R}^d \times \mathbb{R}^d$ are given or estimated by

$$
\begin{aligned}
\|A\|_1 &= \max_{j=1,\dots,d} \sum_{i=1}^{d} |a_{ij}|, \\
\|A\|_\infty &= \max_{i=1,\dots,d} \sum_{j=1}^{d} |a_{ij}|, \\
\|A\|_2 &\leq \sqrt{\|A\|_1 \|A\|_\infty}.
\end{aligned}
$$

Otherwise, $\|\cdot\|_Y$ denotes the norm corresponding to the space Y and $\|\cdot\|_{X\to Y}$ the operator norm for bounded linear mappings from X to Y. The topological dual of spaces Y shall be denoted by Y^*. Further unexplained notation can be found in [JS03] and [AF03].

Chapter 1

Introduction

In this thesis we deal with European options. This is an agreement to pay an amount of money that deterministically depends upon the future price of a certain asset, the so-called underlying, at a fixed point of time T in the future, the so-called maturity. In the generic situation of a European call or put option this can be looked upon as the right to buy or sell the asset at time T for a previously fixed price. For the seller of such an option there are basically three questions which naturally arise in this context.

1. What price to ask?
2. What can be done against the risk?
3. How well can this be done?

The first question is the problem of option pricing which has been studied intensively. The latter questions form the so-called hedging problem. This problem will be focused in the thesis. We will not take into account transaction costs and therefore assume that trading with the asset solely depends upon the associated price at the corresponding point of time. Obviously, the solution of those problems still heavily depends upon the chosen model.

At first, this asset price was modeled with a geometric Brownian motion, cf. [Osb59] and [Sam65]. This leads to a so-called complete market. That means, every contingent claim can be perfectly replicated. In other words, there exists a self-financing trading strategy, the so-called hedging strategy, and an option price such that the following holds. If the seller of the option invests the money he received in the beginning in the underlying and trades according to the hedging strategy without having to invest additional money, he will come up with exactly the amount of money he needs to pay the buyer of the option. This option price and the corresponding self-financing trading strategy are explicitly given by the famous Black-Scholes formulas, cf. [BS73] and [Mer73]. In this setting the solution of the hedging problem is therefore given by those formulas without a remaining risk.

However, this model does not meet important statistical effects, that are observed in real market data, cf. [Con01] or [Sch03, Section 4.1.2]. Main points of critic are the symmetry of the normal distribution, the light tails that are implied by the model and the constant volatility. That means, in real market data gains and losses occur in a asymmetric fashion, the probability of extreme events, such as big price movements, are underestimated in the model and drastic changes of volatility occur in real market data.

To overcome the first two of these shortcomings, more general models which allow for jumps have been introduced. The first to be considered to this end were the so-called Lévy processes. With the jump measure they have an additional parameter and therefore are more flexible. In spite of that they are still tractable processes, since they have stationary and independent increments. However, due to that they are not apt to model the changes of volatility. Nevertheless, they are now widely used in financial mathematics. Examples are the Variance Gamma model, cf. [MS90], the CGMY model as generalization, cf. [CGMY02], the Normal Inverse Gaussian model, cf. [BN95], and its generalization, the Generalized Hyperbolic Processes, cf. [Pra99],[EP02] and [EvH04], and the Meixner model, cf. [Sch01a]. An overview of their application in Finance can be found in [Sch03].

Allowing for jumps leads to an incomplete market. That means, in general it is now not possible to perfectly replicate the option. Therefore other approaches for option pricing and the measurement of the risk involved are necessary. One approach is to consider the market consisting of the option and the underlying and to determine the interval of prices such that the market is free of arbitrage. This is the basis for the so-called Superhedging approach, where the price and the hedging strategy are chosen in such a way that the issuer of the option runs no risk of losing money. That means, one has to choose at least the highest possible option price, cf. [EKQ95]. However, it has been shown in [HKK06] that in most cases this approach does not lead to practicable solutions. Particularly, for exponential Lévy models [EJ97] have shown that there are only trivial Superhedging strategies, meaning that the option price is chosen so high that it suffices to invest all of this money in the underlying and thereafter do no trading.

Another approach is to consider the portfolio with the option and a variable number of positions of the underlying and apply portfolio optimization. That means the option price and the number of positions of the underlying at each point of time are chosen as to optimize the value of the portfolio under a certain utility function, cf. [Kal02] or [FL00]. This yields the optimal price and the optimal hedging strategy.

The tractability of this approach depends upon the choice of the utility function. A very tractable choice is a quadratic utility function. This corresponds to the approach of quadratic hedging, which comprises the (local) risk minimization and the variance-optimal approach. The first allows non-self-financing trading strategies, that means additional investment during the time interval is allowed. But then it demands that the contingent claim is met perfectly. The second holds on to the self-financing property of the hedging strategy. The option price v and the hedging strategy ϑ are now chosen to minimize the following remaining hedging error J_0:

$$J_0 := E((H(S_T) - v + \vartheta \bullet S_T)^2),$$

where S denotes the price process of the underlying and $H(S_T)$ the contingent claim of the European option. An overview over the corresponding literature is given in [Pha00] and [Sch01b].

From an economical point of view the use of the quadratic utility function could be doubted. As long as the trading does not render a higher value than the contingent claim, higher losses lead to a lower utility value, which meets economical understanding of the problem. However, if there exists a strategy which renders a higher value than the contingent claim, the corresponding gain is punished by the quadratic function. But apart from this possible shortcoming the quadratic approach leads to tractable mathematical structures and to linear hedging, i.e. n options can be hedged using $n\vartheta$ as hedging strategy, where ϑ is the hedging strategy for one option, cf. [CV05a]. This property is inherent to the quadratic approach and appreciated in praxis.

This thesis focuses on the variance-optimal hedging problem. In this setting the hedging problem has already been extensively studied. We will only deal with the martingale setting. More specifically, the price process S is assumed to be a square integrable martingale. Roughly speaking, this means that the best guess for the price of S in the future is the current one. Since [FS86] it is known that in this setting the solution of the hedging problem corresponds to a projection in the Hilbert space of square integrable martingales and can be expressed via the Galtchouk-Kunita-Watanabe decomposition, cf. [KW67].

Many different approaches have come up with more or less explicit representations of the solution of the hedging problem. [BL89] have provided an expression using the carré-du-champ operator, the Malliavin derivative is used in [BNLk$^+$03] and Laplace transforms in [HKK06]. In [ČK07] several representations in the general semimartingale setting are given even for the non-martingale case using the semimartingale characteristics. For certain continuous models [HPS01] used a PDE representation. In [CV05a] the option price is derived using a partial integro differential equation (PIDE) approach and the trading strategy is expressed via a singular integral involving the option price.

Based on these representations the solution can be computed numerically. For the integral transformation method of [HKK06] this can be done using a Fast Fourier Transform scheme, which yields the option price, the hedging strategy and the remaining hedging error. However, a numerical study with error analysis has not yet been presented and this approach can not be extended to path-dependent options.

We will in the sequel develop a numerical scheme based upon a new representation of the hedging error as a solution to a PIDE. The representation is similar to the one obtained in [HPS01] but now we allow for jumps. This scheme is efficient in that way that the order of convergence with respect to the complexity performs better than a comparable finite difference scheme similar to the one in [CV05a]. Furthermore, it will be equipped with a detailed error analysis. Unlike the integral transform method the PIDE approach can be extended to path-dependent options like barrier options, cf. [CV05b, Section 2.2]. In this thesis, however, we restrict the consideration to European options to show the principle and use the overlap of results of our method with the ones obtained by the integral transform method to show mutual convergence.

That means, we show that our results converge against the ones obtained with the other method. Thus, we are able to confirm both, the accuracy of our method and of the integral transform approach.

As mentioned before, we will propose a numerical scheme based upon a PIDE approach for both, the option price and the hedging error. As by-product we derive the hedging strategy as well. A PIDE approach for the option price was already numerically exploited by [CV05a] using a finite difference scheme and by [MSW06] using a Galerkin scheme. Making use of the semi-explicit formula for the hedging strategy, in [CTV05] the remaining hedging error is then derived via a Monte-Carlo simulation. That means, a large number of sample paths are generated and trading is done according to the hedging strategy. The resulting hedging errors are duly averaged yielding the estimate for the hedging error. In terms of computation steps a Monte-Carlo simulation is therefore expensive. In general, it takes several hours of computation time in order to come up with an accurate result with while PIDE solvers yield such results within minutes.

Especially with regard to the previous argument in favor of our scheme, it is natural to ask why it is useful to have a fast algorithm for the computation of the variance-optimal hedging error or in other words what this object can be used for in practice. Besides the use as measure for the remaining risk involved with the hedging of a European option it appears in formulas for other financial problems. In [ČK07, Section 1.2] it is shown how it can be used to compute the Sharpe ratio. This is a measure of the excess return per unit of risk in an investment asset or a trading strategy. It is used in practice to rank the performance of portfolios. Furthermore, in [KR08] it is stated that the quadratic hedging error is the key to exponential utility indifference pricing. The value of these two concepts for practical use are generally acknowledged.

The structure of the thesis is as follows. In the first chapter we state for the convenience of the reader some general results of the different mathematical areas that will be needed in the sequel. The following chapter is dedicated to the formulation of the hedging problem and derives a first representation of the solution as functions of the underlying. In chapter 4 the smoothness and integrability properties of these functions are shown which allow for their representation as solutions of PIDEs. This PIDE is taken care of numerically in the ensuing chapter following the method of [MSW06]. After formulation of the numerical scheme some objects remain to be computed which require additional numerical tools. The corresponding approach is presented in chapter 6. The last chapter concludes the analysis with numerical experiments which compare the results with the ones computed with the method of [HKK06]. They show that our results converge to the results obtained via the other method and confirm the order of that convergence. At the beginning of each chapter there will be a short non-rigorous summary of the ensuing with additional remarks. Only reading those “main threads” should suffice to understand the main ideas of the thesis. This is meant to serve as apology to the reader for the many oblong calculations which could not be avoided.

Chapter 2

Preliminaries

Main thread. *Within this thesis we will make use of different representations of the objects of interest. At first the option price, the trading strategy and the hedging error are expressed in terms of a Lévy process. This can be interpreted as a special case of a Markov processes and the same time as a semimartingale. Therefore, we use a representation of the solution via semimartingale characteristics. These describe the local behavior of the corresponding process and are generally easy to be computed using the calculation rules that are given here. Then the Markov property of the Lévy process is used to represent those as deterministic functions of the underlying. The corresponding generator A, which also describes the local dynamics of the Markov process, will basically be the operator that forms the PIDE.*

However, these functions of interest are functions of space and time. The spaces which are usually used in that setting for the analysis of partial differential equations are the Bochner spaces. That means, existence and uniqueness results are given in terms of these spaces. They are a generalization of L^p spaces to spaces of functions which take values in a Banach space. More specifically, functions $f : \mathbb{R}^2 \to \mathbb{R}; (t, x) \mapsto f(t, x)$ will be interpreted as functions $f : \mathbb{R} \to X; t \mapsto f_t$ where X is a corresponding Banach space of functions $\mathbb{R} \to \mathbb{R}$.

Particularly when dealing with polynomial interpolation we have to deal with the usual function spaces C^k and in order to derive sharp estimates we will use a fractional generalization, the so-called Hölder spaces. In order to show that they are solutions of PIDEs, smoothness and integrability of the deterministic functions and therefore of the operators that are used for their representation have to be shown. To this end techniques from the theory of Sobolev spaces are used, because the Sobolev norms just measure smoothness and integrability. The fractional Sobolev spaces that are applied in this analysis can be obtained as generalizations of the integer ones using several different approaches. Either a fractional version of the finite difference is used, which allows for an explicit representation in terms of the function. Or the corresponding fractional multiplier in Fourier space is used, which is apt to the analysis of the bilinear operators we will have to deal with. However, in the later discrete setting this method can not successfully be applied. But the fractional Sobolev spaces can also be obtained using an interpolation method, which allows to estimate the fractional norms with *integer ones. In the discrete spaces this will therefore be the approach of choice.*

As soon as the PIDE is derived it can be further treated with methods of functional analysis. The PIDE will consist of a time derivative and an operator A which only affects space. The solution can then be expressed with Duhamel's formula, that means basically as the image of the exponential of the operator A. During the discretization the operator will be exchanged with discrete ones. That means the error of this step can be estimated via the norm of the difference of the corresponding exponential operators. Thus, the properties of A play the key role in the analysis of the method. And again two distinct approaches will be used. For the non-discrete spaces a Fourier technique can be applied. It turns out that A can then be represented as a pseudo differential operator. The exponential, however, will be represented via a complex contour integral of the resolvent.

In this chapter we summarize for the convenience of the reader some basic results of the different mathematical areas that will be used in the sequel. This presentation shall only state the basic ideas that will be used and will therefore state no proofs. For detailed treatment of these theories we refer to the references that are given in the corresponding section.

2.1 Bochner spaces

The canonical spaces when dealing with parabolic differential equations are the so-called Bochner spaces. They are a generalization of L^p spaces to vector-valued functions. That means we consider functions

$$f : J \to X; t \mapsto f(t),$$

where $J \subset \mathbb{R}$ is a - possibly infinite - interval, and X is some Banach space. For the integral with respect to t the usual Lebesgue integral is too restrictive. Therefore, a new notion of integral has to be introduced, the *Bochner integral.* For references of the definition and the following results we refer to [EN00, VII/ Appendix C], particularly for the definition of Bochner-measurable functions and the Bochner-integral. Here, we only state the basic properties which will be used in the analysis.

Proposition 2.1.1 *A Bochner-measurable function $f : J \to X$ is Bochner-integrable iff $\int_J \|f(s)\|_X ds < \infty$, where $\|\cdot\|_X$ denotes the norm of the Banach space X. Furthermore, we have*

1. *the triangle inequality: $\| \int_J f(s)ds\|_X \leq \int_J \|f(s)\|_X ds$,*
2. *Fubini's theorem,*
3. *Lesbesgue's dominated convergence theorem.*

Proof. See the references after [EN00, VII/ Appendix C.3]. □

Furthermore, we have that Bochner integration and the application of a closed operator, like the one we will introduce in the last section of this chapter, commute, cf. [EN00, VII/ Appendix C.4].

Proposition 2.1.2 *Let $A : D(A) \subset X \to Y$ be a closed operator acting between two Banach spaces X, Y. If $f : J \to X$ is a Bochner integrable function with $f(s) \in D(A)$ for almost all $s \in J$ and if $Af : J \to Y$ given by $(Af)(s) := Af(s)$ is integrable, then $\int_J f(s)ds \in D(A)$ and*

$$A\left(\int_J f(s)ds\right) = \int_J Af(s)ds.$$

The Bochner spaces are now defined as follows, cf. [EN00, VII/ Appendix C.5/C.6]

Definition 2.1.3 If we identify functions $f : J \to X$ that are λ-almost surely equal, where λ is the Lesbesgue measure on J, then the Bochner norms are defined as follows for $1 \leq p < \infty$:

$$\begin{aligned} \|f\|_{L^p(J;X)} &:= \left(\int_J \|f(s)\|_X^p ds\right)^{1/p}, \\ \|f\|_{L^\infty(J;X)} &:= \operatorname*{ess\,sup}_J \|f\|_X. \end{aligned}$$

The corresponding Bochner spaces are consequently defined as follows:

$$\begin{aligned} L^p(J;X) &:= \{f : J \to X; f \text{ is measurable and } \|f\|_{L^p(J;X)} < \infty\}, \\ L^\infty(J;X) &:= \{f : J \to X; f \text{ is measurable and } \|f\|_{L^\infty(J;X)} < \infty\}. \end{aligned}$$

If we set $H^0(J,X) := L^2(J,X)$, then we can define the corresponding higher order Sobolev spaces for $k \in \mathbb{N}$ as follows:

$$\begin{aligned} H^k(J,X) &:= \Big\{f \in H^{k-1}(J,X); f(t) = f(s_0) + \int_{s_0}^t g(s)ds \\ &\qquad \text{for some } s_0 \in J \text{ and } g \in H^{k-1}(J,X)\Big\}. \end{aligned}$$

The corresponding norms are now defined by

$$\|f\|_{H^k(J,X)} := \|f\|_{H^{k-1}(J,X)} + \|g\|_{H^{k-1}(J,X)}.$$

2.2 Hölder and Sobolev spaces

In the analysis we want to measure smoothness and integrability of functions. There are basically two approaches we choose to this end. For the first part of the analysis, that means Chapter 4, we use derivatives in the strong sense. More specifically, we consider the fractional analogues of the space C^r. These are the so-called Hölder spaces. They are defined as follows for $r \geq 0$ and $r = [r] + \{r\}$, where $0 \leq \{r\} < 1$:

$$\begin{aligned} \|f\|_C &:= \sup_{x\in\mathbb{R}} |f(x)|, \\ \|f\|_{C^{[r]}} &:= \sum_{k=0}^{[r]} \|D^k f\|_C, \\ \|f\|_{C^r} &:= \|f\|_{C^{[r]}} + \sup_{y\neq 0} \frac{|D^{[r]}f(x+y) - D^{[r]}f(x)|}{|y|^{\{r\}}}, \\ C^r &:= \left\{ f \in C^{[r]}; \|f\|_{C^r} < \infty \right\}. \end{aligned}$$

The weighted version is accordingly defined via $\|f\|_{C^r_\omega} := \|e^{\omega x} f\|_{C^r}$. For $\omega = 0$ we omit the subscript.

We will deal with functions depending upon time t and will have to study the smoothness properties with respect to time t of functions g of the following form

$$g : [0,T] \times \mathbb{R} \to \mathbb{R}; g(t,x) := \int_{\mathbb{R}} f(t,x,y)dy.$$

The following two propositions are direct consequences of the dominated convergence theorem when g is continuous with respect to t, respectively when differentiation under the integral sign can be done.

Proposition 2.2.1 *Let $J \subset \mathbb{R}$ be a closed interval and $f : J \times \mathbb{R} \to \mathbb{R}$ such that*

- *$f(t,\cdot) \in L^1$ for all $t \in J$,*
- *$f(\cdot,x) : J \to \mathbb{R}$ is continuous in $t_0 \in J$ for almost all $x \in \mathbb{R}$,*
- *there exists a neighborhood U of t_0 and an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that $|f(t,x)| \leq g(x)$ for all $t \in J \cap U$ and almost all $x \in \mathbb{R}$.*

Then the function $F : J \to \mathbb{R}$,

$$F(t) := \int_{\mathbb{R}} f(t,x)dx$$

is continuous in t_0.

Proof. [Els07, IV.5.6]

Proposition 2.2.2 *Let $J \subset \mathbb{R}$ be a closed interval and $f : J \times \mathbb{R} \to \mathbb{R}$ such that*

- *$f(t,\cdot) \in L^1$ for all $t \in J$,*

- *there exists* $\epsilon > 0$ *such that the partial derivative* $\frac{\partial}{\partial t} f(t,x)$ *exists for all* $t \in U := (t_0 - \epsilon, t_0 + \epsilon) \cap J$ *and all* $x \in \mathbb{R}$,
- *there exists an integrable function* $g : \mathbb{R} \to \mathbb{R}$ *such that*

$$\left| \frac{\partial}{\partial t} f(t,x) \right| \leq g(x)$$

for all $t \in J \cap U$ *and all* $x \in \mathbb{R}$.

Then the function $F : J \to \mathbb{R}$,

$$F(t) := \int_{\mathbb{R}} f(t,x)dx$$

is differentiable in t_0, *and we have*

$$F'(t_0) = \int_{\mathbb{R}} \frac{\partial}{\partial t} f(t_0,x)dx.$$

Proof. [Els07, IV.5.7]

For the main part of the analysis this notion of smoothness is too strong. Therefore, we use a weak version of derivatives and their respective fractional analogues. More specifically, we consider the so-called Sobolev spaces. They are commonly used whenever dealing with partial differential equations. They coincide with other function spaces and can be embedded in generalized function space scales. These Sobolev spaces have been intensely studied in [AF03]. An overview over several function space scales and their appearance in the more general Triebel-Lizorkin spaces can be found in [RS96]. Their application in connection with the theory of pseudo differential operators was documented in [BL02] and [Jac96].

Before we start with the introduction of Sobolev spaces we first introduce the Fourier transform which will play a key role for measuring smoothness of functions. Here we follow [Rud91, Chapter 7]. We start by defining the Fourier transform on L^1. Here, L^1 denotes all integrable complex-valued functions. Let $f \in L^1$. We define its *Fourier transform* $\mathcal{F}f := \hat{f}$ by

$$\hat{f}(z) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-izx} f(x)dx.$$

For this transform we have the following well-known properties.

1. The *inversion formula* (cf. [Rud91, Theorem 7.7]): Let $f \in L^1$. If we denote

$$(\mathcal{F}^{-1}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{izx} f(z)dz$$

we have

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}f)(x) \quad \lambda - \text{a.s.}$$

Particularly this implies, that $\mathcal{F}^{-1}(\mathcal{F}f) = f$ for $f \in C^0 \cap L^1$.

2. *Convolution formula* (cf. [Rud91, Theorem 7.2]): For $f, g \in L^1$ define the convolution by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Then we have

$$\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g},$$

and for $f, g \in L^1 \cap L^2$

$$\widehat{fg} = \sqrt{2\pi}\hat{f} * \hat{g}.$$

Since L^1 is dense in L^2, the space of all square integrable complex-valued functions, this transformation $\mathcal{F}$ can be extended to $\mathcal{F} : L^2 \to L^2$, cf. [Rud91, Theorem 7.9 ff.].

Theorem 2.2.3 *There is a linear isometry Ψ of L^2 onto L^2 which is uniquely determined by*

$$\Psi f = \mathcal{F}(f) \quad \forall f \in L^1.$$

Furthermore, we have Parseval's formula for the scalar product $(\cdot,\cdot)_{L^2}$ in L^2:

$$(f, g)_{L^2} = (\Psi f, \Psi g) \quad \forall f, g \in L^2.$$

This linear isometry Ψ is sometimes called the *Fourier-Plancherel* transform and will be considered the extension of $\mathcal{F}$ from $L^1 \cap L^2$ to L^2. In the sequel it will also be denoted by $\mathcal{F}$, respectively by $\hat{\cdot}$. Next, we will further extend the Fourier transform onto the so-called space of tempered distributions. This is defined as follows.

Definition 2.2.4 The space of *rapidly decreasing functions* $\mathscr{S}$ also known as the *Schwartz space* is the space of all functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{\mathscr{S},N} \quad := \quad \sup_{k \le N} \sup_{x \in \mathbb{R}} (1+x^2)^N |D^k f(x)| < \infty$$

for $N \in \mathbb{N}_0$. Together with the countable set of semi-norms $(\|\cdot\|_{\mathscr{S},N})_{N \in \mathbb{N}}$ this forms a locally-convex topological space, cf. [Rud91, Theorem 1.37]. The dual space with respect to this topology,

$$\mathscr{S}' := \{f \in \mathscr{L}(\mathscr{S}, \mathbb{C}); f \text{ continuous }\},$$

is called *space of tempered distributions.*

The Fourier transform for $f \in \mathscr{S}'$ is now defined by

$$\mathcal{F}(f)(\Phi) := f(\mathcal{F}(\Phi)) \quad \forall \Phi \in \mathscr{S}.$$

We have $\mathcal{F} : \mathscr{S}' \to \mathscr{S}'$ is continuous with a continuous inverse, cf. [Rud91, Definition 7.14 ff]. Furthermore, it is an extension of the Fourier-Plancherel transform on $L^2 \subset \mathscr{S}'$ as well as the Fourier transform on $L^1 \subset \mathscr{S}'$. That means the respective definitions coincide on L^2 and L^1.

Now we are able to define Sobolev spaces of fractional order as in [Rud91, Section 8.8]. For $s \in \mathbb{R}$ define

$$H^s \quad := \quad \{f \in \mathscr{S}'; \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s} \quad := \quad \|\mathcal{F}^{-1}\left((1+|\cdot|^2)^{s/2}\mathcal{F}f(\cdot)\right)\|_{L^2}.$$

The space H^s is called *Sobolev space of order s* and the norm is the so-called Bessel potential norm. It is a Hilbert space with scalar product

$$(v,w)_{H^s} := \int_{\mathbb{R}} (1+|z|^2)^s \hat{v}(z)\overline{\hat{w}(z)}dz \quad \forall v,w \in H^s.$$

In the Sobolev setting we have $(H^s)^*$ is isomorph to H^{-s}, where $(H^s)^*$ denotes the topological dual of H^s, cf. [AF03, Theorem 7.63]. More specifically, for every continuous linear form $F \in \mathscr{L}(H^s, \mathbb{C})$ there exists $f \in H^{-s}$ such that for all $v \in H^s$ we have

$$F(v) = (f,v)_{L^2} = \int_{\mathbb{R}} \hat{f}(z)\overline{\hat{v}(z)}dz.$$

On the other hand for every $f \in H^{-s}$ this definition leads to a continuous linear form $F \in \mathscr{L}(H^s, \mathbb{C})$.

Therefore, we can identify $(H^s)^*$ and H^{-s} and define the so-called *duality pairing* $\langle \cdot, \cdot \rangle_{(H^s)^* \times H^s} : H^{-s} \times H^s \to \mathbb{C}$ by

$$\langle f, v \rangle_{(H^s)^* \times H^s} := (f,v)_{L^2} \quad \forall f \in H^{-s}, v \in H^s.$$

However, this is not the only approach that was used to define Sobolev spaces of fractional order. This results in equivalent norms and respective properties that can be deduced for the Bessel potential norm. One equivalent norm is the so-called *Slobodeckij-norm.* Let $r = [r] + \{r\}$, where $0 < \{r\} < 1$, then we have

$$\|f\|_{H^r} \quad \sim \quad \|f\|_{H^{[r]}} + \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(D^{[r]}f(x+y) - D^{[r]}f(x))^2}{|y|^{1+2\{r\}}} dxdy\right)^{1/2}.$$

This norm is similar to the structures we will encounter in the ensuing analysis, up to an additional weighting factor. However, the use of the Bessel potential norm will be favorable with respect to the analysis of the weighted setting that will be introduced in the sequel.

Another way of obtaining Sobolev spaces of fractional order is to use interpolation methods on Sobolev spaces of non-fractional order, cf. [AF03, Section 7.64]. This results in a further equivalent norm which naturally disposes of the so-called interpolation property. That means, this property also holds for the Bessel potential norm.

More specifically, let $0 < \theta < 1$ and $s = (1-\theta)k_1 + \theta l_1, t = (1-\theta)k_2 + \theta l_2$. Let furthermore T be a bounded operator for the following domains and ranges $T : H^{k_1} \to$

H^{k_2} and $T : H^{l_1} \to H^{l_2}$. Then the corresponding operator norm for the intermediate spaces $T : H^s \to H^t$ can be bounded as follows, cf. [AF03, Section 7.64 and Theorem 7.65]:

$$\|T\|_{H^s \to H^t} \quad \leq \quad C\|T\|^{(1-\theta)}_{H^{k_1} \to H^{k_2}} \|T\|^{\theta}_{H^{l_1} \to H^{l_2}}.$$

Particularly, this implies that the respective norms possess the following *interpolation property*:

$$\|f\|_{H^s} \quad \leq \quad C\|f\|^{(1-\theta)}_{H^{k_1}} \|f\|^{\theta}_{H^{l_1}}.$$

The products of two elements of Sobolev spaces have been studied in the setting of the more general Triebel-Lizorkin spaces in [RS96]. We will use the following result in the sequel, which can be found in [RS96, Theorem 4.6.1/1]. Let $0 < s_1 < s_2$, such that $s_2 > 1/2$, then we have the following. There exists a constant $\tilde{C} > 0$ such that

$$\|fg\|_{H^{s_1}} \quad \leq \quad \tilde{C}\|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \quad \forall f \in H^{s_1} \text{ and } \forall g \in H^{s_2}. \tag{2.1}$$

Finally, Hölder spaces can be embedded into Sobolev spaces via the following Sobolev Embedding Theorem also known as Sobolev's Lemma, cf. [BL02, Theorem 15.9 and previous remarks].

Theorem 2.2.5 *Let $s > 1/2$. Then for any $s' < s - 1/2$ we have $H^s \subset C^{s'}$ densely and continuously. This implies that there exists a constant $\tilde{C} \in \mathbb{R}$ such that for $f \in H^s \cap C^{s'}$ we have the following norm estimate.*

$$\|f\|_{C^{s'}} \leq \tilde{C}\|f\|_{H^s}$$

Furthermore, we will use a weighted version of Sobolev spaces, similar to the definitions in [BL02, Section 15.9.2]. To this end we generalize the notion of Schwartz space in the following way.

Definition 2.2.6 Let $\omega \in \mathbb{R}$ and $\mathscr{S}_\omega$ denote the space of C^∞ functions such that $e^{\omega x} u \in \mathscr{S}$. The corresponding topology is induced by the following system of semi-norms:

$$\|u\|_{\mathscr{S}_\omega, N} := \sup_{k \leq N} \sup_{x \in \mathbb{R}} (1 + x^2)^N |D^k(e^{\omega x} f(x))| < \infty.$$

The dual space with respect to this topology shall be denoted $\mathscr{S}'_{-\omega}$.

For $\omega \in \mathbb{R}$ the Fourier transform can be extended to $\mathscr{S}'_{-\omega}$ as follows.

Definition 2.2.7 Let $\omega \in \mathbb{R}$. For $u \in \mathscr{S}'_{-\omega}$ define $\hat{u}(\cdot - i\omega) \in \mathscr{S}'$ by

$$\hat{u}(\cdot - i\omega)(\hat{v}(\cdot + i\omega)) = u(v) \quad \forall v \in \mathscr{S}_\omega.$$

With this we can define following norms and spaces for $\omega \in \mathbb{R}$ and $s \in \mathbb{R}$:

$$H^s_\omega \;:=\; \left\{ f \in \mathscr{S}'_\omega ; \|f\|_{H^s_\omega} < \infty \right\},$$

where

$$\|f\|_{H^s_\omega} \;:=\; \left(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{f}(\xi + i\omega)|^2 d\xi \right)^{1/2}.$$

By [BL02, Theorem 15.29] we have the following properties. Again this is a Hilbert space with scalar product

$$(v, w)_{H^s_\omega} := \int_{\mathbb{R}} (1+|z|^2)^s \hat{v}(z+i\omega)\overline{\hat{w}(z+i\omega)} dz \quad \forall v, w \in H^s_\omega.$$

Let $e^{\omega \cdot} f, e^{\omega \cdot} g \in H^s \cap L^1$ for some $s \in \mathbb{R}$. Since $\mathcal{F}(e^{\omega \cdot} f)(z) = \hat{f}(z+i\omega)$ and for g likewise, we have $(f, g)_{H^s_\omega} = (e^{\omega x} f, e^{\omega x} g)_{H^s}$. For shorter notation we define $\|f\|^2_{H^s_{\omega_1, \omega_2}} := \|f\|^2_{H^s_{\omega_1}} + \|f\|^2_{H^s_{\omega_2}}$ and the spaces correspondingly.

Again, we can identify $(H^s_\omega)^*$ with $H^{-s}_{-\omega}$, and the duality pairing

$$\langle \cdot, \cdot \rangle_{(H^s_\omega)^* \times H^s_\omega} : H^{-s}_{-\omega} \times H^s_\omega \to \mathbb{C}$$

is now given by

$$\langle f, v \rangle_{(H^s_\omega)^* \times H^s_\omega} \;=\; \int_{\mathbb{R}} \hat{f}(z - i\omega)\overline{\hat{v}(z+i\omega)} dz \quad \forall f \in H^{-s}_{-\omega}, v \in H^s_\omega.$$

2.3 Markov processes

The results of this section will not be explicitly used in the ensuing analysis. But they constitute the link between the stochastic formulation involving semimartingale characteristics and the operators that will be used thereafter.

The results of this section are taken out of [Jac96, Chapter 1, 2]. However, in order to be consistent with the notation in this thesis it differs sightly from the one in the reference. More precisely, we denote by $C_0(\mathbb{R}^d)$ the space of continuous functions $\mathbb{R}^d \to \mathbb{R}$ vanishing at infinity and by $C^\infty_c(\mathbb{R}^d)$ the space of arbitrarily often differentiable functions $\mathbb{R}^d \to \mathbb{R}$ with compact support.

We now introduce the notion of Markov processes in the sense of [Jac96, Chapter 1]. A *stochastic process* is the quadruple $(\Omega, \mathcal{F}, P, (X_t)_{t \geq 0})$ where $(\Omega, \mathcal{F}, P)$ is a probability space and $X_t : \Omega \to \mathbb{R}^d$ is a random variable for every $t \geq 0$. A *universal process* is the family $(\Omega, \mathcal{F}, P^x, (X_t)_{t \geq 0})_{x \in \mathbb{R}^d}$ where $(\Omega, \mathcal{F}, P^x, (X_t)_{t \geq 0})$ is a stochastic process for every $x \in \mathbb{R}^d$, for every $B \in \mathcal{F} : x \mapsto P^x(B)$ is a Borel measurable mapping and

$$P^x(X_0 = x) = 1$$

for all $x \in \mathbb{R}^d$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(X_t)_{t\geq 0}$, that means

$$\mathcal{F}_t := \sigma(X_s; s \leq t)$$

is the sub-σ-field of $\mathcal{F}$ generated by the set of random variables $\{X_s; s \leq t\}$.

A universal process is called a *Markov process*, if the following Markov property holds for all Borel sets $A \in \mathcal{B}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$:

$$E^x(1_A(X_{s+t})|\mathcal{F}_s) = E^{X_s}(1_A(X_t)) \quad P^x - a.s.,$$

where $1_A : \mathbb{R}^d \to \{0, 1\}; 1_A(x) = 0 \Leftrightarrow x \in A$ is the usual indicator function and $E^x(Y|\mathcal{F}_s)$ denotes the conditional expectation of the random variable Y with respect to the sub-σ-field $\mathcal{F}_s$ and the probability measure P^x.

In the sequel we will sometimes use the abbreviate notation $((X_t)_{t\geq 0}, P^x)_{x\in\mathbb{R}^d}$ for Markov processes. Given a Markov process $(X_t)_{t\geq 0}$, we can define on $B_b(\mathbb{R}^d)$, the space of bounded Borel functions on $\mathbb{R}^d$, a family $(T_t)_{t\geq 0}$ of operators by

$$T_t : B_b(\mathbb{R}^d) \to B_b(\mathbb{R}^d); T_t u(x) := E^x(u(X_t)).$$

Due to the Markov property, $(T_t)_{t\geq 0}$ is a semigroup of operators, i.e.

$$T_{s+t} = T_s \circ T_t$$

holds for $s, t \geq 0$.

We now focus on the class of Feller processes. A Markov process if called a *Feller process* if the restriction of its corresponding semigroup to $C_0(\mathbb{R}^d)$ is a Feller semigroup. Hereby, a semigroup of operators $(T_t)_{t\geq 0}$ on the Banach space $(C_0(\mathbb{R}^d), \|\cdot\|_C)$ is called a Feller semigroup, if the following conditions are fulfilled:

1. $T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ is a linear contraction, i.e. it is linear and we have $\|T_t u\|_C \leq \|u\|_C$.
2. $\lim_{t\to 0} \|T_t u - u\|_C = 0$, i.e. the semigroup is strongly continuous.
3. $0 \leq u \leq 1$ implies $0 \leq T_t u \leq 1$.

Now we can introduce the central object in studying the properties of Feller semigroups $(T_t)_{t\geq 0}$ on $C_0(\mathbb{R}^d)$, namely its generator. To this end we define the linear subspace

$$D(A) := \{u \in C_0(\mathbb{R}^d); \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists with respect to the topology of } C_0(\mathbb{R}^d)\}$$

and call it the *domain* of the *generator* A of $(T_t)_{t\geq 0}$. The operator A is then defined as follows:

$$A : D(A) \to C_0(\mathbb{R}^d); Au := \lim_{t\to 0} \frac{T_t u - u}{t}.$$

The operator $(A, D(A))$ is a densely defined closed operator on $C_0(\mathbb{R}^d)$ and uniquely determines $(T_t)_{t\geq 0}$.

Under some restrictions the operator A can be expressed as a Pseudo Differential Operator (PDO), cf [Jac96, Theorem 1.3]. This shall be stated in a special case. That means, we further restrict the class of considered processes to the class of Lévy processes. They shall be defined as follows.

Definition 2.3.1 The Markov process $((X_t)_{t\geq 0}, P^x)_{x\in\mathbb{R}^d}$ is called a *Lévy process* if for every $x \in \mathbb{R}^d$ the stochastic process $(\Omega, \mathcal{F}, P^x, (X_t)_{t\geq 0})$ satisfies the following conditions:

1. For any choice $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.
2. $X_0 = x$ a.s.
3. The distribution of $X_t - X_s$ equals the distribution of $X_{t-s} - X_0$ for $0 \leq s < t$.
4. It is stochastically continuous, i.e. for every $t \geq 0$ and $\epsilon > 0$
$$\lim_{s\to t} P^x(|X_s - X_t| > \epsilon) = 0.$$
5. There exists $\Omega_0 \in \mathcal{F}$ with $P^x(\Omega_0) = 1$, such that for every $\omega \in \Omega_0, X_t(\omega)$ is càdlàg (continu à droite, limites à gauche), i.e. it is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

This definition basically followed [Jac96, Definition 2.5] except for the last additional assumption. The first four assumptions already yield the existence of a version that satisfies the last assumption. However, for convenience's sake we followed [Sat99, Definition 1.6] and already assumed the càdlàg property.

For Lévy processes (X_t) the generator restricted to the space $C_c^\infty(\mathbb{R}^d)$ is a pseudo differential operator (PDO) with symbol Ψ^X, where Ψ^X is the *characteristic exponent* of X. That means, according to [Jac96, Equation (1.24)] we have for $u \in C_c^\infty(\mathbb{R}^d)$ and the corresponding version in $\mathbb{R}^d$ of the Fourier transform

$$Au(x) = -\mathcal{F}^{-1}(\Psi^X(\cdot)\mathcal{F}u(\cdot))(x),$$

where $\Psi^X : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\Psi^X(\xi) = -\lim_{t\to 0} \frac{\hat{P}^x_{X_t - x}(\xi) - 1}{t}.$$

Here $P^x_{X_t - x}$ denotes the distribution of the random variable $X_t - x$ under P^x. The Fourier transform of this measure, the so-called *characteristic function* of $X_t - x$ under P^x, is defined by

$$\hat{P}^x_{X_t - x}(\xi) := (2\pi)^{-1/2} E^x(e^{i(X_t - x)^\top \xi}).$$

This characteristic function can be expressed via the so-called Lévy-Khintchine triplet, which corresponds in case of a Lévy process to the differential semimartingale characteristics that will be introduced in the next section.

2.4 Semimartingale characteristics

The aim of semimartingale calculus is to provide a setting, in which analysis of stochastic processes can be done in a similar fashion as the analysis of deterministic functions. The semimartingale characteristics describe the local behavior of a semimartingale. It would go beyond the scope of this thesis to formally introduce the notion of semimartingale characteristics. To this end and for the notation we refer to [JS03]. But we state the results which will be needed for the formulation of the problem and the derivation of the PIDE. These can be found in [JS03] or [Kal06], respectively.

At this point we briefly state two special properties of local martingales that will be used for the derivation of the PIDE.

Proposition 2.4.1 *A local martingale X is a uniformly integrable martingale if and only if it is a process of class (D). That means, the set of random variables*

$$\{X_T; T \text{ finite-valued stopping time}\}$$

is uniformly integrable. Particularly, every bounded local martingale is a uniformly integrable martingale.

Proof. [JS03, I.1.47] □

Lemma 2.4.2 *A continuous local martingale X with finite variation, i.e. $\forall \omega \in \Omega : t \to X_t(\omega)$ has finite variation, is almost surely equal to zero.*

Proof. [JS03, I.4.13/I.4.14] □

Now we start with the semimartingale characteristics. By $h : \mathbb{R}^d \to \mathbb{R}^d$ we now denote in this section a fixed *truncation function*, i.e. h is bounded and $h(x) = x$ in some open neighborhood of 0. Unless otherwise stated, we assume that the characteristics are given with respect to such a truncation function h.

Definition 2.4.3 Let (B, C, ν) be the characteristics of a semimartingale X. If there are predictable processes b,c and a transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathcal{B}^d)$ such that

$$\begin{aligned} B_t &= \int_0^t b_s ds, \\ C_t &= \int_0^t c_s ds, \\ \nu([0,t] \times G) &= \int_0^t F_s(G) ds \quad \forall G \in \mathcal{B}^d, \end{aligned}$$

we call (b, c, F) the *differential characteristics* of X and denote them by ∂X.

If the differential characteristics are deterministic they correspond to the so-called Lévy-Khintchine triplet and therefore determine the characteristic function.

Proposition 2.4.4 *Let $((X_t)_{t\geq 0}, P^x)_{x\in\mathbb{R}}$ be a Markov process such that for every $x \in \mathbb{R}$ the process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P^x,\ X - x)$ is a semimartingale in the sense of [JS03]. Then $((X_t)_{t\geq 0}, P^x)_{x\in\mathbb{R}}$ is a Lévy process if and only if for every $x \in \mathbb{R}^d$ the semimartingale $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P^x,\ X - x)$ admits a version (b, c, F) of the differential characteristics which does not depend on (ω, t). In this case (b, c, F) is equal to the Lévy-Khintchine triplet, i.e. the characteristic function of the random variable $X_t - x$ is given by*

$$E(e^{iu(X_t-x)}) = \exp t\Big(iub - \frac{1}{2}ucu + \int (e^{iuy} - 1 - iuh(y))F(dy)\Big).$$

Proof. [JS03, Corollary II.4.19] □

The following theorem is a consequence of the Itô formula for semimartingales and will be essential to derive the PIDE.

Theorem 2.4.5 *If X is a semimartingale with characteristics (B, C, ν) then for each $f \in C_b^2(\mathbb{R}^d)$, i.e. $f : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and twice continuously differentiable, the process*

$$f(X) - f(X_0) - \sum_{j\leq d} D_j f(X_-) \bullet B^j - \frac{1}{2}\sum_{j,k\leq d} D_{jk} f(X_-) \bullet C^{jk}$$
$$-\Big(f(X_- + x) - f(X_-) - \sum_{j\leq d} D_j f(X_-)h^j(x)\Big) * \nu$$

is a local martingale.

Proof. [JS03, Theorem II.2.42] □

The following calculation rules can be found in [Kal06].

Proposition 2.4.6 *Let X be an $\mathbb{R}^d$-valued semimartingale and H an $\mathbb{R}^{n\times d}$-valued predictable process with $H^{j\cdot} \in L(X), j = 1, \ldots, n$ (i.e. integrable with respect to X). If $\partial X = (b, c, F)$, then the differential characteristics of the $\mathbb{R}^n$-valued integral process*

$$\int_0^{\cdot} H_t dX_t := \Big(\int_0^{\cdot} H_t^{j\cdot} dX_t\Big)_{j=1,\ldots,n}$$

are given by $\partial(\int_0^{\cdot} H_t dX_t) = (\tilde{b}, \tilde{c}, \tilde{F})$, where

$$\begin{aligned}
\tilde{b}_t &= H_t b_t + \int (\tilde{h}(H_t x) - H_t h(x))F_t(dx),\\
\tilde{c}_t &= H_t c_t H_t^{\top},\\
\tilde{F}_t(G) &= \int 1_G(H_t x)F_t(dx) \quad \forall G \in \mathcal{B}^n \text{ with } 0 \notin G.
\end{aligned}$$

Here, $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ denotes the truncation function which is used on $\mathbb{R}^n$.

Proposition 2.4.7 *Let X be an $\mathbb{R}^d$-valued semimartingale with differential characteristics $\partial X = (b, c, F)$. Suppose that $f : U \to \mathbb{R}^n$ is twice differentiable on some open subset $U \subset \mathbb{R}^d$ such that X,X_- are U-valued. Then the $\mathbb{R}^n$-valued semimartingale $f(X)$ has differential characteristics $\partial(f(X)) = (\tilde{b}, \tilde{c}, \tilde{F})$, where*

$$\begin{aligned}
\tilde{b}^i_t &= \sum_{k=1}^{d} D_k f^i(X_{t-}) b^k_t + \frac{1}{2} \sum_{k,l=1}^{d} D_{kl} f^i(X_{t-}) c^{kl}_l \\
&\quad + \int \left(\tilde{h}^i(f(X_{t-} + x) - f(X_{t-})) - \sum_{k=1}^{d} D_k f^i(X_{t-}) h^k(x) \right) F_t(dx), \\
\tilde{c}^{ij}_t &= \sum_{k,l=1}^{d} D_k f^i(X_{t-}) c^{kl}_t D_l f^j(X_{t-}), \\
\tilde{F}_t(G) &= \int 1_G(f(X_{t-} + x) - f(X_{t-})) F_t(dx) \quad \forall G \in \mathcal{B}^n \text{ with } 0 \notin G.
\end{aligned}$$

2.5 Semigroup approach to Parabolic differential equations

After having applied the techniques of the previous sections, the solution of the hedging problem can be represented as the solution of a PIDE. This is cast into a variational formulation and successively substituted by approximate equations. In order to estimate the resulting error, particularly for the spatial semi-discretization, the equation has to be expressed with the means of functional analysis. More specifically, the aim is to represent the solution as image of an exponential of a corresponding operator. The respective properties of such exponentials are then exploited for the error analysis.

To this end, we basically follow [Jür05, chapter 1,2] and state the notions and results that will be needed in Section 5.3. That means we consider the following variational setting.

Let $X \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} X^*$ denote the so-called Gelfand or Evolution triple. That means X and H are Hilbert spaces and $\overset{d}{\hookrightarrow}$ denotes a dense embedding. Furthermore, assume $a : X \times X \to \mathbb{R}$ to be a continuous, X-coercive bilinear form, i.e.

$$|a(v, w)| \leq c_1 \|v\|_X \|w\|_X, \quad v, w \in X,$$

and

$$a(v, v) \geq c_2 \|v\|_X^2 - c_3 \|v\|_X^2, \quad v \in X,$$

for constants $c_1, c_2 > 0$ and $c_3 \in \mathbb{R}$. We are interested in the solution of the following variational equation.

Find $U \in H^1(0, T; X^*) \cap L^2(0, T; X)$ such that for all $v \in X$ we have

$$\begin{aligned}
\frac{d}{dt}(U(t), v) + a(U(t), v) &= (g(t), v), \quad 0 < t \leq T, \\
(U(0), v) &= (u_0, v), \qquad (2.2)
\end{aligned}$$

where $g \in L^2(0,T;X^*)$ and $u_0 \in L^2(0,T;H)$. In order to analyze the properties of the solution, a corresponding sectorial operator A can be defined. Here, a sectorial operator is defined as follows.

Definition 2.5.1 An operator $A : D(A) \subset X_0 \to X_0$ is called *sectorial*, if and only if

1. A is linear, closed, and densely defined in X_0,
2. there exists an angle $\theta \in (0, \pi/2)$, such that

$$G := \{z \in \mathbb{C} : \theta \leq |\arg z| \leq \pi\} \subset \varrho(A),$$

   where $\varrho(A)$ is the resolvent set of A and such that there exists a constant $\tilde{C}$ such that

$$\|(z-A)^{-1}f\|_{X_0} \leq \frac{\tilde{C}}{|z|}\|f\|_{X_0} \quad \forall z \in G.$$

For the variational equation (2.2) we can now define a corresponding sectorial operator as follows, cf. [Jür05, Lemma 1.20].

Lemma 2.5.2 *Let the linear operator $A : X \to X^*$ be defined by*

$$\langle Av, w\rangle_{X^* \times X} := a(v,w), \quad v, w \in X,$$

where a is a continuous and coercive bilinear form. Let furthermore $\tilde{A} := A|_H$ with

$$D(\tilde{A}) := \{f \in X; Af \in H\}.$$

Then $A : X \subset X^* \to X^*$ and $\tilde{A} : D(\tilde{A}) \subset H \to H$ are sectorial operators on the respective spaces for some $\theta_0 \in (0, \pi/2)$. Furthermore, there exists a constant $\tilde{C}$ such that we have the following estimate for all $z \in \varrho(A)$ and $v \in X$, where $\varrho(A)$ denotes the resolvent set:

$$|z|\|v\|_{L^2}^2 + \|v\|_X^2 \leq \tilde{C}|z\|v\|_{L^2}^2 - a(v,v)|.$$

This property permits to define the exponential of the operators A and $\tilde{A}$ via the *Dunford-Cauchy integral* and to derive some properties of the corresponding semigroup, cf. [Jür05, Lemma 1.3].

Lemma 2.5.3 *Let $A : X_0 \to X_0$ be a sectorial operator with $\theta \in (0, \pi/2)$ and let Γ be a piecewise smooth simple curve in G running from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$. Then by*

$$e^{-tA} := \frac{1}{2\pi i}\int_\Gamma e^{-tz}(z-A)^{-1}dz, \quad t > 0$$

a strongly continuous analytic semigroup $(e^{-tA})_{t\geq 0}$ is defined, when setting $e^{0A} := I$. That means, we have

1. *Semigroup:* $e^{-tA}e^{-sA} = e^{-(t+s)A}$ *for all* $t, s > 0$,
2. *Strong continuity:* $\lim_{t\searrow 0} e^{-tA}f = f$ *for all* $f \in X^*$,
3. *Analyticity: The mapping* $t \to e^{-tA}$ *is analytical on* $(0,\infty)$ *in the topology of operators* $X^* \to X^*$.

Furthermore, we have for $f \in X_0$ *and* $k \in \mathbb{N}_0$ *the following:*

1. *The following equality is well-defined:*
$$A^k e^{-tA} f = e^{-tA} A^k f.$$
2. *The function* $t \to e^{-tA}f$ *is in* $C^\infty((0,\infty), X_0)$ *with derivatives*
$$\frac{\partial^k}{\partial t^k} e^{-tA} f = (-1)^k A^k e^{-tA} f.$$
3. *Moreover, there exists a constant* $\tilde{C}$ *independent of* f *and* t *such that we have*
$$\|A^k e^{-tA} f\|_{X_0} \le \tilde{C} t^{-k} \|f\|_{X_0}.$$

Particularly, this means if we identify A and $\tilde{A}$ from Lemma 2.5.2 the following. There exists a constant $\tilde{C}$ such that

$$\begin{aligned}
\left\| \frac{\partial^k}{\partial t^k} e^{-tA} f \right\|_{X^*} &= \|A^k e^{-tA} f\|_{X^*} \le \tilde{C} t^{-k} \|f\|_{X^*} \quad \forall f \in X^*, k \in \mathbb{N}_0, t > 0, \\
\left\| \frac{\partial^k}{\partial t^k} e^{-tA} f \right\|_{H} &= \|A^k e^{-tA} f\|_{H} \le \tilde{C} t^{-k} \|f\|_{H} \quad \forall f \in H, k \in \mathbb{N}_0, t > 0.
\end{aligned}$$

Now we can finally link the operator A and the solution U of (2.2). By *Duhamel's principle* the solution U is given by (cf. [Jür05, Equation (1.1) and (2.2)])

$$U(t) = e^{-tA} u_0 + \int_0^t e^{-(t-\tau)A} g(\tau) d\tau.$$

Chapter 3

Formulation of the problem

Main thread. *Now we will introduce the formal setting, formulate the problem and express the solution in terms of functions of the underlying. We first restrict the class of stochastic processes for the underlying to Regular Lévy Processes of Exponential type with positive order. It comprises most of the popular Lévy models, except for the Variance Gamma model. This choice yields the existence and necessary smoothness of the density of the process which again is needed to show the smoothness of the solutions. Due to the corresponding properties of the density of the VG process, there is not much hope that this approach can be applied to this limiting case as well.*

Secondly, the space of strategies and option prices over which the optimization is performed is chosen in order to end up in the setting of [ČK07]. They have shown among other things that the solution is given in terms of semimartingale characteristics. Next, the European call and put functions are chosen as payoff functions. It is shown that we have a call-put parity. Therefore, it suffices to restrict further analysis to the bounded put case.

However, they do not dispose of enough smoothness for the following analysis. Therefore, we use smooth approximations that will be formally introduced in the next chapter. For those we apply the calculation rules for semimartingale characteristics. It turns out that the characteristics can be expressed via deterministic functions of the underlying. These functions are already given in the notation that will be used in the sequel. More specifically, it will turn out in Section 4.2 that they are given as results of the application of the operators Γ *and* ψ *on the option price function and the exponential function. This resembles the approach that has been taken by [BL89]. The analysis of the operators* Γ *and* ψ *will play a key role for the derivation of the PIDE and the ensuing error analysis. While the additional error due to smoothing will be estimated using other representations of the corresponding functions, we need a representation of the trading strategy* ϑ *in terms of* Γ*. This can finally be given for any* $0 \leq t < T$.

In this thesis we deal with the variance-optimal hedging problem for a European option in the martingale case. As mentioned in the introduction the task thereby is to minimize the variance-optimal hedging error

$$E\left((\tilde{H}(S_T) - v - \int_0^T \vartheta_s dS_s)^2\right)$$

over all admissible hedging strategies ϑ and initial endowments v. The payoff function $\tilde{H}$ and the time horizon T are given as part of the option agreement. In order to define the formal framework we have to choose a model for the discounted asset price S and the space of admissible trading strategies and initial endowments. We start with the formal setting.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P^x, (X_t)_{t\geq 0})_{x\in\mathbb{R}}$ be a one-dimensional Lévy process as defined in the Preliminaries and set $S_t := e^{X_t}$. In the sequel we set $P := P^0$ and the expectation is taken with respect to P if there is no additional superscript. Furthermore, we will omit the details in the sequel and use the abbreviate notation X_t for the Lévy process.

Within this thesis we now restrict the class of possible driving processes to the regular exponential Lévy processes of positive order in the sense of [BL02, Definition 3.4]. They provide a tractable setting in order to analyze the smoothness properties of the (existing) density. Due to the positivity of the order it will turn out that the corresponding density functions are smooth.

Definition 3.0.1 A Lévy process X_t is called *regular Lévy process of exponential type* $[-\eta, \eta]$ of order $\varrho \in (0, 2]$ (RLPE), if for the characteristic exponent Ψ^X of X we have the following. There exists a function $\Phi : \mathbb{C} \to \mathbb{C}$ with

$$\Psi^X(\xi) = -ib\xi + \Phi(\xi)$$

such that Φ is holomorphic in the strip $\{\xi \in \mathbb{C}; \Im\xi \in (-\eta, \eta)\}$ and continuous up to the boundary. Furthermore, there exist $\nu_1, \nu_2 < \varrho$ and constants C_1, C_2 such that for all ξ in that strip the following holds:

$$\begin{aligned} \Phi(\xi) &= C_1|\xi|^\varrho + O(|\xi|^{\nu_1}) \quad \text{for } |\xi| \to \infty, \\ |\Phi'(\xi)| &\leq C_2(1 + |\xi|)^{\nu_2}. \end{aligned}$$

Now we can finally choose the driving process X. The following assumptions will ensure that S is a square integrable martingale. This is necessary in order to come up with a tractable representation of the solution of the hedging problem. Furthermore, we will already impose restrictions upon the jump measure which will be needed for the numerical treatment. That means, we now assume the following:

(A1) X is a RLPE of order $\varrho \in (0, 2]$ and of exponential type $[-\eta, \eta]$ with the Lévy-Khintchine triple (b, σ^2, F) with respect to the 'truncation' function $h(x) = x$.

(A2) To ensure the square integrability of S: We assume $\eta \geq 2$.

(A3) To enforce the martingale property: Let $b = -\frac{1}{2}\sigma^2 - \int_{\mathbb{R}}(e^x - 1 - x)F(dx)$.

(A4) For later numerical analysis: There exists $0 < \nu < 2$ such that we have the following:

- F is absolutely continuous with respect to the Lebesgue measure with kernel function k, i.e. $F(dy) = k(y)dy$.
- k satisfies a Calderón-Zygmund estimate: For all $\alpha \in \mathbb{N}_0$ there exists a constant $C(\alpha)$, such that we have

$$|k^{(\alpha)}(z)| \leq C(\alpha)|z|^{-(1+\nu+\alpha)}, \quad \forall z \in \mathbb{R} \setminus \{0\}.$$

  For the integrability in the weighted case we further assume

$$|k^{(\alpha)}(z)| \leq C(\alpha)e^{-(\eta+\delta)|z|}, \quad \forall z \in \mathbb{R} \setminus [-1, 1].$$

- k is symmetrically bounded from below: There exists $C_- > 0$, such that

$$\forall 0 < |z| < 1 : \quad \frac{1}{2}(k(-z) + k(z)) \geq \frac{C_-}{|z|^{1+\nu}}.$$

The parameters (b, σ^2, F) and η and with them the kernel k and its parameter ν are now fixed for the rest of the thesis.

Remarks.

1. The index ν in this case represents the activity of small jumps. It coincides with the order ϱ if $\sigma^2 = 0$. Therefore, it will constitute the order of the respective operators in Section 5.6.

2. As stated in [BL02], this class of processes includes CGMY processes, Normal Inverse Gaussian processes (NIG), Generalized Hyperbolic Processes (GHP) and Normal Tempered Stable Lévy Processes (NTS) of order $\varrho > 0$. An overview over these models is given in [Sch03].

3. A prominent limiting class which is not included in the ensuing analysis is the Variance-Gamma (VG) process - even though it is a RLPE of order $\varrho = 0$. This is due to the lack of differentiability of the density function in that case. More specifically, a VG process has a selfdecomposable distribution. The corresponding Theorem [Sat99, Theorem 28.4] states that for $t \leq c/2$ the density function is not even continuous with respect to the space variable x. Here, c is the variance rate of the process. Even less so with respect to time. The smooth approximation procedure of the payoff function, which is presented in this thesis, overcomes this first deficiency. But time differentiability solely depends upon the density function. Therefore, it is rather improbable that the ensuing analysis can be extended to that limiting case.

Now that the stochastic process for the asset price S has been chosen we still have to define the meaning of admissibility. For the first representation of the solution to the hedging problem we rely upon [ČK07]. Therefore we use their space of admissible trading strategies and endowments.

Definition 3.0.2 A process ϑ is called *simple* if it is a linear combination of processes of the form $Y1_{]\!]\tau_1,\tau_2]\!]}$, where $\tau_1 \leq \tau_2$ denote stopping times and Y a bounded $\mathcal{F}_{\tau_1}$ - measurable random variable. The pair $(v, \vartheta) \in L^0(\Omega, \mathcal{F}_0, P) \times L(S)$ is called *admissible endowment/strategy pair* if there exist sequences $(v^n)_{n\in\mathbb{N}}$ in $L^2(\Omega, \mathcal{F}_0, P)$ and $(\vartheta^n)_{n\in\mathbb{N}}$ of simple processes such that

$$\begin{aligned} v^n + \vartheta^n \bullet S_t &\rightarrow v + \vartheta \bullet S_t \text{ in probability for any } t \in [0,T] \text{ and} \\ v^n + \vartheta^n \bullet S_T &\rightarrow v + \vartheta \bullet S_T \text{ in } L^2(P). \end{aligned}$$

In the sequel we will compute in log-price. That means, we consider the respective transformed payoff function H. Furthermore, we consider the driving process $x + X$ and set $S^x := e^{x+X}$ for $x \in \mathbb{R}$. Now we are in the setting to apply [ČK07]. To this end we need the following process for $x \in \mathbb{R}$:

$$V_t(x) := E^x(H(X_T)|\mathcal{F}_t)$$

By the Markov property and the stationarity of increments for Lévy processes we have

$$\begin{aligned} V_t(x) &= E^{X_t}(H(X_{T-t})) \quad P^x - a.s. \\ &= E^{x+X_t}(H(X_{T-t})) \quad P - a.s. \end{aligned}$$

Therefore, the hedging problem is solved by computing the following objects:

$$\begin{aligned} V(t,x) &:= E(H(x+X_t)), \\ (v^\star(x), \vartheta^\star(x)) &:= \operatorname{argmin}_{(v,\vartheta) \text{ admissible}} E\left(\left(V(0, x+X_T) - v - \int_0^T \vartheta_{s-} dS_s^x\right)^2\right), \\ J_0(x) &:= E\left(\left(V(0, x+X_T) - v^\star(x) - \int_0^T \vartheta^\star_{s-}(x) dS_s^x\right)^2\right), \end{aligned}$$

because we have $V_t(x) \overset{d}{=} V(T-t, x+X_t)$, i.e. they are equal in distribution.

Now we fix the payoff function which we will consider in the sequel. Natural payoff functions for European options are the European call

$$H_c^K : \mathbb{R} \to \mathbb{R}_+; H_c^K(x) = (e^x - K)^+,$$

respectively the European put

$$H_p^K : \mathbb{R} \to \mathbb{R}_+; H_p^K(x) = (K - e^x)^+.$$

Here, $K \in \mathbb{R}$ is the so-called strike. In the rest of the thesis we will assume $H = H_p^1$. The restriction to strike $K = 1$ does not imply loss of generality. Indeed, we have

$$H^K(x) = KH^1(x - \log K),$$

and the objects of interest depend linearly or bilinearly upon the payoff function, cf. Corollary 5.7.7. Furthermore, we have a Call-Put parity which enables to transfer the results of the put case to the call one. In order to show the parity, we denote the corresponding objects with respect to the put or call function with the subscript/superscript p and c just for now. If we keep in mind that $H_p(x) = H_c(x) + 1 - e^x$, we get

$$\begin{aligned} V^p(t,x) &= E(H_p(x+X_t)) \\ &= E(H_c(x+X_t) + 1 - e^{x+X_t}) \\ &= V^c(t,x) + 1 - e^x E(S_t) \\ &\overset{\text{S martingale}}{=} V^c(t,x) + 1 - e^x. \end{aligned}$$

Let now $\vartheta^{\star,p}(x)$ be the optimal admissible trading strategy with respect to the European put function. We have

$$\begin{aligned} J_0^p(x) &= E\left(\left(V^p(0,x+X_T) - v_p^\star(x) - \int_0^T \vartheta_{s-}^{\star,p}(x) dS_s^x\right)^2\right) \\ &= E\left(\left(V^c(0,x+X_T) + (1 - S_T^x) - v_p^\star(x) - \int_0^T \vartheta_{s-}^{\star,p}(x) dS_s^x\right)^2\right) \\ &= E\left(\left(V^c(0,x+X_T) - (v_p^\star(x) - 1 + S_0^x) - (S_T^x - S_0^x) - \int_0^T \vartheta_{s-}^{\star,p}(x) dS_s^x\right)^2\right) \\ &= E\left(\left(V^c(0,x+X_T) - (v_p^\star(x) - 1 + e^x) - \int_0^T (\vartheta_{s-}^{\star,p}(x) + 1) dS_s^x\right)^2\right). \end{aligned}$$

Furthermore, we have that $(v_p^\star(x) - 1 + e^x, \vartheta^{\star,p}(x) + 1)$ is an admissible endowment/strategy pair. That means, we have $J_0^c(x) \leq J_0^p(x)$. A similar argument yields $J_0^p(x) \leq J_0^c(x)$ and we finally get

$$J_0^c(x) = J_0^p(x), \vartheta^{\star,c}(x) = \vartheta^{\star,p}(x) + 1 \text{ and } v_c^\star(x) = v_p^\star(x) - 1 + e^x.$$

However, these functions do not dispose of sufficient smoothness for the ensuing analysis. Therefore, we will introduce a smooth approximation H^ϵ of H in Section 4.1. We denote the corresponding option price, trading and hedging error function with the superscript ϵ, that means $V^\epsilon, \vartheta^{\epsilon,\star}, J_0^\epsilon$. In Corollary 4.1.5 we will show that $V^\epsilon \in C^2$. This allows to apply the Itô formula for semimartingale characteristics which leads to a representation of the solution in terms of deterministic functions. In order to be consistent with the ensuing analysis we introduce to this end the following operators. Let

$$\begin{aligned} D(\Gamma) := \Bigg\{ &(f_1, f_2) \in C^0 \times C^0;\ \text{left-sided derivatives of } f_1, f_2 \text{ exist and} \\ &\forall x \in \mathbb{R} : \int_{\mathbb{R}} |f_1(x+y) - f_1(x)||f_2(x+y) - f_2(x)| F(dy) < \infty \Bigg\}, \end{aligned}$$

and let $B(\mathbb{R})$ denote the set of Borel functions $\mathbb{R} \to \mathbb{R}$. Then define the bilinear operators $\Gamma : D(\Gamma) \to B(\mathbb{R})$ and $\Psi : D(\Gamma) \to B(\mathbb{R})$ as follows. Here, f' denotes the left sided derivative.

$$\begin{aligned}\Gamma(f_1,f_2)(x) &:= \sigma^2 f_1'(x)f_2'(x) + \int_{\mathbb{R}} (f_1(x+y)-f_1(x))(f_2(x+y)-f_2(x))F(dy)\\ \psi(f_1,f_2)(x) &:= \left(\Gamma(f_1,f_2) - \frac{\Gamma(\exp,f_1)\Gamma(\exp,f_2)}{\Gamma(\exp,\exp)}\right)(x) \qquad (3.1)\end{aligned}$$

In fact, Γ coincides with the carré-du-champs operator on the intersection of the respective domains. This operator has been used in [BL89] to represent the solution of the hedging problem. However, we will not follow along their lines and choose a different approach via semimartingale calculus. Here, we have the advantage that the existence of the corresponding solution functions has already been shown. Furthermore, it provides a possible starting point for a future generalization of this approach to the non-martingale case. Indeed, in [ČK07] this case is mainly treated. In Section 5.6 we will nevertheless make use of the representation of Γ in terms of the generator of X which was used in [BL89]. Here, we now use the representation via semimartingale characteristics and the Itô formula for characteristics in order to derive the following.

Theorem 3.0.3 *The solution of the hedging problem with respect to H^ϵ is given by*

$$\begin{aligned}v^{\epsilon,\star}(x) &= V^\epsilon(T,x),\\ \vartheta_t^{\epsilon,\star}(x) &= \vartheta^\epsilon(T-t,x+X_{t-}),\\ J_0^\epsilon(x) &= J^\epsilon(T,x),\end{aligned}$$

where

$$\begin{aligned}\vartheta^\epsilon(t,x) &:= \left(\frac{\Gamma(V^\epsilon(t,\cdot),\exp)}{\Gamma(\exp,\exp)}\right)(x),\\ J^\epsilon(t,x) &:= E\left(\int_0^t \psi\big(V^\epsilon(t-s,\cdot),V^\epsilon(t-s,\cdot)\big)(x+X_{s-})ds\right).\end{aligned}$$

Proof. In [ČK07, Theorem 4.10] and [ČK07, Theorem 4.12] together with [ČK07, Remarks in paragraph 4.3] it is shown that

$$\begin{aligned}v^{\epsilon,\star}(x) &= V^\epsilon(T,x),\\ \vartheta^{\epsilon,\star} &= \tilde{c}^{SV^\epsilon}(\tilde{c}^S)^{-1},\\ J_0^\epsilon(x) &= E\left(\int_0^T \tilde{c}_s^{V^\epsilon} - (\tilde{c}_s^{SV^\epsilon})^2(\tilde{c}_s^S)^{-1}ds\right),\end{aligned}$$

where $\tilde{c}$ denotes the second part of the modified semimartingale characteristic of the process $\left(\frac{e^{x+X_\cdot}}{V^\epsilon(T-\cdot,x+X_\cdot)}\right)_{T\wedge t}$. Therefore, it remains to be shown that

$$\begin{aligned}\tilde{c}_t^S &= \Gamma(\exp,\exp)(x+X_{t-}),\\ \tilde{c}_t^{SV^\epsilon} &= \Gamma(\exp,V^\epsilon(t,\cdot))(x+X_{t-}),\\ \tilde{c}_t^{V^\epsilon} &= \Gamma(V^\epsilon(t,\cdot),V^\epsilon(t,\cdot))(x+X_{t-})\end{aligned}$$

for $0 \leq t \leq T$. By Corollary 4.1.5 we have $V^\epsilon \in C^2([0,T] \times \mathbb{R})$. That means, we can apply the Itô formula for semimartingale characteristics in Proposition 2.4.7 for the process $\binom{x+X_\cdot}{T-\cdot}_{T\wedge t}$ and the function $f(x_1,x_2) = \binom{e^{x_1}}{V^\epsilon(x_2,x_1)}$. Then we get

$$
\begin{aligned}
\tilde{c}_t^S &= e^{2(x+X_{t-})}\left(\sigma^2 + \int_{\mathbb{R}} (e^y-1)^2 F(dy)\right), \\
\tilde{c}_t^{SV^\epsilon} &= e^{x+X_{t-}}\Big(\sigma^2 D_2 V^\epsilon(T-t, x+X_{t-}) \\
&\quad + \int_{\mathbb{R}} (e^y-1)(V^\epsilon(T-t,x+X_{t-}+y) - V^\epsilon(T-t,x+X_{t-}))F(dy)\Big), \\
\tilde{c}_t^{V^\epsilon} &= \sigma^2\Big(D_2V^\epsilon(T-t,x+X_{t-})\Big)^2 \\
&\quad + \int_{\mathbb{R}} \Big(V^\epsilon(T-t,x+X_{t-}+y) - V^\epsilon(T-t,x+X_{t-})\Big)^2 F(dy)
\end{aligned}
$$

for $0 \leq t \leq T$. This yields the claim. □

However, we will estimate the error caused by this smooth approximation with H^ϵ. For V^ϵ and J^ϵ this is done using other representations of the respective functions. Only for the trading strategy we fall back upon a representation of the kind above. By Corollary 4.1.3 we will have $V \in C^2((0,T] \times \mathbb{R})$. That means we can apply the very same arguments as in the previous proof for the process $\left(\begin{smallmatrix} e^{x+X_\cdot} \\ V(T-\cdot, x+X_\cdot) \end{smallmatrix}\right)_{T-\delta\wedge t}$ for any $\delta > 0$. This yields the following representation for $\vartheta^\star$.

Proposition 3.0.4 *For $0 \leq t < T$ the optimal trading strategy $\vartheta^\star$ can be represented as follows:*

$$
\vartheta_t^\star = \left(\frac{\Gamma(V(t,\cdot), \exp)}{\Gamma(\exp,\exp)}\right)(x+X_{t-}).
$$

Chapter 4

Derivation of the PIDE

Main thread. *This chapter is dedicated to the derivation of the representation of the hedging error function as solution of a PIDE. The uniqueness is dealt with in the next chapter. To this end the properties of the respective functions have to be derived.*

The function V can be represented as convolution of the payoff function H and the distribution P_{-X_t} of $-X_t$. Roughly speaking, the differentiation of V with respect to x can therefore partly be cast upon H and partly upon P_{-X_1}, while differentiation with respect to time t can only be cast upon P_{-X_t}. In Proposition 4.1.2 the resulting two estimates of these two ways are derived. Here, only exploiting the regularity of P_{X_t} yields an estimate which increases as time t tends to zero. But for the derivation of the PIDE and the ensuing numerical analysis we have integrability constraints also with respect to time. However, the alternative of exploiting the regularity of H is also limited, because $H \notin C^2$. Combining both possibilities yields an estimate of the form

$$\|D_1^k(V(t) - \overline{V}^{\epsilon_0}(t))\|_{H^s} \leq C\left(1 + t^{\frac{3/2-s-k(\varrho\vee 1)}{\varrho}}\right),$$

where $\overline{V}^{\epsilon_0}$ is a fixed approximation of V, see below. Unfortunately, this is not sufficient to be able to derive the PIDE. Therefore, we introduce a regularized approximate payoff function H^ϵ. Then the derivatives can be cast upon H^ϵ and the corresponding estimates now only depend upon the approximation parameter ϵ, i.e.

$$\|D_1^k(V^\epsilon(t) - \overline{V}^{\epsilon_0}(t))\|_{H^s} \leq C\left(1 + \epsilon^{3/2-s-k(\varrho\vee 1)}\right).$$

In Lemma 4.1.4 it is shown that the additional error of this approximation tends to zero as $\epsilon \to 0$. Therefore, the convergence order of the overall method will finally be a trade-off between this approximation error and the additional numerical error.

Next, the properties of the operators Γ and Ψ have to be analyzed. With these results the properties of the respective functions can be derived. The structure of $\|\Gamma(f_1, f_2)\|_{L^1}$ is very similar to the Slobodeckij norm, respectively the corresponding scalar product. However, we are interested in the weighted L^2_ω norm. To this end a suitable estimation of the weighted norm of the product of two functions is derived in Proposition 4.2.1.

Furthermore, we have the problem that $V \notin L^2$, but we know $D_2V \in L^2$. It turns out that the norm $\|\Gamma(f,f)\|_{L^2_\omega}$ can be estimated in two ways. Either by applying Fourier *techniques which yields an estimate in terms of the norms $\|\cdot\|_{H^{\varrho/2}_\omega}$ and $\|\cdot\|_{H^{1/2+\delta}_\omega}$. Or by using a more direct approach leading to an estimate via the semi-norm $|\cdot|_{H^1_\omega}$. The first is advantageous in terms of regularity, the second in terms of integrability. In order to exploit both a smooth function $\tilde{f}$ is introduced for each f such that $f-\tilde{f}$ is integrable. The overall estimation problem is then split up into a part involving the difference $f-\tilde{f}$ which is estimated using the first approach and the remaining part involving only $\tilde{f}$ which is estimated using the second approach. This is captured in the estimators $\|f,\tilde{f}\|^w_{(s,\omega)}$, $\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_1}_{(s,\omega_f,\omega)}$ and $\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}$. These estimators allow an acute estimate which is also applicable to the non-integrable functions H and V. The properties of ψ,ψ^ϵ and J,J^ϵ then directly follow from these results. And finally the PIDE can be derived using their regularity and boundedness.*

In order to be able to derive the PIDE the properties of the option price function V have to be studied. It turns out that due to the non-smooth payoff function the objects do not dispose of the necessary regularity and integrability. Therefore, we introduce a version which uses a smooth approximation of the payoff function. This was already used in the previous chapter. The additional error caused by that can be controlled and will be included in the overall error estimation at the end of the next chapter. However, the norm estimates of the original version will be stated as well. This throws a light upon where the obstacles on the way to the PIDE lie.

In the sequel we will thus derive similar results for both, the original V and the approximation V^ϵ and hence for J and J^ϵ. The main difference is that in the case of the original payoff function the regularity is mostly obtained by the properties of the distribution of X_t, while in the other the regularity of the payoff function is exploited. The problem with the first approach is that the norm estimates tend to infinity as t tends to zero. The norm estimates in the second approach, via regularized payoff functions, only depend upon the regularization parameter. This will finally determine the numerical error analysis in the sequel.

4.1 Properties of V and V^ϵ

We start the analysis of the properties of the functions of interest with the analysis of the option price function V. As stated in the introduction, it turns out that it does not dispose of enough regularity for our purposes. Therefore, we introduce a regularized approximation of this function. This is done using a regularized approximate payoff function H^ϵ and considering the corresponding option price function V^ϵ. To this end let $M_p \in \mathbb{N}$ be sufficiently large. Now let $q \in \mathscr{P}_{2M_p-1}$ be the unique polynomial of order $2M_p+1$ such that $q(-1)=q'(-1)=-1$ and $\forall k \in \mathbb{N}: q^{(k+2)}(-1)=q^{(k)}(1)=0$. Define additionally $C_{M_p} := \|q\|_{C^{M_p}([-1,1])}$. Now given a regularization parameter $\epsilon>0$ we define

$$q_l^\epsilon(x) = \epsilon q\left(\frac{e^x-1}{\epsilon}\right).$$

With this the regularized approximative payoff function H^ϵ can be defined as follows:

$$H^\epsilon(x) = \begin{cases} q_l^\epsilon(x) & \text{, if } \log(1-\epsilon) \le x \le \log(1+\epsilon), \\ H(x) & \text{otherwise.} \end{cases}$$

The corresponding functions shall be denoted by the additional superscript ϵ, i.e. $V^\epsilon, \vartheta^\epsilon, \psi^\epsilon$ and J^ϵ. Furthermore, fix a value $\epsilon_0 > 0$ which does not depend on ϵ. For shorter notation we define

$$v^\epsilon(r) := 1 + \epsilon^{3/2-r}.$$

Then the approximate payoff function possesses the following properties.

Lemma 4.1.1 *We have $H^\epsilon \in C^{M_p}(\mathbb{R})$ and for $0 \le s \le 1, \omega \in \mathbb{R}$ we have*

$$\|H - H^\epsilon\|_{H^s_\omega} \le C\epsilon^{3/2-s}.$$

For $0 \le s \le M_p$ and $\omega_1 > 0, \omega_2 \in \mathbb{R}, \omega_3 \in (-1, \infty)$ we have

$$\begin{aligned} \|H^\epsilon\|_{H^s_{\omega_1}} + \|H^\epsilon - H^{\epsilon_0}\|_{H^s_{\omega_2}} &\le Cv^\epsilon(s), \\ \|DH^\epsilon\|_{H^s_{\omega_3}} &\le Cv^\epsilon(s+1). \end{aligned}$$

Proof. The boundary conditions for q yield the required regularity. Furthermore, for $k = 0, 1$ we have

$$\begin{aligned} \sup_{\log(1-\epsilon)\le x\le\log(1+\epsilon)} |D^k H(x)| &\le C\epsilon^{1-k}, \\ \sup_{\log(1-\epsilon)\le x\le\log(1+\epsilon)} |D^k q_l^\epsilon(x)| &\le C_{M_p}\epsilon^{1-k}. \end{aligned}$$

That yields for $k = 0, 1$ with dominated convergence

$$\begin{aligned} \|D^k(H - H^\epsilon)\|^2_{L^2_\omega} &\le C\int_{\log(1-\epsilon)}^{\log(1+\epsilon)} e^{2\omega x}(|D^k H(x)| + |D^k q_l^\epsilon(x)|)^2 dx \\ &\le C(1 + C_{M_p})\epsilon^{2(1-k)}(\log(1+\epsilon) - \log(1-\epsilon)) \\ &\le C(1 + C_{M_p})\epsilon^{2(1-k)}\log\left(\frac{1+\epsilon}{1-\epsilon}\right) \\ &\le C(1 + C_{M_p})\epsilon^{2(1-k)}\log\left(1 + 2\epsilon\frac{1}{1-\epsilon}\right) \\ &\le C(1 + C_{M_p})2\epsilon^{3-2k}\left(\frac{1}{1-\epsilon}\right) \\ &\le C\epsilon^{3-2k}. \end{aligned}$$

The first claim follows by the interpolation property of the Sobolev norms. The same arguments yield the remaining claims. □

Before we start to estimate the specific functions V and V^ϵ we first derive an auxiliary estimate for a generic function of that type. It states two estimates. The first is obtained via the regularity of the distribution of X_t, the second via the regularity of H^ϵ.

Proposition 4.1.2 *Let $f \in C_b(\mathbb{R}) \cap L^1_\omega(\mathbb{R}) \cap H^r_\omega$ for some $\omega \in [-\eta, \eta]$ and $r \geq 0$. Let*

$$g(t,x) := E(f(x+X_t)).$$

Then we have $g \in C^\infty((0,T] \times \mathbb{R})$. Furthermore, let $k \in \mathbb{N}_0, s \geq 0$ and $t > 0$. Then we have

$$\|D_1^k g(t,\cdot)\|_{H^s_\omega} \leq C\|f\|_{H^r_\omega} t^{\frac{r-s-k(\varrho\vee 1)}{\varrho}}.$$

For $k \in \mathbb{N}_0, 0 \leq s + k(\varrho \vee 1) + 1/2 + \delta \leq r$ we have $g \in H^k(0,T;H^s_\omega)$ and for $t \in [0,T]$

$$\|D_1^k g(t,\cdot)\|_{H^s_\omega} \leq C\|f\|_{H^{s+k(\varrho\vee 1)}_\omega}.$$

Proof. In order to study the regularity of g it is convenient to derive a representation that is based upon its Fourier transform. The continuity and boundedness of f yields the continuity of g with respect to x for all $t \in [0,T]$. That means we have

$$g(t,x) = \mathcal{F}^{-1}(\mathcal{F}g(t,\cdot))(x) \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$$

Let Ψ^{-X} denote the characteristic exponent of $-X$. Since X_t is a Lévy process the characteristic function of the distribution of $-X_t$ is given by

$$\widehat{P_{-X_t}}(z) = e^{-t\Psi^{-X}(z)}.$$

Keeping in mind that the characteristic function corresponds to the inverse Fourier transform, this yields the following representation of g:

$$\begin{aligned} g(t,x) &= E(f(x+X_t)) \\ &= \int_{\mathbb{R}} f(x-y) P_{-X_t}(dy) \\ &= \mathcal{F}^{-1}\left(\mathcal{F}\int_{\mathbb{R}} f(\cdot - y) P_{-X_t}(dy)\right)(x) \\ &= \mathcal{F}^{-1}\left(\hat{f}(\cdot) e^{-t\Psi^{-X}(-\cdot)}\right)(x) && (4.1) \\ &= \mathcal{F}^{-1}\left(\hat{f}(\cdot) e^{-t\Psi^{X}(\cdot)}\right)(x). && (4.2) \end{aligned}$$

Now we first study the growth properties of the factors $e^{-t\Psi^X(z)}$ and $\mathcal{F}f(z)$. Let $G := \{z \in \mathbb{C}; |\Im z| \leq \eta\}$. By [BL02, Lemma 3.6] and the definition of the RLPE $-X_t$ we have for all $z \in G$

$$\begin{aligned} |e^{-t\Psi^X(z)}| &\leq Ce^{-t\tilde{c}|z|^\varrho}, \\ |\Psi^X(z)| &\leq C(1+|z|)^{\varrho\vee 1}, \end{aligned}$$

where $\tilde{c} > 0$ is some constant independent of t. Now we show the existence of the time derivatives in the two cases of the assumption. To this end we have to show the existence of integrable functions which dominate the argument of $\mathcal{F}^{-1}$ in (4.1) in order to be able to apply dominated convergence. In the first case we get the following. For $t_0 > 0$ let $U_{t_0} := ({t_0}/2, {3t_0}/2)$. For every $t \in U_{t_0}$ we have for $z, x \in \mathbb{R}$ and $k \in \mathbb{N}_0$

$$\begin{aligned}
&\left| e^{ixz}(1+|z|^2)^{s/2}\frac{\partial^k}{\partial t^k}\Big(\hat{f}(z+i\omega)e^{-t\Psi^X(z+i\omega)}\Big)\right| \\
&= \left|(1+|z|^2)^{s/2}\big(\Psi^X(z+i\omega)\big)^k|\hat{f}(z+i\omega)|e^{-t\Psi^X(z+i\omega)}\right| \\
&\le C(1+|z|)^{s+k(\varrho\vee 1)}|\hat{f}(z+i\omega)|e^{-t_0/2\tilde{c}|z|^\varrho} \\
&\le C\|f\|_{L^1_\omega}(1+|z|)^{s+k(\varrho\vee 1)}e^{-t_0/2\tilde{c}|z|^\varrho}.
\end{aligned}$$

This is an integrable dominating function. In the second case we have

$$0 \le s + k(\varrho \vee 1) + 1/2 + \delta \le r.$$

That means, the following dominating function is integrable for $t \ge 0$:

$$\left| e^{ixz}(1+|z|^2)^{s/2}\frac{\partial^k}{\partial t^k}\Big(\hat{f}(z+i\omega)e^{-t\Psi^X(z+i\omega)}\Big)\right| \le C(1+|z|)^{s+k(\varrho\vee 1)}|\hat{f}(z+i\omega)|.$$

The integrability can be seen as follows:

$$\begin{aligned}
&\int_{\mathbb{R}}(1+|z|)^{s+k(\varrho\vee 1)}|\hat{f}(z+i\omega)|dz \\
&\le C\|(1+|z|)^{s+k(\varrho\vee 1)+1/2+\delta}|\hat{f}(z+i\omega)|\|_{L^2_z}\|(1+|z|)^{-1/2-\delta}\|_{L^2_z} \\
&\le C\|f\|_{H^r_\omega}.
\end{aligned}$$

Therefore, the derivatives with respect to t exist by Proposition 2.2.2 for all $t_0 > 0$ in the first case, and for $t \ge 0$ in the second. The derivatives are then in both cases given by

$$\frac{\partial^k}{\partial t^k}g(t_0,x) = \mathcal{F}^{-1}\left(\mathcal{F}f(\cdot)\Big(\Psi^X(\cdot)\Big)^k e^{-t_0\Psi^X(\cdot)}\right)(x).$$

Now we can use this representation in order to derive the claimed estimates.

$$\begin{aligned}
\|D_1^k g(t,\cdot)\|_{H^s_\omega} &= \left\|(1+|z|^2)^{s/2}\mathcal{F}(D_1^k g(t,\cdot))(z+i\omega)\right\|_{L^2_z} \\
&\le C\left\|(1+|z|^2)^{s/2}(1+|z|)^{k(\varrho\vee 1)}|\hat{f}(z+i\omega)|e^{-t\tilde{c}|z+i\omega|^\varrho}\right\|_{L^2_z} \\
&\le C\left\|(1+|z|)^{s+k(\varrho\vee 1)}|\hat{f}(z+i\omega)|e^{-t\tilde{c}|z+i\omega|^\varrho}\right\|_{L^2_z}.
\end{aligned}$$

Now there are again two ways of proceeding which lead to the two estimates of the claim. The first is to use the integrability due to the exponential $e^{-t\tilde{c}|z|^\varrho}$ which leads to an estimate involving time t as follows. If $t > 0$ we have

$$\begin{aligned}
&\|(1+|z|)^{s+k(\varrho\vee 1)}\hat{f}(z+i\omega)e^{-t\tilde{c}|z+i\omega|^\varrho}\|_{L^2_z} \\
&\le C\|(1+|z|)^r\hat{f}(z+i\omega)\|_{L^2_z}\|(1+|z|)^{s+k(\varrho\vee 1)-r}e^{-t\tilde{c}|z+i\omega|^\varrho}\|_{L^\infty_z} \\
&\le C\|f\|_{H^r_\omega}\|(1+|z|)^{s+k(\varrho\vee 1)-r}e^{-t\tilde{c}|z|^\varrho}\|_{L^\infty_z}.
\end{aligned}$$

If $s+k(\varrho\vee 1)-r\leq 0$ the second term is bounded independently of t and the upper bound follows. Otherwise, a simple calculation shows that the function $y\mapsto y^\alpha e^{-cy^\beta}$ with $\alpha,\beta,c>0$ attains its maximum on $\mathbb{R}_+$ at $y=(c\frac{\beta}{\alpha})^{-1/\beta}$ with value $(c\frac{\beta}{\alpha})^{-\alpha/\beta}e^{-\alpha/\beta}$. Therefore, we get with $\alpha=s+k(\varrho\vee 1)-r,\beta=\varrho$ and $c=\tilde{c}t$

$$\begin{aligned}\|D_1^k g(t,\cdot)\|_{H^s_\omega} &\leq C\|f\|_{H^r_\omega}\|(1+|z|)^{s+k(\varrho\vee1)-r}e^{-t\tilde{c}|z|^\varrho}\|_{L^\infty_z}\\ &\leq C\|f\|_{H^r_\omega}(1+t^{\frac{r-s-k(\varrho\vee1)}{\varrho}}).\end{aligned}$$

The Sobolev Embedding now yields $e^{\omega\cdot}D_1^k g\in L^2(\delta,T;C_b^m)$ for all $k,m\in\mathbb{N}_0$ and $\delta>0$. This shows $g\in C^\infty((0,T]\times\mathbb{R})$. The second way is to directly use the integrability of $\hat{f}$ if $0\leq s+k(\varrho\vee 1)+1/2+\delta\leq r$ and to proceed as follows for $t\geq 0$:

$$\begin{aligned}\left\|(1+|z|)^{s+k(\varrho\vee1)}\hat{f}(z+i\omega)e^{-t\tilde{c}|z|^\varrho}\right\|_{L^2_z} &\leq \left\|(1+|z|)^{s+k(\varrho\vee1)}\hat{f}(z+i\omega)\right\|_{L^2_z}\\ &\leq \|f\|_{H^{s+k(\varrho\vee1)}_\omega}.\end{aligned}$$

This shows $g\in H^k(0,T;H^s_\omega)$. □

Now we can start to derive regularity and integrability properties of V which are due to the regularity of the distribution of X_t. To this end we define for shorter notation the following:

$$v^t(r):=1+t^{\frac{3/2-r}{\varrho}}.$$

Lemma 4.1.3 (Properties of V) *For X_t satisfying (A1) we have*

$$V\in C^\infty((0,T]\times\mathbb{R}),$$

and the following norm estimate for $s>0,t>0$, $0\leq\omega\leq\eta$ and $k\in\mathbb{N}_0$:

$$\|D_1^k(V(t,\cdot)-V^{\epsilon_0}(t,\cdot))\|_{H^s_\omega}\leq Cv^t(s+k(\varrho\vee 1)+\delta).$$

Proof. This follows directly from the first norm estimate of Proposition 4.1.2 for $f:=H-H^{\epsilon_0}$. Let $G:=\{z\in\mathbb{C};|\Im z|\leq\eta\}$. D^2H is an integrable distribution. That means $D^2H-D^2H^{\epsilon_0}\in L^1$ and we can estimate the Fourier transform of $H-H^{\epsilon_0}$ with partial integration as follows for $z\in G$ with $|z|\geq 1$:

$$\begin{aligned}\mathcal{F}(H-H^{\epsilon_0})(z) &= \int_{\mathbb{R}}e^{-izy}(H(y)-H^{\epsilon_0}(y))dy\\ &= \frac{1}{iz}\int_{\mathbb{R}}e^{-izy}(DH(y)-DH^{\epsilon_0}(y))dy\\ &= -\frac{1}{z^2}\int_{\mathbb{R}}e^{-izy}(D^2H(y)-D^2H^{\epsilon_0}(y))dy\\ &\leq C|z|^{-2}.\end{aligned}$$

For $z \in G$ with $|z| \leq 1$ we have

$$\begin{aligned} \mathcal{F}(H - H^{\epsilon_0})(z) &\leq \|H - H^{\epsilon_0}\|_{L^1} \\ &\leq C. \end{aligned}$$

Thus, overall this yields $\mathcal{F}(H - H^{\epsilon_0})(z) \leq C(1 \wedge |z|^{-2})$ for $z \in G$. That means $H - H^{\epsilon_0} \in H_\omega^{3/2-\delta}$ and we can choose $r = 3/2 - \delta$ in Proposition 4.1.2. This yields the claim. □

Now the properties of V^ϵ are also a direct consequence of Proposition 4.1.2. Unlike the norm estimates of V these are independent of time t. They only depend upon the approximation parameter ϵ.

Lemma 4.1.4 *Let $\omega \in [-\eta, \eta]$. For X satisfying (A1) the error of the approximation by the regularized payoff function H^ϵ satisfies for $0 \leq s \leq 1, t \in [0, T]$ and*

$$\|V(t, \cdot) - V^\epsilon(t, \cdot)\|_{H_\omega^s} \leq C\epsilon^{3/2-s}.$$

Now let $s \geq 0$ and $k \in \mathbb{N}_0$ such that $0 \leq s + k(\varrho \vee 1) + 1/2 \leq M_p$. Then we have

$$\|D_1^k(V^\epsilon(t, \cdot) - V^{\epsilon_0}(t, \cdot))\|_{H_\omega^s} \leq Cv^\epsilon(s + k(\varrho \vee 1)).$$

Furthermore, we have the following auxiliary estimates for $\omega_1 \in (0, \eta], \omega_2 \in (-1, \eta]$:

$$\begin{aligned} \|D_1^k V^\epsilon(t, \cdot)\|_{H_{\omega_1}^s} &\leq Cv^\epsilon(s + k(\varrho \vee 1)), \\ \|D_1^k D_2 V^\epsilon(t, \cdot)\|_{H_{\omega_2}^s} &\leq Cv^\epsilon(s + 1 + k(\varrho \vee 1)). \end{aligned}$$

Particularly, we have $V^\epsilon \in H^k(0, T; H_{\omega_1}^s)$.

Proof. This is again a direct consequence of the second norm estimate in Proposition 4.1.2 and Lemma 4.1.1. To this end we apply the proposition for $f = H - H^\epsilon$ and $r = 1$, respectively for $f \in \{H^\epsilon - H^{\epsilon_0}, H^\epsilon\}$ and $r = M_p$. In each case we use the respective weight parameter $\omega, \omega_1, \omega_2$. □

Corollary 4.1.5 *For the option price function V and the approximate option price function V^ϵ we have*

$$\begin{aligned} V &\in C_b^\infty((0, T] \times \mathbb{R}), \\ V^\epsilon &\in C_b^{\tilde{M}_1, \tilde{M}_2}([0, T] \times \mathbb{R}), \end{aligned}$$

where $\tilde{M}_2 + \tilde{M}_1(\varrho \vee 1) + 1/2 + \delta \leq M_p, \tilde{M}_1, \tilde{M}_2 \in \mathbb{N}$.

Proof. These are direct consequences of Lemma 4.1.3 and Lemma 4.1.4. More specifically, the Sobolev Embedding Theorem 2.2.5 yields that

$$V^\epsilon - V^{\epsilon_0} \in C_b^{\tilde{M}_1, \tilde{M}_2}([0, T] \times \mathbb{R}).$$

Furthermore, we have by Lemma 4.1.4 for $0 < \omega < 1$ together with the Sobolev Embedding

$$e^{\omega x} V^{\epsilon_0} \in C_b^{\tilde{M}_1, \tilde{M}_2}([0,T] \times \mathbb{R}).$$

This yields the claim. □

Lemma 4.1.6 *The error of the approximation by the regularized payoff function H^ϵ satisfies*

$$\|J(T,x) - J^\epsilon(T,x)\|_{L^2} \leq C\epsilon^{3/2}.$$

Proof. Here, we use the notation of [JS03]. The hedging error J can be interpreted as projection of V onto a closed subspace, cf. [FS86]. That means, the approximation error of V carries over to the one of J in order of magnitude. To this end consider the Hilbert space $\mathscr{H}^2$ of square integrable martingales with initial value 0 that are stopped at T. Here, the scalar product is given by $\langle X, Y \rangle := E(X_T Y_T)$. Since V^ϵ and V are bounded we have for all $x \in \mathbb{R}$

$$V^\epsilon(T-t, x+X_t) - V^\epsilon(T, x+X_0) \in \mathscr{H}^2 \text{ and } V(T-t, x+X_t) - V(T, x+X_0) \in \mathscr{H}^2.$$

Denote by $Q : \mathscr{H}^2 \to \mathscr{H}^2$ the orthogonal projection with respect to this scalar product onto the closed subspace $\{\vartheta \bullet S; \vartheta \textit{ admissible}\}$. Here, this is a subspace by [ČK07, Corollary 2.5] and it is closed with respect to the scalar product by [ČK07, Corollary 2.9]. Within this framework we have

$$\begin{aligned} J(T,x) &= \|(Id - Q)(V(T-t, x+X_t) - V(T, x+X_0))\|^2_{\mathscr{H}^2}, \\ J^\epsilon(T,x) &= \|(Id - Q)(V^\epsilon(T-t, x+X_t) - V^\epsilon(T, x+X_0))\|^2_{\mathscr{H}^2}. \end{aligned}$$

For orthogonal projections in a Hilbert space we have $\|Id - Q\|_* \leq 1$ where $\|\cdot\|_*$ denotes the induced operator norm, cf. [Wer95, V.3.4]. That yields together with the bound $V, V^\epsilon \leq K = 1$ the following:

$$\begin{aligned} &\|J(T,x) - J^\epsilon(T,x)\|_{L^2} \leq \big\|Id - Q\big\|_* \cdot \\ &\quad \Big\| \big\|V(T-t, x+X_t) - V(T, x+X_0) - (V^\epsilon(T-t, x+X_t) - V^\epsilon(T, x+X_0))\big\|_{\mathscr{H}^2} \\ &\quad \cdot \big\|V(T-t, x+X_t) - V(T, x+X_0) + (V^\epsilon(T-t, x+X_t) - V^\epsilon(T, x+X_0))\big\|_{\mathscr{H}^2} \Big\|_{L^2} \\ &\leq 4 \Big\| \big\|V(0, x+X_T) - V(T, x+X_0) - (V^\epsilon(0, x+X_T) - V^\epsilon(T, x+X_0))\big\|_{\mathscr{H}^2} \Big\|_{L^2} \\ &\leq 4 \Big\| \big\|V(0, x+X_T) - V^\epsilon(0, x+X_T)\big\|_{\mathscr{H}^2} + \big\|V^\epsilon(T, x+X_0) - V(T, x+X_0)\big\|_{\mathscr{H}^2} \Big\|_{L^2}. \end{aligned}$$

The first term we can estimate as follows:

$$\begin{aligned}
&\Big\| \big\| V(0,x+X_T) - V^\epsilon(0,x+X_T) \big\|_{\mathscr{H}^2} \Big\|_{L^2} \\
= &\left\| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \big(H(x+y+z) - H^\epsilon(x+y+z) \big) P_{X_0}(dy) \right)^2 P_{X_T}(dz) \right\|_{L^1}^{1/2} \\
= &\left\| \int_{\mathbb{R}} \big(H(x+z) - H^\epsilon(x+z) \big)^2 P_{X_T}(dz) \right\|_{L^1}^{1/2} \\
\le &\left(\int_{\mathbb{R}} \big\| H(x+z) - H^\epsilon(x+z) \big\|_{L^2}^2 P_{X_T}(dz) \right)^{1/2} \\
= &\| H - H^\epsilon \|_{L^2} \\
\le &\epsilon^{3/2}.
\end{aligned}$$

The second follows along very similar lines:

$$\begin{aligned}
&\Big\| \big\| V(T,x+X_0) - V^\epsilon(T,x+X_0) \big\|_{\mathscr{H}^2} \Big\|_{L^2} \\
= &\left\| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \big(H(x+y+z) - H^\epsilon(x+y+z) \big) P_{X_T}(dy) \right)^2 P_{X_0}(dz) \right\|_{L^1}^{1/2} \\
= &\left\| \int_{\mathbb{R}} \big(H(x+y) - H^\epsilon(x+y) \big) P_{X_T}(dy) \right\|_{L^2} \\
\le &\int_{\mathbb{R}} \big\| H(x+y) - H^\epsilon(x+y) \big\|_{L^2} P_{X_T}(dy) \\
= &\| H - H^\epsilon \|_{L^2} \\
\le &\epsilon^{3/2}.
\end{aligned}$$

This yields the claim. □

4.2 Properties of the operators Γ and ψ

In this section the properties of the bilinear operators Γ and ψ will be determined. They play an important role for the derivation of the PIDE as well as the error estimates in the ensuing numerical treatment.

We start with $\Gamma(\exp, \exp)$. This can be represented in a very tractable manner

$$\Gamma(\exp, \exp)(x) = ce^{2x},$$

where $c := \sigma^2 + \int_{\mathbb{R}} (e^y - 1)^2 k(y) dy$. In the general case we have the problem that V^ϵ is not integrable which impedes a Fourier estimation approach for $\Gamma(V^\epsilon, V^\epsilon)$. However, they are needed in order to derive good estimates. Therefore, we introduce a setting that overcomes this deficiency.

Basically, we use the function V^{ϵ_0} in order to estimate $\Gamma(V^\epsilon, V^\epsilon)$. The estimate is split up in terms involving the difference $(V^\epsilon - V^{\epsilon_0}) \in L^1$ and such that only involve V^{ϵ_0}. The first can be estimated via the Fourier approach. For the others a coarser approach can be taken because they do not depend upon ϵ.

To this end we first define the following space of functions which reflects this strategy:

$$D^w_{s,\omega} := \{(f,\tilde{f}) \in C^1 \times C^1; \|f-\tilde{f}\|_{L^1_\omega} + \|f-\tilde{f}\|_{H^{s+\varrho/2}_\omega} + \|\tilde{f}'\|_{H^s_\omega} < \infty\}.$$

For $(f,\tilde{f}) \in D^w_{s,\omega}$ we can now define the following estimator:

$$\|f,\tilde{f}\|^w_{(s,\omega)} := \|f-\tilde{f}\|_{H^{s+\varrho/2}_\omega} + \|\tilde{f}'\|_{H^s_\omega}.$$

We can now derive the following two preliminary results. The first is a symmetric bound for the norm of a product of two functions. This result is most likely already well-known. However, an explicit reference for it could not be found. The second plays the key role for the estimate of Γ. For the sake of shorter notation we define $\Delta_y f(x) := f(x+y) - f(x)$.

Proposition 4.2.1 *Let $s \geq 0$ and $\omega, \omega_f \in \mathbb{R}$. For*

$$\begin{aligned} f &\in H^s_{\omega_f} \cap H^{1/2+\delta}_{\omega_f} \cap L^1_{\omega_f}, \\ g &\in H^s_{\omega-\omega_f} \cap H^{1/2+\delta}_{\omega-\omega_f} \cap L^1_{\omega-\omega_f} \end{aligned}$$

we have the following estimate of the product:

$$\|fg\|_{H^s_\omega} \leq C(\|f\|_{H^s_{\omega_f}} \|g\|_{H^{1/2+\delta}_{\omega-\omega_f}} + \|f\|_{H^{1/2+\delta}_{\omega_f}} \|g\|_{H^s_{\omega-\omega_f}}).$$

Furthermore, the following estimate holds for $\omega \in [-\eta/2, \eta/2]$ and $(f,\tilde{f}) \in D^w_{(s,\omega)}$:

$$\left\| \|\Delta_y f(\cdot)\|_{H^s_\omega} \sqrt{k(y)} \right\|_{L^2_y} \leq \|f,\tilde{f}\|^w_{(s,\omega)}.$$

Proof. For the first claim we first state the basic inequality for $a, b > 0$ and $s \geq 0$

$$(a+b)^s \leq 2^s(a^s + b^s).$$

Let $f^{\omega_f}(x) := e^{\omega_f x} f(x)$ and $g^{\omega-\omega_f}(x) := e^{(\omega-\omega_f)x} g(x)$ for $x \in \mathbb{R}$. Since $f^{\omega_f}, g^{\omega-\omega_f} \in L^1$ the convolution formula for the Fourier transform holds. Furthermore, we have $fg \in$

L^1_ω and therefore $\|fg\|_{H^s_\omega} = \|e^{\omega\cdot} fg\|_{H^s}$. That means, we have the following:

$$\begin{aligned}
\|fg\|_{H^s_\omega} &= \|f^{\omega_f} g^{\omega-\omega_f}\|_{H^s} \\
&\leq C\|(1+|z|^2)^{s/2}(\widehat{f^{\omega_f} g^{\omega-\omega_f}})(z)\|_{L^2} \\
&\leq C\|(1+|z|)^s(\widehat{f^{\omega_f}} * \widehat{g^{\omega-\omega_f}})(z)\|_{L^2} \\
&= C\left\|\int_{\mathbb{R}}(1+|z|)^s \widehat{f^{\omega_f}}(z-\xi)\widehat{g^{\omega-\omega_f}}(\xi)d\xi\right\|_{L^2} \\
&\leq C\left(\left\|\int_{\mathbb{R}}(1+|z-\xi|)^s \widehat{f^{\omega_f}}(z-\xi)\widehat{g^{\omega-\omega_f}}(\xi)d\xi\right\|_{L^2_z}\right. \\
&\qquad \left. + \left\|\int_{\mathbb{R}} \widehat{f^{\omega_f}}(z-\xi)(1+|\xi|)^s\widehat{g^{\omega-\omega_f}}(\xi)d\xi\right\|_{L^2_z}\right) \\
&\leq C\Bigg(\int_{\mathbb{R}}\left\|(1+|z-\xi|)^s\widehat{f^{\omega_f}}(z-\xi)\right\|_{L^2_z}\left|\widehat{g^{\omega-\omega_f}}(\xi)\right|d\xi \\
&\qquad + \int_{\mathbb{R}}\left|\widehat{f^{\omega_f}}(\xi)\right|\left\|(1+|z-\xi|)^s\widehat{g^{\omega-\omega_f}}(z-\xi)\right\|_{L^2_z} d\xi\Bigg) \\
&\leq C\Big(\|f^{\omega_f}\|_{H^s}\|\widehat{g^{\omega-\omega_f}}\|_{L^1} + \|\widehat{f^{\omega_f}}\|_{L^1}\|g^{\omega-\omega_f}\|_{H^s}\Big) \\
&\overset{(*)}{\leq} C\Big(\|f^{\omega_f}\|_{H^s}\|g^{\omega-\omega_f}\|_{H^{1/2+\delta}} + \|f^{\omega_f}\|_{H^{1/2+\delta}}\|g^{\omega-\omega_f}\|_{H^s}\Big) \\
&\leq C\Big(\|f\|_{H^s_{\omega_f}}\|g\|_{H^{1/2+\delta}_{\omega-\omega_f}} + \|f\|_{H^{1/2+\delta}_{\omega_f}}\|g\|_{H^s_{\omega-\omega_f}}\Big).
\end{aligned}$$

Here, the inequality $(*)$ could be derived similarly to the Sobolev Lemma as follows:

$$\begin{aligned}
\int_{\mathbb{R}}|\widehat{f^{\omega_f}}(\xi)|d\xi &\leq \|(1+|\xi|^2)^{-\frac{1}{2}(1/2+\delta)}\|_{L^2}\|(1+|\xi|^2)^{\frac{1}{2}(1/2+\delta)}\widehat{f^{\omega_f}}(\xi)\|_{L^2} \\
&\leq C\|f^{\omega_f}\|_{H^{1/2+\delta}}.
\end{aligned}$$

The same holds for $g^{\omega-\omega_f}$. With this the first claim follows. The second claim can now be derived with the properties of the kernel k as stated in (A1)-(A4). Let now for the ensuing analysis $(f,\tilde{f}) \in D^w_{(s,\omega)}$. Then we have

$$\left\|\|\Delta_y f\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y} \leq \left\|\|\Delta_y(f-\tilde{f})\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y} + \left\|\|\Delta_y\tilde{f}\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y}.$$

For the first term we have

$$\begin{aligned}
&\left\|\|\Delta_y(f-\tilde{f})(\cdot)\|_{H^s_\omega}\sqrt{k(y)}\right\|^2_{L^2_y} \\
= \quad &\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|z|^2)^s\left|\mathcal{F}(\Delta_y(f-\tilde{f})(\cdot))(z+i\omega)\right|^2 dzk(y)dy \\
\leq \quad &\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|z|^2)^s|e^{i(z+i\omega)y}-1|^2|\mathcal{F}(f-\tilde{f})(z+i\omega)|^2 dzk(y)dy \\
\overset{\text{Fubini}}{=} \quad &\left\|(1+|z|^2)^{s/2}|\mathcal{F}(f-\tilde{f})(z+i\omega)|\|(e^{i(z+i\omega)y}-1)\sqrt{k(y)}\|_{L^2_y}\right\|^2_{L^2_z}.
\end{aligned}$$

For all $z \in \mathbb{C} \setminus \{0\}$ with $|\Im(z)| \leq \eta/2$ we have due to $k(y) \leq C|y|^{-1-\nu}$ and $\nu < 2$ the following:

$$
\begin{aligned}
&\left\|(e^{izy}-1)\sqrt{k(y)}\right\|_{L^2_y}^2 = \int_{\mathbb{R}} |e^{izy}-1|^2 k(y)dy \\
&= \int_{\mathbb{R}\setminus[-1,1]} |e^{izy}-1|^2 k(y)dy + \int_{1\geq|y|\geq 1/|z|} |e^{izy}-1|^2 k(y)dy + \int_{|y|\leq 1/|z|} |e^{izy}-1|^2 k(y)dy \\
&\leq C\Bigg(\int_{\mathbb{R}\setminus[-1,1]} e^{2|\Im(z)||y|}k(y)dy + \int_{1\geq|y|\geq 1/|z|} e^{2|\Im(z)||y|}k(y)dy \\
&\qquad + \int_{|y|\leq 1/|z|} \left|\int_0^1 e^{izy\theta}d\theta\right|^2 |z|^2|y|^2 k(y)dy\Bigg) \\
&\leq C\Bigg(\int_{\mathbb{R}\setminus[-1,1]} e^{\eta|y|}k(y)dy + \int_{1\geq|y|\geq 1/|z|} e^{\eta}|y|^{-1-\nu}dy \\
&\qquad + \int_{|y|\leq 1/|z|} \int_0^1 e^{2|z||y|}d\theta |z|^2|y|^{1-\nu}dy\Bigg) \\
&\leq C\left(1 + \left[y^{-\nu}\right]_{1/|z|}^{1} + e^2|z|^2\left[y^{2-\nu}\right]_0^{1/|z|}\right) \\
&\leq C(1+|z|^{\nu}) \\
&\leq C\left(1+|z|^2\right)^{\nu/2}.
\end{aligned}
$$

Inserting this inequality finally yields

$$
\begin{aligned}
\left\|\|\Delta_y(f-\tilde{f})(\cdot)\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y} &\leq C\left\|(1+|z|^2)^{\frac{s+\nu/2}{2}}\mathcal{F}(f-\tilde{f})(z+i\omega)\right\|_{L^2} \\
&\leq C\|f-\tilde{f}\|_{H^{s+\varrho/2}_\omega}.
\end{aligned}
$$

The second term can now be estimated as follows:

$$
\begin{aligned}
\left\|\|\Delta_y\tilde{f}(\cdot)\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y} &= \left\|\left\|y\int_0^1 \tilde{f}'(\cdot+\theta y)d\theta\right\|_{H^s_\omega}\sqrt{k(y)}\right\|_{L^2_y} \\
&\leq \int_0^1 \left\|\|\tilde{f}'(\cdot+\theta y)\|_{H^s_\omega} y\sqrt{k(y)}\right\|_{L^2_y} d\theta \\
&\leq \|\tilde{f}'\|_{H^s_\omega}\int_0^1 \left\|y\sqrt{k(y)}\right\|_{L^2_y} d\theta \\
&\leq C\|\tilde{f}'\|_{H^s_\omega}.
\end{aligned}
$$

Overall, this shows the second claim. □

The key estimate has now been shown. It remains to transfer the idea and estimate upon the bilinear operator Γ. The operator image will be estimated with respect to L^1

and L^2. Therefore we have to deal with two spaces and the corresponding estimators. That means, we define the following spaces

$$\begin{aligned} D^{\Gamma_1}_{\omega_f,\omega} &:= D^w_{0,\omega_f} \times D^w_{0,\omega-\omega_f}, \\ D^{\Gamma_2}_{s,\omega_f,\omega} &:= (D^w_{s,\omega_f} \cap D^w_{1/2+\delta,\omega_f}) \times (D^w_{s,\omega-\omega_f} \cap D^w_{1/2+\delta,\omega-\omega_f}), \end{aligned}$$

and the corresponding norm estimates for $(f, \tilde{f}, g, \tilde{g}) \in D^{\Gamma_1}_{(\omega_f,\omega)}$

$$\|f, \tilde{f}, g, \tilde{g}\|^{\Gamma_1}_{(\omega_f,\omega)} := \|f, \tilde{f}\|^w_{(0,\omega_f)} \|g, \tilde{g}\|^w_{(0,\omega-\omega_f)},$$

respectively for $(f, \tilde{f}, g, \tilde{g}) \in D^{\Gamma_2}_{(s,\omega_f,\omega)}$

$$\|f, \tilde{f}, g, \tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)} := \|f, \tilde{f}\|^w_{(s,\omega_f)} \|g, \tilde{g}\|^w_{(1/2+\delta,\omega-\omega_f)} + \|f, \tilde{f}\|^w_{(1/2+\delta,\omega_f)} \|g, \tilde{g}\|^w_{(s,\omega-\omega_f)}.$$

With this at hand the results of the previous proposition can be applied to Γ. Thus, the following norm estimates of $\Gamma(f, g)$ can finally be derived.

Lemma 4.2.2 *Let $\omega_f \in [-\eta/2, \eta/2], \omega \in [-\eta, \eta]$, such that $\omega - \omega_f \in [-\eta/2, \eta/2]$. Let further $s \geq 0$ and $(f, \tilde{f}, g, \tilde{g}) \in D^{\Gamma_2}_{s,\omega_f,\omega}$. Then we have the following norm estimates:*

$$\begin{aligned} \|\Gamma(f, g)\|_{H^s_\omega} &\leq C\|f, \tilde{f}, g, \tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}, \\ \|e^{-x}\Gamma(f, \exp)\|_{H^s_{\omega_f}} &\leq C\|f, \tilde{f}\|^w_{(s,\omega_f)}. \end{aligned}$$

And for $(f, \tilde{f}, g, \tilde{g}) \in D^{\Gamma_1}_{\omega_f,\omega}$ we have

$$\|\Gamma(f, g)\|_{L^1_\omega} \leq C\|f, \tilde{f}, g, \tilde{g}\|^{\Gamma_1}_{(\omega_f,\omega)}.$$

Proof. The claim is clear for the terms corresponding to σ^2. For the integral part we can apply Proposition 4.2.1 and get

$$\begin{aligned} \|\Gamma(f, g)\|_{H^s_\omega} &\leq \int_{\mathbb{R}} \|\Delta_y f(\cdot)\Delta_y g(\cdot)\|_{H^s_\omega} k(y) dy \\ &\leq C \int_{\mathbb{R}} \|\Delta_y f(\cdot)\|_{H^s_{\omega_f}} \|\Delta_y g(\cdot)\|_{H^{1/2+\delta}_{\omega-\omega_f}} k(y) dy \\ &\quad + C \int_{\mathbb{R}} \|\Delta_y f(\cdot)\|_{H^{1/2+\delta}_{\omega_f}} \|\Delta_y g(\cdot)\|_{H^s_{\omega-\omega_f}} k(y) dy \\ &\leq C \left\| \|\Delta_y f(\cdot)\|_{H^s_{\omega_f}} \sqrt{k(y)} \right\|_{L^2_y} \left\| \|\Delta_y g(\cdot)\|_{H^{1/2+\delta}_{\omega-\omega_f}} \sqrt{k(y)} \right\|_{L^2_y} \\ &\quad + C \left\| \|\Delta_y f(\cdot)\|_{H^{1/2+\delta}_{\omega_f}} \sqrt{k(y)} \right\|_{L^2_y} \left\| \|\Delta_y g(\cdot)\|_{H^s_{\omega-\omega_f}} \sqrt{k(y)} \right\|_{L^2_y} \\ &\leq C(\|f, \tilde{f}\|^w_{(s,\omega_f)} \|g, \tilde{g}\|^w_{(1/2+\delta,\omega-\omega_f)} + \|f, \tilde{f}\|^w_{(1/2+\delta,\omega_f)} \|g, \tilde{g}\|^w_{(s,\omega-\omega_f)}) \\ &\leq C\|f, \tilde{f}, g, \tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}. \end{aligned}$$

The second claim can be derived along similar lines.

$$\begin{aligned}\|e^{-x}\Gamma(f,exp)\|_{H^s_{\omega_f}} &\leq C\Big\|\|\Delta_y(f)(\cdot)\|_{H^s_{\omega_f}}\sqrt{k(y)}\Big\|_{L^2_y}\Big\|(e^y-1)\sqrt{k(y)}\Big\|_{L^2_y}\\ &\leq C\|f,\tilde{f}\|^w_{(s,\omega_f)}.\end{aligned}$$

The last claim follows along the same lines as the first. Here, we can apply Cauchy's inequality for both estimates.

$$\begin{aligned}\|\Gamma(f,g)\|_{L^1_\omega} &\leq \int_{\mathbb{R}}\|\Delta_y f(\cdot)\Delta_y g(\cdot)\|_{L^1_\omega}k(y)dy\\ &\leq C\Big\|\|\Delta_y f(\cdot)\|_{L^2_{\omega_f}}\sqrt{k(y)}\Big\|_{L^2_y}\Big\|\|\Delta_y g(\cdot)\|_{L^2_{\omega-\omega_f}}\sqrt{k(y)}\Big\|_{L^2_y}\\ &\leq C\|f,\tilde{f}\|^w_{(0,\omega_f)}\|g,\tilde{g}\|^w_{(0,\omega-\omega_f)}\\ &\leq C\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_1}_{(\omega_f,\omega)}.\end{aligned}$$

□

With this we can estimate the norm of $\psi(f,g)$. It satisfies the same norm conditions as $\Gamma(f,g)$. This is reflected in the ensuing lemma.

Lemma 4.2.3 *Let $\omega_f\in[-\eta/2,\eta/2],\omega\in[-\eta,\eta]$, such that $\omega-\omega_f\in[-\eta/2,\eta/2]$. Let further $s\geq 0$ and $(f,\tilde{f},g,\tilde{g})\in D^{\Gamma_2}_{s,\omega_f,\omega}$. Then we have the following norm estimate:*

$$\|\psi(f,g)\|_{H^s_\omega}\leq C\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}.$$

Finally for $(f,\tilde{f},g,\tilde{g})\in D^{\Gamma_1}_{\omega_f,\omega}$ we have

$$\|\psi(f,g)\|_{L^1_\omega}\leq C\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_1}_{(\omega_f,\omega)}.$$

Proof. The previous lemma together with Proposition 4.2.1 directly yield the following:

$$\begin{aligned}\|\psi(f,g)\|_{H^s_\omega} &\leq C\Big(\|\Gamma(f,g)\|_{H^s_\omega}+\|e^{-x}\Gamma(f,\exp)\|_{H^s_{\omega_f}}\|e^{-x}\Gamma(g,\exp)\|_{H^{1/2+\delta}_{\omega-\omega_f}}\\ &\quad+\|e^{-x}\Gamma(f,\exp)\|_{H^{1/2+\delta}_{\omega_f}}\|e^{-x}\Gamma(g,\exp)\|_{H^s_{\omega-\omega_f}}\Big)\\ &\leq C\Big(\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}+\|f,\tilde{f}\|^w_{(s,\omega_f)}\|g,\tilde{g}\|^w_{(1/2+\delta,\omega-\omega_f)}\\ &\quad+\|f,\tilde{f}\|^w_{(1/2+\delta,\omega_f)}\|g,\tilde{g}\|^w_{(s,\omega-\omega_f)}\Big)\\ &\leq C\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_2}_{(s,\omega_f,\omega)}.\end{aligned}$$

Along the same lines as in the previous proof we get the second result using Cauchy's inequality instead of Proposition 4.2.1. □

4.3 Properties of ψ, ψ^ϵ and J, J^ϵ

With the properties of the bilinear operator ψ we can now start to derive norm bounds for ψ, ψ^ϵ. These are the basis for the estimates of the norms of J and J^ϵ.

Lemma 4.3.1 *Let X satisfy (A1), $\omega \in (-2, \eta]$ and $k \in \mathbb{N}_0, s > 0$ such that*

$$(s + \varrho/2 + k(\varrho \vee 1)) \vee (1/2 + \delta + \varrho/2 + k(\varrho \vee 1)) \leq M_p - 1/2.$$

If $t \in [0, T]$ we have

$$\|D_1^k \psi(V^\epsilon, V^\epsilon)(t, \cdot)\|_{H^s_\omega} \leq C(v^\epsilon(s + \varrho/2 + k(\varrho \vee 1)) + v^\epsilon(1/2 + \delta + \varrho/2 + k(\varrho \vee 1))).$$

Furthermore, if additionally $t > 0$ we have

$$\|D_1^k \psi(V, V)(t, \cdot)\|_{H^s_\omega} \leq C(v^t(s + \varrho/2 + k(\varrho \vee 1)) + v^t(1/2 + \delta + \varrho/2 + k(\varrho \vee 1))).$$

Particularly, this implies

$$\psi(V, V) \in C_b^\infty((0, T] \times \mathbb{R}) \text{ and } \psi(V^\epsilon, V^\epsilon) \in C_b^{\tilde{M}}([0, T] \times \mathbb{R})$$

for $3/2 + \tilde{M} + \tilde{M}(\varrho \vee 1) + \varrho/2 \leq M_p$.

Proof. The proof is an application of the properties of ψ in Lemma 4.2.3 and the respective properties of V in Lemma 4.1.3 and of V^ϵ in Lemma 4.1.4. Here, we only show the claim for V^ϵ. The other one follows along the very same lines with $t > 0$. Now, we first consider the existence of the time derivative. For $k = 1$ this can be done with dominated convergence in the sense of Proposition 2.2.2. To this end, it is sufficient to consider $\Gamma(D_1 V^\epsilon, V^\epsilon)(t, x)$ and $\Gamma(D_1 V^\epsilon, \exp)(t, x)$. Similar to the argument in the proof of Lemma 4.2.2 an integrable dominating function can be derived as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} (D_1 V^\epsilon(t, x + y) - D_1 V^\epsilon(t, x))(V^\epsilon(t, x + y) - V^\epsilon(t, x))k(y)dy \\
&= \int_{\mathbb{R}} \int_0^1 D_1 D_2 V^\epsilon(t, x + \theta y)d\theta \int_0^1 D_2 V^\epsilon(t, x + \theta y)d\theta y^2 k(y)dy \\
&\leq Cy^2 k(y)\|D_1 D_2 V^\epsilon(t)\|_C \|D_2 V^\epsilon(t)\|_C.
\end{aligned}$$

By Lemma 4.1.4 these norms exist and bounded independent of t for all $t \in [0, T]$. The same argument holds for $\Gamma(V^\epsilon, \exp)$. Therefore, we can apply dominated convergence in the sense of Proposition 2.2.2 and the time derivative is given by

$$D_1 \psi(V^\epsilon, V^\epsilon) = 2\psi(D_1 V^\epsilon, V^\epsilon).$$

An iteration of the arguments yields together with the Leibniz formula the following representation for $k \in \mathbb{N}_0$:

$$D_1^k \psi(V^\epsilon, V^\epsilon)(t, x) = \sum_{l=0}^{k} \binom{k}{l} \psi(D_1^l V^\epsilon, D_1^{k-l} V^\epsilon)(t, x).$$

Now we can start to derive the norm estimates. We first state that we have for all $m \in \mathbb{N}_0, t \in [0,T]$ and $r \geq 0$ the following by Lemma 4.1.4:

$$\|D_1^m D_2 V^{\epsilon_0}(t,\cdot)\|_{H^r_{\omega/2}} \leq C.$$

Therefore, we have

$$\|D_1^m V^\epsilon(t,\cdot), D_1^m V^{\epsilon_0}(t,\cdot)\|^w_{(r,\omega/2)} \;\leq\; v^\epsilon(r + \varrho/2 + m(\varrho \vee 1)).$$

Now, we can apply the norm estimate of ψ in Lemma 4.2.3 in order to finally get the norm following estimates:

$$\begin{aligned}
&\|\psi(D_1^l V^\epsilon(t,\cdot), D_1^{k-l} V^\epsilon)(t,\cdot)\|_{H^s_\omega} \\
&\leq\; C\|D_1^l V^\epsilon(t,\cdot), D_1^l V^{\epsilon_0}(t,\cdot), D_1^{k-l} V^\epsilon(t,\cdot), D_1^{k-l} V^{\epsilon_0}(t,\cdot)\|^{\Gamma_2}_{(s,\omega/2,\omega)} \\
&\leq\; C(v^\epsilon(s + \varrho/2 + l(\varrho \vee 1))v^\epsilon(1/2 + \delta + \varrho/2 + (k-l)(\varrho \vee 1)) \\
&\qquad + v^\epsilon(s + \varrho/2 + (k-l)(\varrho \vee 1))v^\epsilon(1/2 + \delta + \varrho/2 + l(\varrho \vee 1))).
\end{aligned}$$

Since $\varrho/2 \leq 1$ we can further estimate

$$\begin{aligned}
&v^\epsilon(s + \varrho/2 + l(\varrho \vee 1))v^\epsilon(1/2 + \delta + \varrho/2 + (k-l)(\varrho \vee 1)) \\
&\leq\; 1 + \epsilon^{3/2 - s - \varrho/2 - l(\varrho\vee 1)} + \epsilon^{1 - \delta - \varrho/2 - (k-l)(\varrho \vee 1)} + \epsilon^{5/2 - s - \delta - \varrho - k(\varrho \vee 1)} \\
&\leq\; 1 + \epsilon^{3/2 - s - \varrho/2 - k(\varrho \vee 1)} + \epsilon^{1 - \delta - \varrho/2 - k(\varrho \vee 1)} \\
&\leq\; v^\epsilon(s + \varrho/2 + k(\varrho \vee 1)) + v^\epsilon(1/2 + \delta + \varrho/2 + k(\varrho \vee 1)).
\end{aligned}$$

This yields the claim norm estimates. The Sobolev Embedding yields the remaining claim for $\psi(V^\epsilon, V^\epsilon)$. □

This at last allows to estimate the norms of J and J^ϵ. For J^ϵ there do not turn up new obstacles and the estimates can be derived as direct consequences of the previous lemma. However, for J we have to deal with a new problem. Up to now the norm estimates could be derived for all $t > 0$. And the fact that the estimates tended to infinity as t tended to zero was not fatal. Now, however, this is integrated over time. And due to the time-reversal in J the possible non-integrability of the norm estimates can now occur at every point of time t. Therefore, only a certain degree of regularity can now be shown for J.

Lemma 4.3.2 *Let X satisfy (A1), $\omega \in (-2, \eta]$ and $k \in \mathbb{N}_0, r > 0$ such that*

$$(r + \varrho/2 + k(\varrho \vee 1)) \vee (1/2 + \delta + \varrho/2 + k(\varrho \vee 1)) \leq M_p - 1/2.$$

If $t \in [0,T]$ we have

$$\|D_1^k J^\epsilon(t,\cdot)\|_{H^r_\omega} \leq C(v^\epsilon(r + \varrho/2 + k(\varrho \vee 1)) + v^\epsilon(1/2 + \delta + \varrho/2 + k(\varrho \vee 1))).$$

Furthermore, for $r < (3/2 + \varrho/2)$ we have

$$\|J(t,\cdot)\|_{H^r_\omega} \leq C.$$

Particularly, this implies $J^\epsilon \in C_b^{\tilde{M}}([0,T] \times \mathbb{R})$ for $3/2 + \tilde{M} + \tilde{M}(\varrho \vee 1) + \varrho/2 \leq M_p$.

Proof. Again we start with the existence of the time derivatives. By the previous lemma we have with the Sobolev Embedding

$$D_1^m\psi(V,V)\in C_b((0,T]\times\mathbb{R}) \text{ and } D_1^m\psi(V^\epsilon,V^\epsilon)\in C_b([0,T]\times\mathbb{R})$$

for all $0\le m\le k$. Therefore, we can apply dominated convergence. With a similar argument as in the proof of the previous lemma we can show that the time derivatives of J^ϵ are now given by

$$\begin{aligned} D_1^kJ^\epsilon(t,x) &= E\left(\int_0^t D_1^k\psi(V^\epsilon,V^\epsilon)(t-s,x+X_{s-})ds\right)\\ &\quad+\sum_{l=0}^{k-1}\left(\frac{\partial}{\partial t}\right)^l E(D_1^{k-1-l}\psi(V^\epsilon,V^\epsilon)(0,x+X_{t-})). \end{aligned}$$

Now we estimate the terms of this representation separately. The first term is bounded as follows with the properties of ψ in Lemma 4.3.1:

$$\begin{aligned} \left\|E\left(\int_0^t D_1^k\psi(V^\epsilon,V^\epsilon)(t-s,\cdot+X_{s-})ds\right)\right\|_{H_\omega^r} &\le \int_0^t \|D_1^k\psi(V^\epsilon,V^\epsilon)(t-s)\|_{H_\omega^r}ds\\ &\le C(v^\epsilon(r+\varrho/2+k(\varrho\vee1))+v^\epsilon(1/2+\delta+\varrho/2+k(\varrho\vee1))). \end{aligned}$$

The second term fits the generic function in Proposition 4.1.2 for

$$r=M_p-(k-1-l)(\varrho\vee1)-1/2-\varrho/2 \text{ and } f(x):=D_1^{k-1-l}\psi(V^\epsilon,V^\epsilon)(0,x).$$

This proposition together with the properties of ψ in Lemma 4.3.1 yield

$$\begin{aligned} \left\|\left(\frac{\partial}{\partial t}\right)^l E\Big(D_1^{k-1-l}\psi(V^\epsilon,V^\epsilon)(0,\cdot+X_{t-})\Big)\right\|_{H_\omega^r} &\le \|D_1^{k-1-l}\psi(V^\epsilon,V^\epsilon)(0,\cdot)\|_{H_\omega^{r+l(\varrho\vee1)}}\\ &\le C(v^\epsilon(r+\varrho/2+(k-1)(\varrho\vee1))+v^\epsilon(1/2+\delta+\varrho/2+(k-1-l)(\varrho\vee1))). \end{aligned}$$

The claim for J can be derived along the very same lines. Here, the condition upon r is necessary to ensure the integrability in the following:

$$\begin{aligned} \left\|E\left(\int_0^t \psi(V,V)(t-s,\cdot+X_{s-})ds\right)\right\|_{H_\omega^r} &\le \int_0^t \|\psi(V,V)(t-s)\|_{H_\omega^r}ds\\ &\le C\int_0^t (v^{t-s}(r+\varrho/2)+v^{t-s}(1/2+\delta+\varrho/2))ds\\ &\le C(v^t(r-\varrho/2)+v^t(1/2+\delta-\varrho/2))\\ &\le C. \end{aligned}$$

The Sobolev Embedding yields the remaining claim for J^ϵ. □

If we now combine these estimates we get the following.

Corollary 4.3.3 *The previous estimates imply the following. For $\tilde{M} \in \mathbb{N}$ such that $3/2 + \tilde{M} + \tilde{M}(\varrho \vee 1) + \varrho/2 \leq M_p$ we have*

$$\psi(V^\epsilon, V^\epsilon) \in C_b^{\tilde{M}}([0,T] \times \mathbb{R}) \cap H^{\tilde{M}}(0,T; H^{\tilde{M}}),$$
$$J^\epsilon \in C_b^{\tilde{M}}([0,T] \times \mathbb{R}) \cap H^{\tilde{M}}(0,T; H^{\tilde{M}}).$$

Furthermore, we have for $m \in \mathbb{N}_0$

$$\psi(V,V) \in C_b^\infty((0,T] \times \mathbb{R}) \cap H^m(\delta,T; H^m),$$
$$J \in C_b^{0,1+\varrho/2-\delta}([0,T] \times \mathbb{R}) \cap L^2(0,T; H^{3/2+\varrho/2-\delta}).$$

Proof. This is a direct consequence of the norm estimates of Lemma 4.3.1 and Lemma 4.3.2 as well as the Sobolev Embedding. □

Remarks. On the way to the derivation of the PIDE there are basically two obstacles. The first was the Itô formula for semimartingale characteristics in Theorem 3.0.3. The second is the Itô formula of Theorem 2.4.5 which will be used for the derivation of the PIDE. Both require functions in $C^2([0,T]\times\mathbb{R})$. Neither V nor J satisfy this condition. Here, we have seen that regularity in time represents the greatest obstacle.

4.4 PIDE for J^ϵ

Now we are finally able to derive the PIDE for J^ϵ. Introduce to this end the following operator. For

$$D(A) := \left\{ f \in C^2; \int_{\mathbb{R}} \left| f(\cdot+y) - f(\cdot) - (e^y - 1) Df(\cdot) \right| k(y) dy \in L^1 \text{ and } (D^2 f(\cdot) - Df(\cdot)) \in L^1 \right\}$$

we define $A^X : D(A) \to L^1$ as follows:

$$A^X f(x) \;:=\; \int_{\mathbb{R}} (f(x+y) - f(x) - (e^y - 1) Df(x)) k(y) dy + \frac{1}{2}\sigma^2 (D^2 f(x) - Df(x)).$$

For later use we define the following subset:

$$D_{L^1}(A) \;:=\; \{ f \in D(A); f, Df, D^2 f \in L^1 \}.$$

Remarks.

1. The properties of V^ϵ and J^ϵ in Lemma 4.1.4 and Lemma 4.1.6 show that we have $J^\epsilon(t,\cdot) \in D_{L^1}(A)$ and $V^\epsilon(t,\cdot) - V^{\epsilon_0}(t,\cdot) \in D_{L^1}(A)$. Furthermore, the calculations in Lemma 5.7.1 will show the integrability and regularity conditions for H^ϵ which are necessary for the integral part. They are clear for the part corresponding to σ^2 such that we have $H^\epsilon(t,\cdot) \in D(A)$.

2. This operator coincides with the generator of X as defined in Section 2.3 on the intersection of the respective domains.

Now we can finally show that J^ϵ satisfies a PIDE involving this operator. It strongly resembles the Kolmogorov Backward Equation and is basically a version of its generalization, the Feynmann-Kac formula. Such a generalization for a certain class of continuous models can be found in [HS00]. Here, we used an approach based upon the Itô formula to derive the corresponding equation for our model with jumps. The uniqueness of the solution in a corresponding set is shown in the next chapter, in Corollary 5.1.6.

Theorem 4.4.1 (PIDE for J^ϵ) *For X satisfying (A1)-(A4) the hedging error function J^ϵ solves the following PIDE.*

$$\begin{aligned} D_1 J^\epsilon(t,x) - A^X J^\epsilon(t,x) &= \psi(V^\epsilon, V^\epsilon)(t,x) \quad \forall (t,x) \in (0,T] \times \mathbb{R} \\ J^\epsilon(0,x) &= 0 \quad \forall x \in \mathbb{R} \end{aligned} \tag{4.3}$$

Proof. As before we compute with respect to the 'truncation' function $h(x) = x$. We start by studying the following process for $x \in \mathbb{R}$:

$$M_t^1 := E\left(\int_0^T \psi(V^\epsilon, V^\epsilon)(T-s, x + X_{s-}) ds \middle| \mathcal{F}_t \right) - J^\epsilon(T,x).$$

This is by construction a local martingale with initial value zero. By Corollary 4.3.3 it is bounded and thus a martingale by Proposition 2.4.1. We now proceed by exploiting the Markov nature of X.

$$\begin{aligned} &M_t^1 + J^\epsilon(T,x) \\ &= E\left(\int_t^T \psi(V^\epsilon, V^\epsilon)(T-s, x+X_{s-}) ds \middle| \mathcal{F}_t \right) + \int_0^t \psi(V^\epsilon, V^\epsilon)(T-s, x+X_{s-}) ds \\ &= E^{X_t}\left(\int_0^{T-t} \psi(V^\epsilon, V^\epsilon)(T-t-s, x+X_{s-}) ds \right) + \int_0^t \psi(V^\epsilon, V^\epsilon)(T-s, x+X_{s-}) ds \\ &= J^\epsilon(T-t, x+X_t) + \int_0^t \psi(V^\epsilon, V^\epsilon)(T-s, x+X_{s-}) ds. \end{aligned} \tag{4.4}$$

By Corollary 4.3.3 we know $J^\epsilon \in C^2([0,T] \times \mathbb{R})$. Therefore, we can apply the Itô formula of Theorem 2.4.5 on J^ϵ and the process $\left({}^{T-\cdot}_{x+X_\cdot} \right)_{T\wedge t}$. This yields that

$$\begin{aligned}
M_t^2 \quad := \quad & J^\epsilon(T-t, x+X_t) - J^\epsilon(T, x+X_0) \\
& - \int_0^t \Bigg(- D_1 J^\epsilon(T-s, x+X_{s-}) + D_2 J^\epsilon(T-s, x+X_{s-}) b \\
& + \frac{1}{2} D_{22} J^\epsilon(T-s, x+X_{s-}) \sigma^2 \\
& - \int_{\mathbb{R}} \Big(J^\epsilon(T-s, x+X_{s-}+y) - J^\epsilon(T-s, x+X_{s-}) \\
& - y D_2 J^\epsilon(T-s, x+X_{s-}) \Big) k(y) dy \Bigg) ds \\
= \quad & J^\epsilon(T-t, x+X_t) - J^\epsilon(T, x+X_0) \\
& - \int_0^t \Big(- D_1 J^\epsilon(T-s, x+X_{s-}) + A^X J^\epsilon(T-s, x+X_{s-}) \Big) ds
\end{aligned} \tag{4.5}$$

is a local martingale. Furthermore, with Corollary 4.3.3 we can show that it is bounded. Therefore, we can again conclude that it is a martingale. Indeed, the boundedness follows directly for the functions $J^\epsilon, D_2 J^\epsilon, D_{22} J^\epsilon$ and $D_1 J^\epsilon$. For the remaining term of $A^X J^\epsilon$ this can be seen as follows:

$$\begin{aligned}
& \left\| \int_{\mathbb{R}} (J^\epsilon(t, x+y) - J^\epsilon(t,x) - (e^y - 1) D_2 J^\epsilon(t,x)) k(y) dy \right\|_{C([0,T]\times\mathbb{R})} \\
\leq \quad & \left\| \int_{\mathbb{R}} \int_0^1 \int_0^{\theta_1} D_{22} J^\epsilon(t, x+\theta_2 y) d\theta_2 d\theta_1 y^2 k(y) dy \right\|_{C([0,T]\times\mathbb{R})} \\
& + \int_{\mathbb{R}} |e^y - 1 - y| k(y) dy \|D_2 J^\epsilon\|_{C([0,T]\times\mathbb{R})} \\
\leq \quad & \|D_{22} J^\epsilon\|_{C([0,T]\times\mathbb{R})} \int_{\mathbb{R}} y^2 k(y) dy + C \|D_2 J^\epsilon\|_{C([0,T]\times\mathbb{R})} \\
\leq \quad & C(\|D_{22} J^\epsilon\|_{C([0,T]\times\mathbb{R})} + \|D_2 J^\epsilon\|_{C([0,T]\times\mathbb{R})}).
\end{aligned}$$

We now set

$$\begin{aligned}
Y_s \quad & := - D_1 J^\epsilon(T-s, x+X_{s-}) + A^X J^\epsilon(T-s, x+X_{s-}) + \psi(V^\epsilon, V^\epsilon)(T-s, x+X_{s-}), \\
Z_t \quad & := \int_0^t Y_s ds.
\end{aligned}$$

The representations (4.5) and (4.4) show that we have

$$M_t^1 - M_t^2 = Z_t.$$

That means, Z is a martingale as well. The boundedness of the functions $D_1 J^\epsilon, A^X J^\epsilon$ and $\psi(V^\epsilon, V^\epsilon)$ furthermore yields the boundedness of Y. Therefore, the path $t \mapsto Y_t(\omega)$ is bounded for all $\omega \in \Omega$ and the variation of Z,

$$\mathrm{Var}(Z)_t(\omega) \leq \int_0^t |Y_s(\omega)| ds,$$

is consequently finite. Obviously $t \mapsto Z_t(\omega)$ is continuous for all $\omega \in \Omega$. Thus, Z is a continuous martingale of finite variation and by Lemma 2.4.2 we thus have

$$Z_t = 0 \text{ for all } t \in [0,T] \text{ a.s.}$$

The continuity of $D_1 J^\epsilon$, $A^X J^\epsilon$ and $\psi(V^\epsilon, V^\epsilon)$ yields that Y is càglàd. (continu à gauche, limites à droite), i.e. the paths $t \mapsto Y_t(\omega)$ are continuous from the left with existing limits from the right.

That means for each $\omega \in \Omega$ and $t \in (0,T]$ we have

$$0 = \lim_{h \searrow 0} \frac{1}{h}(Z_{t-h}(\omega) - Z_t(\omega)) = Y_t(\omega).$$

This leads to

$$Y_s = 0 \text{ for all } s \in (0,T] \text{ a.s.}$$

Due to the assumptions on X we can apply [Sat99, Theorem 24.10(i)]. That means, there exists an $\omega \in \Omega$ for each $t \in (0,T]$ and λ-almost all $x \in \mathbb{R}$ such that $X_t(\omega) = x$. Consequently, we have for all $t \in [0,T)$ and λ-almost all $x \in \mathbb{R}$

$$-D_1 J^\epsilon(t,x) + A^X J^\epsilon(t,x) + \psi(V^\epsilon, V^\epsilon)(t,x) = 0.$$

Since the left hand side is continuous in x, the equation holds on the whole real line $\mathbb{R}$. Furthermore, it is continuous in t. Therefore, the equation holds for $(t,x) \in [0,T] \times \mathbb{R}$. This finally yields the claim. □

Chapter 5

Numerical solution of the PIDE

Main thread. *In order to compute the hedging error function a system of two PIDEs has to be solved numerically. The first is the well-known Kolmogorov Backward equation for the option price function, which is used to assemble the second PIDE for the hedging error function. Since both are very similar, only differing in the right hand side, they can be treated nearly the same way. Therefore, it is sufficient to consider a generic PIDE which includes both cases.*

Several steps have to be performed for the numerical computation. Most of them are standard steps when dealing with PDEs numerically. Before starting with those some basic transformations are performed such that the equation is cast into a standard setting. Particularly, the operator has to be transformed in order to ensure coercivity. This is a crucial property which among other things ensures the uniqueness of the solution. Furthermore, the operator is then generalized in three more ways. Firstly, a PDO is defined which extends this operator to a larger domain. This is necessary to be able to cast the resulting PIDE into a variational setting. Furthermore, this enables to show continuity and coercivity for the different settings quite easily. To this end, we further define powers of this operator which is the second generalization. Thus, for every Sobolev norm we now have an equivalent energy norm with respect to the operator above at hand. This property we will use for the spatial error estimate. Finally, we add weights. This allows to work in respective weighted spaces.

Now, the equation can be cast into a variational setting by multiplying the whole equation with test functions and then integrating. To this end the space of test functions has be determined as well as the space of functions where the solution of the variational equation is sought for. In the Galerkin setting these two spaces coincide. The respective space is chosen in such a way that the resulting sesquilinear form is defined, continuous and coercive. This already ensures existence and uniqueness of the solution and yields a priori estimates.

Step by step this space will now be simplified while taking into account numerical errors. For each step the resulting error will be estimated in terms of a respective parameter. In the end these parameters are expressed in terms of a reference parameter. The first simplification is to localize the equation. That means the new space Y *consists only of functions whose support is a subset of a finite interval* Ω. *But the respective* *right hand side of the equation can be shown to decrease exponentially as* x *tends to infinity. Therefore, the localization error can be shown to decrease exponentially as well. The size of the interval will consequently be chosen to depend logarithmically on the reference parameter.*

Next, the interval is subdivided resulting in N *subintervals of width* h*. The parameter* h *will be the reference parameter. That means that the others will be chosen depending upon the same. The overall error estimate will therefore be given in terms of* h*, too. The space is accordingly simplified to the space* Y_h *of all piecewise polynomials of degree* p*. This is a finite-dimensional space. That means, as soon as a basis is chosen the variational equation results in a finite number of linear equations for each point of time* t*. But due to the non-locality of the operator this system is in general fully populated which would lead to a high computation cost. Therefore, a matrix compression technique in the spirit of [Sch98] is applied which results in a sparse matrix* $\tilde{A}$*. The resulting approximations are solutions of respective variational equations with approximate operators* A_h *and* $\tilde{A}_h$*. That means, they can be represented as results of an application of the exponential of the respective operator. Thus, the additional error caused by that can be estimated using functional calculus. This yields an error estimate which is a trade-off between the order of the estimate in terms of* h *and the order of the Sobolev norm that is applied to the data. That means, an increase of the exponent of* h *in the estimate has to be compensated by a higher order of the corresponding norm.*

Therefore, the system of linear equations can now efficiently be solved for a fixed point of time t *using an appropriate algorithm, namely GMRES. In case of the PIDE for the hedging error there remains the assembly of the right hand side. This should be done efficiently in terms of additional computation steps and in terms of additional implementation effort. To this end a decomposition of* Γ *as in [BL89] is applied:*

$$\Gamma(f,g) = A(fg) - fAg - gAf.$$

The operator A *is now substituted by* A_d*. Its application results in a matrix vector multiplication where the matrices involved are sparse. Moreover, the respective matrices have already been implemented or can be assembled easily. To be able to derive an error estimate of this sparse assembly independent of the size of* Ω *a similar technique as for the estimate of the operator* Γ *is applied. That means, the bilinearity of* $\Gamma(f,f)$ *is used to split up the error estimate in a term involving* $f-\tilde{f}$ *and one only involving* $\tilde{f}$*. Here,* $\tilde{f}$ *is again independent of* h *or* ϵ *such that* $f-\tilde{f}$ *is integrable.*

It remains to discretize the time interval $[0,T]$*. To this end a Galerkin scheme is applied as well, namely a specialized discontinuous Galerkin (dG) scheme. This time the interval is subdivided into a geometric mesh and the degree of the piecewise polynomials in time is linearly increasing. That means, in the respective space* $\mathscr{S}^{\underline{r}}(\mathscr{M},Y_h)$ *the interval width and the degree of the polynomial increase as* t *increases. Thus, the analyticity of the solution for* $t>0$ *is exploited in a gradually increasing way. This allows to use an increasingly coarser grid while still maintaining the order of the error estimate. This results in a method whose additional error decreases exponentially with the number of subintervals of* $[0,T]$*. That means, it is sufficient to choose the number of intervals depending logarithmically upon the reference parameter.*

With this method, the option price function can be computed in $O(N(\log N)^8)$ *and the hedging error function in* $O(N\epsilon^{-(6+\delta)\varrho}(\log N)^8)$ *computation steps. The trading strategy can be computed using the approximate operator* Γ_d *from the sparse assembly together with the approximate option price function. This again results in some matrix-vector multiplications with sparse matrices. The overall estimates for the additional errors caused by all those steps can then be derived as a compilation of the particular numerical errors. It turns out that this results in a trade-off between* h*, the reference parameter, and* ϵ*, the approximation parameter. The latter is subsequently set to* $\epsilon := h^s$ *where the optimal exponent is given by a function involving the parameters of the model and the method. That means, finally* h *is the only free parameter left.*

In this chapter we will develop an algorithm to compute the solutions of the system of PIDEs. The first of which is the well-known Kolmogorov Backward Equation for the option price process. The second is the one we derived for the hedging error. To this end we will adapt the method that is given in [MSW06]. That means, we will apply several numerical techniques. However, most of them are standard procedure when dealing with PDEs numerically. But two steps are due to the special structure in this case. The first is the matrix compression which has to be applied. This is due to the non-local behavior of the generator of jump diffusions. The second is the approximate assembly. It allows for a fast and easily applicable procedure to assemble the second PIDE. Along the way there are several parameters with different effects onto the overall approximation error. The central of which shall be the mesh width h. In the last section these parameters will be chosen depending upon h. That means in the end there will be one free parameter left which determines the procedure and the convergence speed.

In order to end up in a tractable parabolic framework where the sesquilinear form due to the operator is continuous and coercive and the resulting energy norm is equivalent to a Sobolev norm we have to apply the following transformations. Additionally, in order to be cast into framework where all functions of interest tend to zero as x tends to infinity we will consider an approximate excess to payoff, $V^\epsilon - H^{\epsilon_0}$.

Let to this end

$$c_1 \;:=\; \begin{cases} \int_{\mathbb{R}}(e^y-1)k(y)dy & \text{, if } 0<\varrho<1, \\ 0 & \text{otherwise.} \end{cases}$$

and $q > 0$ sufficiently large. This means it should meet $q > \eta(\sigma^2/2 + |c_1|) + C$ where the constant C is the one in [BL02, Lemma 3.6(i)] which coincides with the constant γ in [MSW06, Equation 4]. Then we will apply the following transformations:

$$\begin{aligned} \overline{H}^{\epsilon_0}(t,x) &:= e^{-qt}H^{\epsilon_0}\left(x+(\sigma^2/2+c_1)t\right), \\ \overline{V}^{\epsilon}(t,x) &:= e^{-qt}V^{\epsilon}\left(t,x+(\sigma^2/2+c_1)t\right), \\ \overline{\vartheta^{\epsilon}}(t,x) &:= e^{-qt}\vartheta^{\epsilon}\left(t,x+(\sigma^2/2+c_1)t\right), \\ \overline{J}^{\epsilon}(t,x) &:= e^{-qt}J^{\epsilon}\left(t,x+(\sigma^2/2+c_1)t\right). \end{aligned}$$

This leads to the following system. For all $(t,x) \in (0,T] \times \mathbb{R}$ we have

$$\begin{aligned}
D_1(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})(t,x) + A(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})(t,x) &= -A^X \overline{H}^{\epsilon_0}(t,x), &(5.1)\\
(\overline{V}^\epsilon(0) - H^{\epsilon_0})(x) &= (H^\epsilon - H^{\epsilon_0})(x), &(5.2)\\
\overline{\vartheta^\epsilon}(t,x) &= \frac{1}{c} e^{-2x} \Gamma(\overline{V}^\epsilon, \exp)(t,x), &(5.3)\\
D_1 \overline{J}^\epsilon(t,x) + A\overline{J}^\epsilon(t,x) &= e^{qt} \psi(\overline{V}^\epsilon, \overline{V}^\epsilon)(t,x), &(5.4)\\
\overline{J}^\epsilon(0,x) &= 0. &(5.5)
\end{aligned}$$

Here, $A : D(A) \to L^1$ denotes the following operator for $f \in D(A)$:

$$\begin{aligned}
\varrho \geq 1 : Af(x) &= qf(x) - \Big(\int_{\mathbb{R}} (f(x+y) - f(x) - (e^y - 1)Df(x))\, k(y)dy \\
&\quad + \frac{1}{2}\sigma^2 D_{22} f(x) \Big), \\
\varrho < 1 : Af(x) &= qf(x) - \int_{\mathbb{R}} (f(x+y) - f(x))\, k(y)dy. \qquad (5.6)
\end{aligned}$$

This is well-defined due to the remarks after the definition of A^X. In particular we have $Af \in L^1$ for $f \in D_{L^1}(A)$.

Remarks. These transformations have been suggested in [MSW06]. The additional factor e^{-qt} ensures coercivity. As we have seen in the previous chapter, there was a asymmetry of the upper and lower estimate of Ψ^X, the characteristic exponent of X, if $\varrho < 1$. This was due to the drift term izb in $\Psi^X(z)$. The translation $x \mapsto x - \sigma^2/2t - c_1 t$ removes the drift term and thus ensures continuity even for $\varrho < 1$. Here, the drift part corresponding to σ^2 would not have obstructed this continuity because if $\sigma^2 \neq 0$ we have $\varrho = 2$. However, this may have been a possible source of instabilities for the numerical scheme, cf. [MSW06, Remark 7].

The trading strategy function ϑ will be computed along the way. Due to the similarity of the two PIDEs (5.1) and (5.4) it suffices to study the following generic PIDE in the sequel. Find $U \in C^{1,2}([0,T];\mathbb{R})$ with $U(t,\cdot) \in D_{L^1}(A)$ for $t \in (0,T]$ such that

$$\begin{aligned}
\frac{d}{dt} U(t,x) + AU(t,x) &= g(t,x) \quad \forall (t,x) \in (0,T] \times \mathbb{R}, \qquad (5.7)\\
U(0,x) &= u_0(x) \quad \forall x \in \mathbb{R}.
\end{aligned}$$

Here A is the operator defined above. For g and u_0 we assume the following.

(G1) There exists $0 \leq \lambda \leq \eta$ such that for every $|\omega| \leq \lambda$ we have

$$u_0 \in H^{p+1}_\omega \text{ and } g \in L^2(0,T; H^{p+1}_\omega).$$

Here, $p \in \mathbb{N}_0$ denotes the parameter of polynomial approximation that will be introduced in Section 5.3.

(G2) Furthermore, we assume the existence of some $d > 1$ and a constant $\tilde{C}$ such that

$$\|D_t^l g\|_{H^{\varrho/2}} \leq \tilde{C} d^l \Gamma(l+1), \quad t \in [0,T], l \in \mathbb{N}_0.$$

The first assumption is essential for the error estimation of the spatial semi-discretization in Section 5.3. The second is due to the time discretization and taken out of [SS00, Equation (2.11)].

Remarks. We have $\overline{V}^{\epsilon} - \overline{H}^{\epsilon_0}, \overline{J}^{\epsilon} \in C^{1,2}([0,T];\mathbb{R}) \cap D_{L^1}(A)$ by Lemma 4.1.4 and Lemma 4.3.2. Furthermore, we have $H^{\epsilon} - H^{\epsilon_0} \in H_{\omega}^{p+1}$ for every $\omega \in \mathbb{R}$. In Lemma 5.7.1 it will be shown that $A\overline{H}^{\epsilon_0} \in L^2(0,T;H_{\omega}^{p+1})$ for every $|\omega| < \eta - \delta$ and the second assumption holds as well. The corresponding properties for $\psi(\overline{V}^{\epsilon}, \overline{V}^{\epsilon})$ are a consequence of Lemma 4.3.1. Therefore, this generic PIDE comprises both (5.1) and (5.4).

5.1 Variational Formulation

Before we start with the variational formulation. We first have to extend the operator A to a pseudo differential operator $\mathscr{A}$. Then we can define powers of this operator and a respective scale of domains and image spaces. To this end we define the function $\hat{\Psi}$ which will be used as symbol of the PDO $\mathscr{A}$. Furthermore, we can derive a lower and upper bound of $\hat{\Psi}$ which will ensure continuity, coercivity and the equivalence of the resulting energy norm with a respective Sobolev norm. Finally, we show that the function satisfies a Calderón-Zygmund estimate. This is needed in order to finally end up in the setting of [MSW06] and therefore be able to apply the theory of [Sch98] for the matrix compression technique.

Lemma 5.1.1 *Define the following function* $\hat{\Psi} : \{z \in \mathbb{C}; |\Im(z)| \leq \eta\} \to \mathbb{H}_+$, *where* $\mathbb{H}_+ := \{z \in \mathbb{C}; \mathrm{Re} z > 0\}$ *as follows:*

$$\begin{aligned} \varrho \geq 1 : \hat{\Psi}(z) &:= q - \left(\int_{\mathbb{R}} (e^{izy} - 1 - iz(e^y - 1))k(y)dy - \frac{1}{2}\sigma^2 z^2 \right), \\ \varrho < 1 : \hat{\Psi}(z) &:= q - \int_{\mathbb{R}} (e^{izy} - 1)k(y)dy. \end{aligned}$$

There exists $\tilde{C}, \beta > 0$ *such that we have the following for all* $z \in \{\xi \in \mathbb{C}; |\Im(\xi)| \leq \eta\}$*:*

$$\begin{aligned} |\hat{\Psi}(z)| &\leq \tilde{C}(1+|z|^2)^{\varrho/2}, \\ \mathrm{Re}\hat{\Psi}(z) &\geq \beta(1+|z|^2)^{\varrho/2}. \end{aligned}$$

Therefore, $\hat{\Psi}$ *was well-defined. Furthermore, for all* $\alpha \in \mathbb{N}_0$ *there exists* $C(\alpha)$ *such that*

$$\forall z \in \{\xi \in \mathbb{C}; |\Im(\xi)| \leq \eta\} : |D^{\alpha}\hat{\Psi}(z)| \leq C(\alpha)(1+|z|)^{\varrho-\alpha}.$$

Proof. For $z = 0$ the claims are clear. Let $z \in \{\xi \in \mathbb{C}; |\Im(\xi)| \le \eta\} \setminus \{0\}$ and let Ψ^X denote the characteristic exponent of X. The term $iz(\sigma^2/2 + c_1)$ removes the drift of X. Therefore, the definition of an RLPE directly yields the upper bound. By [BL02, Lemma 3.6(i)] and the choice of q there exists some $\tilde{\beta} > 0$ such that

$$\begin{aligned} q - \eta(\sigma^2/2 + |c_1|) + \mathrm{Re}\Psi^X(z) &\ge \tilde{\beta}(1 + |z|)^{\varrho} \\ &\ge \beta(1 + |z|^2)^{\varrho/2}. \end{aligned}$$

Now we can derive the lower bound as follows:

$$\begin{aligned} \mathrm{Re}\hat{\Psi}(z) &= \mathrm{Re}(q + \Psi^X(z) - iz(\sigma^2/2 + c_1)) \\ &= q + \Im(z)(\sigma^2/2 + c_1) + \mathrm{Re}\Psi^X(z) \\ &\ge q - \eta(\sigma^2/2 + |c_1|) + \mathrm{Re}\Psi^X(z) \\ &\ge \beta(1 + |z|^2)^{\varrho/2}. \end{aligned}$$

For the last claim we now use the Calderón-Zygmund and the tail estimate of k in assumption (A4). We first consider $\varrho \ge 1$ and set for abbreviation's sake

$$g(w) := e^{iw} - 1 - iw \quad \text{and} \quad c_2 := \int_{\mathbb{R}} (e^y - 1 - y)k(y)dy.$$

The claim is clear for the diffusion term $\frac{1}{2}\sigma^2 z^2$. For the other term we have

$$D_z^\alpha \left(\int_{\mathbb{R}} (e^{izy} - 1 - iz(e^y - 1))k(y)dy \right) = D_z^\alpha \left(\int_{\mathbb{R}} g(zy)k(y)dy \right) - D_z^\alpha(izc_2).$$

The upper bound for the last term is obvious. For the first term we can proceed as follows. Due to the monotonic bounds of the derivatives of $g(zy)$ and their integrability with respect to $k(y)dy$ we can differentiate under the integral. For $|z| < 1$ we can estimate crudely and get

$$\begin{aligned} \left| D_z^\alpha \left(\int_{\mathbb{R}} g(zy)k(y)dy \right) \right| &= \left| D_z^\alpha \int_{\mathbb{R}} \int_0^1 \int_0^{\theta_1} e^{izy\theta_2} d\theta_2 d\theta_1 (izy)^2 k(y)dy \right| \\ &\le \int_{\mathbb{R}} \int_0^1 \int_0^{\theta_1} |D_z^\alpha(e^{izy\theta_2} z^2)| d\theta_2 d\theta_1 |y|^2 k(y)dy \\ &\overset{|z|<1}{\le} C \int_{\mathbb{R}} (1 + |y|^\alpha) e^{|y|} |y|^2 k(y)dy \\ &\le C. \end{aligned}$$

For $|z| \ge 1$ we now apply assumption (A4):

$$\begin{aligned} &\left| D_z^\alpha \left(\int_{\mathbb{R}} g(zy)k(y)dy \right) \right| = \left| \int_{\mathbb{R}} y^\alpha D^\alpha g(zy)k(y)dy \right| \\ &\overset{p.i.}{=} \left| (-1)^\alpha \int_{\mathbb{R}} z^{-\alpha} g(zy) D^\alpha(y^\alpha k(y))dy \right| \\ &\le C|z|^{-\alpha} \left(\sum_{m=0}^{\alpha} \int_{\mathbb{R}} |g(zy)||y|^{\alpha-m} k^{(\alpha-m)}(y)dy \right) \\ &\overset{(A4)}{\le} C|z|^{-\alpha} \left(\int_{\mathbb{R}\setminus[-1,1]} |g(zy)| e^{-(\eta+\delta)|y|} |y|^\alpha dy + \int_{-1}^1 |g(zy)||y|^{\alpha-m} |y|^{-1-\nu-(\alpha-m)} dy \right). \end{aligned}$$

The first integral can be estimated as follows:

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]} |g(zy)|e^{-(\eta+\delta)|y|}|y|^\alpha dy &\leq C\int_{\mathbb{R}\setminus[-1,1]} (e^{|\Im(z)||y|}+|z||y|)e^{-(\eta+\delta)|y|}|y|^\alpha dy\\ &\leq C\int_{\mathbb{R}\setminus[-1,1]} (e^{-\delta|y|}+|z||y|e^{-(\eta+\delta)|y|})|y|^\alpha dy\\ &\leq C(1+|z|)\\ &\overset{\varrho\geq 1}{\leq} C|z|^\varrho.\end{aligned}$$

Now we can proceed as in the proof of Proposition 4.2.1 and the remaining integral can be estimated as follows:

$$\begin{aligned}\int_{-1}^{1} |g(zy)||y|^{-1-\nu}dy &\leq C\Bigg(\int_{1\geq|y|\geq 1/|z|} (e^\eta+|z||y|)|y|^{-1-\nu}dy\\ &\quad + \int_{|y|\leq 1/|z|}\int_0^1\int_0^{\theta_1} |e^{izy\theta_2}|d\theta_2 d\theta_1|z|^2|y|^2|y|^{-1-\nu}dy\Bigg)\\ &\leq C\left(1+|z|+|z|^\nu+\int_{|y|\leq 1/|z|} e|z|^2|y|^{1-\nu}dy\right)\\ &\leq C\left(1+|z|+|z|^\nu\right)\\ &\leq C|z|^\varrho.\end{aligned}$$

Altogether this yields the claim for $\varrho \geq 1$. The case $\varrho < 1$ follows along the same lines but without the obstructing terms $c_2 D^\alpha(iz)$ or $|z|$ in the calculations above. □

Now we can define the corresponding PDO $\mathscr{A}^\omega$. Due to the properties of $\hat{\Psi}$ we can even define powers $\mathscr{A}^{\omega,s}$ for $s \in \mathbb{R}$ as usual. Indeed, since for $z \in \mathbb{C}$ with $|\Im(z)| \leq \eta$ we have $\hat{\Psi}(z) \in \mathscr{H}_+$, we can use the main branch of the complex logarithm in order to define

$$\hat{\Psi}^r(z) := e^{r\log(\hat{\Psi}(z))}.$$

For notation's sake we further introduce for $\omega \in \mathbb{R}$ a weighting operator E^ω on the scale $(\mathscr{S}'_{\omega^*})_{\omega^*\in\mathbb{R}}$ as follows:

$$E^\omega : \mathscr{S}'_{\omega^*} \to \mathscr{S}'_{\omega^*-\omega}; f \mapsto e^{\omega\cdot} f.$$

With this we finally define the following.

Definition 5.1.2 For $\omega_1, \omega_2 \in [-\eta, \eta]$ such that $\omega_1 + \omega_2 \in [-\eta, \eta]$ define

$$H^{r,\hat{\Psi}}_{\omega_1,\omega_2} := \left\{ f \in \mathscr{S}'_{\omega_2}; \left(\hat{\Psi}^r(\cdot + i(\omega_1+\omega_2))\mathcal{F}f(\cdot + i\omega_2)\right) \in L^2 \right\}.$$

Now we can define the operator $\mathscr{A}^{\omega_1,s}$ for $s \in \mathbb{R}$ acting on the scale $(H^{r,\hat{\Psi}}_{\omega_1,\omega_2})_{r\in\mathbb{R}}$ as follows. For $r \in \mathbb{R}$ let

$$\mathscr{A}^{\omega_1,s} : H^{r,\hat{\Psi}}_{\omega_1,\omega_2} \to H^{r-s,\hat{\Psi}}_{\omega_1,\omega_2}; f \mapsto E^{-\omega_2}\mathcal{F}^{-1}(\hat{\Psi}^s(\cdot + i(\omega_1+\omega_2))\mathcal{F}f(\cdot + i\omega_2)).$$

For notation's sake we set $\mathscr{A}^\omega := \mathscr{A}^{\omega,1}$ and $\mathscr{A} := \mathscr{A}^{0,1}$.

Remarks. The generalization to $\mathscr{A}^{0,1}$ is necessary in order to be able to work in the Sobolev spaces $H^{\varrho/2}$ which will be used in the sequel. The further generalization to fractional powers $\mathscr{A}^{0,s}$ we chose in order to have an operator such that its energy norm is equivalent to a given Sobolev norm. Finally, the generalization corresponding to ω is used to derive weighted estimates for the sparse assembly in Section 5.6.

Due to the transformations we can show that $\mathscr{A}^{\omega_1}$ coincides with

$$A^{\omega_1} := E^{\omega_1} A E^{-\omega_1}$$

on $D_{L^1}(A)$. Furthermore, we have equivalence of the norms of $H^{r,\hat{\Psi}}_{\omega_1,\omega_2}$ and of the ones of the Sobolev scale $H^{r\varrho}_{\omega_2}$. Additionally, we derive a norm bound for the analysis of the properties of $A^X \overline{H}^{\epsilon_0}$.

Lemma 5.1.3 (Properties of $\mathscr{A}$ and A^X) *Let $\omega_1, \omega_2 \in [-\eta, \eta]$ with $|\omega_1 + \omega_2| \leq \eta$. The operator $\mathscr{A}^{\omega_1}$ coincides with A^{ω_1} on $D_{L^1}(A)$. Furthermore, we have the following norm estimates. For $r \in \mathbb{R}$ there is a constant $\tilde{C}$ such that for every $f_1 \in D(A) \cap H^{\varrho+r}_{\omega_2}$, $f_2 \in H^{s\varrho+r}_{\omega_2}$ the following holds:*

$$\begin{aligned} \|A^X f_1\|_{H^r_{\omega_2}} &\leq \tilde{C} \|f_1\|_{H^{\varrho+r}_{\omega_2}}, \\ \|\mathscr{A}^{\omega_1,s} f_2\|_{H^r_{\omega_2}} &\sim \|f_2\|_{H^{s\varrho+r}_{\omega_2}}. \end{aligned}$$

This implies $H^{r,\hat{\Psi}}_{\omega_2} = H^{r\varrho}_{\omega_2}$ and $\mathscr{A}^{\omega_1,s}$ therefore is an operator of order $s\varrho$ as defined in [Rud91, Section 8.8].

Proof. We can follow the argument of [BL02, Lemma 15.2]. For $f \in D_{L^1}(A)$ we have by definition $f, Af \in L^1$. That means, we can apply Fubini and get for $\varrho \geq 1$

$$\begin{aligned} \mathcal{F}(Af)(z) &= \int_{\mathbb{R}} e^{-ixz} Af(x) dx \\ &= q\mathcal{F}(f)(z) - \frac{1}{2}\sigma^2 \mathcal{F}(f'')(z) \\ &\quad - \int_{\mathbb{R}} \int_{\mathbb{R}} (f(x+y) e^{-i(x+y)z} e^{iyz} - f(x) e^{-ixz} - (e^y - 1) f'(x) e^{-ixz}) k(y) dy dx \\ &= \left(q + \frac{1}{2}\sigma^2 z^2\right) \hat{f}(z) \\ &\quad - \int_{\mathbb{R}} \left(e^{izy} \int_{\mathbb{R}} f(x+y) e^{-i(x+y)z} dx - \mathcal{F}(f)(z) - (e^y - 1)\mathcal{F}(f')(z) \right) k(y) dy \\ &= \left(q - \int_{\mathbb{R}} (e^{izy} - 1 - iz(e^y - 1)) k(y) dy + \frac{1}{2}\sigma^2 z^2 \right) \hat{f}(z) \\ &= \hat{\Psi}(z) \hat{f}(z). \end{aligned}$$

Similarly, the results for $\varrho < 1$ and for A^X can be derived. Furthermore, we have with this result

$$\begin{aligned}\mathcal{F}(A^{\omega_1}f)(z) &= \mathcal{F}(E^{\omega_1}A(E^{-\omega_1}f))(z)\\ &= \mathcal{F}(A(E^{-\omega_1}f))(z+i\omega_1)\\ &= \hat{\Psi}(z+i\omega_1)\mathcal{F}(E^{-\omega_1}f)(z+i\omega_1)\\ &= \hat{\Psi}(z+i\omega_1)\hat{f}(z).\end{aligned}$$

That means, $\mathscr{A}^{\omega_1}$ and A^{ω_1} coincide on $D_{L^1}(A)$. By definition we have

$$\|\mathscr{A}^{\omega_1,s}f\|_{H^r_{\omega_2}} = \|(1+|z|)^r\hat{\Psi}(z+i(\omega_1+\omega_2))^s\hat{f}(z+i\omega_2)\|_{L^2}.$$

If we apply the estimates in Lemma 5.1.1 we now get the desired norm equivalence for $\mathscr{A}^{\omega_1,s}$. The claim for A^X follows along the very same lines. □

Now we can start with the numerical procedure by casting the PIDE into a variational framework. That means, the equation is multiplied by test functions and then integrated. In the Galerkin setting the space of test functions and the space where the solution is sought for coincide. In order to end up in a setting where the existence and uniqueness of the solution of the resulting variational equation are already given we choose the Sobolev space $H^{\varrho/2}_{\omega_2}$, where $|\omega_2| \leq \lambda$. For this choice we have the so-called Gelfand triple

$$H^{\varrho/2}_{\omega_2} \overset{d}{\hookrightarrow} L^2_{\omega_2} \sim (L^2_{\omega_2})^* \overset{d}{\hookrightarrow} \left(H^{\varrho/2}_{\omega_2}\right)^*.$$

Let $(\cdot,\cdot)_{L^2_{\omega_2}}$ denote the scalar product in $L^2_{\omega_2}$ and $\langle\cdot,\cdot\rangle_{\left(H^{\varrho/2}_{\omega_2}\right)^*\times H^{\varrho/2}_{\omega_2}}$ the duality pairing as in Section 2.2.

The problem now reads as follows for some $\omega_1 \in [-\eta,\eta]$ such that $|\omega_1+\omega_2| \leq \eta$.

Find $U \in L^2(0,T;H^{\varrho/2}_{\omega_2}) \cap H^1\left(0,T;\left(H^{\varrho/2}_{\omega_2}\right)^*\right)$ such that we have

$$\begin{aligned}\frac{d}{dt}(U(t),v)_{L^2_{\omega_2}} + a^{\omega_1}_{\omega_2}(U(t),v) &= \langle g(t),v\rangle_{\left(H^{\varrho/2}_{\omega_2}\right)^*\times H^{\varrho/2}_{\omega_2}}, \quad \forall v \in H^{\varrho/2}_{\omega_2},\\ U(0) &= u_0,\end{aligned} \tag{5.8}$$

where

$$a^{\omega_1}_{\omega_2}(v,w) := \langle\mathscr{A}^{\omega_1}v,w\rangle_{\left(H^{\varrho/2}_{\omega_2}\right)^*\times H^{\varrho/2}_{\omega_2}}, \quad \forall v,w \in H^{\varrho/2}_{\omega_2}.$$

For convenience's sake we set $a^\omega := a^\omega_0$.

Since $\mathscr{A}$ is an extension of A we have that every solution U of (5.7) with

$$U \in L^2(0,T;H^{\varrho/2}_{\omega_2}) \cap H^1(0,T;\left(H^{\varrho/2}_{\omega_2}\right)^*)$$

solves this equation (5.8) for $\omega_1 = 0$ as well. Due to the choice of $H^{\varrho/2}_{\omega_2}$ the sesquilinear form $a^{\omega_1}_{\omega_2}$ is continuous and coercive.

Lemma 5.1.4 *For $\omega_1, \omega_2 \in [-\eta, \eta]$ with $|\omega_1 + \omega_2| \leq \eta$ the sesquilinear form $a_{\omega_2}^{\omega_1}$ is continuous and coercive. For $s \geq 0$ there exist $\tilde{C}, \beta > 0$ such that we have*

$$\begin{aligned} |a_{\omega_2}^{\omega_1}(u,v)| &\leq \tilde{C}\|u\|_{H_{\omega_2}^{s\varrho}}\|v\|_{H_{\omega_2}^{(1-s)\varrho}} && \forall u \in H_{\omega_2}^{s\varrho}, \forall v \in H_{\omega_2}^{(1-s)\varrho}, \\ \text{Re } a_{\omega_2}^{\omega_1}(u,u) &\geq \beta\|u\|^2_{H_{\omega_2}^{\varrho/2}} && \forall u \in H_{\omega_2}^{\varrho/2}. \end{aligned}$$

Proof. This follows by definition of $\mathscr{A}^{\omega_1}$ and with Lemma 5.1.1. Indeed, we have

$$\begin{aligned} |a_{\omega_2}^{\omega_1}(u,v)| &= \left| \int_{\mathbb{R}} \mathcal{F}(\mathscr{A}^{\omega_1}u)(z+i\omega_2)\overline{\mathcal{F}(v)(z+i\omega_2)}dz \right| \\ &= \left| \int_{\mathbb{R}} \hat{\Psi}(z+i(\omega_1+\omega_2))\mathcal{F}(u)(z+i\omega_2)\overline{\mathcal{F}(v)(z+i\omega_2)}dz \right| \\ &= \left| \int_{\mathbb{R}} \hat{\Psi}^s(z+i(\omega_1+\omega_2))\mathcal{F}(u)(z+i\omega_2)\hat{\Psi}^{(1-s)}(z+i(\omega_1+\omega_2))\overline{\mathcal{F}(v)(z+i\omega_2)}dz \right| \\ &\leq \left\| \hat{\Psi}^s(z+i(\omega_1+\omega_2))\mathcal{F}(u)(z+i\omega_2) \right\|_{L^2} \left\| \overline{\hat{\Psi}^{(1-s)}(z+i(\omega_1+\omega_2))}\mathcal{F}(v)(z+i\omega_2) \right\|_{L^2} \\ &\leq \tilde{C} \left\| (1+|z+i\omega_2|^2)^{s\varrho/2}\hat{u}(z+i\omega_2) \right\|_{L^2} \left\| (1+|z+i\omega_2|^2)^{(1-s)\varrho/2}\hat{v}(z+i\omega_2) \right\|_{L^2} \\ &= \tilde{C}\|u\|_{H_{\omega_2}^{s\varrho}}\|v\|_{H_{\omega_2}^{(1-s)\varrho}}. \end{aligned}$$

The remaining claim follows along the very same lines. Here, we use the lower bound of Lemma 5.1.1. There exists $\beta > 0$ with

$$\begin{aligned} \text{Re } \hat{\Psi}(z+i(\omega_1+\omega_2)) &\geq \beta(1+|z+i(\omega_1+\omega_2)|^2)^{\varrho/2} \\ &\geq \beta(1+|z|^2)^{\varrho/2}. \end{aligned}$$

Therefore, we finally have

$$\begin{aligned} \text{Re } a_{\omega_2}^{\omega_1}(u,u) &= \text{Re} \int_{\mathbb{R}} \mathcal{F}(\mathscr{A}^{\omega_1}u)(z+i\omega_2)\overline{\mathcal{F}(u)(z+i\omega_2)}dz \\ &= \int_{\mathbb{R}} \text{Re } \hat{\Psi}(z+i(\omega_1+\omega_2))\mathcal{F}(u)(z+i\omega_2)\overline{\mathcal{F}(u)(z+i\omega_2)}dz \\ &\geq \beta \int_{\mathbb{R}} (1+|z|^2)^{\varrho/2}|\mathcal{F}(u)(z+i\omega_2)|^2 dz \\ &= \beta\|u\|^2_{H_{\omega_2}^{\varrho/2}}. \end{aligned}$$

□

Now we are in a standard setting where the uniqueness of the solution can be shown along with an additional a priori estimate.

Lemma 5.1.5 *For every $|\omega_2| \leq \lambda$ and $\omega_1 \in [-\eta, \eta]$ with $|\omega_1 + \omega_2| \leq \eta$ there exists a unique solution for the variational equation (5.8) and we have the following a priori estimate:*

$$\|U\|_{L^2(0,T;H_{\omega_2}^{\varrho/2})} + \|\dot{U}\|_{L^2(0,T;(H_{\omega_2}^{\varrho/2})^*)} + \|U\|_{C([0,T];L^2_{\omega_2})} \leq C(\|u_0\|_{L^2_{\omega_2}} + \|g\|_{L^2(0,T;(H_{\omega_2}^{\varrho/2})^*)}).$$

Proof. This is a well-known result from [LM72]. But no explicit proof of this specific result could be found. Therefore, we will present a short proof for the convenience of the reader. The uniqueness of the solution of (5.8) is shown in [DL92, chapter XVI, paragraph 3, (3.83)]. The a priori estimate can be got via the estimate of [DL92, chapter XVIII, paragraph 3, (3.75)]. By that reference there exists $\alpha > 0$ such that the following holds for all $t \in [0,T]$:

$$\frac{1}{2}\|U(t)\|^2_{L^2_{\omega_2}} + \frac{\alpha}{2}\int_0^t \|U(s)\|^2_{H^{\varrho/2}_{\omega_2}} ds \leq \frac{1}{2}\|u_0\|^2_{L^2_{\omega_2}} + \frac{1}{2\alpha}\int_0^T \|g(s)\|^2_{(H^{\varrho/2}_{\omega_2})^*} ds.$$

This directly yields

$$\|U\|_{C(0,T;L^2_{\omega_2})} + \|U\|_{L^2(0,T;H^{\varrho/2}_{\omega_2})} \leq C(\|u_0\|_{L^2_{\omega_2}} + \|g\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)}). \tag{5.9}$$

For the last inequality we directly consider the equation (5.8). The properties of $\mathscr{A}^{\omega_1}$ in Lemma 5.1.3 and the estimate of (5.9) yield the following:

$$\begin{aligned} \|\dot{U}\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)} &= \|g - \mathscr{A}^{\omega_1}U\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)} \\ &\leq \|g\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)} + \|\mathscr{A}^{\omega_1}U\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)} \\ &\leq \|g\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)} + C\|U\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2}))} \\ &\leq C(\|u_0\|_{L^2_{\omega_2}} + \|g\|_{L^2(0,T;(H^{\varrho/2}_{\omega_2})^*)}). \end{aligned}$$

This yields the claim. □

Finally, this allows to show the uniqueness of the solution for the PIDE in the strong sense in a corresponding set.

Corollary 5.1.6 *The function J^ϵ is the unique solution to the PIDE (4.3) in the following set:*

$$L \;:=\; \Big\{ f \in C^{1,2}([0,T]\times\mathbb{R}) \cap L^2(0,T;H^{\varrho/2}); \forall t \in (0,T] : f(t,\cdot) \in D_{L^1}(A) \Big\}.$$

Proof. We first state that by Theorem 4.4.1 J^ϵ solves (4.3). Furthermore, we have by Lemma 4.3.2 that $J^\epsilon \in L$. Every solution $f \in L$ of the PIDE (4.3) solves the PIDE (5.4) after application of the necessary transformations. As stated in the remark after introduction of the generic PIDE $e^{qt}\psi(\overline{V}^\epsilon, \overline{V}^\epsilon)$ satisfies the assumptions (G1) and (G2) with $\lambda = 2-\delta$. This means that f solves the generic equation (5.7) for $g = e^{qt}\psi(\overline{V}^\epsilon, \overline{V}^\epsilon)$ and $u_0 = 0$. Since A and $\mathscr{A}$ coincide on $D_{L^1}(A)$, we have that f further solves the variational equation (5.8) above for $\omega_1 = 0 = \omega_2$. The previous Lemma 5.1.5 now finally provides the uniqueness of such a solution and the claim follows. □

5.2 Localization

The next step is to truncate the whole real line $\mathbb{R}$ to a finite interval $\Omega = (-R, R)$, where $R > 0$ shall denote the truncation parameter. To this end we define the following spaces for $s \geq 0, k \in \mathbb{N}_0$ and $B \subset \mathbb{R}$:

$$\begin{aligned} \tilde{H}^s &:= \{u \in H^s; u|_{\mathbb{R}\setminus\Omega} = 0\}, \\ \tilde{C}^k(B) &:= \{f : \mathbb{R} \to \mathbb{R}; f|_B \in C^k(B) \text{ and } f|_{\mathbb{R}\setminus\overline{B}} = 0\}. \end{aligned}$$

The variational space shall now be given by

$$Y := \tilde{H}^{\varrho/2} \text{ with } \|\cdot\|_Y := \|\cdot\|_{H^{\varrho/2}}.$$

That means for $\frac{\varrho}{2} \neq \frac{1}{2}$ the the restriction of Y onto Ω coincides with the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^{\varrho/2}}$. In case $\frac{\varrho}{2} = \frac{1}{2}$ it coincides with $H_{00}^{1/2}(\Omega)$ as defined in [LM72].

Remarks. Note that unlike the definition in [MSW06, Section 2] we chose not to consider the restrictions of the functions to Ω but to deal with their extensions by zero onto $\mathbb{R}$. Thus, we can use the results involving the Fourier transform of the preliminary chapter 2 and directly apply the operators A and its pseudo-differential extensions like in Lemma 5.3.8. Therefore, all functions in H^s and $\tilde{H}^s$ are defined on the whole real line $\mathbb{R}$, only in the latter case the support is a subset of $\overline{\Omega}$.

Additionally, we introduce the following function for $r > 0$. Let $\Phi_r \in C^\infty$ with $\Phi_r|_{(-r,r)} = 1$ and supp $\Phi_r \subset (-r-\delta, r+\delta)$. Furthermore, we assume $\|\Phi_r\|_{C^{p+3}} \leq c'$. Here, c' shall neither depend upon r nor upon ϵ.

Now, let $(\cdot,\cdot)$ denote the L^2 scalar product and the problem now reads as follows.
Find $U_R \in L^2(0,T;Y) \cap H^1(0,T;Y^*)$ such that we have

$$\begin{aligned} \frac{d}{dt}(U_R(t), v) + a(U_R(t), v) &= \langle g(t), v\rangle_{Y^*\times Y}, \quad \forall v \in Y, \forall t \in (0,T], \\ U_R(0) &= \Phi_{R-\delta} u_0, \end{aligned} \tag{5.10}$$

where $a = a_0^0$. This sesquilinear form clearly still satisfies the continuity and coercivity conditions. Let $Y|_\Omega$ denote the restriction of Y onto Ω and $(Y|_\Omega)^*$ the respective dual space. Then $Y|_\Omega \overset{d}{\hookrightarrow} L^2(\Omega) \sim (L^2(\Omega))^* \overset{d}{\hookrightarrow} (Y|_\Omega)^*$ is a Gelfand triple and therefore the existence and uniqueness of the solution of the restricted problem is still given. Thus, this is also true for its extension by zero.

The additional error can be shown to be exponentially small. This is due to the following two facts. Firstly, by assumption the right hand side decreases exponentially. This leads to exponential tails. Secondly, the same holds true for the kernel k which leads to the overall exponential localization error. More specifically, the following result can be proved. This has already been shown in [MPS03, Theorem 4.1] for a special case. Here, we now use their arguments to show the generalized result for weighted fractional Sobolev spaces.

Theorem 5.2.1 *Let $|\omega| < \lambda$. The localization error can be estimated as follows:*

$$\begin{aligned}\|U(T,\cdot) - U_R(T,\cdot)\|_{L^2_\omega(\mathbb{R})} &\leq Ce^{-(\lambda-|\omega|)R}\left(\|u_0\|_{L^2_{-\lambda,\lambda}} + \|g\|_{L^2(0,T;(H^{\varrho/2}_{-\lambda,\lambda})^*)}\right),\\ \|U - U_R\|_{L^2(0,T;Y_\omega)} &\leq Ce^{-(\lambda-|\omega|)R}\left(\|u_0\|_{L^2_{-\lambda,\lambda}} + \|g\|_{L^2(0,T;(H^{\varrho/2}_{-\lambda,\lambda})^*)}\right).\end{aligned}$$

Here, C does not depend upon R.

Proof. Define $e_R := U - U_R$ and set $\Phi := \Phi_{R-\delta}$. We can now decompose the overall error as follows:

$$\|e_R\|_{L^2(0,T;H^{\varrho/2}_\omega)} \leq \|(1-\Phi)e_R\|_{L^2(0,T;H^{\varrho/2}_\omega)} + \|\Phi e_R\|_{L^2(0,T;H^{\varrho/2}_\omega)}.$$

With the inequality (2.1) the first term can now be estimated as follows:

$$\begin{aligned}\|(1-\Phi)e_R\|_{L^2(0,T;H^{\varrho/2}_\omega)} &= \left\| e^{\omega x}\frac{(e^{-\lambda x} + e^{\lambda x})}{(e^{-\lambda x} + e^{\lambda x})}(1-\Phi)e_R\right\|_{L^2(0,T;H^{\varrho/2})}\\ &\leq C\|e^{\omega x}(e^{-\lambda x} + e^{\lambda x})^{-1}(1-\Phi)\|_{H^1}\|(e^{-\lambda x} + e^{\lambda x})e_R\|_{L^2(0,T;H^{\varrho/2})}\\ &\leq C\|e^{-(\lambda-|\omega|)|x|}(1-\Phi)\|_{H^1}\|e_R\|_{L^2(0,T;H^{\varrho/2}_{-\lambda,\lambda})}\\ &\leq Ce^{-(\lambda-|\omega|)R}\|e_R\|_{L^2(0,T;H^{\varrho/2}_{-\lambda,\lambda})}.\end{aligned}$$

Along the same lines the corresponding bound for $\|(1-\Phi)e_R(T)\|_{L^2}$ can be derived. For the second term we follow partly the lines of [MPS03, Theorem 4.1]. To this end we take the difference of the corresponding variational equations (5.8) and (5.10). Thus, we get for all $v \in Y$ and all $t \in (0,T]$

$$\begin{aligned}\frac{d}{dt}(e_R(t), v) + a(e_R(t), v) &= 0,\\ e_R(0) &= (1-\Phi)u_0.\end{aligned} \tag{5.11}$$

Inserting $v = e^{2\omega x}\Phi^2 e_R$ into (5.11) leads to

$$\frac{d}{dt}\|\Phi e_R\|^2_{L^2_\omega} + a^0_\omega(\Phi e_R, \Phi e_R) = a^0_\omega((\Phi-1)e_R, \Phi e_R) + a^0_\omega(e_R, \Phi(1-\Phi)e_R).$$

The right hand side can now be estimated with the continuity of a^0_ω and the estimate for the first term as follows:

$$\begin{aligned}&a^0_\omega((\Phi-1)e_R, \Phi e_R) + a^0_\omega(e_R, \Phi(1-\Phi)e_R)\\ &\leq C\|(\Phi-1)e_R\|_{H^{\varrho/2}_\omega}\|\Phi e_R\|_{H^{\varrho/2}_\omega} + \|e_R\|_{H^{\varrho/2}_\omega}\|\Phi(1-\Phi)e_R\|_{H^{\varrho/2}_\omega}\\ &\leq C\|(\Phi-1)e_R\|_{H^{\varrho/2}_\omega}(\|\Phi e_R\|_{H^{\varrho/2}_\omega} + \|e_R\|_{H^{\varrho/2}_\omega}\|\Phi\|_{H^1})\\ &\leq Ce^{-(\lambda-|\omega|)R}\|e_R\|^2_{H^{\varrho/2}_{-\lambda,\lambda}}.\end{aligned}$$

Due to the coercivity of a^0_ω we now have the following:

$$\begin{aligned}\frac{d}{dt}\|\Phi e_R\|^2_{L^2_\omega} + \|\Phi e_R\|^2_{H^{\varrho/2}_\omega} &\leq C\left(\frac{d}{dt}\|\Phi e_R\|^2_{L_\omega(\mathbb{R})} + a^0_\omega(\Phi e_R, \Phi e_R)\right)\\ &\leq Ce^{-(\lambda-|\omega|)R}\|e_R\|^2_{H^{\varrho/2}_{-\lambda,\lambda}}.\end{aligned}$$

Integration by t now yields

$$\begin{aligned} \|\Phi e_R(T)\|_{L^2_\omega} + \|\Phi e_R\|_{L^2(0,T;H^{\varrho/2}_\omega)} &\leq C(\|(1-\Phi)u_0\|_{L^2_\omega} + e^{-(\lambda-|\omega|)R}\|e_R\|^2_{H^{\varrho/2}_{-\lambda,\lambda}}) \\ &\leq Ce^{-(\lambda-|\omega|)R}(\|u_0\|_{L^2_{-\lambda,\lambda}} + \|e_R\|^2_{H^{\varrho/2}_{-\lambda,\lambda}}). \end{aligned}$$

The claim finally follows by applying the a priori estimate in Lemma 5.1.5 to U and (5.8) with $H^{\varrho/2}_\lambda$, respectively $H^{\varrho/2}_{-\lambda}$, and the corresponding result to U_R and (5.11) with Y_λ, respectively $Y_{-\lambda}$.

$$\|e_R(T)\|_{L^2_{-\lambda,\lambda}} + \|e_R\|_{L^2(0,T;H^{\varrho/2}_{-\lambda,\lambda})} \leq C(\|u_0\|_{L^2_{-\lambda,\lambda}} + \|g\|_{L^2(0,T;(H^{\varrho/2}_{-\lambda,\lambda})^*)}).$$

□

In the sequel we will therefore always assume that $R := c_R|\log h|$. Here, h is the reference parameter which will be introduced in the next section.

5.3 Spatial semi-discretization

We will now further follow along the lines of [MSW06]. Therefore, we have first to show that the problem fits into the framework of that reference. To this end we define the following Schwartz-kernel $K_{\mathscr{A}^\omega} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ for $\omega \in [-\eta, \eta]$ via the oscillating integral as introduced in [BL02, Section 15.3.4]

$$K_{\mathscr{A}^\omega}(x, y) := \int_{\mathbb{R}} e^{i\xi(x-y)}\hat{\Psi}(\xi + i\omega)d\xi. \tag{5.12}$$

By Lemma 5.1.1 the symbol $\hat{\Psi}$ satisfies the property [Sch98, Equation (3.0.4)]. Therefore, by [Sch98, Lemma 3.0.2] we have the following Calderón-Zygmund property for $K_{\mathscr{A}^\omega}$. For $\alpha, \beta \in \mathbb{N}_0$ there exists $C(\alpha, \beta)$ such that for all $x, y \in \mathbb{R}$ with $x \neq y$ we have

$$|D^\alpha_x D^\beta_y K_{\mathscr{A}^\omega}(x, y)| \leq C(\alpha, \beta)|x - y|^{-1-\varrho-\alpha-\beta}.$$

If we set $K^\omega(x, x - y) := K_{\mathscr{A}^\omega}(x, y)$ we have for $u, v \in Y$

$$\begin{aligned} \langle \mathscr{A}^\omega u, v\rangle_{(H^{\varrho/2})^* \times H^{\varrho/2}} &= \int_{\mathbb{R}}\int_{\mathbb{R}} K^\omega(x, x-y)u(y)\overline{v(x)}dydx \\ &= \int_{\Omega}\int_{\Omega} K^\omega(x, x-y)u(y)\overline{v(x)}dydx. \end{aligned}$$

This shows that the problem fits into the framework of [MSW06] and [Sch98].

Proceeding corresponding to the already mentioned method of [MSW06] we now discretize this finite interval. Let to this end $\mathscr{T}^l, l > 0$ be a sequence of partitions of Ω which have been obtained via bisection. The corresponding spaces of functions on these partitions shall be denoted by Y^l. If $0 \leq \varrho < 1$, Y^l consists of all piecewise polynomials of degree $p \geq 0$. Otherwise if $1 \leq \varrho \leq 2$, Y^l shall consist of all piecewise polynomials of degree $p \geq 1$. If $p \geq 1$ they are further assumed to be continuous on Ω and to vanish on the boundary. Define

$$\gamma := \begin{cases} \frac{1}{2} & , \text{if } p = 0, \\ \frac{3}{2} & , \text{if } p \geq 1. \end{cases} \tag{5.13}$$

With this we have $Y^l \subset H^{\gamma-\delta}$. Now we are in the setting of [Sch98, Chapter 6] which will allow to apply the respective compression technique.

Next, an approximation level $L > 0$ shall be fixed and the space to be used shall be given by $Y_h := (Y^L, \|\cdot\|_Y)$.

We shall denote the mesh width of the partition $\mathscr{T}^L$ by h, i.e. we have $h = CR2^{-L}$. This will be the reference parameter for the overall error analysis. That means, the others will be chosen depending upon h. The number of intervals shall be denoted by $N = C2^L$. Therefore, N is actually proportional to the number of degrees of spatial freedom, that means $\dim Y_h = CpN$. The semi-discrete problem now reads as follows.

Find $U_{R,h} \in H^1(0,T;Y_h)$, such that we have

$$\begin{aligned} \frac{d}{dt}(U_{R,h}(t), v_h) + a(U_{R,h}(t), v_h) &= (g, v_h), \quad \forall v_h \in Y_h, \forall t \in (0,T], \\ u(0) &= P_L(\Phi_{R-\delta} u_0), \end{aligned} \tag{5.14}$$

where P_L denotes the orthogonal projection from Y_h^* to Y_h. It will be formally defined and analyzed in the ensuing section. The continuity and coercivity of the bilinear form a are still preserved, because we have $Y_h \subset Y$. Therefore, the existence and uniqueness of the solution are still given.

5.3.1 Projections P_I and P_L

In the sequel we have to deal with different ways of projecting the objects of interest onto Y_h. The ensuing analysis will be based on the properties of two projections, namely P_L and P_I. The first is the natural projection which arises with variational formulations. It has already been used to formulate the semi-discrete problem. The latter is easy to use for the computations arising in the implementation. Therefore, we will now at first study the properties of these two projections.

To be able to handle the analysis we have to introduce a basis for the finite dimensional space Y_h. Two different bases will be used to this end. The first is the local Lagrangian basis $(\varphi_i^j)_{i,j}$, the second is the biorthogonal wavelet basis $(\psi_j^l)_{i,j}$. The wavelet basis will be introduced when applying the matrix compression in Subsection 5.3.2. The local Lagrangian basis is now defined as follows.

For the definition we need some notation. For $1 = 1, \ldots, N$ let

$$\begin{aligned} l_i &:= -R + ih, \\ T_i &:= [-R + (i-1)h, -R + ih], \\ t_i &:= \sup |T_i|. \end{aligned}$$

We further define for $i = 1, \dots, N$ and $j = 0, \dots, p$

$$x_i^j \quad := \quad -R + \left(i + \frac{j}{p}\right) h.$$

Finally, we define the following reference functions:

$$L_j(x) := \prod_{k=0, k\neq j}^{p} \frac{(\frac{k}{p} - x)}{(\frac{k}{p} - \frac{j}{p})}.$$

Translation and scaling now yield the following local Lagrangian functions.

$$q_i^j(x) := 1_{T_i}(x) L_j\left(\frac{1}{h}(x - x_i^0)\right), \quad j = 0, \dots, p.$$

In order to enforce continuity on Ω for $p \geq 1$ the overall local Lagrangian basis will now be defined as follows. For $i = 1, \dots, N$ we set the following. If $i \neq N$ let the following be defined for $j = 1, \dots, p$, otherwise for $j = 1, \dots, p-1$.

$$\varphi_i^j(x) \quad := \quad \begin{cases} q_i^j(x) & \text{, if } 1 \leq j < p, \\ q_i^p(x) + q_{i+1}^0(x) & \text{, if } j = p \text{ and } i < N. \end{cases}$$

We will now define the following additional norm on the discretized space for the ensuing analysis. It is a kind of dual norm of this basis and consequently depends upon it. Therefore, it is not to be confused with the usual dual norm of Y_h. Now let for $0 \leq s < \gamma$ and $\omega \in \mathbb{R}$

$$\|u\|_{H^{-s}_{\omega,\varphi}} \quad := \quad \max_{i,j} \frac{|(e^{\omega x} u, \varphi_i^j)|}{\|\varphi_i^j\|_{H^s}}.$$

Furthermore, set $\|\cdot\|_{H^0_{\omega,\varphi}} := \|\cdot\|_{L^2_\omega}$. Then we have the following properties of the local Lagrangian basis and this norm.

Proposition 5.3.1 *The set $(\varphi_i^j)_{i,j}$ forms a basis of Y_h. Furthermore, the following norm estimates hold for $0 \leq t < \gamma$, $0 < s < \gamma$ and $\omega \in \mathbb{R}$:*

$$\begin{aligned} \|\varphi_i^j\|_{H^t} &= C_j h^{1/2 - t}, \\ \|u\|_{H^{-s}_{\omega,\varphi}} &\leq C h^{s-t} \|u\|_{H^{-t}_{\omega,\varphi}}, \\ \|u\|_{H^{-s}_{\omega,\varphi}} &\leq C \max_i \|u 1_{T_i}\|_{H^{-s}_{\omega,\varphi}}, \end{aligned}$$

where C_j is a constant only depending upon j and p.

Proof. The basis property and the last inequality are clear. The first equality is due to the dilation property of Sobolev norms for $f \in H^t \cap L^1$ and $1 \geq a > 0$,

$$\|f\|_{H^t} \leq a^{1/2 - t} \|f(ax)\|_{H^t}.$$

This could e.g. be shown as follows. Let $f_a(x) := f(ax)$. An easy observation yields the following well-known property of the Fourier transform for $f \in L^1$:

$$\hat{f}_a(z) = a^{-1}\hat{f}\left(\frac{z}{a}\right).$$

Together with $0 < a \leq 1$ this leads to

$$\begin{aligned}
\|f_a\|_{H^t}^2 &= \int_{\mathbb{R}} (1+|\xi|^2)^t |\hat{f}_a(\xi)|^2 d\xi \\
&= a^{-2} \int_{\mathbb{R}} (1+|\xi|^2)^t \left|\hat{f}\left(\frac{\xi}{a}\right)\right|^2 d\xi \\
&\overset{v:=\xi a^{-1}}{=} a^{-2} a \int_{\mathbb{R}} (1+|av|^2)^t |\hat{f}(v)|^2 dv \\
&\overset{|a|\leq 1}{\geq} a^{-1} \int_{\mathbb{R}} a^{2t} (1+|v|^2)^t |\hat{f}(v)|^2 dv \\
&= a^{2t-1} \|f\|_{H^t}^2.
\end{aligned}$$

This shows the dilation property. With this we get

$$\begin{aligned}
\|\varphi_i^j\|_{H^t} &= \|\varphi_i^j\|_{H^t} \\
&= Ch^{1/2-t} \|\varphi_i^j(hy + x_i^0)\|_{H_y^t} \\
&= C_j h^{1/2-t}
\end{aligned}$$

due to the definition of φ_i^j. The second claim can then be derived as follows for $0 < r < \gamma$:

$$\begin{aligned}
\|u\|_{H^{-s}_{\omega,\varphi}} &= \max_{i,j} \frac{|(e^{\omega x}u, \varphi_i^j)|}{\|\varphi_i^j\|_{H^s}} \\
&\leq Ch^{s-1/2} \max_{i,j} \|u\|_{H^{-r}_{\omega,\varphi}} \|\varphi_i^j\|_{H^r} \\
&\leq Ch^{s-r} \|u\|_{H^{-r}_{\omega,\varphi}}.
\end{aligned}$$

The same argument yields

$$\|u\|_{H^{-s}_{\omega,\varphi}} \leq Ch^s \|u\|_{L^2_\omega}.$$

This yields the claim. □

Let now

$$P_I : C(\Omega) \to Y_h$$

denote the unique piece-wise polynomial interpolation of degree p on Ω. That means, we define

$$P_I f(x) := \sum_{i=1}^{N-1} \sum_{j=1}^{p} f(x_i^j)\varphi_i^j(x) + \sum_{j=1}^{p-1} f(x_i^j)\varphi_N^j(x).$$

That means the representation in this Lagrangian basis is easy to be computed. And it is multiplicative in a sense that it leads to easy implementations of this operation. Furthermore, the usual approximation properties can now be shown even in a weighted framework. The unweighted version is already well-known and based upon the Bramble-Hilbert-Lemma, cf. [Sch98, Remark 6.1.1]. However, they are not explicitly given for the weighted spaces we introduced. Therefore, we opted for a (quite technical) proof on the basis of divided differences. We will see that the derivation of the approximation property uses derivatives in the following domain:

$$\Omega^* := \Omega \setminus l_i; 1 \leq i \leq N.$$

Furthermore, we introduce an inner domain in order to approximate the error of the interpolation of functions which do not vanish in $\mathbb{R} \setminus \Omega$:

$$\Omega_i := (-R+1, R-1).$$

Lemma 5.3.2 *[Properties of P_I]*

P_I is multiplicative in the following sense for $f, g_1, g_2 \in \tilde{C}(\Omega)$ with $\forall x \in \Omega : g_2(x) \neq 0$.

$$\begin{aligned} P_I(fg_1) &= (P_I f) .* (P_I g_1), \\ P_I\left(\frac{f}{g_2}\right) &= (P_I f) .: (P_I g_2), \end{aligned}$$

where $.*$ resp. $.:$ denote the point-wise multiplication respectively quotient of the two resulting vectors of the representation with respect to the local Lagrangian basis. Let $\omega \in \mathbb{R}$ and $f \in \tilde{H}^{p+1}$. If $p \geq 1$ we additionally assume $f \in \tilde{C}^{p+2}(\Omega^*)$. For $0 \leq s \leq \lfloor \gamma \rfloor, 1 \leq t \leq p+1$ we then have

$$\|(Id - P_I)f\|_{H^s_\omega} \leq Ch^{t-s}\|f\|_{H^t_\omega}.$$

For $p \geq 1$ and $0 < r \leq 1, 1/2 < r_1 \leq p+1$ we additionally have

$$\|(Id - P_I)f\|_{H^{-r}_{\omega,\varphi}} \leq Ch^{r+r_1}\|f\|_{H^{r_1}_\omega}.$$

For $g \in C_b^{p+1} \cap L^2$ we can further estimate for $1 \leq m \leq p+1, m \in \mathbb{N}$

$$\begin{aligned} \|(Id - P_I)g\|_{L^\infty} &\leq C(h^m\|g\|_{C^m} + \|g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}), \\ \|(Id - P_I)g\|_{L^2} &\leq C(h^t\|g\|_{H^t} + \|g\|_{L^2(\mathbb{R}\setminus\Omega_i)}). \end{aligned}$$

Proof. The first claim directly follows from the properties of the local Lagrangian basis. For the other claims we use the approach of [DH02, Chapter 7]. For $x \in T_i$ we have by [DH02, Theorem 7.10]

$$1_{T_i}(x)(Id - P_I)f(x) = w_{L^i}(x)[L^i, x]f,$$

where $L^i := \{x_i^0, \ldots, x_i^p\}$, $w_{L^i}(x) := \Pi_{x^l \in L^i}(x^l - x)$ and $[L^i, x]f$ denotes the corresponding divided difference as defined in [DH02, Definition 7.9]. That means, $[L^i, x]f$ is the leading coefficient of the interpolation polynomial for f and the interpolation points in L^i.

We start with the derivation of auxiliary estimates of divided differences. Let therefore $L = \{x^0, \dots, x^l\}$ be a subset of the nodes in $T \in \mathscr{T}^L$. Then we define

$$S^L := \{\theta \in [0,1]^{|L|}; \|\theta\|_1 \le 1\}.$$

Let further $\mathscr{P}_r(L)$ denote the set of all subsets of L of cardinality r. With this notation the following properties of divided differences can be proved for $x^0 \in L$ and $1 \le t \le |L|$:

1. $[L,x]f = \int_{S^L} f^{(|L|)}(\sum_{x^l \in L} \theta_l^L x^l + (1 - \|\theta^L\|_1)x) d\theta^L$,
2. $[L,x]f \le \sum_{L_1 \in \mathscr{P}_t(L)} h^{t-|L|}[L_1, x]f$,
3. $\|[L,x]f\|_{L^2(T)} \le C\|D^{|L|}f\|_{L^2(T)}$,
4. $\|w_{\{x^0\}}(x)D([L,x]f)\|_{L^2(T)} \le C\|D^{|L|}f\|_{L^2(T)}$.

The first can be found in [DH02, Theorem 7.12]. The second is clear for $|L| = 1$. For $|L| > 1$ this is a consequence of [DH02, Lemma 7.11 (iii)]. Indeed, we have for $x^0, x^1 \in L, x^0 \ne x^1$

$$[L,x]f = \frac{1}{x^1 - x^0}([L \setminus x^0, x]f - [L \setminus x^1, x]f).$$

This can be further iterated up to $|L| - 1$ times. This is the reason for the assumption $t \ge 1$ in the claim. Together with $|x^1 - x^0| \le h$ this yields (2.). The estimate (3.) is a direct consequence of (1.). For the estimate (4.) we additionally have to apply a partial integration with respect to θ. Indeed, since by assumption $f^{|L|+1} \in \tilde{C}(\Omega^*)$ and S^L is compact we can differentiate under the integral by dominated convergence and get the following:

$$\begin{aligned}
&\left\| w_{\{x^0\}}(x) D[L,x]f \right\|_{L^2(T)} \\
= &\left\| w_{\{x^0\}}(x) \int_{S^L} f^{(|L|+1)}\Big(\sum_{x^l \in L} \theta_l^L x^l + (1 - \|\theta^L\|_1)x\Big)(1 - \|\theta^L\|_1) d\theta^L \right\|_{L^2(T)} \\
= &\left\| \int_{S^L} f^{(|L|)}\Big(\sum_{x^l \in L} \theta_l^L x^l + (1 - \|\theta^L\|_1)x\Big) d\theta^L \right. \\
&\left. + \int_{S^{L\setminus x^0}} f^{(|L|)}\Big(\sum_{x^l \in L\setminus x^0} \theta_l^{L\setminus x^0} x^l + (1 - \|\theta^{L\setminus x^0}\|_1)x\Big)(1 - \|\theta^{L\setminus x^0}\|_1) d\theta^{L\setminus x^0} \right\|_{L^2(T)} \\
\le &\|f^{(|L|)}\|_{L^2(T)} \left(\int_{S^L} (1 - \|\theta^L\|_1)^{-1/2} d\theta^L + \int_{S^{L\setminus x^0}} (1 - \|\theta^{L\setminus x^0}\|_1)^{1/2} d\theta^{L\setminus x^0} \right) \\
\le &C\|D^{|L|}f\|_{L^2(T)}.
\end{aligned}$$

With these auxiliary estimates we can start with the derivation of the approximation property on each interval T_i. We have $\forall x \in T_i : |w_L(x)| \le h^{|L|}$. Therefore, we get

$$\begin{aligned}
\Big\| D\big(w_{L^i}(x)[L^i,x]f\big)\Big\|_{L^2(T_i)} &\le \Big\| w_{L^i}(x)D[L^i,x]f + \sum_{L_1\in\mathscr{P}_p(L^i)} w_{L_1}(x)[L^i,x]f\Big\|_{L^2(T_i)}\\
&\overset{2.}{\le} C\Big\| \sum_{L_2\in\mathscr{P}_t(L^i)} h^{t-p-1}\Big(w_{L^i}(x)D[L_2,x]f + \sum_{L_1\in\mathscr{P}_p(L^i)} w_{L_1}(x)[L_2,x]f\Big)\Big\|_{L^2(T_i)}\\
&\le Ch^{t-p-1}\sum_{L_2\in\mathscr{P}_t(L^i)}\Big(\|w_{L^i}(x)D[L_2,x]f\|_{L^2(T_i)} + \sum_{L_1\in\mathscr{P}_p(L^i)}\|w_{L_1}(x)[L_2,x]f\|_{L^2(T_i)}\Big)\\
&\overset{3./4.}{\le} Ch^{t-p-1}h^p\|D^tf\|_{L^2(T_i)}\\
&\le Ch^{t-1}\|D^tf\|_{L^2(T_i)}.
\end{aligned}$$

On the whole interval Ω this yields for $k\in\{0,1\}, k\le\lfloor\gamma\rfloor$

$$\begin{aligned}
\|(Id-P_I)f\|^2_{H^k_\omega} &\le C\left(\sum_{i=1}^N e^{\omega t_i}\sum_{l=0}^k \|D^l(w_{L^i}[L^i,x]f)\|^2_{L^2(T_i)}\right)\\
&\le C\left(\sum_{i=1}^N e^{\omega t_i}h^{t-k}\|D^tf\|^2_{L^2(T_i)}\right)\\
&\le Ch^{t-k}\left(\sum_{i=1}^N \|e^{\omega(t_i-x)}\|_{L^\infty(T_i)}\|D^tf\|^2_{L^2_\omega(T_i)}\right)\\
&\le Ch^{t-k}\|f\|^2_{H^t_\omega}.
\end{aligned}$$

The first estimate of the claim now follows via Sobolev interpolation. The second claim now can be shown as follows. For $r_1\ge 1$ this is a direct consequence of the previous result. More specifically, we have in this case with the properties of $\|\cdot\|_{\omega,\varphi}$ the following:

$$\begin{aligned}
\|(Id-P_I)f\|_{H^{-r}_{\omega,\varphi}} &\le Ch^r\|(Id-P_I)f\|_{L^2_\omega}\\
&\le Ch^{r+r_1}\|f\|_{H^{r_1}_\omega}.
\end{aligned}$$

For $1/2 < r_1 < 1$ we have to apply an additional argument. With the inequality (2.) we get

$$\begin{aligned}
\|(Id-P_I)f\|_{H^{-r}_{\omega,\varphi}} &\le Ch^{r+r_1-1}\|(Id-P_I)f\|_{H^{r_1-1}_{\omega,\varphi}}\\
&\overset{2.}{\le} Ch^{r+r_1-1}\max_i\sum_{x^l\in L^i} h^{-p}\|w_{L^i}[x^l,x]f\|_{H^{r_1-1}_{\omega,\varphi}}\\
&\overset{1.}{\le} Ch^{r+r_1}\max_i\sum_{x^l\in L^i}\left\|\int_0^1 f'(x^l+\theta(x-x^l))d\theta\right\|_{H^{r_1-1}_{\omega,\varphi}}\\
&\le Ch^{r+r_1}\max_i\sum_{x^l\in L^i}\int_0^1 \|f'(x^l+\theta(x-x^l))\|_{H^{r_1-1}_{\omega,\varphi}}d\theta.
\end{aligned}$$

Furthermore, we have for $0 < \theta \leq 1$

$$\begin{aligned}
\|f'(x^l + \theta(x - x^l))\|_{H^{r_1-1}_{\omega,\varphi}} &= \max_{i,j} |(f'(x^l + \theta(x - x^l)), e^{\omega x}\varphi_i^j(x))| \|\varphi_i^j\|^{-1}_{H^{1-r_1}} \\
&\leq Ch^{1/2-r_1}\theta^{-1} \max_{i,j} |(f'(x), e^{\omega(x^l+\theta^{-1}(x-x^l))}\varphi_i^j(x^l + \theta^{-1}(x - x^l)))| \\
&\leq Ch^{1/2-r_1}\theta^{-1} \max_{i,j} \|f' 1_{T_i}\|_{H^{r_1-1}} \|e^{\omega(x^l+\theta^{-1}(x-x^l))}\varphi_i^j(x^l + \theta^{-1}(x - x^l))\|_{H^{1-r_1}} \\
&\leq Ch^{1/2-r_1}\theta^{-1}\theta^{1/2+r_1-1} \max_{i,j} \|f' 1_{T_i}\|_{H^{r_1-1}} \|\varphi_i^j\|_{H^{1-r_1}_\omega} \\
&\leq C\theta^{-3/2+r_1} \max_{i} \|f' 1_{T_i}\|_{H^{r_1-1}} e^{\omega t_i} \\
&\leq C\theta^{-3/2+r_1} \|f'\|_{H^{r_1-1}_\omega} \|1_{T_i}(x) e^{\omega(t_i - x)}\|_{L^\infty} \\
&\leq C\theta^{-3/2+r_1} \|f\|_{H^{r_1}_\omega}.
\end{aligned}$$

This finally leads to

$$\begin{aligned}
\|(Id - P_I)f\|_{H^{-r}_{\omega,\varphi}} &\leq Ch^{r+r_1} \int_0^1 \theta^{-3/2+r_1} d\theta \|f\|_{H^{r_1}_\omega} \\
&\leq Ch^{r+r_1} \|f\|_{H^{r_1}_\omega}.
\end{aligned}$$

For the last claim we decompose as follows:

$$\|(Id - P_I)g\|_{L^\infty} \leq \|(Id - P_I)g\|_{L^\infty(\Omega_i)} + \|(Id - P_I)g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}.$$

Now the rest follows along the very same lines as the first claim. □

In order to do the analysis for the orthogonal projection P_L we first define the following matrices with respect to the local Lagrange basis. Let $0 \leq s < \gamma$ and $\omega \in \mathbb{R}$.

$$\begin{aligned}
(M^\omega_s)_{(i,j),(i',j')} &:= (\varphi_i^j, \varphi_{i'}^{j'})_{H^s_\omega} \\
(D_\omega)_{(i,j),(i',j')} &:= e^{\omega t_i}\delta_{ii'jj'},
\end{aligned}$$

where $\delta_{ii'jj'} \in \{0, 1\}$ and $\delta_{ii'jj'} = 1 \Leftrightarrow (i = i')$ and $(j = j')$. For shorter notation let $M := M^0_0$ denote the so-called mass matrix.

These matrices have got the following properties.

Proposition 5.3.3 *For $\omega \in \mathbb{R}$ and $0 \leq s < \gamma$ we have*

$$\|D_{-\omega} M^\omega_s D_{-\omega}\|_2 \leq Ch^{1-2s}.$$

Furthermore for $\omega h - \log\alpha < -\delta < 0$, where $\alpha > 0$ is a constant depending only on p, we have

$$\|D_{-\omega} M^{-1} D_\omega\|_2 \leq Ch^{-1}.$$

Proof. Let for this proof $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the usual matrix norms. Then we have

$$\begin{aligned}
(D_{-\omega} M^\omega_s D_{-\omega})_{(i,j),(i',j')} &= e^{-(t_i+t_{i'})\omega} \left(\varphi_i^j, \varphi_{i'}^{j'}\right)_{H^s_\omega} \\
&\leq Ce^{-(t_i+t_{i'})\omega} \|\varphi_i^j(x)\|_{H^s_\omega} \|\varphi_{i'}^{j'}(x)\|_{H^s_\omega} \\
&\leq Ch^{1-2s}.
\end{aligned}$$

Together with the fact that M_s^ω is a p-band matrix the first claim follows. Indeed, we have

$$\begin{aligned} \|D_{-\omega}M_s^\omega D_{-\omega}\|_2 &\leq \sqrt{\|D_{-\omega}M_s^\omega D_{-\omega}\|_1 \|D_{-\omega}M_s^\omega D_{-\omega}\|_\infty} \\ &\leq C_p h^{1-2s}. \end{aligned}$$

By [CGM85, Remarks after Lemma 3] we have

$$(M^{-1})_{(i,j)(i',j')} \leq (M^{-1})_{(i,j)(i,j)} e^{-\log\alpha|i-i'|} \leq Ch^{-1}e^{-\log\alpha|i-i'|}.$$

Therefore, we have

$$\begin{aligned} \|D_{-\omega}M^{-1}D_\omega\|_\infty &= \max_{(i,j)} \sum_{(i',j')} |(M^{-1})_{(i,j),(i',j')}| e^{h\omega|i-i'|} \\ &\leq \max_{(i,j)} \sum_{(i',j')} |(M^{-1})_{(i,j)(i,j)}| e^{(-\log\alpha + h\omega)|i-i'|} \\ &\overset{\text{geom. series}}{\leq} \frac{C}{h}\frac{2}{1-e^{-\delta}} \\ &\leq \frac{C}{h}. \end{aligned}$$

and $\|D_{-\omega}M^{-1}D_\omega\|_1 \leq Ch^{-1}$ can be shown along the very same lines. This yields the claim. □

With this we can now derive a formulation for the orthogonal projection. Let $P_L : (Y_h)^* \to Y_h$ be defined by the following equation:

$$(P_L f, \varphi_i^j) = (f, \varphi_i^j) \quad \forall (i,j).$$

This can also be expressed via the matrices we have just introduced:

$$P_L f = (\varphi_i^j)_{(i,j)}^\top M^{-1} (f, \varphi_i^j)_{(i,j)}.$$

Now we can derive approximation and stability estimates for P_L. Here, we now use the representation of P_L above. This allows to derive the estimates by exploiting the properties of the matrices M^{-1} and M_s^ω. Additionally, we can use these properties to derive a stability estimate for P_I.

Lemma 5.3.4 *[Properties of P_L] Let $\omega \in \mathbb{R}$ with $\omega h - \log\alpha < -\delta < 0$, where $\alpha > 0$ is some constant only depending upon p. The following stability estimate holds for $0 \leq s_1, t_1 < \gamma$ and $f \in \tilde{H}^0$:*

$$\|P_L f\|_{H_\omega^{s_1}} \leq Ch^{-t_1-s_1}\|f\|_{H_{\omega,\varphi}^{-t_1}}.$$

In the weighted setting we have for $0 \leq s_2 \leq \lfloor\gamma\rfloor$, $1 \leq t_2 \leq p+1$ and $f \in \tilde{H}^{t_2} \cap C^{p+2}(\Omega^*)$ the following approximation property:

$$\|(Id - P_L)f\|_{H_\omega^{s_2}} \leq Ch^{t_2-s_2}\|f\|_{H_\omega^{t_2}}.$$

For $0 \leq s_4 < \gamma, 1/2 < t_4 \leq p+1$ *and* $f \in \tilde{H}^{t_4} \cap C^{p+2}(\Omega^*)$ *we additionally have*

$$\|P_I f\|_{H^{s_4}_\omega} \leq Ch^{-s_4}(h^{t_4}\|f\|_{H^{t_4}_\omega} + \|f\|_{L^2_\omega}).$$

Finally, we have the following approximation and stability estimate for $g \in C_b^{p+1}$ *and* $1 \leq m \leq p+1, m \in \mathbb{N}$*:*

$$\begin{aligned} \|(Id - P_L)g\|_{L^\infty} &\leq C(h^m\|g\|_{C^m} + \|g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}), \\ \|P_L g\|_{L^\infty} &\leq C\|g\|_{L^\infty}. \end{aligned}$$

Proof. For the first estimate we use the representation of P_L via the mass matrix:

$$\begin{aligned} \|P_L f\|^2_{H^{s_1}_\omega} &= (M^{-1}(f,\varphi_i^j)_{(i,j)})^\top M^\omega_{s_1}(M^{-1}(f,\varphi_i^j)_{(i,j)}) \\ &= (M^{-1}(f,\varphi_i^j)_{(i,j)})^\top D_\omega(D_{-\omega}M^\omega_{s_1}D_{-\omega})D_\omega M^{-1}(f,\varphi_i^j)_{(i,j)} \\ &\leq \|D_\omega M^{-1}(f,\varphi_i^j)_{(i,j)}\|_2\|(D_{-\omega}M^\omega_{s_1}D_{-\omega})D_\omega M^{-1}(f,\varphi_i^j)_{(i,j)}\|_2 \\ &\leq \|D_\omega M^{-1}D_{-\omega}\|_2\|D_\omega(f,\varphi_i^j)_{(i,j)}\|_2\|D_{-\omega}M^\omega_{s_1}D_{-\omega}\|_2\|D_\omega M^{-1}D_{-\omega}\|_2\|D_\omega(f,\varphi_i^j)_{(i,j)}\|_2 \\ &\leq Ch^{-1}h^{1-2s_1}h^{-1}\|(e^{\omega x}e^{\omega(t_i-x)}f,\varphi_i^j)\|_2^2 \\ &\leq Ch^{-1-2s_1}\left\|\|e^{\omega(t_i-x)}1_{\mathrm{supp}\ \varphi_i^j}f\|_{H^{-t_1}_{\omega,\varphi}}\|\varphi_i^j\|_{H^{t_1}}\right\|_2^2 \\ &\leq Ch^{-1-2s_1}\left\|\|e^{\omega(t_i-x)}1_{\mathrm{supp}\ \varphi_i^j}\|_{L^\infty}\|f1_{T_i}\|_{H^{-t_1}_{\omega,\varphi}}\|\varphi_i^j\|_{H^{t_1}}\right\|_2^2 \\ &\leq Ch^{-1-2s_1}\|f\|^2_{H^{-t_1}_{\omega,\varphi}}h^{1-2t_1} \\ &\leq Ch^{-2(t_1+s_1)}\|f\|^2_{H^{-t_1}_{\omega,\varphi}}. \end{aligned}$$

This shows the first claim. Together with this result the second claim is a consequence of the previous Lemma 5.3.2, because

$$\begin{aligned} \|(Id-P_L)f\|_{H^{s_2}_\omega} &\leq \|(Id-P_I)f\|_{H^{s_2}_\omega} + \|P_L(Id-P_I)f\|_{H^{s_2}_\omega} \\ &\leq C(\|(Id-P_I)f\|_{H^{s_2}_\omega} + h^{-s_2}\|(Id-P_I)f\|_{L^2_\omega}) \\ &\leq Ch^{t_2-s_2}\|f\|_{H^{t_2}_\omega}. \end{aligned}$$

The next claim can be now derived with these properties and with the previous Lemma 5.3.2 as follows:

$$\begin{aligned} \|P_I f\|_{H^{s_4}_\omega} &= \|P_L P_I f\|_{H^{s_4}_\omega} \\ &\leq Ch^{-2s_4}\|P_I f\|_{H^{-s_4}_{\omega,\varphi}} \\ &\leq Ch^{-2s_4}\big(\|(Id-P_I)f\|_{H^{-s_4}_{\omega,\varphi}} + \|f\|_{H^{-s_4}_{\omega,\varphi}}\big) \\ &\leq Ch^{-s_4}\big(h^{t_4}\|f\|_{H^{t_4}_\omega} + \|f\|_{L^2_\omega}\big). \end{aligned}$$

Finally, we get the stability estimate for L^∞ as follows:

$$\begin{aligned} \|P_L g\|_{L^\infty} &\leq C\|M^{-1}\|_\infty \max_{i,j}|(g,\varphi_i^j)| \\ &\leq C\|g\|_{L^\infty}h^{-1}\max_{i,j}\|\varphi_i^j\|_{L^1} \\ &\leq C\|g\|_{L^\infty}. \end{aligned}$$

Now the final claim follows along the very same lines as the second with the corresponding estimate for P_I. □

5.3.2 Matrix compression

After choosing a basis the variational equation (5.14) turns into a system of linear equations for every $t \in [0, T]$. But due to the non-locality of the sesquilinear form a the resulting matrix is in general densely populated. In order to avoid that a matrix compression is applied. That means, we choose a specific basis which allows to set most of the entries of the resulting matrix to zero without great loss of accuracy. Furthermore, this basis will allow for optimal preconditioning. This results in an additional speed-up. To this end we choose a hierarchical biorthogonal wavelet basis as in [PS03],

$$\{\psi_j^l\}_{j,l}, \quad l \in \mathbb{N}_0,\, j = 1, 2, \cdots, M^l,$$

where $M^l = (dim(Y^l) - dim(Y^{l-1}))$ with the following properties:

(W1) $Y_h = \text{span}\{\psi_j^l; 0 \le l \le L, 1 \le j \le M^l\}$.

(W2) The diameter of the support S_j^l of ψ_j^l is bounded by $C2^{-l}$.

(W3) Wavelets ψ_j^l with $S_j^l \cap \partial\Omega = \varnothing$ have a vanishing moment property up to order p. That means, we have $(\psi_j^l, q) = 0$ for all polynomials q of degree p or less.

(W4) The functions $\psi_j^l, l \ge l_0$ can be obtained by scaling and translation of $\psi_j^{l_0}$.

(W5) For $v \in Y$ there exists a representation in terms of these wavelets

$$v = \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} v_j^l \psi_j^l$$

with $v_j^l = (v, \tilde{\psi}_j^l)$ where $\tilde{\psi}_j^l$ are the corresponding dual wavelets. This series converges in $\tilde{H}^s$ for $0 \le s \le \varrho/2$.

(W6) The following norm equivalence holds for $0 \le s < \gamma$ with γ as defined in (5.13):

$$\sum_{l=0}^{\infty} \sum_{j=1}^{M^l} |v_j^l|^2 2^{2ls} \sim \|v\|_{H^s}^2. \tag{5.15}$$

The following one-sided norm bound holds for $\varrho/2 < s \le p + 1$:

$$\sum_{l=0}^{L} \sum_{j=1}^{M^l} |v_j^l|^2 2^{2ls} \le CL^\kappa \|v\|_{H^s}^2, \tag{5.16}$$

where $\kappa = 0$ if $s < p + 1$ and $\kappa = 1$ otherwise.

(W7) This allows to introduce a further projection $Q_h : Y \to Y_h$ by truncation of the wavelet expansion:

$$Q_h v := \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_j^l.$$

For all $f \in \tilde{H}^t$ with $\varrho/2 \leq t \leq p+1$ and $0 \leq s \leq \varrho/2$ the projection satisfies

$$\|(Id - Q_h)f\|_{H^s} \leq Ch^{t-s}\|f\|_{H^t} \tag{5.17}$$

Examples.

(E1) For $p = 1$ the following continuous, piecewise linear spline wavelets vanishing outside $(0, 1)$ can be used after scaling to Ω. The inner wavelets can be obtained via scaling and translation of the following mother wavelet $\psi(x)$ which takes the values $(0, -1/2, 1, -1/2, 0)$ at $(0, 1/4, 1/2, 3/4, 1)$. That means $\psi_j^l(x) := \psi(2^{l-1}x - (2j-1)2^{-2})$ for $1 \leq j \leq 2^l - 2$ and $l \geq 2$. The boundary wavelets can be obtained accordingly. Here, the mother wavelet ψ_* takes the values $(0, 1, -1/2, 0)$ at $(0, 1/4, 1/2, 3/4)$ and therefore $\psi_0^l(x) = \psi_*(2^{l-1}x)$. Similarly we have the mother wavelet ψ^* which takes the values $(0, -1/2, 1, 0)$ at $(1/4, 1/2, 3/4, 1)$ and leads to $\psi_{2^l-1}^l(x) = \psi^*(2^{l-1}x - 2^{l-1} + 1)$.

(E2) For higher orders p the construction in [DS99] can be used. More specifically, using [DS99, Theorem 3.1] together with the simplification shown in [DS99, Proposition 4.6] the wavelets can be constructed.

(E3) In [Ste03] a further class of wavelets has been introduced. Here, the wavelets usually have a larger support than the first two examples (E1) and (E2). However, in contrast to them the dual wavelets in this case are uniformly local as well.

Remarks.

1. The stability in (W6) of the examples (E1),(E2),(E3) above follow by [DS99, Theorem 2.1]. Here, the first assumption [DS99, (C1)] is trivially met, since we can choose $S_k = S_k^* = Y^k$, where Y^l is the discrete space with respect to the triangulation $\mathscr{T}^l$ as defined in the beginning of spatial discretization. The further assumptions, i.e. the necessary direct and inverse estimate [DS99, (C2),(C3)], are given as stated in [DS99, Remark 4.1]. The upper bound in (W6) follows for $s < p + 1$ by the same reference. For $s = p + 1$ the argument as in [PSS97, Proposition 4.2] yields the claim. The approximation property (W7) is then a consequence of these estimates and of the approximation property of P_I in Lemma 5.3.2.

2. The transformation of the representations of functions in Y_h with respect to the basis (ψ_j^l) into such with respect to (φ_i^j), the so-called inverse wavelet transform, can be done in $O(N)$ steps as described in [DS99, Section 4.4] or [Ste03, Equation (2.11)]. For the wavelet transform we have to exploit the locality of both bases in order to come up with an efficient algorithm, cf. Section 5.6.2.

Let $\omega \in [-\eta, \eta]$. In the semi-discrete setting the sesquilinear form $a_0^\omega : Y_h \times Y_h \to \mathbb{C}$ corresponds to a matrix $\mathbf{A}^\omega$. It is defined by $\mathbf{A}^\omega_{(l,j),(l',j')} = a_0^\omega(\psi_j^l, \psi_{j'}^{l'})$. In [Sch98, Lemma 8.2.1] it has been shown that the entries decrease polynomially. This is due to the vanishing moment property of the biorthogonal wavelet basis and the Calderón-Zygmund estimate of the Schwartz-kernel $K_{\mathscr{A}^\omega}$. Indeed, a Taylor expansion of this kernel shows this property. The terms of lower order vanish due the vanishing moment property. The remaining term can be estimated due to Calderón-Zygmund property of $K_{\mathscr{A}^\omega}$. Then it turns out that with increasing distance of supports the corresponding entries decrease polynomially.

Therefore, we substitute $\mathbf{A}^\omega$ by $\tilde{\mathbf{A}}^\omega$ which is defined as follows:

$$\tilde{\mathbf{A}}^\omega_{(l,j),(l',j')} := \begin{cases} \mathbf{A}^\omega_{(l,j),(l',j')} & \text{, if } \operatorname{dist}(S_j^l, S_{j'}^{l'}) \le \delta_{l,l'} \text{ or } S_j^l \cap \partial\Omega \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Here, the parameter $\delta_{l,l'}$ is given by

$$\delta_{l,l'} := c_0 \max\{2^{-L+\hat{\alpha}(2L-l-l')}, 2^{-l}, 2^{-l'}\},$$

where $c_0 > 0$ and $1 \ge \hat{\alpha} > \frac{2p+2}{2p+2+\varrho}$. This results in a corresponding sesquilinear form $\tilde{a}^\omega$. For convenience's sake we again set $\tilde{a} := \tilde{a}^0$ and $\mathbf{A} := \mathbf{A}^0$, $\tilde{\mathbf{A}} := \tilde{\mathbf{A}}^0$.

Due to the continuity and coercivity of a we can define an equivalent norm as follows:

$$\|u\|_{a^\omega} := (a^\omega(u,u))^{1/2} \sim \|u\|_Y.$$

The compression effect has been estimated in [MSW06, Proposition 2] which is based upon [PS97, Theorem 3.1] and the further studies in [Sch98] for biorthogonal wavelets. Here, the vanishing moment property of the wavelets is used together with a Taylor expansion of the kernel and its Calderón-Zygmund property in (A4) to estimate the matrix norm of the difference. The one-sided bound in (5.16) finally shows the upper norm bound below. The respective constant factor κ of the consistency condition depends upon c_0. If this parameter is chosen big enough κ can be decreased to a sufficiently small value. That means we get the following result.

Lemma 5.3.5 *If the compression factor c_0 is chosen big enough then for $\omega \in [-\eta, \eta]$ there exists some $0 < \kappa < 1$ independent of h such that the following consistency condition is satisfied for all $L > 0$:*

$$|a^\omega(u_h, v_h) - \tilde{a}^\omega(u_h, v_h)| \le \kappa \|u_h\|_{a^\omega} \|v_h\|_{a^\omega} \quad \forall u_h, v_h \in Y_h.$$

Additionally, we can estimate the compression effect. For $\varrho/2 \le s, s' \le p+1, \omega \in [-\eta, \eta]$ there exists a constant $\tilde{C}$ such that for all $u \in \tilde{H}^s, v \in \tilde{H}^{s'}$ we have

$$|a^\omega(Q_h u, Q_h v) - \tilde{a}^\omega(Q_h u, Q_h v)| \le \tilde{C} h^{s+s'-\varrho} \|u\|_{H^s} \|v\|_{H^{s'}}.$$

In [PS03, Proposition 3.2] it has been shown that the properties of a^ω carry over to $\tilde{a}^\omega$ which is a direct consequence of the consistency estimate above. More specifically, we have the following.

Lemma 5.3.6 *If the compression factor c_0 is chosen big enough then for $\omega \in [-\eta, \eta]$ there exist constants $\tilde{\alpha}, \tilde{\beta} > 0$ such that we have*

$$\begin{aligned} |\tilde{a}^\omega(u_h, v_h)| &\leq \tilde{\alpha}\|u_h\|_{a^\omega}\|v_h\|_{a^\omega}, \\ \text{Re } \tilde{a}^\omega(u_h, u_h) &\geq \tilde{\beta}\|u_h\|^2_{a^\omega} \end{aligned}$$

for all $u_h, v_h \in Y_h$.

In [MSW06, Proposition 3] it has been shown that the effectiveness of the compression technique can be expressed as follows.

Lemma 5.3.7 *The number of non-trivial entries of $\tilde{\mathbf{A}}^\omega$ is bounded by*

$$\begin{cases} O(N \log N) & , \text{ if } \hat{\alpha} < 1, \\ O(N(\log N)^2) & , \text{ if } \hat{\alpha} = 1. \end{cases}$$

The perturbed problem now reads as follows.

Find $\tilde{U}_{R,h} \in H^1(0, T; Y_h)$, such that we have

$$\begin{aligned} \frac{d}{dt}\left(\tilde{U}_{R,h}(t), v_h\right) + \tilde{a}(\tilde{U}_{R,h}(t), v_h) &= (g(t), v_h), \quad \forall v_h \in Y_h, \forall t \in (0, T], \\ u(0) &= P_L(\Phi_{R-\delta}u_0). \end{aligned} \tag{5.18}$$

Due to Lemma 5.3.6, the continuity and coercivity of the sesquilinear form is preserved. Thus the problem still permits only one unique solution.

5.3.3 Error estimation of the semi-discretization

The overall error due to spatial discretization and compression can be estimated with the means of [MSW06]. However, in this reference they only dealt with the option price process V at time T. That means, they could afford estimates of order t^{-k}. This way they were able to impose few assumptions upon the regularity of the initial value. However, for our system of equations we need estimates that are square-integrable in time. Therefore, we cannot use those results directly. To this end, we will redo their proofs up to a certain point where we substitute the terms t^{-l} with higher order norms for g and u_0.

The approach chosen in [MSW06] is based upon the semigroup approach which we introduced in Section 2.5. Therefore, we define the following operators which correspond to the sesquilinear forms a and $\tilde{a}$:

$$\begin{aligned} A_h &: Y_h \to Y_h; f_h \mapsto \mathbf{A}f_h, \\ \tilde{A}_h &: Y_h \to Y_h; f_h \mapsto \tilde{\mathbf{A}}f_h. \end{aligned}$$

Exploiting the properties of the sesquilinear forms we have that we deal with sectorial operators. Let

$$G(\theta) := \{z \in \mathbb{C}; \theta \leq |\arg z| \leq \pi\}$$

for $\theta \in (0, \pi/2)$. Then we have the following.

Lemma 5.3.8 *The operators $\mathscr{A}$, A_h and $\tilde{A}_h$ are sectorial. Furthermore, for $\omega_1, \omega_2 \in [-\eta, \eta]$ such that $\omega_1 + \omega_2 \in [-\eta, \eta]$ there exists an angle $\theta_{\omega_2}^{\omega_1}$ with the following property. For any $s \geq 0$ there exists a $\tilde{C} > 0$ such that for all $f \in H_{\omega_2}^s$ we have*

$$\|(z - \mathscr{A}^{\omega_1})^{-1} f\|_{H_{\omega_2}^s} \leq \frac{\tilde{C}}{|z|} \|f\|_{H_{\omega_2}^s} \quad \forall z \in G(\theta_{\omega_2}^{\omega_1}).$$

Proof. The sesquilinear form $a_{\omega_2}^{\omega_1}$ has been shown to be continuous and coercive with respect to $H_{\omega_2}^{\varrho/2}$ in Lemma 5.1.4. By Lemma 2.5.2 the corresponding operator $\mathscr{A}^{\omega_1} : H_{\omega_2}^{\varrho} \to L_{\omega_2}^2$ is sectorial. Furthermore, in Lemma 5.3.6 has been shown that $\tilde{a}^{\omega_1}$ is continuous and coercive. Since A_h is just a restriction to Y_h we have that A_h and $\tilde{A}_h$ are sectorial as well.

Therefore, there exists an angle $\theta_{\omega_2}^{\omega_1}$ and $\tilde{C} > 0$ such that

$$\|(z - \mathscr{A}^{\omega_1})^{-1} f\|_{L_{\omega_2}^2} \leq \frac{\tilde{C}}{|z|} \|f\|_{L_{\omega_2}^2} \quad \forall z \in G(\theta_{\omega_2}^{\omega_1})$$

for all $f \in L_{\omega_2}^2$. Since $\mathscr{A}^{\omega_1}$ is a PDO we can generalize this inequality as follows. Let $f \in H_{\omega_2}^s$ then by definition

$$\mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2} \hat{f}(\cdot)) \in L_{\omega_2}^2.$$

This means

$$\begin{aligned}
\|(z - \mathscr{A}^{\omega_1})^{-1} f\|_{H_{\omega_2}^s} &= \|(1 + |\xi|^2)^{s/2} (z - \hat{\Psi}(\xi + i(\omega_1 + \omega_2)))^{-1} \hat{f}(\xi + i\omega_2)\|_{L_\xi^2} \\
&= \|(z - \hat{\Psi}(\xi + i(\omega_1 + \omega_2)))^{-1} (1 + |\xi|^2)^{s/2} \hat{f}(\xi + i\omega_2)\|_{L_\xi^2} \\
&= \|(z - \mathscr{A}^{\omega_1})^{-1} \mathcal{F}^{-1}((1 + |\cdot - i\omega_2|^2)^{s/2} \hat{f}(\cdot))\|_{L_{\omega_2}^2} \\
&\leq \frac{\tilde{C}}{|z|} \|\mathcal{F}^{-1}((1 + |\cdot - i\omega_2|^2)^{s/2} \hat{f}(\cdot))\|_{L_{\omega_2}^2} \\
&= \frac{\tilde{C}}{|z|} \|f\|_{H_{\omega_2}^s}
\end{aligned}$$

for all $z \in G(\theta_{\omega_2}^{\omega_1})$. □

Following the lines of argument in [MSW06] we now can estimate the overall error of the semi-discretization together with the matrix compression. Due to Duhamel's representation formula, see end of Section 2.5, the error can be estimated via the operator norm of the difference of the respective exponentials. More specifically, we define the following operators:

$$\begin{aligned}
E_h(t) &:= e^{-t\mathscr{A}} - e^{-tA_h} P_L, \\
\tilde{E}_h(t) &:= e^{-tA_h} P_L - e^{-t\tilde{A}_h} P_L.
\end{aligned}$$

Now we have to estimate the operator norms of E_h and $\tilde{E}_h$. To this end we use their Dunford-Taylor integral representation given in Lemma 2.5.3. Choose for example the following simple curve for some $s \geq 0$, where $\theta_A \in (0, \pi/2)$ is the sectorial angle that corresponds to all operators $\mathscr{A}$, A_h and $\tilde{A}_h$. $\Gamma_s = \Gamma_s^1 \cup \Gamma_s^2 \cup \Gamma_s^3$ with

$$\begin{aligned} \Gamma_s^1 &= \{re^{i\theta_A} : \infty \geq r \geq s^{-1}\}, \\ \Gamma_s^2 &= \{s^{-1}e^{ir} : \theta_A \leq r \leq 2\pi - \theta_A\}, \\ \Gamma_s^3 &= \{re^{-i\theta_A} : s^{-1} \leq r \leq \infty\}. \end{aligned} \tag{5.19}$$

Then we have

$$\begin{aligned} E_h(t)f &= \frac{1}{2\pi i}\int_{\Gamma_t} e^{-tz}((z-\mathscr{A})^{-1}f - (z-A_h)^{-1}P_L f)dz, \\ \tilde{E}_h(t)f &= \frac{1}{2\pi i}\int_{\Gamma_t} e^{-tz}((z-A_h)^{-1}P_L f - (z-\tilde{A}_h)^{-1}P_L f)dz. \end{aligned}$$

We now can derive the following norm bounds. The arguments are the same as for the proof of [MSW06, Theorem 1, 2] up to a certain point. Unlike that reference we accept higher order norms on the data while reducing the exponent of time t. Here, we will make use of the projection Q_h in order to be able to use to the more general approximation property as well as to be able to use the compression estimates of Lemma 5.3.5.

Lemma 5.3.9 *Let $\omega \in [0, \eta]$ such that $\omega R \geq (p+1)|\log h|$. Then the following estimate holds true for $0 \leq \theta_1 \leq 1$:*

$$\|E_h(t)f\|_{H^{\theta_1 \varrho/2}} + \|\tilde{E}_h(t)f\|_{H^{\theta_1 \varrho/2}} \leq Ch^{p+1-\varrho/2}t^{-1/2\theta_1}(\|f\|_{H^{p+1}} + \|f\|_{H^{\varrho/2}_{-\omega,\omega}})$$

for all $f \in H^{p+1} \cap H^{\varrho/2}_{-\omega,\omega}$. If additionally $\varrho < \gamma$ and $\omega R \geq (p+1+\varrho)|\log h|$ then the following estimate holds true for $0 \leq \theta_1, \theta_2 \leq 1$:

$$\|E_h(t)f\|_{H^{\theta_1\theta_2\varrho/2}} + \|\tilde{E}_h(t)f\|_{H^{\theta_1\theta_2\varrho/2}} \leq Ch^{p+1-\theta_2\varrho/2}t^{1/2(-1+\theta_2(1-\theta_1))}(\|f\|_{H^{p+1}} + \|f\|_{H^{\varrho}_{-\omega,\omega}})$$

for all $f \in H^{p+1} \cap H^{\varrho}_{-\omega,\omega}$.

Proof. For shorter notation we define the following abbreviations:

$$\begin{aligned} w &:= (z-\mathscr{A})^{-1}f, \\ w_h &:= (z-A_h)^{-1}P_L f, \\ \tilde{w}_h &:= (z-\tilde{A}_h)^{-1}P_L f. \end{aligned}$$

With this we have

$$\begin{aligned} \|E_h(t)f\|_{L^2} + \|\tilde{E}_h(t)f\|_{L^2} &\leq C\int_{\Gamma_t} e^{-tz}(\|w-w_h\|_{L^2} + \|w_h-\tilde{w}_h\|_{L^2})|dz|, \\ \|E_h(t)f\|_{Y} + \|\tilde{E}_h(t)f\|_{Y} &\leq C\int_{\Gamma_t} e^{-tz}(\|w-w_h\|_{Y} + \|w_h-\tilde{w}_h\|_{Y})|dz|. \end{aligned}$$

Now let $|z| \geq t^{-1}$ for some $t \in (0, T]$. In the ensuing analysis we will make use of the following Galerkin orthogonality. For all $v_h \in Y_h$ we have

$$\begin{aligned} z(w - w_h, v_h) - a(w - w_h, v_h) &= ((z - \mathscr{A})w, v_h) - ((z - A_h)w_h, v_h) \\ &= (f, v_h) - (P_L f, v_h) \\ &= 0. \end{aligned}$$

This allows to substitute $w - w_h$ by $w - Q_h(\Phi_{R-\delta} w)$ in the following analysis. This difference can be estimated with the projection property of Q_h and similar to the proof of Theorem 5.2.1 as follows for $0 \leq s \leq \varrho/2$ and $g \in H^{p+1} \cap H^s_{-\omega,\omega}$:

$$\begin{aligned} \|g - Q_h(\Phi_{R-\delta} g)\|_{H^s} &\leq \|\Phi_{R-\delta} g - Q_h(\Phi_{R-\delta} g)\|_{H^s} + \|g - (\Phi_{R-\delta} g)\|_{H^s} \\ &\leq C(h^{p+1-s}\|g\|_{H^{p+1}} + e^{-\omega R}\|g\|_{H^s_{-\omega,\omega}}) \\ &\leq Ch^{p+1-s}(\|g\|_{H^{p+1}} + \|g\|_{H^s_{-\omega,\omega}}). \end{aligned}$$

Using the estimate in Lemma 2.5.2 we now get

$$\begin{aligned} & z\|w - w_h\|^2_{L^2} + \|w - w_h\|^2_Y \\ &\leq C|z(w - w_h, w - w_h) - a(w - w_h, w - w_h)| \\ &= C|z(w - w_h, w - Q_h(\Phi_{R-\delta} w)) - a(w - w_h, w - Q_h(\Phi_{R-\delta} w))| \\ &\leq C(|z|\|w - w_h\|_{L^2}\|w - Q_h(\Phi_{R-\delta} w)\|_{L^2} + \|w - w_h\|_Y\|w - Q_h(\Phi_{R-\delta} w)\|_Y) \\ &\leq Ch^{p+1-\varrho/2}(\sqrt{|z|}(\|w\|_{H^{p+1-\varrho/2}} + \|w\|_{L^2_{-\omega,\omega}}) + \|w\|_{H^{p+1}} + \|w\|_{Y_{-\omega,\omega}}) \\ &\quad \cdot(\sqrt{|z|}\|w - w_h\|_{L^2} + \|w - w_h\|_Y). \end{aligned}$$

Therefore, we have with the property of $(z - \mathscr{A})^{-1}$ in Lemma 5.3.8

$$\begin{aligned} \sqrt{|z|}\|w - w_h\|_{L^2} + \|w - w_h\|_Y &\leq Ch^{p+1-\varrho/2}\sqrt{|z|}(\|w\|_{H^{p+1}} + \|w\|_{Y_{-\omega,\omega}}) \\ &\leq Ch^{p+1-\varrho/2}\frac{1}{\sqrt{|z|}}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}). \end{aligned}$$

Interpolation then yields for $0 \leq \theta_1 \leq 1$

$$\|w - w_h\|_{H^{\theta_1 \varrho/2}} \leq Ch^{p+1-\varrho/2}|z|^{-1+\theta_1/2}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}). \tag{5.20}$$

For $\varrho < \gamma$ the bound for the L^2 norm can be enhanced with Nitsche's trick. Let to this end $v := (\bar{z} - \mathscr{A}^*)^{-1}(w - w_h)$, where $\mathscr{A}^*$ denotes the adjoint operator. Now we can use this result, the unweighted approximation property of Q_h in (5.17) and the asymmetric upper bound of a in Lemma 5.1.4 to derive the following:

$$\begin{aligned}
\|w-w_h\|_{L^2}^2 &= z(w-w_h,v)-a(w-w_h,v)\\
&= z(w-w_h,\Phi_{R-\delta}v-Q_h(\Phi_{R-\delta}v))-a(w-w_h,\Phi_{R-\delta}v-Q_h(\Phi_{R-\delta}v))\\
&\quad +z(w-w_h,(1-\Phi_{R-\delta})v)-a(w-w_h,(1-\Phi_{R-\delta})v)\\
&\leq C\Big((\sqrt{|z|}\|(Id-Q_h)(\Phi_{R-\delta}v)\|_{L^2}+\|(Id-Q_h)(\Phi_{R-\delta}v)\|_Y)\\
&\quad \cdot(\sqrt{|z|}\|w-w_h\|_{L^2}+\|w-w_h\|_Y)\\
&\quad +|z|\|w-w_h\|_{L^2}\|(1-\Phi_{R-\delta})v\|_{L^2}+\|w-w_h\|_{L^2}\|(1-\Phi_{R-\delta})v\|_{H^\varrho}\Big)\\
&\leq C\Big(h^{\varrho/2}(\sqrt{|z|}\|v\|_Y+\|v\|_{H^\varrho})(\sqrt{|z|}\|w-w_h\|_{L^2}+\|w-w_h\|_Y)\\
&\quad +e^{-\omega R}\|w-w_h\|_{L^2}|z|\|v\|_{H^\varrho_{-\omega,\omega}}\Big).
\end{aligned}$$

Since $\mathscr{A}^*$ is a sectorial operator as well we can use the same norm estimate for the operator norm of the resolvent and we get

$$\begin{aligned}
\|v\|_Y^2 &\leq a(v,v)\\
&= (v,(\overline{z}-\mathscr{A}^*)v)-z(v,v)\\
&= (v,w-w_h)-z(v,v)\\
&\leq C(\|w-w_h\|_{L^2}\|v\|_{L^2}+|z|\|v\|_{L^2}^2)\\
&\leq C\frac{1}{|z|}\|w-w_h\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\|v\|_{H^\varrho} &\leq C\|\mathscr{A}^*v\|_{L^2}\\
&\leq C(|z|\|v\|_{L^2}+\|w-w_h\|_{L^2})\\
&\leq C\|w-w_h\|_{L^2}.
\end{aligned}$$

Furthermore, we can use that A_h is sectorial as well and use the stability estimate of P_L to get

$$\begin{aligned}
\|v\|_{H^\varrho_{-\omega,\omega}} &\leq C\frac{1}{|z|}\|w-w_h\|_{H^\varrho_{-\omega,\omega}}\\
&\leq C\frac{1}{|z|}(\|w\|_{H^\varrho_{-\omega,\omega}}+\|w_h\|_{H^\varrho_{-\omega,\omega}})\\
&\leq C\frac{1}{|z|^2}(\|f\|_{H^\varrho_{-\omega,\omega}}+\|P_Lf\|_{H^\varrho_{-\omega,\omega}})\\
&\leq C\frac{1}{|z|^2}(\|f\|_{H^\varrho_{-\omega,\omega}}+h^{-\varrho}\|f\|_{L^2_{-\omega,\omega}})\\
&\leq Ch^{-\varrho}\frac{1}{|z|^2}\|f\|_{H^\varrho_{-\omega,\omega}}.
\end{aligned}$$

Therefore, the following holds for $\omega R \geq (p+1+\varrho)|\log h|$:

$$\begin{aligned}
\|w-w_h\|_{L^2} &\leq Ch^{\varrho/2}(\sqrt{|z|}\|w-w_h\|_{L^2} + \|w-w_h\|_Y) + e^{-\omega R}h^{-\varrho}\frac{1}{|z|^2}\|f\|_{H^{\varrho}_{-\omega,\omega}} \\
&\leq C\frac{1}{\sqrt{|z|}}h^{p+1}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}) + h^{p+1}\frac{1}{|z|^2}\|f\|_{H^{\varrho}_{-\omega,\omega}} \\
&\leq C\frac{1}{\sqrt{|z|}}h^{p+1}(\|f\|_{H^{p+1}} + \|f\|_{H^{\varrho}_{-\omega,\omega}}).
\end{aligned}$$

Now we can again interpolate with the result in (5.20) for $0 \leq \theta_2 \leq 1$ and get

$$\|w-w_h\|_{H^{\theta_1\theta_2\varrho/2}} \leq Ch^{p+1-\theta_2\varrho/2}|z|^{-1/2(1+\theta_2(1-\theta_1))}(\|f\|_{H^{p+1}} + \|f\|_{H^{\varrho}_{-\omega,\omega}}).$$

Finally, the claims for E_h follow with the following estimate for $s \in \mathbb{R}$:

$$\int_{\Gamma_t} e^{-tz}|z|^{-s}|dz| \leq Ct^{-1+s}.$$

Along the same lines the claim with respect to $\tilde{E}_h$ can be shown. Now we use the estimate of Lemma 2.5.2 for $\tilde{A}_h$. To this end we use the approximation property of $\tilde{a}$ in Lemma 5.3.5 and of Q_h in (5.17) and finally the result above for $\|w-w_h\|_Y$.

$$\begin{aligned}
&|z|\|w_h-\tilde{w}_h\|^2_{L^2} + \|w_h-\tilde{w}_h\|^2_Y \\
&\quad\leq C|z(w_h-\tilde{w}_h, w_h-\tilde{w}_h) - \tilde{a}(w_h-\tilde{w}_h, w_h-\tilde{w}_h)| \\
&\quad\leq C|a(w_h, w_h-\tilde{w}_h) - \tilde{a}(w_h, w_h-\tilde{w}_h)| \\
&\quad\leq C|a(w_h-Q_h(\Phi_{R-\delta}w), w_h-\tilde{w}_h) - \tilde{a}(w_h-Q_h(\Phi_{R-\delta}w), w_h-\tilde{w}_h)| \\
&\qquad +C|a(Q_h(\Phi_{R-\delta}w), w_h-\tilde{w}_h) - \tilde{a}(Q_h(\Phi_{R-\delta}w), w_h-\tilde{w}_h)| \\
&\quad\leq C(\|w_h-Q_h(\Phi_{R-\delta}w)\|_Y\|w_h-\tilde{w}_h\|_Y + h^{p+1-\varrho/2}\|w\|_{H^{p+1}}\|w_h-\tilde{w}_h\|_Y) \\
&\quad\leq C\Big(\|w-w_h\|_Y + \|w-Q_h(\Phi_{R-\delta}w)\|_Y + h^{p+1-\varrho/2}\|w\|_{H^{p+1}}\Big)\|w_h-\tilde{w}_h\|_Y \\
&\quad\leq Ch^{p+1-\varrho/2}\left(\frac{1}{\sqrt{|z|}}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}) + \frac{1}{|z|}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}})\right)\|w_h-\tilde{w}_h\|_Y \\
&\quad\leq Ch^{p+1-\varrho/2}\frac{1}{\sqrt{|z|}}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}})\|w_h-\tilde{w}_h\|_Y.
\end{aligned}$$

For $\varrho < \gamma$ and $\omega R \geq (p+1+\varrho)|\log h|$ the estimate for L^2 can again be enhanced with Nitsche's trick, cf. [MSW06, Proof of Theorem 2, Step 3] and the line of argument above. Therefore, we get

$$\begin{aligned}
\|w_h-\tilde{w}_h\|_{L^2} &\leq Ch^{p+1}\frac{1}{\sqrt{|z|}}(\|f\|_{H^{p+1}} + \|f\|_{H^{\varrho}_{-\omega,\omega}}), \\
\|w_h-\tilde{w}_h\|_{L^2} &\leq Ch^{p+1-\varrho/2}\frac{1}{|z|}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}), \\
\|w_h-\tilde{w}_h\|_Y &\leq Ch^{p+1-\varrho/2}\frac{1}{\sqrt{|z|}}(\|f\|_{H^{p+1}} + \|f\|_{Y_{-\omega,\omega}}).
\end{aligned}$$

The claim now follows along the very same lines as for E_h. □

These results now finally allow to estimate the overall error of spatial discretization and compression.

Theorem 5.3.10 (Error of semi-discretization) *The additional error due to spatial semi-discretization and compression can be estimated as follows. If $\varrho < \gamma$ and $\lambda R \geq (p+1+\varrho)|\log h|$ then for every $0 \leq \theta \leq 1$ we have*

$$\|U_R(T) - \tilde{U}_{R,h}(T)\|_{H^{\theta\varrho/2}} \leq Ch^{p+1-\theta\varrho/2}(\|u_0\|_{H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda}} + \|g\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda})}).$$

If additionally $0 < \theta \leq 1$ we have

$$\|U_R - \tilde{U}_{R,h}\|_{L^2(0,T;H^{(1-\delta)\theta\varrho/2})} \leq Ch^{p+1-\theta\varrho/2}(\|u_0\|_{H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda}} + \|g\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda})}).$$

If $u_0 = 0$ this bound is given for every $0 \leq \theta \leq 1$, i.e.

$$\|U_R - \tilde{U}_{R,h}\|_{L^2(0,T;H^{\theta\varrho/2})} \leq Ch^{p+1-\theta\varrho/2}\|g\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda})}.$$

If $\varrho \geq \gamma$ and $\lambda R \geq (p+1)|\log h|$ then the estimates above hold for $\theta = 1$ and $H^{\varrho/2}_{-\lambda,\lambda}$ instead of $H^{\varrho}_{-\lambda,\lambda}$.

Proof. This is a direct consequence of the previous Lemma 5.3.9, because we have the following for every $0 \leq \theta_1, \theta_2 \leq 1$ if $\varrho < \gamma$ and $\lambda R \geq (p+1+\varrho)|\log h|$:

$$\begin{aligned}
&\|U_R(t) - \tilde{U}_{R,h}(t)\|_{H^{\theta_1\theta_2\varrho/2}} \\
&\leq \|(E_h(t) + \tilde{E}_h(t))(\Phi_{R-\delta}u_0)\|_{H^{\theta_1\theta_2\varrho/2}} + \int_0^t \|(E_h(t-\tau) + \tilde{E}_h(t-\tau))g(\tau)\|_{H^{\theta_1\theta_2\varrho/2}} d\tau \\
&\leq Ch^{p+1-\theta_2\varrho/2}\Big(t^{1/2(-1+\theta_2(1-\theta_1))}\|u_0\|_{H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda}} \\
&\quad + \int_0^t \|g(\tau)\|_{H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda}}(t-\tau)^{1/2(-1+\theta_2(1-\theta_1))}d\tau\Big) \\
&\leq Ch^{p+1-\theta_2\varrho/2}\Big(t^{1/2(-1+\theta_2(1-\theta_1))}\|u_0\|_{H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda}} + \|g\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-\lambda,\lambda})}\Big).
\end{aligned}$$

This is square integrable for every $0 \leq \theta_1 < 1$ and $0 < \theta_2 \leq 1$ or if $u_0 = 0$. The claim therefore follows for $\theta_2 = \theta$ and $\theta_1 = 1$, respectively $\theta_1 = 1 - \delta$. For $\varrho \geq \gamma$ the claim follows along the very same lines. □

5.4 Time Discretization

For time discretization we now again use a Galerkin method, namely the discontinuous Galerkin (dG) scheme. This method exploits the analytic behavior of the solution in later time steps. To this end neither the mesh width nor the polynomial degree is homogeneous. More specifically, we will use a geometric mesh together with a linear vector of polynomial degrees. Near the origin at $t = 0$, where the estimates of the norms of the time derivative of the solution tend to infinity, the approximation is done via the mesh width and not via the degree of the polynomial in time. That means, the smallest mesh width together with the smallest polynomial degree is located at the origin. For later time steps the polynomial degree as well as the mesh width increase.

For a time mesh $I_m = (t_{m-1}, t_m)$ with $0 = t_0 < t_1 < \cdots < t_{M_t} = T$ and a vector $\underline{r} := (r_m)_{m=0}^{M_t}$ of polynomial degrees we define the dG scheme as follows. Let $\mathscr{M} := (I_m)_{m=1}^{M_t}$ and $k_m := t_m - t_{m-1}$. As we will not enforce overall continuity in time for the discrete solution we will now introduce jump terms. To this end define the following one-sided limits for

$$u \in H^1(\mathscr{M}, Y_h) = \{v \in L^2(0,T;Y_h) : v|_{I_m} \in H^1(I_m, Y_h), m = 1,2,\ldots,M_t\} :$$

$$\begin{aligned} u_m^+ &:= \lim_{s\to 0^+} u(t_m + s), \quad m = 0,1,\ldots,M_t - 1, \\ u_m^- &:= \lim_{s\to 0^+} u(t_m - s), \quad m = 1,2,\ldots,M_t. \end{aligned}$$

Using this we can define the following jump terms:

$$[\![u]\!]_m := u_m^+ - u_m^-, \quad m = 1,2,\ldots,M_t - 1.$$

The discrete space shall now be the following:

$$\mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h) = \left\{u \in L^2(0,T;Y_h) : u|_{I_m} \in \mathscr{P}_{r_m}(I_m, Y_h), m = 1,2,\ldots,M_t\right\}.$$

Here $\mathscr{P}_{r_m}(I_m)$ denotes the space of all polynomials of degree r_m on I_m. If we now apply Galerkin procedure for time as well the problem reads as follows.

Find $\tilde{U}_{R,h}^{dG} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ such that for all $W \in S^{\underline{r}}(\mathscr{M}, Y_h)$ we have

$$\tilde{B}_{dG}(\tilde{U}_{R,h}^{dG}, W) = F_{dG}(W), \tag{5.21}$$

where

$$\begin{aligned} \tilde{B}_{dG}(U,W) &= \sum_{m=1}^{M_t} \int_{I_m} \Big((\dot{U}, W) + \tilde{a}(U,W)\Big) dt \\ &\quad + \sum_{m=1}^{M_t} ([\![U]\!]_m, W_m^+) + (U_0^+, W_0^+) \end{aligned}$$

and

$$F_{dG}(W) = \sum_{m=1}^{M_t} \int_{I_m} (g(t), W(t))dt + (P_L(\Phi_{R-\delta} u_0), W_0^+).$$

By [SS00, Proposition 2.6] we have the uniqueness of the solution and the following Galerkin orthogonality.

Lemma 5.4.1 *There exists a unique solution $\tilde{U}^{dG}_{R,h} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ of (5.21). The semi-discrete solution $\tilde{U}_{R,h}$ solves the dG scheme and satisfies the following Galerkin orthogonality*

$$\tilde{B}_{dG}(\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}, W) = 0$$

for all $W \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$.

For later use, namely the additional error due to the sparse assembly of Section 5.6, we need a so-called stability estimate. This measures the effect of a perturbation of the data on the solution. To this end we define the following for $v_h \in Y_h$ and $U \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$:

$$\|v_h\|_{\tilde{a}} := |\tilde{a}(v_h, v_h)|^{1/2} \overset{\text{Lemma 5.3.6}}{\sim} \|v_h\|_Y$$

and

$$\|U\|^2_{dG} := \sum_{m=1}^{M_t} \int_{I_m} \|U\|^2_{\tilde{a}} dt + \frac{1}{2}\left(\|U_0^+\|^2_{L^2} + \sum_{m=1}^{M_t-1} \|[\![U]\!]_m\|^2_{L^2} + \|U^-_{M_t}\|^2_{L^2} \right).$$

Now we can prove stability with respect to this notation.

Lemma 5.4.2 *The solution $\tilde{U}^{dG}_{R,h} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ of (5.21) satisfies the following norm bound:*

$$\|\tilde{U}^{dG}_{R,h}\|_{dG} \le C(\|u_0\|_{L^2} + \|g\|_{L^2(0,T;(Y_h)^*)}).$$

Proof. By [MSW05, Proposition 4.8] we have

$$\begin{aligned}
\|\tilde{U}^{dG}_{R,h}\|^2_{dG} &= \tilde{B}_{dG}(\tilde{U}^{dG}_{R,h}, \tilde{U}^{dG}_{R,h}) \\
&= F_{dG}(\tilde{U}^{dG}_{R,h}) \\
&= \int_0^T (g(t), \tilde{U}^{dG}_{R,h}(t))dt + (P_L u_0, (\tilde{U}^{dG}_{R,h})^+(0)) \\
&\le \int_0^T \|g(t)\|_{(Y_h)^*} \|\tilde{U}^{dG}_{R,h}(t)\|_Y dt + \|P_L(\Phi_{R-\delta} u_0)\|_{L^2} \|(\tilde{U}^{dG}_{R,h})^+(0)\|_{L^2} \\
&\le \|g(t)\|_{L^2(0,T;(Y_h)^*)} \|\tilde{U}^{dG}_{R,h}\|_{L^2(0,T;Y)} + \|u_0\|_{L^2} \|(\tilde{U}^{dG}_{R,h})^+(0)\|_{L^2} \\
&\le \left(\|g(t)\|_{L^2(0,T;(Y_h)^*)} + \|u_0\|_{L^2}\right) \|\tilde{U}^{dG}_{R,h}\|_{dG}.
\end{aligned}$$

□

In order to be able to derive an exponentially small additional error we now choose the time mesh and the polynomial degrees as follows. We first choose a constant $\sigma \int (0, 1)$. Furthermore, we set $\mu := c_3 d|\log \sigma|$. Then the time mesh shall be given by

$$t_0 := 0, \quad t_m := T\sigma^{M_t - m}, \quad 1 \le m \le M_t,$$

and the linear polynomial vector with slope μ likewise by

$$r_1 = 0, \quad r_m := \lfloor \mu m \rfloor, \quad 2 \le m \le M_t.$$

With this setting we can now apply a slight adaptation of [SS00, Theorem 5.4] yielding the following error bound for the time discretization.

Theorem 5.4.3 *Let* $M_t := (p+1)\frac{|\log h|}{|\log \sigma|}$ *and* $0 < \sigma < 1$. *Then there exists* $c_3 > 0$ *independent of* d *such that the error of the dG scheme as specified above can be bounded as follows:*

$$\|\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}\|_{L^2(0,T;Y)} + \|\tilde{U}_{R,h}(T) - \tilde{U}^{dG}_{R,h}(T)\|_{L^2} \;\leq\; Cdh^{p+1}.$$

The number of spatial equations to be solved is bounded by $O(d|\log h|^2)$.

Proof. In [PS04, Theorem 4.5] it has been proved that

$$\frac{1}{2}\left\|\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}\right\|_{L^2} + \left\|\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}\right\|_{L^2(0,T;Y)} \leq C\left\|\tilde{U}_{R,h} - \mathcal{I}\tilde{U}_{R,h}\right\|_{L^2(0,T;Y)}.$$

Here, $\mathcal{I}\tilde{U}_{R,h} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ is the interpolant as defined in [PS04] which coincides with [SS00, Definition 3.1]. Furthermore, in the proof of [SS00, Theorem 5.4] is shown that

$$\left\|\tilde{U}_{R,h} - \mathcal{I}\tilde{U}_{R,h}\right\|_{L^2(0,T;Y)} \leq C\mu\sigma^{M_t}$$

under the assumption that

$$\mu > \max\left\{1, \frac{\log\sigma}{\log(f_{\min})}\right\},$$

where

$$f_{\min} = (C_0 d)^{2\alpha_{\min}}\frac{(1-\alpha_{\min})^{(1-\alpha_{\min})}}{(1+\alpha_{\min})^{(1+\alpha_{\min})}} \text{ with } \alpha_{\min} = \frac{1}{\sqrt{1+C_0 d^2}}.$$

Now we can assume

$$C_1 d^{-1} \leq |\alpha_{\min}| \leq C_2 < 1.$$

This leads to

$$\begin{aligned} f_{\min} &\geq e^{2C_1(\log d + \log C_0)d^{-1}}\frac{1-C_2}{(1+C_2)^{1+C_2}} \\ &\geq e^{c_3' d^{-1}} \end{aligned}$$

for some $c_3' > 0$ independent of d. Therefore, we get

$$\begin{aligned} \frac{\log\sigma}{\log(f_{\min})} &\leq (c_3')^{-1} d|\log\sigma| \\ &< c_3 d|\log\sigma| \\ &= \mu \end{aligned}$$

for some constant $c_3 > 0$ independent of d. That means, we can apply the proof of [SS00, Theorem 5.4] and get

$$\begin{aligned} \frac{1}{2}\left\|\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}\right\|_{L^2} + \left\|\tilde{U}_{R,h} - \tilde{U}^{dG}_{R,h}\right\|_{L^2(0,T;Y)} &\leq C\mu\sigma^{M_t} \\ &\leq Cde^{(p+1)\log\sigma\frac{|\log h|}{|\log\sigma|}} \\ &\leq Cdh^{p+1}. \end{aligned}$$

□

5.5 Solution algorithm

This section is taken out of [MSW06, Section 4] and cited here just for the convenience of the reader. Having derived the fully-discrete formulation of the solution we now have to choose a basis of $\mathscr{S}^{\underline{r}}(\mathscr{M}, X_h)$ in order to end up in a system of linear equations. Again, the basis is chosen such that additional speedup techniques can be applied. More specifically, for each $1 \leq m \leq M_t$ we define

$$\left(\Phi_j := \sqrt{j + 1/2} L_j\right)_{j=0,\ldots,r_m},$$

where L_j is the jth Legendre polynomial on $(-1,1)$, normalized such that $L_j(1) = 1$. The additional factor $\sqrt{j + 1/2}$ was chosen such that $\|\Phi\|_{L^2(-1,1)} = 1$. This is a basis of the space $\mathscr{P}_{r_m}(-1,1)$, the space of all polynomials of degree less or equal to r_m. For the transformaion of these basis functions upon the time intervals I_m we define the following mapping:

$$F_m : (-1,1) \to I_m; \quad F_m(\hat{t}) := \frac{1}{2}(t_{m-1} + t_m) + \frac{1}{2} k_m \hat{t}.$$

If we set $\tilde{U}^{dG}_{R,h,m} := \tilde{U}^{dG}_{R,h}|_{I_m}$ and $W_m = W|_{I_m}$ we can use the following representation:

$$\begin{aligned} \tilde{U}^{dG}_{R,h,m}(t,x) &= \sum_{j=0}^{r_m} \tilde{U}^{dG}_{R,h,m,j}(x)(\Phi_j \circ F_m^{-1})(t), \\ W_m(t,x) &= \sum_{j=0}^{r_m} W_{m,j}(x)(\Phi_j \circ F_m^{-1})(t), \end{aligned}$$

where for $j = 0, \ldots, r_m$ we have $\tilde{U}^{dG}_{R,h,m,j}, W_{m,j} \in X_h$. Therefore, the problem now reads as follows.

For each time step $1 \leq m \leq M_t$ find $(\tilde{U}^{dG}_{R,h,m,j})_{j=0}^{r_m} \in (X_h)^{r_m}$ such that the following holds for all $(W_{m,i})_{i=0}^{r_m} \in (X_h)^{r_m}$.

$$\sum_{i,j=0}^{r_m} C_{ij}(\tilde{U}^{dG}_{R,h,m,j}, W_{m,i}) + \frac{k_m}{2} \sum_{i=0}^{r_m} \tilde{a}(\tilde{U}^{dG}_{R,h,m,i}, W_{m,i}) = \sum_{i=0}^{r_m} f_{m,i}(W_{m,i}),$$

where for $i,j = 0, \ldots, r_m$ we have

$$C_{ij} = \sigma_{ij}\sqrt{(i+1/2)(j+1/2)}, \quad \sigma_{ij} = \begin{cases} (-1)^{i+j} & \text{, if } j > i \\ 1 & \text{otherwise} \end{cases}$$

and

$$f_{m,i}(v) = \int_{I_m} (g(t), W_{m,i})(\Phi_i \circ F_m^{-1})(t)dt + \Phi_i(-1)(\tilde{U}^{dG-}_{R,h,m-1}(t_{m-1}), v),$$

where we set $\tilde{U}^{dG-}_{R,h}(0) = P_L(\Phi_{R-\delta} u_0)$.

From now on we fix some time step $1 \leq m \leq M_t$ and for sake of readability drop the subscript m. Recall that by $\tilde{\mathbf{A}}$ we denote the matrix with respect to $\tilde{a}$. That means $\tilde{\mathbf{A}}_{(i,l),(j,l')} = \tilde{a}(\psi_i^l, \psi_j^{l'})$. Furthermore, let $\mathbf{C}$ be the matrix defined above, $\mathbf{I}$ the identity matrix and $\mathbf{M}$ the mass matrix of the wavelet basis. Then we have to solve the following system of linear equations:

$$\left(\mathbf{C} \otimes \mathbf{M} + \frac{k}{2}\mathbf{I} \otimes \tilde{\mathbf{A}}\right) \underline{u} \;=\; \underline{f}, \tag{5.22}$$

where $\underline{u}$ denotes the coefficient vector of $\tilde{U}^{dG}_{R,h,m} \in \mathscr{P}_{r_m}(I_m, X_h)$.

If we set $\tilde{N} := \dim X_h$ this is a linear system of size $(r+1)\tilde{N}$. Now we can decouple the system into $(r+1)$ linear systems of size $\tilde{N}$ as follows. Let $\mathbf{C} = \mathbf{Q}\mathbf{T}\mathbf{Q}^H$ be the Schur decomposition of the $(r+1)\times(r+1)$ matrix $\mathbf{C}$ with a unitary matrix $\mathbf{Q}$ and an upper triangular matrix $\mathbf{T}$. Here, the diagonal of $\mathbf{T}$ contains the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{r+1}$ of $\mathbf{C}$. A multiplication of $\mathbf{Q}^H \otimes \mathbf{I}$ from the left transforms the system (5.22) into

$$\left(\mathbf{T} \otimes \mathbf{M} + \frac{k}{2}\mathbf{I} \otimes \tilde{\mathbf{A}}\right) \underline{w} \;=\; \underline{g} \text{ with } \underline{w} = (\mathbf{Q}^H \otimes \mathbf{I})\underline{u},\ \underline{g} = (\mathbf{Q}^H \otimes \mathbf{I})\underline{f}. \tag{5.23}$$

This system is block-upper-triangular. With $\underline{w} = (\underline{w}_0, \underline{w}_1, \ldots, \underline{w}_r), \underline{w}_j \in \mathbb{C}^{\tilde{N}}$ we can obtain the solution of (5.23) by solving

$$\left(\lambda_{j+1}\mathbf{M} + \frac{k}{2}\tilde{\mathbf{A}}\right) \underline{w}_j = \underline{s}_j \tag{5.24}$$

for $j = r, r-1, \ldots, 0$, where $\underline{s}_j = \underline{g}_j - \sum_{l=j+1}^{r} \mathbf{T}_{j+1,l+1}\mathbf{M}\underline{w}_l$.

Finally, we can use preconditioning. This results in a lower condition number such that iterative solvers work faster. To this end let

$$\mathbf{S} = \left(\mathrm{Re}(\lambda_{j+1})\mathbf{I} + \frac{k}{2}\mathbf{D}\right)^{1/2}.$$

Here, $\mathbf{D}$ is the diagonal matrix with entries $\mathbf{D}_{(i,l),(i,l)} = 2^{l\varrho/2}$. Now for $j = r, r-1, \ldots, 0$ we approximately solve

$$\mathbf{S}^{-1}\left(\lambda_{j+1}\mathbf{M} + \frac{k}{2}\tilde{\mathbf{A}}\right)\mathbf{S}^{-1}(\mathbf{S}\underline{w}_j) = \mathbf{S}^{-1}\underline{s}_j$$

for $\mathbf{S}\underline{w}_j$ via n_G incomplete GMRES(m_0) iterations (restarted every m_0 iterations). Undoing all those transformations we end up with an approximate solution $\tilde{U}^{dG,\mathrm{GMRes}}_{R,h}$.

The additional error caused by this approximative solution of the linear equations can be bounded as follows.

Theorem 5.5.1 *If we use $n_G = c_0 d^{4+\delta}|\log h|^5$ GMRES(m_0) iterations, $M_t = c_2|\log h|$ time steps and $\mu = c_3 d$, the additional error can be bounded as follows, where c_0, c_2 and c_3 are sufficiently large constants which do not depend upon d or h:*

$$\|\tilde{U}^{dG}_{R,h}(T) - \tilde{U}^{dG,\mathrm{GMRes}}_{R,h}(T)\|_{L^2} + \|\tilde{U}^{dG}_{R,h} - \tilde{U}^{dG,\mathrm{GMRes}}_{R,h}\|_{L^2(0,T;Y)} \le Ch^{p+1}(\|u_0\|_{L^2} + \|g\|_{L^2(0,T;Y_h^*)}).$$

The overall number of computation steps is bounded by $O(d^{6+\delta}N(\log N)^8)$.

Proof. Most of this has already been shown in [PS04, Section 5.5.4] but it has not been stated in the version above. However, their arguments fully carry over to this setting and we get as in [PS04, Equation 5.46]

$$\|\tilde{U}^{dG}_{R,h} - \tilde{U}^{dG,\mathrm{GMRes}}_{R,h}\|_{dG} \leq Ch^{-2}k_1^{-1}r^3_{M_t}q^{n_G}(\|P_L(\Phi_{R-\delta}u_0)\|_{L^2} + \|g\|_{L^2(0,T;Y_h^*)})$$

with $q = 1 - c_4 r_{M_t}^{-4}$ for some constant c_4. With $\log(1 - c_4 r_{M_t}^{-4}) \geq -c_5 r_{M_t}^{-4}$ for some other constant c_5 we further get the following:

$$h^{-2}k_1^{-1}r^3_{M_t}q^{n_G} \leq C\exp(-2\log h - \log k_1 + 3\log r_{M_t} - c_5 n_G r_{M_t}^{-4}).$$

Now we have $\log k_1 = M_t \log\sigma = c_2|\log h|\log\sigma$ and $r_{M_t} \leq M_t\mu = c_2c_3d|\log h|$. Therefore, it suffices to choose n_G as follows:

$$\begin{aligned} n_G &\geq \frac{1}{c_5}(c_2c_3d|\log h|)^4(3\log r_{M_t} + |\log h|(2 + c_2|\log\sigma| + p + 1)) \\ &\geq c_0 d^{4+\delta}|\log h|^5 \end{aligned}$$

for some constant c_0 sufficiently large. The complexity is composed as follows. For each time step $1 \leq m \leq M_t$ there are (r_m+1) equations to be solved. This amounts to a total number of $O(r^2_{M_t}) = O(d^2(\log N)^2)$, since $N = C|\log h|h^{-1} \geq Ch^{-1}$. Each approximate solution needs $O(d^{4+\delta}(\log N)^5)$ GMRES(m_0) iterations. Every such iteration needs $O(N(\log N))$ computation steps. Therefore, the total number of computation steps amounts to $O(d^{6+\delta}N(\log N)^8)$. □

5.6 Sparse assembly of $\psi(V^\epsilon, V^\epsilon)$

Assumption. We were only able to derive error estimates for the following approximation in the case $\sigma^2 = 0$. Therefore, we assume $\sigma^2 = 0$ for the rest of the section. That means, we have $\varrho = \nu < 2$. Furthermore, we will restrict our analysis in this section to the case $p \geq 1$.

Up to this point we are able to assemble the left hand side of the two variational equations for V^ϵ and J^ϵ and solve the resulting sparse linear system in $O(d^{6+\delta}N(\log N)^8)$ steps. The only issue left to be dealt with is to assemble the corresponding right hand side. For the equation corresponding to the option price $\overline{V}^\epsilon$ the right hand side is up to basis transformations given by $(A\overline{H}^{\epsilon_0}, \varphi_i^l)_{(i,l)}$. Recall that in this case only the efficient inverse wavelet transform is needed. Therefore, this can be computed in $O(N)$ computation steps. That means [MSW06] did not have to apply special techniques for the speedup of this assembly.

But for the equation corresponding to $\overline{J}^\epsilon$ the situation is different. Here, the right hand side is up to efficient basis transformations given by $(\psi(\overline{V}^\epsilon, \overline{V}^\epsilon), \varphi_i^l)_{(i,l)}$ where $\overline{V}^\epsilon$ is not at hand. We can only use the approximative function $\tilde{\overline{V}} := \tilde{\overline{V}}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}$

for this purpose which is given as linear combination of the basis functions (φ_i^l). A natural approach similar to the one used for the left hand side would be to compute $(\psi(\varphi_{i_1}^{l_1}, \varphi_{i_2}^{l_2}), \varphi_{i_3}^{l_3})$ for all combinations of indices and use the bilinearity of ψ to assemble. However, this would not only mean a lot of additional implementation effort, but as the computation of each $(\psi(\tilde{\bar{V}}, \tilde{\bar{V}}), \varphi_i^l)$ would require the summation of $O(N)$ terms this would also blow up the number of computation steps to $O(N^2)$.

As mentioned before, the Γ-operator corresponds to the carré-du-champ operator on the intersection of domains, cf. [Mey76]. In [BL89, Proposition 2] it has been shown that it can be represented using the corresponding generator on a restricted domain. Here, we now use a decomposition in that sense. More specifically, define the following space:

$$D^2(\Gamma) := \{(f,g) \in D(\Gamma); f, g \in D(A) \text{ such that } fg \in D(A)\}.$$

On this space we have the following equality for $(f,g) \in D^2(\Gamma)$:

$$\Gamma(f,g) \quad = \quad A(fg) - fAg - gAf. \tag{5.25}$$

This equivalence can easily be checked by using the representation of A in (5.6) and the definition of Γ in (3.1). For A we already have an approximation $\tilde{A}$ that results in a sparse matrix with respect to the wavelet basis. The generalized matrix $\tilde{A}^\omega$ for A^ω can be computed using the same approach, see next Chapter. In the sequel we will now introduce the following additional approximate operators $A_d^\omega, \Gamma_d^{\omega_f,\omega}$ and $\psi_d^{\omega_f,\omega}$. These are defined using the 'sparse' operators $\tilde{A}^\omega$ and $\tilde{P}_L$. Furthermore, we make use of the multiplicative property of P_I such that the additional implementation effort is small and the overall assembly is done in $O(N(\log N))$ computation steps.

Remarks. In terms of order of convergence this approach will result in a loss of $\nu/2$ due to the decomposition (5.25) or more specifically due to the fact that the additional error is estimated term-wise. This loss of order is reflected in the fact the norm estimate for $\Gamma(f,g)$ in Lemma 4.2.2 involves terms with Sobolev norms of order $r + \nu/2$ while the ones for A in Lemma 5.1.3 involve such of order $r+\nu$. Furthermore, the use of the operator will P_I result in an additional loss of order $1/2+\delta$. This is due to the estimate $\|P_I f\|_{L^2} \leq \|f\|_{H^{1/2+\delta}}$. Due to the quadratic nature of ψ_d this loss of order is doubled. Nevertheless, these losses are taken into account in exchange for the simplicity and speed of the resulting computation of this sparse assembly.

5.6.1 Approximate operator $\tilde{P}_L$

For the definition of A_d^ω we will make use of the projection operator P_L. As already shown, it can be computed using the representation

$$P_L f = (\varphi_i^l)_{(i,l)}^\top M^{-1} (f, \varphi_j^l)_{(j,l)},$$

where M denotes the mass matrix of the Lagrangian basis. However, despite the sparseness of M the inverse M^{-1} is dense. This leads to high computation costs.

Therefore, we will now introduce an approximate operator $\tilde{P}_L$ using an approximate inverse.

Let C be an arbitrary matrix. As defined in [CGM85, Paragraph 3.2], let $B(C,k)$ denote the matrix that consists of the $2k+1$ main diagonals of C. Let furthermore

$$\gamma_\delta := \left\lceil \frac{(p+1)|\log(1-e^{-\delta})|}{\delta} \right\rceil,$$

and with this define

$$\tilde{P}_L f := (\varphi_i^l)^\top_{(i,l)} B(M^{-1}, \gamma_\delta)(f, \varphi_j^l)_{(j,l)}.$$

This matrix $B(M^{-1}, \gamma_\delta)$ can be assembled directly using for example the formulas in [CGM85, Lemma 2] or in special cases [Meu92, Theorem 2.8].

Example 5.6.1 For $p = 1$ the mass matrix is given by the following:

$$M = -\frac{h}{6} \begin{pmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -4 & -1 \\ & & & -1 & -4 \end{pmatrix}.$$

This is a so-called Toeplitz matrix with parameter -4. Therefore, we can apply [Meu92, Theorem 2.8] and get the following. Let r_- and r_+ denote the solutions of the quadratic equation

$$r^2 + 4r + 1 = 0.$$

Then the inverse is given by

$$(M^{-1})_{i,j} = -\frac{6}{h} \frac{(r_+^i - r_-^i)(r_+^{N-j+1} - r_-^{N-j+1})}{(r_+ - r_-)(r_+^{N+1} - r_-^{N+1})}.$$

The additional approximation error can be estimated as follows.

Lemma 5.6.2 *Let $p \geq 1$ and $\omega \in \mathbb{R}$ with $\omega h - \alpha < -\delta < 0$, where α is some constant only depending upon p. The approximate orthogonal projection $\tilde{P}_L : (Y_h)^* \to Y_h$ satisfies the following approximation and stability estimate for $1 \leq r \leq p+1$ and $f \in \tilde{H}^r \cap \tilde{C}^{p+2}(\Omega^*)$:*

$$\begin{aligned} \|(Id - \tilde{P}_L)f\|_{L^\infty_\omega} &\leq Ch^{r-1/2-\delta}\|f\|_{H^r_\omega}, \\ \|\tilde{P}_L f\|_{L^\infty_\omega} &\leq \frac{C}{h} \max_{(i,l)} |e^{\omega t_i}(f, \varphi_i^l)|. \end{aligned}$$

Finally, we get for $g \in C_b^{p+1}$ and $1 \leq m \leq p+1, m \in \mathbb{N}$

$$\|(Id - \tilde{P}_L)g\|_{L^\infty} \leq C(h^m \|g\|_{C^m} + \|g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}).$$

Proof. We can use similar arguments as for P_L in Lemma 5.3.4 and estimate via the matrix norm of the difference $M^{-1} - B(M^{-1}, \gamma_\delta)$. Indeed, we have

$$\begin{aligned}
\|e^{\omega x}(P_L - \tilde{P}_L)f\|_{L^\infty} &\leq \|(e^{\omega x}\varphi_i^l)_{(i,l)}^\top (M^{-1} - B(M^{-1}, \gamma_\delta))(f, \varphi_i^l)_{(i,l)}\|_{L^\infty} \\
&\leq C\|D_\omega(M^{-1} - B(M^{-1}, \gamma_\delta))D_{-\omega}D_\omega(f, \varphi_i^l)_{(i,l)}\|_\infty \\
&\leq C\|D_\omega(M^{-1} - B(M^{-1}, \gamma_\delta))D_{-\omega}\|_\infty \|f\|_{L^2_\omega} h^{1/2}.
\end{aligned}$$

Here, $\|\cdot\|_\infty$ denoted the usual matrix norm. As in Proposition 5.3.3 we have

$$\begin{aligned}
\|D_\omega(M^{-1} - B(M^{-1}, \gamma_\delta))D_{-\omega}\|_\infty &= \max_i \sum_{|i-j|>\gamma_\delta} |\tilde{m}_{ij}| e^{\omega h|i-j|} \\
&\leq \max_i \sum_{|i-j|>\gamma_\delta} |\tilde{m}_{ii}| e^{(-\alpha+\omega h)|i-j|} \\
&\overset{\text{geom. series}}{\leq} \frac{C}{h} \frac{2e^{-\gamma_\delta \delta}}{1 - e^{-\delta}} \\
&\leq Ch^p.
\end{aligned}$$

Here, $\tilde{m}_{ij}$ denoted the entries of M^{-1}. Together with the approximation property of P_L in Lemma 5.3.4 and the Sobolev Lemma we get the following:

$$\begin{aligned}
\|(Id - \tilde{P}_L)f\|_{L^\infty_\omega} &\leq \|(Id - P_L)f\|_{L^\infty_\omega} + \|(P_L - \tilde{P}_L)f\|_{L^\infty_\omega} \\
&\leq \|(Id - P_L)f\|_{H_\omega^{1/2+\delta}} + Ch^{p+1/2}\|f\|_{L^2_\omega} \\
&\leq Ch^{r-1/2-\delta}\|f\|_{H^r_\omega}.
\end{aligned}$$

This shows the first claim. The second follows directly from the definition of $\tilde{P}_L$. The last follows along the very same lines with the corresponding estimates for P_I and P_L.

$$\begin{aligned}
\|(Id - \tilde{P}_L)g\|_{L^\infty} &\leq \|(Id - P_L)g\|_{L^\infty} + \|(P_L - \tilde{P}_L)g\|_{L^\infty} \\
&\leq C(h^m\|g\|_{C^m} + \|g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)} + h^{p+1}\|g\|_{L^\infty}) \\
&\leq C(h^m\|g\|_{C^m} + \|g\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}).
\end{aligned}$$

□

5.6.2 Wavelet transform

In order to exploit the multiplicative property of P_I we have to work with respect to the Lagrangian basis (φ_i^j). In order to use the speedup of the matrix compression we have to work in the wavelet basis (ψ_j^l). Therefore, we have to implement basis transformations for both directions. As already mentioned, for one direction, namely the inverse wavelet transform, the usual multiscale reconstruction algorithm as described in [Coh00, Equation (14.21)] can be used which leads to a complexity of $O(N)$. However, for the other direction [Coh00, Equation (14.20)] to lead to such a complexity, we have to have a compactly supported dual scaling function and therefore a compactly supported dual wavelet. This is the case for the wavelets in (E3) but not for (E1) and (E2). For those latter wavelets we have to exploit the fact that both, the wavelets and the Lagrange basis are local.

For the multiscale algorithm we have to introduce the following notation. Let

$$\varphi^l := (\varphi_i^{j,l})_{ij}$$

be the local Lagrange basis on the triangulation $\mathscr{T}^l$ and let

$$\psi^l := (\psi_j^k)_{k=0,\ldots,l;j=0,\ldots,M^l}.$$

Since $Y^l \subset Y^{l+1}$ there exists a matrix $\mathbf{T} \in \mathbb{R}^{(2^{l+1}-1)\times(2^{l+1}-1)}$ such that the following holds:

$$\mathbf{T}\varphi^{l+1} = P\begin{pmatrix}\varphi^l\\ \psi^l\end{pmatrix}.$$

Here, P is a permutation matrix such that the right hand side is ordered with respect to the following order:

$$f \prec g :\Leftrightarrow \inf \operatorname{supp}\,(f) \leq \inf \operatorname{supp}\,(g).$$

Since both bases, φ^l and ψ^l, are uniformly local with respect to φ^{l+1} this ordering results in a q-banded matrix $\mathbf{T}$ where q does not depend upon l.

Example 5.6.3 For the wavelets (E1) and the ordering $(\psi_0^l, \varphi_1^l, \psi_1^l, \ldots, \varphi_{2^l-1}^l, \psi_{2^l-1}^l)$ the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \begin{pmatrix} 1 & -1/2 & & & & \\ 1/2 & 1 & 1/2 & & & \\ & -1/2 & 1 & -1/2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1/2 & 1 & 1/2 \\ & & & & -1/2 & 1 \end{pmatrix}.$$

The multiscale decomposition algorithm is now recursively defined as follows. Let $f_h \in Y_h$ with the representations

$$f_h = \sum_{i,j} c_{ij}\varphi_i^j,$$

$$f_h = \sum_{l=0}^{L}\sum_{j=0}^{M^l} d_j^l \psi_j^l.$$

Let $d^l := (d_j^l)_{j=0,\ldots,M^l}$, $c^{L+1} := (c_{ij})_{ij}$ and $c^0 := d_0^0$. The coefficients c^{L+1} and $d^0, \ldots, d^L$ are connected as follows for $l = 0, \ldots, L$.

$$\mathbf{T}P\begin{pmatrix}c^l\\ d^l\end{pmatrix} = c^{l+1} \tag{5.26}$$

That means, for the computation of c^{L+1} given the input $d^0, \dots, d^L$, the inverse wavelet transform, a matrix-vector multiplication is performed at each level l. For the wavelet transform the (banded) system of linear equations (5.26) has to be solved. This can be done using band matrix solvers as in [GVL07, Section 4.3] which result in a complexity of $O(2^{l+1}q^2)$ for each level l and therefore $O(N)$ for the whole transformation.

5.6.3 Approximative operators $A_d^\omega, \Gamma_d^{\omega_f,\omega}$ and $\psi_d^{\omega_f,\omega}$

Now we can subsequently introduce the approximate operators. They are defined via their domain and their representation formula. Additionally, we introduce respective subsets of these domains that are used for the estimate of the approximation error. They comprise those functions with higher regularity and additional properties such that e.g. the decomposition (5.25) holds.

Furthermore, we will apply the same idea of dividing the problem of approximating $\Gamma(f, g)$ into two parts. The first involves the difference $f - \tilde{f}$ where $\tilde{f}$ is some smooth approximation of f not depending upon h or ϵ. This part will be assembled using the sparse approximation of the operators. The remaining term which only involves $\tilde{f}$ will be implemented directly. This will be shown in the next chapter.

Unlike the spaces in Section 4.2 we now work in C^{p+2} and the respective norms. This is the canonical space for the operator P_I. Furthermore, it simplifies the analysis, because $\|fg\|_{L^\infty} = \|f\|_{L^\infty}\|g\|_{L^\infty}$.

In view of the decomposition we will define the following two basic spaces for $\omega \in \mathbb{R}$ and $r \geq 0$. Here, the additional regularity assumption $\tilde{C}^{p+2}(\Omega^*)$ enables to apply the approximation results for P_L and P_I in the weighted case.

$$\begin{aligned} D_{\mathbb{R}}^\omega &:= \{e^{\omega x} f \in C_b^{p+3} \cap D(A) \text{ independent of } \epsilon, h\}, \\ D_\Omega^r &:= \{f \in H^r \cap H^{1/2+\delta} \cap \tilde{C}^{p+2}(\Omega^*); f|_{\mathbb{R}\setminus(-R/2-\delta, R/2+\delta)} = 0\}. \end{aligned}$$

Remarks. The first space comprises the approximative functions that are subtracted from the functions of interest in order to ensure integrability. Particularly, we have $\overline{H}^{\epsilon_0}(t, \cdot) \in D_{\mathbb{R}}^\omega$ for all $t \in [0, T]$ and $\omega \geq 0$. In the second space we have the remaining differences. That means, we have e.g. $\Phi_{R/2}\overline{V}^\epsilon(t, \cdot) \in D_\Omega^r$ for $t \in [0, T], 0 \leq r \leq M_p$ and $\Phi_{R/2} v_h \in D_\Omega^r$ for $v_h \in Y_h$ and $0 \leq r < \gamma$.

For convenience's sake we set $D_\Omega := D_\Omega^0$. With this we can now define the approximative operator $A_d^\omega : D_\Omega \to Y_h$ as follows. For $f \in D_\Omega$ let

$$A_d^\omega f := E^{-\omega} \tilde{P}_L \tilde{A}^\omega P_I (E^\omega f).$$

Here, we implicitly change the basis representations. For the application of $\tilde{A}^\omega$ we need the representation with respect to the wavelet basis. For the other operators we use the representation with respect to the nodal basis. Since these transformations can be done in $O(N)$ steps it does not affect the overall complexity. But then the implementation only uses the matrix $\tilde{\mathbf{A}}^\omega$ and the easy to compute matrix $B(M^{-1}, \gamma_\delta)$. Furthermore, the application of the operators now involves only point-wise vector multiplications and multiplications of sparse matrices with vectors. Therefore, the assembly and application of these operators need $O(N \log N)$ computation steps.

With the norm estimates for the operators involved in the definition of A_d^ω we can now derive a norm bound for the difference of A and A_d^ω. For shorter notation define the following for $r, s \geq 0$:

$$\|f\|_{(r,s,\omega)}^{A_d} := h^{-1/2-\nu}(\|f\|_{H^r_\omega} + h^s\|f\|_{H^{r+s}_\omega}).$$

Lemma 5.6.4 *[Properties of A_d] Let $1 \leq m \leq p+1, m \in \mathbb{N}$ and $\omega \in (-\eta, \eta)$. Furthermore, let $c_R \geq 2\frac{p+1/2-\nu}{\eta-|\omega|}$. Then we have for $f \in D_\Omega^{m+\nu+1/2}$*

$$\|(A - A_d^\omega)f\|_{L^\infty_\omega} \leq Ch^m\|f\|_{(m,\nu+1/2,\omega)}^{A_d}.$$

Additionally, the following stability estimate holds for $f \in D_\Omega$.

$$\|A_d^\omega f\|_{L^\infty_\omega} \leq C\|f\|_{(0,1/2+\delta,\omega)}^{A_d}$$

Proof. We start by stating

$$\begin{aligned}\|(A - A_d^\omega)f\|_{L^\infty_\omega} &= \|E^\omega A^J(E^{-\omega}E^\omega f) - \tilde{P}_L\tilde{A}^\omega P_I(E^\omega f)\|_{L^\infty}\\ &= \|A^\omega f^\omega - \tilde{P}_L\tilde{A}^\omega P_I f^\omega\|_{L^\infty},\end{aligned}$$

where $f^\omega(x) := e^{\omega x}f(x)$. Now the first claim can be decomposed as follows:

$$\begin{aligned}\|A^\omega f^\omega - \tilde{P}_L\tilde{A}^\omega P_I f^\omega\|_{L^\infty} \leq& \|(Id - \tilde{P}_L)A^\omega f^\omega\|_{L^\infty} + \|\tilde{P}_L A^\omega((Id - Q_h)f^\omega)\|_{L^\infty}\\ &+\|\tilde{P}_L\tilde{A}^\omega((P_I - Q_h)f^\omega)\|_{L^\infty} + \|\tilde{P}_L(A^\omega - \tilde{A}^\omega)(Q_h f^\omega)\|_{L^\infty}.\end{aligned}$$

These terms can now be estimated with the previous results.

1. For the first term we use the approximation property of $\tilde{P}_L$ in Lemma 5.6.2 and the property of A^ω in Lemma 5.1.3.

$$\begin{aligned}\|(Id - \tilde{P}_L)A^\omega f^\omega\|_{L^\infty} &\leq C(h^m\|A^\omega f^\omega\|_{C^m} + \|A^\omega f^\omega\|_{L^\infty(\mathbb{R}\setminus\Omega_i)})\\ &\leq C(h^m\|A^\omega f^\omega\|_{H^{m+1/2+\delta}} + \|A^\omega f^\omega\|_{L^\infty(\mathbb{R}\setminus\Omega_i)})\\ &\leq C(h^m\|f\|_{H_\omega^{m+\nu+1/2+\delta}} + \|A^\omega f^\omega\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}).\end{aligned}$$

The second term herein can be estimated with the lower bound of c_R as follows. Let $x \in \mathbb{R}\setminus\Omega_i$. Since $f^\omega \in D_\Omega$, the terms in the definition of A^ω with respect to

x vanish. That means, we get

$$\begin{aligned}
|(A^\omega f^\omega)(x)| &= \left| \int_{-R/2-\delta-x}^{R/2+\delta-x} f^\omega(x+y)e^{-\omega y}k(y)dy \right| \\
&\leq C\|f^\omega\|_{L^\infty} \int_{\{|y|\geq R/2-1\}} e^{-(\eta-|\omega|)|y|}e^{(\eta-|\omega|)|y|}e^{-\omega y}k(y)dy \\
&\leq C\|f^\omega\|_{L^\infty} e^{-(\eta-|\omega|)(R/2-1)} \int_{\mathbb{R}} (1\wedge y^2)e^{\eta|y|}k(y)dy \\
&\leq C\|f\|_{L^\infty_\omega} e^{-(\eta-|\omega|)R/2} \\
&\leq C\|f\|_{H^{1/2+\delta}_\omega} h^{p+1/2-\nu} \\
&\leq C\|f\|_{H^m_\omega} h^{m-1/2-\nu}.
\end{aligned}$$

Therefore, we finally get

$$\|(Id-\tilde{P}_L)A^\omega f^\omega\|_{L^\infty} \leq Ch^m\|f\|^{A_d}_{(m,\nu+1/2,\omega)}.$$

2. The second term can be estimated with the stability estimate of $\tilde{P}_L$ in Lemma 5.6.2 and the continuity property of a^ω in Lemma 5.1.4.

$$\begin{aligned}
\|\tilde{P}_L A^\omega((Id-Q_h)f^\omega)\|_{L^\infty} &\leq Ch^{-1}\max_{(i,l)}|a^\omega((Id-Q_h)f^\omega,\varphi_i^l)| \\
&\leq Ch^{-1/2-\nu/2}\|(Id-Q_h)f^\omega\|_{H^{\nu/2}} \\
&\leq Ch^{m-1/2-\nu}\|f\|_{H^m_\omega}.
\end{aligned}$$

3. The estimate for the last term can be derived additionally using the approximation estimate in Lemma 5.3.5.

$$\begin{aligned}
\|\tilde{P}_L(A^\omega-\tilde{A}^\omega)(Q_hf^\omega)\|_{L^\infty} &\leq Ch^{-1}\max_{(i,l)}|(a^\omega(Q_hf^\omega,\varphi_i^l)-\tilde{a}^\omega(Q_hf^\omega,\varphi_i^l))| \\
&\leq Ch^{m-1/2-\nu}\|f\|_{H^m_\omega}.
\end{aligned}$$

4. The third term can be bounded with the results of the second and the last as follows:

$$\begin{aligned}
&\|\tilde{P}_L\tilde{A}^\omega((P_I-Q_h)f^\omega)\|_{L^\infty} \\
&\quad\leq \|\tilde{P}_L(A^\omega-\tilde{A}^\omega)(Q_h(P_I-Id)f^\omega)\|_{L^\infty} + \|\tilde{P}_LA^\omega(Q_h(P_I-Id)f^\omega)\|_{L^\infty} \\
&\quad\leq Ch^{-1/2-\nu/2}\|Q_h(Id-P_I)f^\omega\|_{H^{\nu/2}} \\
&\quad\leq Ch^{m-1/2-\nu}\|f\|_{H^m_\omega}.
\end{aligned}$$

All those estimates taken into account yield the first claim. The second follows along similar lines with the stability estimate of P_I in Lemma 5.3.4.

$$\begin{aligned}
\|A_d^\omega f\|_{L^\infty_\omega} &= \|\tilde{P}_L \tilde{A}^\omega P_I f^\omega\|_{L^\infty} \\
&\leq Ch^{-1} \max_{(i,l)} |\tilde{a}^\omega(P_I f^\omega, \varphi_i^l)| \\
&\leq Ch^{-1} \max_{(i,l)} (|a^\omega(P_I f^\omega, \varphi_i^l) - \tilde{a}^\omega(P_I f^\omega, \varphi_i^l)| + |a^\omega(P_I f^\omega, \varphi_i^l)|) \\
&\leq Ch^{-1/2-\nu/2} \|P_I f^\omega\|_{H^{\nu/2}} \\
&\leq Ch^{-1/2-\nu}(h^{1/2+\delta}\|f\|_{H_\omega^{1/2+\delta}} + \|f\|_{L^2_\omega}).
\end{aligned}$$

□

Assumption. Now let for the rest of this section

$$\omega, \omega_f \in (-\eta, \eta), \quad \text{such that} \quad \omega - \omega_f \in (-\eta, \eta).$$

For $\omega^* := |\omega| \vee |\omega_f| \vee |\omega - \omega_f|$ we now assume $c_R \geq 2^{\frac{p+1/2-\nu}{\eta-\omega^*}}$.

Using this approximative generator, we can define the corresponding approximative carré-du-champ operator $\hat{\Gamma}_d^{\omega_f,\omega} : D(\hat{\Gamma}_d) \to Y_h$, where

$$D(\hat{\Gamma}_d) := \{(f,g) \in D_\Omega \times D_\Omega; fg \in D_\Omega\}.$$

For $(f,g) \in D(\hat{\Gamma}_d)$ let

$$\hat{\Gamma}_d^{\omega_f,\omega}(f,g) \ := \ A_d^\omega(fg) - fA_d^{\omega-\omega_f} g - gA_d^{\omega_f} f.$$

For the estimation of the difference of $\hat{\Gamma}_d^{\omega_f,\omega}$ and Γ we further have to restrict ourselves to functions for which the representation formula (5.25) holds. To this end we define for $r \geq 0$

$$D_r^{\hat{\Gamma}_d} := \{(f,g) \in D^2(\Gamma) \cap (D_\Omega^r \times D_\Omega^r); fg \in D_\Omega^r\}.$$

For shorter notation of the difference estimate define the following for $r, s \geq 0$:

$$\|f,g\|^{\hat{\Gamma}_d}_{(r,s,\omega_f,\omega)} := \|f\|^{A_d}_{(r,s,\omega_f)} \|g\|_{H^{1/2+\delta}_{\omega-\omega_f}} + \|f\|_{H^{1/2+\delta}_{\omega_f}} \|g\|^{A_d}_{(r,s,\omega-\omega_f)}.$$

With this notation the additional error due to $\hat{\Gamma}_d^{\omega_f,\omega}$ can now be bounded as follows.

Lemma 5.6.5 *[Properties of $\hat{\Gamma}_d^{\omega_f,\omega}$] Let $1 \leq m \leq p+1, m \in \mathbb{N}$ and $(f,g) \in D^{\hat{\Gamma}_d}_{m+\nu+1/2}$. Then we have*

$$\|(\Gamma - \hat{\Gamma}_d^{\omega_f,\omega})(f,g)\|_{L^\infty_\omega} \ \leq \ Ch^m \|f,g\|^{\hat{\Gamma}_d}_{(m,\nu+1/2,\omega_f,\omega)}.$$

For $(f,g) \in D(\hat{\Gamma}_d)$ we have the stability estimate

$$\|\hat{\Gamma}_d^{\omega_f,\omega}(f,g)\|_{L^\infty_\omega} \ \leq \ C\|f,g\|^{\hat{\Gamma}_d}_{(0,1/2+\delta,\omega_f,\omega)}.$$

Proof. This is a direct consequence of the previous Lemma 5.6.4. Indeed, together with the Sobolev Embedding and the estimate of Sobolev norms of products as in (2.1) we get

$$
\begin{aligned}
&\|(\Gamma - \hat{\Gamma}_d^{\omega_f,\omega})(f,g)\|_{L^\infty_\omega} \\
&\leq \Big(\|(A - A_d^\omega)(fg)\|_{L^\infty_\omega} + \|f(A - A_d^{\omega-\omega_f})g\|_{L^\infty_\omega} + \|g(A - A_d^{\omega_f})f\|_{L^\infty_\omega}\Big) \\
&\leq Ch^m\Big(\|fg\|^{A_d}_{(m,\nu+1/2,\omega)} + \|f\|_{L^\infty_{\omega_f}}\|g\|^{A_d}_{(m,\nu+1/2,\omega-\omega_f)} + \|g\|_{L^\infty_{\omega-\omega_f}}\|f\|^{A_d}_{(m,\nu+1/2,\omega_f)}\Big) \\
&\leq Ch^m\Big(\|f\|_{H^{1/2+\delta}_{\omega_f}}\|g\|^{A_d}_{(m,\nu+1/2,\omega-\omega_f)} + \|g\|_{H^{1/2+\delta}_{\omega-\omega_f}}\|f\|^{A_d}_{(m,\nu+1/2,\omega_f)}\Big) \\
&\leq Ch^m\|f,g\|^{\hat{\Gamma}_d}_{(m,\nu+1/2,\omega_f,\omega)}.
\end{aligned}
$$

The second claim can be derived along the very same lines with the second result of Lemma 5.6.4. □

Now we are still confronted with the problem that among others $\exp \notin D_\Omega$. Therefore, we apply a similar technique as in Section 4.2. That means, for every function f that depends upon ϵ or h we introduce a smooth function $\tilde{f} \in D_\mathbb{R}^{\omega_f}$ for which $f - \tilde{f} \in L^2$. Then we use the bilinearity of Γ in order to decompose its representation into terms involving $f - \tilde{f}$ and such only depending upon $\tilde{f}$. The terms corresponding to $\tilde{f}$ will be associated with the operator A while the others will be computed with A_d^ω.

More specifically, we define the following operator $\Gamma_d : D(\Gamma_d) \to C_b$, where

$$D(\Gamma_d) := \Big\{(f,\tilde{f},g,\tilde{g}) \in \bigcup_{-\eta\leq\omega_1,\omega_2\leq\eta} (C(\mathbb{R}) \times D_\mathbb{R}^{\omega_1} \times C(\mathbb{R}) \times D_\mathbb{R}^{\omega_2-\omega_1}); f - \tilde{f}, g - \tilde{g} \in D_\Omega\Big\}.$$

For $(f,\tilde{f},g,\tilde{g}) \in D(\Gamma_d)$ we define

$$
\begin{aligned}
\Gamma_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g}) \;:=\; & \hat{\Gamma}_d^{\omega_f,\omega}(f-\tilde{f},g-\tilde{g}) \\
& + \Big(A_d^\omega(\tilde{f}(g-\tilde{g})) - \tilde{f}A_d^{\omega-\omega_f}(g-\tilde{g}) - (g-\tilde{g})A^J\tilde{f}\Big) \\
& + \Big(A_d^\omega(\tilde{g}(f-\tilde{f})) - \tilde{g}A_d^{\omega_f}(f-\tilde{f}) - (f-\tilde{f})A^J\tilde{g}\Big) + \Gamma(\tilde{f},\tilde{g}).
\end{aligned}
$$

Similarly to the domains and function spaces corresponding to the operators Γ and ψ we now define for $r \geq 0$

$$
\begin{aligned}
D^{\Gamma_d}_{(r,\omega_f,\omega)} \;:=\; & \Big\{(f,\tilde{f},g,\tilde{g}) \in (C(\mathbb{R}) \times D_\mathbb{R}^{\omega_f} \times C(\mathbb{R}) \times D_\mathbb{R}^{\omega-\omega_f}); (f-\tilde{f}, g-\tilde{g}) \in D^r_\Omega \\
& \text{and } (f,g),(\tilde{f},g),(f,\tilde{g}),(\tilde{f},\tilde{g}) \in D^2(\Gamma)\Big\}.
\end{aligned}
$$

If we now again introduce the corresponding estimate for $r, s \geq 0$,

$$
\begin{aligned}
\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_d}_{(r,s,\omega_f,\omega)} \;:=\; & \|f-\tilde{f},g-\tilde{g}\|^{\hat{\Gamma}_d}_{(r,s,\omega_f,\omega)} + h^{-\delta}\|\tilde{f}\|_{C^{p+1}_{\omega_f}}\|g-\tilde{g}\|^{A_d}_{(r,s,\omega-\omega_f)} \\
& + h^{-\delta}\|\tilde{g}\|_{C^{p+1}_{\omega-\omega_f}}\|f-\tilde{f}\|^{A_d}_{(r,s,\omega_f)},
\end{aligned}
$$

we can derive the following error estimate.

Lemma 5.6.6 *[Properties of $\Gamma_d^{\omega_f,\omega}$] Let $1 \leq m \leq p+1, m \in \mathbb{N}$. Then we have the following for $(f, \tilde{f}, g, \tilde{g}) \in D^{\Gamma_d}_{(m+\nu+1/2,\omega_f,\omega)}$:*

$$\|\Gamma(f,g) - \Gamma_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g})\|_{L^\infty_\omega} \leq Ch^m \|f,\tilde{f},g,\tilde{g}\|^{\Gamma_d}_{(m,\nu+1/2,\omega_f,\omega)}.$$

For $(f,\tilde{f},g,\tilde{g}) \in D(\Gamma_d)$ we have the following stability estimate:

$$\begin{aligned}\|\Gamma_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g})\|_{L^\infty_\omega} \leq\ & C\big(\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_d}_{(0,1/2+\delta,\omega_f,\omega)} + \|g-\tilde{g}\|_{H^{1/2+\delta}_{\omega-\omega_f}}\|A\tilde{f}\|_{L^2_{\omega_f}} \\ & + \|f-\tilde{f}\|_{H^{1/2+\delta}_{\omega_f}}\|A\tilde{g}\|_{L^2_{\omega-\omega_f}} + \|\Gamma(\tilde{f},\tilde{g})\|_{H^{1/2+\delta}_\omega}\big).\end{aligned}$$

Proof. Again, the first claim is a consequence of the previous Lemmas. With the representation formula (5.25) we have the following:

$$\begin{aligned}\Gamma(f,g) - \Gamma_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g}) =\ & (\Gamma - \hat{\Gamma}_d^{\omega_f,\omega})(f-\tilde{f}, g-\tilde{g}) \\ & + \Big((A - A_d^\omega)(\tilde{f}(g-\tilde{g})) - \tilde{f}(A - A_d^{\omega_f-\omega})(g-\tilde{g})\Big) \\ & + \Big((A - A_d^\omega)(\tilde{g}(f-\tilde{f})) - \tilde{g}(A - A_d^{\omega_f})(f-\tilde{f})\Big).\end{aligned}$$

We have by definition $R = c_R|\log h|$. Together with the inequality (2.1) we have

$$\begin{aligned}\|\tilde{f}(g-\tilde{g})\|^{A_d}_{(m,\nu+1/2,\omega)} \leq\ & \|\Phi_R\tilde{f}\|_{H^{p+1}_{\omega_f}}\|g-\tilde{g}\|^{A_d}_{(m,\nu+1/2,\omega)} \\ \leq\ & C|\log h|\|\Phi_R\|_{C^{p+1}}\|\tilde{f}\|_{C^{p+1}_{\omega_f}}\|g-\tilde{g}\|^{A_d}_{(m,\nu+1/2,\omega)} \\ \leq\ & Ch^{-\delta}\|\tilde{f}\|_{C^{p+1}_{\omega_f}}\|g-\tilde{g}\|^{A_d}_{(m,\nu+1/2,\omega)}.\end{aligned}$$

Along the very same lines we get

$$\|\tilde{g}(f-\tilde{f})\|^{A_d}_{(m,\nu+1/2,\omega-\omega_f)} \leq Ch^{-\delta}\|\tilde{g}\|_{C^{p+1}_{\omega-\omega_f}}\|f-\tilde{f}\|^{A_d}_{(m,\nu+1/2,\omega-\omega_f)}.$$

Altogether this yields the first claim with the previous Lemma 5.6.5 and the properties of A_d^ω in Lemma 5.6.4. For the second we now only have to apply the stability estimate of A_d^ω and additionally consider the terms that canceled out in the previous calculation. This way the claim follows directly with the Sobolev Embedding. □

This definition now allows to introduce the corresponding approximate operator $\psi_d^{\omega_f,\omega}$. For $(f,\tilde{f},g,\tilde{g}) \in D(\Gamma_d)$ let it be defined as follows:

$$\begin{aligned}&\psi_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g}) := \\ &\quad \Gamma_d^{\omega_f,\omega}(f,\tilde{f},g,\tilde{g}) - \frac{1}{ce^{2x}}\Gamma_d^{\omega_f,\omega_f-1}(f,\tilde{f},exp,exp)\Gamma_d^{\omega-\omega_f,\omega-\omega_f-1}(g,\tilde{g},exp,exp).\end{aligned}$$

This is well-defined, because the domains have been chosen such that it is easy to see that $(f,\tilde{f},\exp,\exp), (g,\tilde{g},\exp,\exp) \in D(\Gamma_d)$. Again introducing the corresponding notation for $r, s \geq 0$,

$$\begin{aligned}\|f,\tilde{f},g,\tilde{g}\|^{\psi_d}_{(r,s,\omega_f,\omega)} :=\ & \|f,\tilde{f},g,\tilde{g}\|^{\Gamma_d}_{(r,s,\omega_f,\omega)} + \|f-\tilde{f}\|^{A_d}_{(r,s,\omega_f)}\|g,\tilde{g}\|^w_{(1/2+\delta,\omega-\omega_f)} \\ & + \|g-\tilde{g}\|^{A_d}_{(r,s,\omega-\omega_f)}(h^{1/2+\nu}\|f-\tilde{f}\|^{A_d}_{(1/2+\nu+\delta,\nu+1/2,\omega_f)} + \|f,\tilde{f}\|^w_{(1/2+\delta,\omega_f)}),\end{aligned}$$

we can now estimate the error due to this approximation.

Lemma 5.6.7 (Properties of $\psi_d^{\omega_f,\omega}$) *Let $1 \le m \le p+1, m \in \mathbb{N}$. Then we have the following for $(f,\tilde f,g,\tilde g) \in D^{\Gamma_d}_{(m+\nu+1/2,\omega_f,\omega)}$:*

$$\|\psi(f,g) - \psi_d^{\omega_f,\omega}(f,\tilde f,g,\tilde g)\|_{L^\infty_\omega} \le Ch^m \|f,\tilde f,g,\tilde g\|^{\psi_d}_{(m,\nu+1/2,\omega_f,\omega)}.$$

Furthermore, we have the following stability estimate:

$$\begin{aligned}\|\psi_d^{\omega_f,\omega}(f,\tilde f,g,\tilde g)\|_{L^\infty_\omega} \le C\Big(&\|f,\tilde f,g,\tilde g\|^{\Gamma_d}_{(0,1/2+\delta,\omega_f,\omega)} + \|g-\tilde g\|_{H^{1/2+\delta}_{\omega-\omega_f}}\|A\tilde f\|_{L^2_{\omega_f}} \\ &+\|f-\tilde f\|_{H^{1/2+\delta}_{\omega_f}}\|A\tilde g\|_{L^2_{\omega-\omega_f}} + \|\Gamma(\tilde f,\tilde g)\|_{H^{1/2+\delta}_\omega} \\ &+\big(\|f-\tilde f\|^{A_d}_{(0,1/2+\delta,\omega_f)} + \|f,\tilde f\|^w_{(1/2+\delta,\omega_f)}\big)\big(\|g-\tilde g\|^{A_d}_{(0,1/2+\delta,\omega-\omega_f)} + \|g,\tilde g\|^w_{(1/2+\delta,\omega-\omega_f)}\big)\Big).\end{aligned}$$

Proof. The difference satisfies the following equality:

$$\begin{aligned}&\psi(f,g) - \psi_d^{\omega_f,\omega}(f,\tilde f,g,\tilde g) \\ &= \Gamma(f,g) - \Gamma_d^{\omega_f,\omega}(f,\tilde f,g,\tilde g) - \frac{1}{ce^{2x}}\Big((\Gamma(f,exp) - \Gamma_d^{\omega_f,\omega_f-1}(f,\tilde f,exp,exp))\Gamma(g,exp) \\ &\quad +(\Gamma(g,exp) - \Gamma_d^{\omega-\omega_f,\omega-\omega_f-1}(g,\tilde g,exp,exp))\Gamma(f,exp) \\ &\quad -(\Gamma(g,exp) - \Gamma_d^{\omega-\omega_f,\omega-\omega_f-1}(g,\tilde g,exp,exp))(\Gamma(f,exp) - \Gamma_d^{\omega_f,\omega_f-1}(f,\tilde f,exp,exp))\Big).\end{aligned}$$

Furthermore, we have

$$\|f,\tilde f,\exp,\exp\|^{\Gamma_d}_{(1/2+\nu+\delta,\nu+1/2,\omega_f,\omega_f-1)} \le C\|f-\tilde f\|^{A_d}_{(1/2+\nu+\delta,\nu+1/2,\omega_f)}.$$

Thus, the first claim directly follows from the previous Lemma 5.6.6 together with the estimate for Γ in Lemma 4.2.2. Again for the second claim we have just to additionally consider the terms that have canceled out in the previous calculation. Here, we have

$$\|\Gamma_d^{\omega_f,\omega}(f,\tilde f,\exp,\exp)\|_{L^\infty_{\omega_f-1}} \le C(\|f-\tilde f\|^{A_d}_{(0,1/2+\delta,\omega_f)} + \|f,\tilde f\|^w_{(1/2+\delta,\omega_f)}).$$

The same holds for $g,\tilde g$ and therefore the second claim now follows with the second part of the previous Lemma. □

Now we have defined all necessary approximate operators. In the computation we will now substitute $\psi(f,g)$ by $P_I\psi_d^{0,0}(f,\tilde f,g,\tilde g)$. As already mentioned, the latter can be implemented without much additional effort as it only involves point-wise multiplications of vectors and matrix-vector multiplications. The only matrices needed to this end are $\tilde A^\omega$ and $B(M^{-1},\gamma_\delta)$. Additionally, the terms with respect to the already given functions $\tilde f$ and $\tilde g$ have to be implemented. This problem will be dealt with in Chapter 6. Since the matrices are sparse, the overall assembly can be achieved in $O(N\log N)$ steps as was proclaimed. Now, in order to be able to estimate the resulting error we have to consider the following difference and norm.

Theorem 5.6.8 (Error of sparse assembly) *Let $1 \leq m, m_1 \leq p+1, m, m_1 \in \mathbb{N}$. Then we have the following estimate for $(f,\tilde{f},g,\tilde{g}) \in D^{\Gamma_d}_{(m+\nu+1/2,0,0)}$ with $\psi(f,g) \in L^\infty$:*

$$\begin{aligned}&\|\psi(f,g) - P_I\psi_d^{0,0}(f,\tilde{f},g,\tilde{g})\|_{Y_h^*}\\ &\leq\ Ch^m\|f,\tilde{f},g,\tilde{g}\|^{\psi_d}_{(m,\nu+1/2,0,0)} + h^{m_1}\|\psi(f,g)\|_{C^{m_1}} + \|\psi(f,g)\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}\big).\end{aligned}$$

For $(f,\tilde{f},g,\tilde{g}) \in D(\Gamma_d)$ we have the following stability estimate:

$$\begin{aligned}&\|P_I\psi_d^{0,0}(f,\tilde{f},g,\tilde{g})\|_{Y_h^*}\\ &\leq\ Ch^{-\delta}\Big(\|f,\tilde{f},g,\tilde{g}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + \|g-\tilde{g}\|_{H^{1/2+\delta}}\|A\tilde{f}\|_{L^2}\\ &\quad+\|f-\tilde{f}\|_{H^{1/2+\delta}}\|A\tilde{g}\|_{L^2} + \|\Gamma(\tilde{f},\tilde{g})\|_{H^{1/2+\delta}}\\ &\quad+\big(\|f-\tilde{f}\|^{A_d}_{(0,1/2+\delta,0)} + \|f,\tilde{f}\|^{w}_{(1/2+\delta,0)}\big)\big(\|g-\tilde{g}\|^{A_d}_{(0,1/2+\delta,0)} + \|g,\tilde{g}\|^{w}_{(1/2+\delta,0)}\big)\Big).\end{aligned}$$

The overall assembly can be done within $O(N\log N)$ steps.

Proof. This is a direct consequence of the previous lemma, because we have the following with $R = c_R|\log h| \leq Ch^{-\delta}$. By [Els07, Chapter VI, Paragraph 2, Theorem 2.10] we have for $f_1 \in Y_h^* \cap L^\infty$

$$\begin{aligned}\|f_1\|_{Y_h^*} &\leq \sup_{v_h\in Y_h} |(f_1,v_h)|\|v_h\|_Y^{-1}\\ &\leq \|f_1\|_{L^\infty}\sup_{v_h\in Y_h}\|v_h\|_{L^1}\|v_h\|_Y^{-1}\\ &\leq CR^{1/2}\|f_1\|_{L^\infty}\sup_{v_h\in Y_h}\|v_h\|_{L^2}\|v_h\|_Y^{-1}\\ &\leq Ch^{-\delta}\|f_1\|_{L^\infty}.\end{aligned}$$

That means, we can estimate as follows:

$$\begin{aligned}&\|\psi(f,g) - P_I\psi_d^{0,0}(f,\tilde{f},g,\tilde{g})\|_{Y_h^*}\\ &\leq\ Ch^{-\delta}(\|P_I(\psi(f,g) - \psi_d^{0,0}(f,\tilde{f},g,\tilde{g}))\|_{L^\infty} + \|(Id-P_I)\psi(f,g)\|_{L^\infty})\\ &\leq\ Ch^{-\delta}(\|\psi(f,g) - \psi_d^{0,0}(f,\tilde{f},g,\tilde{g})\|_{L^\infty} + h^{m_1}\|\psi(f,g)\|_{C^{m_1}} + \|\psi(f,g)\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}).\end{aligned}$$

Similarly, we get the stability estimate. □

5.7 Numerical error estimate

Now all approximation steps for the model problem have been introduced and studied. Therefore, we can now apply these results to the equations (5.1) and (5.4) in order to estimate the overall error of all those steps. The problems we have to study now read as follows.

Find $\tilde{U}^{\epsilon,dG}_{R,h} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ such that for all $W \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ we have

$$\tilde{B}_{dG}(\tilde{U}^{\epsilon,dG}_{R,h}, W) = \int_0^T (A\overline{H}^{\epsilon_0}(t), W(t))dt + (P_L \Phi_{R-\delta}(H^\epsilon - H^{\epsilon_0}), W(0)^+). \quad (5.27)$$

The approximate solution of (5.27) via the GMRES method shall be denoted by $\tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}$. The computed approximation of the transformed option price function is now given by the following:

$$\tilde{\overline{V}} := \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h} + \overline{H}^{\epsilon_0}.$$

Since $\tilde{\overline{V}} - \overline{H}^{\epsilon_0} \notin D_\Omega$, we introduce the following functions for the application of ψ_d for $r > 0$:

$$\tilde{\overline{V}}_r := \Phi_r \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h} + \overline{H}^{\epsilon_0},$$
$$\tilde{\overline{V}}^\epsilon{}_r := \Phi_r(\overline{V}^\epsilon - \overline{H}^{\epsilon_0}) + \overline{H}^{\epsilon_0}.$$

Note that $\tilde{\overline{V}}_R = \tilde{\overline{V}}$. The approximate trading strategy is then computed by

$$\tilde{\overline{\vartheta}} = P_I \left(\frac{1}{ce^{2x}} \Gamma_d^{0,-1}(\tilde{\overline{V}}_{R/2} - \overline{H}^{\epsilon_0}, 0, \exp, \exp) \right) + \frac{1}{ce^{2x}} \Gamma(\overline{H}^{\epsilon_0}, \exp). \quad (5.28)$$

Now, the problem for the hedging error function reads as follows.

Find $\tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h} \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ such that for all $W \in \mathscr{S}^{\underline{r}}(\mathscr{M}, Y_h)$ we have

$$\tilde{B}_{dG}(\tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h}, W) = \int_0^T (P_I \psi_d^{0,0}(\tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0})(t), W(t))dt. \quad (5.29)$$

This is well-defined, because we have $(\tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}) \in D(\Gamma_d)$. The approximate solution of (5.29) via the GMRES method shall again be denoted by $\tilde{\overline{J}}^{\epsilon,dG,\Delta,\mathrm{GMRes}}_{R,h}$. The computed approximation of the transformed hedging error function is now given by

$$\tilde{\overline{J}} := \tilde{\overline{J}}^{\epsilon,dG,\Delta,\mathrm{GMRes}}_{R,h}.$$

Before we can estimate the overall computation error for the option price function $\tilde{\overline{V}}$ we have to derive some properties of the corresponding right hand side $A^X \overline{H}^{\epsilon_0}$.

Lemma 5.7.1 *For $0 \le s \le M_p - \lceil \varrho \rceil$ and $\omega \in [-\eta + \delta, \eta - \delta]$ the function $A^X \overline{H}^\epsilon$ satisfies the following norm estimate:*

$$\|A^X \overline{H}^\epsilon\|_{L^\infty(0,T;H^s_\omega)} \le C(1 + \epsilon^{1-\lceil s \rceil} + \epsilon^{3/2 - \lceil s \rceil - \varrho}).$$

Furthermore, we have for $0 \le l \le M_p - \lceil 3\varrho/2 \rceil, l \in \mathbb{N}_0$

$$\|D_t^l A^X \overline{H}^\epsilon\|_Y \le C(1 + \epsilon^{1-l-\lceil \varrho/2 \rceil} + \epsilon^{3/2 - l - \lceil \varrho/2 \rceil - \varrho}).$$

Proof. For the first claim it is obviously sufficient to consider AH^ϵ. These estimates are similar to those derived in Lemma 4.1.1. The integrability of $A^X H^\epsilon$, however, is due to the close resemblance of H^ϵ to $1-e^x$ for $x \to -\infty$ and the fact, that $A^X(1-e^x)=0$. Let $0 \le k \le M_p - \lceil \varrho \rceil$ and decompose the problem as follows. Here, we can use dominated convergence due to the properties of H^ϵ and get

$$\begin{aligned}
\sum_{l=0}^{k} \|D^l(A^X H^\epsilon)\|_{L^2_\omega} &= \sum_{l=0}^{k} \|A^X(D^l H^\epsilon)\|_{L^2_\omega} \\
&\le \sum_{l=0}^{k} \left(\|A^X(D^l H^\epsilon)\|_{L^2_\omega([-1,1])} + \|A^X(D^l H^\epsilon)\|_{L^2_\omega(\mathbb{R}\setminus[-1,1])}\right) \\
&\le \sum_{l=0}^{k} \left(\|D^l H^\epsilon\|_{H^\varrho_\delta} \|e^{(\omega-\delta)x}\|_{L^\infty([-1,1])} + \|A^X(D^l H^\epsilon)\|_{L^2_\omega(\mathbb{R}\setminus[-1,1])}\right) \\
&\le C(1+\epsilon^{3/2-k-\varrho}) + \sum_{l=0}^{k} \|A^X(D^l H^\epsilon)\|_{L^2_\omega(\mathbb{R}\setminus[-1,1])}.
\end{aligned}$$

Let $H(x) = (1-e^x)^+$ be the usual put function. Then we have for $x \in (-\infty,-1)$

$$\begin{aligned}
|A^X D^l H^\epsilon(x)| &= \left| \int_{-\infty}^{\log(1-\epsilon)-x} (D^l H(x+y) - D^l H(x) - (e^y-1) D^{l+1} H(x)) k(y) dy \right. \\
&\quad \left. + \int_{\log(1-\epsilon)-x}^{\log(1+\epsilon)-x} D^l q_l^\epsilon(x+y) k(y) dy \right| \\
&= \left| \int_{\log(1-\epsilon)-x}^{\log(1+\epsilon)-x} D^l q_l^\epsilon(x+y) k(y) dy \right| \\
&\le \epsilon \|D^l q_l^\epsilon\|_{L^\infty((\log(1-\epsilon),\log(1+\epsilon)))} \|k\|_{L^\infty((\log(1-\epsilon)-x,\log(1+\epsilon)-x))} \\
&\le C\epsilon(1+\epsilon^{1-l}) e^{-\eta|x|}.
\end{aligned}$$

Similarly, we have for $x \in (1,\infty)$

$$\begin{aligned}
|A^X D^l H^\epsilon(x)| &= \left| \int_{-\infty}^{\log(1+\epsilon)-x} D^l H^\epsilon(x+y) k(y) dy \right| \\
&\le C \|D^l H^\epsilon\|_{L^\infty} \|e^{-\delta y} k(y)\|_{L^\infty_y((-\infty,\log(1+\epsilon)-x))} \int_{-\infty}^{0} e^{\delta y} dy \\
&\le C(1+\epsilon^{1-l}) e^{-(\eta-\delta)|x|}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|A^X H^\epsilon\|_{H^k_\omega} &\le C(1+\epsilon^{3/2-k-\varrho} + (1+\epsilon^{1-k})\|e^{-(\eta-\delta)|x|}\|_{L^2_\omega}) \\
&\le C(1+\epsilon^{3/2-k-\varrho} + \epsilon^{1-k}).
\end{aligned}$$

This yields the first claim. The second claim is now a straightforward application of this result, because we have

$$\|D_t^l A^X \overline{H}^\epsilon\|_Y \le C \|A^X \overline{H}^\epsilon\|_{H^{\varrho/2+l}}.$$

□

With the results of the previous sections we can now finally estimate the overall error which accumulated along the various approximation steps. We start with the overall error corresponding to the option price function. The first result of the ensuing theorem is a version of the error bound already obtained in [MSW06, Theorem 4/5]. But the third is the one that will be needed for the error analysis of $\tilde{\overline{\vartheta}}$ and $\tilde{\overline{J}}$. For notation's sake we introduce the following indicator:

$$\kappa \;:=\; \begin{cases} 1 & \text{, if } \varrho \geq \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.7.2 (Overall error of the option price function $\tilde{\overline{V}}$) *Now let the parameter $M_t := (p+1)\frac{|\log h|}{|\log \sigma|}$. Then the error can be estimated as follows if $c_R \geq \frac{p+1}{\eta-\delta}$:*

$$\|\overline{V}^\epsilon - \tilde{\overline{V}}\|_{L^2(0,T;H^{(1-\delta)\varrho/2})} \;\leq\; Ch^{p+1-\varrho/2}(1+\epsilon^{1/2-p}).$$

For $c_R \geq \frac{p+1+(1-\kappa)\varrho}{\eta-\delta}$ we have

$$\|\overline{V}(T) - \tilde{\overline{V}}(T)\|_{L^2} \;\leq\; C(\epsilon^{3/2} + h^{p+1-\kappa\varrho/2}(1+\epsilon^{1/2-p})),$$
$$\|\overline{V}^\epsilon - \tilde{\overline{V}}\|_{L^2(0,T;L^2)} \;\leq\; Ch^{p+1-(1-\kappa)\delta-\kappa\varrho/2}(1+\epsilon^{1/2-p}).$$

The overall number of computation steps is of order $O(N(\log N)^8)$.

Proof. The equation (5.1) for $\overline{V}^\epsilon - \overline{H}^{\epsilon_0}$ fits the model problem (5.7) with $g = A^X\overline{H}^{\epsilon_0}$ and $u_0 = H^\epsilon - H^{\epsilon_0}$. By Lemma 5.7.1 applied to $A^X\overline{H}^{\epsilon_0}$ we see that the model parameter d does not depend upon ϵ or h. Therefore, the overall computation cost follows by Theorem 5.5.1 with $d = C$. Furthermore, we have $\lambda = \eta$. The first error can now be decomposed as follows. With $U^\epsilon := \overline{V}^\epsilon - \overline{H}^{\epsilon_0}$ we have

$$\begin{aligned} \|\overline{V}(T) - \tilde{\overline{V}}(T)\|_{L^2} \;\leq\; & \|\overline{V}(T) - \overline{V}^\epsilon(T)\|_{L^2} + \|U^\epsilon(T) - U^\epsilon_R(T)\|_{L^2} \\ & + \|U^\epsilon_R(T) - \tilde{U}^\epsilon_{R,h}(T)\|_{L^2} + \|\tilde{U}^\epsilon_{R,h}(T) - \tilde{U}^{\epsilon,dG}_{R,h}(T)\|_{L^2} \\ & + \|\tilde{U}^{\epsilon,dG}_{R,h}(T) - \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}(T)\|_{L^2}. \end{aligned}$$

Similarly we get for the second error

$$\begin{aligned} & \|\overline{V}^\epsilon - \tilde{\overline{V}}\|_{L^2(0,T;H^{(1-\delta)\varrho/2})} \\ & \quad \leq \; \|U^\epsilon - U^\epsilon_R\|_{L^2(0,T;H^{\varrho/2})} + \|U^\epsilon_R - \tilde{U}^\epsilon_{R,h}\|_{L^2(0,T;H^{(1-\delta)\varrho/2})} \\ & \qquad + \|\tilde{U}^\epsilon_{R,h} - \tilde{U}^{\epsilon,dG}_{R,h}\|_{L^2(0,T;Y)} + \|\tilde{U}^\epsilon_{R,h} - \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}\|_{L^2(0,T;Y)}. \end{aligned}$$

By Lemma 4.1.4 we have

$$\|\overline{V}(T) - \overline{V}^\epsilon(T)\|_{L^2} \;\leq\; C\epsilon^{3/2}.$$

Now we can use the results of the previous sections in order to estimate the respective partial errors. The respective properties of the terms $H^\epsilon - H^{\epsilon_0}$ and $A^X\overline{H}^{\epsilon_0}$ are taken out of Lemma 4.1.1 and Lemma 5.7.1.

1. The localization error can be estimated with Theorem 5.2.1:

$$
\begin{aligned}
&\|U^\epsilon(T) - U^\epsilon(T)_R\|_{L^2} + \|U^\epsilon - U^\epsilon_R\|_{L^2(0,T;H^{\varrho/2})} \\
\leq\ & Ce^{-\frac{\eta-\delta}{\eta-\delta}(p+1)|\log h|}\big(\|H^\epsilon - H^{\epsilon_0}\|_{L^2_{-\eta+\delta,\eta-\delta}} + \|A^X\overline{H}^{\epsilon_0}\|_{L^2(0,T;(H^{\varrho/2}_{-\eta+\delta,\eta-\delta})^*)}\big) \\
\leq\ & Ch^{p+1}\big(\|H^\epsilon - H^{\epsilon_0}\|_{L^2_{-\eta+\delta,\eta-\delta}} + \|A^X\overline{H}^{\epsilon_0}\|_{L^2(0,T;(L^2_{-\eta+\delta,\eta-\delta})^*)}\big) \\
\leq\ & Ch^{p+1}.
\end{aligned}
$$

2. The error due to spatial semi-discretization can be bounded with Theorem 5.3.10 as follows:

$$
\begin{aligned}
&\|U^\epsilon(T)_R - \tilde{U}^\epsilon(T)_{R,h}\|_{L^2} \\
\leq\ & Ch^{p+1-\kappa\varrho/2}\big(\|H^\epsilon - H^{\epsilon_0}\|_{H^{p+1}\cap H^{\varrho}_{-\eta+\delta,\eta-\delta}} + \|A^X\overline{H}^{\epsilon_0}\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-\eta+\delta,\eta-\delta})}\big) \\
\leq\ & Ch^{p+1}(1+\epsilon^{1/2-p})
\end{aligned}
$$

and

$$
\begin{aligned}
\|U^\epsilon_R - \tilde{U}^\epsilon_{R,h}\|_{L^2(0,T;L^2)} &\leq Ch^{p+1-(1-\kappa)\delta-\kappa\varrho/2}(1+\epsilon^{1/2-p}) \\
\|U^\epsilon_R - \tilde{U}^\epsilon_{R,h}\|_{L^2(0,T;H^{(1-\delta)\varrho/2})} &\leq Ch^{p+1-\varrho/2}(1+\epsilon^{1/2-p}).
\end{aligned}
$$

3. The time discretization error can be bounded with Theorem 5.4.3:

$$
\begin{aligned}
&\|\tilde{U}^\epsilon_{R,h}(T) - \tilde{U}^{\epsilon,dG}_{R,h}(T)\|_{L^2} + \|\tilde{U}^\epsilon_{R,h} - \tilde{U}^{\epsilon,dG}_{R,h}\|_{L^2(0,T;Y)} \\
\leq\ & Ch^{p+1}\big(\|H^\epsilon - H^{\epsilon_0}\|_{L^2} + \|A^X\overline{H}^{\epsilon_0}\|_{L^2(0,T;(Y_h)^*)}\big) \\
\leq\ & Ch^{p+1}.
\end{aligned}
$$

4. The error due to the approximative solution of the linear system can be estimated with Theorem 5.5.1:

$$
\begin{aligned}
&\|\tilde{U}^{\epsilon,dG}_{R,h}(T) - \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}(T)\|_{L^2} + \|\tilde{U}^{\epsilon,dG}_{R,h} - \tilde{U}^{\epsilon,dG,\mathrm{GMRes}}_{R,h}\|_{L^2(0,T;Y)} \\
\leq\ & Ch^{p+1}\big(\|H^\epsilon - H^{\epsilon_0}\|_{L^2} + \|A^X\overline{H}^{\epsilon_0}\|_{L^2(0,T;Y_h^*)}\big) \\
\leq\ & Ch^{p+1}.
\end{aligned}
$$

Altogether this yields the claim. □

In order to derive norm estimates for $\tilde{\overline{\vartheta}}$ and $\tilde{\overline{J}}$ we have to make use of the functions $\tilde{\overline{V}}$, $\tilde{\overline{V}}_R$ and $\tilde{\overline{V}}^\epsilon_R$. Therefore, we now derive auxiliary norm estimates of the differences that will appear in the error estimation of $\tilde{\overline{\vartheta}}$ and $\tilde{\overline{J}}$. For abbreviation's sake we define the following:

$$
\tilde{v}(r) \ := \ h^{p+1-r-\kappa\varrho/2-\delta}v^\epsilon(p+1).
$$

This enables to estimate the following differences.

Lemma 5.7.3 *Let* $\frac{p+1+(1-\kappa)\varrho}{\eta}|\log h| + 2\delta \leq r \leq R$ *be given. The following norm estimates hold for* $0 \leq s_1 \leq p+1, \omega \in [-\eta, \eta]$ *and* $0 \leq s_2 < \gamma$*:*

$$\begin{aligned} \|\overline{V}^\epsilon - \overline{H}^{\epsilon_0}\|_{H^{s_1}_\omega} + h^\delta \|\overline{V}^\epsilon_r - \overline{H}^{\epsilon_0}\|_{H^{s_1}_\omega} &\leq C v^\epsilon(s_1), \\ h^\delta \|\tilde{\overline{V}}_r - \overline{V}^\epsilon_r\|_{H^{s_2}} + \|\tilde{\overline{V}} - \overline{V}^\epsilon\|_{H^{s_2}} &\leq C\tilde{v}(s_2), \\ h^\delta \|\tilde{\overline{V}}_r - \overline{H}^{\epsilon_0}\|_{H^{s_2}} &\leq C\tilde{v}(s_2). \end{aligned}$$

Furthermore, we have the following approximation property for $\omega \in (-\eta+\delta, \eta-\delta)$*:*

$$\|\overline{V}^\epsilon - \overline{V}^\epsilon_r\|_{H^{s_1}_\omega} \leq Ce^{-(\eta-|\omega|-\delta)(r-\delta)} v^\epsilon(s_1).$$

Proof. For the first estimate we use the Dunford-Taylor representation of $\overline{V}^\epsilon - \overline{H}^{\epsilon_0}$ and apply the same estimation procedure as in the proof for the spatial semi-discretization error. More specifically, we have

$$(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})(t,\cdot) = e^{-t\mathscr{A}}(H^\epsilon - H^{\epsilon_0}) + \int_0^t e^{-(t-\tau)\mathscr{A}}(A^X \overline{H}^{\epsilon_0})(\tau)d\tau.$$

Using the curve of (5.19) we get for $f \in \{H^\epsilon - H^{\epsilon_0}, A^X\overline{H}^{\epsilon_0}\}$ together with the properties of the resolvent of $\mathscr{A}^\omega$ in Lemma 5.3.8 the following:

$$\begin{aligned} \|e^{-t\mathscr{A}} f\|_{H^{s_1}_\omega} &= \|e^{-t\mathscr{A}^\omega}(e^{\omega x} f)\|_{H^{s_1}} \\ &\leq \int_{\Gamma_t} e^{-tz} \|(z-\mathscr{A}^\omega)^{-1}(e^{\omega x} f)\|_{H^{s_1}} |dz| \\ &\leq C\|f\|_{H^{s_1}_\omega}. \end{aligned}$$

This yields with the properties of $H^\epsilon - H^{\epsilon_0}$ and $A^X\overline{H}^{\epsilon_0}$

$$\begin{aligned} \|\overline{V}^\epsilon - \overline{H}^{\epsilon_0}\|_{H^{s_1}_\omega} &\leq C\|H^\epsilon - H^{\epsilon_0}\|_{H^{s_1}_\omega} + \int_0^t \|A^X \overline{H}^{\epsilon_0}(\tau)\|_{H^{s_1}_\omega} d\tau \\ &\leq C v^\epsilon(s_1). \end{aligned}$$

The second estimate is a direct consequence of this result. Indeed, we have

$$\begin{aligned} \|\overline{V}^\epsilon_r - \overline{H}^{\epsilon_0}\|_{H^{s_1}_\omega} &= \|\Phi_r(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{H^{s_1}_\omega} \\ &\leq C\|\Phi_r\|_{H^{p+1}} \|\overline{V}^\epsilon - \overline{H}^{\epsilon_0}\|_{H^{s_1}_\omega} \\ &\leq Cr\|\Phi_r\|_{C^{p+1}} v^\epsilon(s_1) \\ &\leq Ch^{-\delta} v^\epsilon(s_1). \end{aligned}$$

For the third estimate we use the shift property of $P_{\underline{L}}$ of Lemma 5.3.4 in order to be able to apply the result of the previous theorem, $\|\tilde{\overline{V}} - \overline{V}^\epsilon\|_{L^2} \leq Ch^{p+1-\delta} v^\epsilon(p+1)$. However, since $\overline{V}^\epsilon - \overline{H}^{\epsilon_0} \notin \tilde{H}^{p+1}$ we have first to modify by multiplication of $\Phi_{R-\delta}$. For the additional terms we use the first estimate as follows for $0 \leq s \leq p+1$:

$$\begin{aligned} \|(1-\Phi_r)(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{H^s} &\leq \|(1-\Phi_r)e^{-\eta|x|}\|_{H^{p+1}} \|\overline{V}^\epsilon - \overline{H}^{\epsilon_0}\|_{H^s_{-\eta,\eta}} \\ &\leq Ce^{-(r-\delta)\eta} v^\epsilon(s) \\ &\leq Ch^{p+1} v^\epsilon(s). \end{aligned}$$

With this we can now proceed as described:

$$\begin{aligned}
\|\tilde{\overline{V}} - \overline{V}^\epsilon\|_{H^{s_2}} &\leq \|(1-\Phi_{R-\delta})(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{H^{s_2}} + \|(Id - P_L)\Phi_{R-\delta}(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{H^{s_2}} \\
&\quad + \|P_L(\Phi_{R-\delta}(\overline{V}^\epsilon - \overline{H}^{\epsilon_0}) - (\tilde{\overline{V}} - \overline{H}^{\epsilon_0}))\|_{H^{s_2}} \\
&\leq C(h^{p+1}v^\epsilon(s_2) + \tilde{v}(s_2) + h^{-s_2}\|\Phi_{R-\delta}(\overline{V}^\epsilon - \overline{H}^{\epsilon_0}) - (\tilde{\overline{V}} - \overline{H}^{\epsilon_0})\|_{L^2}) \\
&\leq C(\tilde{v}(s_2) + h^{-s_2}(\|(1-\Phi_{R-\delta})(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{L^2} + \|\overline{V}^\epsilon - \tilde{\overline{V}}\|_{L^2})) \\
&\leq C(\tilde{v}(s_2) + h^{p+1-s_2}v^\epsilon(0) + \tilde{v}(s_2)) \\
&\leq C\tilde{v}(s_2).
\end{aligned}$$

The approximation property we get similarly as above:

$$\begin{aligned}
\|\overline{V}^\epsilon - \overline{V}_r^\epsilon\|_{H_\omega^{s_1}} &= \|(1-\Phi_r)(\overline{V}^\epsilon - \overline{H}^{\epsilon_0})\|_{H_\omega^{s_1}} \\
&\leq C\|e^{-(\eta-|\omega|)|x|}(1-\Phi_r)\|_{H^{p+1}}\|\overline{V}^\epsilon - \overline{H}^{\epsilon_0}\|_{H^{s_1}_{-\eta,\eta}} \\
&\leq Ce^{-(\eta-|\omega|-\delta)(r-\delta)}v^\epsilon(s_1).
\end{aligned}$$

The remaining norm estimates now follow from these results:

$$\begin{aligned}
\|\tilde{\overline{V}}_r - \overline{V}_r^\epsilon\|_{H^{s_2}} &\leq C\|\Phi_r\|_{H^{p+1}}\|\tilde{\overline{V}} - \overline{V}^\epsilon\|_{H^{s_2}} \\
&\leq Ch^{-\delta}\tilde{v}(s_2).
\end{aligned}$$

And finally we have

$$\begin{aligned}
\|\tilde{\overline{V}}_r - \overline{H}^{\epsilon_0}\|_{H^{s_2}} &\leq C\|\overline{V}_r^\epsilon - \overline{H}^{\epsilon_0}\|_{H^{s_2}} + \|\tilde{\overline{V}}_r - \overline{V}_r^\epsilon\|_{H^{s_2}} \\
&\leq Ch^{-\delta}\tilde{v}(s_2).
\end{aligned}$$

□

These results now allow to estimate the error for the approximate trading strategy function $\tilde{\overline{\vartheta}}$. Here, we have to deal with weighted versions of the norm estimates of the previous lemma. But we do not dispose of weighted versions for the discretization errors. Therefore, we have to apply a crude method to deal with this problem. That means, we estimate via the supremum of the weight function. Since the weight we deal with is e^{-x} and the parameter R can be chosen proportional to η^{-1}, we still get convergence for most model parameter sets. But the order of the estimate is less than that which is suggested by the experiments in Chapter 7.

Theorem 5.7.4 (Overall error of the trading strategy function $\tilde{\overline{\vartheta}}$) *Now let the parameters be given by $c_R \geq 2\left(\frac{p+1/2-\nu}{\eta-1} \vee \frac{p+1+(1-\kappa)\varrho}{\eta-\delta}\right)$ and $M_t := (p+1)\frac{|\log h|}{|\log \sigma|}$. Additionally, we assume $h \leq \epsilon, \sigma^2 = 0$ and $p \geq 1$. Then the overall error can be estimated as follows:*

$$\|\overline{\vartheta} - \tilde{\overline{\vartheta}}\|_{L^2(0,T;L^2)} \leq C(\epsilon^{3/2} + h^{p+1/2-(1+1/2\kappa)\nu - c_R/2 - 2\delta}\epsilon^{1/2-p}).$$

The overall computation cost is of order $O(N(\log N)^8)$.

Proof. In the ensuing analysis we need weighted versions of the computation error with respect to $\tilde{\overline{V}}$. Since we do not dispose of such results we have to do the following crude estimation:

$$\begin{aligned} \|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}\|_{H^s_{-1}} &\leq \|\Phi_{R/2}e^{-x}\|_{H^{p+1}}\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}\|_{H^r} \\ &\leq Ch^{-c_{R/2}}\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}\|_{H^r} \\ &\leq Ch^{-c_{R/2}-\delta}\tilde{v}(r). \end{aligned}$$

In Proposition 3.0.4 it has been shown that for $0 \leq t < T$ we have

$$\vartheta(t,x) = \frac{1}{ce^{2x}}\Gamma(V, \exp)(t,x).$$

That means, the error can now again be decomposed as follows:

$$\begin{aligned} \|\overline{\vartheta} - \tilde{\overline{\vartheta}}\|_{L^2} \leq\ & \left\| \frac{1}{ce^{2x}}\Gamma(\overline{V} - \overline{V}^{\epsilon}_{R/2}, \exp)\right\|_{L^2} + \left\|(Id - P_I)\left(\frac{1}{ce^{2x}}\Gamma(\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon_0}, \exp)\right)\right\|_{L^2} \\ &+ \left\|P_I\left(\frac{1}{ce^{2x}}\left(\Gamma(\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon_0}, \exp) - \Gamma^{0,-1}_d(\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon_0}, 0, \exp, \exp)\right)\right)\right\|_{L^2} \\ &+ \left\|P_I\left(\frac{1}{ce^{2x}}\Gamma^{0,-1}_d(\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}, 0, \exp, \exp)\right)\right\|_{L^2}. \end{aligned}$$

These terms can now be estimated with the results of the previous Lemma as follows.

1. With the property of Γ in Lemma 4.2.2, the approximation property of $\overline{V}^{\epsilon}$ in Lemma 4.1.4 and the previous Lemma 5.7.3 we have

$$\begin{aligned} \left\|\frac{1}{ce^{2x}}\Gamma(\overline{V} - \overline{V}^{\epsilon}_{R/2}, \exp)\right\|_{L^2} &\leq C(\|\overline{V} - \overline{V}^{\epsilon}, 0\|^w_{(0,-1)} + \|\overline{V}^{\epsilon} - \overline{V}^{\epsilon}_{R/2}, 0\|^w_{(0,-1)}) \\ &\leq C(\|\overline{V} - \overline{V}^{\epsilon}\|_{H^{\varrho/2}_{-1}} + \|\overline{V}^{\epsilon} - \overline{V}^{\epsilon}_{R/2}\|_{H^{\varrho/2}_{-1}}) \\ &\leq C(\epsilon^{3/2} + h^{(\eta-1)c_{R/2}}) \\ &\leq C(\epsilon^{3/2} + h^{p+1/2-\nu}). \end{aligned}$$

2. For the second term we have again the problem that $\Gamma(V^{\epsilon}_{R/2} - \overline{H}^{\epsilon_0}, \exp) \notin \tilde{H}^{p+1}$. The approximation property of P_I in Lemma 5.3.2 and the norm estimate for Γ

in Lemma 4.2.2 therefore yield the following:

$$\begin{aligned}
&\left\|(Id-P_I)\left(\frac{1}{ce^{2x}}\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)\right)\right\|_{L^2}\\
&\leq\ C(h^{p+1}\|e^{-2x}\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)\|_{H^{p+1}}+\|e^{-2x}\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)\|_{L^2(\mathbb{R}\setminus\Omega_i)})\\
&\leq\ C(h^{p+1}\|\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},0\|^{w}_{(p+1,-1)}\\
&\qquad+\|e^{-x}e^{-\eta|x|}\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}\|e^{-x}\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)\|_{L^2_{-\eta,\eta}})\\
&\leq\ C(h^{p+1-\delta}v^{\epsilon}(p+1+\varrho/2)+e^{(\eta-1)(R-1)}\|\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0}\|_{Y_{-\eta,\eta}}\\
&\leq\ C(h^{p+1-\delta}v^{\epsilon}(p+1+\varrho/2)+h^{p+1/2-\nu}v^{\epsilon}(\varrho/2))\\
&\overset{h\leq\epsilon}{\leq}\ C(\tilde{v}(\varrho/2)+\tilde{v}(1/2+\nu))\\
&\overset{\nu=\varrho}{\leq}\ C\tilde{v}(1/2+\nu).
\end{aligned}$$

3. With the estimate of the error due to Γ_d in Lemma 5.6.6 we further get

$$\begin{aligned}
&\left\|P_I\left(\frac{1}{ce^{2x}}\left(\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)-\Gamma_d^{0,-1}(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},0,\exp,\exp)\right)\right)\right\|_{L^2}\\
&\leq\ Ch^{-\delta}\left\|\Gamma(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},\exp)-\Gamma_d^{0,-1}(\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0},0,\exp,\exp)\right\|_{L^\infty_{-2}}\\
&\leq\ Ch^{p+1-\delta}\|\overline{V}^{\epsilon}_{R/2}-\overline{H}^{\epsilon_0}\|^{A_d^J}_{(p+1,\nu+1/2,-1)}\\
&\overset{h\leq\epsilon}{\leq}\ Ch^{p+1/2-\nu-\delta}v^{\epsilon}(p+1)\\
&\leq\ C\tilde{v}(1/2+\nu).
\end{aligned}$$

4. Finally, we use the stability estimate in Lemma 5.6.6 and the crude estimate above. Thus, we get

$$\begin{aligned}
&\left\|P_I\left(\frac{1}{ce^{2x}}\Gamma_d^{0,-1}(\overline{V}^{\epsilon}_{R/2}-\tilde{\tilde{V}},0,\exp,\exp)\right)\right\|_{L^2}\leq Ch^{-\delta}\|\overline{V}^{\epsilon}_{R/2}-\tilde{\tilde{V}}\|^{A_d^J}_{(0,1/2+\delta,-1)}\\
&\leq Ch^{-1/2-\nu-\delta}(\|\overline{V}^{\epsilon}_{R/2}-\tilde{\tilde{V}}\|_{L^2_{-1}}+h^{1/2+\delta}\|\overline{V}^{\epsilon}_{R/2}-\tilde{\tilde{V}}\|_{H^{1/2+\delta}_{-1}})\\
&\leq C\tilde{v}(1/2+\nu+c_R/2+\delta).
\end{aligned}$$

These estimates combined yield the claim:

$$\begin{aligned}
\|\overline{\vartheta}-\tilde{\tilde{\vartheta}}\|_{L^2}\ &\leq\ C(\epsilon^{3/2}+h^{p+1/2-\nu}+\tilde{v}(1/2+\nu)+\tilde{v}(1/2+\nu)+\tilde{v}(1/2+\nu+c_R/2+\delta))\\
&\leq\ C(\epsilon^{3/2}+\tilde{v}(1/2+\nu+c_R/2+\delta)).
\end{aligned}$$

□

Before we now finally are able to estimate the error for the hedging error function $\tilde{\tilde{J}}$ we first have to derive the following error bound and stability estimate for the right hand side of equation (5.29).

Lemma 5.7.5 *The difference of the right hand side and its sparse version can be bounded as follows for* $c_R \geq 2\left(\frac{p+1+(1-\kappa)\varrho}{2-\delta} \vee \frac{p+1/2-\nu}{\eta-1}\right)$, $h \leq \epsilon$, $\sigma^2 = 0$ *and* $p \geq 1$*:*

$$\|\psi(\overline{V}^\epsilon, \overline{V}^\epsilon) - P_I\psi_d^{0,0}(\tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0})\|_{L^2(0,T;Y_h^*)} \leq C\tilde{v}(1+2\nu+\delta).$$

The norm of the approximate right hand side is bounded as follows:

$$\|P_I\psi_d^{0,0}(\tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0})\|_{L^2(0,T;Y_h^*)} \leq Ch^{-1-2\nu-\delta}.$$

Proof. For the first claim we can decompose as follows:

$$\begin{aligned}
&\|\psi(\overline{V}^\epsilon, \overline{V}^\epsilon) - P_I\psi_d^{0,0}(\tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\overline{V}}_{R/2}, \overline{H}^{\epsilon_0})\|_{Y_h^*} \leq \\
&\quad C\Big(\|\psi(\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon + \overline{V}^\epsilon_{R/2})\|_{Y_h^*} + \|\psi(\overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon_{R/2}) - P_I\psi_d^{0,0}(\overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0}, \overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0})\|_{Y_h^*} \\
&\quad +\|P_I\psi_d^{0,0}(\overline{V}^\epsilon_{R/2} - \tilde{\overline{V}}_{R/2}, 0, \overline{V}^\epsilon_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon_0})\|_{Y_h^*}\Big).
\end{aligned}$$

Now the respective terms can be bounded with the following. Since $h \leq \epsilon$ we have

$$v^\epsilon(r) + \tilde{v}(r) \leq C, \quad \forall 0 \leq r \leq 3/2.$$

Furthermore, due to $\sigma^2 = 0$ we can assume $1/2 + \varrho/2 + \delta \leq 3/2$. This observation allows to neglect most of the terms that will appear in the ensuing analysis.

1. For the first term we can use the norm estimate for ψ in Lemma 4.2.3 and the results of Lemma 5.7.3:

$$\begin{aligned}
&\|\psi(\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon + \overline{V}^\epsilon_{R/2})\|_{Y_h^*} \leq C\|\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}, 0, \overline{V}^\epsilon + \overline{V}^\epsilon_{R/2}, 2\overline{H}^{\epsilon_0}\|^{\Gamma_2}_{(0,0,0)} \\
&\leq C\Big(\|\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}, 0\|^w_{(0,0)}\|\overline{V}^\epsilon + \overline{V}^\epsilon_{R/2}, 2\overline{H}^{\epsilon_0}\|^w_{(1/2+\delta,0)} \\
&\qquad +\|\overline{V}^\epsilon + \overline{V}^\epsilon_{R/2}, 2\overline{H}^{\epsilon_0}\|^w_{(0,0)}\|\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}, 0\|^w_{(1/2+\delta,0)}\Big) \\
&\leq C\Big(\|\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}\|_{H^{\varrho/2}}(\|\overline{V}^\epsilon + \overline{V}^\epsilon_{R/2} - 2\overline{H}^{\epsilon_0}\|_{H^{\varrho/2+1/2+\delta}} + \|2D\overline{H}^{\epsilon_0}\|_{H^{1/2+\delta}}) \\
&\qquad +\|\overline{V}^\epsilon - \overline{V}^\epsilon_{R/2}\|_{H^{\varrho/2+1/2+\delta}}(\|\overline{V}^\epsilon + \overline{V}^\epsilon_{R/2} - 2\overline{H}^{\epsilon_0}\|_{H^{\varrho/2}} + \|2D\overline{H}^{\epsilon_0}\|_{L^2})\Big) \\
&\leq C\Big(h^{p+1-\delta}v^\epsilon(\varrho/2)v^\epsilon(\varrho/2 + 1/2 + \delta) + h^{p+1-\delta}v^\epsilon(\varrho/2 + 1/2 + \delta)v^\epsilon(\varrho/2)\Big) \\
&\overset{\sigma^2=0}{\leq} Ch^{p+1-\delta}.
\end{aligned}$$

2. The second term is estimated with the error estimate for ψ_d in Theorem 5.6.8:

$$\begin{aligned}
&\|\psi(\overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon_{R/2}) - P_I\psi_d^{0,0}(\overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0}, \overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0})\|_{Y_h^*} \\
&\leq Ch^{p+1}\big(h^{-\delta}\|\overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0}, \overline{V}^\epsilon_{R/2}, \overline{H}^{\epsilon_0}\|^{\psi_d}_{(p+1,\nu+1/2,0,0)} + \|\psi(\overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon_{R/2})\|_{C^{p+1}}\big) \\
&\quad +C\|\psi(\overline{V}^\epsilon_{R/2}, \overline{V}^\epsilon_{R/2})\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}.
\end{aligned}$$

Here, we have

$$\begin{aligned}
&\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}, \overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{\psi_d}_{(p+1,\nu+1/2,0,0)} \\
\leq\quad & C\|\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon 0}\|^{A_d^J}_{(p+1,\nu+1/2,0)}\Big(\|\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon 0}\|_{H^{1/2+\delta}} + h^{-\delta} \\
& +\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{w}_{(1/2+\delta,0)} + h^{1/2+\nu}\|\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon 0}\|^{A_d^J}_{(1/2+\delta,\nu+1/2,0)})\Big) \\
\overset{\sigma^2=0}{\leq}\quad & Ch^{-\delta}\|\overline{V}^{\epsilon}_{R/2} - \overline{H}^{\epsilon 0}\|^{A_d^J}_{(p+1,\nu+1/2,0)} \\
\overset{h\leq\epsilon}{\leq}\quad & Ch^{-1/2-\nu-2\delta}v^{\epsilon}(p+1)
\end{aligned}$$

and

$$\begin{aligned}
\|\psi(\overline{V}^{\epsilon}_{R/2}, \overline{V}^{\epsilon}_{R/2})\|_{C^{p+1}} \quad\leq\quad & C\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}, \overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{\Gamma_2}_{(p+3/2+\delta,0,0)} \\
\leq\quad & C\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{w}_{(p+3/2+\delta,0)}\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{w}_{(1/2+\delta,0)} \\
\leq\quad & Ch^{-\delta}v^{\epsilon}(p+3/2+\delta+\varrho/2).
\end{aligned}$$

Finally, the last term can be estimated as usual with the property of ψ in Lemma 4.2.3:

$$\begin{aligned}
&\|\psi(\overline{V}^{\epsilon}_{R/2}, \overline{V}^{\epsilon}_{R/2})\|_{L^\infty(\mathbb{R}\setminus\Omega_i)} \leq C\|e^{-(2-\delta)|x|}\|_{L^\infty(\mathbb{R}\setminus\Omega_i)}\|\psi(\overline{V}^{\epsilon}_{R/2}, \overline{V}^{\epsilon}_{R/2})\|_{L^\infty_{-2+\delta,2-\delta}} \\
\leq\quad & Ce^{-(2-\delta)(R-1)}(\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}, \overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{\Gamma_2}_{(1/2+\delta,-1+\delta/2,-2+\delta)} \\
& +\|\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}, \overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}\|^{\Gamma_2}_{(1/2+\delta,1-\delta/2,2-\delta)}) \\
\leq\quad & Ch^{p+1}v^{\epsilon}(1/2+\delta+\varrho/2)^2 \\
\leq\quad & Ch^{p+1}.
\end{aligned}$$

Overall, this yields

$$\begin{aligned}
&\|\psi(\overline{V}^{\epsilon}_{R/2}, \overline{V}^{\epsilon}_{R/2}) - P_I\psi_d^{0,0}(\overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0}, \overline{V}^{\epsilon}_{R/2}, \overline{H}^{\epsilon 0})\|_{Y_h^*} \\
\leq\quad & Ch^{p+1-\delta}(1+h^{-1/2-\nu-2\delta}+\epsilon^{-1/2-\delta-\varrho/2})v^{\epsilon}(p+1) \\
\leq\quad & Ch^{p+1}h^{-1/2-\nu-3\delta}v^{\epsilon}(p+1) \\
\leq\quad & C\tilde{v}(1/2+\nu+3\delta).
\end{aligned}$$

3. The estimate for the last term follows by the stability estimate in Theorem 5.6.8:

$$\begin{aligned}
&\|P_I\psi_d^{0,0}(\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon 0})\|_{Y_h^*} \\
\leq\quad & Ch^{-\delta}\Big(\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon 0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + \|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}\|_{H^{1/2+\delta}} \\
& +\big(\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}\|^{A_d^J}_{(0,1/2+\delta,0)} + \|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}, 0\|^{w}_{(1/2+\delta,0)}\big) \\
& \cdot\big(\|\overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2} - 2\overline{H}^{\epsilon 0}\|^{A_d^J}_{(0,1/2+\delta,0)} + \|\overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon 0}\|^{w}_{(1/2+\delta,0)}\big)\Big) \\
\leq\quad & Ch^{-\delta}\Big(\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon 0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + \tilde{v}(1/2+\delta) \\
& +\big(\tilde{v}(1/2+\nu) + \tilde{v}(1/2+\varrho/2+\delta)\big)h^{-1/2-\nu}\Big) \\
\leq\quad & Ch^{-\delta}\Big(\|\overline{V}^{\epsilon}_{R/2} - \tilde{\overline{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\overline{V}}_{R/2}, 2\overline{H}^{\epsilon 0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + \tilde{v}(1+2\nu)\Big).
\end{aligned}$$

The first term can now be estimated similar to the previous step as follows:

$$\begin{aligned}
&\|\overline{V}^{\epsilon}_{R/2} - \tilde{\tilde{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\tilde{V}}_{R/2}, 2\overline{H}^{\epsilon_0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} \\
\leq\ & C\Big(\|\overline{V}^{\epsilon}_{R/2} - \tilde{\tilde{V}}_{R/2}\|^{A^J_d}_{(0,1/2+\delta,0)}(\|\overline{V}^{\epsilon}_{R/2} + \tilde{\tilde{V}}_{R/2} - 2\overline{H}^{\epsilon_0}\|_{H^{1/2+\delta}} + h^{-\delta}) \\
&+\|\overline{V}^{\epsilon}_{R/2} + \tilde{\tilde{V}}_{R/2} - 2\overline{H}^{\epsilon_0}\|^{A^J_d}_{(0,1/2+\delta,0)}\|\overline{V}^{\epsilon}_{R/2} - \tilde{\tilde{V}}_{R/2}\|_{H^{1/2+\delta}}\Big) \\
\leq\ & Ch^{-1/2-\nu-\delta}(\tilde{v}(0) + \tilde{v}(1/2+\delta) + \tilde{v}(0)) \\
\leq\ & C\tilde{v}(1+\nu+2\delta).
\end{aligned}$$

Overall this yields

$$\begin{aligned}
&\|P_I\psi^{0,0}_d(\overline{V}^{\epsilon}_{R/2} - \tilde{\tilde{V}}_{R/2}, 0, \overline{V}^{\epsilon}_{R/2} + \tilde{\tilde{V}}_{R/2}, 2\overline{H}^{\epsilon_0})\|_{Y^*_h} \\
\leq\ & C(\tilde{v}(1+\nu+3\delta) + \tilde{v}(1+2\nu+\delta)) \\
\leq\ & C\tilde{v}(1+2\nu+\delta).
\end{aligned}$$

Combining all those estimates the first claim follows:

$$\begin{aligned}
&\|\psi(\overline{V}^{\epsilon}, \overline{V}^{\epsilon}) - P_I\psi^{0,0}_d(\tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0})\|_{L^2(0,T;Y^*_h)} \\
\leq\ & C(h^{p+1-\delta} + \tilde{v}(1/2+\nu+3\delta) + \tilde{v}(1+2\nu+\delta)) \\
\leq\ & C\tilde{v}(1+2\nu+\delta).
\end{aligned}$$

The second claim can now be derived using similar calculations and using the stability estimate of Theorem 5.6.8:

$$\begin{aligned}
&\|P_I\psi^{0,0}_d(\tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0})\|_{L^2(0,T;Y^*_h)} \\
&\leq Ch^{-\delta}\Big(\|\tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + \|\tilde{\tilde{V}}_{R/2} - \overline{H}^{\epsilon_0}\|_{H^{1/2+\delta}} + \|\Gamma(\tilde{\overline{H}}^{\epsilon_0}, \tilde{\overline{H}}^{\epsilon_0})\|_{H^{1/2+\delta}} \\
&\qquad + \big(\|\tilde{\tilde{V}}_{R/2} - \overline{H}^{\epsilon_0}\|^{A^J_d}_{(0,1/2+\delta,0)} + \|\tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}\|^{w}_{(1/2+\delta,0)}\big)^2\Big) \\
&\leq Ch^{-\delta}\Big(\|\tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}, \tilde{\tilde{V}}_{R/2}, \overline{H}^{\epsilon_0}\|^{\Gamma_d}_{(0,1/2+\delta,0,0)} + h^{-1-2\nu}\Big) \\
&\leq Ch^{-\delta}\Big(\|\tilde{\tilde{V}}_{R/2} - \overline{H}^{\epsilon_0}\|^{A^J_d}_{(0,1/2+\delta,0)}(\|\tilde{\tilde{V}}_{R/2} - \overline{H}^{\epsilon_0}\|_{H^{1/2+\delta}} + h^{-\delta}) + h^{-1-2\nu}\Big) \\
&\leq Ch^{-\delta}(h^{-1/2-\nu-\delta} + h^{-1-2\nu}) \\
&\leq Ch^{-1-2\nu-\delta}.
\end{aligned}$$

□

Finally we can estimate the overall computation error with respect to the hedging error function $\tilde{\tilde{J}}$.

Theorem 5.7.6 (Overall error of the hedging error function $\tilde{\tilde{J}}$) *Choose the parameters as follows,* $c_R := 2\left(\frac{p+1+(1-\kappa)\varrho}{2-\delta} \vee \frac{p+1/2-\nu}{\eta-1}\right)$, $\mu = \epsilon^{-\varrho}$ *and* $M_t := (p+1)\frac{|\log h|}{|\log \sigma|}$.

We further assume $p \geq 1, \sigma^2 = 0$ and $h \leq \epsilon$. Then the overall computation error can be estimated as follows:

$$\|\overline{J}(T) - \tilde{\overline{J}}(T)\|_{L^2} \;\leq\; C(\epsilon^{3/2} + h^{p-(2+1/2\kappa)\nu-\delta}\epsilon^{1/2-p}).$$

The overall computation cost is bounded by $O(\epsilon^{-(6+\delta)\varrho}N(\log N)^8)$.

Proof. The equation for $\overline{J}^\epsilon$ fits the model problem (5.7) with $g = e^{qt}\psi(\overline{V}^\epsilon, \overline{V}^\epsilon)$ and $u_0 = 0$. By Lemma 4.3.1 we have the following:

$$\begin{aligned}
\|D_t^l\psi(\overline{V}^\epsilon, \overline{V}^\epsilon)\|_Y \;&\leq\; C\sum_{k=0}^{l} \|D_t^{l-k}\overline{V}^\epsilon, D_t^{l-k}\overline{V}^{\epsilon_0}, D_t^k\overline{V}^\epsilon, D_t^k\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(\varrho/2,0,0)} \\
&\leq\; C\sum_{k=0}^{l}\Big(\|D_t^{l-k}(\overline{V}^\epsilon - \overline{V}^{\epsilon_0})\|^w_{(\varrho/2,0)}\|D_t^k(\overline{V}^\epsilon - \overline{V}^{\epsilon_0})\|^w_{(1/2+\delta,0)} \\
&\qquad +\|D_t^k(\overline{V}^\epsilon - \overline{V}^{\epsilon_0})\|^w_{(\varrho/2,0)}\|D_t^{l-k}(\overline{V}^\epsilon - \overline{V}^{\epsilon_0})\|^w_{(1/2+\delta,0)}\Big).
\end{aligned}$$

Now we can use the difference of the representations via Duhamel's formula of $\overline{V}^\epsilon - \overline{H}^{\epsilon_0}$ and of $\overline{V}^{\epsilon_0} - \overline{H}^{\epsilon_0}$ and get the following:

$$(\overline{V}^\epsilon - \overline{V}^{\epsilon_0})(t) \;=\; e^{-t\mathscr{A}}(H^\epsilon - H^{\epsilon_0}).$$

By Lemma 2.5.3 the time derivative can now be estimated as follows:

$$\begin{aligned}
\|D_t^k(\overline{V}^\epsilon - \overline{H}^\epsilon)\|_{H^r} \;&\leq\; C\|\mathscr{A}^k e^{-t\mathscr{A}}(H^\epsilon - H^{\epsilon_0})\|_{H^r} \\
&\leq\; C\|e^{-t\mathscr{A}}\mathscr{A}^k(H^\epsilon - H^{\epsilon_0})\|_{H^r} \\
&\leq\; C\|\mathscr{A}^k(H^\epsilon - H^{\epsilon_0})\|_{H^r} \\
&\leq\; C\|H^\epsilon - H^{\epsilon_0}\|_{H^{k\varrho+r}} \\
&\leq\; Cv^\epsilon(k\varrho + r).
\end{aligned}$$

Therefore, we finally get

$$\begin{aligned}
\|D_t^l\psi(\overline{V}^\epsilon, \overline{V}^\epsilon)\|_Y \;&\leq\; \sum_{k=0}^{l} v^\epsilon(k\varrho + \varrho/2)v^\epsilon((l-k)\varrho + 1/2 + \delta) \\
&\leq\; C\epsilon^{-l\varrho}.
\end{aligned}$$

For $D_t^l(e^{qt}\psi(\overline{V}^\epsilon, \overline{V}^\epsilon))$ this holds likewise. That means, we have $d = \epsilon^{-\varrho}$. Since the computation cost for the sparse assembly is bounded by $O(N\log N)$ the overall number of computation steps is therefore bounded by $O(\epsilon^{-(6+\delta)\varrho}N(\log N)^8)$ by Theorem 5.5.1. Recall for the sequel that due to $\sigma^2 = 0$ we have $\nu = \varrho < 2$. The computation error can now be decomposed as follows:

$$\begin{aligned}
\|\overline{J}(T) - \tilde{\overline{J}}(T)\|_{L^2} \;&\leq\; \|\overline{J}(T) - \overline{J}^\epsilon(T)\|_{L^2} + \|\overline{J}^\epsilon(T) - \overline{J}^\epsilon_R(T)\|_{L^2} \\
&\quad +\|\overline{J}^\epsilon_R(T) - \tilde{\overline{J}}^\epsilon_{R,h}(T)\|_{L^2} + \|\tilde{\overline{J}}^\epsilon_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG}_{R,h}(T)\|_{L^2} \\
&\quad +\|\tilde{\overline{J}}^{\epsilon,dG}_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h}(T)\|_{L^2} + \|\tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG,\Delta,\mathrm{GMRes}}_{R,h}(T)\|_{L^2}.
\end{aligned}$$

By Lemma 4.2.3 we have that $D_2\overline{V}^\epsilon \in L^\infty_{\omega/2}$ leads to $e^{qt}\psi(\overline{V}^\epsilon, \overline{V}^\epsilon) \in L^2(0,T;(Y_\omega)^*)$. That means, we have $\lambda = 2-\delta$. The following norm estimates of the right hand side are all due to the properties stated in that Lemma and the auxiliary estimates in the previous Lemma 5.7.5. With this the respective terms can now be bounded with the results of the previous sections.

1. By Lemma 4.1.6 we have $\|\overline{J}(T) - \overline{J}^\epsilon(T)\|_{L^2} \le C\epsilon^{3/2}$.

2. The localization error can be bounded with Theorem 5.2.1 as follows:
$$\begin{aligned}
&\|\overline{J}^\epsilon(T) - \overline{J}^\epsilon_R(T)\|_{L^2} \le Ce^{-\frac{2-\delta}{2-\delta}(p+1)|\log h|}\|\psi(\overline{V}^\epsilon,\overline{V}^\epsilon)\|_{L^2(0,T;H^{\varrho/2}_{-2+\delta,2-\delta})}\\
&\le\ Ch^{p+1}\Big(\|\overline{V}^\epsilon,\overline{V}^{\epsilon_0},\overline{V}^\epsilon,\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(0,-1+\delta/2,-2+\delta)} + \|\overline{V}^\epsilon,\overline{V}^{\epsilon_0},\overline{V}^\epsilon,\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(0,1-\delta/2,2-\delta)}\Big)\\
&\le\ Ch^{p+1}v^\epsilon(\varrho/2)v^\epsilon(\varrho/2+1/2+\delta)\\
&\le\ Ch^{p+1}.
\end{aligned}$$

3. The bound for the error due to spatial semi-discretization follows with Theorem 5.3.10:
$$\begin{aligned}
&\|\overline{J}^\epsilon_R(T) - \tilde{\overline{J}}^\epsilon_{R,h}(T)\|_{L^2} \le Ch^{p+1-\delta}\|\psi(\overline{V}^\epsilon,\overline{V}^\epsilon)\|_{L^\infty(0,T;H^{p+1}\cap H^{\varrho}_{-2+\delta,2-\delta})}\\
&\le\ Ch^{p+1-\delta}\Big(\|\overline{V}^\epsilon,\overline{V}^{\epsilon_0},\overline{V}^\epsilon,\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(p+1,0,0)} + \|\overline{V}^\epsilon,\overline{V}^{\epsilon_0},\overline{V}^\epsilon,\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(\varrho,-1+\delta/2,-2+\delta)}\\
&\qquad +\|\overline{V}^\epsilon,\overline{V}^{\epsilon_0},\overline{V}^\epsilon,\overline{V}^{\epsilon_0}\|^{\Gamma_2}_{(\varrho,1-\delta/2,2-\delta)}\Big)\\
&\le\ Ch^{p+1-\delta}v^\epsilon(p+1+\varrho/2)\Big)\\
&\le\ C\epsilon^{-\varrho/2}\tilde{v}(0).
\end{aligned}$$

4. By Theorem 5.4.3 the error due to time discretization can be bounded as follows:
$$\begin{aligned}
\|\tilde{\overline{J}}^\epsilon_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG}_{R,h}(T)\|_{L^2} &\le Cdh^{p+1}\\
&\le Ch^{p+1}\epsilon^{-\varrho}.
\end{aligned}$$

5. The error due to sparse assembly can now be estimated using the stability estimate of the dG-scheme in Lemma 5.4.2. The estimate then follows with the previous Lemma 5.7.5:
$$\begin{aligned}
\|\tilde{\overline{J}}^{\epsilon,dG}_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h}(T)\|_{L^2} &\le \|\tilde{J}^{\epsilon,dG}_{R,h} - \tilde{J}^{\epsilon,dG,\Delta}_{R,h}\|_{dG}\\
&\le \|\psi(\overline{V}^\epsilon,\overline{V}^\epsilon) - P_I\psi^{0,0}_d(\tilde{\overline{V}}_{R/2},\overline{H}^{\epsilon_0},\tilde{\overline{V}}_{R/2},\overline{H}^{\epsilon_0})\|_{L^2(0,T;Y_h^*)}\\
&\le C\tilde{v}(1+2\nu+\delta).
\end{aligned}$$

6. The error due to the approximate solution via the GMRES scheme is estimated with Theorem 5.5.1 and the stability estimate of the previous Lemma 5.7.5:
$$\begin{aligned}
&\|\tilde{\overline{J}}^{\epsilon,dG,\Delta}_{R,h}(T) - \tilde{\overline{J}}^{\epsilon,dG,\Delta,\mathrm{GMRes}}_{R,h}(T)\|_{L^2}\\
&\le\ Ch^{p+1}\|P_I\psi^{0,0}_d(\tilde{\overline{V}}_{R/2},\overline{H}^{\epsilon_0},\tilde{\overline{V}}_{R/2},\overline{H}^{\epsilon_0})\|_{L^2(0,T;Y_h^*)}\\
&\le\ Ch^{p+1}h^{-1-2\nu-\delta}\\
&\le\ C\tilde{v}(1+2\nu+\delta).
\end{aligned}$$

These estimates all combined yield the claim:

$$\begin{aligned}\|\overline{J}(T) - \tilde{\overline{J}}(T)\|_{L^2} &\leq C(\epsilon^{3/2} + h^{p+1} + \epsilon^{-\varrho/2}\tilde{v}(0) + h^{p+1}\epsilon^{-\varrho} + 2\tilde{v}(1+2\nu+\delta)) \\ &\leq C(\epsilon^{3/2} + \tilde{v}(1+2\nu+\delta)).\end{aligned}$$

□

In order to have a scheme that depends upon only one parameter we still have to choose ϵ depending upon h. Here, we choose a dependence of the following form $\epsilon := h^s$. The exponent s will now be chosen such that the error estimate due to regularization coincides in order with the one by discretization. Furthermore, we undo all transformations that have been applied up to now.

Corollary 5.7.7 *The approximate solutions without transformations are given by*

$$\begin{aligned}\tilde{V}(T,x) &:= Ke^{qT}\tilde{\overline{V}}\left(T, x - (\sigma^2/2 + c_1)T + \log K\right), \\ \tilde{\vartheta}(t,x) &:= Ke^{qt}\tilde{\overline{\vartheta}}\left(t, x - (\sigma^2/2 + c_1)t + \log K\right), \\ \tilde{J}(T,x) &:= K^2 e^{qT}\tilde{\overline{J}}\left(T, x - (\sigma^2/2 + c_1)T + \log K\right).\end{aligned}$$

Let $\epsilon := Ch^s$ and assume $p \geq 1$. If the functions are computed separately, then we choose s for the corresponding function as follows:

$$\begin{aligned}s^V &:= 1, \\ s^\vartheta &:= \frac{p + 1/2 - (1 + 1/2\kappa)\nu - c_R/2 - 2\delta}{p+1}, \\ s^J &:= \frac{p - (2 + 1/2\kappa)\nu - \delta}{p+1}.\end{aligned}$$

If we denote by M the number of computation steps, this choice leads under the assumptions of the respective theorem to the following error bounds in terms of M:

$$\begin{aligned}\|V(T) - \tilde{V}(T)\|_{L^2} &\leq CM^{(-1+\delta)3/2s^V}, \\ \|\vartheta - \tilde{\vartheta}\|_{L^2(0,T;L^2)} &\leq CM^{(-1+\delta)3/2s^\vartheta}, \\ \|J(T) - \tilde{J}(T)\|_{L^2} &\leq CM^{-\frac{3/2s^J}{1+(6+\delta)\nu s+\delta}}.\end{aligned}$$

Proof. For the following choice of s the term $\epsilon^{3/2}$ dominates the error estimate of the respective theorem:

$$\begin{aligned}3/2s^V &\leq p + 1 + s^V(1/2 - p), \\ 3/2s^\vartheta &\leq p + 1/2 - \nu - c_R/2 - 2\delta + s^\vartheta(1/2 - p), \\ 3/2s^J &\leq p - 2\nu - \delta + s^J(1/2 - p).\end{aligned}$$

The minimal bound is achieved when choosing the maximal value for the respective exponent s. For the complexity we have $N(\log N)^8 \leq Ch^{-1-\delta}$ and therefore $\epsilon^{-(6+\delta)\varrho}N(\log N)^8 \leq Ch^{-(6+\delta)\varrho s-1-\delta}$. This yields the claim. □

In order to be able to get a feeling for these error estimates Table 5.1 shows the order of convergence in terms of number of computation steps. Here, we used a set of model parameters which will be used for the numerical experiments in Chapter 7. For convenience's sake we omitted the value of δ.

p	ν	η	s^V	s^ϑ	s^J	Order for $\tilde{\overline{V}}$	Order for $\tilde{\overline{\vartheta}}$	Order for $\tilde{\overline{J}}$
1	0,14	9,97	1	0,63	0,36	1,5	0,94	0,41
2	0,14	9,97	1	0,74	0,57	1,5	1,1	0,58
3	0,14	9,97	1	0,79	0,68	1,5	1,18	0,64
4	0,14	9,97	1	0,82	0,74	1,5	1,23	0,68
5	0,14	9,97	1	0,84	0,79	1,5	1,26	0,7
6	0,14	9,97	1	0,86	0,82	1,5	1,29	0,72

Table 5.1: Order of convergence with respect to M

If we compare these results with a finite difference scheme such as the one that was suggested in [CV05a] we have the following. Such a scheme needs $O(h^{-2})$ time steps in order to ensure convergence. In each time step $O(h^{-2})$ operations have to be performed. The error is then bounded by Ch. That means, the order of convergence in terms of computation steps is $1/4$. Therefore, for such small jump activities the results above still show better order of convergence.

Chapter 6

Implementation

Main thread. *For the actual implementation there are still three issues that have to be dealt with. These are the explicit computation of* $\Gamma(H^{\epsilon_0}, H^{\epsilon_0})$*,* $\Gamma(H^{\epsilon_0}, \exp)$ *and the assembly of* $\tilde{A}^{\omega}$*. This should be accomplished in a way that the implementation can be generalized to other kernels with a minimal additional effort. To this end the whole computation is based upon solely two functions which depend upon the kernel and therefore have to be implemented anew for a new kernel. These are* $k^{-1}_{\Delta\eta,\Delta\nu}$ *and* $\hat{k}^{-1}_{\Delta\eta,\Delta\nu}$ *which represent antiderivatives with respect to a mixture of polynomial and exponential moments of the kernel. The singularity involved is dealt with by using the Hadamard integral which allows to treat summands of the existing singular integral separately. Finally, the implementation of these functions is shown for the case of a* $CGMY$ *kernel.*

In this chapter we present methods for the assembly of the objects that have not been taken care of up to now. These are $\Gamma(H^{\epsilon_0}, H^{\epsilon_0})$, $\Gamma(H^{\epsilon_0}, \exp)$ and the computation of $\tilde{A}^{\omega}$. The latter has already been efficiently implemented by Matache, Schwab, Wihler and others of the research group of Prof. Schwab. The content of this section is therefore only a slight adaptation of these results. Nevertheless, their methods will be presented in a nutshell at this point in order to show the following. Firstly, it is easy to add the hedging error module to an existing implementation for the option price following this method. It essentially only uses already existing functions. Secondly, the necessary implementation effort in order to transfer the program onto a further kernel is reduced to implementing basic finite-part integrals of the kernel with respect to a monomial or an exponential. Thus, an implementation using this method can be generalized to further kernels with a minimal additional effort.

In the sequel we will again only deal with the pure jump part, i.e. we assume $\sigma^2 = 0$. The terms that correspond to this Brownian motion can be dealt with with standard methods. For the remaining parts we have to apply methods that allow for a treatment of hypersingular integrals. To this end we introduce the notion of finite-part integrals in the sense of a Hadamard integral. They allow to compute the miscellaneous summands of the singular integral separately. An overview of their usage for the computation of such hypersingular integrals can be found in [ML98].

A finite-part integral in this sense is defined as follows.

Definition 6.0.1 [Finite-part integral] Let $b > 0$. For $\alpha \in \mathbb{R}$ the *finite-part integral* shall for monomials be defined as follows:

$$⨎_0^b x^\alpha dx := \begin{cases} \log b & \text{, if } \alpha = -1, \\ \frac{b^{\alpha+1}}{\alpha+1} & \text{otherwise.} \end{cases}$$

For $\alpha \leq -1, m > -\alpha - 2$ and $g \in C^{(m+1)}[0, b)$ the corresponding finite-part integral shall be defined as follows:

$$⨎_0^b g(x)x^\alpha dx \quad := \quad \int_0^b x^\alpha \left(g(x) - \sum_{k=0}^m \frac{g^{(k)}(0)}{k!} x^k \right) dx + \sum_{k=0}^m \frac{g^{(k)}(0)}{k!} ⨎_0^b x^{\alpha+k} dx.$$

For functions $g \in C^{(m+1)}[0, \infty)$, such that for all $k = 0, 1, \ldots, m+1$ and $l \geq 1$ we have

$$\left| \int_0^\infty g^{(k)}(x) x^{l-1} dx \right| < \infty \tag{6.1}$$

the finite-part integral shall finally be defined as follows:

$$⨎_0^\infty g(x)x^\alpha dx := ⨎_0^b g(x)x^\alpha dx + \int_b^\infty g(x)x^\alpha dx.$$

Remarks. This definition clearly states that this notion of an integral is equivalent to the usual Riemann integral if the latter is finite. That means

$$\int_0^\infty f(x)dx < \infty \quad \Rightarrow \quad ⨎_0^\infty f(x)dx = \int_0^\infty f(x)dx.$$

For notation's sake we will therefore set $⨎_a^b f(x)dx := \int_a^b f(x)dx$ if $0 \notin [a, b]$. Furthermore, we can generalize onto $\mathbb{R}$ by setting

$$⨎_{\mathbb{R}} g(x)|x|^\alpha dx := ⨎_0^\infty g(-x)x^\alpha dx + ⨎_0^\infty g(x)x^\alpha dx$$

for functions $g \in C^{m+1}(\mathbb{R})$ satisfying (6.1) on $\mathbb{R}$. This integral is again a linear functional, cf. [KU98, Section 1.4.2], and the usual integration-by-parts rule still remains valid if the order of the singularity is not an integer, cf. [ML98, Theorem 2.7] or [KU98, Theorem 1.4.2].

The two functions, k^{-1} and $\hat{k}^{-1}$, on which we will base the ensuing method can now be introduced. They are the only ones that have to be implemented anew if the program should be transfered to a new kernel.

$$\begin{aligned} k_{\Delta\eta,\Delta\nu}(y) &:= e^{y\Delta\eta} y^{\Delta\nu} k(y), \\ k^{-1}_{\Delta\eta,\Delta\nu}(x) &:= \begin{cases} -\int_x^\infty k_{\Delta\eta,\Delta\nu}(y)dy & \text{, if } x > 0, \\ \int_{-\infty}^x k_{\Delta\eta,\Delta\nu}(y)dy & \text{, if } x < 0, \end{cases} \\ \hat{k}^{-1}_{\Delta\eta,\Delta\nu}(x) &:= \begin{cases} ⨎_0^x k_{\Delta\eta,\Delta\nu}(y)dy & \text{, if } x > 0, \\ -⨎_x^0 k_{\Delta\eta,\Delta\nu}(y)dy & \text{, if } x < 0. \end{cases} \end{aligned}$$

For shorter notation we furthermore introduce the following:

$$c_{\Delta\eta,\Delta\nu}(x) \quad := \quad \fint_{-\infty}^{x} k_{\Delta\eta,\Delta\nu}(y)dy.$$

This function can be computed with $k^{-1}_{\Delta\eta,\Delta\nu}$ and $\hat{k}^{-1}_{\Delta\eta,\Delta\nu}$ as follows:

$$c_{\Delta\eta,\Delta\nu}(x) \quad = \quad \begin{cases} k^{-1}_{\Delta\eta,\Delta\nu}(x) & \text{, if } x \leq -1, \\ c_{\Delta\eta,\Delta\nu}(-1) + (\hat{k}^{-1}_{\Delta\eta,\Delta\nu}(x) - \hat{k}^{-1}_{\Delta\eta,\Delta\nu}(-1)) & \text{, if } -1 < x \leq 1, \\ c_{\Delta\eta,\Delta\nu}(1) - (k^{-1}_{\Delta\eta,\Delta\nu}(1) - k^{-1}_{\Delta\eta,\Delta\nu}(x)) & \text{, if } 1 < x < \infty, \\ c_{\Delta\eta,\Delta\nu}(1) - k^{-1}_{\Delta\eta,\Delta\nu}(1) & \text{, if } x = \infty. \end{cases}$$

6.1 Computation of $\Gamma(H^{\epsilon_0}, H^{\epsilon_0}), \Gamma(H^{\epsilon_0}, \exp)$ and AH^{ϵ_0}

In case of these smooth functions the integrals which we are about to consider exist in the usual sense. Therefore, we can split up the singular integrals into sums of finite-part integrals and compute those separately. More specifically, we now consider payoff functions of the following form:

$$H^{\epsilon_0}(x) = (1 - e^x)1_{(-\infty,-\epsilon_0)}(x) + \tilde{q}(x)1_{[-\epsilon_0,\epsilon_0]}(x),$$

where we have $\epsilon_0 > 0$, and $\tilde{q}$ shall denote a polynomial such that $H^{\epsilon_0} \in C^{p+1}(\mathbb{R})$. The restriction to a strike $K = 1$ can be done without loss of generality as was shown right in the beginning. Therefore, this model payoff function comprises the approximate put function that has been introduced in the previous chapters. By definition of Γ we have to compute the following:

$$\Gamma(f, g) = A(fg) - fAg - gAf.$$

Here, we show the computation for $\varrho \geq 1$, the other case follows along the very same lines. That means, we consider the following operators:

$$\begin{aligned} Af(x) &= \int_{\mathbb{R}} (f(x+y) - f(x) - (e^y - 1)f'(x))k(y)dy, \\ \overline{A}f(x) &= \fint_{\mathbb{R}} (f(x+y) - f(x) - (e^y - 1)f'(x))k(y)dy. \end{aligned}$$

If $f \in C^2(\mathbb{R})$ the integrals exist in the usual sense, and we have $Af = \overline{A}f$.

This is the case for $\Gamma(H^{\epsilon_0}, H^{\epsilon_0})$ and $\Gamma(H^{\epsilon_0}, \exp)$. It is therefore now sufficient to compute terms of the form $\overline{A}g$, where

$$g(x) = e^{mx}1_{(-\infty,-\epsilon_0)} \text{ or } g(x) = e^{mx}p(x)1_{[-\epsilon_0,\epsilon_0]}$$

for $m \in \mathbb{N}, m \leq \eta$ and p denotes some polynomial. Considering those functions separately formally leads to Dirac distributions $\delta_{-\epsilon_0}(x)$ due to the jump at $-\epsilon_0$. However, due to the regularity of the overall payoff function H^{ϵ_0} these terms will cancel out when recombining the results. Thus, in the actual computation they can be neglected.

Formally, for the first term we have the following:

$$
\begin{aligned}
&\overline{A}(e^{mx}1_{(-\infty,-\epsilon_0)}) + (c_{1,0}(\infty) - c_{0,0}(\infty))\delta_{-\epsilon_0}(x)e^{mx} \\
&= \fint_{\mathbb{R}} \Big(e^{m(x+y)}1_{(-\infty,-\epsilon_0)}(x+y) - e^{mx}1_{(-\infty,-\epsilon_0)}(x)(1+m(e^y-1))k(y)\Big)dy \\
&= e^{mx}\left(\fint_{-\infty}^{-\epsilon_0-x} e^{my}k(y)dy - 1_{(-\infty,-\epsilon_0)}(x)\fint_{\mathbb{R}}(1+m(e^y-1))k(y)dy\right) \\
&= e^{mx}\left(c_{m,0}(-\epsilon_0-x) - 1_{(-\infty,-\epsilon_0)}(x)\Big((1-m)c_{0,0}(\infty)+mc_{1,0}(\infty)\Big)\right).
\end{aligned}
$$

The second term can be computed as follows:

$$
\begin{aligned}
&\overline{A}(e^{mx}p(x)1_{[-\epsilon_0,\epsilon_0]}(x)) - (c_{1,0}(\infty) - c_{0,0}(\infty))\delta_{-\epsilon_0}(x)e^{mx}p(x) \\
&= e^{mx}\fint_{\mathbb{R}}\Bigg(e^{my}p(x+y)1_{[-\epsilon_0,\epsilon_0]}(x+y) \\
&\qquad -1_{[-\epsilon_0,\epsilon_0]}(x)\Big(p(x)+(e^y-1)(mp(x)+p'(x))\Big)\Bigg)k(y)dy.
\end{aligned}
$$

The last terms evaluate to

$$
\begin{aligned}
&e^{mx}1_{[-\epsilon_0,\epsilon_0]}(x)\fint_{\mathbb{R}}\Big(p(x)+(e^y-1)(mp(x)+p'(x))\Big)k(y)dy \\
&= e^{mx}1_{[-\epsilon_0,\epsilon_0]}(x)\Big(c_{0,0}(\infty)((1-m)p(x)-p'(x)) + c_{1,0}(\infty)(mp(x)+p'(x))\Big).
\end{aligned}
$$

For the first term we use the Taylor expansion of p:

$$
\begin{aligned}
&e^{mx}\fint_{\mathbb{R}} e^{my}p(x+y)1_{[-\epsilon_0,\epsilon_0]}(x+y)k(y)dy = e^{mx}\fint_{-\epsilon_0-x}^{\epsilon_0-x} e^{my}p(x+y)k(y)dy \\
&= e^{mx}\left(\fint_{-\epsilon_0-x}^{\epsilon_0-x}\sum_{k=0}^{\deg p}\frac{p^{(k)}(x)}{k!}y^k e^{my}k(y)dy\right) \\
&= e^{mx}\left(\sum_{k=0}^{\deg p}\frac{p^{(k)}(x)}{k!}\Big(c_{m,k}(\epsilon_0-x) - c_{m,k}(-\epsilon_0-x)\Big)\right).
\end{aligned}
$$

Thus, the computation of $\Gamma(H^{\epsilon_0}, H^{\epsilon_0})$ and $\Gamma(H^{\epsilon_0}, \exp)$ can be reduced to a sum of evaluations of the function $c_{\Delta\eta,\Delta\nu}$.

6.2 Computation of (AH, φ_i^l) and assembly of $\tilde{A}^\omega$

In this section we have to consider non-smooth functions H and ψ_i^l, where H is the original put function and ψ_i^l a wavelet basis function. That means that the singular integrals are distributions and therefore do not necessarily exist at all evaluation points. Because of that we can only compute the projection via P_L which suffices in our application.

The term (AH, φ_i^l) is actually not used in our method. However, we already know by [MSW06] that the option price can be computed directly with this method without previous regularization. Furthermore, even though a PIDE representation for J could not be proved we can implement and compute the solution of the resulting system of equations. Then, this solution can be compared to the real hedging error in this case in order to study the obstacles that impede the theoretical treatment.

As already mentioned we cannot apply the method introduced in the previous section, because $AH \notin C(\mathbb{R})$. Thus, we will apply partial integration several times to come up with a representation which only uses function evaluations of $k^{-1}_{\Delta\eta,\Delta\nu}$. To this end we introduce the following extension to antiderivatives of higher order for $n \in \mathbb{N}$:

$$k^{-(n+1)}_{\Delta\eta,\Delta\nu}(x) \quad := \quad \begin{cases} -⨍_x^\infty e^{y\Delta\eta} y^{\Delta\nu} k^{-n}_{0,0}(y)dy & \text{, if } x \geq 0, \\ ⨍_{-\infty}^x e^{y\Delta\eta} y^{\Delta\nu} k^{-n}_{0,0}(y)dy & \text{, if } x < 0. \end{cases}$$

These functions have the following properties, which allow to apply partial integration in the ensuing analysis. Particularly, we derive a representation of those antiderivatives based upon $k^{-1}_{\Delta\eta,\Delta\nu}$.

Proposition 6.2.1 *The functions $k^{-(n+1)}_{\Delta\eta,0}$ are antiderivatives of $k^{-1}_{\Delta\eta,0}$ in the following sense. For $n \in \mathbb{N}, m \in \mathbb{Z}$ with $-\eta \leq m \leq \eta$ and $x \neq 0$ we have*

$$D_x(e^{mx} k^{-(n+1)}_{m,0}(-x)) = -e^{mx} k^{-n}_{m,0}(-x).$$

Additionally, we have the following growth estimates:

$$\begin{aligned} k^{-n}_{0,0}(x) &\leq C|x|^{n-1-\nu} \quad \forall x \in [-1,1] \setminus \{0\}, \\ k^{-n}_{0,0}(x) &\leq Ce^{-\eta|x|} \quad \forall x \in \mathbb{R} \setminus [-1,1]. \end{aligned}$$

Finally, we have the following equivalences for $m \neq 0$:

$$\begin{aligned} e^{mx} k^{-(n+1)}_{m,0}(-x) &= -\sum_{l=1}^{n} k^{-(n+1-l)}_{0,0}(-x) \left(-\frac{1}{m}\right)^l + e^{mx} \left(-\frac{1}{m}\right)^n k^{-1}_{m,0}(-x), \\ k^{-(n+1)}_{0,0}(x) &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k k^{-1}_{0,n-k}(x). \end{aligned}$$

Proof. The first statement is clear by definition for $m = 0$, so are the growth estimates due to the growth assumptions for k. For $m \neq 0$ we have the following for $x > 0$:

$$
\begin{aligned}
D_x(e^{mx}k_{m,0}^{-(n+1)}(-x)) &= D_x\left(e^{mx}\int_{-\infty}^{-x} e^{my}k_{0,0}^{-n}(y)dy\right)\\
&= me^{mx}\int_{-\infty}^{-x} e^{my}k_{0,0}^{-n}(y)dy - k_{0,0}^{-n}(-x)\\
&= me^{mx}\frac{1}{m}\left([e^{my}k_{0,0}^{-n}(y)]_{-\infty}^{-x} - \int_{-\infty}^{-x} e^{my}k_{0,0}^{-(n-1)}(y)dy\right) - k_{0,0}^{-n}(-x)\\
&= -e^{mx}\int_{-\infty}^{-x} e^{my}k_{0,0}^{-(n-1)}(y)dy\\
&= -e^{mx}k_{m,0}^{-n}(-x).
\end{aligned}
$$

Along the very same lines the claim follows for $x < 0$. The first equivalence now follows similarly with partial integration. As just seen, we have

$$
\begin{aligned}
e^{mx}k_{m,0}^{-(n+1)}(-x) &= e^{mx}\frac{1}{m}\left([e^{my}k_{0,0}^{-n}(y)]_{-\infty}^{-x} - \int_{-\infty}^{-x} e^{my}k_{0,0}^{-(n-1)}(y)dy\right)\\
&= \frac{1}{m}\left(k_{0,0}^{-n}(-x) - e^{mx}k_{m,0}^{-n}(-x)\right).
\end{aligned}
$$

Iterating this formula yields the claim. Finally, the representation of $k_{0,0}^{-(n+1)}$ can be derived via the antiderivative property. We have

$$
\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^k k_{0,n-k}^{-1}(x) = \begin{cases} -⨏_x^{\infty}\frac{1}{n!}(x-y)^n k(y)dy & \text{, if } x \geq 0,\\ ⨏_{-\infty}^{x}\frac{1}{n!}(x-y)^n k(y)dy & \text{, if } x < 0.\end{cases}
$$

For $n = 1$ this is clearly an antiderivative of $k_{0,0}^{-1}$ which vanishes at infinity. Since antiderivatives are unique up to a constant and the limits at infinity coincide, we have the desired equivalence. The claim now follows by induction. □

With these functions we can now start to compute (AH, φ_i^l). Similar to the computation in Lemma 5.7.1 we get the following representation of AH for $x \neq 0$:

$$
AH(x) = 1_{(-\infty,0)}(x)\int_{-x}^{\infty}(e^{x+y}-1)k(y)dy - 1_{(0,\infty)}(x)\int_{-\infty}^{-x}(e^{x+y}-1)k(y)dy.
$$

With this we get the following with the properties of $e^{mx}k_{m,0}(-x)$ in Proposition 6.2.1 by applying partial integration:

$$
\begin{aligned}
&\int_{-R}^{0} AH(x)\varphi_i^l(x)dx = \int_{-R}^{0}\int_{-x}^{\infty}(e^{x+y}-1)k(y)dy\varphi_i^l(x)dx\\
&= -\varphi_i^l(0-)\int_0^{\infty}(e^y-1)k_{0,0}^{-1}(y)dy + \int_{-R}^{0}\int_{-x}^{\infty}(e^{x+y}-1)k_{0,0}^{-1}(y)dyD\varphi_i^l(x)dx\\
&= D^{n-1}\varphi_i^l(0-)\sum_{n=1}^{p+1}\left((-1)^n\int_0^{\infty}(e^y-1)k_{0,0}^{-n}(y)dy\right)\\
&\quad +(-1)^{p+2}\int_{-R}^{0}\int_{-x}^{\infty}(e^{x+y}-1)k_{0,0}^{-(p+1)}(y)dyD^{p+1}\varphi_i^l(x)dx.
\end{aligned}
$$

Here, $D^{n-1}\varphi_i^l(0-) := \lim_{x\nearrow 0} D^{n-1}\varphi_i^l(x)$. Along the same lines a similar result can be derived for $x > 0$ with $D^{n-1}\varphi_i^l(0+) := \lim_{x\searrow 0} D^{n-1}\varphi_i^l(x)$. Let now

$$\begin{aligned} c_n^+ &:= \int_0^\infty (e^y - 1)k_{0,0}^{-n}(y)dy, \\ c_n^- &:= \int_{-\infty}^0 (e^y - 1)k_{0,0}^{-n}(y)dy. \end{aligned}$$

These terms can be computed using the formulas in Proposition 6.2.1. Due to the existence of the integrals the limit at $x = 0$ is given by the corresponding sum of finite-part integrals. Combining these results, we get the following:

$$\begin{aligned} (AH, \varphi_i^l) &= \sum_{n=1}^{p+1}(-1)^n c_n^+ D^{n-1}\varphi_i^l(0-) \\ &\quad +(-1)^{p+2}\int_{-R}^0\int_{-x}^\infty (e^{x+y}-1)k_{0,0}^{-(p+1)}(y)dyD^{p+1}\varphi_i^l(x)dx \\ &\quad -\sum_{n=1}^{p+1} c_n^- D^{n-1}\varphi_i^l(0+) \\ &\quad -\int_0^R\int_{-\infty}^{-x}(e^{x+y}-1)k_{0,0}^{-(p+1)}(y)dyD^{p+1}\varphi_i^l(x)dx. \end{aligned}$$

Since $D^{p+1}\varphi_i^l(x)$ is just a sum of Dirac distributions and their derivatives, this integral can be computed via a sum of evaluations of $k_{1,0}^{-n} - k_{0,0}^{-n}$.

The assembly of $\tilde{A}$ now follows along similar lines. Here, we have the advantage that all functions have bounded support and vanish at the boundaries. Therefore, applying partial integration does not produce boundary terms. Formally, the integral does not necessarily exist in the usual sense at $x = 0$. Therefore, we first apply Fubini and then partial integration with respect to x. Hereby we denote the antiderivative by D^{-1}. The resulting integral exists in the usual sense and therefore coincides with the finite-part integral. Consequently, we can split the finite-part integral into separate terms. Finally, we use the fact that Fubini and partial integration are applicable for this kind of integral:

$$\begin{aligned} a(\psi_i^l, \psi_{i'}^{l'}) &= \int_\Omega\int_\mathbb{R}(\psi_i^l(x+y) - \psi_i^l(x) - (e^y-1)D\psi_i^l(x))k(y)dy\psi_{i'}^{l'}(x)dx \\ &= \fint_\mathbb{R}\int_\Omega(D^{-1}\psi_i^l(x+y) - D^{-1}\psi_i^l(x) - (e^y-1)\psi_i^l(x))k(y)D\psi_{i'}^{l'}(x)dxdy \\ &= \int_\Omega\fint_\mathbb{R}\psi_i^l(x+y)k(y)dy\psi_{i'}^{l'}(x)dx - c_{0,0}(\infty)(\psi_i^l, \psi_{i'}^{l'}) \\ &\quad -(c_{1,0}(\infty) - c_{0,0}(\infty))(D\psi_i^l, \psi_{i'}^{l'}) \\ &= \int_\Omega\fint_\mathbb{R} D^{p+1}\psi_i^l(x+y)k_{0,0}^{-2(p+1)}(y)dyD^{p+1}\psi_{i'}^{l'}(x)dx - c_{0,0}(\infty)\mathbf{M}_{(l,i),(l',i')} \\ &\quad -(c_{1,0}(\infty) - c_{0,0}(\infty))\mathbf{C}_{(l,i),(l',i')}, \end{aligned}$$

where $\mathbf{M}$ denotes the mass matrix and $\mathbf{C}$ the so-called cross matrix with respect to the wavelet basis. This enables to finally assemble the corresponding matrix for the generalized version A^ω. To this end we can do the following decomposition for $f \in Y$:

$$\begin{aligned}
(E^\omega A(E^{-\omega} f))(x) &= \int_{\mathbb{R}} (f(x+y)e^{-\omega y} - f(x) - (e^y - 1)(-\omega f(x) + f'(x)))k(y)dy \\
&= Af(x) + \int_{\mathbb{R}} (f(x+y)(e^{-\omega y} - 1) + \omega(e^y - 1)f(x))k(y)dy \\
&= Af(x) + \int_{\mathbb{R}} (f(x+y) - f(x))(e^{-\omega y} - 1)k(y)dy \\
&\quad + f(x) \int_{\mathbb{R}} (e^{-\omega y} - 1 + \omega(e^y - 1))k(y)dy \\
&= Af(x) + (c_{0,0} - c_{-\omega,0} + c')f(x) + \fint_{\mathbb{R}} f(x+y)(k_{-\omega,0}(y) - k(y))dy,
\end{aligned}$$

where

$$c' := \int_{\mathbb{R}} (e^{-\omega y} - 1 + \omega(e^y - 1))k(y)dy.$$

This term is finite, because

$$\begin{aligned}
\int_{\mathbb{R}} (e^{-\omega y} - 1 + \omega(e^y - 1))k(y)dy &= \int_{\mathbb{R}} \omega y \left(\int_0^1 e^{\theta y} - e^{-\omega \theta y} d\theta \right) k(y)dy \\
&= \int_{\mathbb{R}} \omega y^2 \left(\int_0^1 \int_{-\omega}^1 \theta_1 e^{\theta_2 \theta_1 y} d\theta_2 d\theta_1 \right) k(y)dy \\
&< \infty.
\end{aligned}$$

Therefore, we can compute c' as follows:

$$c' = c_{-\omega,0} - (1+\omega)c_{0,0} + \omega c_{1,0}.$$

Altogether, this yields

$$\begin{aligned}
a^\omega(\psi_i^l, \psi_{i'}^{l'}) &= a(\psi_i^l, \psi_{i'}^{l'}) + \omega(c_{1,0} - c_{0,0})\mathbf{M}_{(l,i),(l',i')} \\
&\quad + \int_\Omega \fint_{\mathbb{R}} D^{p+1}\psi_i^l(x+y)(k_{-\omega,0}^{-2(p+1)}(y) - k_{0,0}^{-2(p+1)}(y))dy D^{p+1}\psi_{i'}^{l'}(x)dx.
\end{aligned}$$

6.3 Implementation of CGMY kernel

Finally, we show the implementation of the functions $k^{-1}_{\Delta\eta,\Delta\nu}$,$\hat{k}^{-1}_{\Delta\eta,\Delta\nu}$ for the exemplary case of a CGMY process. Here, we can use the fact that the modified kernel $k_{\Delta\eta,\Delta\nu}$ is again a CGMY kernel, however with different parameters.

More precisely, a CGMY process is a pure jump Lévy process, where the kernel of the jump measure is given by

$$k(x) := C \begin{cases} \frac{e^{-Mx}}{x^{1+Y}} & \text{, if } x > 0, \\ \frac{e^{Gx}}{(-x)^{1+Y}} & \text{, if } x < 0 \end{cases}$$

for some $C, G > 0, M > 1$ and $Y < 2$. If $G > 2$, $M > 2$ and $Y > 0$, then the corresponding process meets the assumptions (A1)-(A4). As mentioned before, the corresponding functions can be computed via the antiderivative of this kernel function. Define to this end the following auxiliary functions for $x > 0$:

$$\begin{aligned}
\tilde{k}_{\eta,\nu}(x) &:= x^{-1-\nu}e^{-\eta x}, \\
\tilde{k}^{-1}_{\eta,\nu}(x) &:= -\int_x^\infty y^{-1-\nu}e^{-\eta y}dy, \\
\hat{\tilde{k}}^{-1}_{\eta,\nu}(x) &:= \fint_0^x y^{-1-\nu}e^{-\eta y}dy.
\end{aligned}$$

With this the functions $k^{-1}_{\Delta\eta,\Delta\nu}(x)$ are given by

$$\begin{aligned}
k_{\Delta\eta,\Delta\nu}(y) &= C(1_{(-\infty,0)}(y)\tilde{k}_{G+\Delta\eta,Y-\Delta\nu}(-y) + 1_{(0,\infty)}(y)\tilde{k}_{M-\Delta\eta,Y-\Delta\nu}(y)), \\
k^{-1}_{\Delta\eta,\Delta\nu}(x) &= C(1_{(-\infty,0)}(x)\tilde{k}^{-1}_{G+\Delta\eta,Y-\Delta\nu}(-x) + 1_{(0,\infty)}(x)\tilde{k}^{-1}_{M-\Delta\eta,Y-\Delta\nu}(x)), \\
\hat{k}^{-1}_{\Delta\eta,\Delta\nu}(x) &= C(1_{(-\infty,0)}(x)\hat{\tilde{k}}^{-1}_{G+\Delta\eta,Y-\Delta\nu}(-x) + 1_{(0,\infty)}(x)\hat{\tilde{k}}^{-1}_{M-\Delta\eta,Y-\Delta\nu}(x)).
\end{aligned}$$

Therefore, it is sufficient to implement $\tilde{k}^{-1}_{\eta,\nu}$ and $\hat{\tilde{k}}^{-1}_{\eta,\nu}$. The first can be expressed via the already given functions

$$\begin{aligned}
\Gamma_{inc}(a,x) &= \int_x^\infty t^{a-1}e^{-t}dt, \\
E_i(x) &= \int_x^\infty t^{-1}e^{-t}dt.
\end{aligned}$$

Here, Γ_{inc} is defined for $x \in [0,\infty)$, if $a > 0$. Using change of variables and partial integration, we get for $0 \neq \nu \neq 1$

$$\begin{aligned}
\tilde{k}^{-1}_{\eta,\nu}(x) &= \int_x^\infty y^{-\nu-1}e^{-\eta y}dy \\
&= \eta^{\nu+1}\eta^{-1}\int_{\eta x}^\infty y^{-\nu-1}e^{-y}dy \\
&= \eta^\nu \frac{1}{-\nu}\left([y^{-\nu}e^{-y}]_{\eta x}^\infty - \int_{\eta x}^\infty y^{-\nu}e^{-y}dy\right) \\
&= \frac{1}{\nu}x^{-\nu}e^{-\eta x} + \eta^\nu\frac{1}{\nu(1-\nu)}\left([y^{1-\nu}e^{-y}]_{\eta x}^\infty - \int_{\eta x}^\infty y^{1-\nu}e^{-y}dy\right) \\
&= e^{-\eta x}\left(\frac{1}{\nu}x^{-\nu} - \frac{\eta}{\nu(1-\nu)}x^{1-\nu}\right) - \frac{\eta^\nu}{\nu(1-\nu)}\Gamma_{inc}(2-\nu,\eta x).
\end{aligned}$$

Analogously, we get for $\nu = 1$

$$\tilde{k}^{-1}_{\eta,\nu}(x) = \frac{e^{-\eta x}}{x} + \eta E_i(\eta x).$$

And for $\nu = 0$ we directly have

$$\tilde{k}^{-1}_{\eta,\nu}(x) = E_i(\eta x).$$

The computation of $\hat{\tilde{k}}_{\eta,\nu}^{-1}(x)$ is now done via numerical integration. By definition of the finite-part integral we have with the transformation $v = \frac{2}{x}y - 1$ and the notation $t(v) := \frac{x}{2}(v+1)$ the following:

$$\begin{aligned}\hat{\tilde{k}}_{\eta,\nu}^{-1}(x) &= \int_0^x (e^{-\eta y} - 1 + \eta y)|y|^{-1-\nu} dy + \frac{1}{-\nu} x^{-\nu} - \eta \frac{1}{1-\nu} x^{1-\nu} \\ &= \int_{-1}^1 \left(e^{-\eta t(v)} - 1 + \eta t(v)\right) |t(v)|^{-2} |t(v)|^{1-\nu} \frac{x}{2} dv - \left(\frac{1}{\nu} x^{-\nu} + \frac{\eta}{1-\nu} x^{1-\nu}\right).\end{aligned}$$

The remaining singular integral exists in the usual sense. Therefore, we can apply the Gauß-Jacobi quadrature formula with respect to the weight function

$$w(x) = (1-x)^\alpha (1+x)^\beta \text{ with } \alpha = 0, \beta = 1-\nu.$$

Chapter 7

Numerical experiments

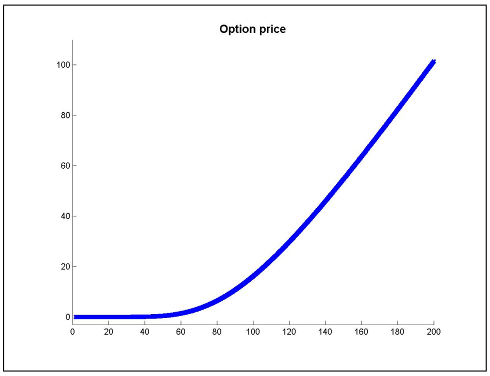
In this chapter we will show the results of the implementation and compare the solutions with those derived by the integral transformation method of [HKK06]. To this end we use a CGMY process with the following exemplary parameters:

$$C = 9.61, G = 9.97, M = 16.51, Y = 0.143.$$

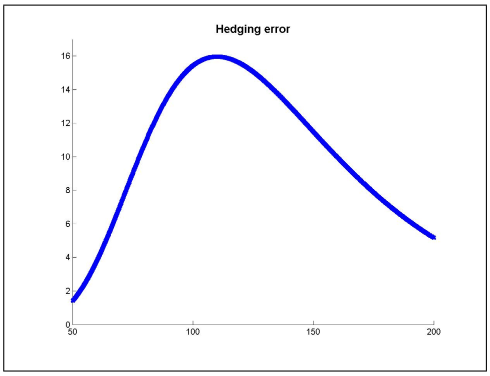
These have been suggested in [CGMY02, Table 3]. Furthermore, we consider a European call function with strike $K = 100$. The compression factors are chosen $\hat{\alpha} = 0.8$ and $c_0 = 1$. The slope for the dG scheme at level $L = 9$ is set to $\mu = 1$ and the grading factor to $\sigma = 0.5$. The computations are done with $\epsilon = 0.5h^{0.7}$. That means we used the average of the exponents s^V and s^J.

Then a computation at approximation level $L = 9$ yields the functions that are displayed in the Figures 7.1(a),7.1(b) and 7.1(c) and together in Figure 7.1(d). At that level the computation takes only a few minutes - about 8 minutes on a Laptop with a processor at 2.0 GHz. For higher levels we encountered problems with the implementation. More specifically, the solutions began to oscillate for large initial values x. But, if these oscillations are ignored the resulting functions show the same order of convergence that is shown in Figure 7.2(d). This suggests that these oscillations are caused by cancellation effects due to rounding errors. Therefore, they should not represent a general drawback connected with the method. However, in the sequel we will only present results that have been obtained directly without further corrections.

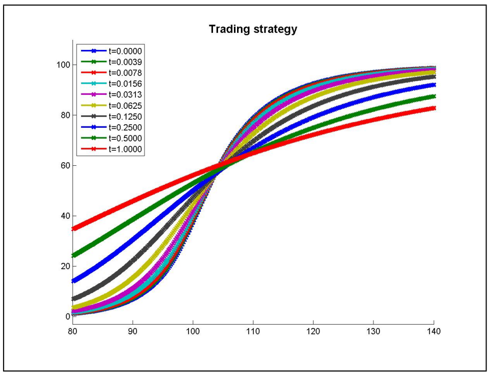
If we compare the results with the reference functions computed via the integral transformation method up to that level $L = 9$ we see that we already obtain reliable approximations. The respective procedure of approximation can be seen in Figure 7.2(a) for V, in Figure 7.2(b) for ϑ and finally in Figure 7.2(c) for J. Here, only a clipping has been shown in order to demonstrate the approximation. However, if we compute the L^2 errors of those functions with respect to the reference we see that there is convergence on the whole real line. Indeed, the order of this convergence is given in Figure 7.2(d). It represents the development of the L^2 difference at maturity for V and J and the difference with respect to the norm $\|\cdot\|_{L^2(0,T;L^2)}$ in case of ϑ. The error is given in log-scale against the mesh width which is given in log-scale as well. The



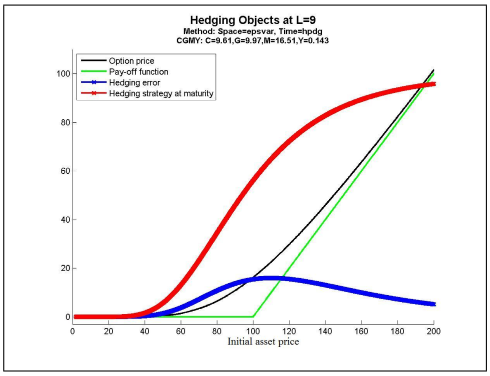
(a) Option price $\tilde{V}(T)$ at level $L = 9$



(b) Hedging error $\tilde{J}(T)$ at level $L = 9$



(c) Trading strategy $\tilde{\vartheta}$ at level $L = 9$



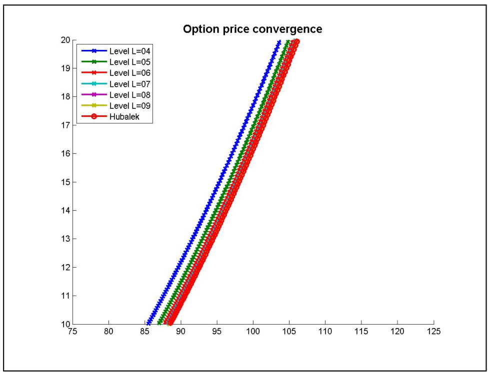
(d) Hedging objects $\tilde{V}, \tilde{\vartheta}, \tilde{J}$ at $L = 9$

Figure 7.1: Solutions of the hedging problem for a CGMY process with model parameters $C = 9.61, G = 9.97, M = 16.51, Y = 0.143$

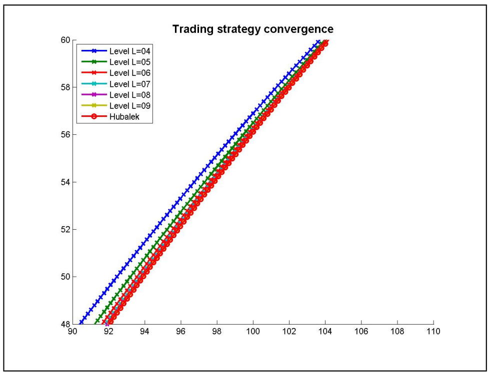
average slope that is given in the figure therefore represents the exponent of the mesh width h in the error approximation. That means for instance that

$$\|\tilde{J}(T) - J(T)\|_{L^2} \sim Ch^{1.18}$$

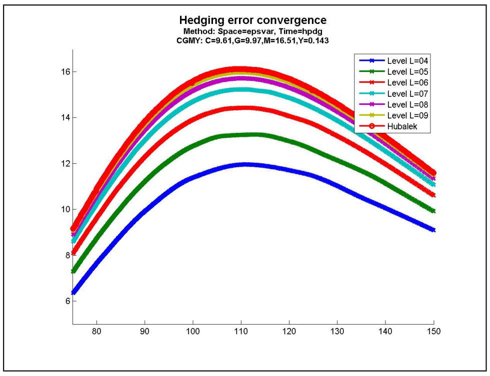
is suggested. These results meet the error bounds that are induced by Table 5.1 for the choice of the exponent s. For the hedging error J it is even far better than suggested by the error bound.



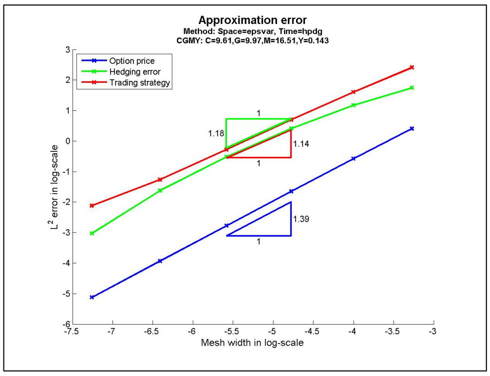
(a) Convergence of $\tilde{V}(T)$



(b) Convergence of $\tilde{J}(T)$



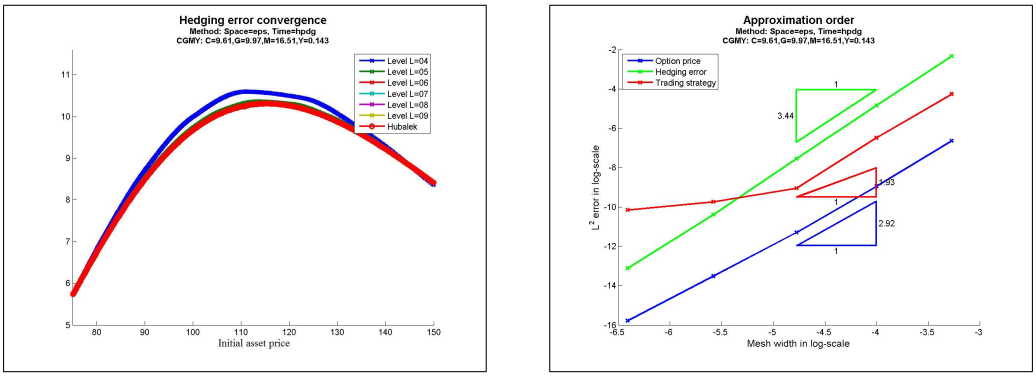
(c) Convergence of $\tilde{\vartheta}(T)$



(d) Order of convergence

Figure 7.2: Convergence of option price, hedging error and trading strategy for CGMY with $C = 9.61, G = 9.97, M = 16.51, Y = 0.143$

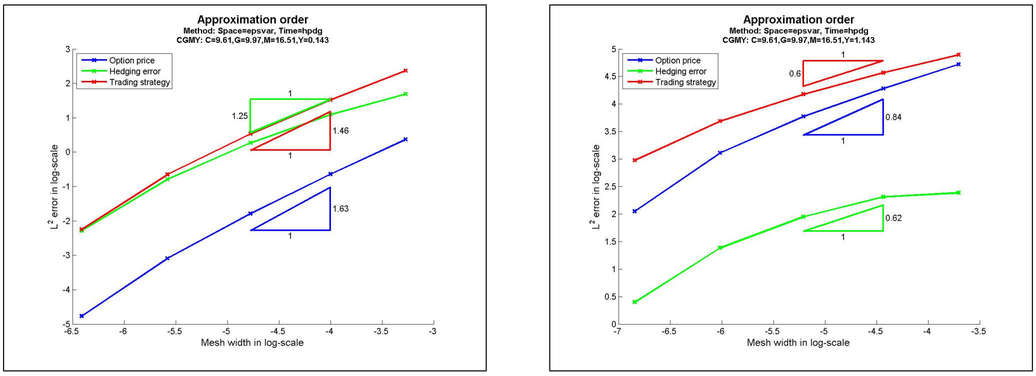
The exponent s in the definition of ϵ in Corollary 5.7.7 was chosen such that the approximation error due to regularization dominates the error estimate. If we choose a smooth payoff function right from the beginning we do not have to apply the regularized approximation. This is done in Figure 7.3(a) and Figure 7.3(b) for the approximate put function H^{ϵ_0} with $\epsilon_0 = 0.5$. In order to determine the order of convergence we use the results at $L = 9$ as reference function. This is necessary, because the result via the integral transformation method is not reliable at that small scale. That means, the implementation that was used for the computation via that method yields oscillations in the order of magnitude of the distance we are about to consider. As expected, we can see that the resulting errors are far smaller and the order of convergence is far higher than with additional regularization.



(a) Convergence of $\tilde{J}$ for $H^{0.5}$ (b) Order of convergence for $H^{0.5}$

Figure 7.3: Convergence for CGMY with $C = 9.61, G = 9.97, M = 16.51, Y = 0.143$ and the approximate put function $H^{0.5}$

The most effectful parameter for the error estimation is the activity of small jumps, that means Y. If we choose a substantially higher activity, namely $Y = 1.143$, we see in Figure 7.4(b) that the order of convergence is less than for $Y = 0.143$. But still it remains higher than the theoretical estimate suggests. Here, we again used the solution for $L = 9$ as reference for the same reasons as before.



(a) Order of convergence for $Y = 0.143$ (b) Order of convergence for $Y = 1.143$

Figure 7.4: Order of convergence for CGMY with $C = 9.61, G = 9.97, M = 16.51, Y = 0.143$ and $C = 9.61, G = 9.97, M = 16.51, Y = 1.143$

Conclusion. The experiments show that convergence takes place and its rate meet the theoretical bounds derived in the previous analysis. While the experimental convergence rate for the option price is quite close to the theoretical upper bound, the ones for the trading strategy and the hedging error are far better. The reasons for this are probably the cancellation effects that have not been taken into account in the theoretical analysis. More specifically, for J^ϵ the analysis was based upon the properties of Γ. The cancellation effects in

$$\psi(V^\epsilon, V^\epsilon) = \Gamma(V^\epsilon, V^\epsilon) - \frac{\Gamma(V^\epsilon, \exp)^2}{\Gamma(\exp, \exp)}$$

have not been considered. Likewise, the error bound of Γ_d is based upon the error bound of A_d. Here, the cancellation effect of the difference

$$\Gamma(f, g) = A(fg) - fAg - gAf$$

has not been considered. These effects, however, take place in the computation which could cause the discrepancy.

Nevertheless, in total the experiments show the convergence of the scheme and the competitive convergence rate. The more involved theoretical analysis including these cancellation effects is left to possible future research.

Acknowledgements

At this point I would like to thank those people and institutions that helped and supported during the research for this thesis. Firstly, thanks go to my supervisor professor Jan Kallsen for his commitment on my behalf. Without him I would not have gone into stochastic analysis. Particularly, I would like to thank for his patience and the most valuable ideas especially when progress stalled. But without the support of professor Christoph Schwab and his PhD students this project would not have been undertaken. Particularly, without receiving the code for the computation of the option price the implementation for the hedging error would not have been possible. The visits to Zurich to this end have partly been financed by a research grant from AMAMEF for which I am thankful. Grateful I am also to TopMath who supported me since the intermediate diploma. Among other things they provided the financial support for the participation in international conferences which gave valuable new insights. Many thanks also go to Johannes Muhle-Karbe for his valuable comments. Finally, there are the people outside academia for whose support I would like to thank. Among others, this is my family who had to suffer my varying moods. Finally, I use the last words in this thesis to express gratefulness to my relatives in Munich who provided shelter for the last month of this research.

Nomenclature

$\mathscr{A}^{\omega_1,s}$ generalization of A 56

$\mathbf{A}^\omega, \mathbf{A}, \tilde{\mathbf{A}}^\omega, \tilde{\mathbf{A}}$ matrices for the corresponding linear forms with respect to the wavelet basis 75

$\tilde{a}^\omega(\cdot,\cdot), \tilde{a}(\cdot,\cdot)$ compressed sesquilinear form, $\tilde{a} = \tilde{a}^0$ 75

A transformed generator 53

$a(\cdot,\cdot)$ sesquilinear form $a = a_0^0$ 61

A_d^ω sparse approximation of A^ω 93

A^X operator which coincides with the generator for X on the intersection of domains 46

A^{ω_1} transformed version of A, $A^{\omega_1} = e^{\omega_1 x} A e^{-\omega_1 x}$ 57

$a_{\omega_2}^{\omega_1}, a^\omega$ sesquilinear form with respect to $\mathscr{A}^{\omega_1}$ and $H_{\omega_2}^{\varrho/2}$, $a^\omega = a_0^\omega$ 58

$A_h, \tilde{A}_h$ discrete operators for a and $\tilde{a}$ 76

$(\cdot,\cdot), (\cdot,\cdot)_{L^2}$ scalar product in L^2

$(\mathcal{F}_t)_{t\geq 0}$ filtration generated by X_t 14

$(\Omega, \mathcal{F}, P, (X_t)_{t\geq 0})$ stochastic process 14

$(\Omega, \mathcal{F}, P^x, (X_t)_{t\geq 0})_{x\in\mathbb{R}^d}$ universal process 14

(b, σ^2, F) Lévy-Khintchine triple of X_t 23

$(x)^+$ $= x$ if $x \geq 0$ and 0 otherwise

$⨍ f(x)dx$ finite part integral 116

$\langle\cdot,\cdot\rangle_{(H^s)^*\times H^s}$ duality pairing for Sobolev spaces 12

$\stackrel{d}{=}$ equality in distribution

$\stackrel{d}{\hookrightarrow}$ dense embedding

$\Im$ imaginary part

κ indicator depending upon ϱ 103

K strike 25

$k(y)$ kernel of jump measure F 23

K^ω kernel in the sense of [MSW06] 63

$K_{\mathscr{A}^\omega}$ Schwartz kernel of $\mathscr{A}^\omega$ 63

$k_{\Delta\eta,\Delta\nu}, k^{-1}_{\Delta\eta,\Delta\nu}, \hat{k}^{-1}_{\Delta\eta,\Delta\nu}$ basic functions for implementation, compute exponential and polynomial moments of k 117

$k^{-(n+1)}_{\Delta\eta,\Delta\nu}$ higher order antiderivatives of $k_{\Delta\eta,\Delta\nu}$ 120

$\mathscr{L}(X,Y)$ set of continuous linear mappings from X to Y

L approximation level 64

$L^p(J;X), L^\infty(J;X), H^k(J;X)$ Bochner spaces 8

M^ω_s, M^ω, M weighted mass matrices for the local Lagrange basis 70

M_p sufficiently large parameter such that $H^\epsilon \in C^{M_p}$ 31

M_t number of time steps 83

$\mathbb{N}$ $= \{1, 2, \dots\}$

$\mathbb{N}_0$ $= \{0, 1, 2, \dots\}$

ν activity of small jumps, $\nu = \varrho$ if $\sigma^2 = 0$ 23

N number of intervals in Y_h 64

n_G number of GMRes iterations 87

Ω finite interval $\Omega = (-R, R)$ 61

Ω^* $= \Omega \setminus l_i; 1 \leq i \leq N$ 66

Ω_i $= (-R+1, R-1)$ 66

$(\psi^l_j)_{j,l}$ biorthogonal wavelet basis 74

$\hat{\Psi}$ symbol of $\mathscr{A}$ 54

φ^j_i local Lagrangian basis of Y_h 65

Φ_r smooth indicator function for $(-r, r)$ 61

P_I polynomial interpolation operator 66

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