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# On the Representation of $P_n$-positive definite Functions and Applications

Kristine Ey

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| Vorsitzender: | | Univ.-Prof. Dr. Gregor Kemper |
| Prüfer der Dissertation: | 1. | Univ.-Prof. Dr. Rupert Lasser |
| | 2. | Prof. Dr. Ryszard Szwarc,<br>Universität Wrocław, Polen |
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**Zusammenfassung**

Positiv-definite Funktionen treten in verschiedenen Fragestellungen in der reinen und angewandten Mathematik auf, wie zum Beispiel im Bereich der orthogonalen Polynome, der numerischen Quadratur, sowie der Zeitreihenanalyse. In diesen Gebieten liegt dem Begriff der positiven Definitheit üblicherweise eine Gruppen- bzw. Halbgruppenstruktur zugrunde. Wir verallgemeinern zentrale Sätze über positiv-definite Funktionen auf allgemeinere algebraische Strukturen, die von polynomialen Folgen induziert werden. Insbesondere zeigen wir, dass jede solche positiv-definite Funktion die Transformierte eines positiven endlichen Borel-Maßes auf den reellen Zahlen ist, und finden Voraussetzungen, unter denen die Beschaffenheit des Trägers dieses Maßes genauer bestimmt werden kann. Zur Veranschaulichung und Anwendung der Ergebnisse werden stationäre Folgen und bestimmte nicht-autonome lineare Volterra-Differenzengleichungen betrachtet. Im letzteren Fall erhalten wir Aussagen über die Existenz unbeschränkter Lösungen.

**Abstract**

Positive definite functions arise in various areas in pure and applied mathematics, such as orthogonal polynomials, numerical integration, and time series analysis. In these applications, the notion of positive definiteness is depending on an underlying group or semigroup structure. We extend some central results on positive definite functions to more general algebraic structures, which are induced by polynomial sequences. In particular, we show that every positive definite function of this type is the transform of a positive finite Borel measure on the reals, and find conditions which yield more information on the character of the support of this measure. For illustration and application of our results, we consider stationary sequences and certain nonautonomous linear Volterra difference equations. In the latter case, statements on the existence of unbounded solutions are obtained.

# Contents

# Introduction

Positive definite sequences arise in various classical questions in pure and applied mathematics, such as the moment problem and time series analysis. These fields share the problem that a measure representing the positive definite sequence is required. In the theory of orthogonal polynomials and the moment problem, this measure is the orthogonalizing measure, in the theory of stationary sequences, it is the spectral measure. We will deal with both areas.

Positive definiteness is depending on the underlying algebraic structure on $\mathbb{N}_0$ or $\mathbb{Z}$. For example, the group $(\mathbb{Z},+)$ can be studied, occurring in time series analysis, cp. [BD02], or the semigroup $(\mathbb{N}_0,+)$, as in the theory of orthogonal polynomials and the moment problem, cp. [BCR84]. We will concentrate on polynomial hypergroups $(\mathbb{N}_0,\omega)$ and more general structures on $\mathbb{N}_0$ defined by polynomial sequences, which contain the semigroup $(\mathbb{N}_0,+)$ as a special case.

For instance, in time series analysis, the covariance function of a weakly stationary stochastic process $\big(X_n\big)_{n\in\mathbb{Z}}$ satisfies

$$\psi(m,n) := \mathrm{Cov}(X_m;X_n) = \mathrm{Cov}(X_{m-n};X_0).$$

Hence $\psi\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{C}$ is only depending on the value of $m-n$. The abbreviated covariance function $\varphi\colon \mathbb{Z}\to\mathbb{C},\ h\mapsto\psi(h,0)$ is positive definite and can be represented by Herglotz's theorem, compare [BD02, Theorem 4.3.1], by

$$\begin{aligned}\varphi(h) &= \int_{(-\pi;\pi]} e^{ih\nu}\,dF(\nu),\\ \psi(m,n) &= \int_{(-\pi;\pi]} e^{i(m-n)\nu}\,dF(\nu).\end{aligned}$$

We will extend this theorem to the previously mentioned algebraic structures on $\mathbb{N}_0$. This establishes the possibility to analyze more general data.

The following question is commonly known as the moment problem: Does there exist a positive measure $\mu$ such that a given sequence $\big(\mu_n\big)_{n\in\mathbb{N}_0}$, $\mu_n \in \mathbb{R}$, can be represented as

$$\mu_n = \int_{\mathbb{R}} x^n \, d\mu \qquad \forall\, n \in \mathbb{N}_0?$$

This question can be answered positively if and only if $\big(\mu_n\big)_{n\in\mathbb{N}_0}$ is a positive definite sequence, see [Cho69, 34.9 Theorem]. If the monomials are substituted by a more general polynomial sequence – for example an orthogonal polynomial sequence – this question is called modified moment problem. It is of certain interest in numerical analysis and time series analysis with appropriate covariance properties. For the purposes of numerical integration, modified moments lead to a stabilization of the Chebyshev algorithm, cf. [CZ93], which computes the recurrence coefficients of the orthogonal polynomials corresponding to the underlying measure.

In the context of polynomial hypergroups and signed polynomial hypergroups an approach to the representation of positive definite functions has already been made. An orthogonal polynomial sequence $\big(R_n\big)_{n\in\mathbb{N}_0}$ with the properties

$$\begin{aligned} R_n(1) &= 1 && \forall\, n \in \mathbb{N}_0, \\ g(m,n;k) &\geq 0 && \forall\, m,n \in \mathbb{N}_0,\ |m-n| \leq k \leq m+n, \end{aligned} \tag{P}$$

where $g(m,n;k)$ denote the linearization coefficients of the product $R_mR_n$, namely

$$R_mR_n = \sum_{k=|m-n|}^{m+n} g(m,n;k) R_k,$$

induces a polynomial hypergroup $(\mathbb{N}_0, \omega)$. The convolution $\omega$ is defined via the linearization coefficients. Bochner's theorem characterizes the bounded positive definite functions on the polynomial hypergroup as transforms of a positive measure supported on the set of all real bounded characters of $(\mathbb{N}_0, \omega)$, which is homeomorphic to the set

$$D := \{x \in \mathbb{R} \colon |R_n(x)| \leq 1 \ \forall\, n \in \mathbb{N}_0\} \subseteq [-1;1].$$

For corresponding reference on hypergroups and the Bochner theorem see for example [BH95].

Since orthogonal polynomials always obey a three-term recurrence relation, there is a close relationship to the theory of difference equations. We will show that

Bochner's theorem yields conditions for the boundedness or unboundedness of solutions of time-depending linear difference equations.

Property (P) is a very strong condition, which we would like to loosen. Hence we are interested in positive definite functions, where the convolution is not necessarily induced by a polynomial hypergroup but by a polynomial sequence which is as general as possible. This generalization enables us to understand the anatomy of Bochner's theorem for polynomial hypergroups. Since our approach does not depend on positivity and orthogonality assumptions any more, we will be be able to make statements on a wider class of linear difference equations and stationary sequences than with polynomial hypergroups.

The outline of this thesis is as follows: In Chapter 1 we introduce polynomial hypergroups and the Bochner theorem as well as some measure theoretical notation and the spectral theorem for unbounded normal operators.
Chapter 2 is devoted to the representing measures of $P_n$-positive definite sequences. Existence and uniqueness questions will be treated. As the assumptions of Bochner's theorem are rather strong, we are interested in the question, which assumptions are actually necessary in order to obtain single statements of the theorem. A large step in this direction has already been made by Margit Rösler [Rös95]. She was able to show that replacing assumption (P) by

$$\sum_{k=|m-n|}^{m+n} |g(m,n;k)| \leq C \qquad \forall\, m,n \in \mathbb{N}_0, \tag{S}$$

where $C \geq 0$, leads to a very similar theorem, where only the set $D$ has to be replaced by

$$\{x \in \mathbb{C} \colon \sup_{n\in\mathbb{N}_0} |R_n(x)| < \infty\}.$$

In order to analyze the different parts of the Bochner theorem, we will start with a general polynomial sequence, which satisfies only weak properties. After that we will gradually add to the assumptions. As a next step, we characterize those functions which can be represented by a singular or absolutely continuous measure. Finally, Stieltjes' and Haviland's modified moment problem will be solved, i.e. we provide a sufficient and necessary condition for $P_n$-positive definite sequences such that a representing measure exists which is supported on $[0;\infty)$ and $[-1;1]$, respectively.
Chapter 3 comes up with various examples of $P_n$-positive definite functions. As in the case of the monomials on the unit circle, the $P_n$-positive definite functions

can be characterized as covariance functions of weakly stationary processes. We discuss certain aspects of those processes, such as spectral measure, generator and imaginary part. We conclude the chapter with some specific examples.
In our last chapter we turn towards an application of our results to nonautonomous linear difference equations and certain Volterra difference equations.

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# 1 Preliminaries

In preparation of our results in the following chapters, we provide some mathematical tools. Apart from two lemmata in Section 1.2 no proofs will be given, since all results are well-known from the literature. Before we present the necessary theorems, we define some important vector spaces and sets.

Throughout the thesis, we abbreviate the complex, real, and natural numbers with $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$, respectively. The integers will be written as $\mathbb{Z}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will denote the polynomials of exact degree $n \in \mathbb{N}_0$ with real and complex coefficients by $\mathbb{P}_n[\mathbb{R}]$ and $\mathbb{P}_n[\mathbb{C}]$, respectively. Polynomials of arbitrary degree will be denoted by $\mathbb{P}[\mathbb{R}]$ resp. $\mathbb{P}[\mathbb{C}]$. We will abbreviate the most important sequence spaces on $\mathbb{N}_0$ in the following way:

- the space of all sequences

$$\ell(\mathbb{N}_0) := \{x \colon \mathbb{N}_0 \to \mathbb{C}\},$$

- the space of all bounded sequences

$$\ell^\infty(\mathbb{N}_0) := \left\{x \in \ell(\mathbb{N}_0) : \sup_{n \in \mathbb{N}_0} |x(n)| < \infty \right\},$$

  which becomes a Banach space with the supremum norm

$$\|x\|_\infty := \sup_{n \in \mathbb{N}_0} |x(n)| \qquad \forall\, x \in \ell^\infty(\mathbb{N}_0),$$

- the space of all absolutely summable sequences

$$\ell^1(\mathbb{N}_0) := \left\{x \in \ell(\mathbb{N}_0) : \sum_{n=0}^{\infty} |x(n)| < \infty \right\},$$

which becomes a Banach space with the norm

$$\|x\|_1 := \sum_{n=0}^{\infty} |x(n)| \qquad \forall\, x \in \ell^1(\mathbb{N}_0),$$

- the space of all square summable sequences

$$\ell^2(\mathbb{N}_0) := \left\{ x \in \ell(\mathbb{N}_0) : \sum_{n=0}^{\infty} |x(n)|^2 < \infty \right\},$$

which becomes a Hilbert space with the scalar product

$$\langle x; y \rangle := \sum_{n=0}^{\infty} x(n)\overline{y(n)} \qquad \forall\, x, y \in \ell^2(\mathbb{N}_0)$$

and the norm

$$\|x\|_2 := \sqrt{\langle x; x \rangle} \qquad \forall\, x \in \ell^2(\mathbb{N}_0).$$

## 1.1 Measure Theory and Representation Theorems

For our representation theorems, we need suitable sets of bounded measures supported on closed subsets of the real line. Hence the appropriate $\sigma$-Algebra is the Borel $\sigma$-Algebra $\mathcal{B}$. We will call a measure $\mu\colon \mathcal{B} \to \mathbb{C}$, where $-\infty \leq a < b \leq \infty$,

- *positive Borel measure on* $[a;b]$, if there exists a bounded monotonously increasing right-continuous function $F\colon [a;b] \to [0;\infty)$ with

$$\lim_{x \searrow a} F(x) = 0$$

such that $d\mu(x) = dF(x)$, namely $\mu\big((c;d]\big) = F(d) - F(c)$ for all $a \leq c \leq d \leq b$. We denote the space of all positive Borel measures on $[a;b]$ by $\mathcal{M}^+([a;b])$.

- *signed Borel measure on* $[a;b]$, if there exist $\mu^+, \mu^- \in \mathcal{M}^+([a;b])$ with $\mu = \mu^+ - \mu^-$. We denote the space of all signed Borel measures on $\mathbb{R}$ by $\mathcal{M}([a;b])$.

- *complex Borel measure on* $[a;b]$, if $\Re\mu$ and $\Im\mu$ are signed regular Borel measures on $[a;b]$. We denote the space of all complex Borel measure on $[a;b]$ by $\mathcal{M}_{\mathbb{C}}([a;b])$.

Note that $\mathcal{M}^+([a;b]) \subset \mathcal{M}([a;b]) \subset \mathcal{M}_{\mathbb{C}}([a;b])$, since we postulate $\mu(\mathbb{R}) < \infty$ for all positive measures. For a complex Borel measure $\mu$, we define its *variation* $|\mu| \in \mathcal{M}^+(\mathbb{R})$ for all $B \in \mathcal{B}$ by

$$|\mu|(B) := \sup\left\{\sum_{n=1}^{\infty} |\mu(B_n)|\colon B_n \in \mathcal{B},\ \bigcup_{n=1}^{\infty} B_n = B,\ B_n \text{ pairwise disjoint}\right\}$$

There exists a Borel measurable function $f$ with $|f(x)| = 1$ for all $x \in \mathbb{R}$, such that $|\mu|(B) = \int_B f\, d\mu$ for all $B \in \mathcal{B}$. $\|\mu\| := |\mu|(\mathbb{R})$ is called *total variation of* $\mu$. Note that the total variation every complex Borel measures is finite. In fact, $(\mathcal{M}_{\mathbb{C}}(\mathbb{R}), \|\cdot\|)$ is a Banach space. We define the support $\operatorname{supp}\mu$ of a complex Borel measure $\mu \in \mathcal{M}_{\mathbb{C}}([a;b])$ by

$$\operatorname{supp}\mu := \overline{\Big\{x \in [a;b]\colon |\mu|\big([x-\varepsilon; x+\varepsilon]\big) > 0 \text{ for all } \varepsilon > 0\Big\}}.$$

A positive Borel measure $\mu$ induces several function spaces. We will abbreviate for $1 \le p < \infty$ the space of all Borel measurable $p$-integrable functions by

$$\mathcal{L}^p\big([a;b], \mu\big) := \left\{f\colon [a;b] \to \mathbb{C}\colon f \text{ measurable}, \int_{[a;b]} |f|^p\, d\mu < \infty\right\}.$$

Unfortunately, the mapping $f \mapsto \left(\int_{[a;b]} |f|^p\, d\mu\right)^{\frac{1}{p}}$ is only a seminorm for many choices of positive measures $\mu$, since it is not necessarily positive definite. In order to avoid this problem, we define the equivalence relation

$$f \sim g \qquad :\Longleftrightarrow \qquad \mu\left(\big\{x \in [a;b]\colon f(x) \neq g(x)\big\}\right) = 0 \quad \forall\, f, g \in \mathcal{L}^p([a;b], \mu),$$

and the equivalence class

$$[f] := \{g \in \mathcal{L}^p([a;b], \mu)\colon f \sim g\} \qquad\qquad \forall\, f \in \mathcal{L}^p([a;b], \mu).$$

Now we are able to define the Banach space of all $p$-integrable equivalence classes

$$L^p\big([a;b], \mu\big) := \big\{[f]\colon f \in \mathcal{L}^p\big([a;b], \mu\big)\big\},$$

which is equipped with the norm

$$\left\|[f]\right\|_p := \left( \int_{[a;b]} |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

In particular, the space $L^2([a;b];\mu)$ is a Hilbert space with the scalar product

$$\langle f; g \rangle := \int_{[a;b]} f\overline{g} \, d\mu \qquad \forall\, f, g \in L^2([a;b], \mu).$$

Strictly speaking, we would always have to denote the elements of $L^p([a;b],\mu)$ by $[f]$ for $f \in \mathcal{L}^p([a;b],\mu)$. For abbreviation, we will simply refer to them by $f$.

In Chapter 2, we will compare complex measures $\mu$ and $\nu$ on the measurable space $(\Omega, \mathcal{A})$, where $\Omega$ is a set – usually a subset of $\mathbb{R}$ – and $\mathcal{A}$ a $\sigma$-algebra on $\Omega$. For that, we need the following notions:

(i) We call $\mu$ *absolutely continuous with respect to* $\nu$, if for every measurable set $A \in \mathcal{A}$ holds

$$\nu(A) = 0 \qquad \Longrightarrow \qquad \mu(A) = 0.$$

We abbreviate $\mu \ll \nu$.

(ii) We say that $\mu$ *and* $\nu$ *are mutually singular*, if there are two sets $A_\mu, A_\nu \in \mathcal{A}$ with $A_\mu \cap A_\nu = \varnothing$ and $A_\mu \cup A_\nu = \Omega$, such that

$$\mu\big|_{A_\nu} \equiv 0 \equiv \nu\big|_{A_\mu}.$$

We abbreviate $\mu \perp \nu$.

**Theorem 1.1 (Radon-Nikodym and Lebesgue's decomposition theorem):** *Let* $(\Omega, \mathcal{A}, \nu)$ *be a finite measure space and* $\mu$ *a* $\sigma$*-finite complex measure on* $\Omega$*. Then there exist two* $\sigma$*-finite measures* $\mu_s$ *and* $\mu_{ac}$ *such that*

$$\mu = \mu_s + \mu_{ac} \qquad\qquad \mu_s \perp \nu, \qquad\qquad \mu_{ac} \ll \nu.$$

*Moreover, there exists a function* $f \in L^1(\Omega, \mu)$ *such that* $d\mu_{ac} = f \, d\nu$*.*

A proof of this theorem can be found e.g. in [Lan93, Thm. VII.2.4]. Note that for $\nu \in \mathcal{M}^+(\mathbb{R})$, this theorem can be applied to all $\mu \in \mathcal{M}_{\mathbb{C}}(\mathbb{R})$, since we assume $\nu(\mathbb{R}), |\mu|(\mathbb{R}) < \infty$.

We now turn to a representation theorem by Choquet which will be the main basis for our results in Chapter 2. Let $X$ be a locally compact Hausdorff space and $C(X, \mathbb{K})$ the vector space of continuous maps $f\colon X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and $C^+(X, \mathbb{R})$ the cone of nonnegative continuous functions $X \to \mathbb{R}$, i.e.

$$C^+(X, \mathbb{R}) := \{f \in C(X, \mathbb{R})\colon f \geq 0\}\,.$$

Let $Y$ be a subspace of $C(X, \mathbb{R})$ and abbreviate $Y^+ := Y \cap C^+(X, \mathbb{R})$. The space $Y$ is called *adapted*, if

1. $Y = Y^+ - Y^+$,

2. for all $x \in X$ there is a $f \in Y^+$ such that $f(x) > 0$,

3. for every $f \in Y^+$ and $\varepsilon > 0$ there exists some $g \in Y^+$ and a compact set $C \subseteq X$ such that
$$f(x) \leq \varepsilon g(x) \qquad \forall\, x \in X \setminus C.$$

We point out that $\mathbb{P}[\mathbb{R}] \subset C(\mathbb{R}, \mathbb{R})$ is an adapted space. Recall the identity $\mathbb{P}[\mathbb{R}]^+ = \mathbb{P}[\mathbb{R}]^2 + \mathbb{P}[\mathbb{R}]^2$, cf. [BCR84, 2.1 Lemma]:

**Lemma 1.2:** *Let $P \in \mathbb{P}[\mathbb{R}]^+$, i.e. $P(x) \geq 0$ for all $x \in \mathbb{R}$. Then there exist $Q_1, Q_2 \in \mathbb{P}[\mathbb{R}]$ with $P = Q_1^2 + Q_2^2$.*

Now we have prepared the necessary vocabulary for Choquet's representation theorem, see e.g. [Cho69, 34.6 Thm.]:

**Theorem 1.3 (Choquet's representation theorem):** *Let $X$ be a locally compact Hausdorff space and $Y \subseteq C(X, \mathbb{R})$ an adapted space. Suppose $\Phi\colon Y \to \mathbb{R}$ is a positive linear functional, i.e. $\Phi(Y^+) \subseteq [0; \infty)$. Then there exists a $\mu \in M^+(X)$ such that*

$$\Phi(f) = \int_X f\, d\mu \qquad \forall\, f \in Y$$

*and every $f \in Y$ is $\mu$-integrable. Here, $M^+(X)$ denotes the set of all positive – possibly unbounded – Borel measures on $X$.*

As we will see in Section 1.3, this theorem provides the solution of the moment problem.

On a compact set, continuous functions can be uniformly approximated by polynomials, cf. [Lan93, Thm. III.1.1]:

**Theorem 1.4 (Weierstraß' approximation theorem):** *Let $B \in \mathcal{B}$ be compact and $f \in C(\mathbb{R}, \mathbb{C})$. For every $\varepsilon > 0$ there exists a polynomial $P_\varepsilon \in \mathbb{P}[\mathbb{C}]$ such that*

$$\sup_{x \in B} \left| f(x) - P_\varepsilon(x) \right| \leq \varepsilon.$$

## 1.2 Linear Operators on Hilbert Spaces

In this section, we introduce the basic notation of linear operators and present some necessary tools from spectral theory. We use [Con90] as basic reference. Let $(\Omega, \mathcal{A})$ be a measurable space – again $\Omega$ is a set and $\mathcal{A}$ a $\sigma$-algebra on $\Omega$ – and $\mathcal{H}$ a Hilbert space equipped with the scalar product $\langle \cdot ; \cdot \rangle$. A linear operator $A \colon \mathcal{H} \to \mathcal{H}$ on $\mathcal{H}$ will be called *bounded* if

$$\|A\| := \sup_{x \in \mathcal{H}} \frac{\|Ax\|}{\|x\|} < \infty.$$

We denote the densely defined linear and bounded linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$, respectively. For $A \in \mathcal{L}(\mathcal{H})$, we denote the dense domain of $A$ in $\mathcal{H}$ by $\mathcal{D}(A)$ and the range of $A$ by $\mathcal{R}(A)$. We abbreviate the *graph of* $A \in \mathcal{L}(\mathcal{H})$ by

$$\Gamma(A) := \{(x, Ax) \in \mathcal{H} \times \mathcal{H} \colon x \in \mathcal{D}(A)\}.$$

We call $A \in \mathcal{L}(\mathcal{H})$ closed, if $\Gamma(A)$ is closed in $\mathcal{H} \times \mathcal{H}$. If $A, B \in \mathcal{L}(\mathcal{H})$ with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ax = Bx$ for all $x \in \mathcal{D}(A)$, we abbreviate this by $A \subseteq B$. Define

$$\mathcal{D}(A^*) := \{y \in \mathcal{H} \colon \langle Ax; y \rangle = \langle x; y^* \rangle \text{ for some } y^* \in \mathcal{H} \text{ and all } x \in \mathcal{D}(A)\}.$$

The linear operator defined by $A^* \colon \mathcal{D}(A^*) \to \mathcal{H},\ y \mapsto y^*$ is called *adjoint of* $A$. We call $A \in \mathcal{L}(\mathcal{H})$

- *symmetric*, if $A \subseteq A^*$, in particular

$$\langle Ax; y \rangle = \langle x; Ay \rangle \qquad \forall\, x, y \in \mathcal{D}(A),$$

- *self-adjoint*, if $A$ and $A^*$ are symmetric, in particular $A = A^*$,
- *normal*, if $A$ is closed and $AA^* = A^*A$.

Note that if $A \in \mathcal{B}(\mathcal{H})$ – and in particular $\mathcal{D}(A) = \mathcal{H}$, then $A$ is symmetric if and only if $A$ is self-adjoint.

If for some $A \in \mathcal{L}(\mathcal{H})$ there exists a bounded operator $B \in \mathcal{B}(\mathcal{H})$, such that

$$B\colon \mathcal{H} \to \mathcal{D}(A), \qquad AB = \mathrm{id}, \qquad BA \subseteq \mathrm{id},$$

then $A$ is called *boundedly invertible*. We define the spectrum $\sigma(A)$ of $A$ by

$$\sigma(A) := \{\lambda \in \mathbb{C}\colon A - \lambda\,\mathrm{id} \text{ is boundedly invertible}\}.$$

A *spectral measure* on $(\Omega, \mathcal{A}, \mathcal{H})$ is a function $E\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that

(i) for every $A \in \mathcal{A}$, $E(A)$ is an orthogonal projection,

(ii) $E(\varnothing) = 0$ and $E(\Omega) = \mathrm{id}_{\mathcal{H}}$,

(iii) $E(A \cap B) = E(A)E(B)$ for all $A, B \in \mathcal{A}$,

(iv) for any sequence $\big(A_n\big)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$ and all $x \in \mathcal{H}$ holds

$$E\left(\bigcup_{n=1}^{\infty} A_n\right) x = \sum_{n=1}^{\infty} E(A_n)x.$$

For a spectral measure $E$ and $x, y \in \mathcal{H}$ we define the function $E_{x,y}\colon \mathcal{A} \to \mathbb{C}$ by $E_{x,y}(A) := \langle E(A)x; y\rangle$. $E_{x,y}$ is a complex measure on $\mathcal{A}$ with total variation less or equal to $\|x\|\|y\|$, cf. [Con90]. Integration with respect to a spectral measure can be defined pointwise as we will see in the following theorem.

**Theorem 1.5 (Spectral Theorem):** *Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator, $\mathcal{H}$ a Hilbert space. Then there exists a unique spectral measure $E$ on $(\mathbb{C}, \mathcal{B}, \mathcal{H})$ such that*

*(i)* $N = \int z\, dE(z)$,

*(ii)* $E(A) = 0$ *if* $A \cap \sigma(N) = \varnothing$*,*

*(iii)* *if* $U$ *is an open subset of* $\mathbb{C}$ *and* $U \cap \sigma(N) \neq \varnothing$*, then* $E(U) \neq 0$*.*

*For every Borel function* $f\colon \sigma(A) \to \mathbb{C}$ *a densely defined linear operator* $N_f \in \mathcal{L}(\mathcal{H})$ *can be defined by*

$$\langle N_f x; y \rangle = \int f \, dE_{x,y}$$

*for all* $x, y \in \mathcal{D}(N_f)$*, where*

$$\mathcal{D}\big(N_f\big) := \left\{ x \in \mathcal{H}\colon \int |f|^2 \, dE_{x,x} < \infty \right\}.$$

*We abbreviate* $N_f := \int f \, dE$*. The spectrum of* $N_f$ *satisfies*

$$\sigma(N_f) = f\big(\sigma(N)\big).$$

It remains to remark that for $N \in \mathcal{L}(\mathcal{H})$ normal and every polynomial $P \in \mathbb{P}[\mathbb{C}]$ holds

$$P(N) \subseteq \int P \, dE = N_P.$$

If $A$ is self-adjoint and $P \in \mathbb{P}[\mathbb{R}]$, then

$$P(A) = \int P \, dE = A_P.$$

We sketch proofs of the next two lemmata in order to emphasize that they are not depending on the Spectral Theorem. Later, we will use these lemmata together with a representation theorem to obtain the Spectral Theorem for bounded self-adjoint operators.

**Lemma 1.6:** *Let* $\mathcal{H}$ *be a Hilbert space,* $P \in \mathbb{P}[\mathbb{C}]$*, and* $N \in \mathcal{B}(\mathcal{H})$ *normal. Then*

$$\sigma\big(P(N)\big) = P\big(\sigma(N)\big).$$

*Proof.* Let $\lambda \in \mathbb{C}$ and denote the $n$ zeros of $P - \lambda$ by $z_1, \dots, z_n$. Then

$$\begin{aligned} P(N) - \lambda \,\mathrm{id} = c \prod_{k=1}^{n} (N - z_k \,\mathrm{id}) \text{ is invertible} \\ \iff \quad & N - z_k \,\mathrm{id} \ \text{ is invertible for all } 1 \le k \le n \\ \iff \quad & z_k \notin \sigma(N) \text{ for all } 1 \le k \le n \\ \iff \quad & \text{there is no } z \in \sigma(N) \text{ such that } P(z) = \lambda. \end{aligned}$$ □

**Lemma 1.7:** *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ self-adjoint. For any $P \in \mathbb{P}[\mathbb{C}]$ with $\|P\|_{\sigma(A)} \le 1$ holds $\|P(A)\| \le 1$.*

*Proof.* For any complex polynomial $P$ holds $\sigma\big(P(A)\big) = P\big(\sigma(A)\big)$ by the preceding lemma. Since $P(A)$ is a normal element of the $C^*$-algebra $\mathcal{B}(\mathcal{H})$, the spectral radius and the norm of $P(A)$ coincide and hence $\|P(A)\| \le 1$. □

If $N \in \mathcal{L}(\mathcal{H})$ is a normal operator, then a vector $x_0 \in \mathcal{H}$ is called *cyclic vector* for $N$ if for all $m, n \in \mathbb{N}_0$ we have $x_0 \in \mathcal{D}(N^{*m} N^n)$ and

$$\mathcal{H} = \overline{\operatorname{span}} \left\{ N^{*m} N^n x_0 \colon m, n \in \mathbb{N}_0 \right\}.$$

Every normal operator that possesses a cyclic vector is "similar" to a multiplication operator, cf. [Con90, X.4.18 Thm.]. We will state this theorem for self-adjoint operators only, since we are only interested in measures which are supported on the real line.

**Theorem 1.8:** *Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint and $x_0 \in \mathcal{H}$ a cyclic vector for $A$. Then there exists a positive measure $\mu \in \mathcal{M}^+(\mathbb{R})$ with $\mathbb{P}[\mathbb{C}] \subset L^2(\mathbb{R}, \mu)$ and an isomorphism $U \colon \mathcal{H} \to L^2(\mathbb{R}, \mu)$ with $U x_0 = 1$ and $U N U^{-1} = M_\mu$, where*

$$M_\mu \colon \mathcal{D}(M_\mu) \subseteq L^2(\mathbb{R}, \mu) \to L^2(\mathbb{R}, \mu), \qquad f \mapsto x f$$

*and*

$$\mathcal{D}(M_\mu) := \left\{ f \in L^2(\mathbb{R}, \mu) \colon x f \in L^2(\mathbb{R}, \mu) \right\}.$$

*In particular $\mu = E_{x_0, x_0}$ and $\operatorname{supp} \mu = \sigma(A)$, where $E$ denotes the spectral measure of $A$.*

## 1.3 Orthogonal Polynomials

As basic reference on orthogonal polynomials we used [Sze67, Chi78].

For any positive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$ there exists a sequence of real polynomials $\big(R_n\big)_{n\in\mathbb{N}_0}$ with $\deg R_n = n$ and

$$\int_{\mathbb{R}} R_m R_n \, d\mu = \alpha_n \delta_{m,n}$$

for all $m, n \in \mathbb{N}_0$, where $\alpha_n \geq 0$. If $\mu$ has infinite support, the parameters $\alpha_n$ are positive. We call $\big(R_n\big)_{n\in\mathbb{N}_0}$ an *orthogonal polynomial sequence with respect to* $\mu$. According to Favard's theorem, a polynomial sequence $\big(R_n\big)_{n\in\mathbb{N}_0}$ is an orthogonal polynomial sequence with respect to a positive Borel measure $\mu$ with infinite support if and only if $\big(R_n\big)_{n\in\mathbb{N}_0}$ satisfies a three-term recurrence relation

$$xR_n(x) = A_n R_{n+1}(x) + B_n R_n(x) + C_n R_{n-1}(x) \qquad \forall\, n \in \mathbb{N}_0,$$

where $R_{-1} \equiv 0,\ R_0 \equiv A_{-1} > 0$ and $B_n \in \mathbb{R},\ A_n \cdot C_{n+1} > 0$ for all $n \in \mathbb{N}_0$.

A famous example for an orthogonal polynomial family are the Jacobi polynomials which are depending on two parameters $\alpha, \beta > -1$. They are given by the recurrence relation

$$\begin{aligned}
P_1^{(\alpha,\beta)}(x) &= \frac{\alpha+\beta+2}{2}x - \frac{\beta-\alpha}{2}, \qquad\qquad P_0^{(\alpha,\beta)}(x) = 1,\\
xP_n^{(\alpha,\beta)}(x) &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)}(x)\\
&\quad + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(x)\\
&\quad + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

They satisfy $P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}$, where $(a)_n := a\cdot(a+1)\cdots(a+n-1)$, $n \in \mathbb{N}$, denotes the *Pochhammer symbol*. The Jacobi polynomials form an orthogonal polynomial family with respect to the measure $(1-x)^\alpha(1+x)^\beta\, dx$. The leading coefficient of the Jacobi polynomials equals $\frac{(n+\alpha+\beta+1)_n}{2^n n!}$ and the following important asymptotic formula is valid, cf. [Sze67, Thm 8.21.8]:

**Theorem 1.9:** *Let $\alpha, \beta > -1$ and $0 < \theta < \pi$. Then*

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-\frac{1}{2}} k(\theta) \cos(N\theta + \gamma) + f_n(\theta),$$

*where*

$$k(\theta) = \pi^{-\frac{1}{2}} \left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}, \quad N = n + \frac{\alpha+\beta+1}{2}, \quad \gamma = -\frac{(\alpha+\frac{1}{2})\pi}{2}$$

*and $f_n \colon (0;\pi) \to \mathbb{R}$ satisfy $\sup_{n\in\mathbb{N}} |f_n(\theta)| \cdot n^{\frac{3}{2}} < \infty$ for all $\theta \in (0;\pi)$. The bound for this error term holds uniformly on $[\varepsilon; \pi - \varepsilon]$ for every $\varepsilon > 0$.*

An asymptotic formula for $x \in \mathbb{R}\setminus[-1;1]$ is the following, cf. [Sze67, Thm 8.21.7]:

**Theorem 1.10:** *Let $\alpha, \beta > -1$. Then for all $x \in \mathbb{R} \setminus [-1;1]$*

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) \simeq{}& |x-1|^{-\frac{\alpha}{2}} |x+1|^{-\frac{\beta}{2}} \left[\sqrt{|x+1|} + \sqrt{|x-1|}\right]^{\alpha+\beta} \\ &\cdot (2\pi n)^{-\frac{1}{2}} (x^2-1)^{-\frac{1}{2}} \left[x + (x^2-1)^{\frac{1}{2}}\right]^{n+\frac{1}{2}}, \end{aligned}$$

*where the positive branches of the roots are chosen. $a_n \simeq b_n$ denotes*

$$b_n \neq 0 \qquad \text{and} \qquad \left|\frac{a_n}{b_n}\right| \xrightarrow{n\to\infty} 1.$$

An important special case of the Jacobi polynomials are the Chebyshev polynomials of first kind $\left(T_n\right)_{n\in\mathbb{N}_0}$, which correspond to $\alpha = \beta = -\frac{1}{2}$. For convenience, they are usually normalized by $T_n(1) = 1$. In particular we have

$$T_n(x) := \frac{n!}{\left(\frac{1}{2}\right)_n} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) \qquad \forall\, n \in \mathbb{N}_0.$$

For $x = \cos\theta \in [-1;1]$ they can be represented as

$$T_n(\cos\theta) = \cos(n\theta) \qquad \forall\, n \in \mathbb{N}_0.$$

From this formula one can easily see that $T_n$ has $n$ distinct real zeros, namely

$$T_n(\cos\theta) = 0 \qquad \Longleftrightarrow \qquad \theta \in \left\{\frac{(k-\frac{1}{2})\pi}{n} : k = 1, \ldots, n\right\}.$$

The minima and maxima attain the values $-1$ and 1, respectively. The leading coefficient of $T_n$ is $2^{n-1}$. The Chebyshev polynomials of first kind are of great importance for polynomial inequalities, since they are extremal in many ways, cf. [BE95].

**Lemma 1.11:** *Choose $P \in \mathbb{P}_n[\mathbb{R}]$ for some $n \in \mathbb{N}$. For all $x \in \mathbb{R} \setminus [-1;1]$ holds*

$$|P(x)| \leq |T_n(x)| \cdot \sup_{x \in [-1;1]} |P(x)|.$$

*Equality holds if and only if $P = cT_n$ for some $c \in \mathbb{R} \setminus \{0\}$.*

### 1.3.1 The Moment Problem

The moment problem is a classical question which is closely related to orthogonal polynomials since its answer provides the existence of the desired orthogonalizing measure.

**The $I$-moment problem:** Let $I \subseteq \mathbb{R}$ be a closed generalized interval and $\varphi \in \ell(\mathbb{N}_0)$ be an arbitrary sequence. Which condition is necessary and sufficient for the existence of a positive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$ such that

$$\varphi(n) = \int_I x^n \, d\mu \qquad \forall\, n \in \mathbb{N}_0?$$

Under which conditions is this solution unique?

This question was first posed – and answered – by Stieltjes in 1894 for the case $I = [0;\infty)$. Hence the $[0;\infty)$-moment problem is also known as *Stieltjes moment problem.* In 1919–21, Hamburger generalized this approach to $I = \mathbb{R}$. Consequently, the $\mathbb{R}$-moment problem is today known as *Hamburger moment problem*, which possesses a solution if and only if $\varphi$ is a real sequence and

$$\det\left(\varphi(i+j)\right)_{i,j=0,\dots,n} \geq 0 \qquad \forall\, n \in \mathbb{N}_0.$$

This follows from Choquet's representation theorem, cf. Theorem 1.3, and the identity $\mathbb{P}[\mathbb{R}^+] = \mathbb{P}[\mathbb{R}]^2 + \mathbb{P}[\mathbb{R}]^2$, cf. Lemma 1.2, since $\mathbb{P}[\mathbb{R}]$ is an adapted space. There exists a solution of Stieltjes' moment problem if and only if

$$\det\left(\varphi(i+j)\right)_{i,j=0,\dots,n} \geq 0 \qquad \text{and} \qquad \det\left(\varphi(i+j+1)\right)_{i,j=0,\dots,n} \geq 0 \qquad \forall\, n \in \mathbb{N}_0.$$

If $I$ is bounded, the uniqueness of $\mu$ follows by the Weierstraß approximation theorem, cf. Theorem 1.4. Otherwise, this question is not easily answered. If

$$\sum_{n=1}^{\infty} \big(\varphi(2n)\big)^{-\frac{1}{2n}} = \infty \tag{1.3.1}$$

the solution of the moment problem is unique. Equation (1.3.1) is called *Carleman's criterion.*

### 1.3.2 Polynomial Hypergroups and Bochner's Theorem

For a proof of the following results on polynomial hypergroups and hypergroups, see [Las83, BH95].

In the following, we suppose $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. In this case, $\big(R_n\big)_{n\in\mathbb{N}_0}$ satisfies the three-term recurrence relation

$$\begin{aligned} &R_0 \equiv 1, \qquad R_1(x) = \frac{1}{a_0}(x - b_0), \\ &R_1(x)R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \end{aligned} \tag{1.3.2}$$

where $a_n, c_n > 0,\ b_n \in \mathbb{R}$ and $a_0 + b_0 = 1,\ a_n + b_n + c_n = 1$. By the orthogonality of the polynomial sequence it follows immediately that there exist coefficients $g(m, n; k) \in \mathbb{R}$ with

$$R_m(x)R_n(x) = \sum_{k=|m-n|}^{m+n} g(m, n; k)R_k(x).$$

Suppose $g(m, n; k) \geq 0$ for all $m, n, k \in \mathbb{N}_0$ and define

$$\begin{aligned} \omega\colon \mathbb{N}_0 \times \mathbb{N}_0 &\to \ell^1(\mathbb{N}_0) \\ (m, n) &\mapsto \sum_{k=|m-n|}^{m+n} g(m, n; k)\delta_k, \end{aligned}$$

where $\delta_k = \big(\delta_{kn}\big)_{n\in\mathbb{N}_0}$ and $\delta_{kn}$ denotes the Kronecker symbol. For abbreviation, $\omega(m, n)$ is often denoted by $m * n$. Then the pair $(\mathbb{N}_0, \omega)$ is called the *polynomial hypergroup generated by* $\big(R_n\big)_{n\in\mathbb{N}_0}$ and we say that $\big(R_n\big)_{n\in\mathbb{N}_0}$ *induces a polynomial*

*hypergroup.* Polynomial hypergroups are special cases of discrete hypergroups, cf. [Las05].
An example for a polynomial hypergroup is provided by the Jacobi polynomials. Normalizing $p_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomials induce a polynomial hypergroup if $\beta \leq \alpha$ and $\alpha + \beta + 1 \geq 0$.

Define the sets

$$D := \left\{ z \in \mathbb{C} \colon \big(R_n(z)\big)_{n\in\mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0) \right\}, \qquad D_s := D \cap \mathbb{R}.$$

If $\big(R_n\big)_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup, then

$$D = \{ z \in \mathbb{C} \colon \; |R_n(z)| \leq 1 \text{ for all } n \in \mathbb{N}_0 \}.$$

For $m \in \mathbb{N}_0$, we define a translation operator $\mathcal{T}_m$ by

$$\begin{aligned} \mathcal{T}_m \colon \; & \ell(\mathbb{N}_0) \to \ell(\mathbb{N}_0) \\ & \big(\varphi(n)\big)_{n\in\mathbb{N}_0} \mapsto \left( \sum_{k=|m-n|}^{m+n} g(m,n;k)\varphi(k) \right)_{n\in\mathbb{N}_0}. \end{aligned}$$

A function $\varphi \colon \mathbb{N}_0 \to \mathbb{C}$ is called *character of* $(\mathbb{N}_0, \omega)$, if

$$\varphi(m)\varphi(n) = \mathcal{T}_m\varphi(n) \qquad \forall\, m,n \in \mathbb{N}_0.$$

The following characterization of these functions can be found e.g. in [BH95].

**Lemma 1.12:** *Let $\big(R_n\big)_{n\in\mathbb{N}_0}$ induce a polynomial hypergroup. A function $\varphi$ is a character of $(\mathbb{N}_0, \omega)$ if and only if there exists a $z \in \mathbb{C}$ such that*

$$\varphi(n) = R_n(z) \qquad \forall\, n \in \mathbb{N}_0.$$

*In particular, the set of all bounded characters is homeomorphic to $D$.*

Now we introduce a central notion for this thesis. A function $\varphi \colon \mathbb{N}_0 \to \mathbb{C}$ is called *positive definite with respect to* $\big(R_n\big)_{n\in\mathbb{N}_0}$ – or short *$R_n$-positive definite*, if for all $n \in \mathbb{N}_0, \; c_0, \dots, c_n \in \mathbb{C}$ holds

$$\sum_{k=0}^{n} \sum_{j=0}^{n} c_k \overline{c_j} \mathcal{T}_k \varphi(j) \geq 0.$$

In the group case $(\mathbb{Z},+)$ positive definiteness implies boundedness, but this is not the case on the semigroup $(\mathbb{N}_0,+)$ or for $R_n$-positive definiteness. The following theorem characterizes the bounded $R_n$-positive definite functions.

**Theorem 1.13 (Bochner's theorem):** *Let $\left(R_n\right)_{n\in\mathbb{N}_0}$ be a polynomial sequence which induces a polynomial hypergroup and choose $\varphi \in \ell^\infty(\mathbb{N}_0)$. The function $\varphi$ is positive definite with respect to $\left(R_n\right)_{n\in\mathbb{N}_0}$ if and only if there exists a positive Borel measure $\mu \in \mathcal{M}^+(D_s)$ such that*

$$\varphi(n) = \int_{D_s} R_n \, d\mu \qquad \forall\, n \in \mathbb{N}_0.$$

*In particular, $\|\varphi\|_\infty = \varphi(0)$.*

We already mentioned in the introduction that there exists a generalization of this theorem, which involves signed hypergroups, cf. [Rös95]. Signed hypergroups are a generalization of hypergroups mainly in the sense that the convolution does not need to map positive measures on positive measures. In this context, the assumptions

$$g(m,n;k) \geq 0 \qquad \text{and} \qquad R_n(1) = 1$$

are replaced by

$$\sum_{k=|m-n|}^{m+n} |g(m,n;k)| \leq C \qquad \forall\, m,n \in \mathbb{N}_0$$

for some $C > 0$. The proof of these theorems in the context of hypergroups and signed hypergroups is based on harmonic analysis. As we will see in Chapter 2, this result can be achieved by exploiting solely the structure of the polynomial sequence and we will be able to drop the orthogonality condition.

# 2 Representation of $P_n$-positive definite Sequences

In the first place, this chapter is devoted to generalizing the solution of the moment problem to more general polynomial sequences than monomials. Since the monomials do not induce a signed polynomial hypergroup, Bochner's theorem is not applicable. Still, there is a result which is equivalent to Bochner's theorem, cf. [BCR84, 2.8 Thm.]:

*Suppose $\varphi$ is the moment sequence of a positive Borel measure $\mu \in \mathcal{M}^+(\mathbb{R})$ and hence a positive definite function on the semigroup $(\mathbb{N}_0, +)$. Then $\varphi$ is bounded if and only if* $\operatorname{supp} \mu \subseteq [-1; 1]$.

We want to understand where the common basis of these two theorems lies and which steps are necessary to achieve the whole result. For this purpose, we extend the definition of positive definiteness to polynomial sequences which satisfy only weak properties.

In the following, we always assume $\left(P_n\right)_{n\in\mathbb{N}_0}$ to be a real polynomial sequence with $P_n \in \mathbb{P}_n[\mathbb{R}]$, $P_0 \equiv 1$. For abbreviation, we will call such a polynomial sequence $\left(P_n\right)_{n\in\mathbb{N}_0}$ a *real polynomial family.* There exist real coefficients $g(m,n;k)$ with

$$P_m(x)P_n(x) = \sum_{k=0}^{m+n} g(m,n;k)P_k(x).$$

For $m \in \mathbb{N}_0$, we define a translation operator $\mathcal{T}_m$ by

$$\begin{aligned} \mathcal{T}_m\colon\ \ell(\mathbb{N}_0) &\to \ell(\mathbb{N}_0) \\ \varphi &\mapsto \left(\sum_{k=0}^{m+n} g(m,n;k)\varphi(k)\right)_{n\in\mathbb{N}_0}. \end{aligned}$$

Having this available, we denote a function $\varphi$ as *positive definite with respect to* $\big(P_n\big)_{n\in\mathbb{N}_0}$ – or short *$P_n$-positive definite*, if for all $n \in \mathbb{N}_0$, $c_0, \ldots, c_n \in \mathbb{C}$

$$\sum_{i=0}^{n}\sum_{j=0}^{n} c_i\overline{c_j}\ \mathcal{T}_i\varphi(j) \geq 0.$$

In case the inequality is strict for all $(c_0, \ldots, c_n) \neq 0$, we call $\varphi$ *strictly positive definite with respect to* $\big(P_n\big)_{n\in\mathbb{N}_0}$. These definitions are analogous to the case where $\big(P_n\big)_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup. In this chapter, we want to characterize the convex cone of positive definite functions.
For any sequence $\varphi \in \ell(\mathbb{N}_0)$, we define the linear functional $\Phi_\varphi$ by

$$\Phi_\varphi \colon \mathbb{P}[\mathbb{C}] \to \mathbb{C}, \qquad P = \sum_{n=0}^{N} a_n P_n \mapsto \sum_{n=0}^{N} a_n \varphi(n).$$

As $\{P_n \colon n \in \mathbb{N}_0\}$ is a basis of $\mathbb{P}[\mathbb{C}]$, this functional is well-defined. Note that the definition of $\mathcal{T}_m$, $\Phi_\varphi$ and the notion of positive definiteness are strongly dependent of the choice of the real polynomial family $\big(P_n\big)_{n\in\mathbb{N}_0}$.
The functional $\Phi_\varphi$ and the translation operators $\mathcal{T}_n$ are connected via the following lemma.

**Lemma 2.1:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\varphi \in \ell(\mathbb{N}_0)$. Then the translation operator $\mathcal{T}_n$ and the functional $\Phi_\varphi$ satisfy*

$$\mathcal{T}_n\varphi(k) = \Phi_\varphi(P_nP_k) \qquad \textit{and} \qquad \mathcal{T}_m\mathcal{T}_n\varphi(k) = \Phi_\varphi(P_mP_nP_k) \tag{2.0.1}$$

*for all $m, n, k \in \mathbb{N}_0$.*

*Proof.* From the relation

$$\begin{aligned}\mathcal{T}_n\varphi(k) &= \sum_{j=0}^{n+k} g(n,k;j)\varphi(j)\\ &= \sum_{j=0}^{n+k} g(n,k;j)\Phi_\varphi(P_j)\\ &= \Phi_\varphi(P_nP_k)\end{aligned}$$

we see that the first part of equation (2.0.1) is true. Second part follows from

$$\begin{aligned}\mathcal{T}_m\mathcal{T}_n\varphi(k) &= \sum_{j=0}^{m+k} g(m,k;j)\mathcal{T}_n\varphi(j)\\ &= \sum_{j=0}^{m+k} g(m,k;j)\Phi_\varphi(P_nP_j)\\ &= \Phi_\varphi(P_mP_nP_k).\end{aligned}$$

□

Now we are able to show a convenient property of translation operators:

**Proposition 2.2:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family. Then the translation operators satisfy*

$$\mathcal{T}_n = P_n\big(P_1^{-1}(\mathcal{T}_1)\big) = P_n\big(a_0\mathcal{T}_1 + b_0\big) \qquad \forall\, n \in \mathbb{N}_0, \tag{2.0.2}$$

*where $a_0$ and $b_0$ are given by $P_1(x) = \frac{1}{a_0}x - \frac{b_0}{a_0}$.*

*Proof.* We abbreviate $\mathcal{T} := P_1^{-1}(\mathcal{T}_1) = a_0\mathcal{T}_1 + b_0$ and show the claim via induction. For $n = 0, 1$ equation (2.0.2) is straightforward. Now suppose it has already been shown for $0 \le k \le n \in \mathbb{N}$ and let $\varphi \in \ell(N_0),\ m \in \mathbb{N}_0$. Then with equation (2.0.1) and Lemma 2.1

$$\begin{aligned}g(n,1;n+1)P_{n+1}(\mathcal{T})\varphi(m) &= P_1(\mathcal{T})P_n(\mathcal{T})\varphi(m) - \sum_{k=0}^{n} g(n,1;k)P_k(\mathcal{T})\varphi(m)\\ &= \mathcal{T}_1\mathcal{T}_n\varphi(m) - \sum_{k=0}^{n} g(n,1;k)\mathcal{T}_k\varphi(m)\\ &= \Phi_\varphi(P_1P_nP_m) - \sum_{k=0}^{n} g(n,1;k)\Phi_\varphi(P_kP_m)\\ &= g(n,1;n+1)\Phi_\varphi(P_mP_{n+1}) = g(n,1;n+1)\mathcal{T}_{n+1}\varphi(m)\end{aligned}$$

gives us the claim for $n+1$, as $g(n,1;n+1) \neq 0$ for all $n \in \mathbb{N}$ due to the ascending degrees of the polynomials. □

Let $\big(R_n\big)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial family, which induces a polynomial hypergroup. As already discussed in Section 1.3.2, there exists a representation theorem for positive definite functions with respect to $\left(R_n\right)_{n\in\mathbb{N}_0}$. Therefore, the orthogonal polynomial sequence has to satisfy the rather strong condition that all linearization coefficients $g(m,n;k)$ are nonnegative. Since this condition is in general not easily verified, we are interested in dropping this assumption. This leads us to the following theorem:

**Theorem 2.3:** *A function $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ is positive definite with respect to $\left(P_n\right)_{n\in\mathbb{N}_0}$ if and only if there exists a $\mu \in \mathcal{M}^+(\mathbb{R})$ with*

$$\varphi(n) = \int_{\mathbb{R}} P_n \, d\mu \qquad \forall\, n \in \mathbb{N}_0.$$

In this case, we call $\mu$ a *representing measure of $\varphi$*. Since $\mu$ is not necessarily uniquely determined, there might be a whole class of representing measures. We will refer to the elements of this class by $\hat{\varphi}$. We point out that $\mu \in \mathcal{M}^+(\mathbb{R})$ is a representing measure of a $P_n$-positive definite function if and only if all moments of $\mu$ are finite, namely

$$\int_{\mathbb{R}} x^n \, d\mu < \infty \qquad \forall\, n \in \mathbb{N}_0,$$

as in this case all polynomials are $\mu$-integrable. Since $\mu \in \mathcal{M}^+(\mathbb{R})$ guarantees $\mu(\mathbb{R}) < \infty$ only, not every positive measure is a representing measure.

The proof of this theorem follows the same a idea as in [BR02], where the same assertion is shown for the case of a real polynomial family inducing a polynomial hypergroup. In this paper, the correspondence between the polynomial hypergroup $(\mathbb{N}_0, \omega)$ and the semigroup $(\mathbb{N}_0, +)$ is exploited, so that the moment problem yields the existence of the desired measure.

*Proof.* Suppose $\varphi$ is positive definite. By Choquet's representation theorem 1.3, it suffices to show that for any nonnegative polynomial $Q \in \mathbb{P}[\mathbb{R}]$, i.e. $Q(x) \geq 0$ for all $x \in \mathbb{R}$, holds $\Phi_\varphi(Q) \geq 0$, since $\mathbb{P}[\mathbb{R}]$ is an adapted space.
If $Q \in \mathbb{P}[\mathbb{R}]$ is nonnegative on $\mathbb{R}$, by Lemma 1.2 there exist real polynomials $Q_1, Q_2$ such that $Q = Q_1^2 + Q_2^2$, where

$$Q_i = \sum_{j=0}^{n} c_{ij} P_j, \qquad i = 1, 2,$$

with coefficients $c_{ij} \in \mathbb{R}$. This implies

$$\begin{aligned} Q &= \sum_{i,j=0}^{n} c_{1i}c_{1j}P_iP_j + \sum_{i,j=0}^{n} c_{2i}c_{2j}P_iP_j \\ &= \sum_{i,j=0}^{n} c_{1i}c_{1j} \sum_{l=0}^{i+j} g(i,j;l)P_l + \sum_{i,j=0}^{n} c_{2i}c_{2j} \sum_{l=0}^{i+j} g(i,j;l)P_l, \end{aligned}$$

and therefore

$$\begin{aligned} \Phi_\varphi(Q) &= \sum_{i,j=0}^{n} c_{1i}c_{1j} \sum_{l=0}^{i+j} g(i,j;l)\Phi_\varphi(P_l) + \sum_{i,j=0}^{n} c_{2i}c_{2j} \sum_{l=0}^{i+j} g(i,j;l)\Phi_\varphi(P_l) \\ &= \sum_{i,j=0}^{n} c_{1i}c_{1j}\mathcal{T}_i\varphi(j) + \sum_{i,j=0}^{n} c_{2i}c_{2j}\mathcal{T}_i\varphi(j) \geq 0. \end{aligned}$$

For the converse direction, let $\mu \in \mathcal{M}^+(\mathbb{R})$ be a representing measure of $\varphi$, $n \in \mathbb{N}_0$ and $c_0, \dots, c_n \in \mathbb{C}$. Then

$$\begin{aligned} \sum_{i,j=0}^{n} c_i\overline{c_j}\mathcal{T}_i\varphi(j) &= \sum_{i,j=0}^{n} c_i\overline{c_j} \int_{\mathbb{R}} P_iP_j \, d\mu \\ &= \int_{\mathbb{R}} \left( \sum_{i=0}^{n} c_iP_i \right)^2 d\mu \geq 0 \end{aligned}$$

gives us the $P_n$-positive definiteness of $\varphi$. □

From this theorem follow some properties of $P_n$-positive functions which are valid for every choice of the real polynomial family.

**Corollary 2.4:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\varphi \neq 0$ a $P_n$-positive definite function. Then $\varphi(0) > 0$ and for all $n \in \mathbb{N}_0$*

$$\varphi(n) \in \mathbb{R}, \qquad\qquad \varphi(n)^2 \leq \mathcal{T}_n\varphi(n) \cdot \varphi(0).$$

*In particular, if $\varphi$ is unbounded, then $\mathcal{T}_n\varphi$ is unbounded, too.*

*Proof.* Let $\mu$ be the measure given by Theorem 2.3. Then by the Cauchy-Schwartz inequality

$$\begin{aligned}
\varphi(0) &= \int 1\,d\mu = \mu(\mathbb{R}) > 0,\\
\varphi(n) &= \int_{\mathbb{R}} P_n\,d\mu \in \mathbb{R},\\
\varphi(n)^2 &= \left(\int_{\mathbb{R}} P_n\,d\mu\right)^2 \leq \int_{\mathbb{R}} P_n^2\,d\mu \cdot \int_{\mathbb{R}} 1\,d\mu = \mathcal{T}_n\varphi(n)\cdot\varphi(0).
\end{aligned}$$

□

Using Theorem 2.3, we can characterize the strictly $P_n$-positive definite functions as well. The following corollary is a slight generalization of the main theorem in [Las84].

**Corollary 2.5:** *A function $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ is strictly positive definite with respect to $\big(P_n\big)_{n\in\mathbb{N}_0}$ if and only if there exists a $\mu \in \mathcal{M}^+(\mathbb{R})$ with infinite support and*

$$\varphi(n) = \int_{\mathbb{R}} P_n\,d\mu \qquad \forall\, n \in \mathbb{N}_0.$$

*Proof.* Suppose $\varphi$ is strictly $P_n$-positive definite. By Theorem 2.3, there exists a representing measure $\mu \in \mathcal{M}^+(\mathbb{R})$ of $\varphi$. If $\operatorname{supp}\mu = \{x_1, \dots, x_N\}$ is finite, then

$$\int_{\mathbb{R}} \prod_{n=1}^{N} (x - x_n)^2\,d\mu = 0.$$

We abbreviate

$$P(x) := \prod_{n=1}^{N}(x - x_n) = \sum_{n=0}^{N} c_n P_n(x),$$

where $c_n \in \mathbb{C}$ are uniquely determined and not all equal to zero as $\{P_n\colon n\in\mathbb{N}_0\}$ is a basis of $\mathbb{P}[\mathbb{R}]$. In particular, $\Phi_\varphi(P^2) = 0$ and hence

$$\sum_{k,j} c_k\overline{c_j}\mathcal{T}_j\varphi(j) = \int_{\mathbb{R}} P^2\,d\mu = 0,$$

which contradicts the strict $P_n$-positive definiteness of $\varphi$.

Suppose $\mu$ has infinite support. Choose $N \in \mathbb{N}_0$ and $(c_0, \ldots, c_N) \in \mathbb{C}^{N+1} \setminus \{0\}$ arbitrarily, and define a polynomial by

$$P := \sum_{n=0}^{N} c_n P_n \in \mathbb{P}[\mathbb{C}].$$

As we have

$$\sum_{k,j} c_k \overline{c_j} \mathcal{T}_k \varphi(j) = \int_{\mathbb{R}} P^2 \, d\mu > 0,$$

the function $\varphi$ is strictly $P_n$-positive definite. $\square$

By dropping some of the assumptions for the real polynomial family and the boundedness of $\varphi$, we loose an important property of any representing measure $\mu$, namely the characterization of the support of $\mu$. In Section 2.1, we are going find conditions under which $\operatorname{supp} \mu$ is bounded.

**Remark 2.6:** In general, the representing measure of a $P_n$-positive definite function $\varphi$ is not unique. The problem is equivalent to solving the moment problem for the sequence $\big(\Phi_\varphi(x^n)\big)_{n \in \mathbb{N}_0}$, cf. [BR02]. If the sequence $\big(\Phi_\varphi(x^n)\big)_{n \in \mathbb{N}_0}$ is positive definite, there exists either exactly one or uncountably many solutions $\mu \in \mathcal{M}^+(\mathbb{R})$, cf. [Akh65]. Some of them – the so-called *N-extremal* solutions – have the convenient property that the polynomials $\mathbb{P}[\mathbb{C}]$ are dense in $L^2(\mathbb{R}, \mu)$. If $\varphi$ is strictly $P_n$-positive definite, the Gram-Schmidt process applied to $\{x^n \colon n \in \mathbb{N}_0\}$ yields an orthonormal polynomial sequence $\big(Q_n\big)_{n \in \mathbb{N}_0}$ with respect to $\Phi_\varphi$, cf. [Sze67]. We denote the sequence space of all finite sequences on $\mathbb{N}_0$ by $\ell_{00}(\mathbb{N}_0)$, i.e.

$$\ell_{00}(\mathbb{N}_0) := \{x \in \ell(\mathbb{N}_0) \colon x(n) = 0 \text{ for almost all } n \in \mathbb{N}_0\}$$

The linear symmetric operator

$$\begin{aligned} &A \colon \ell_{00}(\mathbb{N}_0) \subset \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0), \\ &\quad \delta_n \mapsto \Phi_\varphi(x Q_n Q_{n+1}) \delta_{n+1} + \Phi_\varphi(x Q_n Q_n) \delta_n + \Phi_\varphi(x Q_n Q_{n-1}) \delta_{n-1} \end{aligned}$$

plays a central role in the analysis of the moment problem. This kind of tridiagonal operator is usually called *Jacobi operator* in the literature, cf. [Tes99]. It turns out that the N-extremal solutions of the moment problem correspond to the self-adjoint extensions of $A$ in the following sense: A solution $\mu$ of the moment problem is N-extremal if and only if there exists a self-adjoint extension $\tilde{A}$ of $A$ such that $\mu = E_{\delta_0,\delta_0}(\tilde{A})\varphi(0)$, i.e.

$$\langle P_n(\tilde{A})\delta_0;\delta_0\rangle \cdot \varphi(0) = \int_{\sigma(\tilde{A})} P_n \, dE_{\delta_0,\delta_0}(\tilde{A}) = \varphi(n),$$

where $E(\tilde{A})$ is the spectral measure of $\tilde{A}$, cf. [Sim98, Prop. 4.15]. Since $\delta_0$ is a cyclic vector for $\tilde{A}$, we have $\sigma(\tilde{A}) = \operatorname{supp} E_{\delta_0,\delta_0}(\tilde{A})$. If $A$ is a bounded operator, then it can be extended in a unique way to $\ell^2(\mathbb{N}_0)$ and the moment problem possesses the unique solution $E_{\delta_0,\delta_0}(A)\varphi(0)$.
It remains to remark that in case $\varphi$ is not strictly $P_n$-positive definite there exists a finite orthonormal polynomial system and a symmetric matrix with the same properties as above can be found in the same way.

## 2.1 Regaining properties of Bochner's theorem

Paying tribute to our weak assumptions in Theorem 2.3, we lost all information on the support of the representing measure in the case of $\varphi \in \ell^\infty(\mathbb{N}_0)$ in comparison to Theorem 1.13. We need some further definitions in order to regain this information step by step.

For an arbitrary real polynomial family $\big(P_n\big)_{n\in\mathbb{N}_0}$ we define the sets

$$\begin{aligned} D^\infty &:= \left\{z \in \mathbb{C}\colon\ (P_n(z))_{n\in\mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0)\right\}, \\ D^\alpha &:= \left\{z \in \mathbb{C}\colon\ \left\|(P_n(z))_{n\in\mathbb{N}_0}\right\|_\infty \le \alpha\right\} \qquad \forall\, \alpha > 0. \end{aligned}$$

If $\alpha \le \beta$, then $D^\alpha \subseteq D^\beta$ is obvious. Since all measures in this section are supported on the real line, we are interested in the intersections $D_s^\infty := D^\infty \cap \mathbb{R}$ and $D_s^\alpha := D^\alpha \cap \mathbb{R}$. Due to

$$D^\alpha = \bigcap_{n=0}^{\infty} \{z \in \mathbb{C}\colon |P_n(z)| \le \alpha\},$$

$D^\alpha$ resp. $D_s^\alpha$ are compact in $\mathbb{C}$ resp. $\mathbb{R}$. For our first result on bounded $P_n$-positive definite functions, we need the interior of $D_s^\alpha$:

$$\begin{aligned} U_s^\alpha &:= \{x \in D_s^\alpha\colon (x-\varepsilon; x+\varepsilon) \subset D_s^\alpha \text{ for some } \varepsilon > 0\}, \\ U_s^\infty &:= \bigcup_{\alpha \ge 1} U_s^\alpha. \end{aligned}$$

As $U_s^\infty$ is the union of open sets, $U_s^\infty$ is open, too. These sets play a crucial role in our analysis because they provide criteria for the boundedness or unboundedness of a $P_n$-positive definite function.

**Proposition 2.7:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\mu \in \mathcal{M}^+(\mathbb{R})$ a positive measure with finite moments. Define a $P_n$-positive definite function $\varphi$ by*

$$\varphi(n) := \int_{\mathbb{R}} P_n \, d\mu \qquad \forall\, n \in \mathbb{N}_0.$$

*Then we have:*

*(i) If $\operatorname{supp}\mu \subseteq D_s^\alpha$ for some $\alpha \geq 1$, then $\varphi$ is bounded and $\|\varphi\|_\infty \leq \alpha\varphi(0)$.*

*(ii) If $\operatorname{supp}\mu \subset U_s^\infty$ is compact, then there exists a constant $\alpha \geq 1$ with*

$$\operatorname{supp}\mu \subset U_s^\alpha \subset D_s^\alpha \qquad \text{and} \qquad \|\varphi\|_\infty \leq \alpha\varphi(0).$$

*Proof.* The first assertion follows immediately from

$$|\varphi(n)| \leq \int_{D_s^\alpha} |P_n| \, d\mu \leq \alpha \int_{D_s^\alpha} 1 \, d\mu = \alpha\varphi(0) \qquad \forall\, n \in \mathbb{N}_0.$$

We proceed to the second claim. Since $\operatorname{supp}\mu \subseteq U_s^\infty$, the family $\left\{U_s^n \colon n \in \mathbb{N}\right\}$ is an open cover of $\operatorname{supp}\mu$. As $\operatorname{supp}\mu$ is compact, there exists $N \in \mathbb{N}$ with

$$\operatorname{supp}\mu \subseteq \bigcup_{n=1}^{N} U_s^n = U_s^N \subset D_s^N.$$

Now $\|\varphi\|_\infty \leq N\varphi(0)$ follows from (i). $\square$

There are classical real polynomial families such that $D_s^\alpha \subsetneq D_s^\infty$ for all $\alpha > 0$, and $D_s^\infty$ is bounded.

**Example 2.1:** Consider the Jacobi polynomials $p_n^{(\alpha,\beta)}$ which are depending on two parameters $\alpha, \beta > -1$. They are given by the recurrence relation

$$\begin{aligned}
p_1^{(\alpha,\beta)}(x) &= \frac{\alpha+\beta+2}{2(\alpha+1)}x - \frac{\beta-\alpha}{2(\alpha+1)}, \qquad\qquad p_0^{(\alpha,\beta)}(x) = 1,\\
p_1^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) &= \frac{(n+\alpha+\beta+1)(n+\alpha+1)(\alpha+\beta+2)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(\alpha+1)}p_{n+1}^{(\alpha,\beta)}(x)\\
&\quad + \frac{\alpha-\beta}{2(\alpha+1)}\left[1 - \frac{(\alpha+\beta+2)(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}\right]p_n^{(\alpha,\beta)}(x)\\
&\quad + \frac{n(n+\beta)(\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)(\alpha+1)}p_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}$$

By definition, cp. Section 1.3, they satisfy $p_n^{(\alpha,\beta)}(1) = 1$ for all $n \in \mathbb{N}_0$ and hence we have $1 \in D_s^1$. We assume $-\frac{1}{2} < \alpha < \beta$. Since

$$(-1)^n p_n^{(\alpha,\beta)}(-1) = \prod_{k=1}^{n} \frac{\beta+k}{\alpha+k} \sim n^{\beta-\alpha} \qquad \forall\, n \in \mathbb{N}_0,$$

cf. [Sze67, p.58f], we have $|p_n^{(\alpha,\beta)}(-1)| \to \infty$ as $n \to \infty$ and thus $-1 \notin D_s^\infty$. Here, $a_n \sim b_n$ denotes $\frac{|a_n|}{|b_n|} \to C > 0$ as $n \to \infty$ for complex sequences $\big(a_n\big)_{n\in\mathbb{N}}$, $\big(b_n\big)_{n\in\mathbb{N}}$. For all $x \in (-1;1)$, Theorem 1.9 yields

$$\begin{aligned}
p_n^{(\alpha,\beta)}(x) &= \frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(\cos\theta) \sim n^{-\alpha-\frac{1}{2}}k(\theta)\cos(N\theta+\gamma) + n^{-\alpha}f_n(\theta)\\
&= n^{-\alpha-\frac{1}{2}}\Big(k(\theta)\cos(N\theta+\gamma) + n^{\frac{1}{2}}f_n(\theta)\Big) \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus $x \in D_s^\infty$ for all $x \in (-1;1)$. Since the convergence is uniform on bounded subintervals of $(-1;1)$, we have $x \in U_s^\infty$. By Theorem 1.10,

$$p_n^{(\alpha,\beta)}(x) \sim n^{-\frac{1}{2}-\alpha}\left[x + (x^2-1)^{\frac{1}{2}}\right]^{n+\frac{1}{2}} \sim n^{-\frac{1}{2}-\alpha}x^n$$

and hence $x \notin D_s^\infty$ for all $x \in \mathbb{R} \setminus [-1;1]$.
Summing up, $D_s^\infty = (-1;1]$ is not closed, yet bounded, and $U_s^\infty = (-1;1)$. In particular, given a positive measure $\mu$ the function $n \mapsto \int P_n\, d\mu$ is bounded if $\operatorname{supp}\mu \subseteq (-1;1)$.

On the other hand it is possible to find a real polynomial family with $D^\infty = \mathbb{C}$.

**Example 2.2:** Define $P_n(x) := \frac{x^n}{n^n},\ n \in \mathbb{N}$. Then for all $z \in \mathbb{C}$ holds

$$|P_n(z)| = \left(\frac{|z|}{n}\right)^n \leq \begin{cases} 1 & n \geq |z| \\ |z|^n & n < |z|. \end{cases}$$

Hence $|P_n(z)|$ is bounded by 1 for large $n$ and thus $D^\infty = \mathbb{C}$.

By this example, we also see that the size of $D^\infty$ and $D_s^\infty$ is largely influenced by the leading coefficients of the real polynomial family. We will take a closer look at this dependence. Unfortunately, there does not exist a lower bound for $D_s^\infty$ which depends only on the leading coefficients, as $D_s^\infty$ can be empty for every choice of those. An upper bound can be given with help of the following theorem.

**Theorem 2.8:** *Let $P \in \mathbb{P}_n[\mathbb{C}]$ for some $n \in \mathbb{N}$. Denote the leading coefficient of $P$ by $\ell_n(P) \neq 0$ and let $\alpha > 0$. Then*

$$\lambda\left(\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}\right) \leq 4\sqrt[n]{\frac{\alpha}{2|\ell_n(P)|}},$$

*where $\lambda$ denotes the one-dimensional Lebesgue measure. Equality holds if and only if $P(x) = \alpha T_n\left(\sqrt[n]{\frac{2|\ell_n(P)|}{\alpha}} \cdot \dfrac{x}{2} + c\right)$ for some $c \in \mathbb{R}$.*

*Proof.* We assume $\ell_n(P) > 0$ w.l.o.g and denote for $k = 1, \ldots, n$ the complex zeros of $P$ by $a_k + ib_k,\ a_k, b_k \in \mathbb{R}$. As

$$\prod_{k=1}^n |x - a_k - ib_k| = \prod_{k=1}^n \sqrt{(x-a_k)^2 + b_k^2} \geq \prod_{k=1}^n |x - a_k| \qquad \forall\, x \in \mathbb{R}$$

we can assume that $b_k = 0$ for all $k$. In particular, this assumption implies $P \in \mathbb{P}[\mathbb{R}]$. Suppose $\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}$ consists of two or more disjoint intervals – there can be at most $n$ of them. We want to move the zeros of $P$ in a convenient way in order to close the gaps. For this purpose, we assume $a_1 \leq \cdots \leq a_n$ and $\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\} = \bigcup_{k=1}^n [c_k; d_k]$ with $a_k \in [c_k; d_k] =: A_k$ for $k = 1, \ldots, n$. By assumption there exists some $j$ with $A_j \cap A_{j+1} = \varnothing$, i.e. $c_{j+1} - d_j =: \varepsilon_j > 0$. Define a polynomial $Q \in \mathbb{P}_n[\mathbb{R}]$ by

$$Q(x) := \ell_n(P) \cdot \prod_{k=1}^j (x - a_k) \cdot \prod_{k=j+1}^n (x - (a_k - \varepsilon_j)).$$

It is straightforward to see that $|Q(x)| < \alpha$ for all $x \in A_1, \dots, A_j$ and for all $x \in A_{j+1} - \varepsilon_j, \dots, A_n - \varepsilon_j$. In particular, we have

$$\lambda\left(\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}\right) < \lambda\left(\{x \in \mathbb{R} \colon |Q(x)| \leq \alpha\}\right)$$

and the gap between $A_j$ and $A_{j+1}$ is closed. Since we can close all gaps in this manner we can assume that $\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\} = [a; b]$ for some $a, b \in \mathbb{R}$.
Since

$$|P(x)| \leq \alpha \qquad \Longleftrightarrow \qquad \left|\frac{P(x)}{\ell_n(P)}\right| \leq \frac{\alpha}{\ell_n(P)},$$

we define

$$\tilde{P} := \frac{P}{\ell_n(P)} \qquad \text{and} \qquad \tilde{\alpha} = \frac{\alpha}{\ell_n(P)}$$

and examine $\left\{x \in \mathbb{R} \colon |\tilde{P}(x)| \leq \tilde{\alpha}\right\}$ instead of $\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}$. In particular

$$\lambda\Big(\left\{x \in \mathbb{R} \colon |\tilde{P}(x)| \leq \tilde{\alpha}\right\}\Big) = \lambda\left(\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}\right).$$

Note that $\tilde{P}$ is monic. For convenience we want to get rid of $\tilde{\alpha}$, hence we define

$$\hat{P}(x) := \frac{2}{\tilde{\alpha}} \tilde{P}\left(\sqrt[n]{\frac{\tilde{\alpha}}{2}} x\right) \qquad \forall\, x \in \mathbb{R},$$

which is again a monic polynomial. We have the identity

$$\sqrt[n]{\frac{2}{\tilde{\alpha}}} \cdot \lambda\left(\left\{x \in \mathbb{R} \colon |\tilde{P}(x)| \leq \tilde{\alpha}\right\}\right) = \lambda\left(\left\{x \in \mathbb{R} \colon |\hat{P}(x)| \leq 2\right\}\right).$$

Now suppose that $b - a = 4 + 2\varepsilon$ for some $\varepsilon > 0$. We can assume $[a; b] = [-2-\varepsilon; 2+\varepsilon]$ w.l.o.g. and define $Q(x) := \frac{1}{2}\hat{P}\left((2+\varepsilon)x\right)$. Since $|\hat{P}(x)| \leq 2$ for all $x \in [-2-\varepsilon; 2+\varepsilon]$, we have $|Q(x)| \leq 1$ for all $x \in [-1; 1]$. By Lemma 1.11, we have $|Q(x)| \leq |T_n(x)|$ for all $x \in \mathbb{R} \setminus [-1; 1]$. But this is a contradiction, since the leading coefficient of $Q$ equals $\frac{(2+\varepsilon)^n}{2}$, whereas the leading coefficient of $T_n$ equals $2^{n-1}$. In particular

$$\begin{aligned} 4 \geq \lambda\left(\left\{x \in \mathbb{R} \colon |\hat{P}(x)| \leq 2\right\}\right) &= \sqrt[n]{\frac{2}{\tilde{\alpha}}} \lambda\left(\left\{x \in \mathbb{R} \colon |\tilde{P}(x)| \leq \tilde{\alpha}\right\}\right) \\ &= \sqrt[n]{\frac{2}{\tilde{\alpha}}} \lambda\left(\{x \in \mathbb{R} \colon |P(x)| \leq \alpha\}\right). \end{aligned}$$

It remains to show that equality only holds for $\hat{P}(x) = 2T_n\left(\frac{x}{2}\right)$. Consider the polynomial $R(x) := 2T_n\left(\frac{x}{2}\right) - \hat{P}(x)$. Denote the extremal points of $T_n(\frac{x}{2})$ by $x_{n-1} < x_{n-2} < \cdots < x_1$. We know that $2T_n(\frac{x_k}{2}) = (-1)^{k-1}2$. In each interval $[2x_k; 2x_{k-1}]$ lies a zero of $R$. If $x_k$ is a zero of $R$ for some $k$, then $\hat{P}$ has an extremal point at $x_k$, too, and the zero $x_k$ has at least multiplicity 2. Hence $R$ has at least $n-2$ – counting multiplicities – zeros in the interval $[x_{n-1}; x_1]$. Together with the two zeros at $-2$ and 2 the polynomial $R$ has at least $n$ zeros. Since $\deg R = n-1$, this implies $R = 0$. Resubstitution yields

$$2T_n\left(\frac{x}{2}\right) = \hat{P}(x) = \frac{2}{\tilde{\alpha}}\tilde{P}\left(\sqrt[n]{\frac{\tilde{\alpha}}{2}}x\right) = \frac{2}{\alpha}P\left(\sqrt[n]{\frac{\tilde{\alpha}}{2}}x\right). \qquad \square$$

Now an upper bound for the size of $D_s^\alpha$ and $D_s^\infty$ – and hence for the size of $U_s^\alpha$ and $U_s^\infty$ – is straightforward. The size of $D^\infty$ and $D^\alpha$ is of less importance in our setting, since all representing measures are supported on $\mathbb{R}$.

**Corollary 2.9:** *Let $\alpha \geq 1$, $a > 0$, and $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family.*

*(i) If $|\ell_n(P_n)| \geq a^n$ for all $n \in \mathbb{N}_0$, then $\lambda(D_s^\alpha) \leq \frac{4}{a}\min\left\{\frac{\alpha}{2}; 1\right\}$ and $\lambda(D_s^\infty) \leq \frac{4}{a}$.*

*(ii) If $\sqrt[n]{|\ell_n(P_n)|} \to a$ as $n \to \infty$, then $\lambda(D_s^\alpha) \leq \lambda(D_s^\infty) \leq \frac{4}{a}$.*

*Proof.* (i) By Theorem 2.8 we know

$$\lambda\left(\{x \in \mathbb{R} \colon |P_n(x)| \leq \alpha\}\right) \leq 4\sqrt[n]{\frac{\alpha}{2\ell_n(P_n)}} \leq \frac{4}{a}\sqrt[n]{\frac{\alpha}{2}} =: a_n$$

for all $n \in \mathbb{N}_0$. For $1 \leq \alpha \leq 2$ the sequence $\left(a_n\right)_{n\in\mathbb{N}_0}$ is increasing, hence

$$\lambda(D_s^\alpha) \leq \inf_{n\in\mathbb{N}} a_n = a_1 = \frac{2\alpha}{a}.$$

If $\alpha \geq 2$, then the sequence $\left(a_n\right)_{n\in\mathbb{N}}$ is decreasing and we have

$$\lambda(D_s^\alpha) \leq \inf_{n\in\mathbb{N}} a_n = \lim_{n\to\infty} a_n = \frac{4}{a}.$$

As $D_s^\infty = \bigcup_{\alpha\geq 1} D_s^\alpha$ and $D_s^\beta \subseteq D_s^\alpha$ if $\beta \leq \alpha$ we have

$$\lambda(D_s^\infty) \leq \sup_{\alpha\geq 1}\lambda(D_s^\alpha) \leq \frac{4}{a}.$$

(ii) As $\sqrt[n]{\frac{2}{\alpha}} \to 1$ as $n \to \infty$, we have

$$\sqrt[n]{\frac{2}{\alpha}|\ell_n(P_n)|} \overset{n\to\infty}{\longrightarrow} a. \tag{2.1.1}$$

If $\sqrt[N]{\frac{2}{\alpha}|\ell_N(P_N)|} \geq a$ for some $N \in \mathbb{N}$, then by Theorem 2.8

$$\lambda(D_s^\alpha) \leq \lambda\left(\{x \in \mathbb{R}\colon |P_N(x)| \leq \alpha\}\right) \leq 4\sqrt[N]{\frac{\alpha}{2|\ell_N(P_N)|}} \leq \frac{4}{a}.$$

Hence we can assume $\sqrt[n]{\frac{2}{\alpha}|\ell_n(P_n)|} < a$ for all $n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$0 < a - \sqrt[N]{\frac{2}{\alpha}|\ell_N(P_N)|} < \varepsilon.$$

This leads to

$$\lambda(D_s^\alpha) \leq \lambda\left(\{x \in \mathbb{R}\colon |P_N(x)| \leq \alpha\}\right) \leq 4\sqrt[N]{\frac{\alpha}{2|\ell_N(P_N)|}} < \frac{4}{a-\varepsilon}.$$

Since $\lambda(D_s^\alpha) < \frac{4}{a-\varepsilon}$ for all $\varepsilon > 0$ and $\alpha \geq 1$, we have $\lambda(D_s^\alpha) \leq \lambda(D_s^\infty) \leq \frac{4}{a}$. □

We want to remark that the second inequality is sharp, since equality holds in case of the Jacobi polynomials $\left(p_n^{(\alpha,\beta)}\right)_{n\in\mathbb{N}_0}$, where $-\frac{1}{2} < \alpha < \beta$. From Example 2.1 we know that $D_s^\infty = (-1;1]$, hence $\lambda(D_s^\infty) = 2$. The leading coefficients are given by $\ell_n(p_n^{(\alpha,\beta)}) = \frac{(n+\alpha+\beta+1)_n}{2^n(\alpha+1)_n}$ and satisfy $\sqrt[n]{\ell_n(p_n^{(\alpha,\beta)})} \to 2$ as $n \to \infty$. Thus the preceding corollary gives 2 as an upper estimate of the size of $D_s^\infty$.

For a polynomial hypergroup $(\mathbb{N}_0, \omega)$ Theorem 2.8 also yields some information on the interdependence between the leading coefficients of the underlying polynomial sequence and the set $D_s^\infty$, which corresponds to the set of real bounded characters.

**Corollary 2.10:** *Let $\left(R_n\right)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence which satisfies property* (P) *and hence induces a polynomial hypergroup. The following hold:*

*(i) If the leading coefficients satisfy $\ell_n(R_n) \geq a^n$ for all $n \in \mathbb{N}$ and some $a > 0$, then*

$$\lambda(D_s^\infty) = \lambda(D_s^1) \leq \frac{2}{a}.$$

*(ii) If $\lambda(D_s^\infty) = \lambda(D_s^1) \geq \frac{4}{a}$, then $\frac{a^n}{2} \geq \ell_n(R_n) \geq q^n$ for some $0 < q \leq a$ and all $n \in \mathbb{N}_0$.*

*Proof.* (i) As $D_s^\infty = D_s^1$ for every polynomial hypergroup, the claim follows from Corollary 2.9 (i).

(ii) If $\ell_N(R_N) > \frac{a^N}{2}$ for some $N \in \mathbb{N}$, then by Theorem 2.8

$$\begin{aligned}\lambda(D_s^1) &\leq \lambda\left(\{x \in \mathbb{R} \colon |R_N(x)| \leq 1\}\right) \leq 4\sqrt[N]{\frac{1}{2\ell_N(R_N)}}\\ &< 4\sqrt[N]{\frac{1}{a^N}} = \frac{4}{a},\end{aligned}$$

which is a contradiction to $\lambda(D_s^1) \geq \frac{4}{a}$. Hence $\ell_n(R_n) \leq \frac{a^n}{2}$ for all $n \in \mathbb{N}_0$. We turn to the second inequality. One can show via induction

$$\ell_n(R_n) = \frac{1}{a_0^n}\prod_{k=1}^{n-1}\frac{1}{a_k} \qquad \forall\, n \in \mathbb{N}_0,$$

where the coefficients $a_k$ are defined as in (1.3.2). Since $\left(R_n\right)_{n\in\mathbb{N}_0}$ satisfies property (P), we have $0 < a_k = g(1, k; k+1) < 1$ for all $k \in \mathbb{N}$. In particular,

$$\ell_n(R_n) = \frac{1}{a_0^n}\prod_{k=1}^{n-1}\frac{1}{a_k} \geq \frac{1}{a_0^n} =: q^n. \qquad \square$$

As a next step, we will specify the location of $D_s^\alpha$ and $\operatorname{supp}\hat{\varphi}$ for bounded $\varphi$.

**Theorem 2.11:** *Let $\varphi \colon \mathbb{N}_0 \to \mathbb{C}$ be positive definite with respect to the real polynomial family $\left(P_n\right)_{n\in\mathbb{N}_0}$ and $\hat{\varphi}$ a representing measure of $\varphi$. Suppose there exists a constant $\alpha > 0$ and some $n_0 \in \mathbb{N}$ such that the absolute sums of the linearization coefficients of $P_m P_{n_0}$ are uniformly bounded, i.e.*

$$\sum_{k=0}^{m+n_0} |g(m, n_0; k)| \leq \alpha \qquad \forall\, m \in \mathbb{N}_0.$$

*If $\varphi$ is bounded, then $\operatorname{supp}\hat{\varphi} \subseteq P_{n_0}^{-1}\left([-\alpha; \alpha]\right)$ and $|\varphi(n_0)| \leq \alpha \cdot \varphi(0)$. In particular, $\hat{\varphi}$ is uniquely determined and $D_s^\infty \subseteq P_{n_0}^{-1}([-\alpha; \alpha])$.*

*Proof.* We assume w.l.o.g. $\varphi \neq 0$ and recall $\|\varphi\|_\infty := \sup_{n\in\mathbb{N}_0} |\varphi(n)| > 0$. For all $k \in \mathbb{N}_0$ holds $\|\mathcal{T}_{n_0}^k\varphi\|_\infty \leq \alpha^k\|\varphi\|_\infty$, since via induction

$$\begin{aligned} |\mathcal{T}_{n_0}^{k+1}\varphi(m)| &= \left|\sum_j g(m,n_0;j)\left(\mathcal{T}_{n_0}^k\varphi\right)(j)\right| \\ &\leq \sum_j |g(m,n_0;j)| \cdot \left|\mathcal{T}_{n_0}^k\varphi(j)\right| \leq \alpha^{k+1}\|\varphi\|_\infty. \end{aligned}$$

Now suppose there exists a $\lambda \in \operatorname{supp}\hat{\varphi} \setminus P_{n_0}^{-1}([-\alpha;\alpha])$. Then $|P_{n_0}(\lambda)| > \alpha$. Hence a compact set $C$ can be found with $\lambda \in C$, $C \cap P_{n_0}^{-1}([-\alpha;\alpha]) = \varnothing$, $\hat{\varphi}(C) > 0$, and $|P_{n_0}(x)| \geq c > \alpha$ for all $x \in C$. Then

$$\alpha^{2k-1}\|\varphi\|_\infty \geq |T_{n_0}^{2k-1}\varphi(n_0)| = \int_{\mathbb{R}} P_{n_0}^{2k}\, d\hat{\varphi} \geq \int_C P_{n_0}^{2k}\, d\hat{\varphi} \geq \int_C c^{2k}\, d\hat{\varphi} = c^{2k}\hat{\varphi}(C).$$

This is a contradiction, if $k$ is sufficiently large, i.e. for

$$k > \frac{\ln\left(\frac{\alpha\hat{\varphi}(C)}{\|\varphi\|_\infty}\right)}{\ln\left(\frac{\alpha^2}{c^2}\right)}. \qquad \square$$

Note that, if $\left(P_n\right)_{n\in\mathbb{N}_0}$ is an orthonormal polynomial sequence, then the sums $\sum_{k=0}^{n+1} |g(n,1;k)|$ are uniformly bounded if and only if the support of the orthogonalizing measure $\nu$ is bounded. Hence in that case the boundedness of $\varphi$ implies uniqueness of the representing measure and $\operatorname{supp}\hat{\varphi} \subseteq P_1^{-1}([-\alpha;\alpha])$ for some $\alpha > 0$. In particular, $\operatorname{supp}\nu \subseteq P_1^{-1}([-\alpha;\alpha])$, since $n \mapsto \delta_{n0}$ is a bounded $P_n$-positive definite function.

Bochner's theorem for signed hypergroups, cf. [Rös95], is – in the polynomial case – depending on the assumption that $\left(P_n\right)_{n\in\mathbb{N}_0}$ is an orthogonal polynomial sequence. With the help of the preceding Theorem 2.11, we see that this strong property of the real polynomial family is not necessary.

**Theorem 2.12:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family, $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ a $P_n$-positive definite function and $\hat{\varphi}$ a representing measure of $\varphi$. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_mP_n$, i.e.*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m,n \in \mathbb{N}_0. \tag{2.1.2}$$

*Then $\varphi$ is bounded if and only if $\operatorname{supp}\hat{\varphi} \subseteq D_s^\alpha$. In that case, $\varphi$ is bounded by $\alpha\varphi(0)$.*

*Proof.* Suppose $\varphi$ is bounded. Since the absolute sums of the linearization coefficients of $P_m P_n$ are uniformly bounded by $\alpha$ for all $n \in \mathbb{N}$ and $\alpha \geq 1$, we have

$$\operatorname{supp} \hat{\varphi} \subseteq \bigcap_{n \in \mathbb{N}} P_n^{-1}([-\alpha; \alpha]) \cap \mathbb{R} = D^\alpha \cap \mathbb{R} = D_s^\alpha$$

by Theorem 2.11. The converse direction is obviously true. $\square$

This theorem gives us two corollaries which possess analogs in the hypergroup case. In the second corollary, we return to signed polynomial hypergroups.

**Corollary 2.13:** *Let $\left(P_n\right)_{n \in \mathbb{N}_0}$ be a real polynomial family. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_m P_n$, i.e.*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m,n \in \mathbb{N}_0.$$

*Then for all $x \in \mathbb{R}$ the sequence $\big(P_n(x)\big)_{n \in \mathbb{N}_0}$ is either bounded by $\alpha$ or unbounded, i.e. $D_s^\alpha = D_s^\infty$.*

*Proof.* For $x \in \mathbb{R}$ define a $P_n$-positive definite function $\varphi_x \colon \mathbb{N}_0 \to \mathbb{R},\ n \mapsto P_n(x)$. By Theorem 2.12, $\varphi_x$ is either bounded by $\alpha\varphi_x(0) = \alpha$, if $\operatorname{supp} \hat{\varphi}_x = \{x\} \subseteq D_s^\alpha$, or unbounded, if $\operatorname{supp} \hat{\varphi}_x = \{x\} \not\subseteq D_s^\alpha$. $\square$

**Corollary 2.14:** *Let $\left(P_n\right)_{n \in \mathbb{N}_0}$ be an orthogonal polynomial family with orthogonalizing measure $\nu \in \mathcal{M}^+(\mathbb{R})$. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_m P_n$, i.e.*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m,n \in \mathbb{N}_0.$$

*Then the support of $\nu$ is contained in $D_s^\alpha$, i.e.*

$$\operatorname{supp} \nu \subseteq D_s^\alpha = D_s^\infty.$$

*In particular, the support of $\nu$ is compact.*

*Proof.* Consider the $P_n$-positive definite function $\varphi\colon \mathbb{N}_0 \to \mathbb{R},\ n \mapsto \delta_{n0}\nu(\mathbb{R})$ which has $\nu$ as a representing measure. Since $\varphi$ is bounded,

$$\operatorname{supp}\hat{\varphi} = \operatorname{supp}\nu \subseteq D_s^\alpha = D_s^\infty$$

by Theorem 2.12 and Corollary 2.13. As $D_s^\alpha$ is compact, $\operatorname{supp}\nu$ is compact, too. $\square$

The main difference between polynomial hypergroups and signed polynomial hypergroups is the nonnegativity of the linearization coefficients. Since this property also leads to a Banach algebra structure, attention has been paid to this question. A number of criteria have been developed which guarantee $g(m, n; k) \geq 0$ for all $m, n, k \in \mathbb{N}_0$, cf. [Szw95]. Using the tools developed in this section, we will present a case in which nonnegativity of the linearization coefficients is impossible.

**Corollary 2.15:** *Let $\left(R_n\right)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence with respect to $\nu \in \mathcal{M}^+(\mathbb{R})$, where $\operatorname{supp}\nu = [-1; 1]$, and $R_n(1) = 1$ for all $n \in \mathbb{N}_0$. If $\ell_N(R_N) > 2^{N-1}$ for some $N \in \mathbb{N}$, then $\left(R_n\right)_{n\in\mathbb{N}_0}$ does not admit nonnegative linearization.*

*Proof.* If $g(m, n; k) \geq 0$ for all $m, n, k \in \mathbb{N}_0$, then $\left(R_n\right)_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup and hence $\operatorname{supp}\nu \subseteq D_s^1 = D_s^\infty$ by Corollary 2.14. But this is a contradiction to Theorem 2.8, since

$$\begin{aligned} 2 = \lambda(\operatorname{supp}\nu) \leq \lambda(D_s^1) \leq \lambda\left(\{x \in \mathbb{R}\colon |R_N(x)| \leq 1\}\right) &\leq 4\sqrt[N]{\frac{1}{2\ell_N(R_N)}} \\ &< 4\sqrt[N]{\frac{1}{2^N}} = 2. \end{aligned}$$

In particular, $g(m, n; k) < 0$ for some choice of $m, n, k \in \mathbb{N}_0$ and hence $\left(R_n\right)_{n\in\mathbb{N}_0}$ does not admit nonnegative linearization. $\square$

If a real polynomial family induces a polynomial hypergroup, then $D^\infty = D^1$. This implies that all bounded characters of a polynomial hypergroup are bounded by 1. We are now going to see that this property is due to the uniform boundedness of the absolute sums of the linearization coefficients, i.e. $\sum\limits_{k=|m-n|}^{m+n} |g(m, n; k)| \leq \alpha$ for all $m, n \in \mathbb{N}_0$. Under this assumption, Theorem 2.12 yields $D^\infty \cap \mathbb{R} = D^\alpha \cap \mathbb{R}$, and as we will see $D^\infty = D^\alpha$ is also true. In the terminology of polynomial hypergroups this means that every bounded character is bounded by $\alpha$, if the real polynomial family satisfies (2.1.2).

**Proposition 2.16:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_m P_n$, i.e.*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m,n \in \mathbb{N}_0. \tag{2.1.3}$$

*Then $D^\infty = D^\alpha$.*

*Proof.* For $m \in \mathbb{N}$ define a linear operator on the Banach space $\ell^\infty(\mathbb{N}_0)$ by

$$\begin{aligned} A_m \colon \ell^\infty(\mathbb{N}_0) &\to \ell^\infty(\mathbb{N}_0) \\ \big(x(n)\big)_{n\in\mathbb{N}_0} &\mapsto \left(\sum_{k=0}^{m+n} g(m,n;k)x(k)\right)_{n\in\mathbb{N}_0}. \end{aligned}$$

For all $m \in \mathbb{N}$, $A_m$ is well-defined, bounded, and $\|A_m\| \leq \alpha$ due to (2.1.3). For $\lambda \in D^\infty$, the sequence $\big(P_n(\lambda)\big)_{n\in\mathbb{N}_0}$ is bounded and

$$A_m\big(P_n(\lambda)\big)_{n\in\mathbb{N}_0} = \big(P_m(\lambda)P_n(\lambda)\big)_{n\in\mathbb{N}_0}.$$

Hence $P_m(\lambda) \in \sigma_p(A_m)$ and thus $|P_m(\lambda)| \leq \alpha$ for all $m \in \mathbb{N}_0$. □

By the same proof technique one can show the following:

**Corollary 2.17:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_1 P_n$, i.e.*

$$\sum_{k=0}^{n+1} |g(1,n;k)| \leq \alpha \qquad \forall\, n \in \mathbb{N}_0.$$

*Then $D^\infty \subseteq P_1^{-1}(B_\alpha(0))$, where $B_\alpha(0) := \{z \in \mathbb{C} \colon |z| \leq \alpha\}$.*

## 2.2 Stieltjes' and Haviland's Modified Moment Problem

Beyond boundedness of the positive definite function and analysis of the real polynomial family, there is another possibility to guarantee the existence of a representing measure with restricted support. Thereto, we will treat Stieltjes' and Haviland's modified moment problem.

For this purpose, we define two functions associated to a a $P_n$-positive definite function $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ by

$$\begin{aligned} \varphi^{(1)}\colon \mathbb{N}_0 &\to \mathbb{C} & \varphi^{(2)}\colon \mathbb{N}_0 &\to \mathbb{C} \\ n &\mapsto \Phi_\varphi(xP_n), & n &\mapsto \Phi_\varphi(x^2P_n). \end{aligned}$$

Now we are able to solve Stieltjes' and Haviland's modified moment problem:

**Theorem 2.18:** *Let $\varphi$ be a positive definite function with respect to a real polynomial family $\left(P_n\right)_{n\in\mathbb{N}_0}$. Define a second real polynomial family by $Q_{2n}(x) := P_n(x^2)$ and $Q_{2n+1}(x) := xP_n(x^2)$ for all $n \in \mathbb{N}_0$. The following are equivalent:*

1. *There exists a representing measure $\mu$ of $\varphi$ whose support is contained in $[0;\infty)$, i.e.*
$$\operatorname{supp}\mu \subseteq [0;\infty).$$
2. *The function $\varphi^{(1)}$ is $P_n$-positive definite.*
3. *The function $\psi\colon \mathbb{N}_0 \to \mathbb{C},\ n \mapsto \begin{cases} \varphi\left(\frac{n}{2}\right) & n \text{ even,} \\ 0 & n \text{ odd,} \end{cases}$ is $Q_n$-positive definite.*

*Proof.* **1.⇒2.:** This is clear since $\varphi^{(1)}$ can be represented by the positive measure $xd\mu$.

**2.⇒3.:** For some $N \in \mathbb{N}$, choose $c_1, \dots, c_N \in \mathbb{C}$. For every odd polynomial $Q$ we have $\Phi_\psi(Q) = 0$ and thus

$$\begin{aligned} \sum_{i,j=0}^{N} c_i\overline{c_j}\Phi_\psi(Q_iQ_j) &= \sum_{\substack{i,j=0\\ i,j\in 2\mathbb{N}_0}}^{N} c_i\overline{c_j}\Phi_\psi(Q_iQ_j) + \sum_{\substack{i,j=0\\ i,j\in 2\mathbb{N}_0+1}}^{N} c_i\overline{c_j}\Phi_\psi(Q_iQ_j) \\ &= \sum_{i,j=0}^{\left[\frac{N}{2}\right]} c_{2i}\overline{c_{2j}}\Phi_\varphi(P_iP_j) + \sum_{i,j=0}^{\left[\frac{N-1}{2}\right]} c_{2i+1}\overline{c_{2j+1}}\Phi_\varphi(xP_iP_j) \geq 0. \end{aligned}$$

**3.⇒1.:** By Theorem 2.3, there exists a measure $\nu \in \mathcal{M}^+(\mathbb{R})$ such that for all $n \in \mathbb{N}_0$ holds

$$\psi(2n) = \varphi(n) = \int_{\mathbb{R}} Q_{2n}\, d\nu,$$

hence we have

$$\varphi(n) = \int_{[0;\infty)} P_n\, d\mu,$$

where $\mu$ arises by transformation of $\nu$ with the function $x \mapsto x^2$. □

We see that the solution of Stieltjes' modified moment problem is an extension of the solution of the classical Stieltjes' moment problem, compare [BCR84]. The same holds for Haviland's modified moment problem.

**Theorem 2.19:** *Let $\varphi$ be a positive definite function with respect to a real polynomial family $\left(P_n\right)_{n\in\mathbb{N}_0}$. The following are equivalent:*

1. *The support of the unique representing measure $\mu$ of $\varphi$ is contained in $[-1;1]$, i.e.*

$$\operatorname{supp}\mu \subseteq [-1;1].$$

2. *The function $\varphi - \varphi^{(2)}$ is $P_n$-positive definite.*

3. *The functions $\varphi - \varphi^{(1)}$ and $\varphi + \varphi^{(1)}$ are $P_n$-positive definite.*

A weaker version of this theorem has already been shown in [Las95]. In this paper, the author shows that Haviland's modified moment problem has a solution if and only if (2) and (3) are satisfied.

*Proof.* **2.⇒1.:** Suppose there exists a representing measure $\mu$ of $\varphi$ with $\operatorname{supp}\mu \not\subseteq [-1;1]$. Then there exists an interval $[a;b]$ with $[a;b] \cap [-1-\varepsilon; 1+\varepsilon] = \varnothing$ for some $\varepsilon > 0$ and $\mu([a;b]) > 0$. Abbreviate

$$C_0 := \int_{[-1;1]} 1 - x^2\, d\mu \qquad \text{and} \qquad C_1 := \int_{[a;b]} x^2 - 1\, d\mu > 0.$$

For every $n \in \mathbb{N}_0$ holds

$$\begin{aligned}
\Phi_{\varphi-\varphi^{(2)}}(x^{2n}) &= \int_{\mathbb{R}} x^{2n}(1-x^2)\,d\mu \\
&= \int_{[-1;1]} x^{2n}(1-x^2)\,d\mu - \int_{\mathbb{R}\setminus[-1;1]} x^{2n}(x^2-1)\,d\mu \\
&\leq C_0 - \int_{[a;b]} x^{2n}(x^2-1)\,d\mu \\
&\leq C_0 - C_1(1+\varepsilon)^{2n}.
\end{aligned}$$

If $C_0 = 0$, then $\Phi_{\varphi-\varphi^{(2)}}(x^{2n}) \leq -C_1(1+\varepsilon)^{2n} < 0$ for all $n \in \mathbb{N}_0$. Otherwise choose some $N > \frac{\ln \frac{C_0}{C_1}}{2\ln(1+\varepsilon)}$. Then $\Phi_{\varphi-\varphi^{(2)}}(x^{2N}) < 0$, which is a contradiction since $x^{2N} \geq 0$ for all $x \in \mathbb{R}$.

**1.⇒2.:** Since $(1-x^2)d\mu$ is a positive measure on $[-1;1]$ representing $\varphi - \varphi^{(2)}$, this function is positive definite.

**3.⇒1.:** We derive the claim from Theorem 2.18. Since $\varphi + \varphi^{(1)}$ is $P_n$-positive definite, there exists a representing measure $\mu$ of $\varphi$ with $\operatorname{supp}\mu \subseteq [-1;\infty)$. Now we assume $\operatorname{supp}\mu \not\subseteq [-1;1]$. As in 2. ⇒ 1. there exists an interval $[a:b] \subset [0;\infty)$ with $[a;b] \cap [-1;1+\varepsilon] = \varnothing$ for some $\varepsilon > 0$ and $\mu([a;b]) > 0$. Abbreviate

$$D_0 := \int_{[-1;1]} 1 - x\,d\mu \qquad \text{and} \qquad D_1 := int_{[a;b]} x - 1\,d\mu > 0.$$

For every $n \in \mathbb{N}_0$ holds

$$\begin{aligned}
\Phi_{\varphi-\varphi^{(1)}}(x^{2n}) &= \int_{[} -1;\infty) x^{2n}(1-x)\,d\mu \\
&= \int_{[-1;1]} x^{2n}(1-x)\,d\mu - \int_{(1;\infty)} x^{2n}(x-1)\,d\mu \\
&\leq D_0 - \int_{[a;b]} x^{2n}(x-1)\,d\mu \\
&\leq D_0 - D_1(1+\varepsilon)^{2n}.
\end{aligned}$$

If $D_0 = 0$, then $\Phi_{\varphi-\varphi^{(1)}}(x^{2n}) \leq -D_1(1+\varepsilon)^{2n} < 0$ for all $n \in \mathbb{N}_0$. Otherwise choose some $N > \frac{\ln \frac{D_0}{D_1}}{\ln(1+\varepsilon)}$. Then $\Phi_{\varphi-\varphi^{(1)}}(x^{2N}) < 0$, which is a contradiction to the $P_n$-positive definiteness of $\varphi - \varphi^{(1)}$ since $x^{2N} \geq 0$ for all $x \in \mathbb{R}$.

**1.⇒3.:** $\varphi \pm \varphi^{(1)}$ can be represented by the positive measure $(1 \pm x)d\mu$, hence those functions are positive definite. □

Combining Theorem 2.19 with our results in Section 2.1 yields some interesting implications. Suppose $\big(P_n\big)_{n\in\mathbb{N}_0}$ is a real polynomial family with

$$\sum_{n=0}^{\infty} |g(n,1;k)| \leq \alpha$$

for all $n \in \mathbb{N}_0$ and some $\alpha > 0$. Choose a bounded $P_n$-positive definite function $\varphi\colon \mathbb{N}_0 \to \mathbb{R}$. By Theorem 2.11, we have $\operatorname{supp}\hat{\varphi} \subseteq P_1^{-1}([-\alpha;\alpha])$ and in particular

$$\operatorname{supp}\hat{\varphi} \subseteq [b_0 - \alpha a_0; b_0 + \alpha a_0],$$

where $a_0$ and $b_0$ are given by $P_1(x) = \frac{1}{a_0}(x - b_0)$. Then Theorem 2.19 gives us the $P_n$-positive definiteness of the functions $\varphi^{(1)} - (b_0 - \alpha a_0)\varphi$ and $-\varphi^{(1)} - (b_0 + \alpha a_0)\varphi$.

If the real polynomial family $\big(R_n\big)_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup with $D_s^1 = [-1;1]$, then a $R_n$-positive definite function $\varphi$ is bounded if and only if $\varphi - \varphi^{(2)}$ and $\varphi \pm \varphi^{(1)}$ are $R_n$-positive definite functions.

## 2.3 Composition of the Support of the Representing Measure

The following results are inspired by the work of R. Doss, cf. [Dos67, Dos68, Dos71]. For a compact set $B \subset \mathbb{R}$, we denote by $\|.\|_B$ the supremum norm over $B$, i.e.

$$\|f\|_B := \sup_{x\in B} |f(x)| \qquad \forall\, f \in C(B,\mathbb{C}).$$

After analyzing size and location of the support of representing measures, we are now interested in the composition of the support. Since positivity is not essential for these results, we generalize Theorem 2.3 to all sequences $\varphi \in \ell(\mathbb{N}_0)$ with continuously extendable $\Phi_\varphi$ – which is equivalent to the boundedness of the support of $\hat{\varphi}$ – and present a Bochner-Schönberg-Eberlein-type theorem.

The following lemma is well known for the semigroup $(\mathbb{N}_0, +)$, cf. [ST63, p.103].

**Lemma 2.20:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a sequence of real polynomials with $\deg P_n = n$ and let $\varphi\colon \mathbb{N}_0 \to \mathbb{R}$. Suppose there exists a compact set $B \subseteq \mathbb{R}$ and a constant $A \geq 0$ such that $\Phi_\varphi$ is a bounded linear functional on $\mathbb{P}[\mathbb{R}] \subseteq C(B,\mathbb{R})$ with $\|\Phi_\varphi\|_B \leq A$, i.e.*

$$\sup_{x\in B}\left|\sum_{k=0}^{n} c_k P_k(x)\right| \leq 1 \quad \Rightarrow \quad \left|\sum_{k=0}^{n} c_k \varphi(k)\right| \leq A. \tag{2.3.1}$$

*Then there exist $P_n$-positive definite functions $\varphi^+, \varphi^-\colon \mathbb{N}_0 \to \mathbb{R}$ with $\varphi = \varphi^+ - \varphi^-$.*

*Proof.* The linear functional $\Phi_\varphi$ can be extended continuously to $C(B)$, we denote the extension by $\Phi_\varphi$, too. For $f \in C^+(B,\mathbb{R})$ – where $C^+(B,\mathbb{R})$ is defined as in Section 1.1 – define

$$\Phi_\varphi^+(f) := \sup\big\{\Phi_\varphi(h) : h \in C^+(B,\mathbb{R}),\ h \leq f\big\}.$$

The existence of the limit follows immediately by

$$|\Phi_\varphi(h)| \leq A\|h\|_B \leq A\|f\|_B \quad \Rightarrow \quad |\Phi_\varphi^+(f)| \leq A\|f\|_B. \tag{2.3.2}$$

It remains to show the linearity of $\Phi_\varphi^+$. Let $f, g \in C^+(B,\mathbb{R})$ and choose some $h, k \in C^+(B,\mathbb{R})$ with $h \leq f,\ k \leq g$. Then the linearity of $\Phi_\varphi$ yields

$$\Phi_\varphi(h) + \Phi_\varphi(k) = \Phi_\varphi(h + k) \leq \Phi_\varphi^+(f + g),$$

hence we can conclude

$$\Phi_\varphi^+(f) + \Phi_\varphi^+(g) \leq \Phi_\varphi^+(f + g).$$

In order to show the opposite order relation, choose $h \in C^+(B,\mathbb{R}),\ h \leq f + g$. Define for all $x \in B$

$$p(x) := \max\{h(x) - g(x), 0\} \qquad\qquad q(x) := \min\{h(x), g(x)\}.$$

Then $p \leq f,\ q \leq g,\ p + q = h$ and $p, q \in C^+(B,\mathbb{R})$. Now follows

$$\begin{aligned} \Phi_\varphi^+(f) + \Phi_\varphi^+(g) &\geq \Phi_\varphi(p) + \Phi_\varphi(q) = \Phi_\varphi(p + q) = \Phi_\varphi(h) \\ &\Rightarrow \Phi_\varphi^+(f) + \Phi_\varphi^+(g) \geq \Phi_\varphi^+(f + g). \end{aligned}$$

The homogeneity of $\Phi_\varphi^+$ is straightforward. $\Phi_\varphi^+$ can be extended in a unique way to a bounded linear functional $\Phi_\varphi^+ \colon C(B,\mathbb{R}) \to \mathbb{R}$. Now define

$$\varphi^+(n) := \Phi_\varphi^+(P_n), \qquad\qquad \varphi^-(n) := \Phi_\varphi^+(P_n) - \Phi_\varphi(P_n).$$

Since $\Phi_\varphi^+(P) \geq 0$ and $\Phi_\varphi^+(P) \geq \Phi_\varphi(P)$ for all $P \in \mathbb{P}[\mathbb{R}]$, $P\big|_{\mathbb{R}} \geq 0$, $\varphi^+, \varphi^-$ are positive definite by Theorem 2.3 and obviously $\varphi = \varphi^+ - \varphi^-$. $\square$

With help of this result, we can represent every function $\varphi \colon \mathbb{N}_0 \to \mathbb{C}$ which satisfies the boundedness condition (2.3.1).

**Theorem 2.21:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family, $B \subset \mathbb{R}$ a compact set and $\varphi \colon \mathbb{N}_0 \to \mathbb{C}$. There exists a unique complex measure $\mu$ whose support is contained in $B$ and*

$$\varphi(n) = \int\limits_B P_n \, d\mu, \qquad \forall\, n \in \mathbb{N}_0,$$

*if and only if there exists a constant $A \geq 0$ such that*

$$\sup_{x\in B} \left| \sum_{k=0}^{n} c_k P_k(x) \right| \leq 1 \quad \Rightarrow \quad \left| \sum_{k=0}^{n} c_k \varphi(k) \right| \leq A. \tag{2.3.3}$$

*Proof.* Suppose there exists a measure $\mu$ satisfying the assumption. Then for every $P = \sum_{k=0}^n c_k P_k \in \mathbb{P}[\mathbb{C}]$ with $\|P\|_B \leq 1$

$$\left| \sum_{k=0}^{n} c_k \varphi(k) \right| = \left| \int_B \sum_{k=0}^{n} c_k P_k \, d\mu \right| \leq \int_B |P| \, d|\mu| \leq |\mu|(B) =: A < \infty.$$

For the converse direction, suppose $\varphi$ satisfies (2.3.3) with $A \geq 0$ and define

$$\begin{aligned} \varphi_{\Re} \colon \mathbb{N}_0 &\to \mathbb{R} \\ n &\mapsto \Re\varphi(n) \end{aligned} \qquad\qquad \begin{aligned} \varphi_{\Im} \colon \mathbb{N}_0 &\to \mathbb{R} \\ n &\mapsto \Im\varphi(n). \end{aligned}$$

For $P = \sum_{k=0}^n c_k P_k \in \mathbb{P}[\mathbb{R}]$, $\|P\|_B \leq 1$,

$$\left| \sum_{k=0}^{n} c_k \varphi_{\Re,\Im}(k) \right| \leq \left| \sum_{k=0}^{n} c_k \varphi_{\Re}(k) + i c_k \varphi_{\Im}(k) \right| \leq A \tag{2.3.4}$$

by (2.3.3). Hence Lemma 2.20 can be applied and there exists a decomposition

$$\varphi = \varphi_{\Re}^{+} - \varphi_{\Re}^{-} + i\varphi_{\Im}^{+} - i\varphi_{\Im}^{-},$$

where $\varphi_{\Re}^{+}, \varphi_{\Re}^{-}, \varphi_{\Im}^{+}, \varphi_{\Im}^{-}$ are $P_n$-positive definite. Theorem 2.3 yields

$$\varphi(n) = \int_{\mathbb{R}} P_n \, d\mu = \int_{\mathbb{R}} P_n \, d(\mu_{\Re}^{+} - \mu_{\Re}^{-} + i\mu_{\Im}^{+} - i\mu_{\Im}^{-}).$$

It remains to show $\operatorname{supp}\mu \subseteq B$. Suppose there exists $x \in \operatorname{supp}\mu \setminus B$. Then there can be easily found a $f \in C(\operatorname{supp}\mu \cup B, \mathbb{C})$ with $\|f\|_B = 0$ and $f(x) = 1$, such that $\left|\int n \cdot f \, d\mu_{\Re}\right| \to \infty$ as $n \to \infty$, which contradicts (2.3.4).
The uniqueness of the representing complex measure is straightforward, since the support is bounded as $\operatorname{supp}\mu \subseteq B$. □

We point out that the Spectral Theorem for bounded self-adjoint operators on Hilbert spaces can be derived almost directly from Theorem 2.21. A similar approach has been used in [RR98, Thm. 2], where a more general version of the following theorem has been shown.

**Theorem 2.22:** *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ self-adjoint. Then there exists a spectral measure $E$ on $\sigma(A)$ such that*

$$A = \int_{\sigma(A)} t \, dE(t).$$

*In particular, for every $x, y \in \mathcal{H}$ there exists a measure $E_{x,y} \in \mathcal{M}_{\mathbb{C}}(\sigma(A))$ with total variation $\leq \|x\|\|y\|$ such that*

$$\langle Ax; y\rangle = \int_{\sigma(A)} t \, dE_{x,y}(t).$$

*Proof.* Choose $P_n \equiv x^n$ for $n \in \mathbb{N}_0$. Let $x, y \in \mathcal{H}$ and define $\varphi_{x,y} \colon \mathbb{N} \to \mathbb{C}$ by $\varphi_{x,y}(n) := \langle A^n x; y\rangle$. Then for every $P \in \mathbb{P}[\mathbb{C}]$ with $\|P\|_{\sigma(A)} \leq 1$ holds by Lemma 1.7 and the Cauchy-Schwartz inequality

$$\left|\Phi_{\varphi_{x,y}}(P)\right| = \langle P(A)x; y\rangle \leq \|P(A)\|\|x\|\|y\| \leq \|x\|\|y\|.$$

Hence, there exists a unique representing measure $E_{x,y} \in \mathcal{M}_{\mathbb{C}}(\sigma(A))$ of $\varphi_{x,y}$ by Theorem 2.21 since $\sigma(A) \subseteq \mathbb{R}$ is bounded. We define a mapping $\rho\colon B(\sigma(A)) \to$

$\mathcal{B}(\mathcal{H})$, where $B(\sigma(A))$ denotes the bounded Borel measurable functions on $\sigma(A)$, by

$$\langle \rho(f)x; y\rangle := \int f\, dE_{x,y} \qquad \forall\, x, y \in \mathcal{H}.$$

We will show that $\rho$ is a $*$-representation: Since all $f \in B(\sigma(A))$ are $E_{x,y}$-integrable for all $x, y \in \mathcal{H}$, $\rho$ is well-defined, and as the mapping $(x, y) \mapsto E_{x,y}$ is sesquilinear, the linearity of $\rho(f)$ for all $f \in B(\sigma(A))$ is obvious. The identity $E_{x,y} = \overline{E_{y,x}}$ gives us $\rho(\bar{f}) = \rho(f)^*$ for all $f \in B(\sigma(A))$. By the definition of $E_{x,y}$, we have $\rho(P) = P(A)$ for all polynomials $P \in \mathbb{P}[\mathbb{C}]$. In particular we have $\rho(PQ) = \rho(P)\rho(Q)$ for all polynomials $P, Q \in \mathbb{P}[\mathbb{C}]$. As $\mathbb{P}[\mathbb{C}]$ is dense in $L^1(\sigma(A), E_{x,y})$ for all $x, y \in \mathcal{H}$, the identity $\rho(fg) = \rho(f)\rho(g)$ holds for all $f, g \in B(\sigma(A))$, too.

We define $E\colon \mathcal{B} \to \mathcal{B}(\mathcal{H}),\ B \mapsto \rho(\chi_B)$. Exploiting the properties of $\rho$ it is straightforward to show that $E$ is a spectral measure. □

For a suitable choice of $\big(P_n\big)_{n\in\mathbb{N}_0}$, there exists a large class of functions which satisfy the requirements of Theorem 2.21. The following example shows that for an orthonormal polynomial sequence the $\ell^2$-sequences are such a class.

**Example 2.3:** Let $\big(R_n\big)_{n\in\mathbb{N}_0}$ be an orthonormal polynomial sequence with respect to $\nu \in \mathcal{M}^+(\mathbb{R})$, where $\nu$ has bounded support, and let $\varphi \in \ell^2(\mathbb{N}_0)$ be a real resp. complex square summable sequence. Then holds for all $N \in \mathbb{N}$ and $P \in \mathbb{P}_N[\mathbb{C}]$ by the Cauchy-Schwartz inequality

$$\begin{aligned}|\Phi_\varphi(P)| &= \left|\sum_{k=0}^{N} \varphi(k) \int_{\mathbb{R}} P \cdot R_k\, d\nu\right| \le \sqrt{\int_{\mathbb{R}} |P|^2\, d\nu} \cdot \sqrt{\sum_{k=0}^{N} |\varphi(k)|^2} \\ &\le \sqrt{\nu(\mathbb{R})} \cdot \|P\|_{\operatorname{supp}\nu} \cdot \|\varphi\|_2\end{aligned}$$

and hence there exists a signed resp. complex Borel measure $\mu$ with $\operatorname{supp}\mu \subseteq \operatorname{supp}\nu$ and

$$\varphi(n) = \int_{\mathbb{R}} R_n\, d\mu \qquad \forall\, n \in \mathbb{N}_0.$$

Actually, $\mu$ is given by

$$\mu(A) = \int_A \sum_{n=0}^{\infty} \varphi(n) R_n\, d\nu$$

for all Borel sets $A$. In particular, $\mu$ is absolutely continuous with respect to $\nu$, since $\sum_{n=0}^{\infty} \varphi(n) R_n \in L^2(\mathbb{R}, \nu) \subset L^1(\mathbb{R}, \nu)$.

Motivated by this example, we turn to the question under which conditions the representing measure is absolutely continuous or singular with respect to an arbitrary positive measure.

**Theorem 2.23:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family, $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ a function, and $\nu \in \mathcal{M}^+(\mathbb{R})$. There exists a complex measure $\mu \in \mathcal{M}_{\mathbb{C}}(\mathbb{R})$ with compact support and $\mu \perp \nu$ such that*

$$\varphi(n) = \int P_n \, d\mu, \qquad \forall\, n \in \mathbb{N}_0, \tag{2.3.5}$$

*if and only if there exists a constant $A \geq 0$ and a compact set $B$ such that $\varphi$ satisfies* (2.3.3) *and for every $\varepsilon > 0$ there exists a polynomial $p \in \mathbb{P}[\mathbb{C}]$ such that*

$$\|p\|_B \leq 1, \qquad \int_B |p| \, d\nu < \varepsilon, \qquad |\Phi_\varphi(p)| \geq A - \varepsilon.$$

*Proof.* Suppose $\varphi$ can be represented by a singular measure $\mu$ with bounded support. We assume w.l.o.g. that $\nu(\mathbb{R}) \leq 1$. Define $B := \operatorname{supp}\mu$ and $A := |\mu|(B)$. Hence (2.3.3) is satisfied. Now let $\varepsilon > 0$. There exists a $\mu$-integrable function $f$ with $|f(x)| = 1$ for all $x \in \mathbb{R}$ and $\int_B f \, d\mu = |\mu|(B)$, cp. Chapter 1. As $C(B)$ is dense in $L^1(B, \mu)$, there exists a continuous function $g$ with $\|g\|_B \leq 1$ and $\|f - g\|_1 < \varepsilon$.

$$\left|\int_B g \, d\mu\right| \geq \left|\int_B f \, d\mu\right| - \left|\int_B f - g \, d\mu\right| \geq A - \|f - g\|_1 > A - \varepsilon.$$

As $\mu \perp \nu$, there exists a Borel set $N_\nu \subseteq \mathbb{R}$ with $\mu_{|\mathbb{R}\setminus N_\nu} \equiv 0 \equiv \nu_{|N_\nu}$. Since $\mu$ is regular, there exists a compact set $K \subseteq B \cap N_\nu$ with

$$|\mu(K)| \geq |\mu|(B \cap N_\nu) - \varepsilon = A - \varepsilon.$$

By the regularity of $\nu$, there exists an open set $U \supset K$ with

$$\nu(U) \leq \nu(K) + \varepsilon \leq \nu(B \cap N_\nu) + \varepsilon = \varepsilon.$$

As $L := B \setminus V$ and $K$ are bounded and disjoint the Lemma of Urysohn gives us a continuous function $\eta\colon B \to [0;1]$ with $\eta_{|L} \equiv 0$ and $\eta_{|K} \equiv 1$. In particular there exists a complex polynomial $p$ such that

$$\|\eta g - p\|_B \leq \min\left\{\frac{\varepsilon}{A}; \varepsilon\right\}.$$

Altogether we have $\|p\|_B \leq 1 + \varepsilon$ and the inequalities

$$\int_B |p|\,d\nu = \int_L |p|\,d\nu + \int_{K\cap U} |p|\,d\nu \leq \varepsilon\nu(L) + \nu(U) \leq 2\varepsilon,$$
$$\left|\int_B p\,d\mu\right| \geq \left|\int_B g\,d\mu\right| - \int_B |g-p|\,d|\mu| \geq \left|\int_B g\,d\mu\right| - \int_K |\eta g - p|\,d|\mu|$$
$$\geq A - \varepsilon - |\mu|(K)\frac{\varepsilon}{A} = A - 2\varepsilon.$$

We now turn towards the converse direction. Theorem 2.21 gives us the existence of a representing measure $\mu$. It remains to show $\mu \perp \nu$.
Thereto let $\varepsilon > 0$. We have to show that there exists a set $B \subset \mathbb{R}$, such that $\nu(B) < \varepsilon$ and $|\mu|(B) \geq A - \varepsilon$. By assumption, there exists a compact set $E$, a positive number $A \geq 0$ and a polynomial $p \in \mathbb{P}[\mathbb{C}]$ with

$$\|p\|_E \leq 1, \qquad \int_E |p|\,d\nu \leq \frac{\varepsilon^2}{2A}, \qquad |\Phi_\varphi(p)| = \left|\int_E p\,d\mu\right| \geq A - \frac{\varepsilon}{2}.$$

Now for $B := \left\{x \in E\colon |p(x)| > \frac{\varepsilon}{2A}\right\}$ we obtain

$$\nu(B) = \int_B 1\,d\nu < \int_B \frac{2A}{\varepsilon}|p|\,d\nu \leq \frac{2A}{\varepsilon}\cdot\frac{\varepsilon^2}{2A} = \varepsilon,$$

and conclude

$$|\mu|(B) \geq \left|\int_B p\,d\mu\right| \geq \left|\int_E p\,d\mu\right| - \left|\int_{E\setminus B} p\,d\mu\right|$$
$$\geq A - \frac{\varepsilon}{2} - \frac{\varepsilon}{2A}\int_{E\setminus B} d|\mu| = A - \varepsilon. \qquad \square$$

**Theorem 2.24:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family, $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ and let $\nu \in \mathcal{M}^+(\mathbb{R})$. There exists a complex measure $\mu \ll \nu$, $\mu \in \mathcal{M}_\mathbb{C}(\mathbb{R})$ with compact support and*

$$\varphi(n) = \int P_n\,d\mu, \qquad \forall\, n \in \mathbb{N}_0, \tag{2.3.6}$$

*if and only if there exists a compact set $B \subset \operatorname{supp}\nu$ such that for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that for any polynomial $p \in \mathbb{P}[\mathbb{C}]$ holds*

$$\|p\|_B \leq 1, \int_B |p|d\nu < \delta \;\Rightarrow\; |\Phi_\varphi(p)| < \varepsilon. \tag{2.3.7}$$

*Proof.* Suppose we have a measure $\mu \ll \nu$ satisfying (2.3.6). Define $B := \operatorname{supp}\mu$ and let $\varepsilon > 0$. Since $\mu$ is absolutely continuous, there exists a function $m \in L^1(B,\nu)$ with $d\mu = m\,d\nu$, see Theorem 1.1. There exists a continuous function $f \in C(B)$ with

$$\int_B |m-f|d\nu < \frac{\varepsilon}{2}.$$

Choose $\delta := \frac{\varepsilon}{2\|f\|_B}$. Now let $p \in \mathbb{P}[\mathbb{C}]$ with $\|p\|_B \leq 1$, $\int_B |p|d\nu < \delta$. Then

$$\begin{aligned}|\Phi_\varphi(p)| = \left|\int_B pm\,d\nu\right| &\leq \int_B \left|p\big(m-f\big)\right| d\nu + \int_B |pf|\,d\nu \\ &< \frac{\varepsilon}{2} + \|f\|_B \cdot \int_B |p|d\nu \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

For the converse direction, we have to show the existence of $\mu$ first. Let $\varepsilon > 0$. By assumption, there exists a $\delta > 0$ such that (2.3.7) holds. Choose some $\alpha > 0$, such that $\alpha \leq 1$ and $\alpha\nu(B) < \delta$. Then for $p = \sum c_k P_k \in \mathbb{P}[\mathbb{C}]$ with $\|p\|_B \leq 1$ holds

$$\int_B |\alpha p|d\nu \leq \alpha\nu(B) < \delta.$$

Hence by assumption

$$\left|\sum_{k=0}^n c_k\varphi(k)\right| < \frac{\varepsilon}{\alpha}$$

and Theorem 2.21 – with $A = \frac{\varepsilon}{\alpha}$ – yields the desired measure $\mu$.
It remains to show that $\mu$ is absolutely continuous with respect to $\nu$. By the Radon-Nikodym theorem, cp. Theorem 1.1, there exists a decomposition $d\mu = d\mu_s + f d\nu$, where $\mu_s \perp \nu$ and $f \in L^1(B,\nu)$. Suppose $\mu_s \neq 0$ and define $A_s := |\mu_s|(\mathbb{R})$, $\varepsilon := \frac{A_s}{4}$ and

$$\tilde{\varphi}(n) := \int_B P_n f\,d\nu.$$

Hence there exists $\tilde{\delta} > 0$ such that for any $p \in \mathbb{P}[\mathbb{C}]$ holds

$$\|p\|_B \leq 1,\ \int_B |p|d\nu < \tilde{\delta} \ \Rightarrow\ |\Phi_{\tilde{\varphi}}(p)| < \varepsilon.$$

On the other hand, by Theorem 2.23, there is a $p$ such that $\|p\|_B \leq 1$ and

$$\int_B |p|\,d\nu \leq \min\{\delta;\tilde{\delta}\} \qquad \text{and} \qquad |\Phi_{\varphi-\tilde{\varphi}}(p)| > A_s - \varepsilon.$$

Summing up we get

$$|\Phi_\varphi(p)| \geq |\Phi_{\varphi-\tilde{\varphi}}(p)| - |\Phi_{\tilde{\varphi}}(p)| \geq A_s - \varepsilon - \varepsilon \geq 2\varepsilon,$$

which is a contradiction to (2.3.7). Hence $\mu_s = 0$ and thus $\mu \ll \nu$. $\square$

# 3 Examples of $P_n$-positive definite Functions

In the previous chapter, we characterized $P_n$-positive definite functions as the transformation of positive Borel measures and dealt with the support of these representing measures. Now, we want to give some examples of $P_n$-positive definite functions and their appearance.
We continue to assume $\big(P_n\big)_{n\in\mathbb{N}_0}$ to be a real polynomial family with linearization coefficients $g(m,n;k)$. Throughout this chapter, $\mathcal{H}$ will be a Hilbert space which is equipped with the scalar product $\langle\cdot;\cdot\rangle$. In time series analysis, we usually have $\mathcal{H} = L^2(\Omega,\mu)$, where $\mu$ is a probability measure.

## 3.1 $P_n$-stationary Sequences on Hilbert Spaces

In the group case, positive definite functions arise as covariance functions of weakly stationary time series. Over the past years, there has been some effort to extend the theory of time series on the group $(\mathbb{Z},+)$ to hypergroups and polynomial hypergroups, cf. [HL92, HL03, Hös98, LL89, Lei91]. For any sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ in $\mathcal{H}$, we call

$$\psi\colon \mathbb{N}_0\times\mathbb{N}_0 \to \mathbb{C}, \qquad (m,n)\mapsto \langle x_m;x_n\rangle$$

the *covariance function of* $\big(x_n\big)_{n\in\mathbb{N}_0}$.

We call $\big(x_n\big)_{n\in\mathbb{N}_0}$ $P_n$*-stationary*, if

$$\psi(m,n) = \mathcal{T}_n\big(\psi(k,0)\big)_{k\in\mathbb{N}_0}(m) \qquad \forall\, m,n\in\mathbb{N}_0.$$

In the following, we will formally abbreviate

$$x_{m*n} := \sum_{k=0}^{m+n} g(m,n;k)x_k.$$

This notation is motivated by the polynomial hypergroup case, where the convolution $\omega$ is often abbreviated in the same manner. By the definition of the translation operators $\mathcal{T}_n$ this is equivalent to

$$\psi(m,n) = \langle x_m; x_n \rangle = \sum_{k=0}^{m+n} g(m,n;k)\langle x_k; x_0\rangle = \langle x_{m*n}; x_0\rangle.$$

Hence, we define $\varphi(n) := \psi(n,0)$ for all $n \in \mathbb{N}_0$. For simplicity reasons, we also call $\varphi$ the *covariance function* of $\left(x_n\right)_{n\in\mathbb{N}_0}$, if this sequence is $P_n$-stationary. We abbreviate the subspace of $\mathcal{H}$ which is generated by the sequence $x = \left(x_n\right)_{n\in\mathbb{N}_0}$ by $\mathcal{H}(x)$, i.e.

$$\mathcal{H}(x) := \operatorname{span}\left\{x_n \colon n \in \mathbb{N}_0\right\}. \tag{3.1.1}$$

The following lemma provides a possibility to calculate examples of $P_n$-stationary sequences. As we will see in the following, all $P_n$-stationary sequences are of this form.

**Lemma 3.1:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family and $A \in \mathcal{L}(\mathcal{H})$ a self-adjoint operator. For any $x_0 \in \mathcal{H}$ with $x_0 \in \mathcal{D}(A^n)$ for all $n \in \mathbb{N}$, the sequence $\left(P_n(A)x_0\right)_{n\in\mathbb{N}_0}$ is $P_n$-stationary.*

*Proof.* For all $m,n \in \mathbb{N}_0$ the operator $P_n(A)$ is self-adjoint and one has

$$\begin{aligned}\langle P_m(A)x_0; P_n(A)x_0\rangle &= \langle (P_mP_n)(A)x_0; x_0\rangle \\ &= \sum_{k=0}^{m+n} g(m,n;k)\left\langle P_k(A)x_0; x_0\right\rangle.\end{aligned}$$

Hence, the sequence $\left(P_n(A)x_0\right)_{n\in\mathbb{N}_0}$ is $P_n$-stationary. □

The following theorem provides an analogon of the Herglotz theorem for weakly stationary processes, cf. [BD02]:

**Theorem 3.2:** *Let $\left(P_n\right)_{n\in\mathbb{N}_0}$ be a real polynomial family. A sequence $\varphi \in \ell(\mathbb{N}_0)$ is the covariance function of a $P_n$-stationary sequence if and only if $\varphi$ is positive* *definite with respect to* $\big(P_n\big)_{n\in\mathbb{N}_0}$. *In particular, there exists a self-adjoint operator* $A \in \mathcal{L}(\ell^2(\mathbb{N}_0))$ *with spectral measure* $E$,

$$\varphi(n) = \langle P_n(A)\delta_0; \delta_0\rangle \cdot \varphi(0) = \int_{\sigma(A)} P_n \, dE_{\delta_0,\delta_0} \cdot \varphi(0),$$

*and* $\operatorname{supp} E_{\delta_0,\delta_0} = \sigma(A)$, *where* $\delta_0 = (\delta_{n0})_{n\in\mathbb{N}_0}$.

*Proof.* Suppose $\varphi$ is $P_n$-positive definite. By Theorem 2.3 and Remark 2.6, there exists a measure $\mu \in \mathcal{M}^+(\mathbb{R})$ and a self-adjoint operator $A \in \mathcal{L}(\ell^2(\mathbb{N}_0))$, such that $\mu$ is a representing measure of $\varphi$ and

$$\langle P(A)\delta_0; \delta_0\rangle \cdot \varphi(0) = \int_{\mathbb{R}} P \, d\mu$$

for all $P \in \mathbb{P}[\mathbb{C}]$. Since $\delta_0$ is a cyclic vector for $A$ we have $\sigma(A) = \operatorname{supp}\mu$ by Theorem 1.8. Define $x_n := P_n(A)\delta_0 \cdot \sqrt{\varphi(0)}$. Then for all $m, n \in \mathbb{N}_0$

$$\langle x_n; x_0\rangle = \langle P_n(A)\delta_0; \delta_0\rangle \cdot \varphi(0) = \int_{\mathbb{R}} P_n \, d\mu = \varphi(n)$$

and

$$\langle x_m; x_n\rangle = \langle P_m(A)P_n(A)\delta_0; \delta_0\rangle \cdot \varphi(0) = \langle x_{m*n}; x_0\rangle.$$

For the opposite direction, suppose $\big(x_n\big)_{n\in\mathbb{N}_0}$ is a $P_n$-stationary sequence. By definition, $\varphi(n) = \langle x_n; x_0\rangle$. Let $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$. Then

$$\begin{aligned}\sum_{i,j=1}^{n} c_i\overline{c_j} c T_i\varphi(j) &= \sum_{i,j=1}^{n} c_i\overline{c_j}\langle x_i; x_j\rangle \\ &= \left\langle \sum_{i=1}^{n} c_i x_i; \sum_{j=1}^{n} c_j x_j \right\rangle \geq 0\end{aligned}$$

gives us the $P_n$-positive definiteness of $\varphi$. □

From this theorem follows a commutativity and associativity result which corresponds to the theory of polynomial hypergroups.

**Corollary 3.3:** *Any $P_n$-stationary sequence* $\big(x_n\big)_{n\in\mathbb{N}_0}$ *in* $\mathcal{H}$ *satisfies*

$$\langle x_{k*m}; x_n\rangle = \langle x_{m*k}; x_n\rangle = \langle x_m; x_{n*k}\rangle \qquad \forall\, k, m, n \in \mathbb{N}_0.$$

*Proof.* By Theorem 3.2, there exists a self-adjoint operator $A \in \mathcal{L}(\ell^2(\mathbb{N}_0))$ with $\langle P_n(A)\delta_0; \delta_0\rangle = \langle x_n; x_0\rangle$ for all $n \in \mathbb{N}_0$. It follows for $m, n, k \in \mathbb{N}_0$

$$\begin{aligned}\langle x_{m*k}; x_n\rangle &= \langle P_m(A)P_k(A)\delta_0; P_n(A)\delta_0\rangle \\ &= \langle P_k(A)P_m(A)\delta_0; P_n(A)\delta_0\rangle = \langle x_{k*m}; x_n\rangle\end{aligned}$$

and

$$\begin{aligned}\langle x_{m*k}; x_n\rangle &= \langle P_m(A)P_k(A)\delta_0; P_n(A)\delta_0\rangle \\ &= \langle P_m(A)\delta_0; P_n(A)P_k(A)\delta_0\rangle = \langle x_m; x_{n*k}\rangle.\end{aligned}$$

□

A theorem from Chapter 2 gives us a characterization of bounded $P_n$-stationary sequences, if we have certain restrictions on the real polynomial family:

**Proposition 3.4:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\big(x_n\big)_{n\in\mathbb{N}_0}$ a $P_n$-stationary sequence with covariance function $\varphi$. Suppose that there exists a uniform bound $\alpha \geq 1$ for the absolute sums of the linearization coefficients of $P_mP_n$, i.e.*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m, n \in \mathbb{N}_0.$$

*The sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ is bounded if and only if $\varphi$ is bounded. In this case, we have $\|x_n\| \leq \alpha\|x_0\|$ and $|\varphi(n)| \leq \alpha\varphi(0)$ for all $n \in \mathbb{N}_0$.*

*Proof.* By Theorem 3.2, $\varphi$ is a $P_n$-positive definite function and hence a representing measure $\hat{\varphi}$ exists. If $\operatorname{supp}\hat{\varphi} \subseteq D_s^\alpha$, then $\varphi$ is bounded by $\alpha\varphi(0)$ by Theorem 2.12 and

$$\begin{aligned}\|x_n\|^2 = \langle x_{n*n}; x_0\rangle &= \sum_{k=0}^{2n} g(n,n;k)\varphi(k) \\ &= \int_{D_s^\alpha} P_n^2\, d\hat{\varphi} \leq \alpha^2\|x_0\|^2.\end{aligned}$$

If $\operatorname{supp}\hat{\varphi} \not\subseteq D_s^\alpha$, then the covariance function $\varphi$ is unbounded by Theorem 2.12. Using the Cauchy-Schwartz inequality

$$\|x_n\|\|x_0\| \geq |\langle x_n; x_0\rangle| = |\varphi(n)|$$

and hence the $P_n$-stationary sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ is unbounded, too. □

### 3.1.1 Generator and spectral measure

We already know that for every self-adjoint $A \in \mathcal{L}(\mathcal{H})$ and $x_0 \in \bigcap_n \mathcal{D}(A^n)$ the sequence $\big(P_n(A)x_0\big)_{n\in\mathbb{N}_0}$ is $P_n$-stationary. The reverse would give us a self-adjoint operator $A \in \mathcal{L}(\mathcal{H})$ corresponding to a given $P_n$-stationary sequence and we could apply the Spectral Theorem. Hence, for a given $P_n$-stationary sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$, we call a self-adjoint operator $A \in \mathcal{L}\left(\overline{\mathcal{H}(x)}\right)$ – where $\mathcal{H}(x)$ is given by (3.1.1) – which satisfies $x_n = P_n(A)x_0$ for all $n \in \mathbb{N}$, *generator of* $\big(x_n\big)_{n\in\mathbb{N}_0}$. The following theorem gives a positive answer to the question, whether every $P_n$-stationary sequence possesses a generator.

**Theorem 3.5:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family. A sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ is $P_n$-stationary if and only if there exists a self-adjoint operator $A \in \mathcal{L}\left(\overline{\mathcal{H}(x)}\right)$ with $x_0 \in \mathcal{D}(A^n)$ – which implies $\mathcal{H}(x) \subseteq \mathcal{D}(A)$ – and $x_n = P_n(A)x_0$ for all $n \in \mathbb{N}_0$. In particular, $E_{x_0,x_0}$ is a representing measure of the covariance function of $\big(x_n\big)_{n\in\mathbb{N}_0}$ and* $\operatorname{supp} E_{x_0;x_0} = \sigma(A)$, *where $E$ denotes the spectral measure of $A$.*

*Proof.* Suppose that $\big(x_n\big)_{n\in\mathbb{N}_0}$ is $P_n$-stationary. By Theorem 3.2, there exists a self-adjoint linear operator $B \in \mathcal{L}(\ell^2(\mathbb{N}_0)$ with $\varphi(n) = \langle P_n(B)\delta_0; \delta_0\rangle \cdot \varphi(0)$. We abbreviate $y_n := P_n(B)\delta_0 \cdot \sqrt{\varphi(0)}$ and define a linear mapping by

$$\begin{aligned} \Psi\colon \overline{\operatorname{span}}\,\{y_n\colon n \in \mathbb{N}_0\} &\subseteq \ell^2(\mathbb{N}_0) \to \overline{\mathcal{H}(x)} \\ y_n &\mapsto x_n. \end{aligned} \tag{3.1.2}$$

$\Psi$ is well-defined since $\sum_{n=0}^{\infty} a_n y_n = \sum_{n=0}^{\infty} b_n y_n \in \ell^2(\mathbb{N}_0)$ implies for all $m \in \mathbb{N}_0$

$$\begin{aligned} \left\langle \sum_{n=0}^{\infty} a_n x_n; x_m \right\rangle &= \sum_{n=0}^{\infty} a_n \langle x_n; x_m\rangle = \sum_{n=0}^{\infty} a_n \langle x_{m*n}; x_0\rangle \\ &= \sum_{n=0}^{\infty} a_n \langle y_{m*n}; y_0\rangle = \sum_{n=0}^{\infty} b_n \langle y_{m*n}; y_0\rangle \\ &= \left\langle \sum_{n=0}^{\infty} b_n x_n; x_m \right\rangle. \end{aligned}$$

$\Psi$ is an isometric isomorphism, since it is surjective and for all $m, n \in \mathbb{N}_0$

$$\begin{aligned}\langle \Psi y_m; \Psi y_n \rangle &= \left\langle \Psi\big(P_m(B)\delta_0\sqrt{\varphi(0)}\big); \Psi\big(P_n(B)\delta_0\sqrt{\varphi(0)}\big) \right\rangle \\ &= \langle x_m; x_n \rangle = \mathcal{T}_m\varphi(n) \\ &= \left\langle P_m(B)\delta_0\sqrt{\varphi(0)}; P_n(B)\delta_0\sqrt{\varphi(0)} \right\rangle \\ &= \langle y_m; y_n \rangle.\end{aligned}$$

Hence, $A := \Psi B \Psi^{-1}$ is a self-adjoint linear operator on $\overline{\mathcal{H}(x)}$ with $\mathcal{D}(A) = \Psi\big(\mathcal{D}(B)\big)$. Due to $A(\mathcal{H}(x)) \subseteq \mathcal{H}(x)$ one has $\mathcal{H}(x) \subseteq \mathcal{D}(A^n)$.

It remains to show $x_n = P_n(A)x_0$ for all $n \in \mathbb{N}_0$ which is an immediate consequence of the equality

$$\begin{aligned}P_n(A)x_0 &= \Psi P_n(B)\Psi^{-1}x_0 = \Psi P_n(B)\delta_0 \cdot \sqrt{\varphi(0)} \\ &= \Psi y_n = x_n.\end{aligned}$$

The backward direction follows from

$$\langle x_m; x_n \rangle = \langle P_m(A)P_n(A)x_0; x_0 \rangle = \langle x_{m*n}; x_0 \rangle \qquad \forall\, m, n \in \mathbb{N}_0. \qquad \square$$

If $A$ is bounded, then the uniqueness of the generator of a $P_n$-stationary sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ is straightforward, since the operator is uniquely determined by its actions on $\mathcal{H}(x)$. We call the measure $E_{x_0;x_0}$ *spectral measure of* $\big(x_n\big)_{n\in\mathbb{N}_0}$. As in classical time series analysis, it is a representing measure of the covariance function.

Our results from Chapter 2 yield a boundedness condition on the generator of a $P_n$-stationary sequence.

**Proposition 3.6:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\big(x_n\big)_{n\in\mathbb{N}_0}$ be a $P_n$-stationary sequence in $\mathcal{H}$ with bounded covariance function. Suppose there exists a constant $\alpha > 0$ such that the linearization coefficients of the polynomial sequence $\big(P_n\big)_{n\in\mathbb{N}_0}$ satisfy*

$$\sum_{k=0}^{n+1} |g(n,1;k)| \leq \alpha \qquad \forall\, n \in \mathbb{N}_0.$$

*Then the generator $A$ of $\big(x_n\big)_{n\in\mathbb{N}_0}$ is bounded, i.e. $A \in \mathcal{B}\left(\overline{\mathcal{H}(x)}\right)$ and*

$$\|A\| \leq \max_{x\in[-\alpha;\alpha]} \left|P_1^{-1}(x)\right|.$$

*Proof.* Let $\varphi\colon \mathbb{N}_0 \to \mathbb{R}$ denote the covariance function of $\big(x_n\big)_{n\in\mathbb{N}_0}$. Referring to Theorem 3.5, there exists a self-adjoint $A \in \mathcal{L}(\overline{\mathcal{H}(x)})$ with

$$\varphi(n) = \langle P_n(A)x_0; x_0\rangle \cdot \varphi(0) \qquad \forall\, n \in \mathbb{N}_0,$$

and $\varphi$ is a $P_n$-positive definite function. Since $x_0$ is a cyclic vector we have the identity $\sigma(A) = \operatorname{supp} E_{x_0,x_0}$ by Theorem 1.8. By Theorem 2.11, $\sigma(A)$ is contained in $P_1^{-1}([-\alpha;\alpha])$. Since $A$ is self-adjoint, spectral radius and operator norm coincide, hence the claim is shown. □

The boundedness of its generator does not necessarily imply the boundedness of a $P_n$-stationary sequence. Yet, we can make statements on the boundedness of $P_n$-stationary sequences if we know the spectrum of the generator.

**Proposition 3.7:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be a real polynomial family and $\big(x_n\big)_{n\in\mathbb{N}_0}$ a $P_n$-stationary sequence in $\mathcal{H}$ with generator $A \in \mathcal{B}(\mathcal{H})$. If the spectrum of $A$ is contained in $U_s^\infty$, i.e. $\sigma(A) \subset U_s^\infty$, then $\big(x_n\big)_{n\in\mathbb{N}_0}$ is bounded.*

*Proof.* As $\sigma(A)$ is the spectrum of a bounded operator, it is compact. By Proposition 2.7 (ii), there exists a constant $\alpha \geq 1$ such that $\sigma(A) \subseteq D_s^\alpha$. This implies the inequality

$$\|x_n\|^2 = \int_{\sigma(A)} P_n^2 \, d\mu \leq \alpha^2 \|x_0\|^2,$$

where $\mu$ denotes the spectral measure of $\big(x_n\big)_{n\in\mathbb{N}_0}$. Hence the sequence $\big(x_n\big)_{n\in\mathbb{N}_0}$ is bounded by $\alpha\|x_0\|$. □

### 3.1.2 Imaginary Part

In this subsection, we analyze the correspondence between $P_n$-stationary sequences and weakly stationary sequences in the sense of time series analysis. A sequence $\big(z_n\big)_{n\in\mathbb{Z}}$ in $\mathcal{H}$ is called *weakly stationary*, if

$$\langle z_m; z_n\rangle = \langle z_{m-n}; z_0\rangle \qquad \forall\, m, n \in \mathbb{Z}.$$

Since a weakly stationary sequence corresponds to the trigonometric polynomial sequence $\left(e^{int}\right)_{n\in\mathbb{Z}} = \left(\cos(nt) + i\sin(nt)\right)_{n\in\mathbb{Z}}$, the idea to analyze $T_n$- resp. $U_n$-stationary sequences is self-evident. By $T_n$ resp. $U_n$ we denote the Chebyshev polynomials of first resp. second kind. They are given by the recurrence relations

$$\begin{aligned} xT_n(x) &= \frac{1}{2}T_{n+1}(x) + \frac{1}{2}T_{n-1}(x), \\ xU_n(x) &= \frac{n+2}{2n+2}U_{n+1}(x) + \frac{n}{2n+2}U_{n-1}(x), \end{aligned}$$

for $n \in \mathbb{N}$, with initial conditions $T_0, U_0 \equiv 1,\ T_1(x), U_1(x) = x$. They also possess a well-known representation in terms of trigonometric functions:

$$\begin{aligned} T_n(\cos\theta) &= \cos(n\theta), \\ U_n(\cos\theta) &= \frac{\sin\left((n+1)\theta\right)}{(n+1)\sin\theta}. \end{aligned}$$

Note that $T_n$ and $U_n$ induce a polynomial hypergroup on $\mathbb{N}_0$, since for all $m, n \in \mathbb{N}_0$, $m \leq n$,

$$\begin{aligned} T_m(x)T_n(x) &= \frac{1}{2}T_{m+n}(x) + \frac{1}{2}T_{|m-n|}(x), \\ U_m(x)U_n(x) &= \sum_{j=0}^{m} \frac{m+n+1-2j}{(n+1)(m+1)}U_{n+m-2j}(x). \end{aligned}$$

Let $\left(x_n\right)_{n\in\mathbb{N}_0}$ be a $T_n$-stationary sequence in $\mathcal{H}$. In the following, we say that $\left(x_n\right)_{n\in\mathbb{N}_0}$ *allows an imaginary part*, if there exists a $U_n$-stationary sequence $\left(y_n\right)_{n\in\mathbb{N}_0}$ in $\mathcal{H}$, such that $\left(x_{|n|} + iny_{|n|-1}\right)_{n\in\mathbb{Z}}$ is weakly stationary.

**Proposition 3.8:** *Let $\left(x_n\right)_{n\in\mathbb{N}_0}$ be a $T_n$-stationary sequence in $\mathcal{H}$. $\left(x_n\right)_{n\in\mathbb{N}_0}$ allows an imaginary part in $\overline{\mathcal{H}(x)}$ if and only if the covariance function $\varphi$ is bounded. In this case, the imaginary part is given by*

$$y_n = U_n(A)B(x_0) \tag{3.1.3}$$

*for all $n \in \mathbb{N}_0$, where $A \in \mathcal{L}(\overline{\mathcal{H}(x)})$ denotes the unique bounded and self-adjoint generator of $\left(x_n\right)_{n\in\mathbb{N}_0}$ and $B$ is a self-adjoint solution of $A^2 + B^2 = \mathrm{id}$, such that $A$ and $B$ commute.*

*Proof.* Let $A$ be the generator of $\big(x_n\big)_{n\in\mathbb{N}_0}$. $A$ is bounded by Proposition 3.6, since $T_1(x) = x$ and $\sum |g(n,1;k)| = 1$ for all $n \in \mathbb{N}$. The uniqueness of $A$ is evident. Let $E$ be the spectral measure of $A$ which is given by the Spectral Theorem, cf. Theorem 1.5. Since $\|A\| \leq 1$, we have $\sigma(A) = \operatorname{supp} E \subseteq [-1;1]$. Hence

$$B := \int_{\sigma(A)} \sqrt{1-x^2}dE(x)$$

is a bounded self-adjoint operator. For $n \in \mathbb{N}_0$, define

$$y_n := U_n(A)Bx_0.$$

Then, setting $y_{-1} := 0,\ U_{-1} \equiv 0$, it holds for $m, n \in \mathbb{Z}$

$$\begin{aligned}
&\langle x_{|m|} + imy_{|m|-1}; x_{|n|} + iny_{|n|-1}\rangle \\
&\qquad = \int_{-1}^{1} \big(T_{|m|}(x) + imU_{|m|-1}(x)\sqrt{1-x^2}\big)\big(T_{|n|}(x) - inU_{|n|-1}(x)\sqrt{1-x^2}\big)dE_{x_0;x_0}(x) \\
&\qquad = \int_{-\pi}^{\pi} \big(\cos(m\theta) + i\sin(m\theta)\big)\big(\cos(n\theta) - i\sin(n\theta)\big)d\nu(\theta) \\
&\qquad = \int_{-\pi}^{\pi} e^{i(m-n)\theta}\, d\nu(\theta) = \langle x_{|m-n|} + i(m-n)y_{|m-n|-1}; x_0\rangle,
\end{aligned}$$

where $d\nu(\theta) = \sin(\theta)d\mu(\cos\theta)$. $\square$

Note that the imaginary part $\big(y_n\big)_{n\in\mathbb{N}_0}$ is uniquely determined by the choice of $B$. As long as we assume $y_n \in \overline{\mathcal{H}(x)}$ for all $n \in \mathbb{N}_0$, the only possible choices for $B$ are $\pm\sqrt{1-A^2}$.

## 3.2 Specific examples

While we were dealing with a rather abstract occurrence of $P_n$-positive definite functions, namely covariance functions, in the previous section, we will now discuss specific functions, e.g. $n \mapsto \delta_{n0}$ and $n \mapsto 1$. Since there is no general answer possible whether these functions are $P_n$-positive definite, we will handle some particular real polynomial families. Here, the focus is on polynomial sequences which are not orthogonal.

### 3.2.1 Plancherel measure

In the harmonic analysis of hypergroups, the Plancherel measure $\pi$ plays a central role. It is characterized by the identity

$$\int_K |f(x)|^2 \, dm(x) = \int_{\hat{K}} |\hat{f}(\alpha)|^2 \, d\pi(\alpha), \tag{3.2.1}$$

where $m$ is the Haar measure on the hypergroup $K$ with dual space $\hat{K}$ and $\hat{}$ : $L^1(K) \to C_0(\hat{K})$ denotes the Fourier transform. If $K$ is a polynomial hypergroup induced by the orthogonal polynomial sequence $\big(R_n\big)_{n\in\mathbb{N}_0}$, then equation (3.2.1) takes the shape

$$\sum_{n=0}^{\infty} |\varphi(n)|^2 h(n) = \int_{D^1} \left| \sum_{n=0}^{\infty} \varphi(n) R_n(x) h(n) \right| \, d\mu(x),$$

where $\mu$ is the orthogonalizing measure of the polynomial sequence, the Haar weights are given by $h(n)^{-1} := \int R_n^2 \, d\mu$ and

$$\varphi \in \ell^1(h, \mathbb{N}_0) := \left\{ (\varphi(n))_{n\in\mathbb{N}_0} : \sum_{n=0}^{\infty} |\varphi(n)| \, h(n) < \infty \right\}.$$

Therefore, the Plancherel measure of a polynomial hypergroup is the orthogonalizing measure $\mu$. Since $\mu$ is a representing measure of the – consequently $R_n$-positive definite – sequence $(\delta_{0n})_{n\in\mathbb{N}_0}$, we pose the question: Are there polynomial systems such that $(\delta_{0n})_{n\in\mathbb{N}_0}$ is $P_n$-positive definite, but which are not orthogonal? This question can be answered positively and we are now going to give two examples.

**Example 3.1:** Consider the real polynomial family $\big(P_n\big)_{n\in\mathbb{N}_0} := \big(x^n\big)_{n\in\mathbb{N}_0}$ given by the monomials. Then obviously

$$\int P_n \, d\delta_0 = \delta_{n0} \qquad \forall \, n \in \mathbb{N}_0$$

and hence $n \mapsto \delta_{n0}$ is a $x^n$-positive definite function.

Since this example is rather trivial, we want to give another example.

**Example 3.2:** The Bernoulli polynomials are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

They obey the difference equation

$$B_n(x+1) - B_n(x) = nx^{n-1}.$$

Calculation of the partial derivative $\frac{\partial}{\partial x}\frac{te^{xt}}{e^t-1}$ yields $B_n'(x) = nB_{n-1}(x)$. Hence

$$\int_0^1 B_n(x)\,dx = \frac{1}{n+1}\big(B_{n+1}(1) - B_{n+1}(0)\big) = 0$$

for all $n \in \mathbb{N}$. Altogether we have

$$\int_0^1 B_n(x)\,dx = \delta_{0n} \qquad\qquad \forall\, n \in \mathbb{N}_0,$$

and $n \mapsto \delta_{n0}$ is a $B_n$-positive definite function.

This second example is more interesting, since the support of the representing measure is of infinite cardinality, and hence Haar weights could be calculated. Still, the existence of a Haar measure is depending on the orthogonality of the real polynomial family. We also remark that the sequence $\big(\delta_{0n}\big)_{n\in\mathbb{N}_0}$ is not necessarily $P_n$-positive definite.

For arbitrary $n_0 \in \mathbb{N}$, the function $n \mapsto \delta_{nn_0}$ is never $P_n$-positive definite, since every $P_n$-positive definite function satisfies $\varphi(0) > 0$.

### 3.2.2 The unit sequence

Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence with orthogonalizing measure $\nu \in \mathcal{M}^+([-1;1])$. As in the case of polynomial hypergroups, we assume $P_n(1) = 1$. Hence

$$1 = \int_{[-1;1]} P_n\,d\delta_1$$

and the unit sequence $n \mapsto 1$ is $P_n$-positive definite. The point measure $\delta_1$ is absolutely continuous with respect to $\nu$ if and only if $\nu(\{1\}) > 0$. In this case, the Radon-Nikodym derivative of $\delta_1$ with respect to $\nu$ equals

$$0 \not\equiv \chi_{\{1\}} \cdot \frac{1}{\nu(\{1\})} \in L^1([-1;1],\nu).$$

Since $\chi_{\{1\}}^2 = \chi_{\{1\}}$, this function is in $L^2([-1;1],\nu)$, too, and we have the expansion

$$\begin{aligned}\chi_{\{1\}} &= \sum_{n=0}^{\infty} \left\langle \chi_{\{1\}}; \frac{P_n}{\|P_n\|} \right\rangle_{\nu} \cdot \frac{P_n}{\|P_n\|} = \sum_{n=0}^{\infty} \frac{1}{\|P_n\|} \cdot \int_{\{1\}} P_n \, d\nu \cdot \frac{P_n}{\|P_n\|} \\ &= \sum_{n=0}^{\infty} \frac{\nu(\{1\})}{\|P_n\|} \cdot \frac{P_n}{\|P_n\|}.\end{aligned}$$

Thus, we have the identity

$$\frac{d\delta_1}{d\nu} = \sum_{n=0}^{\infty} \frac{1}{\|P_n\|} \cdot \frac{P_n}{\|P_n\|},$$

which gives us $\left(\frac{1}{\|P_n\|}\right)_{n\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$ if $\nu(\{1\}) > 0$.

If on the other hand we have $\left(\frac{1}{\|P_n\|}\right)_{n\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$, we can show $\delta_1 \ll \nu$ with the help of Theorem 2.24. Therefore let $\varepsilon > 0$ and abbreviate $C := \sum_{n=0}^{\infty} \frac{1}{\|P_n\|}$ and $\delta := \frac{\varepsilon}{C}$. For any polynomial $p \in \mathbb{P}[\mathbb{C}]$ with $\int |p| \, d\nu < \delta$ and $\|p\|_{\operatorname{supp}\nu} \leq 1$ holds

$$p = \sum_{n=0}^{\infty} \langle p; \frac{P_n}{\|P_n\|} \rangle_{\nu} \cdot \frac{P_n}{\|P_n\|}.$$

Applying the Cauchy Schwartz inequality and Parseval's identity we obtain

$$\begin{aligned}|\Phi_1(p)| = |p(1)| &= \left| \sum_{n=0}^{\infty} \left\langle p; \frac{P_n}{\|P_n\|} \right\rangle_{\nu} \cdot \frac{P_n(1)}{\|P_n\|} \right| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{\|P_n\|^2} \cdot \sum_{n=0}^{\infty} \left| \left\langle p; \frac{P_n}{\|P_n\|} \right\rangle \right|^2 \\ &= C \cdot \|p\|_{\nu}^2 = C \cdot \int \underbrace{|p|^2}_{\leq |p|} \, d\nu < C \cdot \delta = \varepsilon.\end{aligned}$$

Hence the unique representing measure of $n \mapsto 1$, namely $\delta_1$, is absolutely continuous with respect to $\nu$. Altogether we have that $\left(\frac{1}{\|P_n\|}\right)_{n\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$ if and only if $\nu(\{1\}) > 0$.

If we assume $\left(P_n\right)_{n\in\mathbb{N}_0}$ to induce a polynomial hypergroup, then

$$0 < \|P_n\|^2 = \int_{[-1;1]} \sum_{k=0}^{2n} g(n,n;k) P_k \, d\nu = g(n,n;0) \leq 1.$$

Hence, $\left(\frac{1}{\|P_n\|}\right)_{n\in\mathbb{N}_0} \notin \ell^2(\mathbb{N}_0)$ and thus $\nu(\{1\}) = 0$. This implies for every polynomial hypergroup that if 1 is contained in the support of the Plancherel measure, then it is an accumulation point.

Again there are examples of real polynomial families $\left(P_n\right)_{n\in\mathbb{N}_0}$ which are not orthogonal, although the unit sequence is $P_n$-positive definite:

$$1 = \int x^n \, d\delta_1 \qquad \text{and} \qquad 1 = \int_0^1 n x^{n-1} \, dx = \int_0^1 B_n(x+1) - B_n(x) \, dx,$$

where $B_n$ denote the Bernoulli polynomials. As in the previous subsection the second example seems to be of more interest, since there is no point $x_0$ where the polynomials are all equal to 1.

# 4 Application to Linear Difference Equations

As in Chapter 3, we will always assume $\mathcal{H}$ to be a Hilbert space. In this chapter, we will analyze linear difference equations of the form

$$\sum_{k=0}^{n+1} a_{n,k} x(k) = Ax(n) \qquad \forall\, n \in \mathbb{N}, \tag{4.0.1}$$

with initial condition $x(0) = x_0$, where $x(n) \in \mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$ with $A^n x_0 \in \mathcal{D}(A)$, $a_{n,k} \in \mathbb{R}$ for all $n, k \in \mathbb{N}_0$. We assume $a_{n,n+1} \neq 0$ for all $n \in \mathbb{N}$. Equation 4.0.1 belongs to the class of *linear Volterra difference equation*, see [Ela05].

In the following, we attach a polynomial sequence $\left(P_n\right)_{n\in\mathbb{N}_0}$ to (4.0.1), which is recursively defined by

$$xP_n = \sum_{k=0}^{n+1} a_{n,k} P_k \tag{4.0.2}$$

with initial condition $P_0 \equiv 1$. We will formally abbreviate equation (4.0.1) by $x(n * 1) = Ax(n)$. This notation is inspired by the theory of hypergroups, where the convolution $\omega$ is often abbreviated in the same manner. By this approach, we have the advantage of dealing with a formally autonomous equation instead of a time-dependent difference equation.

## 4.1 Representation of the Solutions

The solution of the equation $x(n + 1) = Ax(n)$ is given by $x(n) = A^n x_0$. Taking into account that this equation is connected to the monomials $P_n(x) = x^n$, there exists an equivalent formula for the solution of the more general nonautonomous problem (4.0.1):

**Proposition 4.1:** *Let $A \in \mathcal{L}(\mathcal{H})$. The sequence $\big(x(n)\big)_{n\in\mathbb{N}_0}$ in $\mathcal{H}$ is a solution of $x(n*1) = Ax(n)$ with initial value $x(0) = x_0 \in \mathcal{H}$ if and only if $x(n) = P_n(A)x_0$ for all $n \in \mathbb{N}_0$. In particular, if $A^n x_0 \notin \mathcal{D}(A)$ for some $n \in \mathbb{N}_0$, then there does not exist a solution of the initial value problem.*

*Proof.* Define $x(n) := P_n(A)x_0$. By definition $x(0) = x_0$. For all $n \geq 1$ holds:

$$\sum_{k=0}^{n+1} a_{n,k}x(k) = \sum_{k=0}^{n+1} a_{n,k}P_k(A)x_0 = AP_n(A)x_0 = Ax(n).$$

Hence $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is a solution of $x(n*1) = Ax(n)$.

Now suppose that a solution of $x(n*1) = Ax(n)$ is given by $\big(x(n)\big)_{n\in\mathbb{N}_0}$. The identity $x(0) = P_0(A)x_0$ is obviously true and we have the solution identity

$$\begin{aligned} x(N+1) &= \frac{1}{a_{N,N+1}} \cdot \left( Ax(N) - \sum_{n=0}^{N} a_{N,n}x(n) \right) \\ &= \frac{1}{a_{N,N+1}} \cdot \left( P_1(A)P_N(A)x_0 - \sum_{n=0}^{N} a_{N,n}P_n(A)x_0 \right) \\ &= P_{N+1}(A)x_0. \end{aligned}$$ □

It follows immediately:

**Corollary 4.2:** *The solution of $x(n*1) = Ax(n)$ is a $P_n$-stationary sequence, if and only if $A\big|_{\mathcal{H}(x)}$ is a symmetric operator.*

**Corollary 4.3:** *Let $A \in \mathcal{L}(\mathcal{H})$ be a normal operator and $\big(x(n)\big)_{n\in\mathbb{N}_0}$ a solution of $x(n*1) = Ax(n)$. Then by the Spectral Theorem 1.5, there exists a set $B \subseteq \sigma(A)$ and a measure $\mu$ with*

$$\langle x(n); x(0)\rangle = \int_B P_n \, d\mu.$$

**Example 4.1:** Let $\mathcal{H} = \mathbb{C}^m$, $A \in \mathbb{C}^{m\times m}$ be normal. Consider the nonautonomous equation

$$\frac{n+2}{2n+2}x(n+1) + \frac{n}{2n+2}x(n-1) = Ax(n) \tag{4.1.1}$$

with initial conditions $x(0) = x_0 \in \mathbb{C}^m$, $x(1) = Ax(1)$. The polynomial sequence associated to this equation, are the Chebyshev polynomials $\big(U_n\big)_{n\in\mathbb{N}_0}$ of second kind. They satisfy

$$\lim_{n\to\infty} |U_n(z)| = \infty \qquad \text{for all } z \in \mathbb{C} \setminus [-1;1].$$

If $\sigma(A) \not\subseteq [-1;1]$, then there exists a solution $\big(x(n)\big)_{n\in\mathbb{N}_0}$ of (4.1.1), which is unbounded. In particular, for normal but non-Hermitian $A \in \mathbb{C}^{m\times m}$, there exists an unbounded solution of $x(n * 1) = Ax(n)$.

Up to now, we are restricted to normal matrices. We are interested in the representation of the solution of equations of type (4.0.1) with arbitrary $A \in \mathbb{C}^{n\times n}$. In her 2006 paper [Oro06], Á. Orosz analyzed the solutions of the following type of one dimensional equations of higher order:

$$Q(\mathcal{T}_1)f(n) = 0, \qquad \text{with} \quad Q \in \mathbb{P}[\mathbb{C}],\ f\colon \mathbb{N}_0 \to \mathbb{C}, \tag{4.1.2}$$

where $\mathcal{T}_1$ denotes the translation operator defined in Chapter 2. The author arrives at the following theorem:

**Theorem 4.4 ([Oro06]):** *Suppose $Q \in \mathbb{P}_n[\mathbb{C}]$ for some $n \in \mathbb{N}$. Let $\lambda_1, \dots, \lambda_r$ be the distinct complex roots of $Q$ and let $m_j$ be the multiplicity of $\lambda_j$. Then the solutions of* (4.1.2) *form a $n$-dimensional linear space. The solution space is spanned by $n \mapsto P_n^{(k)}(\lambda_j)$, where $1 \le j \le r$ and $0 \le k \le m_j - 1$.*

It is straightforward that the approach of [Oro06] is related to ours in the following way:

**Proposition 4.5:** *Suppose $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is a solution of the difference equation*

$$\mathcal{T}_1^N f(n) + a_{N-1}\mathcal{T}_1^{N-1} f(n) + \cdots + a_0 f(n) = 0. \tag{4.1.3}$$

*Then $y(n) := \big(x(n), \mathcal{T}_1 x(n), \dots, \mathcal{T}_1^{N-1}x(n)\big)^T \in \mathbb{C}^N$ is a a solution of*

$$y(n * 1) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{N-2} & -a_{N-1} \end{pmatrix} y(n). \tag{4.1.4}$$

An equation of type $x(n*1) = Ax(n)$ can be transferred into an equation of type (4.1.3) if and only if $A$ is similar to a companion matrix as in (4.1.4). Still, there is a possibility of using the explicit representation of A. Orosz for matrices, which are not similar to a companion matrix.

**Example 4.2:** Consider the equation

$$x(n*1) = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix}}_{=:A} x(n), \qquad x(0) = x_0 \in \mathbb{C}^3.$$

$A$ is not similar to a companion matrix, since the characteristic polynomial of $A$ does not equal its minimal polynomial, cf. [Bra64]. Its Jordan canonical form $J$ and its transition matrix $B$ are given by

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix},$$

hence $A = BJB^{-1}$. The Jordan blocks of $J$ are similar to companion matrices and hence $J$ is similar to a matrix $C$ where the transition matrix is given by $D$, i.e. $C = DJD^{-1}$:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying Theorem 4.4 we receive the general solution

$$x(n) = B \cdot D^{-1} \cdot \begin{pmatrix} a \cdot P_n(2) + b \cdot P_n'(2) \\ a \cdot P_1 P_n(2) + b \cdot (P_1 P_n)'(2) \\ c \cdot P_n(2) \end{pmatrix}.$$

This implies

$$P_n(A) = BD^{-1} \begin{pmatrix} P_n(2) - 2P_n'(2) & P_n'(2) & 0 \\ P_1 P_n(2) - 2(P_1 P_n)'(2) & (P_1 P_n)'(2) & 0 \\ 0 & 0 & P_2(2) \end{pmatrix} DB^{-1}$$

under the additional assumption, that $P_1(x) = x$.

In the special case $P_n = T_n$ all solutions of the equation $x(n*1) = Ax(n)$ are bounded if and only if $A$ is a diagonalizable matrix and $\sigma(A) \subseteq [-1;1]$.

## 4.2 Boundedness and Unboundedness

As Example 4.1 shows, results on real polynomial families and $P_n$-positive definite functions can help to show the existence of bounded or unbounded solutions. We are now going to concentrate on this question.

**Proposition 4.6:** *Let $A \in \mathcal{B}(\mathcal{H})$ be similar to a self-adjoint operator $D \in \mathcal{B}(\mathcal{H})$ and suppose $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is a solution of $x(n*1) = Ax(n)$ with initial condition $x(0) = x_0 \in \mathcal{H}$. Then there exists a complex bounded measure $\mu \in \mathcal{M}_\mathbb{C}(\sigma(A))$ with*

$$\langle x(n); x_0\rangle = \int_{\sigma(A)} P_n \, d\mu.$$

*Proof.* By assumption, there exists an invertible $S \in \mathcal{B}(\mathcal{H})$ with $S^{-1}AS = D$. Now choose $P \in \mathbb{P}[\mathbb{C}]$ with $\|P\|_{\sigma(A)} \leq 1$. Note that $\sigma(A) = \sigma(D) \subseteq \mathbb{R}$. We abbreviate $\varphi(n) := \langle x(n); x_0\rangle$ for all $n \in \mathbb{N}_0$. By Theorem 2.21, we have to show that $|\Phi_\varphi(P)| \leq C$ for a constant $C \geq 0$ which is independent of the choice of $P$. Since $\|P\|_{\sigma(D)} \leq 1$, it follows with Lemma 1.7 and the Cauchy-Schwartz inequality

$$\begin{aligned}|\Phi_\varphi(P)| &= \left|\sum_{k=0}^{n} c_k \langle SP_k(D)S^{-1}x_0; x_0\rangle\right| \\ &\leq \|S\|\|S^{-1}\|\|P(D)\|\|x_0\|^2 \\ &\leq \|S\|\|S^{-1}\|\|x_0\|^2 =: C.\end{aligned}$$

Hence Theorem 2.21 gives us the existence of the required measure. □

Now follows immediately with Proposition 2.7:

**Corollary 4.7:** *Let $A \in \mathcal{B}(\mathcal{H})$ be similar to a self-adjoint bounded operator. If $\sigma(A) \subseteq U_s^\infty$, then every solution $\big(x(n)\big)_{n\in\mathbb{N}_0}$ of $x(n*1) = Ax(n)$ is bounded.*

*Proof.* From the proof of Proposition 2.7 follows the existence of some $\alpha \geq 1$ with $\sigma(A) \subseteq D_s^\alpha$. Hence for all $n \in \mathbb{N}_0$

$$\|x(n)\|^2 = \langle x(n*n); x_0\rangle = \int_{\sigma(A)} P_n^2 \, d\mu \leq \alpha^2 \|x(0)\|^2.$$ □

Using results from Chapter 2, we can give more criteria for the existence of bounded and unbounded solutions.

**Theorem 4.8:** *Let $A \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator leaving $\mathcal{D}(A)$ invariant, i.e. $A\big(\mathcal{D}(A)\big) \subseteq \mathcal{D}(A)$ and suppose for some $\alpha > 0$ holds*

$$\sum_{k=0}^{n+1} |a_{n,k}| \leq \alpha |a_{0,1}| - |a_{0,0}| \qquad \forall\, n \in \mathbb{N}.$$

*If $\sigma(A) \not\subseteq P_1^{-1}([-\alpha;\alpha])$, then there exists an unbounded solution of $x(n*1) = Ax(n)$.*

*Proof.* Since the absolute sums of the linearization coefficients of $xP_n$ are bounded by $\alpha|a_{0,1}| - |a_{0,0}|$, the absolute sums of the linearization coefficients of $P_1P_n$ are bounded by $\alpha$. We abbreviate $\Sigma := \operatorname{supp}\hat{\varphi} \setminus P_1^{-1}([-\alpha;\alpha])$. Let $E$ be the spectral measure of $A$. Hence $E(\Sigma) \neq 0$ is a projection, cf. Theorem 1.5. Choose some $x_0 \in \mathcal{R}(E(\Sigma))$, such that $x_0 \neq 0$ and define $x(n) := P_n(A)x_0$. By Proposition 4.1, $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is a solution of $x(n*1) = Ax(n)$ with initial condition $x(0) = x_0$. The function $\varphi\colon \mathbb{N}_0 \to \mathbb{C}$ defined by $\varphi(n) := \langle x(n); x_0\rangle$ is $P_n$-positive definite since

$$\begin{aligned}\langle x(n); x_0\rangle &= \langle P_n(A)x_0; x_0\rangle = \langle P_n(A)E(\Sigma)x_0; x_0\rangle \\ &= \langle (P_n\chi_\Sigma)(A)x_0; x_0\rangle = \int_\Sigma P_n \, dE_{x_0;x_0}.\end{aligned}$$

In particular, $E_{x_0;x_0}$ is a representing measure of $\mu$ and $\operatorname{supp} E_{x_0;x_0} \not\subseteq P_1^{-1}([\alpha;\alpha])$. Hence $\varphi$ is unbounded by Theorem 2.11 and by the Cauchy-Schwartz inequality $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is an unbounded sequence in $\mathcal{H}$. □

Theorem 2.12 gives us the following dichotomy:

**Proposition 4.9:** *Let $\big(P_n\big)_{n\in\mathbb{N}_0}$ be the real polynomial family which arises from (4.0.2) and $A \in \mathcal{L}(\mathcal{H})$ a self-adjoint operator with spectral measure $E$ leaving $\mathcal{D}(A)$ invariant, i.e. $\mathcal{R}(A) \subseteq \mathcal{D}(A)$. Suppose there exists a constant $\alpha \geq 1$ such that*

$$\sum_{k=0}^{m+n} |g(m,n;k)| \leq \alpha \qquad \forall\, m,n \in \mathbb{N}_0.$$

*Then holds:*

1. *If* $x_0 \in \mathcal{R}\big(E(D^\alpha)\big)$, *then the solution of* $x(n*1) = Ax(n)$ *with initial condition* $x(0) = x_0$ *is bounded by* $\alpha\|x_0\|$.

2. *If* $x_0 \notin \mathcal{R}\big(E(D^\alpha)\big)$, *then the solution of* $x(n*1) = Ax(n)$ *with initial condition* $x(0) = x_0$ *is unbounded.*

*Proof.* Suppose $x(0) = x_0 \in \mathcal{R}(E(D^\alpha))$. Then

$$\begin{aligned}\|x(n)\|^2 &= \langle P_n^2(A)x_0; x_0\rangle = \langle (P_n^2\chi_{D^\alpha})(A)x_0; x_0\rangle \\ &= \int_{D^\alpha} P_n^2 \, dE_{x_0;x_0} \le \alpha^2\|x_0\|^2.\end{aligned}$$

If on the other hand $x(0) = x_0 \notin \mathcal{R}(E(D^\alpha))$, then follows as in the proof of Theorem 4.8

$$\langle P_n(A)x_0; x_0\rangle = \int_{\mathbb{R}} P_n(A) \, dE_{x_0,x_0}$$

with $\operatorname{supp} E_{x_0,x_0} \not\subseteq D^\alpha$. By Theorem 2.12 and the Cauchy-Schwartz inequality the sequence $\big(x(n)\big)_{n\in\mathbb{N}_0}$ is unbounded. □

**Example 4.3:** We consider the discrete wave equation, where the spatial Laplacian has been discretized using central difference quotients:

$$\ddot{u}_i(t) = u_{i-1}(t) - 2u_i(t) + u_{i+1}(t), \qquad i \in \mathbb{Z}, \tag{4.2.1}$$

with some suitable initial condition on $\big(u_i(0)\big)_{i\in\mathbb{Z}}$ and $\big(\dot{u}_i(0)\big)_{i\in\mathbb{Z}}$. Discretization in time via the central difference quotients gives us the equation

$$u(k+1) - 2u(k) + u(k-1) = h^2 Ju(k), \tag{4.2.2}$$

with initial condition $u(0) = u_0 \in \ell^2(\mathbb{Z})$, some step size $h > 0$ and the tridiagonal operator

$$J\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad \big(x(n)\big)_{n\in\mathbb{Z}} \mapsto \big(x(n-1) - 2x(n) + x(n+1)\big)_{n\in\mathbb{Z}}.$$

This operator is well-defined and $J \in \mathcal{B}(\ell^2(\mathbb{Z}))$ with $\sigma(J) = [-4; 0]$. The real polynomial family which is induced by (4.2.2) is a translation and dilation of the Chebyshev polynomials of first kind, namely $P_n(x) = T_n(\frac{x+2}{2})$. In particular, we have $D_s^\infty = D_s^1 = [-4; 0]$ and thus $\sigma(J) \subseteq D_s^1$. Hence for every initial condition $x_0 \in \ell^2(\mathbb{Z})$, the solution of (4.2.2) is bounded by $\|x_0\|$ as soon as $h \le 1$.

# Notation

| | |
|---|---|
| $\mathbb{N}$ | the natural numbers |
| $\mathbb{N}_0$ | $\mathbb{N} \cup \{0\}$ |
| $\mathbb{R}$ | the real numbers |
| $\mathbb{C}$ | the complex numbers |
| $\ell(\mathbb{N}_0)$ | space of all complex sequences on $\mathbb{N}_0$, cf. p.1 |
| $\ell^p(\mathbb{N}_0)$ | Banach space of all $p$-summable sequences in $\ell(\mathbb{N}_0)$, cf. p.1 |
| $\ell^\infty(\mathbb{N}_0)$ | space of all bounded sequences in $\ell(\mathbb{N}_0)$, cf. p.1 |
| $L^p(\Omega, \mu)$ | Banach space of all measurable, $p$-$\mu$-integrable functions on $\Omega \subseteq \mathbb{R}$, cf. p.4 |
| $L^\infty(\Omega, \mu)$ | Banach space of all measurable bounded functions on $\Omega \subseteq \mathbb{R}$, cf. p.4 |
| $\mathbb{P}[\mathbb{K}]$ | space of all polynomials with coefficients in the field $\mathbb{K}$ |
| $\mathbb{P}_n[\mathbb{K}]$ | space of all polynomials of exact degree $n \in \mathbb{N}_0$ with coefficients in the field $\mathbb{K}$ |
| $(\mathcal{H}, \langle \cdot ; \cdot \rangle)$ | Hilbert space with scalar product |
| $\mathcal{H}(x)$ | subspace of $\mathcal{H}$ generated by a sequence $x = \big(x_n\big)_{n \in \mathbb{N}_0}$, cf. p.50 |
| $\mathcal{L}(X)$ | densely defined linear operators on a space $X$, cf. p.6 |
| $\mathcal{B}(X)$ | bounded linear operators on a space $X$, cf. p.6 |
| $\mathcal{M}(\Omega)$ | the signed bounded Borel measures on $\Omega \subseteq \mathbb{R}$, cf. p.2 |
| $\mathcal{M}^+(\Omega)$ | cone of positive bounded Borel measures on $\Omega \subseteq \mathbb{R}$, cf. p.2 |
| $\mathcal{M}_{\mathbb{C}}(\Omega)$ | the complex Borel measures on $\Omega \subseteq \mathbb{C}$, cf. p.3 |
| $\mathcal{D}(A)$ | dense domain of a linear operator $A \in \mathcal{L}(\mathcal{H})$, cf. p.6 |
| $\mathcal{R}(A)$ | range of a linear operator $A \in \mathcal{L}(\mathcal{H})$, cf. p.6 |
| $D^\alpha$ | set where a real polynomial family is bounded by $\alpha > 0$, cf. p.24 |
| $D^\infty$ | set where a real polynomial family is bounded, cf. p.24 |
| $D^\alpha_s$ | $D^\alpha \cap \mathbb{R}$, cf. p.24 |
| $D^\infty_s$ | $D^\infty \cap \mathbb{R}$, cf. p.24 |
| $U^\alpha_s$ | interior of $D^\alpha_s$, cf. p.24 |
| $U^\infty_s$ | union of all $U^\alpha_s$, cf. p.24 |

| | |
|---|---|
| $P_n^{(\alpha,\beta)}$ | Jacobi polynomials depending on the parameters $\alpha, \beta > -1$, cf. p.10 |
| $T_n$ | Chebyshev polynomials of first kind, cf. p.11 |
| $U_n$ | Chebyshev polynomials of second kind, cf. p.56 |
| $g(m, n; k)$ | linearization coefficients of the product $P_m P_n$, cf. p.17 |
| $\Phi_\varphi$ | linear functional, cf. p.18 |
| $\mathcal{T}_n$ | translation operator, cf. p.17 |
| $\mu \perp \nu$ | $\mu$ and $\nu$ are mutually singular , cf. p.4 |
| $\mu \ll \nu$ | $\mu$ is absolutely continuous w.r.t. $\nu$, cf. p.4 |
| $\hat{\varphi}$ | representing measure of the sequence $\varphi$, cf. p.20 |

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