Technische Universität München
Fakultät für Physik

Wave Equations for Low Frequency Waves in Hot Magnetically Confined Plasmas.

Roman Kochergov

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Abstract

The investigation of wave propagation and instabilities in plasmas requires the knowledge of the constitutive relation, i.e. the relation between oscillating wave electric field and current in the plasma. The constitutive relation in a hot nonuniform plasma has an integral non-local form: the current at a given point depends on the fields at other points. Explicit expressions for the constitutive relation were previously obtained only for very special cases: restrictive approximations were mostly introduced at the early stages of derivation to simplify the form of the constitutive relation.

In the present work, the constitutive relation of a hot magnetised plasma is derived directly from the linearised Vlasov equation for the distribution function of plasma particles without making any assumption other than the validity of the drift approximation for the description of the particle orbits in the static magnetic configuration. In the integrals which define the oscillating plasma current, a change of integration variables from the position of particles to the position of the guiding centres of particles has allowed us to apply mathematical techniques similar to those of the uniform plasma limit to perform the expansion in harmonics of the particle cyclotron frequency. The constitutive relation is written in integral form as a convolution of Fourier components in each direction of plasma inhomogeneity. Since the general Fourier representation for the wave electromagnetic field is used, the wave equations obtained are valid in a wide range of frequencies and wavelengths.

The general constitutive relation has been specialised to obtain the wave equations describing low frequency drift and shear Alfvén waves, which play an important role in tokamak plasma stability, providing a mechanism for the generation of plasma microturbulence. These wave equations generalise those of the gyro-kinetic theory, based on a simpler gyro-kinetic equation derived by averaging of the Vlasov equation on the timescale of the fast particle gyro-motion. Exploiting the fact that these waves propagate mostly in the diamagnetic direction (the direction perpendicular to the directions of the equilibrium magnetic field $\vec{B}_0$ and to its gradient $\vec{\nabla} B_0$), the integro-differential equations have been simplified and put into a form which is essentially equivalent to the wave equations of the gyro-kinetic theory. Namely, the equations obtained are differential in the radial variable and take into account the finite Larmor radius effects to all orders along the diamagnetic direction, since the wavelengths can be of the order of the thermal ion Larmor radius.

The wave equations obtained in this way are in a form suitable for numerical solution with standard methods, for example with finite elements in the radial variable, and thus offer a good starting point for applications.

Contents

Chapter 1

Introduction

1.1 Overview

The effort to realise controlled thermonuclear fusion is underway since the end of 1950's. It was observed that the masses of nuclei are always smaller than the sum of the proton and neutron masses which constitute the nucleus. This mass difference corresponds to the nuclear binding energy and can be obtained from Einstein's energy-mass relation $E = \Delta mc^2$. For the different nuclei the binding energy per nucleon has a different value increasing from 1 MeV (^{2}H) to the maximal 8.6 MeV (^{56}Fe) and then again decreasing to 7.5 MeV (^{235}U)[1]. As a result, if two light nuclei fuse into one, the difference in binding energy will be released in the form of the kinetic energy of the reaction products, which then can be used to produce the electrical power. Fusion of a nucleus of deuterium (D) with a nucleus of tritium (T) gives an α-particle (^{4}He) and a neutron together with the usable energy up to 17.6 MeV per reaction. In macroscopic terms, just 1 kg of this fuel would release 10^8 kWh of energy and would provide the requirements of a 1 GW electrical power station for a day [2].

The main obstacle to realise fusion is the Coulomb repulsion of nuclei. In order to induce fusion of the nuclei it is necessary to overcome their mutual repulsion due to the positive charges. Inside stars the huge gravity helps to overcome this repulsion, but on the Earth we must look for another way to realise the fusion requirements. The cross-sections (probability of the reaction) for the nuclear fusion reactions are small at low energies, but increase with energy. Appreciable amounts of the fusion energy can be obtained only if nuclei with sufficiently high energy are made to react. These nuclei must remain in the reacting region and retain their energy for a sufficient time. In other words, the product of the confinement time and density of the reacting high energy particles must be sufficiently large to get an efficient thermonuclear reactor.

The most promising way to supply the required criterion is to heat the fuel to a high temperature while holding the particles in a closed volume. The thermal energy of the nuclei must be about 10 KeV, that implies a temperature around 10^8K. It is obvious that the fuel is fully ionised at such temperatures. The electrostatic charge of the nuclear ions is neutralised by the presence of an equal number of electrons, and the resulting gas is called a plasma. Although globally a plasma is electrically neutral, there may, however, exist transient local concentrations of charge or external potentials; due to free charges, a plasma can carry an electrical current. The basic thermodynamic parameters of the plasma are temperature T and density of the charged particles n, free oscillations of the plasma particles are characterised by the plasma frequency ω_p.

Since the high temperature of the fusion plasma precludes confinement by material walls, another method of confinement is needed. The most successful candidate for the thermonuclear reactor, the tokamak, uses a magnetic scheme of plasma confinement. The word tokamak originates from abbreviation of the Russian name of the device '**TO**roidalnaja **KA**mera s **MA**gnitnimi **K**atushkami' that means 'toroidal chamber with magnetic coils'. The tokamak works as a large transformer (see Fig.1.1). Fuel (usually a D-T mixture) is introduced in a toroidal vacuum chamber, which is used as the secondary winding of a transformer. The current ramping up in the primary winding of the transformer causes a power gas discharge in the secondary winding, which ionises the fuel and creates a high temperature plasma.

Confinement of the plasma inside the chamber is realised by means of a strong magnetic field created by external poloidal coils and by plasma current. The poloidal coils create the toroidal magnetic field, the plasma current creates the poloidal magnetic field, so that the total magnetic field in the tokamak is helical. The helical magnetic field lines of a tokamak form an infinite set of nested toroidal magnetic surfaces. It is well-known that in a strong magnetic field charged particles travel in a first approximation freely along the magnetic field lines, gyrating in small orbits with a cyclotron frequency Ω_c around them. More precisely, in the inhomogeneous magnetic field plasma particles experience additionally slow drifts perpendicular to the field lines; the poloidal magnetic field $\vec{B}_\theta$ produced by the current in the plasma itself directs these drifts along the tangent to the magnetic surface directions. Thus, charged particles remain near one of the magnetic surfaces until collisions with other particles bring them to an adjacent surface. The collisions of the particles in high temperature plasmas are rare enough, as a result, the tokamak plasma particles travel a distance millions of times the dimensions of the vessel before reaching the wall due to collisions. Transversal transport of heat and energy to the wall are greatly reduced in a such way, and plasma-wall interaction is highly restricted.

Heating of the tokamak plasma occur firstly by its own discharge current and then by different external sources. Scientists heat a plasma, for example, using the wave-particles resonances occurring when the frequency of the electromagnetic wave emitted by an external antenna coincides with a natural frequency of the motion of the particles in the quasiperiodic configuration. The electron cyclotron resonance heating (ECRH) and ion cyclotron resonance heating (ICRH) are examples of such method [2]. Alternatively

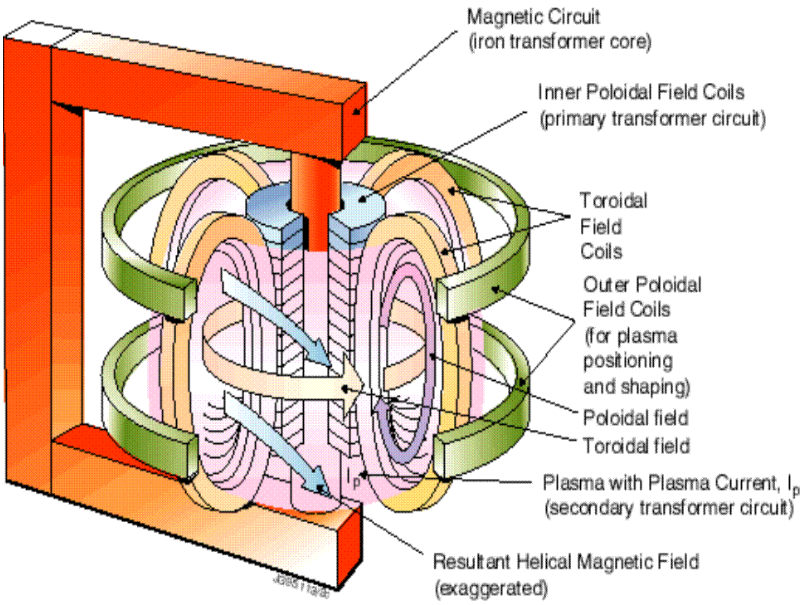

Figure 1.1: Tokamak configuration.

the plasma can be heated by means of a high energy beam of neutrals injected into the plasma, neutral beam injection (NBI) [2]. The neutrals are easily ionised in the plasma, they release energy due to Coulomb scattering, and the temperature of the plasma is increased.

The temperature, density and confinement time required for the efficient thermonuclear reactor have all been already obtained in tokamaks, but not simultaneously. This will happen, hopefully, in ITER[1] if it will be decided to proceed with construction of this device. ITER should for the first time give a positive balance between the fusion energy produced in the reactor and the energy spent to support the fusion conditions in the plasma. Physicists hope, that thermonuclear fusion will one day be able to replace other power sources if they will become exhausted or too expensive.

1.2 Role of instabilities in fusion plasmas

The properties of fully ionised magnetised plasma have been intensively investigated theoretically and experimentally for the last fifty years. The inhomogeneity of the tokamak plasma gives rise to a set of instabilities driven by free energy of gradients both in the density and temperature. These instabilities have a low frequency on the timescale of the plasma and cyclotron frequencies of the particles, ω_p and Ω_c, respectively. The largest-

[1] International Thermonuclear Experimental Reactor - the huge tokamak of the next generation.

scale in comparison with Larmor radius of plasma ions ρ_i low-frequency instabilities are called macroscopic instabilities, and are studied by means of the magnetohydrodynamics theory (MHD). Examples are the so called Mirnov oscillations, sawtooths and tearing modes instabilities [2], etc. Growth of these macroinstabilities can lead to disruption of the plasma discharge, that is collapsing of the plasma current in an uncontrollable way. Physicists have understood the behaviour of the macroinstabilities to avoid the most destructive of them, but small-scale gradient driven microinstabilities are still a serious obstacle to efficient of a plasma.

Gradient driven microinstabilities were first discovered in the end of 50's - beginning of 60's [5], [6], [7]. They are described by models which include non zero (finite) Larmor radius (FLR) and collisionless dissipation effects in a magnetised nonuniform plasma. The transport of heat observed in tokamak high-temperature plasma is well above that associated with classical Coulomb scattering of particles. It is believed that this 'anomalous' transport of heat is due to small scale turbulence in the plasma. Gradient driven microinstabilities provide a mechanism for the generation of this turbulence. The small scale turbulence is a subject of great attention of the scientists. Investigation of turbulence must include nonlinear effects; studying the characteristics of the linear modes, however, is useful to identify possible driving mechanisms and the conditions for this turbulence. Time and space scales of the microinstabilities are relevant to characterise the turbulent state and to estimate the plasma transport.

The most common gradient driven instabilities belong to the "drift branch", which can be analysed in the electrostatic limit [8], [9]. Examples of these "drift" modes are the electron drift mode ("universal" instability), the ion temperature gradient (ITG) mode, and the trapped[2] electron and the trapped ion modes [10]. Shear Alfvén modes, which are essentially electromagnetic, can also be destabilised in a tokamak by gradients if the ratio of plasma pressure to the pressure of the poloidal magnetic field, β_p, exceeds unity [11]. These instabilities have been considered by different authors in many papers, where different mathematical methods have been used (see the review [12]); nevertheless some of their features remain still to be understood.

1.3 Motivation and content of the present work

Investigation of wave propagation and instabilities in plasmas requires the knowledge of the constitutive relation, i.e. the relation between oscillating electric field and current in the plasma [3], [4]. The constitutive relation is obtained from the perturbed distribution function of particles which describes response of the plasma to the oscillating electromagnetic fields. In turn, the perturbed distribution function of particles is calculated by integration of the linearised Vlasov equation along unperturbed orbits of the parti-

[2]The trapped particles oscillate in regions of minimum magnetic field strength at the outer part of torus if ratio of parallel component of velocity to perpendicular component of velocity is small enough.

cles in the equilibrium electromagnetic field. The constitutive relation closes Maxwell equations, from which the wave equations are obtained. A simple form of this approach was used initially to study gradient driven instabilities in a nonuniform plasma [13]-[15].

A hot plasma, however, is a very complex medium, since it is dispersive both in time and space. Dispersion in time means that the propagation velocity of waves depends on their frequencies, in a plasma this is due to wave-particles resonances with characteristic frequencies in the motion of charged particles in the static magnetic field. Dispersion in space means that the propagation velocity of waves depends also on the wavevectors; in a plasma this is due to the thermal spread of the velocities of charged particles. In the uniform plasma limit all these properties can be summarised in a conductivity tensor. The derivation of the conductivity tensor of a hot, magnetised, uniform and unbounded plasma from the linearised Vlasov equation can be found in textbooks on plasma physics [3], [4], [16].

Generalisation of the concepts of plane waves and conductivity tensor to non-uniform laboratory plasmas, however, is far from trivial. In particular, space dispersion implies that the constitutive relation of hot plasma is non-local, i.e. the current at a given point depends on the fields at other points. Only recently wave equations taking into account space dispersion and valid in the ion cyclotron frequency range and above in axisymmetric plasmas have been obtained from the Vlasov equation [17], [18]. These derivations do not take into account effects which are important at lower frequencies, in particular, diamagnetism, which provides the free energy for gradient driven instabilities, and the influence of the particle drift motion on wave-particles resonances. Since these instabilities have frequencies much lower than the ion cyclotron frequency, they have been studied starting from the so called 'gyro-kinetic' equation (firstly introduced in [19] and [20]), i.e. a simplified kinetic equation obtained by averaging the Vlasov equation over the 'fast' gyro-motion of the charge particles. A comprehensive recent derivation of these equations in the linear limit can be found in [21].

The object of this thesis is to derive the constitutive relation of a hot, magnetised plasma directly from the linearised Vlasov equation, without making any assumption other then the validity of the drift approximation for the description of the particle orbits in the static magnetic configuration. This requires only that the particle gyro-radius should be small compared to the typical gradient length of the equilibrium magnetic field. The derivation is made for a plane-stratified plasma in a sheared magnetic field (Chapt.2), and for an axisymmetric toroidal plasma of arbitrary poloidal cross-section (Chapt.4). In the last case, which is the most interesting for us, the wave field is represented as an arbitrary superposition of toroidal and poloidal Fourier components; the amplitude of each partial wave, moreover, is further Fourier decomposed in the radial direction. This 'spectral decomposition' has the advantage that each component has a well-definite local wavevector which varies only on the scale of the equilibrium, and thus slowly in compared to the Larmor radius of the particles. The plasma response to each Fourier components, therefore, is found to be locally similar to that of uniform plasma, but all components are coupled to each other by the poloidal inhomogeneity. Accordingly, this response can be evaluated using mathematical techniques which are relatively straightforward generalisations of those used to obtain the conductivity tensor in the uniform plasma limit, although the algebra is considerably heavier. Since the plasma is dispersive in space, the constitutive relation obtained is non-local, i.e. it is an integral relation involving, in addition to the integrations over velocity space which depend on the distribution functions, two additional integrations (one in Fourier and one in real space) in each variable which is not ignorable. The kernel of this integral relation is clearly related to the conductivity tensor of the uniform plasma, and contains additional terms and corrections which can be recognised as due to diamagnetic effects and to the drift motion of charged particles in the non-uniform configuration.

The constitutive relation obtained in this way is in principle valid at all frequencies, but is of course much too complicated to be used as it is to solve practical problems. Its value consists in the first place in the possibility of deriving from it simpler equations to be used for concrete problems, by making approximations whose range of validity can be explicitely stated and fully tested in each case. It thus provides a unifying approach to the whole literature of waves in tokamak plasmas. Since it was clearly not possible to cover the whole frequency range of interest, in this thesis a particular efforts has been devoted to obtain wave equations valid for description of low frequency gradient driven instabilities of the drift and shear Alfvén wave types.

The low frequency limit of our approach (Chapt.3 and Chapt.5 for plane-stratified and toroidal plasmas, respectively) should coincide with the gyro-kinetic theory. Both these approaches take into account kinetic effects in the plasma and must give the same results if the same ordering and assumptions are used. The difference is that we approximate the formal solution of the Vlasov equation, rather than first averaging the Vlasov operator and then solving the kinetic equation. The gyro-kinetic equations are independent from the gyro-angle, i.e. they have a reduced number of variables in comparison with the Vlasov equation. The perturbed distribution function obtained as a solution of these equations depends only on the 'slow' drift motion.

A large amount of useful information on low frequency stability has been obtained from the gyro-kinetic equations; nevertheless the gyro-kinetic approach is not completely satisfactory. It was discovered that the simplest forms of the gyro-kinetic equation were not consistent with Liouville theorem, i.e. they do not describe a volume-conserving flux in phase space [22]. In addition, different authors derive gyro-kinetic equations differing in more or less important details, depending on assumptions and simplifications not always explicitly stated.

In the Vlasov derivation and gyro-kinetic approach different approximations are used. It is, therefore, useful to understand whether, and under what conditions, known results from the gyro-kinetic theory can be reproduced by the Vlasov derivation, and vice-versa, in order to clarify the range of validity of the results obtained with both theories. Detailed comparison of the results obtained by these two approaches, unfortunately, have been only partially possible. Recent efforts of scientists have mostly concentrated on the derivation of nonlinear gyro-kinetic equations consistent with Liouville theorem, i.e. conserving the particle flux in phase space [22]. These equations are needed for studying the plasma turbulence. The solution of the gyro-kinetic equation in the linear approximation to obtain the constitutive relation has been relatively neglected. As a consequence, it has been possible to reproduce only results based on the old 'heuristic' gyro-kinetic equations, which is mathematically not always rigorous.

To investigate the physics described by the constitutive relation which we have derived, we have first applied the local approximation (Sect.3.4-3.5 for plane-stratified and Sect.4.9 for toroidal plasmas). In this approximation the wavelengths of the waves considered are supposed much smaller than the characteristic lengths of the equilibrium gradients. The local approximation is similar to the well-known Wentzel-Kramer-Brillouin (WKB) approximation developed for the quantum mechanics [23]. The perturbed fields are written in the Eikonal form, i.e. as nearly plane waves with slowly varying amplitudes and rapidly varying phases. This scale separation greatly simplifies form of the constitutive relation since the variation of the electromagnetic field amplitude is considered as a small parameter. To the lowest order the wave equations assume an algebraic form; the solvability condition of this system of the algebraic equations is the local dispertion relation for low frequency waves.

Our local dispersion relation describes compressional, shear Alfvén wave and quasi-electrostatic waves of drift kind. Drift waves propagate mainly in the diamagnetic direction, i.e. perpendicularly to the direction of an equilibrium magnetic field $\vec{B}_0$ in the magnetic surface, and have a group velocity of the order of the diamagnetic[3] velocity. They can be easily destabilised by free energy of gradients at very low values of β, the ratio of plasma pressure to the pressure of the total magnetic field. The shear Alfvén waves have a group velocity of order of the Alfvén speed v_A[4] and propagate along the lines of the equilibrium magnetic field. These waves can also become unstable due to resonance with the electrons if magnitude of the Alfvén speed is less than the value of the thermal velocity of the plasma electrons. This is true if the condition $\beta > m_e/m_i$ is satisfied [4]. The compressional Alfvén waves propagate nearly perpendicularly to the equilibrium magnetic field with a velocity of order of the Alfvén speed v_A, and can hardly be de-stabilised by free energy of the gradients. The examination of the local dispertion relation gives us check that the method developed covers the correct physics.

As mentioned above, the wave equations obtained are integro-differential and very complicated. They can be put in a more manageable form taking advantage of the the following observation. Gradient driven waves can have wavelengths comparable with the Larmor radius of thermal ions, and propagate mostly in the diamagnetic direction.

[3]The diamagnetic velocity is proportional to the logarithmic density gradient and is defined by eq.(3.25).

[4]$v_A = 2.18 \times 10^{15} B_0 (A_i n_i)^{-1/2}\, cm/s$, where A_i is an atomic number of ions, n_i is their density.

This means, that the diamagnetic component of wavevector k_η is much larger than any other components, in particular, than the radial component of wavevector, $k_\eta \gg k_\psi$. If value $k_\eta \rho_i$ is order of unity, therefore, $k_\psi \rho_i \ll 1$. The finite Larmor radius (FLR) approximation based on the expansion in the small value $k_\psi \rho_i$ is used to simplify the general integro-differential wave equations (Sect. 3.6 for plane-stratified and Sect. 5.5 for toroidal plasmas). In this way we obtain the wave equations differential in the radial variables, while keeping all orders in the thermal Larmor radius in the diamagnetic direction.

The compressional Alfvén waves can be eliminated from consideration by assuming that the vector potential of the wave field is parallel to the magnetic field (for example, [24]), (Sect. 3.5-3.6, 5.5). Proceeding in this way we obtain two coupled wave equations for the drift and shear Alfvén waves which are essentially equivalent to those used in the literature based on the gyro-kinetic approach. One can easily recognise in these two equations the charge neutrality and the parallel component of Amper's law.

In the equation for charge neutrality, the polarisation charge due to the perpendicular dynamic of the ions in slowly varying electric field is compensated by the longitudinal motion of the electrons. It is interesting to note that in many early papers describing the drift waves the polarisation current of ions was missed. This current was 'rediscovered' later by Lee [25]. In the Vlasov derivation polarisation current of ions appears in a natural way by taking into account that the sum of the contributions of all non-resonant cyclotron harmonics to the plasma conductivity is not-negligible even in the low frequency limit, a fact which was overlooked in the early literature.

In the discussion of drift instabilities the parellel electron motion is usually split into an adiabatic part, representing the motion of the electrons in phase with the potential, and a non-adiabatic part describing the small corrections due to the finite inertia of the electrons and to dissipative effects in the plasma. The natural representation of the Vlasov approach does not require this splitting; however, one can easily separate the adiabatic and the non-adiabatic parts with help of an appropriate integration by parts.

The method proposed in this work is quite general; in principle, it is suitable for the investigation of plasma waves and instabilities at all frequencies and wavelengths. The wave equations obtained for low frequency instabilities, moreover, are in a form suitable for numerical solution with standard methods, and thus offer an interesting starting point for applications.

Chapter 2

Slab Geometry

2.1 The basic equations

The basic equations of the kinetic theory of waves in plasmas are Maxwell equations for the macroscopic electrical and magnetic fields $\vec{E}$ and $\vec{B}$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \tag{2.1}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \tag{2.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.3}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{2.4}$$

together with linearised Vlasov equation for each species α of charged particles in plasma:

$$\begin{aligned} \frac{df_\alpha}{dt} &= \frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\alpha + \frac{eZ_\alpha}{m_\alpha}\left(\vec{E}_0 + \frac{\vec{v}}{c} \times \vec{B}_0\right) \cdot \frac{\partial f_\alpha}{\partial v} \\ &= -\frac{eZ_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial F_\alpha}{\partial v} \end{aligned} \tag{2.5}$$

In these equations eZ_α and m_α are charge and mass of the particles, $F_\alpha(\vec{v}, \vec{r})$, $\vec{E}_0(\vec{r})$, $\vec{B}_0(\vec{r})$ denote the 'equilibrium' distribution function and the steady-state electric and magnetic fields, while $f_\alpha(\vec{v}, \vec{r}, t)$, $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$ are the respective perturbations. The momenta of f_α give the perturbed macroscopic charge and current densities

$$\rho = \sum_\alpha eZ_\alpha \int f_\alpha d\vec{v} \qquad \vec{j} = \sum_\alpha eZ_\alpha \int \vec{v} f_\alpha d\vec{v} \tag{2.6}$$

which appear as sources in Maxwell's equations.

If the equilibrium fields and equilibrium distribution functions are specified, eqs (2.1)-(2.6) constitute a closed system which, with appropriate initial and/or boundary conditions, uniquely specifies the wave fields and the perturbation of the distribution functions associated with the wave propagation.

The equation (2.5) can be solved, in principle, by the integrating the r.h. side, regarded as known, along the characteristics of the first order partial differential operator on the l.h. side, which are orbits of the charged particles in the static fields $\vec{E}_0$ and $\vec{B}_0$. This formal solution has the same structure as in the homogeneous case [3], namely

$$f_\alpha(\vec{r},\vec{v},t) = -\frac{Z_\alpha e}{m_\alpha}\int_{-\infty}^{t}\left(\vec{E}(\vec{r}\,',t\,') + \frac{\vec{v}\,'}{c}\times\vec{B}(\vec{r}\,',t\,')\right)\cdot\frac{\partial}{\partial\vec{v}\,'}[F_\alpha(\vec{r}\,',\vec{v}\,')]dt\,' \tag{2.7}$$

where $\vec{r}\,'$, $\vec{v}\,'$ are the solutions of the 'unperturbed' equation of motion

$$\frac{d\vec{r}\,'}{dt\,'} = \vec{v}\,';\quad \frac{d\vec{v}\,'}{dt\,'} = \frac{Z_\alpha e}{m_\alpha}\left(\vec{E}_0(\vec{r}\,') + \frac{\vec{v}\,'}{c}\times\vec{B}_0(\vec{r}\,')\right) \tag{2.8}$$

for time variable $t\,' \le t$, satisfying the 'final' conditions

$$\vec{v}\,'(t\,'=t) = \vec{v};\quad \vec{r}\,'(t\,'=t) = \vec{r}. \tag{2.9}$$

We can solve exactly the equations of motion of the charge particles in the electromagnetic field only in the some special cases, and the motion of the particles in the nonuniform magnetised plasma do not belong to them. But under usual conditions the characteristic lengths of the gradients of the equilibrium magnetic field are much greater then the Larmor radii of the particles, hence the particle orbits can be obtained in the drift approximation and the integration can be performed with the required accuracy.

From equation (2.7) the perturbed current and density can be evaluated as

$$\rho(\vec{r},t) = -\sum_\alpha \frac{e^2{Z_\alpha}^2}{m_\alpha}\int d\vec{v}\int_{-\infty}^{t}\left(\vec{E}(\vec{r}\,',t\,') + \frac{\vec{v}\,'}{c}\times\vec{B}_0(\vec{r}\,',t\,')\right)\cdot\frac{\partial F_\alpha}{\partial\vec{v}\,'}dt\,'$$
$$\vec{j}(\vec{r},t) = -\sum_\alpha \frac{e^2{Z_\alpha}^2}{m_\alpha}\int d\vec{v}\vec{v}\int_{-\infty}^{t}\left(\vec{E}(\vec{r}\,',t\,') + \frac{\vec{v}\,'}{c}\times\vec{B}_0(\vec{r}\,',t\,')\right)\cdot\frac{\partial F_\alpha}{\partial\vec{v}\,'}dt\,'$$

For an electromagnetic wave with harmonic time dependence $e^{-i\omega t}$, where ω is a frequency of oscillations, Maxwell equations can be summarised in the single equation

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \frac{\omega^2}{c^2}\left(\vec{E} + \frac{4\pi i}{\omega}\vec{j}\right) \tag{2.10}$$

together with the expression for the current

$$\vec{j}(\vec{r},t) = -e^{-i\omega t}\sum_\alpha \frac{e^2{Z_\alpha}^2}{m_\alpha}\int d\vec{v}\vec{v}\int_{-\infty}^{t} dt\,' e^{i\omega(t-t\,')}\left(\vec{E}(\vec{r}\,') + \frac{\vec{v}\,'}{c}\times\vec{B}_0(\vec{r}\,')\right)\cdot\frac{\partial F_\alpha}{\partial\vec{v}\,'} \tag{2.11}$$

In these equations $\gamma = Im(\omega) > 0$ to satisfy the causality. As in the homogeneous case, the results obtained under this assumption have finally to be continued analytically for $Im(\omega) < 0$.

To derive the wave equations for low frequency instabilities we have to specify the configuration of the plasma system. We will take first a simple nonuniform plasma system, to extend and develop the existing mathematical formalism of traditional integration of the Vlasov equation, and then apply the results obtained to a realistic toroidal plasma to obtain the differential wave equations describing drift and sheared Alfven waves instabilities.

2.2 The equilibrium configuration

The simplest nonuniform configuration in which relevant wave propagation can be studied is a plane stratified plasma in a sheared magnetic field [26], [27] and other. In this case two coordinates are ignorable, so the wave equation can be formulated as a one-dimensional problem. We assume that all macroscopic plasma parameters (density, temperature of all charged species, intensity of the static magnetic field) depend only on x.

A tokamak plasma can be idealised in this way by identifying x, y and z with the radial, poloidal and toroidal directions, respectively. The equilibrium magnetic field can be written in a form

$$\vec{B_0} = B_y(x)\vec{u}_y + B_z(x)\vec{u}_z = B_0(x)\left(\sin\Theta \cdot \vec{u}_y + \cos\Theta \cdot \vec{u}_z\right) \tag{2.12}$$

where we take into account shear of the magnetic field (pitch angle Θ); B_y and B_z can roughly simulate the poloidal and toroidal components of the tokamak magnetic field.

Although a rather rough idealisation, this geometry allows to investigate efficiently many important aspects of wave propagation in tokamak plasmas. In particular, this approximation is adequate for study of antennas for launching high frequency waves, for auxiliary heating and current drive. The present simplified model also allows to understand also the basic properties of drift waves, and, to some extent, to predict which features of the real equilibrium will most influence its stability.

For the solution of the linearised Vlasov equation, in addition to the laboratory system of coordinates (x, y, z) with base unit vectors ($\vec{u}_x$, $\vec{u}_y$, $\vec{u}_z$) it is convenient to define a reference local frame (ξ, η, ζ) with base unit vectors ($\vec{u}_\xi$, $\vec{u}_\eta$, $\vec{u}_\zeta$), such that $\vec{u}_\zeta$ is directed at each point parallel to the static magnetic field, $\vec{u}_\xi$ coincides with $\vec{u}_x$ and $\vec{u}_\eta$ goes in the diamagnetic direction (Fig. 2.1). The coordinate transformation from the laboratory frame to the local frame is

$$\begin{aligned} \xi &= x \\ \eta &= y\cos\Theta - z\sin\Theta \\ \zeta &= y\sin\Theta + z\cos\Theta \end{aligned} \tag{2.13}$$

and the corresponding rotation of the basis is

$$
\begin{aligned}
\vec{u}_\xi &= \vec{u}_x \\
\vec{u}_\eta &= \vec{u}_y \cos\Theta - \vec{u}_z \sin\Theta \\
\vec{u}_\zeta &= \vec{u}_y \sin\Theta + \vec{u}_z \cos\Theta
\end{aligned}
\tag{2.14}
$$

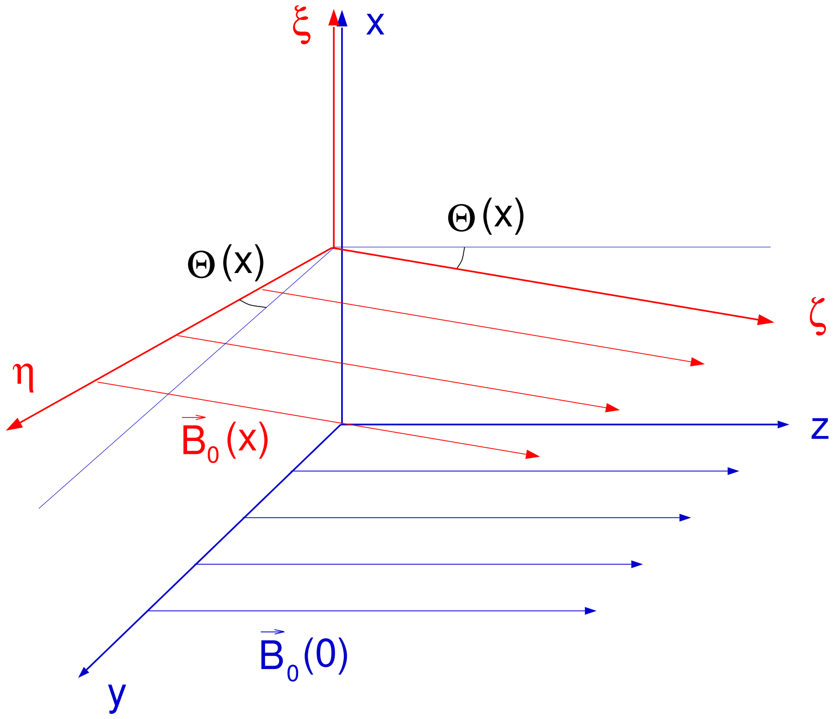

Figure 2.1: The frames used in the slab geometry case.

The coordinates ξ, η, ζ will be called 'field aligned' coordinates further in the work. In the uniform limit they are the coordinates, in which the plasma dielectric tensor takes the standard form, as defined by Stix [3], if the x-axis directed parallel to the perpendicular component of the wavevector $\vec{k}_\perp$. The importance of using field-aligned coordinates in wave propagation problem is due to the different nature of the motion of charged particles along and perpendicularly to the static magnetic field, well known in the uniform plasma limit.

It is worth to note some properties of the transformation (2.13). In the presents of shear $\Theta_x = d\Theta/dx$ this transformation depends on x, so that (2.13) is not a rigid rotation. To regarded transformation (2.13) as a 'local' rigid rotation, we have to neglect the terms quadratic in the shear, or proportional to the second derivatives of Θ. This is always justified if the system to be simulated is a tokamak, where the characteristic length of shear, $L_S = (d\log(\Theta)/dx)^{-1}$, is much greater than the linear dimensions of the system in the ignorable dimensions $L_S^2 \gg L_z^2 + L_y^2$.

2.3 Solution of the equation of motion

The advantage of the slab geometry, described above is that solution of the equation of motion (2.8) in the static field (2.12) can be easily obtained in the drift approximation, without recourse to the general theory of drift motion. It is intuitively clear, that in this geometry both the parallel velocity and the total energy, hence also the modulus of the perpendicular velocity, are constant of the motion. Working in the field aligned coordinates, we can write

$$\vec{v}\,' = v_{\|}\vec{u}_\zeta' + \vec{v}_\perp' \tag{2.15}$$

$$\vec{r}\,' = \vec{r}_g' + \vec{\rho}\,' = \vec{r}_g' + \frac{\vec{u}_\zeta' \times \vec{v}_\perp'}{\Omega_g'} \tag{2.16}$$

where $\vec{r}_g'$ denotes the 'guiding center' position, $\Omega_g = ZeB_0/mc$ is a cyclotron frequency of the particles, c is a light velocity. We recall that primed quantities are the running values, while unprinted ones are the 'final' conditions. The perpendicular velocity $\vec{v}_\perp'$ and gyroradius $\vec{\rho}\,'$ will be obtained as a perturbative series

$$\begin{aligned} \vec{v}_\perp' &= \vec{v}_{0\perp}' + \vec{v}_{1\perp}' + \vec{v}_{2\perp}' + \cdots \\ \vec{\rho}\,' &= \vec{\rho}_1' + \vec{\rho}_2' + \cdots \end{aligned} \tag{2.17}$$

in the small expansion parameter

$$\epsilon_d = \left| (\vec{\rho} \cdot \vec{\nabla}) \frac{\vec{B}}{B} \right| \simeq \frac{\rho}{L_B} \ll 1 \tag{2.18}$$

where L_B is the characteristic length of the gradient of the magnetic field intensity (in the tokamak $L_B \simeq R_0$, the major radius). We also assume $\vec{E}_0$, Θ_x are order of ϵ_D, in agreement with the condition mentioned above, so that shear has no explicit influence on the motion to the order required for derivation of the wave equations.

Substituting (2.18) into (2.8) and expanding the x-dependent quantities in Taylor series around the guiding center position, we easily recognise that the zero-order velocity satisfies the same equation as in a uniform magnetic field. It is convenient to write the well known solution of this equation in circularly polarised components. We introduce the complex unit vectors

$$\vec{u}_\pm = \frac{1}{\sqrt{2}} (\vec{u}_\xi \mp i\vec{u}_\eta) \tag{2.19}$$

into our consideration.

The 'final condition' for the gyration motion in the rotational frame are, to the lowest order

$$\vec{v}'_{\perp}(t'=t) = \vec{v}_{\perp} = \frac{v_{\perp}}{\sqrt{2}}\left(e^{+i\phi_v}\vec{u}_{+} + e^{-i\phi_v}\vec{u}_{-}\right) \tag{2.20}$$

$$\vec{\rho}'(t'=t) = \vec{\rho} = -\frac{\vec{v}_{\perp}\times\vec{u}_{\zeta}}{\Omega_g} = i\frac{v_{\perp}}{\sqrt{2}\Omega_g}\left(e^{+i\phi_v}\vec{u}_{+} - e^{-i\phi_v}\vec{u}_{-}\right) \tag{2.21}$$

where ϕ_v is the 'gyration angle' in velocity cylindrical coordinates. Taking this into account, the lowest order velocity can be written

$$\vec{v}'_{\perp 0} = \frac{v_{\perp}}{\sqrt{2}}\left(e^{+i(\phi_v+\Omega_g(t'-t))}\vec{u}_{+} + e^{-i(\phi_v+\Omega_g(t'-t))}\vec{u}_{-}\right) \tag{2.22}$$

and for gyroradius we obtain

$$\vec{\rho}'_1 = i\frac{v_{\perp}}{\sqrt{2}\Omega_g}\left(e^{+i(\phi_v+\Omega_g(t'-t))}\vec{u}_{+} - e^{-i(\phi_v+\Omega_g(t'-t))}\vec{u}_{-}\right) \tag{2.23}$$

The guiding centers move on a magnetic field line; to this order we have

$$\vec{r}'_g = v_{\parallel}\vec{u}_{\zeta}(t'-t) + \vec{r}_g \tag{2.24}$$

Hence, $\Omega'_g = \Omega_g$, and all quantities which depend only on x do not need a prime.

To next order in ϵ_d one has to solve

$$\frac{d\vec{v}'_{\perp 1}}{dt'} + \Omega_g(\vec{u}_{\zeta}\times\vec{v}'_{\perp 1}) = \frac{Ze}{m}\vec{E}_0 + \left[(\vec{\rho}'_1\cdot\vec{\nabla})\Omega_g\right](\vec{v}'_{\perp 0}\times\vec{u}_{\zeta}) \tag{2.25}$$

If the last term evaluated using the lower order results, the r.h. side contains constant terms and terms oscillating with the frequencies Ω_g and $2\Omega_g$. Separating $\vec{v}'_{\perp 1}$ into an averaged and oscillating part, one find easily for the $\vec{E}\times\vec{B}$ and 'drift' velocities, $\vec{v}_E$ and $\vec{v}_D$, the well known results

$$\vec{v}'_{\perp 1} = \vec{v}_E + \vec{v}_D \tag{2.26}$$

where

$$\vec{v}_E = c\frac{\vec{E}_0\times\vec{B}_0}{B_0^2} \tag{2.27}$$

characterises the drift of charged particles in cross electrical and magnetic fields and

$$\vec{v}_D = \frac{v_{\perp}^2}{2\Omega_g^2}\left(\vec{u}_{\zeta}\times\vec{\nabla}\Omega_g\right) = \frac{v_{\perp}^2}{2\Omega_g^2}\frac{d\Omega_g}{dx}\vec{u}_{\eta} \tag{2.28}$$

characterises *grad* $\vec{B}$ drift of the plasma particle. In slab geometry the latter is the only magnetic drift, since the magnetic field is straight. In the magnetic field of a tokamak or other magnetic confinement devices, however, there is a 'curvature' drift attributable to the centrifugal force experienced by the particles as they travel along curved magnetic lines. The total magnetic drift is then [2]

$$\vec{v}_D = \frac{v_\perp^2 + 2v_\parallel^2}{2\Omega_g^2}\left(\vec{u}_\zeta \times \vec{\nabla}\Omega_g\right) \tag{2.29}$$

It is occasionally worthwhile keeping the curvature drift in the motion of the particles, even if the static magnetic configuration is approximated by a straight magnetic field.

As a consequence of the drift, to this order eqs (2.20) and (2.24) should be replaced by

$$\vec{v}_\perp'(t' = t) = \vec{v}_\perp + \vec{v}_E + \vec{v}_D \tag{2.30}$$

and

$$\vec{r}_g' - \vec{r}_g = (v_\parallel \vec{u}_\zeta + \vec{v}_E + \vec{v}_D)(t' - t) \tag{2.31}$$

The guiding center now moves across field lines, but, in the present geometry, still in a plane of constant x. Thus, it is again justified to omit primes on Ω_g, $\vec{u}_\eta$ and $\vec{u}_\zeta$.

In conclusion, we evaluate the orbits of plasma particles in the presented geometry as

$$\begin{aligned} \vec{r}' &= \vec{r} + \left(v_D \vec{u}_\eta + v_\parallel \vec{u}_\zeta\right)(t' - t) - \frac{(\vec{v}_\perp' - \vec{v}_\perp) \times \vec{u}_\zeta}{\Omega_g} \\ \vec{v}' &= \vec{v}_\perp' + \vec{v}_D \vec{u}_\eta + v_\parallel \vec{u}_\zeta \end{aligned} \tag{2.32}$$

In the field aligned frame

$$\begin{aligned} x' &= x - \frac{v_\perp}{\Omega_g}\left[\sin\left(\phi_v + \Omega_g\left(t - t'\right)\right) - \sin\phi_v\right] \\ \eta' &= \eta - \frac{v_\perp}{\Omega_g}\left[\cos\left(\phi_v + \Omega_g\left(t - t'\right)\right) - \cos\phi_v\right] + v_D\left(t - t'\right) \\ \zeta' &= \zeta + v_\parallel\left(t - t'\right) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} v_x' &= v_\perp \cos\left(\phi_v + \Omega_g\left(t - t'\right)\right) \\ v_\eta' &= v_\perp \sin\left(\phi_v + \Omega_g\left(t - t'\right)\right) + v_D \\ v_\zeta' &= v_\parallel \end{aligned} \tag{2.34}$$

Except for the drift velocity $v_D \vec{u}_\eta$, these equations are identical to those valid in the uniform plasma limit. In this simplified geometry, moreover, $v_D \vec{u}_\eta$ is constant in time, although a function of $v_\perp$ (and of $v_\parallel$ if the curvature drift is retained).

2.4 The equilibrium distribution function

The unperturbed distribution function F_α can be found from steady-state kinetic equations

$$\vec{v} \cdot \vec{\nabla} F_\alpha + \frac{eZ_\alpha}{m_\alpha} \left(\vec{E}_0 + \frac{\vec{v}}{c} \times \vec{B}_0 \right) \cdot \frac{\partial F_\alpha}{\partial \vec{v}} = C(F_\alpha) \tag{2.35}$$

where C is the Fokker-Planck collisional operator, $\vec{E}_0$ and $\vec{B}_0$ satisfy the equation

$$\vec{\nabla} \times \vec{E}_0 = 0 \qquad \vec{\nabla} \cdot \vec{E} = 0 \tag{2.36}$$

$$\vec{\nabla} \times \vec{B}_0 = \frac{4\pi}{c} \vec{j}_0 \qquad \vec{\nabla} \cdot \vec{B}_0 = 0 \tag{2.37}$$

We now specify the unperturbed distribution function F_α. In a tokamak a magnitude of the static electrical field is far less than that of the static magnetic field. To the leading order, F_α must satisfy [12]:

$$\vec{v} \cdot \vec{\nabla} F_\alpha - \Omega_{g\alpha} \cdot \frac{\partial F_\alpha}{\partial \phi_v} = 0 \tag{2.38}$$

This equation allows F_α to be an arbitrary function of the constants of motion $v_\perp$ and $v_\parallel$. Moreover, F_α depends on x through the position of the guiding centers $\vec{r}_g$.

$$\vec{r}_g = \vec{r} - \vec{\rho} = \vec{r} + \frac{\vec{v}_\perp}{\Omega_g} \times \vec{b}_0 \tag{2.39}$$

where $\vec{b}_0 = \vec{B}_0 / B_0$. Thus, in absence of collisions F_α will have the form

$$F_\alpha(\vec{r}, \vec{v}) = F_\alpha \left(v_\parallel, v_\perp^2, x + \frac{v_\eta}{\Omega_g} \right) \tag{2.40}$$

Neglecting collisions altogether, however, is hardly acceptable for steady magnetically confined plasmas. On the contrary, one can often assume that local thermal equilibrium prevails at each instant, at least separately for ions and electrons. In this case

$$F_\alpha(\vec{r}, \vec{v}) = F_{M\alpha} \left(v^2, x + \frac{v_\eta}{\Omega_g} \right) = \frac{n_\alpha}{(\pi v_{th\alpha}^2)^{3/2}} \exp \left(- \frac{v_\alpha^2}{v_{th\alpha}^2} \right) \tag{2.41}$$

where $v_{th\alpha} = \sqrt{2T_\alpha / m_\alpha}$ is a thermal velocity for each species α of plasma particles. $F_{M\alpha}$ are Maxwellian distribution functions whose density and temperature are function of the position of the guiding centers $\vec{r}_{g\alpha}$ rather than the position of the particles. We will limit ourselves to locally Maxwellian plasmas. More general distribution functions, however, can also be of interest, and generalisation is not difficult.

Associated with gradients of density and temperature is so called equilibrium diamagnetic current, which flows perpendicular to both the static magnetic field and the gradients, i.e. in the diamagnetic direction.

$$\vec{J}_D(\vec{r}) = \sum_\alpha Z_\alpha e \int F_{M\alpha}\left(v^2, x + \frac{v_\eta}{\Omega_g}\right) \vec{v} dv = J_D(x)(\vec{b_0} \times \vec{u}_\zeta) \tag{2.42}$$

Taylor-developing density and temperature around the observation point $\vec{r}$ we obtain

$$\begin{aligned} J_D(x) &\simeq \sum_\alpha Z_\alpha e \int \frac{v_\eta^2}{\Omega_{g\alpha}} F_{M\alpha}(v^2) \frac{1}{L_{n\alpha}} \left[1 + \eta_{T\alpha}\left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right] d\vec{v} \\ &= \sum_\alpha \frac{Z_\alpha e n_\alpha v_{th\alpha}^2}{2\Omega_{g\alpha} L_{n\alpha}} (1 + \eta_{T\alpha}) \end{aligned} \tag{2.43}$$

where we have defined

$$\frac{1}{L_{n\alpha}} = \frac{d \log n_\alpha}{dx} \qquad \eta_{T\alpha} = \frac{d \log T_\alpha}{d \log n_\alpha} \tag{2.44}$$

The characteristic lengths $L_{n\alpha}$ and the ratio $\eta_{T\alpha}$ are in principle function of the position; implicit in the notation, however, is the fact that, in practice, they vary only little over one Larmor radius, and will be considered as constants in most of the following considerations.

The diamagnetic current is not associated to any material flow. The particles which at a given instant cross a plane $x = const$ with opposite perpendicular velocity have their gyrocenters on opposite sides of this plane. If there is a density gradient, therefore, there will be more particles with velocities in one direction than in the opposite one, resulting in a net particle flux across any plane $\eta = const$, without any motoin of the guiding centers in this direction. Since electrons gyrate in the counerclockwise and ions in the clockwise direction, the resulting diamagnetic fluxes are also of opposite signs, so that the two contributions to the diamagnetic current add. A temperature gradient has the similar effect, because the average perpendicular velocity of particles with the guiding centers to the right and to the left of a given x plane is different.

By contrast, the particle drifts do gives rise to microscopic flows.

2.5 The formal solution of the linearised Vlasov equation

The formal solution of the linearised Vlasov equation (2.11) can now be easily specialised to the case of a Maxwellian plasma

$$\vec{j}(\vec{r},t) = \sum_{\alpha} \frac{2e^2 Z_\alpha^2}{m_\alpha} \int d\vec{v} F_{M\alpha}(v^2, \vec{r}_{g\alpha}) \frac{\vec{v}}{v_{th\alpha}} \int_{-\infty}^{t} dt' \tag{2.45}$$
$$\left[\left(\frac{\vec{v}}{v_{th\alpha}} - \frac{v_{th\alpha}}{2\Omega_{ga}} \vec{b} \times \vec{\nabla} \log F_{M\alpha} \right) \cdot \left(\vec{E}(\vec{r}', t') + \frac{\vec{v}'}{c} \times \vec{B}(\vec{r}', t') \right) \right]$$

where for brevity

$$\vec{\nabla} \log F_{M\alpha} = \frac{1}{L_n} \left[1 + \eta_{T\alpha} \left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2} \right) \right] \vec{u}_x \tag{2.46}$$

The evaluation of the integrals in (2.45) is greatly facilitated in the present geometry by fact that, since the equilibrium parameters vary only in the x-direction, the electromagnetic field can be Fourier decomposed along the ignorable coordinates y and z

$$\vec{E}(x, y, z, t) = \sum_{k_y} \sum_{k_z} \vec{E}(k_y, k_z, x) e^{i(k_y y + k_z z - \omega t)} \tag{2.47}$$

A similar decomposition holds, of course, also for $\vec{B}$, for current $\vec{j}$ and for perturbed distribution function f_α. Each term in this sum will be called a 'partial wave', and is identified by its wavenumber k_y and k_z. In the case of toroidal geometry the admissible values of k_y and k_z are determined by the periodicity conditions in the y and z directions, respectively. If the plasma is assumed unbounded in one or both of these directions, on other hands, the sum in (2.47) can be interpreted as a short-hand notation for Fourier integrals.

We now can proceed to evaluate the constitutive relation of a plane-stratified plasma explicitly. Although x is not an ignorable coordinate, it is always possible to write the amplitude of each partial wave in (2.47) as a Fourier sum (or integral)

$$\vec{E}(k_y, k_z, x) = \sum_{k_x} \vec{E}(k_y, k_z, k_x) e^{i(k_x x)} \tag{2.48}$$

Substituting into (2.47), we can then write the total fields as

$$\vec{E} = \sum_{\vec{k}} \vec{E}(\vec{k_\perp}, k_z) e^{i\left(\vec{k}\vec{r} - \omega t\right)} \tag{2.49}$$
$$\vec{B} = \frac{c}{\omega} \sum_{\vec{k}} \left(\vec{k} \times \vec{E}(\vec{k_\perp}, k_z) \right) e^{i\left(\vec{k}\vec{r} - \omega t\right)}$$

For convenience, all components of $\vec{k}$ have been treated here on the same footing, although k_x is always a dummy summation variable, while k_y and k_z are good wavenumbers, so that partial waves with given values of k_y and k_z could be treated independently from each other.

When expressions (2.49), together with the orbits obtained in the previously section, are substituted in (2.45), the current becomes

$$\vec{j}(\vec{r}, t) = \sum_{\vec{k}} \vec{j}(x; \vec{k}) e^{i\left(\vec{k}\vec{r} - \omega t\right)} \tag{2.50}$$

with

$$\begin{aligned} \vec{j}(x; \vec{k}) = \sum_{\alpha} \frac{2e^2 Z_\alpha^2}{m_\alpha} \int d\vec{v} F_{M\alpha} \left(\vec{r}_\perp + \frac{\vec{v}_\perp \times \vec{u}_\zeta}{\Omega_{g\alpha}}\right) \frac{v_\parallel \vec{u}_\zeta + \vec{v}_\perp}{v_{th\alpha}} \\ \int_{-\infty}^{t} dt\,' e^{i(\omega - k_\zeta v_\parallel - k_\eta v_{D\alpha})(t - t\,')} \left\{ \frac{v_\parallel \vec{u}\,'_\zeta + \vec{v}\,'_\perp + v_{D\alpha} \vec{u}\,'_\eta}{v_{th\alpha}} - \frac{v_{th\alpha}}{2\Omega_{g\alpha} L_{n\alpha}} \cdot \right. \\ \left. \left[1 + \eta_{T\alpha} \left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right] \vec{u}\,'_\eta \right\} \cdot \vec{E}(\vec{k}_\perp, k_z) e^{-i \frac{\vec{k}_\perp \cdot (\vec{v}\,'_\perp - \vec{v}_\perp) \times \vec{u}_\zeta}{\Omega_{g\alpha}}} \end{aligned} \tag{2.51}$$

In the presence of gradients, however, the amplitude $\vec{j}(x; \vec{k})$ in eqn (2.50) is not the Fourier transform of the current. Inverting the Fourier transform, the latter is found to be

$$\begin{aligned} \vec{j}(\vec{k}_\perp; k_z) = \sum_{\alpha} \frac{2e^2 Z_\alpha^2}{m_\alpha} \int \frac{d\vec{r}_\perp}{(2\pi)^2 S} e^{-i\vec{k}_\perp \vec{r}_\perp} \int d\vec{v} F_\alpha \left(\vec{r}_\perp + \frac{\vec{v}_\perp \times \vec{u}_\zeta}{\Omega_{g\alpha}}\right) \\ \frac{v_\parallel \vec{u}_\zeta + \vec{v}_\perp}{v_{th\alpha}} \sum_{\vec{k}_\perp^b} e^{i\vec{k}_\perp^b \vec{r}_\perp} \int_{-\infty}^{t} dt\,' e^{i(\omega - k_\zeta v_\parallel - \bar{k}_\eta v_{D\alpha})(t - t\,')} \\ \left\{ \frac{v_\parallel \vec{u}\,'_\zeta + \vec{v}\,'_\perp + v_{D\alpha} \vec{u}\,'_\eta}{v_{th\alpha}} - \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_{n\alpha}} \cdot \left[1 + \eta_{T\alpha} \left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right] \vec{u}\,'_\eta \right\} \cdot \\ \vec{E}(\vec{k}_\perp^b, k_z) e^{-i \frac{\vec{k}_\perp^b \cdot (\vec{v}\,'_\perp - \vec{v}_\perp) \times \vec{u}_\zeta}{\Omega_{g\alpha}}} \end{aligned} \tag{2.52}$$

Here for convenience $\vec{r}_\perp = (x, y)$, and S is the cross-section of the plasma perpendicular to the static magnetic field, $\vec{k}_\perp^b \equiv (\bar{k}_x, \bar{k}_y)$ denotes a dummy summation variable. We continue to treat the two perpendicular coordinates on the same footing to facilitate the generalisation to more complicated geometries; in the present case, however, the integration over y is trivial, and gives a δ-function $\delta(k_y - \bar{k}_y)$, as expected.

It is now convenient to take into account that the equilibrium distribution function depends only on the position of the guiding centres (2.40). Changing the integration variable for the Fourier transform from $\vec{r}_\perp$ to $\vec{R}_\perp = \vec{r}_\perp + (\vec{v}_\perp \times \vec{u}_\zeta)/\Omega_{g\alpha}$ allows us to obtain the more symmetric expressions.

$$\vec{j}(\vec{k}_\perp, k_z) = \sum_\alpha \frac{2e^2 Z_\alpha^2}{m_\alpha} \sum_{\vec{k}_\perp^b} \int \frac{d\vec{R}_\perp}{(2\pi)^2 S_\perp} e^{i(\vec{k}_\perp^b - \vec{k}_\perp)\vec{R}_\perp} \int d\vec{v} F_{M\alpha}(\vec{v}, \vec{R}_\perp)$$
$$\frac{v_\parallel \vec{u}_\zeta + \vec{v}_\perp}{v_{th\alpha}} e^{i\left(\frac{\vec{k}_\perp \cdot \vec{v}_\perp \times \vec{u}_\zeta}{\Omega_{g\alpha}}\right)} \int_{-\infty}^{t} dt' e^{i(\omega - k_\zeta v_\parallel - \bar{k}_\eta v_{D\alpha})(t-t\,')} \tag{2.53}$$
$$\left\{ \frac{v_\parallel \vec{u}_\zeta' + \vec{v}_\perp' + v_{D\alpha}\vec{u}_\eta'}{v_{th\alpha}} - \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_{n\alpha}} \left[1 + \eta_{T\alpha}\left(\frac{v^2}{v_{th\alpha}^2} - \frac{3}{2}\right)\right] \vec{u}_\eta' \right\}$$
$$\vec{E}(\vec{k}_\perp^b, k_z) e^{-i\frac{\vec{k}_\perp^b \cdot \vec{v}_\perp\,' \times \vec{u}_\zeta}{\Omega_{g\alpha}}}$$

This equation is of the form

$$\vec{j}(\vec{k}_\perp, k_z) = \sum_{\vec{k}_\perp^b} \underline{\underline{\Sigma}}(\vec{k}_\perp, \vec{k}_\perp^b, k_z) \cdot \vec{E}(\vec{k}_\perp^b)$$
$$\underline{\underline{\Sigma}} = \int \frac{d\vec{R}_\perp}{(2\pi)^2 S} e^{i(\vec{k}_\perp^b - \vec{k}_\perp)\vec{R}_\perp} \underline{\underline{\sigma}}(\vec{k}, \vec{k}^b, \vec{R}_\perp) \tag{2.54}$$

where, inserting the explicit form of the Maxwellian

$$\frac{4\pi i}{\omega} \underline{\underline{\sigma}}(\vec{k}, \vec{k}^b, \vec{R}_\perp) = -2 \sum_\alpha \frac{\omega_{p\alpha}^2(\vec{R}_\perp)}{\omega^2} \int_0^\infty \frac{e^{-w^2}}{\pi} w dw \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du$$
$$\times \int_0^{2\pi} d\phi_v G_\alpha(\vec{k}, \vec{k}^b, \vec{v}, \vec{R}) \left(\underline{\underline{\Pi}}_\alpha^0 + \underline{\underline{\Pi}}_\alpha^D + \underline{\underline{\Pi}}_\alpha^B \right) \tag{2.55}$$

In (2.55) dimensionless velocities have been introduced, $\vec{u} = \vec{v}_\parallel / v_{th\alpha}$ and $\vec{w} = \vec{v}_\perp / v_{th\alpha}$; the 'generalised propagator' G_α has the form

$$G_\alpha(\vec{k}, \vec{k}^b, \vec{v}, \vec{R}) = -i\omega e^{i\frac{\vec{k}_\perp \cdot \vec{v}_\perp \times \vec{u}_\zeta}{\Omega_{g\alpha}}} \int_{-\infty}^{t} dt' e^{i(\omega - k_\zeta v_\parallel - \bar{k}_\eta v_{D\alpha})(t-t')} e^{-i\frac{\vec{k}_\perp^b \cdot \vec{v}_\perp\,' \times \vec{u}_\zeta}{\Omega_{g\alpha}}} \tag{2.56}$$

and three terms in the brackets are defined as

$$
\begin{aligned}
\underline{\underline{\Pi}}^0_\alpha &= (u\vec{u}_\zeta + \vec{w}) : (u\vec{u}'_\zeta + \vec{w}\,') \\
\underline{\underline{\Pi}}^D_\alpha &= -\frac{v_{th\alpha}}{2\Omega_{g\alpha} L_{n\alpha}} \left[1 + \eta_{T\alpha} \left(\frac{v^2}{v^2_{th\alpha}} - \frac{3}{2} \right) \right] (u\vec{u}_\zeta + \vec{w}) : \vec{u}'_\eta \\
\underline{\underline{\Pi}}^B_\alpha &= \frac{v_{D\alpha}}{v_{th\alpha}} (u\vec{u}_\zeta + \vec{w}) : \vec{u}'_\eta
\end{aligned}
\tag{2.57}
$$

The equations (2.54)-(2.57) are used to evaluate explicitly the conductivity operator. These equations can be regarded as the constitutive relation for plane-stratified non-uniform plasmas in the Fourier representation.

The introduction of the new variable (2.53) is one of the key features of the method proposed. This transformation was first considered by Catto [28], who retained finite gyro-radius effects in a far simpler manner than in the earlier papers on the gyro-kinetic theory [29]-[31]. Changing the variable $\vec{r}_\perp \to \vec{R}_\perp$ was used also in [32] in a very special case (cylindrical layer of plasma). This change of variables makes our derivation essentially equivalent to the gyro-kinetic one. At the same time, it allows us to use mathematical techniques which are simple extensions of those used in the uniform limit.

Of the three contribution to $\underline{\underline{\sigma}}$, the first remains finite in the uniform plasma limit. We will call $\underline{\underline{\Pi}}^0_\alpha$ term as a 'bulk' part of the conductivity. Other terms exist only in the inhomogeneous plasma limit. The second one arised because of the space dependence of the equillibrium density and temperature; by analogy with (2.43) we will call $\underline{\underline{\Pi}}^D_\alpha$ the 'diamagnetic' conductivity. The third term is due to the drift motion of particles, and we will, therefore, call it the 'drift' conductivity.

2.6 Bessel function expansion

Although in field aligned coordinates the physical components of $\vec{k}$ depend on x

$$
\frac{d\vec{k}}{dx} = \frac{d\Theta}{dx}(k_\zeta \vec{u}_\eta - k_\eta \vec{u}_\zeta) \tag{2.58}
$$

the primed phase

$$
\begin{aligned}
\vec{k}\cdot\vec{r}\,' - \omega t' &= \vec{k}\cdot\vec{r} - \omega t + \Phi' \\
\Phi' &= (k_\eta v_{D\alpha} + k_\zeta v_\parallel - \omega)(t' - t) - \frac{\vec{k}_\perp \cdot (\vec{v}\,'_\perp - \vec{v}_\perp) \times \vec{u}_\zeta}{\Omega_{g\alpha}}
\end{aligned}
\tag{2.59}
$$

has the same form as in truly Cartesian frame, namely

$$\Phi' = -\frac{k_\xi v_\perp}{\Omega_{g\alpha}}[\sin(\phi_v + \Omega_{g\alpha}(t-t')) - \sin\phi_v]$$
$$+k_\eta \left[\frac{v_\perp}{\Omega_{g\alpha}}[\cos(\phi_v + \Omega_{g\alpha}(t-t')) - \cos\phi_v] + v_{D\alpha}(t'-t)\right] + k_\zeta v_\parallel (t'-t) \tag{2.60}$$

We recall that due to changing of variables (2.53) one can use the mathematica of the uniform limit [4], [33]. We can, therefore, expand the factor depending on the gyration velocity in harmonics of the cyclotron frequency as in homogeneous plasma case:

$$e^{i\frac{\vec{k}_\perp\cdot\vec{v}_\perp\times\vec{u}_\zeta}{\Omega_{g\alpha}}} e^{-i\frac{\vec{k}^b_\perp\cdot\vec{v}_\perp{}'\times\vec{u}_\zeta}{\Omega_{g\alpha}}} = e^{i\nu w\sin(\phi_v-\delta)} e^{-i\bar{\nu} w\sin(\phi_v+\Omega_{g\alpha}(t-t')-\bar{\delta})}$$
$$= \sum_{p=-\infty}^{\infty} J_p(\nu w) e^{ip(\phi_v-\delta)} \sum_{\bar{p}=-\infty}^{\infty} J_{\bar{p}}(\bar{\nu} w) e^{i\bar{p}[\phi_v+\Omega_{g\alpha}(t-t')-\bar{\delta}]} \tag{2.61}$$

where

$$\nu_\alpha = \frac{k_\perp v_{th\alpha}}{\Omega_{g\alpha}} \qquad \bar{\nu}_\alpha = \frac{\bar{k}_\perp v_{th\alpha}}{\Omega_{g\alpha}} \tag{2.62}$$

and

$$\delta = \arctan k_\eta / k_\xi \qquad \bar{\delta} = \arctan \bar{k}_\eta / \bar{k}_\xi \tag{2.63}$$

We will omit index α and write simply ν and $\bar{\nu}$ further.

Using this expansion in (2.55) and performing the integration over the gyroangle ϕ_v then gives

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}} = -4\sum_\alpha \frac{\omega^2_{p\alpha}(\vec{R}_\perp)}{\omega^2} \int_0^\infty \frac{e^{-w^2}}{\pi} w dw \int_{-\infty}^\infty \frac{e^{-u^2}}{\sqrt{\pi}} du$$
$$\times(-i\omega) \sum_{p=-\infty}^{\infty} \int_{-\infty}^t dt' e^{i(\omega - p\Omega_{g\alpha} - k_\zeta v_\parallel - \bar{k}_\eta v_{D\alpha})(t-t')} \left(\underline{\underline{\Pi}}^{0,p}_\alpha + \underline{\underline{\Pi}}^{D,p}_\alpha + \underline{\underline{\Pi}}^{B,p}_\alpha\right) e^{ip(\bar{\delta}-\delta)} \tag{2.64}$$

The explicit evaluation of the three matrices $\underline{\underline{\Pi}}^{0,p}_\alpha$, $\underline{\underline{\Pi}}^{D,p}_\alpha$ and $\underline{\underline{\Pi}}^{B,p}_\alpha$ is a somewhat lengthly but straightforward task. Using recurrence relations of Bessel function, we can obtain the matrices components in the local field aligned frame

$$\Pi^{0,p}_{\xi\xi} = w^2 \left[\frac{p}{\nu w} J_p(\nu w)\cos\delta + iJ'_p(\nu w)\sin\delta\right] \left[\frac{p}{\bar{\nu} w} J_p(\bar{\nu} w)\cos\bar{\delta} - iJ'_p(\bar{\nu} w)\sin\bar{\delta}\right]$$
$$\Pi^{0,p}_{\xi\eta} = iw^2 \left[\frac{p}{\nu w} J_p(\nu w)\cos\delta + iJ'_p(\nu w)\sin\delta\right] \left[J'_p(\bar{\nu} w)\cos\bar{\delta} - \frac{ip}{\bar{\nu} w} J_p(\bar{\nu} w)\sin\bar{\delta}\right]$$
$$\Pi^{0,p}_{\xi\zeta} = uw \left[\frac{p}{\nu w} J_p(\nu w)\cos\delta + iJ'_p(\nu w)\sin\delta\right] J_p(\bar{\nu} w)$$

$$
\begin{aligned}
\Pi^{0,p}_{\eta\xi} &= -iw^2 \left[J'_p(\nu w)\cos\delta + \frac{ip}{\nu w}J_p(\nu w)\sin\delta\right]\left[\frac{p}{\bar\nu w}J_p(\bar\nu w)\cos\bar\delta - iJ'_p(\bar\nu w)\sin\bar\delta\right] \quad (2.65)\\
\Pi^{0,p}_{\eta\eta} &= w^2 \left[J'_p(\nu w)\cos\delta + \frac{ip}{\nu w}J_p(\nu w)\sin\delta\right]\left[J'_p(\bar\nu w)\cos\bar\delta - \frac{ip}{\bar\nu w}J_p(\bar\nu w)\sin\bar\delta\right]\\
\Pi^{0,p}_{\eta\zeta} &= -iuw \left[J'_p(\nu w)\cos\delta + \frac{ip}{\nu w}J_p(\nu w)\sin\delta\right] J_p(\bar\nu w)\\
\Pi^{0,p}_{\zeta\xi} &= uwJ_p(\nu w)\left[\frac{p}{\bar\nu w}J_p(\bar\nu w)\cos\bar\delta - iJ'_p(\bar\nu w)\sin\bar\delta\right]\\
\Pi^{0,p}_{\zeta\eta} &= iuwJ_p(\nu w)\left[J'_p(\bar\nu w)\cos\bar\delta - \frac{ip}{\bar\nu w}J_p(\bar\nu w)\sin\bar\delta\right]\\
\Pi^{0,p}_{\zeta\zeta} &= u^2 J_p(\nu w)J_p(\bar\nu w)
\end{aligned}
$$

In the diamagnetic term the only nonvanishing term are

$$
\begin{aligned}
\Pi^{D,p}_{\xi\eta} &= \Lambda_{D\alpha}(u,w)\left[\frac{p}{\nu w}J_p(\nu w)\cos\delta + iJ'_p(\nu w)\sin\delta\right] J_p(\bar\nu w)\\
\Pi^{D,p}_{\eta\eta} &= -i\Lambda_{D\alpha}(u,w)w\left[J'_p(\nu w)\cos\delta + \frac{ip}{\nu w}J_p(\nu w)\sin\delta\right] J_p(\bar\nu w) \qquad (2.66)\\
\Pi^{D,p}_{\zeta\eta} &= \Lambda_{D\alpha}(u,w)uJ_p(\nu w)J_p(\bar\nu w)
\end{aligned}
$$

where for brevity

$$
\Lambda_{D\alpha}(u,w) = \frac{v_{th\alpha}}{2\Omega_{g\alpha}L_{n\alpha}}\left[1 + \eta_{T\alpha}\left(\frac{v^2}{v^2_{th\alpha}} - \frac{3}{2}\right)\right]
$$

while all other terms are equal to zero.
Finally, in the drift term the nonvanishing elements are

$$
\begin{aligned}
\Pi^{B,p}_{\xi\eta} &= \frac{v_{D\alpha}}{v_{th\alpha}}w\left[\frac{p}{\nu w}J_p(\nu w)\cos\delta + iJ'_p(\nu w)\sin\delta\right] J_p(\bar\nu w)\\
\Pi^{B,p}_{\eta\eta} &= -i\frac{v_{D\alpha}}{v_{th\alpha}}w\left[J'_p(\nu w)\cos\delta + \frac{ip}{\nu w}J_p(\nu w)\sin\delta\right] J_p(\bar\nu w) \qquad (2.67)\\
\Pi^{B,p}_{\zeta\eta} &= \frac{v_{D\alpha}}{v_{th\alpha}}uJ_p(\nu w)J_p(\bar\nu w)
\end{aligned}
$$

while the other components are equal to zero.

A more compact form for these rather complicated expressions can be obtained as follows. Let

$$
\underline{\underline{R}}(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta & 0 \\ \sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k_x/k_\perp & -k_y/k_\perp & 0 \\ k_y/k_\perp & k_x/k_\perp & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.68)
$$

be the rotation which brings the x-axis into the direction of $\vec{k}_\perp = k_\xi \vec{u}_\xi + k_\eta \vec{u}_\eta$. Obviously, $\underline{\underline{R}}^{-1}(\delta) = \underline{\underline{R}}(-\delta)$. Then we will have

$$\Pi_\alpha^{0,p} = \underline{\underline{R}}(\delta) \cdot \vec{\pi}^p(\nu w) : \vec{\pi}^{p^\dagger}(\bar{\nu} w) \cdot \underline{\underline{R}}(-\bar{\delta})$$
$$\Pi_\alpha^{D,p} = \underline{\underline{R}}(\delta) \cdot \vec{\pi}^p(\nu w) : \tau_\alpha^p(\bar{\nu} w) \vec{u}_\eta' \tag{2.69}$$
$$\Pi_\alpha^{B,p} = \underline{\underline{R}}(\delta) \cdot \vec{\pi}^p(\nu w) : \Upsilon_\alpha^{p^\dagger}(\bar{\nu} w) \vec{u}_\eta'$$

with

$$\pi_\xi^p(\nu w) = \frac{p}{\nu} J_p(\nu w) \qquad \pi_\xi^{p^\dagger}(\bar{\nu} w) = \frac{p}{\bar{\nu}} J_p(\bar{\nu} w)$$
$$\pi_\eta^p(\nu w) = -iw J_p'(\nu w) \qquad \pi_\eta^{p^\dagger}(\nu' w) = -iw J_p'(\bar{\nu} w) \tag{2.70}$$
$$\pi_\zeta^p(\nu w) = u J_p(\nu w) \qquad \pi_\zeta^{p^\dagger}(\bar{\nu} w) = u J_p(\bar{\nu} w)$$

$$\tau_\alpha^p(\bar{\nu} w) = \Lambda_{D\alpha}(u, w) J_p(\bar{\nu} w) \qquad \Upsilon_\alpha^{p^\dagger}(\bar{\nu} w) = \frac{v_{D\alpha}}{v_{th\alpha}} J_p(\bar{\nu} w) \tag{2.71}$$

We recall, finally, that these expressions are written in the local field aligned frame. To obtain the conductivity operator in the laboratory frame, it is necessary to apply the rotation, defined by eqs (2.13)-(2.14). It is clear, that the complete constitutive relation in the laboratory frame will be quite complicated. In practice, it is often more convenient to write the wave equation in the local field aligned frame throughout. This is possible by writing of the differential operators, which occur in the vacuum part of Maxwell's equations in the local field aligned frame.

2.7 The constitutive relation in the space representation

We construct now the conductivity tensor operator in real space using in eqn (2.54) the definition

$$\vec{E}(\vec{k}_\perp^b, k_z) = \frac{1}{S_\perp} \int \frac{d\vec{R}_\perp'}{(2\pi)^2} \vec{E}(\vec{R}_\perp', k_z) e^{-i\vec{k}_\perp^b \vec{R}_\perp'} \tag{2.72}$$

and performing the requiring superposition over perpendicular wavenumbers. In this way we obtain

$$\vec{j}(\vec{r}_\perp, k_z) = \frac{1}{(2\pi)^2 S_\perp} \int \underline{\underline{\Sigma}}(\vec{r}_\perp, \vec{r}_\perp'; k_z) \cdot \vec{E}(\vec{r}_\perp'; k_z) d\vec{r}_\perp' \tag{2.73}$$

with

$$\underline{\underline{\Sigma}}(\vec{r}_\perp, \vec{r}'_\perp, k_z) = \sum_{\vec{k}_\perp} \sum_{\vec{k}^b_\perp} e^{i(\vec{k}_\perp \vec{r}_\perp - \vec{k}^b_\perp \vec{r}'_\perp)} \int \frac{d\vec{R}_\perp}{S_\perp} e^{i(\vec{k}^b_\perp - \vec{k}_\perp)\vec{R}_\perp} \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}^b_\perp, \vec{R}_\perp) \tag{2.74}$$

These equations together with the expressions for $\underline{\underline{\sigma}}$ obtained above, are the constitutive relation of a hot plane stratified plasma in the space representation, which can be used in practice to close Maxwell equations.

The relation of the conductivity operator obtained in the previous section with conductivity tensor of the uniform plasma can be put into evidence by assuming that the equilibrium plasma parameters vary sufficiently slowly. Writing the perturbed electrical field as a nearly plane wave

$$\vec{E}(\vec{r}_\perp, k_z) = \vec{E}_0(\vec{r}_\perp, k_z) e^{i\vec{k}^0_\perp \cdot \vec{r}_\perp} \tag{2.75}$$

with a slowly varying amplitude and rapid varying phase and substituting this into eqs (2.73)-(2.74) we obtain

$$\begin{aligned} \vec{j}(\vec{r}_\perp, k_z) &= \sum_{\vec{k}_\perp} \sum_{\vec{k}^b_\perp} \int \frac{d\vec{r}'_\perp}{(2\pi)^2 S_\perp} e^{i(\vec{k}_\perp \vec{r}_\perp - \vec{k}^b_\perp \vec{r}'_\perp)} \\ \int \frac{d\vec{r}_\perp}{(2\pi)^2 S_\perp} & e^{i(\vec{k}^b_\perp - \vec{k}_\perp)\vec{r}_\perp} \underline{\underline{\sigma}}(\vec{k}_\perp, \vec{k}^b_\perp; \vec{r}_\perp) \cdot \vec{E}_0(\vec{r}'_\perp, k_z) e^{i\vec{k}^0_\perp \cdot \vec{r}'_\perp} \end{aligned} \tag{2.76}$$

If the spatial dependence of $\underline{\underline{\sigma}}$ and of $\vec{E}_0(\vec{r}_\perp, k_z)$ (and of Θ, which enters in the definitions of k_η and k_ζ) are negligible on the scale of the typical wavelength $\left|\vec{k}^0_\perp\right|^{-1}$, we can perform both space integrations to lowest order as if these quantities were constants, obtaining two δ-functions: the integration over $\vec{r}'_\perp$ imposes $\vec{k}^b_\perp = \vec{k}^0_\perp$, and the integration over $\vec{r}_\perp$ then imposes also $\vec{k}_\perp = \vec{k}^0_\perp$. The final result is

$$\vec{j}(\vec{r}_\perp, k_\zeta) = \underline{\underline{\sigma}}(\vec{k}^0_\perp, \vec{k}^0_\perp, \vec{r}_\perp) \cdot \vec{E}_0(\vec{r}_\perp, k_z) \tag{2.77}$$

Thus, the 'diagonal part' (in the wavevector space) $\underline{\underline{\sigma}}(\vec{k}^0_\perp, \vec{k}^0_\perp, \vec{r}_\perp)$ of the conductive kernel $\underline{\underline{\sigma}}$ plays role of the local conductivity tensor in the 'local' approximation. In the present geometry, the local dispersion relation

$$\begin{aligned} det \left\{ \frac{c^2}{\omega^2} [k^0_i k^0_j - (k^0)^2 \delta_{ij}] + \epsilon_{ij}(\vec{k}^0, \omega; \vec{r}_\perp) \right\} &= 0 \\ \epsilon_{ij}(\vec{k}^0, \omega; \vec{r}_\perp) = \delta_{ij} + \frac{4\pi i}{\omega} \sigma_{ij}(\vec{k}^0_\perp, \vec{k}^0_\perp, \vec{r}_\perp) & \end{aligned} \tag{2.78}$$

specifies the admissible value k_x^0 for a given frequency and given values of ignorable components k_y^0 and k_z^0 of the wavevector.

Here we can make a short summary of the intermediate results obtained. The general integral constitutive relation for non-uniform plasmas is obtained in convolution form in the inhomogeneity direction. We can use either Fourier or real space representation of the constitutive relation. In the literature the fact, that, in principle, wave equations in non-uniform plasmas should be integral equations was considered, for example, in the papers [27], [34]-[37]. In these works, however, only some of authors [36], [27] kept the distinction between k and k'. Moreover, all authors almost immediately eliminate the convolution either by developing for small Larmor radii, or by introducing an Eikonal Ansatz. Both procedures transform the wave equations into a simpler, purely differential form.

The greatest difficulty of both the Vlasov derivation and the gyro-kinetic approache is estimation of the drift motion contribution to the wave-particle resonance. This estimation cannot be done analytically in a general case; often v_D is considered just as a constant. It has been shown numerically [24], [38]-[40] that retaining the drift velocity in the plasma-wave resonance significantly changes the thresholds of the instabilities excitation and the values of their growth rates. However, untill now the problem of efficient evaluation of the drift contribution to the plasma-wave resonance is still not completely solved.

2.8 Constitutive relation at high frequencies.

The complexity of the constitutive relation described by the conductivity operator obtained in the previous section becomes obvious from a look to the integrations which must be performed. The time integrations in eqn (2.64) are elementary, so that the integrals over parallel velocities take the form

$$
\begin{aligned}
U_{p\alpha}^{(\kappa)} &= -i\omega \int_{-\infty}^{\infty} du u^{\kappa} \frac{e^{-u^2}}{\sqrt{\pi}} \int_{-\infty}^{t} dt' e^{i(\omega - p\Omega_{g\alpha} - k_\zeta v_{th\alpha} u - \bar{k}_\eta v_{D\alpha})(t-t')} = \\
&= \int_{-\infty}^{\infty} \frac{\omega}{\omega - p\Omega_{g\alpha} - \bar{k}_\eta v_{D\alpha} - k_\zeta v_{th\alpha} u} \frac{e^{-u^2}}{\sqrt{\pi}} u^{\kappa} du
\end{aligned}
\tag{2.79}
$$

where κ is a small integer, and v_D is the total drift velocity. We recall that this definitions hold to begin with for $Im(\omega) > 0$, and must be analytically continued for $Im(\omega) < 0$. It is convenient to rewrite the function $U_n^{(\kappa)}$ as

$$
U_{p\alpha}^{(\kappa)} = U^{(\kappa)}(x_{p\alpha}^D) = -x_{0\alpha} \int_{-\infty}^{\infty} \frac{u^{\kappa}}{u - x_{p\alpha}^D} \frac{e^{-u^2}}{\sqrt{\pi}} du \tag{2.80}
$$

with

$$x_{p\alpha} = \frac{\omega - p\Omega_{g\alpha}}{k_\zeta v_{th\alpha}} \qquad\qquad x_{p\alpha}^D = \frac{\omega - p\Omega_{g\alpha} - \bar{k}_\eta v_{D\alpha}}{k_\zeta v_{th\alpha}} \tag{2.81}$$

The integration over velocities, however, are considerably more complicated, because of the fact that the drift velocity v_D is itself a function of velocity. If the slab geometry is taken strictly, and the curvature drift is omitted, v_D depends only on $v_\perp^2$; the integrations over $u = v_\parallel / v_{th}$ can then also be performed in terms of the Plasma Dispersion Function (PDF) and its derivatives

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{u - \zeta} du + i\sigma\sqrt{\pi} e^{-\zeta^2} \tag{2.82}$$

with

$$\sigma = \begin{cases} 0 & \text{for } \mathrm{Im}(\zeta) > 0 \\ 1 & \text{for } \mathrm{Im}(\zeta) = 0 \\ 2 & \text{for } \mathrm{Im}(\zeta) < 0 \end{cases} \tag{2.83}$$

We note, that by adding and subtracting $x_{p\alpha}^D$ in the numerator of the integrand we can obtain

$$\begin{aligned} U^{(0)}(x_{p\alpha}^D) &= -x_{0\alpha} Z(x_{p\alpha}^D) \\ U^{(1)}(x_{p\alpha}^D) &= \frac{x_{0\alpha}}{2} Z'(x_{p\alpha}^D) \\ U^{(2)}(x_{p\alpha}^D) &= \frac{x_{0\alpha} x_{p\alpha}^D}{2} Z'(x_{p\alpha}^D) \\ U^{(3)}(x_{p\alpha}^D) &= -\frac{x_{0\alpha}}{2} \left[1 - (x_{p\alpha}^D)^2 Z'(x_{p\alpha}^D) \right] \end{aligned} \tag{2.84}$$

The argument of PDF, however, depends now on $w^2 = v_\perp^2 / v_{th}^2$, so that the integration over the perpendicular velocity cannot be performed in closed form. If the curvature drift is retained in order to better simulate a toroidal configuration, on the other hand, v_D depends also on u^2. In this case the integration over the parallel velocity is possible only numerically.

However, in some particular cases the integration over velocities can be perform in a closed form; of course some simplifications must be applied. First of all, we will consider the high frequency limit, when

$$|\omega - p\Omega_{g\alpha} - k_\eta v_D| >> \omega_{*B} = \frac{k_\eta v_{th\alpha}^2}{2\Omega_{g\alpha}} \frac{d\Omega_{g\alpha}}{dx} \tag{2.85}$$

is valid. The waves satisfying the condition (2.85) will be called 'bulk' plasma waves; propagation of the 'bulk' plasma waves is similar to that of uniform plasma limit, only slightly modified by gradients. For this approximation v_D in the denominator of (2.79) and drift conductivity (2.67) can be neglected and all velocity integrals can be performed in closed form. We need few Bessel function identities, which easily follow from Weber integrals of the second kind [41]:

$$2\int_0^\infty e^{-\zeta w^2} J_p(\nu w) J_p(\bar{\nu} w) w dw = \frac{1}{\zeta} I_p\left(\frac{\nu\bar{\nu}}{2\zeta}\right) exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4\zeta}\right)\right\} \tag{2.86}$$

valid for $Re(\zeta) > 0$ and real index $p > -1$. Putting $\zeta \to 1$ and taking appropriate derivatives with respect to ν, $\bar{\nu}$, one can obtains the integrals which enter in $\underline{\underline{\sigma}}$:

$$\begin{aligned}
&S^p_\alpha(\nu,\bar{\nu}) \equiv 2\int_0^\infty e^{-w^2} J_p(\nu w) J_p(\bar{\nu} w) w dw = \\
&= I_p\left(\frac{\nu\bar{\nu}}{2}\right) exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4}\right)\right\} \\
&D^p_\alpha(\nu,\bar{\nu}) \equiv 2\int_0^\infty e^{-w^2} J_p(\nu w) J'_p(\bar{\nu} w) w^2 dw = \frac{\partial S^p(\nu,\bar{\nu})}{\partial \bar{\nu}} \\
&= \left[\frac{\nu}{2} I'_p\left(\frac{\nu\bar{\nu}}{2}\right) - \frac{\bar{\nu}}{2} I_p\left(\frac{\nu\bar{\nu}}{2}\right)\right] exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4}\right)\right\} \\
&T^p_\alpha(\nu,\bar{\nu}) \equiv 2\int_0^\infty e^{-w^2} J'_p(\nu w) J'_p(\bar{\nu} w) w^3 dw = \frac{\partial D^p(\nu,\bar{\nu})}{\partial \bar{\nu}} \\
&= \left[\left(\frac{p^2}{\nu\bar{\nu}} + \frac{\nu\bar{\nu}}{2}\right) I_p\left(\frac{\nu\bar{\nu}}{2}\right) - \frac{\nu^2+\bar{\nu}^2}{4} I'_p\left(\frac{\nu\bar{\nu}}{2}\right)\right] exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4}\right)\right\}
\end{aligned} \tag{2.87}$$

Deriving first respect to ζ and then proceeding as above, one obtains the additional integrals in the diamagnetic conductivity:

$$\begin{aligned}
&Y^p_\alpha(\nu,\bar{\nu}) \equiv 2\int_0^\infty e^{-w^2} J_p(\nu w) J_p(\bar{\nu} w) w^3 dw = \\
&= \left[\left(1 - \frac{\nu^2+\bar{\nu}^2}{4}\right) I_p\left(\frac{\nu\bar{\nu}}{2}\right) + \frac{\nu\bar{\nu}}{2} I'_p\left(\frac{\nu\bar{\nu}}{2}\right)\right] exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4}\right)\right\} \\
&W^p_\alpha(\nu,\bar{\nu}) \equiv 2\int_0^\infty e^{-w^2} J'_p(\nu w) J_p(\bar{\nu} w) w^4 dw = \frac{\partial Y^p(\nu,\bar{\nu})}{\partial \nu} = \\
&= \left[\left(\frac{p^2}{\nu} + \frac{\nu}{2}\frac{\nu^2+3\bar{\nu}^2}{4} - \nu\right) I_p\left(\frac{\nu\bar{\nu}}{2}\right) + \frac{\bar{\nu}}{2}\left(1 - \frac{3\nu^2+\bar{\nu}^2}{4}\right) I'_p\left(\frac{\nu\bar{\nu}}{2}\right)\right] \\
&\times exp\left\{-\left(\frac{\nu^2+\bar{\nu}^2}{4}\right)\right\}
\end{aligned} \tag{2.88}$$

The conductivity kernel can then be written

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}} = -\sum_\alpha \frac{\omega^2_{p\alpha}(\vec{R}_\perp)}{\omega^2} \sum_{p=-\infty}^{\infty} \underline{\underline{R}}(\delta) \cdot \left[\underline{\underline{\pi}}^{0,p}_\alpha(\nu,\bar{\nu}) \underline{\underline{R}}(\bar{\delta}) + \frac{v_{th\alpha}}{2\Omega_{c\alpha} L_{n\alpha}} \underline{\underline{\pi}}^{D,p}_\alpha(\nu,\bar{\nu})\right] e^{-ip(\delta-\bar{\delta})} \tag{2.89}$$

with

$$
\begin{aligned}
\pi^{0,p}_{\xi\xi} &= \frac{2p^2}{\nu\bar{\nu}} S^p_\alpha(\nu,\bar{\nu})(-x_{0\alpha}Z(x_{p\alpha})) \\
\pi^{0,p}_{\xi\eta} &= 2i\frac{p}{\nu} D^p_\alpha(\nu,\bar{\nu})(-x_{0\alpha}Z(x_{p\alpha})) \\
\pi^{0,p}_{\xi\zeta} &= \frac{p}{\nu} S^p_\alpha(\nu,\bar{\nu})(x_{0\alpha}Z'(x_{p\alpha})) \\
\pi^{0,p}_{\eta\xi} &= -2i\frac{p}{\bar{\nu}} D^p_\alpha(\bar{\nu},\nu)(-x_{0\alpha}Z(x_{p\alpha})) \\
\pi^{0,p}_{\eta\eta} &= 2T^p_\alpha(\bar{\nu},\nu)(-x_{0\alpha}Z(x_{p\alpha})) \\
\pi^{0,p}_{\eta\zeta} &= -iD^p_\alpha(\bar{\nu},\nu)(x_{0\alpha}Z'(x_{p\alpha})) \\
\pi^{0,p}_{\zeta\xi} &= \frac{p}{\bar{\nu}} S^p_\alpha(\nu,\bar{\nu})(x_{0\alpha}Z'(x_{p\alpha})) \\
\pi^{0,p}_{\zeta\eta} &= iD^p_\alpha(\nu,\bar{\nu})(x_{0\alpha}Z'(x_{p\alpha})) \\
\pi^{0,p}_{\zeta\zeta} &= S^p_\alpha(\nu,\bar{\nu})(x_{0\alpha}x_{p\alpha}Z'(x_{p\alpha}))
\end{aligned}
\tag{2.90}
$$

and

$$
\begin{aligned}
&\pi^{D,p}_{\xi\eta} = \frac{2p}{\nu}\left\{\left[\left(1-\frac{3}{2}\eta_T\right)S^p_\alpha(\nu,\bar{\nu})+\eta_T Y^p_\alpha(\nu,\bar{\nu})\right](-x_{0\alpha}Z(x_{p\alpha})) \right.\\
&\left.+\frac{\eta_T}{2}S^p_\alpha(\nu,\bar{\nu})x_{0\alpha}x_{p\alpha}Z'(x_{p\alpha})\right\} \\
&\pi^{D,p}_{\eta\eta} = -2i\left\{\left[\left(1-\frac{3}{2}\eta_T\right)D^p_\alpha(\bar{\nu},\nu)+\eta_T W^p_\alpha(\nu,\bar{\nu})\right](-x_{0\alpha}Z(x_{p\alpha})) \right.\\
&\left.+\frac{\eta_T}{2}D^p_\alpha(\bar{\nu},\nu)x_{0\alpha}x_{p\alpha}Z'(x_{p\alpha})\right\} \\
&\pi^{D,p}_{\zeta\eta} = \left[\left(1-\frac{3}{2}\eta_T\right)S^p_\alpha(\nu,\bar{\nu})+\eta_T Y^p_\alpha(\nu,\bar{\nu})\right](x_{0\alpha}Z'(x_{p\alpha})) \\
&-\eta_T S^p_\alpha(\nu,\bar{\nu})x_{0\alpha}[1-x^2_{p\alpha}Z'(x_{p\alpha})] \\
&\pi^{D,p}_{\xi\xi} = \pi^{D,p}_{\xi\zeta} = \pi^{D,p}_{\eta\xi} = \pi^{D,p}_{\eta\zeta} = \pi^{D,p}_{\zeta\xi} = \pi^{D,p}_{\zeta\zeta} = 0
\end{aligned}
\tag{2.91}
$$

contributes to the conductivity tensor.

It is now easy to recover the limit of a uniform plasma (or the lowest order in the WKB approximation), assuming that the field is a plane wave and that $\bar{\nu} \to \nu$ and $\bar{\delta} \to \delta$.

In the absence of gradients, we can, moreover, without restriction of the generality take $\delta = 0$. Defining $\lambda_\alpha = \nu^2/2 = k_\perp^2 v_{th_\alpha}^2/2\Omega_{g_\alpha}^2$, we easily obtain

$$
\begin{aligned}
\frac{4\pi i}{\omega}\sigma_{\xi\xi} &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \sum_{p=-\infty}^{+\infty} \frac{p^2}{\lambda_\alpha} I_p(\lambda_\alpha) e^{-\lambda_\alpha}(-x_{0\alpha} Z(x_{p\alpha})) \\
\frac{4\pi i}{\omega}\sigma_{\xi\eta} &= -\frac{4\pi i}{\omega}\sigma_{\eta\xi} = -i\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \sum_{p=-\infty}^{+\infty} p[I_p'(\lambda_\alpha) - I_p(\lambda_\alpha)] e^{-\lambda_\alpha}(-x_{0\alpha} Z(x_{p\alpha})) \\
\frac{4\pi i}{\omega}\sigma_{\xi\zeta} &= -\frac{4\pi i}{\omega}\sigma_{\zeta\xi} = -\frac{n_\parallel n_\perp}{2} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{c\alpha}} \frac{v_{th\alpha}^2}{c^2} \sum_{p=-\infty}^{+\infty} \frac{p}{\lambda_\alpha} I_p(\lambda_\alpha) e^{-\lambda_\alpha}(x_0^2 Z'(x_{p\alpha})) \\
\frac{4\pi i}{\omega}\sigma_{\eta\eta} &= \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \sum_{p=-\infty}^{+\infty} \left[\frac{p^2}{\lambda_\alpha} I_p(\lambda_\alpha) - 2\lambda_\alpha (I_p'(\lambda_\alpha) - I_p(\lambda_\alpha))\right] e^{-\lambda_\alpha}(-x_{0\alpha} Z(x_{p\alpha})) \\
\frac{4\pi i}{\omega}\sigma_{\eta\zeta} &= -\frac{4\pi i}{\omega}\sigma_{\zeta\eta} = i\frac{n_\parallel n_\perp}{2} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{c\alpha}} \frac{v_{th\alpha}^2}{c^2} \sum_{p=-\infty}^{+\infty} [I_p'(\lambda_\alpha) - I_p(\lambda_\alpha)] e^{-\lambda_\alpha}(x_0^2 Z'(x_{p\alpha})) \\
\frac{4\pi i}{\omega}\sigma_{\zeta\zeta} &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \sum_{p=-\infty}^{+\infty} I_p(\lambda_\alpha) e^{-\lambda_\alpha}(x_{0\alpha} x_{p\alpha} Z'(x_{p\alpha}))
\end{aligned}
\tag{2.92}
$$

where $n_\parallel = k_\parallel c/\omega$, $n_\perp = k_\perp c/\omega$.

It is easily established that in this limit $\underline{\underline{\sigma}}(\vec{k}, \vec{k})$ becomes identical with the conductivity tensor of an infinite, homogeneous plasma having the same parameters (density, temperature, static magnetic field) as the real plasma at the point $\vec{r}_\perp$ [4], [42]. The equation (2.78) is an equation for $k_\perp$ as a function of k_ζ and ω (k_ξ is obtained from the definition $k_\perp^2 = k_\xi^2 + k_\eta^2$).

Chapter 3

The Low Frequency Approximation

The general constitutive relation is given in the previous chapter by the expressions (2.64), (2.69)-(2.71) together with (2.73) and (2.74). Using an appropriate Ansatz for the perturbed fields and specifying the low frequency limit of the general constitutive relation we now obtain wave equations describing propagation of the gradient driven waves.

Firstly we will calculate the low frequency limit of the conductivity operator (2.64). If the condition

$$\omega \ll \Omega_i \tag{3.1}$$

is satisfied, the only resonant wave-particle interaction occurs at the Cherenkov resonance $p = 0$, i.e. when the phase velocity of the wave is of the order of the thermal velocity of the particles. It can be expected that as the frequency becomes comparable with the diamagnetic frequency diamagnetic effects will play an important role; moreover, Cherenkov resonances can be substantially influenced by the drift velocity of the particles. We will calculate now the most important contributions to the conductivity operator and estimate all terms (2.69) using condition (3.1). Including the drift velocity in the denominator of the resonant term ($p = 0$) and the diamagnetic contribution to the conductivity operator, which usually is neglected at higher frequencies, makes the analysis more realistic and useful.

3.1 Polarisation and $\vec{E} \times \vec{B}_0$ current

It has been already said, that Vlasov approach was initially used to describe low frequency instabilities in the magnetised plasma. In early works on drift waves (see review [43]), however, only the resonant $p = 0$ contributions to the conductivity operator were taken into account. The condition (3.1) was taken as a justification of such simplification. We will now show that, in contrast to this assumption, the sum of all non resonant higher harmonic contributions is also non-negligible in the limit of $\omega \ll \Omega_i$. Indeed, calculating this sum one obtains the polarisation and $\vec{E} \times \vec{B}_0$ currents, which were missed in the early papers.

Because of (3.1) in the harmonics $p \geq 1$ of (2.64) the parallel velocity integrals can be evaluated asymptotically. We can obviously neglect the effect of $v_{D\alpha}$ in the denominator. Therefore, the eqs (2.89)-(2.90) together with $|x_p| \gg 1$ can be used to calculate the contribution of the polarisation and $\vec{E} \times \vec{B}_0$ currents to the conductivity operator. For this purpose, we can use approximations for $p \geq 1$

$$
\begin{aligned}
-x_{0\alpha} Z(x_{p\alpha}) \sim -x_{0\alpha} x_{p\alpha} Z'(x_{p\alpha}) &\sim \frac{\omega}{\omega - p\Omega_{g\alpha}} \\
-x_{0\alpha}^2 [1 - (x_{p\alpha})^2] Z'(x_{p\alpha}) &\sim \frac{3}{4} \left(\frac{\omega}{\omega - p\Omega_{g\alpha}} \right)^2
\end{aligned}
\tag{3.2}
$$

In spite of condition (3.1), the contribution of the harmonics $|p| \geq 1$ is not simply negligible: the terms in which a factor p or p^2 compensates the inverse p-dependence in the denominators give a finite contribution also in the limit $\omega/\Omega_{g\alpha} \to 0$. We will call these contributions the 'dominant' ones. The dominant sum can be evaluated explicitly using available formulas for Bessel functions. To obtain reasonably simple results, however, we will exploit the fact that the perpendicular wavelength of low frequency waves is always much shorter than the parallel one. This imposes an approximate relation between the values of the 'poloidal' and 'toroidal' components of the wavevector

$$
k_y = -k_z \cot \Theta + \Delta k_y \qquad |\Delta k_y| \ll |k_y| \tag{3.3}
$$

If we assume also that the range of relevant values of the 'radial' component k_x, $\bar{k}_x$ are also centered on some average value with a spread which is not to large, then we obtain to a good approximation

$$
\bar{\delta} = \delta \qquad |\nu - \bar{\nu}| \ll \nu, \bar{\nu} \tag{3.4}
$$

Using (3.2) and first of (3.4) and defining for brevity

$$
\lambda_{1\alpha} = \frac{\nu \bar{\nu}}{2} \qquad \lambda_{2\alpha} = \frac{\nu^2 + \bar{\nu}^2}{4} \tag{3.5}
$$

we obtain to lowest order in $\omega/\Omega_{g\alpha}$

$$
\begin{aligned}
\sum_{p=-\infty}^{\infty} p^2 S_\alpha^p(\nu, \bar{\nu}) \frac{\omega}{\omega - p\Omega_{g\alpha}} &\to -\frac{\omega^2}{\Omega_{g\alpha}^2} [e^{\lambda_{1\alpha}} - I_0(\lambda_{1\alpha})] e^{-\lambda_{2\alpha}} \\
\sum_{p=-\infty}^{\infty} p D_\alpha^p(\nu, \bar{\nu}) \frac{\omega}{\omega - p\Omega_{g\alpha}} &\to \frac{\omega}{\Omega_{g\alpha}} \left\{ \frac{\nu}{2} [I_0'(\lambda_{1\alpha}) - I_0(\lambda_{1\alpha})] \right. \\
&\left. \quad - \frac{\nu - \bar{\nu}}{\nu} [e^{\lambda_{1\alpha}} - I_0(\lambda_{1\alpha})] \right\} e^{-\lambda_{2\alpha}}
\end{aligned}
\tag{3.6}
$$

To get (3.6) the expressions have been used [44]:

$$\sum_{p=-\infty}^{\infty} I_n(\lambda)e^{-\lambda} = 1 \qquad \sum_{p=-\infty}^{\infty} nI_n(\lambda)e^{-\lambda} = 0 \tag{3.7}$$

Taking into account the expressions (3.6) we obtain

$$\underline{\underline{\sigma}}_{pol}(\vec{k}, \vec{k}^b; \vec{R}_g) = \underline{\underline{R}}(\delta) \cdot \underline{\underline{\sigma}}_{pol}^{Smpl} \cdot \underline{\underline{R}}^{-1}(\bar{\delta}) \tag{3.8}$$

with

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{pol}^{Smpl} = \begin{pmatrix} \hat{S} & -i\hat{D} & 0 \\ i\hat{D}^{\dagger} & \hat{S} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

where

$$\begin{aligned} \hat{S} &= \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2} \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \\ \hat{D} &= -\sum_i \frac{\omega_{pi}^2}{\omega\Omega_{ci}} \left\{ [I_0'(\lambda_1) - I_0(\lambda_1)]e^{-\lambda_2} - \frac{\nu - \bar{\nu}}{\nu}[e^{\lambda_1} - I_0(\lambda_1)]e^{-\lambda_2} + 1 \right\} \\ \hat{D}^{\dagger} &= -\sum_i \frac{\omega_{pi}^2}{\omega\Omega_{ci}} \left\{ [I_0'(\lambda_1) - I_0(\lambda_1)]e^{-\lambda_2} - \frac{\bar{\nu} - \nu}{\bar{\nu}}[e^{\lambda_1} - I_0(\lambda_1)]e^{-\lambda_2} + 1 \right\} \end{aligned} \tag{3.10}$$

In $\hat{S}$ (polarisation current) only the ion contribution need to be retained, since the contribution of the electrons is $\sim m_e/m_i$ times smaller. In $\hat{D}$ (due to $\vec{E} \times \vec{B}_0$ drift motion) the contribution of the electron can be simplified due to $\nu_e \ll 1$, $\bar{\nu}_e \ll 1$, and charge neutrality.

In the field aligned frame the non zero components of $\underline{\underline{\sigma}}_{pol}$ after performing of the 'rotations' $\underline{\underline{R}}(\delta)$, $\underline{\underline{R}}^{-1}(\bar{\delta})$ are

$$
\begin{aligned}
\frac{4\pi i}{\omega}\underline{\underline{\sigma}}^{pol}_{\xi\xi} &= \frac{(k_\xi \bar{k}_\xi + k_\eta^2)\hat{S} + ik_\eta(k_\xi \hat{D} - \bar{k}_\xi \hat{D}^\dagger)}{k_\perp \bar{k}_\perp} \\
\frac{4\pi i}{\omega}\underline{\underline{\sigma}}^{pol}_{\xi\eta} &= \frac{-i(k_\xi \bar{k}_\xi \hat{D} + k_\eta^2 \hat{D}^\dagger) + (k_\xi - \bar{k}_\xi)k_\eta \hat{S}}{k_\perp \bar{k}_\perp} \\
\frac{4\pi i}{\omega}\underline{\underline{\sigma}}^{pol}_{\eta\xi} &= \frac{i(k_\xi \bar{k}_\xi \hat{D}^\dagger + k_\eta^2 \hat{D}) + (k_\xi - \bar{k}_\xi)k_\eta \hat{S}}{k_\perp \bar{k}_\perp} \\
\frac{4\pi i}{\omega}\underline{\underline{\sigma}}^{pol}_{\eta\eta} &= \frac{(k_\xi \bar{k}_\xi + k_\eta^2)\hat{S} - ik_\eta(\bar{k}_\xi \hat{D} - k_\xi \hat{D}^\dagger)}{k_\perp \bar{k}_\perp}
\end{aligned}
\tag{3.11}
$$

In the limit of zero Larmor radius we can simplify the obtained expressions using second of (3.4) to the familiar form

$$
\frac{4\pi i}{\omega}\underline{\underline{\sigma}}^{Smpl}_{pol} = \begin{pmatrix} \hat{S}_0 & -i\hat{D}_0 & 0 \\ i\hat{D}_0 & \hat{S}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{3.12}
$$

with

$$
\begin{aligned}
\hat{S}_0 &= \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2} \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \\
\hat{D}_0 &= -\sum_i \frac{\omega_{pi}^2}{\omega \Omega_{ci}} \left\{ [I_0'(\lambda_1) - I_0(\lambda_1)]e^{-\lambda_2} + 1 \right\}
\end{aligned}
\tag{3.13}
$$

3.2 Resonant terms in the bulk conductivity

In the $p = 0$ terms of (2.64) (Landau resonance of propagating waves modified by the drift motion of particles) the integrals over velocities cannot be evaluated in the closed form without approximations. These integrals are of the following types:

$$
\begin{aligned}
\mathcal{I}^{r,s}_{i,j} = &-\frac{2t_r x_0}{(s+i+j)!}\frac{2^{i+j}}{\nu^i \bar{\nu}^j}\int_0^\infty e^{-w^2} w^{2s+i+j+1} J_i(\nu w) J_j(\bar{\nu} w) dw \\
&\times \int_{-\infty}^{\infty} du \frac{e^{-u^2}}{\sqrt{\pi}} \frac{u^r}{u - x_0[1 - \varpi_B(w^2 + u^2)]}
\end{aligned}
\tag{3.14}
$$

for $i, j = 0, 1$, with

$$t_0 = 1 \quad t_1 = 2x_0 \quad t_2 = 2 \quad t_3 = \frac{4x_0}{3} \tag{3.15}$$

(the coefficients are chosen so that for $s = 0$ these integrals tend to unity in the limit $x_0 \to \infty$ and $\nu, \bar{\nu} \to 0$). Moreover, we have defined

$$\varpi_B = \frac{\omega_B}{\omega} = \frac{k_\eta v_{th\alpha}^2}{2L_B \Omega_{c\alpha}\omega} \; ; \qquad\qquad \frac{1}{L_B} = \frac{1}{B_0}\frac{dB_0}{dx} \tag{3.16}$$

With these notation

$$\underline{\underline{\sigma}}_{Ld} = \underline{\underline{R}}(\delta) \cdot \underline{\underline{\sigma}}_{Ld}^{Smpl} \cdot \underline{\underline{R}}^{-1}(\bar{\delta}) \tag{3.17}$$

with

$$\frac{4\pi i}{\omega} \underline{\underline{\sigma}}_{Ld}^{Smpl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2n_\perp \bar{n}_\perp \hat{\tau}_0 & i n_\parallel n_\perp \hat{\xi} \\ 0 & -i n_\parallel \bar{n}_\perp \hat{\xi}^\dagger & \hat{P} \end{pmatrix} \tag{3.18}$$

where $\vec{n} = c\vec{k}/\omega$ and $\vec{n}^b = c\vec{k}^b/\omega$ and

$$\begin{aligned} \hat{\tau} &= \frac{1}{2} \sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2} \frac{v_{th\alpha}^2}{c^2} \mathcal{I}_{1,1}^{0,0}(\nu, \bar{\nu}) \\ \hat{\xi} &= -\frac{1}{2} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega \Omega_{c\alpha}} \frac{v_{th\alpha}^2}{c^2} \mathcal{I}_{1,0}^{1,0}(\nu, \bar{\nu}) \\ \hat{\xi}^\dagger &= -\frac{1}{2} \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega \Omega_{c\alpha}} \frac{v_{th\alpha}^2}{c^2} \mathcal{I}_{0,1}^{1,0}(\nu, \bar{\nu}) \\ \hat{P} &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \mathcal{I}_{0,0}^{2,0}(\nu, \bar{\nu}) \end{aligned} \tag{3.19}$$

Here $\hat{\tau}$ (magnetic pumping), $\hat{\xi}$ and $\hat{\xi}^\dagger$ (mixed terms) are proportional to the normalized plasma pressure β_p, while $\hat{P} = O((m_i^2/m_e^2) \times (\Omega_{c\alpha}^2/\omega^2))$ (electron Landau damping) is by far the lagest element of the conductivity operator. In most cases, $\hat{\xi}$, $\hat{\xi}^\dagger$ and $\hat{P}$ can be simplified by retaining only the electron contribution, in which the limit of vanishing Larmor radius can be taken. Performing the transformation to the ξ, η, ζ

frame explicitly, we find

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{Ld} = \begin{pmatrix} -2n_\eta \bar{n}_\eta \hat{\tau} & 2\bar{n}_\xi n_\eta \hat{\tau} & -in_\eta n_\| \hat{\xi} \\ \\ 2n_\xi \bar{n}_\eta \hat{\tau} & -2n_\xi \bar{n}_\xi \hat{\tau} & in_\xi n_\| \hat{\xi} \\ \\ i\bar{n}_\eta n_\| \hat{\xi}^\dagger & -i\bar{n}_\xi n_\| \hat{\xi}^\dagger & \hat{P} \end{pmatrix} \tag{3.20}$$

3.3 The diamagnetic conductivity

Diamagnetic contribution to the conductivity operator will be derived from equations (2.69) (resonant part) and (2.91) (non-resonant part), respectively. Using the mathematical technique mentioned above together with (3.4), we obtain the diamagnetic conductivity in the form

$$\underline{\underline{\sigma}}_D = \underline{\underline{R}}(\delta) \cdot \underline{\underline{\sigma}}_D^{Smpl} \tag{3.21}$$

with

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_D^{Smpl} = \begin{pmatrix} 0 & \mathcal{A}^D & 0 \\ \\ 0 & -i(\mathcal{B}^D + \mathcal{H}^D) & 0 \\ \\ 0 & \dfrac{n_\zeta}{\bar{n}_\perp}\mathcal{K}^D & 0 \end{pmatrix} \tag{3.22}$$

The resonant terms are

$$\begin{aligned} \mathcal{H}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega \Omega_{c\alpha}} \frac{\bar{\omega}_{*\alpha}}{\omega} \left\{ \left(1 - \frac{3}{2}\eta_T\right) \mathcal{I}_{1,0}^{0,0} + \eta_T \left[2\mathcal{I}_{1,0}^{0,1} + \frac{1}{2}\mathcal{I}_{1,0}^{2,0} \right] \right\} \\ \mathcal{K}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\bar{\omega}_{*\alpha}}{\omega} \left\{ \left(1 - \frac{3}{2}\eta_T\right) \mathcal{I}_{0,0}^{1,0} + \eta_T \left[\mathcal{I}_{0,0}^{1,1} + \frac{3}{2}\mathcal{I}_{0,0}^{3,0} \right] \right\} \end{aligned} \tag{3.23}$$

And for non-resonant terms we have

$$\begin{aligned} \mathcal{A}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega \Omega_{c\alpha}} \frac{\bar{\omega}_{*\alpha}}{\Omega_{c\alpha}} \left\{ \left(1 + \frac{\lambda_1 - \lambda_2}{\lambda_1}\eta_T\right) \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \right. \\ & \left. -\eta_T \left[I_0'(\lambda_1) - I_0(\lambda_1) \right] e^{-\lambda_2} \right\} \\ \mathcal{B}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2} \frac{\bar{\omega}_{*\alpha}}{\Omega_{c\alpha}} \eta_T \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \end{aligned} \tag{3.24}$$

in $\mathcal{A}^D$ and $\mathcal{B}^D$ the electron contribution can be neglected because of its smallness. Here

$$\bar{\omega}_{*\alpha} = \frac{\bar{k}_\perp v_{th\alpha}^2}{2L_n\Omega_{c\alpha}} \tag{3.25}$$

is the diamagnetic frequency of species α, evaluated with the wavevector of the electrical field. In the field-aligned laboratory frame we obtain

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_D = \begin{pmatrix} 0 & \dfrac{n_\xi}{n_\perp}\mathcal{A}^D + i\dfrac{n_\eta}{n_\perp}(\mathcal{B}^D + \mathcal{H}^D) & 0 \\ 0 & \dfrac{n_\eta}{n_\perp}\mathcal{A}^D - i\dfrac{n_\xi}{n_\perp}(\mathcal{B}^D + \mathcal{H}^D) & 0 \\ 0 & \dfrac{n_\zeta}{\bar{n}_\perp}\mathcal{K}^D & 0 \end{pmatrix} \tag{3.26}$$

3.4 The local approximation

To devise approximations which allow to simplify the complicated integral equation obtained, it is instructive to write the conductivity assuming that the fields can be written in Eikonal form with slowly varying amplitude and fast varying phase. The local approximation implies that the gradients of equilibrium fields are small and one can perform the velocity integrals neglecting the drift velocity modification of the resonance. This means that in this section we take the limit

$$\nu = \bar{\nu} \qquad\qquad \frac{\nu\bar{\nu}}{2} = \frac{\nu^2 + \bar{\nu}^2}{4} = \lambda_\alpha \tag{3.27}$$

and that we drop the distinction between x_0 and x_0^D. In this limit the conductivity operator becomes algebraic and the integrals over velocity space can be performed in closed form. The results obtained in this way can be compared with those of the literature, where v_D in resonance also is neglected.

The polarisation and $\vec{E}\times\vec{B}_0$ current. With the simplifications (3.27) just introduced, polarisation and $\vec{E}\times\vec{B}$ conductivity have the form

$$\begin{aligned} \hat{S} \to \bar{S} &= \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2}\frac{1-\Gamma_0(\lambda_\alpha)}{\lambda_\alpha} \\ \hat{D}, \hat{D}^\dagger \to \bar{D} &= -\sum_i \frac{\omega_{pi}^2}{\omega\Omega_{ci}}[\Gamma_0'(\lambda_\alpha)] + 1] \end{aligned} \tag{3.28}$$

where we have defined

$$\Gamma_0(\lambda_i) = I_0(\lambda_i)e^{-\lambda_i} \qquad\qquad \Gamma_1(\lambda_i) = -\Gamma_0'(\lambda_i) = [I_0(\lambda_i) - I_0'(\lambda_i)]e^{-\lambda_i} \tag{3.29}$$

and thus

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{pol} = \begin{pmatrix} \bar{S} & -i\bar{D} & 0 \\ \\ i\bar{D} & \bar{S} & 0 \\ \\ 0 & 0 & 0 \end{pmatrix} \tag{3.30}$$

independently from the direction of $\vec{k}_\perp$.

Landau conductivity. As it was mentioned above, omitting the modification of the resonant denominator due to drift motion of particles we obtain the Landau terms in closed form. To write compactly the resonant terms in the present limit, it is useful to define the auxiliary functions

$$W_0(x) = -xZ(x) \quad W_1(x) = x^2 Z'(x) \quad W_2(x) = -\frac{4}{3}x^2[1 - x^2 Z'(x)] \tag{3.31}$$

The coefficients of the Landau conductivity in the present approximation can then be written

$$\begin{aligned} \hat{\tau} \rightarrow \bar{\tau} &= \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2}\frac{v_{th\alpha}^2}{c^2}\Gamma_1(\lambda_\alpha)W_0(x_\alpha) \\ \hat{\xi}, \hat{\xi}^\dagger \rightarrow \bar{\xi} &= -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{c\alpha}}\frac{v_{th\alpha}^2}{c^2}\Gamma_1(\lambda_\alpha)W_0(x_\alpha) \\ \hat{P} \rightarrow \bar{P} &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\Gamma_0(\lambda_\alpha)W_1(x_\alpha) \end{aligned} \tag{3.32}$$

so that in the field-aligned frame we have

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{Ld} = \begin{pmatrix} -2n_\eta^2\bar{\tau} & 2n_\xi n_\eta\bar{\tau} & -in_\eta n_\parallel\bar{\xi} \\ \\ 2n_\xi n_\eta\bar{\tau} & -2n_\xi^2\bar{\tau} & in_\xi n_\parallel\bar{\xi} \\ \\ in_\eta n_\parallel\bar{\xi} & -i\bar{n}_\xi n_\parallel\bar{\xi} & \bar{P} \end{pmatrix} \tag{3.33}$$

The diamagnetic current. In this part it is convenient to redefine the diamagnetic frequency as

$$\omega_{*\alpha} = \frac{k_\eta v_{th\alpha}^2}{2L_n\Omega_{c\alpha}} \tag{3.34}$$

proportional to the component of the wavevector perpendicular both to the static magnetic field and to the equilibrium gradients. Accordingly, we define

$$\mathcal{A}^D \to \frac{n_\perp}{n_\eta}\bar{A}^D \quad \mathcal{B}^D \to \frac{n_\perp}{n_\eta}\bar{B}^D \quad \mathcal{H}^D \to \frac{n_\perp}{n_\eta}\bar{H}^D \quad \mathcal{K}^D \to \frac{n_\perp}{n_\eta}\bar{K}^D \tag{3.35}$$

where the resonant terms become in the present limit

$$\begin{aligned}
\bar{H}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{c\alpha}}\frac{\omega_{*\alpha}}{\omega}\left\{(1-\eta_T)\,\Gamma_1(\lambda_\alpha)W_0(x_0)\right.\\
&\left.+\frac{\eta_T}{2}\left[\Gamma_1(\lambda_\alpha)W_1(x_0)+[\Gamma_0(\lambda_\alpha)-2\lambda_\alpha\Gamma_1(\lambda_\alpha)]W_0(x_0)\right]\right\}\\
\bar{K}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\frac{\omega_{*\alpha}}{\omega}\left\{\left(1-\frac{1}{2}\eta_T\right)\Gamma_0(\lambda_\alpha)]W_1(x_0)\right.\\
&\left.+\eta_T\left[\frac{3}{2}\Gamma_0(\lambda_\alpha)W_2(x_0)-\lambda_\alpha\Gamma_1(\lambda_\alpha)W_1(x_0)\right]\right\}
\end{aligned} \tag{3.36}$$

and the non-resonant contribution are

$$\begin{aligned}
\bar{A}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{c\alpha}}\frac{\omega_{*\alpha}}{\Omega_{c\alpha}}\left\{\frac{1-\Gamma_0(\lambda_\alpha)}{\lambda_\alpha}+\eta_T\Gamma_1(\lambda_\alpha)\right\}\\
\bar{B}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2}\frac{\omega_{*\alpha}}{\Omega_{c\alpha}}\eta_T\frac{1-\Gamma_0(\lambda_\alpha)}{\lambda_\alpha}
\end{aligned} \tag{3.37}$$

With substitutions (3.35) eq. (3.26) becomes

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_D = \begin{pmatrix} 0 & \dfrac{n_\xi}{n_\eta}\bar{A}^D + i(\bar{B}^D+\bar{H}^D) & 0 \\ 0 & \bar{A}^D - i\dfrac{n_\xi}{n_\eta}(\bar{B}^D+\bar{H}^D) & 0 \\ 0 & \dfrac{n_\zeta}{n_\eta}\bar{K}^D & 0 \end{pmatrix} \tag{3.38}$$

In summary, the eqs. (2.77), (3.30), (3.33) and (3.38) give the plasma current in the local approximation to the lowest order in the field-aligned frame as

$$\begin{aligned}
\frac{4\pi i}{\omega}J_\xi &= (\bar{S} - 2n_\eta^2\bar{\tau})E_\xi \\
&\quad + \left[-i\bar{D} + 2n_\xi n_\eta\bar{\tau} + \frac{n_\xi}{n_\eta}\bar{A}^D + i(\bar{B}^D + \bar{H}^D)\right]E_\eta - in_\eta n_\zeta\bar{\xi}E_\zeta \\
\frac{4\pi i}{\omega}J_\eta &= (i\bar{D} - 2n_\xi n_\eta\bar{\tau})E_\xi \\
&\quad + \left[\bar{S} - 2n_\xi^2\bar{\tau} + \bar{A}^D - i\frac{n_\xi}{n_\eta}(\bar{B}^D + \bar{H}^D)\right]E_\eta + in_\xi n_\zeta\bar{\xi}E_\zeta \\
\frac{4\pi i}{\omega}J_\zeta &= in_\eta n_\zeta\bar{\xi}E_\xi + \left[-in_\xi n_\zeta\bar{\xi} + \frac{n_\zeta}{n_\eta}\bar{K}^D\right]E_\eta + \bar{P}E_\zeta
\end{aligned} \tag{3.39}$$

It is not difficult to write this current in vector form. For the polarisation current we have

$$\frac{4\pi i}{\omega}(\vec{J}_{pol} + \vec{J}_E) = \bar{S}\vec{E}_\perp + i\bar{D}(\vec{b}\times\vec{E}_\perp) \tag{3.40}$$

The Landau current is

$$\begin{aligned}
\frac{4\pi i}{\omega}\vec{J}_{Ld} = &\left\{2\bar{\tau}\left[\vec{b}\cdot(\vec{n}_\perp\times\vec{E}_\perp)\right] - in_\zeta\bar{\xi}(\vec{E}\cdot\vec{b})\right\}(\vec{n}_\perp\times\vec{E}_\perp) \\
&+ \left\{-in_\zeta\bar{\xi}\left[\vec{b}\cdot(\vec{n}_\perp\times\vec{E}_\perp)\right] + \bar{P}(\vec{E}\cdot\vec{b})\right\}\vec{b}
\end{aligned} \tag{3.41}$$

The diamagnetic current has the form

$$\frac{4\pi i}{\omega}\vec{J}_D = \left[(\vec{b}\times\vec{u}_\xi)\cdot\vec{E}\right]\left[\bar{A}^D\frac{\vec{n}_\perp}{n_\eta} - i(\bar{B}^D + \bar{H}^D)\left(\vec{b}\times\frac{\vec{n}_\perp}{n_\eta}\right) + \frac{n_\zeta}{n_\eta}\bar{K}^D\vec{b}\right] \tag{3.42}$$

It is useful also to write down the divergencies of the currents to apply them further for derivation of the charge neutrality equation. Using the eqs. (3.40)-(3.42) we obtain:

$$\begin{aligned}
\frac{4\pi i}{\omega}\nabla(\vec{J}_{pol} + \vec{J}_E) &= iS\vec{k}_\perp\cdot\vec{E}_\perp - \bar{D}\vec{k}_\perp\cdot(\vec{b}\times\vec{E}_\perp) \\
\frac{4\pi i}{\omega}\nabla\vec{J}_{Landau} &= ik_\zeta\left\{-in_\zeta\bar{\xi}\left[\vec{b}\cdot(\vec{n}_\perp\times\vec{E}_\perp)\right] + \bar{P}(\vec{E}\cdot\vec{b})\right\} \\
\frac{4\pi i}{\omega}\nabla\vec{J}_{Diam} &= i\left\{\vec{k}_\perp\frac{\vec{n}_\perp}{n_\eta}\bar{A}^D + k_\zeta\frac{n_\zeta}{n_\eta}\bar{K}^D\right\}\left[(\vec{b}\times\vec{u}_\xi)\cdot\vec{E}\right]
\end{aligned} \tag{3.43}$$

3.5 The dispersion relation in local approximation

Now we will use the results of previous section to obtain the wave equations and dispersion relation for drift and sheared Alfvén waves in the local approximation. Using (3.39) and (2.1)-(2.2) we obtain the wave equations for propagating low frequency waves in the algebraical form, namely

$$\underline{\underline{M}} \cdot \vec{E} = 0 \tag{3.44}$$

where

$$\begin{aligned}
M_{\xi\xi} &= n_\eta^2 + n_\zeta^2 - (\bar{S} - 2n_\eta^2 \bar{\tau}) \\
M_{\xi\eta} &= -n_\xi n_\eta + \left[i\bar{D} - 2n_\xi n_\eta \bar{\tau} - \frac{n_\xi}{n_\eta} \bar{A}^D - i(\bar{B}^D + \bar{H}^D) \right] \\
M_{\xi\zeta} &= -n_\zeta (n_\xi - i n_\eta \bar{\xi}) \\
M_{\eta\xi} &= -n_\xi n_\eta - \left[i\bar{D} + 2n_\xi n_\eta \bar{\tau} \right] \\
M_{\eta\eta} &= n_\xi^2 + n_\zeta^2 - \left[\bar{S} - 2n_\xi^2 \bar{\tau} + \bar{A}^D - i\frac{n_\xi}{n_\eta}(\bar{B}^D + \bar{H}^D) \right] \\
M_{\eta\zeta} &= -n_\zeta (n_\eta + i n_\xi \bar{\xi}) \\
M_{\zeta\xi} &= -n_\zeta (n_\xi + i n_\eta \bar{\xi}) \\
M_{\zeta\eta} &= -n_\zeta \left(n_\eta - i n_\xi \bar{\xi} + \frac{n_\zeta}{n_\eta} \bar{K}^D \right) \\
M_{\zeta\zeta} &= n_\xi^2 + n_\eta^2 - \bar{P}
\end{aligned} \tag{3.45}$$

The local dispersion relation is finally obtained equating to zero the determinant of the matrix $\underline{\underline{M}}$. Exploiting the fact, that $|\bar{P}| \sim (m_i^2/m_e^2)$, while all other terms are at most $O(m_i/m_e)$, to lowest order in m_e/m_i we can obtain one root of the dispersion relation by letting $|\bar{P}| \to \infty$:

$$n_\perp^2 \simeq -\frac{(n_\zeta^2 - R)(n_\zeta^2 - L)}{(n_\zeta^2 - \bar{S})(1 + 2\bar{\tau})} \tag{3.46}$$

where, as usual, $R = \bar{S} + \bar{D}$, $L = \bar{S} - \bar{D}$. However, determination the other roots of the dispersion relation by developing in m_e/m_i is more difficult. The root (3.46) was found and investigated in the uniform cold plasma case [3], [4] of course term $\bar{\tau}$ was neglected there. Equation (3.46) describes propagation of the compressional Alfvén waves in plasma.

The compressional Alfvén waves can be hardly made unstable by the free energy of gradients, because they propagate nearly perpendicular to the magnetic field lines and there is no wave-particle resonance. It is convenient to factorise out the compressional Alfvén wave by an appropriate Ansatz [24], simplifying the form of the dispersion relation. Let us assume a field of the form

$$\vec{E} = -\vec{\nabla}\Phi + i\frac{\omega}{c}A_\zeta\vec{b}_0 \tag{3.47}$$

where $A_\zeta\vec{b}_0$ is the vector potential of the electrical field. The compressional Alfvén waves have another structure of the vector potential $\vec{A}$, therefore, we describe by this Ansatz the drift and shear Alfven waves. If we substitute (3.47) into Maxwell equation (2.4) we obtain

$$-\nabla^2\Phi + i\frac{\omega}{c}\vec{\nabla}\cdot(A_\zeta\vec{b}_0) = 4\pi\sum_\alpha \rho_\alpha \tag{3.48}$$

From (2.2) we have

$$\vec{\nabla}\times\left(\vec{\nabla}\times(A_\zeta\vec{b}_0)\right) = -i\frac{\omega}{c}\vec{E} + \frac{4\pi}{c}\sum_\alpha \vec{j}_\alpha \tag{3.49}$$

In practice, eqn (3.48) can be simplified to charge neutrality

$$0 = \sum_\alpha \rho_\alpha \tag{3.50}$$

while in eqn (3.49) the displacement current can be neglected. It is easily seen, that the perturbation satisfies $|\vec{b}_0\cdot\vec{\nabla}A_\zeta| \ll |\vec{\nabla}A_\zeta|$ (perpendicular wavelength much shorter than parallel wavelength). Then equation (3.49) becomes

$$\nabla^2 A_\zeta = -\frac{4\pi}{c}\sum_\alpha \vec{j}_\alpha\cdot\vec{b}_0 = -\frac{4\pi}{c}j_\alpha \tag{3.51}$$

in which only parallel component of the current plays a role. Finally, rather then evaluating the perturbed charge densities of individual species, it is convenient to replace (3.50) by

$$\frac{\omega}{c}\tilde{\rho} = \vec{\nabla}\cdot\vec{j} = 0 \tag{3.52}$$

Inserting the low-frequency conductivity obtained above, charge neutrality becomes

$$0 = \left[n_\perp^2(\bar{S}+\bar{A}^D) + n_\parallel^2\bar{K}^D\right]\Phi + n_\parallel\bar{P}(n_\parallel\Phi - A_\zeta) \tag{3.53}$$

while (3.49) gives

$$0 = n_\parallel(\bar{P}+\bar{K}^D)\Phi + (n_\perp^2 - \bar{P})A_\zeta \tag{3.54}$$

The dispersion relation, corresponding to these two equations is

$$(n_\perp^2 - \bar{P})(\bar{S} + \bar{A}^D) + n_\parallel^2(\bar{P} + \bar{K}^D) = 0 \tag{3.55}$$

Two limits of this dispersion relation are interesting. The electrostatic limit can be obtained directly from (3.53) (charge neutrality) by letting $A_\zeta \to 0$

$$n_\perp^2(\bar{S} + \bar{A}^D) + n_\parallel^2(\bar{P} + \bar{K}^D) = 0 \tag{3.56}$$

In the homogeneous limit $\bar{A}^D = \bar{K}^D = 0$ we recognise in eq.(3.56) the dispersion relation satisfied by 'resonance cone' waves, ion Bernstein waves, and other electrostatic waves. At frequencies below the ion cyclotron frequency in the homogeneous limit this equation does not possess propagative solution when $T_i \sim T_e$. The additional diamagnetic terms in the presence of gradients of density and temperature allow for a new class of solutions, namely, the electrostatic drift waves with frequencies of order ω_*.

The eq.(3.56) meet in the literature [24], [45]. However, the polarisation charge is there missed and, in our notation, this results can be obtained from the equation $\bar{P} + \bar{K}^D = 0$ assuming that perpendicular dynamics is neglected.

The second limiting case of interest is so-called MHD limit, $\omega \to 0$, which is obtained by letting $\bar{P} \to \infty$

$$n_\parallel^2 \simeq (\bar{S} + \bar{A}^D) \tag{3.57}$$

Recalling that $\bar{S} \simeq \omega_{pi}^2/\Omega_{ci}^2 = c^2/v_A^2$ where v_A is the Alfven speed, this equation is easily recognised as the dispersion relation of the shear Alfven waves, with corrections due to FLR and diamagnetic effects. In the MHD limit the parallel electric field vanishes, $n_\parallel \Phi - A_\zeta = 0$.

3.6 Wave equation for drift and shear Alfven waves

In the results of previous section we have supposed that the wavelengths of the propagating waves are small in comparison with characteristic lengths of the gradients. This approach helps to understand the main features of the gradient driven instabilities, but is not always valid for these waves. To make more realistic consideration, we now take into account another circumstance. Gradient driven instabilities have wavelengths comparable with the thermal Larmor radii of plasma ions. These waves propagate mostly in the diamagnetic direction, that means that $k_\eta \gg k_\xi$. The value $k_\eta \rho_i$ is order of unity, therefore, $k_\xi \rho_i \ll 1$. Smallness of the parameter $k_\xi \rho_i$ allow us to use FLR approximation in the radial direction. As a consequence, the wave equations for the gradient driven instabilities obtained in this section are differential in the radial variable, while keeping all orders in the diamagnetic direction.

We will agian use the Ansatz (3.47) of the previous section to exclude the compressional Alfvén waves from our consideration. It is convenient to express all perturbed value in accordance with [27] and [34] as

$$\begin{pmatrix} \phi(\vec{r},t) \\ \\ A_\zeta(\vec{r},t) \\ \\ \vec{j}(\vec{r},t) \end{pmatrix} = \sum_{k_y,k_z} \begin{pmatrix} \phi(x|k_y,k_z) \\ \\ A_\zeta(x|k_y,k_z) \\ \\ \vec{j}(x|k_y,k_z) \end{pmatrix} e^{i(k_y y + k_z z - \omega t)} \tag{3.58}$$

where

$$\begin{pmatrix} \phi(x|k_y,k_z) \\ \\ A_\zeta(x|k_y,k_z) \\ \\ \vec{j}(x|k_y,k_z) \end{pmatrix} = \sum_{\bar{k}_x} \begin{pmatrix} \phi(\bar{k}_x|k_y,k_z) \\ \\ A_\zeta(\bar{k}_x|k_y,k_z) \\ \\ \vec{j}(\bar{k}_x|k_y,k_z) \end{pmatrix} e^{i(\bar{k}_x x)} \tag{3.59}$$

and, inversely

$$\begin{pmatrix} \phi(\bar{k}_x|k_y,k_z) \\ \\ A_\zeta(\bar{k}_x|k_y,k_z) \\ \\ \vec{j}(\bar{k}_x|k_y,k_z) \end{pmatrix} = \int \frac{d\bar{x}}{L} \begin{pmatrix} \phi(x|k_y,k_z) \\ \\ A_\zeta(x|k_y,k_z) \\ \\ \vec{j}(x|k_y,k_z) \end{pmatrix} e^{-i(\bar{k}_x \bar{x})} \tag{3.60}$$

The arguments which precede the vertical bar refer to the coordinate x in which the plasma is inhomogeneous. Those following the bar refer to the ignorable coordinates y and z; they could be equivalently written in terms of the field-aligned coordinates η and ζ. In the following, we will omit these quantities, unless required for clarity.

With this notations, the general constitutive relation is

$$\vec{j}(x) = \sum_{k_x} e^{ik_x x} \sum_{\bar{k}_x} \underline{\underline{\Sigma}}(k_x,\bar{k}_x) \cdot \vec{E}(\bar{k}_x) \tag{3.61}$$

Substituting here the fields derived from the above potentials, we obtain in this case

$$\begin{aligned} \begin{pmatrix} \vec{\nabla}\cdot\vec{j} \\ \\ j_z \end{pmatrix} &= \sum_{k_x} e^{ik_x x} \sum_{\bar{k}_x} \begin{pmatrix} \vec{k} \\ \\ -i\vec{b} \end{pmatrix} \cdot \underline{\underline{\Sigma}}(k_x,\bar{k}_x|k_y,k_z) \cdot \\ &\cdot \int \frac{d\bar{x}}{L} \left[\vec{k}^b \phi(\bar{x}|k_y,k_z) - k_0 \vec{b} A_\zeta(\bar{x}|k_y,k_z) \right] e^{-i\bar{k}_x \bar{x}} \end{aligned} \tag{3.62}$$

Compact form of the wave equation, therefore, is obtained by defining of the four scalar operators:

$$
\begin{aligned}
\mathcal{X}(k_x,\bar{k}_x|k_y,k_z) &= \frac{4\pi i}{\omega}\;\frac{\vec{k}}{k}\cdot\underline{\underline{\Sigma}}(k_x,\bar{k}_x|k_y,k_z)\cdot\frac{\vec{k}^b}{k}\\
\mathcal{Y}(k_x,\bar{k}_x|k_y,k_z) &= \frac{4\pi i}{\omega}\;\frac{\vec{k}}{k}\cdot\underline{\underline{\Sigma}}(k_x,\bar{k}_x|k_y,k_z)\cdot\vec{b}\\
\mathcal{Y}^\dagger(k_x,\bar{k}_x|k_y,k_z) &= \frac{4\pi i}{\omega}\;\vec{b}\cdot\underline{\underline{\Sigma}}(k_x,\bar{k}_x|k_y,k_z)\cdot\frac{\vec{k}^b}{k}\\
\mathcal{Z}(k_x,\bar{k}_x|k_y,k_z) &= \frac{4\pi i}{\omega}\;\vec{b}\cdot\underline{\underline{\Sigma}}(k_x,\bar{k}_x|k_y,k_z)\cdot\vec{b}
\end{aligned}
\tag{3.63}
$$

In the uniform limit $\mathcal{X}$ becomes the dielectric constant of the plasma, while $\mathcal{Z}$ is proportional to $\sigma_{\zeta\zeta}$. With these definitions, the wave equations in the Fourier representation can be written:

$$
\begin{aligned}
0 = -\frac{4\pi}{k^2}\rho(k_x) &= \sum_{\bar{k}_x}\left[\mathcal{X}(k_x,\bar{k}_x)\phi(\bar{k}_x) - \frac{\omega}{ck}\mathcal{Y}(k_x,\bar{k}_x)A_\zeta(\bar{k}_x)\right]\\
-\frac{c^2k_\perp^2}{\omega^2}A_\zeta(k_x) &= \sum_{\bar{k}_x}\left[\frac{c\bar{k}}{\omega}\mathcal{Y}^\dagger(k_x,\bar{k}_x)\phi(\bar{k}_x) - \mathcal{Z}(k_x,\bar{k}_x)A_\zeta(\bar{k}_x)\right]
\end{aligned}
\tag{3.64}
$$

for each k_x. In turn, if $\mathcal{T}$ is any of the quantities $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Y}^\dagger$ or $\mathcal{Z}$, we can write

$$
\mathcal{T}(k_x,\bar{k}_x|k_y,k_z) = \int\frac{d\hat{x}}{L}e^{i(\bar{k}_x-k_x)\hat{x}}\mathcal{T}(k_x,\bar{k}_x|k_y,k_z;\hat{x}) \tag{3.65}
$$

with

$$
\begin{aligned}
\tilde{\mathcal{X}}(k_x,\bar{k}_x|k_y,k_z;\hat{x}) &= \frac{1}{k^2}\left[(k_x\bar{k}_x+k_\eta^2)(\hat{S}+\mathcal{A}^D) - i(k_x-\bar{k}_x)k_\eta\hat{D} + k_z^2(\hat{P}+\mathcal{K}^D)\right]\\
\tilde{\mathcal{Y}}(k_x,\bar{k}_x|k_y,k_z;\hat{x}) &= \frac{k_z}{k}\hat{P}\\
\tilde{\mathcal{Y}}^\dagger(k_x,\bar{k}_x|k_y,k_z;\hat{x}) &= \frac{k_z}{k}(\hat{P}+\mathcal{K}^D)\\
\tilde{\mathcal{Z}}(k_x,\bar{k}_x|k_y,k_z;\hat{x}) &= \hat{P}
\end{aligned}
\tag{3.66}
$$

These integral wave equations can be substantially simplified by assuming again potentials in the Eikonal form

$$
\begin{pmatrix}\phi(x)\\ A_\zeta(x)\end{pmatrix} = \begin{pmatrix}\phi(\epsilon x)\\ A_\zeta(\epsilon x)\end{pmatrix}e^{ik_0x} \tag{3.67}
$$

(with $\epsilon \ll 1$), so that

$$\begin{pmatrix} \phi(k_x) \\ \\ A_\zeta(k_x) \end{pmatrix} = \int \frac{b\bar{x}}{L} e^{-i(\bar{k}_x - k_0)\bar{x}} \begin{pmatrix} \phi(\epsilon\bar{x}) \\ \\ A_\zeta(\epsilon\bar{x}) \end{pmatrix} e^{ik_0 x} \tag{3.68}$$

Current and charge densities will have similar form. Substituting this Ansatz into (3.64) we obtain

$$\vec{U}(\epsilon x) = \sum_{k_x} e^{i(k_x - k_0)x} \sum_{\bar{k}_x} \int \frac{d\hat{x}}{L} e^{i(\bar{k}_x - k_x)\hat{x}} \underline{\underline{\tilde{W}}}(k_x, \bar{k}_x; \epsilon\hat{x}) \int \frac{d\bar{x}}{L} e^{-i(\bar{k}_x - k_0)\bar{x}} \vec{V}(\epsilon\bar{x}) \tag{3.69}$$

where we have defined

$$\vec{U} = \begin{pmatrix} -\dfrac{4\pi}{k^2}\rho \\ \\ -\dfrac{4\pi i}{\omega} j_z \end{pmatrix} \quad \underline{\underline{\tilde{W}}} = \begin{pmatrix} \tilde{X} & -\dfrac{\omega}{ck}\tilde{Y} \\ \\ \dfrac{c\bar{k}}{\omega}\tilde{Y}^\dagger & -\tilde{Z} \end{pmatrix} \quad \begin{pmatrix} \phi \\ \\ A_\zeta \end{pmatrix} \tag{3.70}$$

In the limit $\epsilon \to 0$ the integration over $\bar{x}$ gives $\delta(\bar{k}_x - k_0)$, and the summation over k_x gives $\delta(\hat{x} - x)$; hence, to lowest order in the WKB approximation we obtain the local equation

$$\begin{pmatrix} 0 \\ \\ -\dfrac{ck_\perp^2}{\omega^2} A_\zeta(x) \end{pmatrix} = \begin{pmatrix} \tilde{X}(k_0, k_0; x) & -\dfrac{\omega}{ck}\tilde{Y}(k_0, k_0; x) \\ \\ \dfrac{c\bar{k}}{\omega}\tilde{Y}^\dagger(k_0, k_0; x) & -\tilde{Z}(k_0, k_0; x) \end{pmatrix} \begin{pmatrix} \phi(x) \\ \\ A_\zeta(x) \end{pmatrix} \tag{3.71}$$

To this order we easily recover the local dispersion relation. To obtain differential equations for the envelopes, we must take into account terms of higher order in ϵ.

For this purpose, we rearrange first sums and integrations as

$$\vec{U}(\epsilon x) = \sum_{k_x} \sum_{\bar{k}_x} \int \frac{d\bar{x}}{L} \int \frac{d\hat{x}}{L} e^{i(k_x - k_0)(x - \hat{x})} \underline{\underline{\tilde{W}}}(k_x, \bar{k}_x; \epsilon\hat{x}) \cdot e^{-i(\bar{k}_x - k_0)(\bar{x} - \hat{x})} \vec{V}(\epsilon\bar{x}) \tag{3.72}$$

Taking advantage of the localisation of the WKB wavepackets in k_x-space, we next develop the matrix $\underline{\underline{\tilde{W}}}$ around k_0

$$\begin{aligned} \underline{\underline{\tilde{W}}}(k_x, \bar{k}_x; \epsilon\hat{x}) &\simeq \underline{\underline{\tilde{W}}}_0 + (k_x - k_0)\frac{\partial \underline{\underline{\tilde{W}}}_0}{\partial k_x} + (\bar{k}_x - k_0)\frac{\partial \underline{\underline{\tilde{W}}}_0}{\partial \bar{k}_x} \\ &+ \frac{(k_x - k_0)^2}{2}\left(\frac{\partial^2 \underline{\underline{\tilde{W}}}}{\partial k_x^2}\right)_0 + (k_x - k_0)(\bar{k}_x - k_0)\left(\frac{\partial^2 \underline{\underline{\tilde{W}}}}{\partial k_x \partial \bar{k}_x}\right)_0 + \frac{(\bar{k}_x - k_0)^2}{2}\left(\frac{\partial^2 \underline{\underline{\tilde{W}}}}{\partial \bar{k}_x^2}\right)_0 + \ldots \end{aligned} \tag{3.73}$$

where the subscript 0 denoted quantities taken at $k_x = \bar{k}_x = k_0$. Now we note the identities

$$
\begin{aligned}
(k_x - k_0)e^{i(k_x-k_0)(x-\hat{x})} &= i\frac{\partial}{\partial \hat{x}}\left(e^{i(k_x-k_0)(x-\hat{x})}\right) \\
(\bar{k}_x - k_0)e^{-i(\bar{k}_x-k_0)(\bar{x}-\hat{x})} &= i\frac{\partial}{\partial \bar{x}}\left(e^{-i(\bar{k}_x-k_0)(\bar{x}-\hat{x})}\right)
\end{aligned}
\tag{3.74}
$$

Substituting these identities in eq.(3.72) and integrating by parts gives

$$
\begin{aligned}
&\vec{U}(\epsilon x)) = \sum_{k_x}\sum_{\bar{k}_x}\int\frac{d\bar{x}}{L}\int\frac{d\hat{x}}{L}e^{i(k_x-k_0)(x-\hat{x})} \\
&\times\left\{\underline{\underline{\tilde{W}}}_0(\epsilon\hat{x})\cdot\vec{V}(\epsilon\bar{x}) - i\left[\frac{\partial}{\partial\hat{x}}\left(\frac{\partial\underline{\underline{\tilde{W}}}}{\partial k_x}\cdot\vec{V}\right)_0 + \left(\frac{\partial\underline{\underline{\tilde{W}}}}{\partial\bar{k}_x}\right)_0\cdot\frac{\partial\vec{V}}{\partial\bar{x}}\right]\right. \\
&-\frac{1}{2}\left[\frac{\partial^2}{\partial\hat{x}^2}\left(\frac{\partial^2\underline{\underline{\tilde{W}}}}{\partial k_x^2}\right)_0\cdot\vec{V} + 2\frac{\partial}{\partial\hat{x}}\left(\frac{\partial^2\underline{\underline{\tilde{W}}}}{\partial k_x^2} + \frac{\partial^2\underline{\underline{\tilde{W}}}}{\partial k_x\partial\bar{k}_x}\right)_0\cdot\frac{\partial\vec{V}}{\partial\bar{x}}\right. \\
&\left.\left.+\left(\frac{\partial^2\underline{\underline{\tilde{W}}}}{\partial k_x^2} + \frac{2\partial^2\underline{\underline{\tilde{W}}}}{\partial k_x\partial\bar{k}_x} + \frac{\partial^2\underline{\underline{\tilde{W}}}}{\partial\bar{k}_x^2}\right)_0\cdot\frac{\partial^2\vec{V}}{\partial\bar{x}^2}\right]\right\}e^{-i(\bar{k}_x-k_0)(\bar{x}-\hat{x})} + \ldots
\end{aligned}
\tag{3.75}
$$

Now the summation over k_x gives $\delta(\bar{x}-\hat{x})$, and the summation over $\bar{k}_x$ gives $\delta(\hat{x}-x)$. Thus, omitting the expansion parameter ϵ and the subscript 0, writing again k_x instead of k_0 for the mean wavevector of the wavepackets, we finally obtain

$$
\begin{aligned}
&\vec{U}(x) = \underline{\underline{W}}_0(x)\cdot\vec{V}(x) - i\left[\frac{\partial}{\partial x}\left(\frac{\partial\underline{\underline{W}}}{\partial k_x}\cdot\vec{V}\right) - \frac{\partial\underline{\underline{W}}}{\partial\bar{k}_x}\cdot\frac{\partial\vec{V}}{\partial x}\right]_{\bar{k}_x=k_x} \\
&-\frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\underline{\underline{W}}}{\partial k_x^2}\cdot\vec{V}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial^2\underline{\underline{W}}}{\partial k_x\partial\bar{k}_x}\cdot\frac{\partial\vec{V}}{\partial x}\right) + \frac{\partial^2\underline{\underline{W}}}{\partial\bar{k}_x^2}\cdot\frac{\partial^2\vec{V}}{\partial x^2}\right]_{\bar{k}_x=k_x}
\end{aligned}
\tag{3.76}
$$

Once the derivatives are performed and the distinction between barred and unbarred wavevector is droped, k_x plays the role of a parameters, to be chosen, for example, such, that the growth rate $Im(\omega)$ is maximised. As a particular case, if we now take the limit $k_x \to 0$ (of course, after the derivatives with respect to k_x and $\bar{k}_x$ have been taken), eq. (3.76) becomes a finite radius approximation in the radial direction x, with all orders in the Larmor radius kept in the poloidal direction η. This particular approximation applies to many low frequency waves, for which $|k_x| \ll |k_\eta|$ is usually satisfied. When this is the case, $\delta \sim \bar{\delta} \sim \pi/2$, as antisipated earlier.

When the wavenumbers derivatives are taken using the expressions (3.66) for the coefficients, it becomes clear that there are only three leading terms which survives in the zero Larmor radius approximation $k_x \rho_i \ll 1$, namely

$$\left.\frac{\partial \tilde{\mathcal{X}}}{\partial k_x}\right|_{k_x=0} = -i\frac{k_\eta}{k^2}\hat{D} \qquad \left.\frac{\partial \tilde{\mathcal{X}}}{\partial \bar{k}_x}\right|_{k_x=0} = i\frac{k_\eta}{k^2}\hat{D}$$
$$\left.\frac{\partial^2 \tilde{\mathcal{X}}}{\partial k_x \partial \bar{k}_x}\right|_{k_x=0} = i\frac{1}{k^2}(\hat{S} + \mathcal{A}^D) \tag{3.77}$$

while all other contributions are finite Larmor radius corrections. If we keep only these terms, the wave equations are reduced to

$$0 = -\frac{1}{k^2}\frac{d}{dx}\left(\left(\hat{S} + \mathcal{A}^D\right)\frac{d\phi}{dx}\right) + \tilde{\mathcal{X}}(x)\phi - \frac{\omega}{ck}\tilde{\mathcal{Y}}(x)A_\zeta$$
$$0 = \frac{c^2}{\omega^2}\left(-\frac{d^2 A_\zeta}{dx^2} + k_\eta^2 A_\zeta\right) + \frac{c\bar{k}}{\omega}\tilde{\mathcal{Y}}^\dagger(x)\phi - \tilde{\mathcal{Z}}(x)A_\zeta \tag{3.78}$$

By assuming that solution can be written in WKB form, the local dispersion relation corresponding to these wave equations is

$$\tilde{\mathcal{X}}(n_\perp^2 - \tilde{\mathcal{Z}}) + \tilde{\mathcal{Y}}\tilde{\mathcal{Y}}^\dagger = 0 \tag{3.79}$$

which in the electrostatic limit reduces to

$$\tilde{\mathcal{X}} = 0 \tag{3.80}$$

These two equations are easily seen to be identical with (3.55) and (3.56), except that while $n_\perp^2 = n_x^2 + n_\eta^2$ when it is written explicitly, in the argument of the Bessel functions λ_α is defined as

$$\lambda_\alpha = k_\eta^2 v_{th_\alpha}^2 / 2\Omega_{g\alpha}^2 \tag{3.81}$$

In the MHD limit ($\hat{P} \to \infty$) we obtain the equation (3.57).

By comparison of the coefficients of the (3.80) or (3.56) it is easily seen, that plasma ions mostly contribute to the polarisation current, determining the perpendicular dynamic of electrostatic drift waves. The parallel dynamic is defined mostly by the longitudinal motion of the electrons. In the early papers the polarisation current of ions was missed, it was rediscovered only in 1982 [25], improving the understanding of the drift waves structure.

Chapter 4

Toroidal Plasma

4.1 The tokamak

In this chapter we describe the tokamak plasma, which is simple and to this day the most successful means to obtain fusion power. Of course, to investigate propagation of the waves in the tokamak plasma one should have at least a superficial knowledge of the tokamak plasma physics. We will briefly describe in the next few sections main features of the toroidal plasma consideration.

The equations describing the tokamak plasma equilibrium are those of the ideal steady-state magneto-hydrodynamic (MHD):

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} \tag{4.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4.2}$$

$$\vec{j} \times \vec{B} = c\vec{\nabla} p \tag{4.3}$$

The first two are just Maxwell equations in the appropriate limit; note that (4.1) implies the steady-state form of charge conservation, namely

$$\vec{\nabla} \cdot \vec{j} = 0 \tag{4.4}$$

Equation (4.3) on the other hand is the condition for equilibrium between magnetic and pressure forces at each point in the plasma, under the assumption of isotopic pressure. From this force balance, we can immediately derive important information about the general structure of MHD plasma equilibria. Taking the scalar product of both side with $\vec{B}$ and $\vec{J}$, we have:

$$\vec{B} \cdot \vec{\nabla} p = 0 \qquad \vec{j} \cdot \vec{\nabla} p = 0 \tag{4.5}$$

According to these equations, both the magnetic field lines and the current lines must lie on surfaces of constant pressure. In particular, therefore, if the plasma has to be confined in a closed configuration, magnetic field lines must lie entirely within the plasma volume. A toroidal configuration of the vessel allows easily to satisfy this criterion.

The simplest toroidal configurations are those which are axisymmetric around a vertical axis. Axisymmetry allows a rigorous proof that magnetic field lines form magnetic surfaces, hence, that surfaces of constant pressure do exist. Among axisymmetric configuration, the tokamak is characterised by a strong toroidal magnetic field, created by a set of external coils surrounding the vacuum vessel, and a weaker, mainly poloidal, magnetic field due to the current flowing in the plasma itself. The resulting helical magnetic field lines form an infinite set of nested toroidal magnetic surfaces. The main toroidal field is a good approximation to the vacuum field. Its magnitude increases toward the inside approximately as the inverse of the distance R from the vertical axis:

$$\vec{B} \simeq \vec{B}_0 \frac{R_0}{R} \tag{4.6}$$

In addition, external poloidal coils producing a still weaker vertical field to ensure the equilibrium of the plasma column with respect to horizontal expansion.

Of course a real tokamak is not a perfect structure; here, however, we will not consider the consequences of the resulting small deviations from axisymmetry. Since $B_{pol} \ll B_{tor}$, magnetic field lines encircle the magnetic axis with a relatively long pitch, making several turns in the toroidal direction for one in poloidal. In the simplest case of a circular tokomak with large aspect ratio (ratio of the toroidal radius R_0 to the plasma radius a), the pitch can be characterised by the rotational transform

$$\iota = \frac{R_0 B_{pol}}{r B_{tor}} \tag{4.7}$$

where r is the distance from the magnetic axis. Its inverse is safety factor

$$q = \frac{r B_{tor}}{R_0 B_{pol}} \tag{4.8}$$

The value of q at the plasma edge $r = a$ is related to the total current I in the plasma: if toroidicity is neglected and the discharge is assimilated to a straight circular cylinder one finds

$$q_{cyl}(a) = 5 \frac{a^2 B_{tor}}{R_0 I} \tag{4.9}$$

(R_0 and a in m, I in MA, B_{tor} in Tesla). The value q is one of the most important parameters, defining the stability of the tokamak plasma. The tokamak plasma is unstable with respect to ideal MHD modes if the total current exceed the value which corresponds to $q(0) = 1$, i.e. the safety factor at the magnetic axis cannot be made smaller than about unity. As a consequence $q_{cyl}(a)$ lies typically between 2 and 5.

4.2 Toroidal coordinates

In this section we introduce the curvilinear coordinates appropriate to the description of the plasma behaviour in the tokamak equilibrium magnetic field. The tokamak magnetic field structure and the main features of the noncartesian coordinate systems used are shown in the [46].

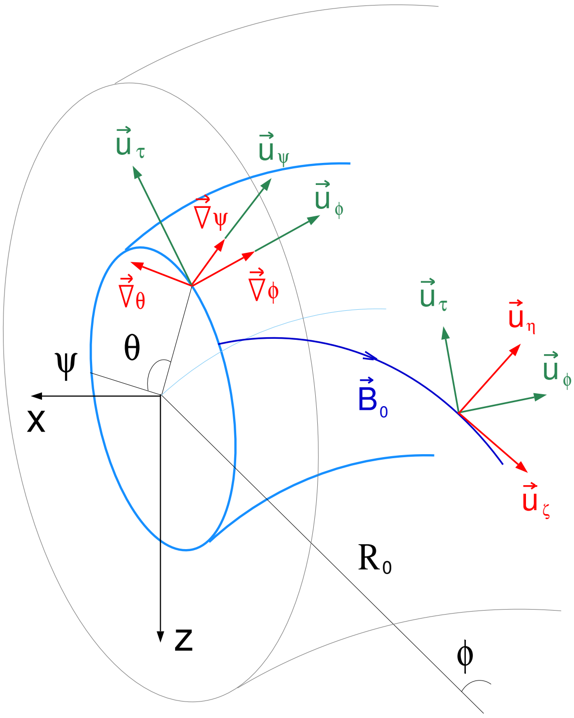

Figure 4.1: Coordinates in the toroidal geometry.

A general axisymmetric equilibrium can be conveniently described by giving the two functions

$$X = X(\psi, \theta) \qquad Z = Z(\psi, \theta) \tag{4.10}$$

or their inverse

$$\psi = \psi(X, Z) \qquad \theta = \theta(X, Z) \tag{4.11}$$

Here X, Z are horizontal and vertical Cartesian coordinates in the poloidal cross-section, with origin at a distance R_0 from the vertical axis. Thus $R = R_0 + X$, Z, ϕ are quasi-cylindrical coordinate system. The parameter ψ, which labels the magnetic surfaces, the poloidal angle θ, and the toroidal angle $\varphi = -\phi$ will be called 'magnetic coordinates' (Fig. 4.1). The change in sign is needed to make both system right-handed if θ is counted, as it is customary, from the outer equatorial point. The 'poloidal' Jacobian

$$J_p = \left(\frac{\partial X}{\partial \psi}\right)\left(\frac{\partial Z}{\partial \theta}\right) - \left(\frac{\partial Z}{\partial \psi}\right)\left(\frac{\partial X}{\partial \theta}\right) \tag{4.12}$$

vanished only on the magnetic axis (and at the hyperbolic points on the separatrix, if present. Therefore, we can use the magnetic coordinates in the whole plasma volume excepting the regions mentioned above.

The covariant metric for the magnetic coordinates (ψ, θ, φ) has the view

$$g_{ij} = \begin{pmatrix} N_\psi^2 & G_p & 0 \\ G_p & N_\theta^2 & 0 \\ 0 & 0 & P \end{pmatrix} \tag{4.13}$$

with determinant

$$g = (det(g_{ij}))^{1/2} = RJ_p \tag{4.14}$$

Here N_θ^2 (which is essentially the 'small radius' squared) is given by

$$N_\theta^2 = \left(\frac{\partial X}{\partial \theta}\right)^2 + \left(\frac{\partial Z}{\partial \theta}\right)^2 \tag{4.15}$$

and

$$\begin{aligned} N_\psi^2 &= \left(\frac{\partial X}{\partial \psi}\right)^2 + \left(\frac{\partial Z}{\partial \psi}\right)^2 \\ G_p &= \left(\frac{\partial X}{\partial \psi}\right)\left(\frac{\partial X}{\partial \theta}\right) + \left(\frac{\partial Z}{\partial \psi}\right)\left(\frac{\partial Z}{\partial \theta}\right) \end{aligned} \tag{4.16}$$

In general the coordinates ψ, θ, φ are not orthogonal, although in axisymmetric configurations they satisfy the "partial-orthogonality" conditions

$$\vec{\nabla}\psi \cdot \vec{\nabla}\varphi = 0 \qquad \vec{\nabla}\theta \cdot \vec{\nabla}\varphi = 0 \tag{4.17}$$

The constitutive relation in the plasma is most conveniently expressed in terms of 'physical' representation of vectors. Let us construct a local orthonormal basis of this representation. At each point, the unit vector $\vec{u}_\psi = \vec{\nabla}\psi/|\vec{\nabla}\psi|$ is orthogonal to the magnetic surface and lies in the poloidal cross-section of the tokamak. This vector is orthohonal to the second unit vector $\vec{u}_\varphi$ pointing in the toroidal direction. The third unit vector we obtain as $\vec{u}_\tau = \vec{u}_\varphi \times \vec{u}_\psi$, it is tangent to the magnetic surface (since the coordinates in the poloidal cross-section are not orthogonal, $\vec{u}_\tau$ is not parallel to $\vec{\nabla}\theta$). The expressions for $\vec{u}_\psi$ and $\vec{u}_\tau$ can be written

$$
\begin{aligned}
\vec{u}_\psi &= \frac{1}{N_\theta}\left(\frac{\partial Z}{\partial \theta}\vec{u}_x - \frac{\partial X}{\partial \theta}\vec{u}_z\right) \\
\vec{u}_\tau &= \frac{1}{N_\theta}\left(\frac{\partial X}{\partial \theta}\vec{u}_x + \frac{\partial Z}{\partial \theta}\vec{u}_z\right)
\end{aligned}
\tag{4.18}
$$

Together with the unit vector in the toroidal direction, $\vec{u}_\varphi$, they constitute a convenient local orthogonal basis. Using definitions of the gradients, eqs.(4.18) can be written in the form

$$
\begin{aligned}
\vec{u}_\psi &= \frac{J_p}{N_\theta}\vec{\nabla}\psi \\
\vec{u}_\tau &= N_\theta\left(\vec{\nabla}\theta + \frac{G_p}{N_\theta^2}\vec{\nabla}\psi\right) \\
\vec{u}_\varphi &= R\vec{\nabla}\varphi
\end{aligned}
\tag{4.19}
$$

As in the slab geometry case, we will use a field aligned frame ($\vec{u}_\xi$, $\vec{u}_\eta$, $\vec{u}_\zeta$) to separate the parallel and perpendicular motions of plasma particles. This frame is obtained by rotating of the two tangent to the magnetic surface unit vectors $\vec{u}_\tau$ and $\vec{u}_\varphi$ around $\vec{u}_\xi$ so that one of the rotated unit vectors becomes parallel to the static magnetic field:

$$
\begin{aligned}
\vec{u}_\xi &= \vec{u}_\psi \\
\vec{u}_\eta &= \cos\Theta\vec{u}_\tau - \sin\Theta\vec{u}_\phi \\
\vec{u}_\zeta &= \sin\Theta\vec{u}_\tau + \cos\Theta\vec{u}_\phi
\end{aligned}
\tag{4.20}
$$

where

$$
\tan\theta = \frac{B_\tau}{B_\phi} = \frac{N_\theta}{qR} \tag{4.21}
$$

is the local pitch of the static magnetic field (the second expression holds only in the so called flux coordinates). Thus $\vec{u}_\zeta = \vec{B}_0/B_0$ is the unit vector tangent to the magnetic field line. The direction of $\vec{u}_\eta$, orthogonal to $\vec{B}_0$ and tangent to the magnetic surface, will be called the 'diamagnetic direction' by analogy with the plane stratified plasma limit.

Using the physical components of the velocity in the field-aligned frame, we can introduce a convenient set of coordinates in velocity space, which separate explicitly the rapidly varying gyroangle ϕ_v from the specific energy ϵ_v, which is a constant of the motion, and μ_v, which is a adiabatically invariant.

$$\epsilon_v = \frac{1}{2}(v_\parallel^2 + v_\perp^2) \qquad \mu_v = \frac{v_\perp^2}{2B_0} \qquad \phi_v = \tan^{-1}(v_\eta/v_\xi) \tag{4.22}$$

The total velocity and the position of a particle can be written (see Sec. 2.3)

$$\begin{aligned} \vec{v} &= \vec{v}_D + v_\parallel \vec{u}_\zeta + v_\perp(\vec{u}_\xi \cos\phi_v + \vec{u}_\eta \sin\phi_v) \\ \vec{r} &= \vec{r}_g - \frac{v_\perp}{\Omega_c}(\vec{u}_\xi \sin\phi_v - \vec{u}_\eta \cos\phi_v) \end{aligned} \tag{4.23}$$

respectively, where $\vec{v}_D$ and $\vec{r}_g$ are the drift velocity and the position of the guiding center, and the parallel and perpendicular velocity are expressed in the terms of ϵ_v and μ_v by inverting (4.22):

$$v_\perp = \sqrt{2\mu_v B_0} \qquad\qquad v_\parallel = \sigma_v\sqrt{2(\epsilon_v - \mu_v B_0)} \tag{4.24}$$

where σ_v is the sign of the parallel velocity. We recall also that the Jacobian of the drift velocity coordinates is

$$J_D = \frac{\partial(v_\xi, v_\eta, v_\zeta)}{\partial(\mu_v, \epsilon_v, \phi_v)} = \frac{B_0}{v_\parallel} \tag{4.25}$$

4.3 The tokomak magnetic field.

The most general axisymmetric static magnetic field can be written

$$\vec{B} = \mathcal{F}_p(\psi)\vec{\nabla}\varphi \times \vec{\nabla}\psi + \mathcal{G}(\psi)\vec{\nabla}\varphi \tag{4.26}$$

where $\mathcal{F}_p(\psi)$ and $\mathcal{G}(\psi)$ are in general arbitrary function of ψ parameter. Then the equation $\vec{\nabla} \cdot \vec{B} = 0$ and the first of conditions (4.5), namely $B_\psi = 0$, are by (4.26) automatically satisfied. These conditions alone would allow $\mathcal{G} = \mathcal{G}(\psi, \theta)$; restricting $\mathcal{G} = \mathcal{G}(\psi)$ to depend only on ψ ensures $j_\psi = 0$, so that the second of eqn. (4.5) is also satisfied. The two terms of (4.26) corresponds to the decomposition of $\vec{B}$ into a poloidal and a toroidal component:

$$B_\tau = \frac{N_\theta}{RJ_p}\mathcal{F}_p(\psi) \qquad\qquad B_\varphi = \frac{\mathcal{G}(\psi)}{R} \tag{4.27}$$

We can rewrite (4.26) in the contravariant form

$$\vec{B} = \mathcal{F}_p(\psi)\vec{\nabla}\varphi \times \vec{\nabla}\psi + \frac{J_p}{R}\mathcal{G}(\psi)\vec{\nabla}\psi \times \vec{\nabla}\theta \tag{4.28}$$

so that the contravariant components are found to be

$$\tilde{B}^{\psi} = 0 \qquad \tilde{B}^{\theta} = \frac{\mathcal{F}_p(\psi)}{RJ_p} \qquad \tilde{B}^{\varphi} = \frac{\mathcal{G}(\psi)}{R^2} \tag{4.29}$$

For the covariant representation we obtain

$$\vec{B} = \frac{\mathcal{F}_p(\psi)}{RJ_p}\left(G_p\vec{\nabla}\psi + N_\theta^2\vec{\nabla}\theta\right) + \mathcal{G}(\psi)\vec{\nabla}\varphi \tag{4.30}$$

which show, that the covariant components of $\vec{B}$ are

$$\tilde{B}_{\psi} = \frac{G_p}{RJ_p}\mathcal{F}_p(\psi) \qquad \tilde{B}_{\theta} = \frac{N_\theta^2}{RJ_p}\mathcal{F}_p(\psi) \qquad \tilde{B}_{\varphi} = \mathcal{G}(\psi) \tag{4.31}$$

We also note that the function $\mathcal{F}_p(\psi)$ can always be made identically unity by using the poloidal flux $\bar{\psi}_p$ as a label for the magnetic surfaces, thereby simplifying eq. (4.26)

$$\vec{B} = \vec{\nabla}\varphi \times \vec{\nabla}\bar{\psi}_p + \mathcal{G}(\bar{\psi}_p)\vec{\nabla}\varphi \tag{4.32}$$

The tokamak magnetic field (4.26) often is written using the safety factor (4.8). The relation between $\mathcal{F}_p(\psi)$, $\mathcal{G}(\psi)$ and safety factor q can be found in a following way [2]. Let us evaluated the toroidal and poloidal fluxes of the field (4.26). The toroidal flux is the flux of $\vec{B}$ across a poloidal cross-section between the magnetic axis and the magnetic surface ψ; similarly, the poloidal flux is the flux of $\vec{B}$ across a strip of a surface $\theta = const$ between the magnetic axis and the magnetic surface ψ. They can be written

$$\begin{aligned} \Psi_T(\psi) &= \int_{\in\psi} \vec{B}\cdot d\vec{S}_{\varphi} = \frac{1}{2\pi}\int (\vec{B}\cdot\vec{\nabla}\varphi)dV \\ \Psi_P(\psi) &= \int_{\in\psi} \vec{B}\cdot d\vec{S}_{\theta} = \frac{1}{2\pi}\int (\vec{B}\cdot\vec{\nabla}\theta)dV \end{aligned} \tag{4.33}$$

The second form is easily proved by noting that $\vec{B}\cdot\vec{\nabla}\varphi = \vec{\nabla}\cdot(\vec{B}\varphi)$, $\vec{B}\cdot\vec{\nabla}\theta = \vec{\nabla}\cdot(\vec{B}\theta)$. Thus, we obtain

$$\begin{aligned} \bar{\psi}_T &= \frac{\Psi_T(\psi)}{2\pi} = \int_0^{\psi} \mathcal{G}(\psi)\int_0^{2\pi}\frac{J_p}{R}d\theta d\psi = \int_0^{\psi}\langle\frac{J_p}{R}\rangle\mathcal{G}(\psi)d\psi \\ \bar{\psi}_P &= \frac{\Psi_T(\psi)}{2\pi} = \int_0^{\psi}\mathcal{F}_p(\psi)d\psi \end{aligned} \tag{4.34}$$

where angular brackets indicate averaging over θ. Now we write expression for the safety factor

$$q(\psi) = \frac{d\bar{\psi}_T}{d\bar{\psi}_P} = \langle\frac{J_p}{R}\rangle\frac{\mathcal{G}(\psi)}{\mathcal{F}_p(\psi)} \tag{4.35}$$

4.4 Flux coordinates

There is clearly a large degree of arbitrariness in the way toroidal coordinates can be chosen. Many problems in the theory of toroidal plasma simplify greatly by using 'flux coordinates', which are characterised by the fact that magnetic field lines are 'straight' lines. Using the contravariant components of $\vec{B}$, eq.(4.29), the equation for the field lines is

$$\frac{d\varphi}{d\theta} = \frac{\tilde{B}^{\varphi}}{\tilde{B}^{\theta}} = \frac{J_p}{R}\frac{\mathcal{G}(\psi)}{\mathcal{F}_p(\psi)} \tag{4.36}$$

In the flux coordinates the r.h. side of this equation must be a surface function. Clearly, this is case if, and only if

$$\frac{\partial}{\partial\theta}\left(\frac{J_p}{R}\right) = 0 \tag{4.37}$$

If this conditions satisfied then

$$\bar{\psi}_T(\psi) = \int_0^{\psi} \frac{J_p}{R}\mathcal{G}(\psi)d\psi \qquad \bar{\psi}_P(\psi) = \int_0^{\psi} \mathcal{F}_p(\psi)d\psi \tag{4.38}$$

so that

$$q(\psi) = \frac{d\bar{\psi}_T}{d\bar{\psi}_P} = \frac{J_p}{R}\frac{\mathcal{G}(\psi)}{\mathcal{F}_p(\psi)} \tag{4.39}$$

Accordingly, the contravariant representation of $\vec{B}$ becomes

$$\vec{B} = \mathcal{F}_p(\psi)\left\{\vec{\nabla}\varphi \times \vec{\nabla}\psi + q(\psi)\vec{\nabla}\psi \times \vec{\nabla}\theta\right\} \tag{4.40}$$

Rewritten as

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\left\{\mathcal{F}_p(\psi)(q(\psi)\theta - \varphi)\right\} \tag{4.41}$$

it is easily recognised as the Clebsch representation of $\vec{B}$. The contravariant component of $\vec{B}$ in the flux coordinates are

$$\tilde{B}^{\psi} = 0 \qquad \tilde{B}^{\theta} = \frac{\mathcal{F}_p(\psi)}{RJ_p} \qquad \tilde{B}^{\varphi} = q(\psi)\tilde{B}^{\theta} \tag{4.42}$$

For the sake of physical transparency, it is usually preferable to write Vlasov and Maxwell equation using the physical components of fields and currents. The physical components of $\vec{B}$ are

$$\tilde{B}_{\psi} = 0 \qquad \tilde{B}_{\tau} = N_{\theta}\frac{\mathcal{F}_p(\psi)}{RJ_p} \qquad \tilde{B}_{\varphi} = q(\psi)R\frac{\mathcal{F}_p(\psi)}{RJ_p} \tag{4.43}$$

4.5 The toroidal equilibrium distribution function

The most general equilibrium function satisfying the time independent Vlasov equation is an arbitrary function of the constants of the motion in the static magnetic and electric field. Two such constants are the specific energy and magnetic moments, introduced above, ϵ_v and μ_v from the eq. (4.22). To describe gradients in space, we need a third constant of the motion depending explicitly on the radial coordinate ψ. In axisymmetric configurations the natural choice is the toroidal momentum of the guiding center

$$\frac{P_\varphi}{m_\alpha} = R v_\parallel \cos\Theta - \frac{Z_\alpha e}{m_\alpha c}\bar{\psi}_{Pg} \tag{4.44}$$

Let us, therefore, assume

$$F_\alpha = F_\alpha(\epsilon_v, \mu_v, \frac{P_\varphi}{m_\alpha}) \tag{4.45}$$

To obtain F_α at the particle position, it is sufficient to develop

$$\bar{\psi}_{Pg} = \bar{\psi}_P - \vec{\rho}\cdot\vec{\nabla}\bar{\psi}_P = \bar{\psi}_P + \frac{(\vec{b}\times\vec{\nabla}\bar{\psi}_P)\cdot\vec{v}_\perp}{\Omega_g} \tag{4.46}$$

i.e.

$$\frac{P_\varphi}{m_\alpha} = R v_\parallel \cos\Theta - \frac{\Omega_g}{B_0}\left[\bar{\psi}_P + \frac{(\vec{b}\times\vec{\nabla}\bar{\psi}_P)\cdot\vec{v}_\perp}{\Omega_g}\right] \tag{4.47}$$

To obtain the distribution function at the position of the particles, it is sufficient to develop $F_\alpha(P_\varphi/m_\alpha)$ to first order in the Larmor radius.

$$F_\alpha(\epsilon_v, \mu_v, P_\varphi/m_\alpha) = F_\alpha(\psi) + \frac{\vec{K}_B\cdot\vec{v}}{\Omega_g}\frac{\partial F_\alpha}{\partial\psi} \tag{4.48}$$

where

$$\vec{K}_B = \vec{b}\times\vec{\nabla}\psi - \frac{RB_0}{\mathcal{F}_p}\cos\Theta\,\vec{b} \tag{4.49}$$

with $\mathcal{F}_p = d\bar{\psi}_P/d\psi$. The first term of $\vec{K}_B$ is the familiar diamagnetic contribution which exist also in the straight limit; the second term is a correction due to toroidicity, which exists due to the plasma particles trapped on the outer part of the magnetic surfaces. Such particles have small values of the longitudinal velocity; moving along the magnetic field lines to the inner part of the magnetic surfaces they are reflected at the certain points backwards by the increasing magnetic field (of course, if their velocities are small enough). As a result trapped particles oscillate at the outer part of tokamak giving new set of instabilities.

The second term (4.48) is sometimes called the 'neoclassic' diamagnetic contribution. Once the diamagnetic contributions are taken into account, ψ can be treated as a constant of the motion in the equilibrium distribution function. Using (4.48) we can easily to evaluate the velocity derivative of F_α which appears on the r.h. side of the linearised Vlasov equation:

$$\frac{\partial F_\alpha}{\partial \vec{v}} = \vec{v}\frac{\partial F_\alpha}{\partial \epsilon_v} + \frac{\vec{v}_\perp}{B_0}\frac{\partial F_\alpha}{\partial \mu_v} + \frac{\vec{K}_B}{\Omega_{g\alpha}}\frac{\partial F_\alpha}{\partial \psi} \tag{4.50}$$

The equilibrium distribution function F_α must be normalised so that

$$2\pi \sum_{\sigma_v=\pm 1} \int_0^\infty d\epsilon_v \int_0^{\epsilon_v/B_0} \frac{B_0}{|v_\parallel|} F_\alpha(\epsilon_v, \mu_v) d\mu_v = n_\alpha(\psi) \tag{4.51}$$

Of particular importance is, of course, the case in which F_α is locally a Maxwellian,

$$F_{M\alpha}(\epsilon_v) = n_\alpha \frac{e^{-2\epsilon_v/v_{th\alpha}^2}}{\pi^{3/2} v_{th\alpha}^3} \tag{4.52}$$

where n_α and $v_{th\alpha}^2 = 2T_\alpha/m_\alpha$ are function of ψ via P_φ/m_α. And we can calculate,

$$\frac{\partial \log F_{M\alpha}}{\partial \psi} = \frac{a}{L_{n\alpha}}\left[1 + \eta_{T\alpha}\left(\frac{2\epsilon_v}{v_{th\alpha}^2} - \frac{3}{2}\right)\right] \tag{4.53}$$

with

$$\frac{a}{L_{n\alpha}} = \frac{\partial \log n_\alpha}{\partial \psi} \qquad \eta_{T\alpha} = \frac{d \log T_\alpha}{d \log n_\alpha} \tag{4.54}$$

where a is a distance used to make the radial variable ψ dimensionless, in practice, the plasma radius. The quantities $L_{n\alpha}$ and $\eta_{T\alpha}$ in turn depend on ψ via P_φ/m_α, however, this variation is usually very slow on the scale of the particle Larmor radius,so that for most purpose we can regard $L_{n\alpha}$ and $\eta_{T\alpha}$ as constant.

With these notations

$$F_{M\alpha}(\epsilon_v, \psi) \simeq n_\alpha(\epsilon_v)\frac{e^{-2\epsilon_v/v_{th\alpha}^2}}{\pi^{3/2} v_{th\alpha}^3(\epsilon_v)}\left\{1 + \frac{a}{L_{n\alpha}}\left[1 + \eta_{T\alpha}\left(\frac{2\epsilon_v}{v_{th\alpha}^2} - \frac{3}{2}\right)\right]\frac{\vec{K}_B\vec{v}}{\Omega_{g\alpha}}\right\} \tag{4.55}$$

the diamagnetic current $\vec{J}_D = J_D\vec{u}_\eta$ flows in the diamagnetic direction and can be evaluated as

$$\begin{aligned} J_D = \sum_\alpha Z_\alpha e n_\alpha \sum_{\sigma_v} \int_0^{2\pi} d\phi_v \int_0^\infty d\epsilon_v \int \frac{B_0}{|v_\parallel|} d\mu_v \frac{e^{-2\epsilon_v/v_{th\alpha}^2}}{\pi^{3/2} v_{th\alpha}^3}\vec{v} \\ \times \frac{a}{L_{n\alpha}}\left[1 + \eta_{T\alpha}\left(\frac{\epsilon_v}{v_{th\alpha}^2} - \frac{3}{2}\right)\right]\frac{(\vec{b}\times\vec{\nabla}\psi)\cdot\vec{v}}{\Omega_{g\alpha}} = \sum_\alpha Z_\alpha e n_\alpha \frac{v_{th\alpha}^2}{2\Omega_{g\alpha} L_{n\alpha}}(1 + \eta_{T\alpha}) \end{aligned} \tag{4.56}$$

that is identical to that calculated in the straight field limit. In the equilibrium for toroidal geometry there is also a 'neoclassic' diamagnetic current coming from second addition of $\vec{K}_B$, which is contributed by trapped particles and flows along magnetic field lines. The neoclassic diamagnetic currents also is not assosiated with a material flow: the physical explanation of the neoclassical part is the same as for classic one only in terms of finite banana orbits combined with density and temperature gradients.

4.6 The formal solution of the Vlasov equation in the toroidal plasma.

For an electromagnetic wave with harmonic time dependence $e^{-i\omega t}$ the formal solution of the integral Vlasov equation is

$$f_\alpha(\vec{r},\vec{v},t) = -e^{-i\omega t}\frac{Z_\alpha e}{m_\alpha}\int_{-\infty}^{t} e^{-i\omega(t-t')}\left[\vec{E}(\vec{r}\,') + \frac{\vec{v}\,'}{c}\times\vec{B}(\vec{r}\,')\right]\cdot\frac{\partial F_\alpha(\vec{r}\,')}{\partial\vec{v}\,'}dt\,' \tag{4.57}$$

Maxwell equation can be summorized in the single equation

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \frac{\omega^2}{c^2}\left(\vec{E}+\frac{4\pi}{c}\vec{j}\right) \tag{4.58}$$

In inhomogeneous configurations the orbits will be determined in the drift approximation. If the particle velocity is decomposed in the usual way

$$\vec{v}\,' = v_\parallel'\vec{b}_0' + \vec{v}_\perp' + \vec{v}_D' \tag{4.59}$$

and if we have Maxwellian distribution function, the total current can be split into several contributions of different physical origin:

$$\vec{j}(\vec{r},t) = [\vec{j}_M(\vec{r}) + \vec{j}_D(\vec{r}) + \vec{j}_B(\vec{r})]e^{-i\omega t} \tag{4.60}$$

with the bulk term

$$\vec{j}_M(\vec{r}) = \sum_\alpha \frac{2e^2Z_\alpha^2}{m_\alpha}\int d\vec{v}\frac{\vec{v}_\parallel\vec{b}_0+\vec{v}_\perp}{v_{th\alpha}}F_{M\alpha}\int_{-\infty}^{t}\frac{v_\parallel'\vec{b}_0'+\vec{v}_\perp'}{v_{th\alpha}}\cdot\vec{E}(\vec{r}\,')e^{-i\omega(t\,'-t)}dt\,' \tag{4.61}$$

the diamagnetic term

$$\begin{aligned}\vec{j}_D(\vec{r}) = \sum_\alpha \frac{2e^2Z_\alpha^2}{m_\alpha}\int d\vec{v}\frac{v_\parallel\vec{b}_0+\vec{v}_\perp}{v_{th\alpha}}F_{M\alpha}\frac{a}{L_{n\alpha}}\left[1+\eta_{T\alpha}\left(\frac{v'^2}{v_{th\alpha}^2}-\frac{3}{2}\right)\right]\\ \times\int_{-\infty}^{t}\frac{\vec{K}_B'v_{th\alpha}}{\Omega_{g\alpha}'}\cdot\left[\vec{E}(\vec{r}\,')+\frac{\vec{v}\,'}{c}\times\vec{B}(\vec{r}\,')\right]e^{-i\omega(t\,'-t)}dt\,'\end{aligned} \tag{4.62}$$

and the particle drift term

$$\vec{j}_B(\vec{r}) = \sum_\alpha \frac{2e^2 Z_\alpha^2}{m_\alpha} \int d\vec{v} \frac{v_\parallel \vec{b}_0 + \vec{v}_\perp}{v_{th\alpha}} F_{M\alpha} \int_{-\infty}^{t} \frac{\vec{v}_{D\alpha}'}{v_{th\alpha}} \times \left[\vec{E}(\vec{r}\,') + \frac{v_\parallel'}{c} \vec{b}_0' \times \vec{B}(\vec{r}\,') \right] e^{-i\omega(t\,'-t)} dt\,' \tag{4.63}$$

The mean-stream of the Vlasov derivation in the toroidal geometry is similar to that in the slab geometry already developed in the Chap. 2, differing in details.

4.7 The spectral Ansatz.

A concrete Ansatz for the wave field must now be introduced. In toroidal configurations the most general wave field can be written [47]

$$\begin{pmatrix} \vec{E}(\vec{r},t) \\ \\ \vec{B}(\vec{r},t) \end{pmatrix} = \sum_{m,n} \begin{pmatrix} \vec{E}^{mn}(\psi) \\ \\ \vec{B}^{mn}(\psi) \end{pmatrix} e^{i(m\theta + n\varphi - \omega t)} \tag{4.64}$$

with m and n integers. We will adopt this form, since it is completely general, and automatically satisfies the required periodicity conditions in both the poloidal and toroidal angle. In addition, it turns out that this Ansatz allows to take best advantage of the splitting of the unperturbed particles motion into the fast gyromotion and the slower motion of the guiding center. Introducing further an appropriate representation for the coefficient $\vec{E}^{mn}(\psi)$ and $\vec{B}^{mn}(\psi)$, we will be able to perform the integration over the gyroangle in the expressions for the perturbed current and charge densities analytically, and in a way as similar as possible to the familiar uniform limit. This advantage is paid by the fact that the individual Fourier components in the sums are not eigenfunctions of Maxwell equation. In axisymmetric configurations the wave equations will be diagonal in the toroidal wavenumber n, so that sum over the toroidal wavenumber n could be omitted and the toroidal Fourier components could be considered one at time. Since θ is not ignorable, on the other hand, the wave equations will not be diagonal in the poloidal wavenumber m, so that poloidal Fourier components will be all coupled to each other; sum over the poloidal wavenumber m is essential.

If we substitute the (4.64) in the results of the previous subsection, the current density can be written

$$\vec{j}(\vec{r},t) = \sum_{m,n} \vec{j}^{mn}(\psi,\theta) e^{i(m\theta + n\varphi - \omega t)} \tag{4.65}$$

with

$$\frac{4\pi i}{\omega}\vec{j}^{mn}(\psi,\theta)=\sum_{\alpha}\frac{4\pi e^2 Z_\alpha^2}{\omega^2 m_\alpha}\int d\vec{v}(v_\parallel\vec{b}_0+\vec{v}_\perp)F_{M\alpha}$$

$$\times\Bigg\{(-i\omega)\int_{-\infty}^{t}e^{i[m(\theta'-\theta)+n(\varphi'-\varphi)-\omega(t'-t)]} \tag{4.66}$$

$$\times\sum_{J=1,3}\left(\frac{1}{F_{M\alpha}}\frac{\partial F_{M\alpha}}{\partial K_J}\right)\frac{\partial K_J}{\partial\vec{v}\,'}\left[\vec{E}^{mn}(\psi')+\frac{\vec{v}\,'}{c}\times\vec{B}^{mn}(\psi')\right]dt'\Bigg\}$$

where for brevity, $K_1=\epsilon_v$, $K_2=\mu_v$, $K_3=\bar{\psi}$. However, equation (4.65) does not represent $\vec{j}$ as a Fourier series in the poloidal angle, since the coefficients of $\vec{E}^{mn}$ and $\vec{B}^{mn}$ on the r.h. side depend on θ. If the multiple integrals over orbits and velocity space can be performed, eqs. (4.65)-(4.66) are, nevertheless, a perfectly acceptable constitutive relation, and, indeed, they are often used as starting point for further approximations. For our present purpose, however, it will be of advantage to evaluate the true poloidal Fourier expansion of $\vec{j}$

$$\vec{j}(\vec{r},t)=\sum_{m,n}\vec{j}^{mn}(\psi)e^{i(m\theta+n\varphi-\omega t)} \tag{4.67}$$

The coefficients of this expansion can be written in convolution form

$$\vec{j}^{mn}(\psi)=\sum_{\bar{m}}\underline{\underline{\Sigma}}(m-\bar{m}|\bar{m},n;\psi)\cdot\vec{E}^{\bar{m}n} \tag{4.68}$$

where kernel $\underline{\underline{\Sigma}}(m|\bar{m},n;\psi)$ is the linear integral

$$\frac{4\pi i}{\omega}\underline{\underline{\Sigma}}(m-\bar{m}|\bar{m},n;\psi)\vec{E}^{\bar{m}n}=\oint\frac{d\theta}{2\pi}e^{-i(m-\bar{m})\theta}\sum_{\alpha}\frac{4\pi e^2 Z_\alpha^2}{\omega^2 m_\alpha}$$

$$\times\int d\vec{v}(v_\parallel\vec{b}_0+\vec{v}_\perp)F_{M\alpha}\Bigg\{(-i\omega)\int_{-\infty}^{t}e^{i[\bar{m}(\theta'-\theta)+n(\varphi'-\varphi)-\omega(t'-t)]} \tag{4.69}$$

$$\times\sum_{J=1,3}\left(\frac{1}{F_{M\alpha}}\frac{\partial F_{M\alpha}}{\partial K_J}\right)\frac{\partial K_J}{\partial\vec{v}\,'}\left[\vec{E}^{\bar{m}n}(\psi')+\frac{\vec{v}\,'}{c}\times\vec{B}^{\bar{m}n}(\psi')\right]dt'\Bigg\}$$

To prepare for the gyroaverage, we now complete the spectral Ansatz by introducing a Fourier representation of the current and fields also in the 'radial' direction.

$$\begin{pmatrix}\vec{j}^{mn}(\psi)\\ \vec{E}^{mn}(\psi)\\ \vec{B}^{mn}(\psi)\end{pmatrix}=\int\begin{pmatrix}\vec{j}^{mn}(\kappa)\\ \vec{\mathcal{E}}^{mn}(\kappa)\\ \vec{\mathcal{B}}^{mn}(\kappa)\end{pmatrix}e^{i\kappa\psi}d\kappa \tag{4.70}$$

The linear relation between $\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa})$ and $\vec{\mathcal{B}}^{\bar{m}n}(\bar{\kappa})$ has the form

$$\vec{\mathcal{B}}^{mn}(\kappa) = \sum_{\bar{m}} \left[\bar{\kappa}\underline{\underline{\mathcal{R}}}^{(1)}(m-\bar{m},\kappa-\bar{\kappa}) - i\underline{\underline{\mathcal{R}}}^{(0)}(m-\bar{m},\kappa-\bar{\kappa})\right]\cdot\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa}) \tag{4.71}$$

with

$$\underline{\underline{\mathcal{R}}}^{(j)}(m,\kappa) = \int \underline{\underline{\mathcal{R}}}^{(j)}(m|\psi)e^{-i\kappa\psi}d\psi \qquad j=0,1 \tag{4.72}$$

where $\underline{\underline{\mathcal{R}}}^{(j)}(m|\psi)$ consists of radial derivatives, k_η, k_ζ components of wave vector and elements of metric.

Performing Fourier transform as in the plane straghtified limit, current density can be written as a double convolution

$$\begin{aligned}\vec{j}^{mn}(\kappa) = &-\sum_\alpha \frac{e^2Z_\alpha^2}{m_\alpha}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^1 d\psi\sum_{\bar{m}} e^{i[(\bar{\kappa}-\kappa)\psi+(\bar{m}-m)\theta]}\\ &\times\int d\vec{v}(v_\parallel\vec{b}_0+\vec{v}_\perp)F_{M\alpha}\int_{-\infty}^t e^{i[\bar{\kappa}(\psi'-\psi)+\bar{m}(\theta'-\theta)+n(\varphi'-\varphi)-\omega(t'-t)]}\\ &\times\sum_{J=1,3}\left(\frac{1}{F_{M\alpha}}\frac{\partial F_{M\alpha}}{\partial K_J}\right)\frac{\partial K_J}{\partial\vec{v}\,'}\cdot\left[\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa})+\frac{\vec{v}\,'}{c}\times\vec{\mathcal{B}}^{\bar{m}n}(\bar{\kappa})\right]dt'\end{aligned} \tag{4.73}$$

This equation can be considered as a constitutive relation in the Fourier representation. It can be written compactly as

$$\vec{j}^{mn}(\kappa) = \sum_{\bar{m}}\int d\bar{\kappa}\underline{\underline{\Sigma}}(\vec{k},\vec{k}_b)\cdot\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa}) \tag{4.74}$$

where

$$\begin{aligned}\vec{k} &= \kappa\vec{\nabla}\psi + m\vec{\nabla}\theta + n\vec{\nabla}\varphi\\ \vec{k}_b &= \bar{\kappa}\vec{\nabla}\psi + \bar{m}\vec{\nabla}\theta + n\vec{\nabla}\varphi\end{aligned} \tag{4.75}$$

are the wavevector in the covariant representation. In turn, the kernel $\underline{\underline{\Sigma}}$ of this integral equation is of the form

$$\underline{\underline{\Sigma}}(\vec{k},\vec{k}_b) = \int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^1 d\psi e^{i[(\bar{\kappa}-\kappa)\psi+(\bar{m}-m)\theta]}\underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\theta) \tag{4.76}$$

where, by comparison with eqn (4.73),

$$\begin{aligned}\underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\theta)\cdot\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa}) = &-\sum_\alpha\frac{e^2Z_\alpha^2}{m_\alpha}\int d\vec{v}(v_\parallel\vec{b}_0+\vec{v}_\perp)F_{M\alpha}\int_{-\infty}^t e^{i[\vec{k}_b\cdot(\vec{r}'-\vec{r})-\omega(t'-t)]}\\ &\times\sum_{J=1,3}\left(\frac{1}{F_{M\alpha}}\frac{\partial F_{M\alpha}}{\partial K_J}\right)\frac{\partial K_J}{\partial\vec{v}\,'}\cdot\left[\vec{\mathcal{E}}^{\bar{m}n}(\bar{\kappa})+\frac{\vec{v}\,'}{c}\times\vec{\mathcal{B}}^{\bar{m}n}(\bar{\kappa})\right]dt'\end{aligned} \tag{4.77}$$

The integral constitutive relation in real space is obtained by inverting the FTs as appropriate:

$$
\begin{aligned}
\vec{j}(\vec{r},t) = \sum_n e^{i(n\varphi-\omega t)} \sum_m e^{im\theta} \int d\kappa\, e^{i\kappa\psi} \\
\times \sum_{\bar{m}} \int d\bar{\kappa}\, \underline{\underline{\Sigma}}(\vec{k},\vec{k}_b) \cdot \int d\bar{\psi}\, e^{-i\bar{\kappa}\bar{\psi}} \int \frac{d\bar{\theta}}{2\pi}\, e^{-im\bar{\theta}} \vec{E}(\bar{\psi},\bar{\theta})
\end{aligned}
\tag{4.78}
$$

This equation can be written as an integral equation relating the amplitude of each poloidal component of the current density all poloidal components of the electrical field:

$$
\vec{j}^{mn}(\psi) = \sum_{\bar{m}} \int_0^1 d\bar{\psi}\, \underline{\underline{\Sigma}}(m,\bar{m},n;\psi,\bar{\psi}) \cdot \vec{E}^{\bar{m},n}(\bar{\psi}) \tag{4.79}
$$

with

$$
\begin{aligned}
\underline{\underline{\Sigma}}(m,\bar{m},n;\psi,\bar{\psi}) = \int d\kappa\, e^{i\kappa\psi} \int d\bar{\kappa}\, e^{-i\bar{\kappa}\bar{\psi}} \underline{\underline{\Sigma}}(\vec{k},\vec{k}_b) = \\
\int d\kappa\, e^{i\kappa\psi} \int d\bar{\kappa}\, e^{-i\bar{\kappa}\bar{\psi}} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^1 d\hat{\psi}\, e^{i[(\bar{\kappa}-\kappa)\hat{\psi}+(\bar{m}-m)\hat{\theta}]} \underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\hat{\psi},\hat{\theta})
\end{aligned}
\tag{4.80}
$$

The structure of the resulting wave equation is thus clear: for each toroidal mode number n one has in principle to solve an infinite set of couple integro-differential equations for the poloidal modes $\vec{E}^{mn}(\psi)$. Isolating in turn each Fourier harmonics of $\vec{\nabla} \times \vec{\nabla} \times \vec{E}$, these equations can be written

$$
\begin{aligned}
\int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\, e^{-im\bar{\theta}} \left\{ e^{-in\varphi} \vec{\nabla} \times \left[\vec{\nabla} \times \left(\sum_{\bar{m}} \vec{E}^{\bar{m}n}(\psi) e^{i(\bar{m}\bar{\theta}+n\varphi)} \right) \right] \right\} \\
= \frac{\omega^2}{c^2} \left[\vec{E}^{mn} + \frac{4\pi i}{\omega} \sum_{\bar{m}} \int_0^1 d\bar{\psi}\, \underline{\underline{\Sigma}}(m,\bar{m},n;\psi,\bar{\psi}) \cdot \vec{E}^{\bar{m}n}(\psi) \right]
\end{aligned}
\tag{4.81}
$$

There is one such equation for each $-\infty < m < \infty$.

4.8 The role of wavevectors.

The main advantage of the spectral formulation is that it alow to define unambiguously the physical components of the wavevectors. In the local reference frame with the ζ coordinate aligned with the static magnetic field, and η in the magnetic surface perpendicular to $\vec{B}_0$, these components are

$$
\begin{aligned}
k_\xi^{mn}(\psi,\theta) &= \frac{N_\theta}{J_p}\left(\kappa - \frac{G_p}{N_\theta^2}m\right) && \textit{radial} \\
k_\eta^{mn}(\psi,\theta) &= \frac{m}{N_\theta}\cos\Theta - \frac{n}{R}\sin\Theta && \textit{diamagnetic} \\
k_\zeta^{mn}(\psi,\theta) &= \frac{m}{N_\theta}\sin\Theta + \frac{n}{R}\cos\Theta && \textit{parallel}
\end{aligned}
\tag{4.82}
$$

where $\tan\Theta = B_{pol}/B_{tor}$. These definitions hold of course both for $\vec{k}$ (the wavevector of the Fourier decomposition of the current) and $\vec{k}_b$ (the wavevector of the Fourier decomposition of the field, distinguished by 'barred' radial and poloidal wavenumbers). In particular, in eqn (4.77) the 'propagator' of the linearised Vlasov equation, i.e. the exponential factor, which represents the phase of the wave as seen by the particle along its trajectory, is expressed in terms of the barred wavevector just as in uniform plasma case. Although function of the position, $k_\xi^{\bar{m}n}$, $k_\eta^{\bar{m}n}$ and $k_\zeta^{\bar{m}n}$ are slowly varying on the scale of the gyromotion, since their variation is entirely due to the metrics, i.e. to gradients of equilibrium quantities only. As a consequence, we can be confident that the plasma response and the physics of particle-wave interactions will depend on the parallel wavenumber $\bar{k}_\parallel$ and of the perpendicular wavenumber $\bar{k}_\perp$

$$
\bar{k}_\perp^2 = k_\xi^{\bar{m}n\,2} + k_\eta^{\bar{m}n\,2} \qquad\qquad \bar{k}_\parallel = k_\zeta^{\bar{m}n} \tag{4.83}
$$

and the separation of the fast gyromotion from the smoother motion of the guiding center is possible just as in the uniform limit, the only condition being the validity of the drift approximation in the equilibrium configuration.

4.9 The WKB and FLR approximations.

As mentioned above, the spectral approach allows to describe the physics of wave propagation in axisymmetric plasmas in a way which is a natural extension of the uniform plasma case. This is paid, however, by the fact that the constitutive relations obtained is a complicated integral equation, non local in both the poloidal and radial coordinates. Fortunately, it is seldom required to solve the wave equations using this general form. With vary few exceptions, the constitutive relation can be put in the differential form at least in the radial variables. There are two situations in which such a reduction is possible. One is when the equilibrium parameters vary slowly on the scale of the local radial wavelength, so that the solution can be written in Eikonal form; the other is when the typical radial wavelength are much large than the thermal Larmor radius of plasma particles.

Let us assume, that the radial dependence of the field can be written in Eikonal form

$$\begin{aligned} \vec{E} &= \sum_{mn} \vec{E}^{mn}(\epsilon\psi) \exp i \left[\frac{S(\epsilon\psi)}{\epsilon} + m\theta + n\varphi - \omega t \right] \\ &\simeq \sum_{mn} \vec{E}^{mn}(\epsilon\psi)\; e^{i[\kappa_0\psi + m\theta + n\varphi - \omega t]} \end{aligned} \tag{4.84}$$

with $\kappa_0 = dS/d\psi$. Inserting this into the integral constitutive relation we obtain

$$\vec{j} = \sum_{mn} \vec{j}^{mn}(\epsilon\psi)\; e^{i[\kappa_0\psi + m\theta + n\varphi - \omega t]} \tag{4.85}$$

with

$$\begin{aligned} \vec{j}^{mn}(\epsilon\psi) &= \sum_{\bar{m}} \int d\kappa\; e^{i(\kappa-\kappa_0)\psi} \int d\bar{\kappa} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi} \int_0^1 d\hat{\psi}\; e^{i[(\bar{\kappa}-\kappa)\hat{\psi} + (\bar{m}-m)\bar{\theta}]} \\ &\times \underline{\underline{\sigma}}(\vec{k}, \vec{k}_b, \epsilon\hat{\psi}, \bar{\theta}) \cdot \int \vec{E}^{\bar{m}n}(\epsilon\bar{\psi}) e^{i(\kappa_0 - \bar{\kappa})\bar{\psi}} d\bar{\psi} \end{aligned} \tag{4.86}$$

To lowest order in ε, the integration over $\bar{\psi}$ gives $\delta(\bar{\kappa} - \kappa_0)$, and the integration over κ gives $\delta(\hat{\psi} - \psi)$, just as in uniform limit. Omitting the ordering parameter ε, this gives

$$\vec{j}^{mn}(\psi) = \sum_{\bar{m}} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\; e^{i(\bar{m}-m)\bar{\theta}} \underline{\underline{\sigma}}(\vec{k}_b, \psi, \bar{\theta}) \cdot \vec{E}^{\bar{m}n}(\psi) \tag{4.87}$$

Expressing the curl in the same approximation, the wave equations become

$$\begin{aligned} 0 = \sum_{\bar{m}} \oint_0^{2\pi} \frac{d\theta}{2\pi}\; e^{i(\bar{m}-m)\theta} & \left[\frac{c^2}{\omega^2} \left(\vec{k}^{\bar{m}n} : \vec{k}^{\bar{m}n} - k^{\bar{m}n\,2} \underline{\underline{I}} \right) \right. \\ & \left. + \underline{\underline{I}} + \frac{4\pi i}{\omega} \underline{\underline{\sigma}}(\vec{k}, \vec{k}_b, \psi, \theta) \right] \cdot \vec{E}^{\bar{m}n}(\psi) \end{aligned} \tag{4.88}$$

for each m. Truncated at an appropriate upper poloidal number$|m| \leq M$, this is a linear homogeneous algebraic system for the poloidal Fourier components of the field; the condition for the existence of a nontrivial solution, i.e. the vanishing of the determinant of its coefficients, plays the role of 'local dispersion relation'.

This procedure fails, however, to give an equation for the envelopes. To obtain one, we begin by rearranging the integrations as

$$\begin{aligned} \vec{j}^{mn}(\epsilon\psi) &= \sum_{\bar{m}} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\; e^{i(\bar{m}-m)\bar{\theta}} \int d\kappa \int d\bar{\kappa} \int_0^1 d\bar{\psi} \int_0^1 d\hat{\psi} \\ & \left[e^{i(\kappa-\kappa_0)(\psi-\hat{\psi})} \underline{\underline{\sigma}}(\vec{k}, \vec{k}_b, \epsilon\hat{\psi}, \bar{\theta}) e^{i(\kappa_0-\bar{\kappa})(\bar{\psi}-\hat{\psi})} \right] \cdot \vec{E}^{\bar{m}n}(\epsilon\bar{\psi}) \end{aligned} \tag{4.89}$$

Taking advantage of the localisation of the WKB wavepackets in κ-space, we next Taylor develop the kernel around κ_0

$$
\begin{aligned}
&\underline{\underline{\sigma}}(\vec{k}, \vec{k}_b; \epsilon\hat{\psi}, \theta) \simeq \underline{\underline{\tilde{\sigma}}}_0 + (\kappa - \kappa_0)\frac{\partial \underline{\underline{\sigma}}_0}{\partial \kappa} + (\bar{\kappa} - \kappa_0)\frac{\partial \underline{\underline{\sigma}}_0}{\partial \bar{\kappa}} \\
&+\frac{(\kappa - \kappa_0)^2}{2}\frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \kappa^2} + (\kappa - \kappa_0)(\bar{\kappa} - \kappa_0)\frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \kappa \partial \bar{\kappa}} + \frac{(\bar{\kappa} - \kappa_0)^2}{2}\frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \bar{\kappa}^2} + \ldots
\end{aligned}
\tag{4.90}
$$

We can now use the identities

$$
\begin{aligned}
(\kappa - \kappa_0)e^{i(\kappa - \kappa_0)(\psi - \hat{\psi})} &= i\frac{\partial}{\partial \hat{\psi}}\left(e^{i(\kappa - \kappa_0)(\psi - \hat{\psi})}\right) \\
(\bar{\kappa} - \kappa_0)e^{i(\bar{\kappa} - \kappa_0)(\bar{\psi} - \hat{\psi})} &= -i\frac{\partial}{\partial \bar{\psi}}\left(e^{i(\bar{\kappa} - \kappa_0)(\bar{\psi} - \hat{\psi})}\right)
\end{aligned}
\tag{4.91}
$$

After appropriate integration by parts, we can as before perform the $\bar{\psi}$ integration to obtain $\delta(\bar{\kappa} - \kappa_0)$, followed by the κ integration which gives $\delta(\psi - \hat{\psi})$. The result is

$$
\begin{aligned}
&\vec{j}^{mn}(\psi) = \sum_{\bar{m}} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\, e^{i(\bar{m} - m)\bar{\theta}} \Bigg\{ \underline{\underline{\sigma}}_0(\vec{k}, \vec{k}_b, \psi, \theta) \cdot \vec{E}^{\bar{m}n}(\psi) \\
&-i\left[\frac{\partial}{\partial \psi}\left(\frac{\partial \underline{\underline{\sigma}}_0}{\partial \kappa} \cdot \vec{E}^{\bar{m}n}\right) + \frac{\partial \underline{\underline{\sigma}}_0}{\partial \bar{\kappa}} \cdot \frac{\partial \vec{E}^{\bar{m}n}}{\partial \psi}\right] - \frac{1}{2}\left[\frac{\partial^2}{\partial \psi^2}\left(\frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \kappa^2} \cdot \vec{E}^{\bar{m}n}\right)\right. \\
&\left.+2\frac{\partial}{\partial \psi}\left(\frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \kappa \partial \bar{\kappa}} \cdot \frac{\partial \vec{E}^{\bar{m}n}}{\partial \psi}\right) + \frac{\partial^2 \underline{\underline{\sigma}}_0}{\partial \bar{\kappa}^2} \cdot \frac{\partial^2 \vec{E}^{\bar{m}n}}{\partial \psi^2}\right] + \ldots \Bigg\}
\end{aligned}
\tag{4.92}
$$

where for brevity the subscript 0 indicates $\kappa = \kappa_0$ and barred wavevectors

$$
\begin{aligned}
\vec{k} &= \kappa_0 \vec{\nabla}\psi + m\vec{\nabla}\theta + n\vec{\nabla}\varphi \\
\vec{k}_b &= \kappa_0 \vec{\nabla}\psi + \bar{m}\vec{\nabla}\theta + n\vec{\nabla}\varphi
\end{aligned}
\tag{4.93}
$$

now differ only for the poloidal component.

According to the above derivation, the equation obtained by substituting (4.92) into Maxwell equations must be interpreted as an equation for the amplitude of the wave electric field in the Eikonal representation. In this case $\kappa_0 = \kappa_0(\psi)$ and ω must satisfy the local dispersion relation defined by (4.88), and it must be verified *a posteriori* that the conditions for validity of the Eikonal representation are respected. in addition to the usual conditions on the radial equilibrium gradients, they require that the relative variation of κ_0 should be small, $d\log(\kappa_0)/d\psi \ll 1$, and thus exclude cuttofs, wave resonances, and confluences of two roots of the dispersion relation, from the radial region of interest.

Alternatively, after performing the derivatives as indicated, we can set $\kappa_0 = 0$ in the r.h. side of eqn (4.92): the whole procedure then amounts to expand the integral constitutive relation to second order in the Larmor radius in the radial direction. In this case the equations obtained by substituting (4.92) into Maxwell equations are the Finite Larmor Radius (FLR) approximations of the Eikonal form with respect to the radial variable. For consistency, it will be necessary to verify *a posteriorly* that the radial wavevector k_ψ satisfying this dispersion relation satisfied also the condition $k_\psi v_{th}/\Omega_c \ll 1$ at all points of the radial domain of interest.

4.10 Expansion in cyclotron harmonics

The integral over the gyrophase angle ϕ_v in velocity space in eq.(4.77) can be performed in full generality, without any assumption other then the validity of the drift approximation for the unperturbed particle orbits in the confining magnetic field. For this purpose it is essential to rearrange the phase of the propagator in such a way that the effect of the rapid gyromotion is clearly separated from the effect of the motion of the guiding center. This can be done by a straightward generalisation of the procedure already followed in straight geometry.

If eqn (4.77) is insert in eqn (4.76), the complete exponential function is the product of two factors

$$H(\vec{k}, \vec{k}_b, \vec{r}) G_p(\vec{k}_b \cdot (\vec{r}\,' - \vec{r})) \tag{4.94}$$

where

$$H(\vec{k}, \vec{k}_b, \vec{r}) = e^{i[(\bar{\kappa}-\kappa)\psi + (\bar{m}-m)\theta]} = e^{i(\vec{k}_b - \vec{k})\cdot\vec{r}} \tag{4.95}$$

is the factor of the Fourier convolutions (which has no φ dependency because of axisymmetry), and

$$\begin{aligned} G_p(\vec{k}_b \cdot (\vec{r}\,' - \vec{r})) &= e^{i[\bar{\kappa}(\psi\,' - \psi) + \bar{m}(\theta\,' - \theta) + n(\varphi\,' - \varphi) - \omega(t\,' - t)]} \\ &= e^{i[\vec{k}_b \cdot (\vec{r}\,' - \vec{r}) - \omega(t\,' - t)]} \end{aligned} \tag{4.96}$$

is the 'particle orbits propagator'. Splitting, as usual, $\vec{r} = \vec{R}_g + \vec{\rho}$, $\vec{r}\,' = \vec{R}_g\,' + \vec{\rho}\,'$, the argument of the propagator becomes

$$\vec{k}_b \cdot (\vec{r}\,' - \vec{r}) = \vec{k}_b \left(\vec{R}_g\,' - \vec{R}_g + \frac{\vec{b}\,' \times \vec{v}_\perp\,'}{\Omega_{c\alpha}'} - \frac{\vec{b} \times \vec{v}_\perp}{\Omega_{c\alpha}} \right) \tag{4.97}$$

Similarly, the argument of the external factor becomes

$$(\vec{k}_b - \vec{k}) \cdot \vec{r} = (\vec{k}_b - \vec{k}) \cdot \left(R_g + \frac{\vec{b} \times \vec{v}_\perp}{\Omega_{c\alpha}} \right) \tag{4.98}$$

The equilibrium distribution functions, on the other hand, depend on the position of the guiding center (neglecting for the moment neoclassical corrections). As in the straight case, therefore, it is convenient to change the variables of the space integrations in eqn (4.76) from the particle coordinates ψ, θ to the guiding center coordinates ψ_g, θ_g. The Jacobian of this transformations is unity to the required accuracy, hence this change of integration variables is completely transparent (boundary conditions would be an exception; they are anyhow problematic for integral equations). Simplifying the contributions between the two factors, we can then rearrange the exponentials as

$$\mathcal{H}\mathcal{G}\,' = \mathcal{H}_g \mathcal{G}_g\,' e^{i(\vec{k}_b \cdot \vec{\rho}\,' - \vec{k} \cdot \vec{\rho})} \tag{4.99}$$

with an external phase factor normally identical to the original one except for the name of the integration variables,

$$\mathcal{H}(\vec{k}, \vec{k}_b, \vec{R}_g) = e^{i[(\bar{\kappa} - \kappa)\psi_g + (\bar{m} - m)\theta_g]} = e^{i(\vec{k}_b - \vec{k}) \cdot \vec{R}_g} \tag{4.100}$$

and the 'guiding center propagator'

$$\begin{aligned} \mathcal{G}_g\,'(\vec{k}_b, \vec{R}_g, \vec{R}_g\,') &= e^{i[\bar{\kappa}(\psi_g\,' - \psi_g) + \bar{m}(\theta_g\,' - \theta_g) + n(\varphi_g\,' - \varphi_g) - \omega(t\,' - t)]} \\ &= e^{i[\vec{k}_b \cdot (\vec{R}_g\,' - \vec{R}_g) - \omega(t\,' - t)]} \end{aligned} \tag{4.101}$$

which depends only on the orbits of the guiding centers. The change of integration variables has symmetrised the phase factor which depends on the gyromotion with respect to the wavevectors $\vec{k}$ and $\vec{k}_b$. As a consequence, it has displaced part of the $\vec{k}$ - dependence from the external FTs to the definition of the conductivity kernel $\underline{\underline{\sigma}}$, which now depends on both $\vec{k}$ and $\vec{k}_b$. This might appear at first sight as an additional a complication, but essential to perform the gyroaverage in a consistent way.

The phase of the exponential which depends on the gyromotion is now

$$\vec{k}_b \cdot \vec{\rho}\,' - \vec{k} \cdot \vec{\rho} = \frac{\bar{k}_\perp v_\perp'}{\Omega\,'} \sin(\phi_v' - \bar{\delta}\,') - \frac{k_\perp v_\perp}{\Omega} \sin(\phi_v - \delta) \tag{4.102}$$

where

$$k_\perp^2 = k_\xi^2 + k_\eta^2 \qquad \delta = \arctan(k_\eta / k_\xi) \qquad \bar{\delta} = \arctan(\bar{k}_\eta / \bar{k}_\xi) \tag{4.103}$$

and

$$\phi_v' = \phi_v + \int_t^{t\,'} \Omega_\alpha(\tau) d\tau \tag{4.104}$$

The familiar Bessel function identities used in the evaluation of the uniform plasma dielectric tensor allow then develop the 'particle propagator' as

$$\mathcal{G}_g' e^{i(\vec{k}_b\cdot\vec{\rho}\,'-\vec{k}\cdot\vec{\rho})} = \sum_{p,q} J_q(\vec{k}\cdot\vec{\rho}) J_p(\vec{k}_b\cdot\vec{\rho}\,') G_p'(\vec{k}_b, \vec{R}_g, \vec{R}_g') e^{-i[(p-q)\phi_v - p\bar{\delta}+q\delta]} \tag{4.105}$$

where

$$G_p'(\vec{k}_b, \vec{R}_g, \vec{R}_g') = e^{i\left[\bar{\kappa}(\psi_g'-\psi_g)+\bar{m}(\theta_g'-\theta_g)+n(\varphi_g'-\varphi_g)+\int_t^{t'}[p\Omega_{g\alpha}(\tau)-\omega]d\tau\right]} \tag{4.106}$$

is the 'guiding center propagator' for the p-th cyclotron harmonics and

$$\vec{k}_b\cdot\vec{\rho}\,' - \vec{k}\cdot\vec{\rho} = \frac{\bar{k}_\perp v_\perp}{\Omega}\sin(\phi_v' - \bar{\delta}) - \frac{k_\perp v_\perp}{\Omega}\sin(\phi_v - \delta) \tag{4.107}$$

4.11 Evaluation of the gyrophase integrals

The integrations over gyrophase ϕ_v now will be performed for the Maxwellian plasma. Recalling, that the Jacobian of drift variables in velocity space is $B/v_\parallel$, and splitting the conductivity into the bulk, diamagnetic and drift contributions we can write

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k}, \vec{k}_b, \vec{R}_g) = -4\pi \sum_\alpha \frac{\omega_{p\alpha}^2(\vec{R}_g)}{\omega^2}\int_0^\infty d\epsilon_v \int \frac{B_0}{v_\parallel} F_\alpha d\mu_v \sum_K \hat{\underline{\underline{\Pi}}}^K(\vec{k}, \vec{k}_b) \tag{4.108}$$

with

$$\begin{aligned}\hat{\underline{\underline{\Pi}}}^K(\vec{k}, \vec{k}_b) &= \int_0^{2\pi}\frac{d\phi_v}{2\pi}\sum_{p,q} J_q\left(\frac{k_\perp v_\perp}{\Omega_{g\alpha}}\right) e^{iq(\phi_v-\delta)} \\ &\times \left\{-i\omega\int_{-\infty}^t \mathcal{G}_g'(\vec{k}_b, \vec{R}_g, \vec{R}_g') J_p\left(\frac{k_\perp' v_\perp'}{\Omega_{g\alpha}'}\right) e^{-ip(\phi_v'-\bar{\delta})}\underline{\underline{\mathcal{P}}}^K\right\} dt'\end{aligned} \tag{4.109}$$

and

$$\begin{aligned}\underline{\underline{\mathcal{P}}}^M &= \frac{v_\parallel \vec{b}_0 + \vec{v}_\perp}{v_{th\alpha}} : \frac{v_\parallel' \vec{b}_0' + \vec{v}_\perp'}{v_{th\alpha}} \\ \underline{\underline{\mathcal{P}}}^D &= -\frac{a}{2L_{n\alpha}}\Lambda_{D\alpha}(\epsilon_v)\frac{v_\parallel \vec{b}_0 + \vec{v}_\perp}{v_{th\alpha}} : \frac{\vec{K}_B' v_{th\alpha}}{\Omega_{g\alpha}'} \\ \underline{\underline{\mathcal{P}}}^B &= \frac{v_\parallel \vec{b}_0 + \vec{v}_\perp}{v_{th\alpha}} : \frac{\vec{v}_{D\alpha}'}{v_{th\alpha}}\end{aligned} \tag{4.110}$$

All factors, which depend on ϵ_v, μ_v, ϕ_v, $\vec{R}_g$, $\vec{R}'_g$ are the same as those entering in the dielectric tensor of the uniform plasma, or simple generalisations thereof, so that the integrations over ϕ_v can now be performed in much the same way as in the uniform case.

To write the results of these integrations in a convenient way, we introduce the normalised parallel velocities

$$u = \frac{v_\parallel}{v_{th\alpha}} \qquad\qquad w = \frac{v_\perp}{v_{th\alpha}} \tag{4.111}$$

with the corresponding primed quantities, and the notations

$$\nu = \frac{k_\perp v_{th\alpha}}{\Omega_{g\alpha}} \qquad\qquad \bar{\nu}\,' = \frac{\bar{k}_\perp v_{th\alpha}}{\Omega'_{g\alpha}} \tag{4.112}$$

It is also useful to use the matrices $\underline{\underline{R}}(\delta)$, $\underline{\underline{R}}^{-1}(\delta) = \underline{\underline{R}}(-\delta)$, as in a case of slab geometry, where δ, $\bar{\delta}$, are the angle between $\vec{k}_\perp$, $\vec{k}_{\perp b}$ and the direction perpendicular to the magnetic surface, respectively. The reason to introduce these matrices becomes clear by comparison with the uniform plasma case. It is well-known that in this limit the hot-plasma dielectric tensor takes its simplest form in reference frame in which one of the two axes perpendicular to the static magnetic field is oriented along the perpendicular component of the wavevector, so that $\vec{k} = k_\perp \vec{u}_x + k_\parallel \vec{b}_0$. In a nonuniform configuration such a choice of the axes orientation is not possible, since the directions perpendicular to $\vec{B}_0$ are not all equivalent. It is possible, however, to define some transformation for each $\vec{k}$ and each $\vec{k}_b$ separately, and exploit it to simplify the evaluation of the conductivity tensor operator. Since this transformation is different for different wavevectors, the rotational matrices $\underline{\underline{R}}(\delta)$ and $\underline{\underline{R}}(\bar{\delta})$ will appear explicitly in the conductivity tensor.

Further we proceed as in the case of the slab geometry (sec. 2.6). Mathematically, the procedure is the same, we write only the results. Let us define two vectors $\vec{\pi}^p$ and $\vec{\pi}^{p^\dagger}$ as

$$\begin{aligned}
\pi^p_x(\nu w) &= \frac{p}{\nu} J_p(\nu w) & \qquad \pi^{p^\dagger}_x(\bar{\nu}\,' w') &= \frac{p}{\bar{\nu}\,'} J_p(\bar{\nu}\,' w') \\
\pi^p_y(\nu w) &= -iwJ'_p(\nu w) & \qquad \pi^{p^\dagger}_y(\bar{\nu}\,' w') &= -iw' J'_p(\bar{\nu}\,' w') \\
\pi^p_z(\nu w) &= uJ_p(\nu w) & \qquad \pi^{p^\dagger}_z(\bar{\nu}\,' w') &= uJ_p(\bar{\nu}\,' w')
\end{aligned} \tag{4.113}$$

With this notation after integration over ϕ_v the tensor $\hat{\underline{\underline{\Pi}}}^K$ takes the form

$$\hat{\underline{\underline{\Pi}}}^K = -i\omega \int_{-\infty}^{t} \sum_{p=-\infty}^{\infty} G'_p(\vec{k}_b, \vec{R}_g, \vec{R}'_g) e^{-ip\bar{\delta}} \underline{\underline{\Pi}}^K e^{ip\delta} dt' \tag{4.114}$$

where the individual contributions are
a) In the 'bulk' conductivity

$$\underline{\underline{\Pi}}^{pM} = \underline{\underline{R}}(\delta) \cdot \vec{\pi}^{p}(\nu w) : \vec{\pi}^{p^{\dagger}}(\bar{\nu}\,'w') \cdot \underline{\underline{R}}^{-1}(\bar{\delta}) \tag{4.115}$$

b) In the diamagnetic conductivity

$$\begin{aligned}
\underline{\underline{\Pi}}^{pD} &= -\Lambda_{D\alpha}(\epsilon_v)\underline{\underline{R}}(\delta) \cdot \vec{\pi}^{p}(\nu w) : \vec{\tau}'^{pD}(\bar{\nu}\,'w') \\
\vec{\tau}'^{pD}(\bar{\nu}\,'w') &= \frac{a v_{th\alpha}}{2L_{n\alpha}\Omega'_{g\alpha}} J_p(\bar{\nu}\,'w')\vec{K}'_B
\end{aligned} \tag{4.116}$$

c) In the drift conductivity

$$\begin{aligned}
\underline{\underline{\Pi}}^{pB} &= \underline{\underline{R}}(\delta) \cdot \vec{\pi}^{p}(\nu w) : \vec{\Upsilon}^{p}(\bar{\nu}\,'w') \\
\vec{\Upsilon}^{p}(\bar{\nu}\,'w') &= \frac{\vec{v}'_{D\alpha}}{v_{th\alpha}} J_p(\bar{\nu}\,'w')
\end{aligned} \tag{4.117}$$

4.12 The high frequency limit

In the case of the Maxwellian plasma, it is possible to perform in closed form also the integrations over the perpendicular velocity, provided that the dependence of the drift velocity $\vec{v}_D$ on $v_\perp$ is disregarded. The most common situation in which this is justified is at the frequencies of the ion cyclotron frequency or higher. The ratio between the diamagnetic and bulk contributions to the conductivity tensor is easily shown to be of the order of

$$\frac{|\underline{\underline{\sigma}}^{D}|}{|\underline{\underline{\sigma}}^{M}|} \sim \frac{\omega_{*\alpha}}{\omega} \tag{4.118}$$

where

$$\omega_{*\alpha} = \frac{k_\perp v_{th\alpha}^2}{2L_n\Omega_{c\alpha}}; \qquad\qquad \frac{1}{L_n} = \frac{d\log n}{d\psi} \tag{4.119}$$

is a 'diamagnetic' frequency. Practically in all situations

$$\frac{\omega_{*i}}{\Omega_{ci}} = k_\perp\rho_i \times \frac{\rho_i}{L_n} \ll 1 \tag{4.120}$$

It follow that at frequencies of the order of the ion cyclotron frequency or higher the diamagnetic contribution to the conductivity can be neglected. Since in toroidal plasmas the characteristic lengths of the gradients of the static magnetic field are longer (by the aspect ratio) than those of density and temperature, the drift conductivity can also be neglected. Finally, at these frequencies $\vec{v}_D$ is mostly negligible also in the propagator: this is the case if

$$k_\eta v_D \ll Max(|\omega - p\Omega_{ci}|, k_\zeta v_{th\alpha}) \tag{4.121}$$

When conditions (4.120) and (4.121) are satisfied, it is possible , and convenient, to revert to w and u as integration variables in velocity space. Moreover, exploiting the identity $v'_\perp = \sqrt{2\mu_v B'_0} = (B'_0/B_0)^{1/2} v_\perp$, we include the primed factor of the Bessel functions in the definition of $\bar{\nu}\,'$. The definition (4.112) will therefore be replaced by

$$\nu = \frac{k_\perp v_{th\alpha}}{\Omega_\alpha} \qquad \bar{\nu}\,' = \frac{\bar{k}_\perp v_{th\alpha}}{\Omega'_\alpha}\left(\frac{B'_0}{B_0}\right)^{1/2} \tag{4.122}$$

The required integrals are the same as those already defined in the straight limit (2.87)-(2.88); we write here the results for convenience. Defining as in the straight limit

$$\lambda_1 = \frac{\nu\bar{\nu}'}{2} \qquad \lambda_2 = \frac{\nu^2 + \bar{\nu}'^{\,2}}{4} \tag{4.123}$$

we obtain the relations

$$\begin{aligned} S^p(\nu,\bar{\nu}\,') &= I_p(\lambda_1)\, e^{-\lambda_2} \\ D^p(\nu,\bar{\nu}\,') &= \left[\frac{\nu}{2} I'_p(\lambda_1) - \frac{\bar{\nu}\,'}{2} I_p(\lambda_1)\right] e^{-\lambda_2} \\ D^p(\bar{\nu}\,',\nu) &= \left[\frac{\bar{\nu}\,'}{2} I'_p(\lambda_1) - \frac{\nu}{2} I_p(\lambda_1)\right] e^{-\lambda_2} \\ T^p(\nu,\bar{\nu}\,') &= \left[\left(\frac{p^2}{2\lambda_1} + \lambda_1\right) I_p(\lambda_1) - \lambda_2 I'_p(\lambda_1)\right] e^{-\lambda_2} \end{aligned} \tag{4.124}$$

and, in the diamagnetic conductivity:

$$\begin{aligned} Y^p(\nu,\bar{\nu}\,') &= \left[(1-\lambda_2)\, I_p(\lambda_1) + \lambda_1 I'_n(\lambda_1)\right] e^{-\lambda_2} \\ X_p(\bar{\nu}\,',\nu) &= \frac{\bar{\nu}'}{2}\left[\left(\frac{p^2}{\lambda_1} + \frac{\nu}{\bar{\nu}'}\left(\lambda_2 + \frac{\bar{\nu}'^{\,2}}{2} - 2\right)\right) I_p(\lambda_1) + \right. \\ &\left. + \left(1 - \lambda_2 - \frac{\nu^2}{2}\right) I'_p(\lambda_1)\right] e^{-\lambda_2} \end{aligned} \tag{4.125}$$

With the help of these identities we can write

$$\begin{aligned} \frac{4\pi i}{\omega} \underline{\underline{\sigma}}(\vec{k},\vec{k}_b;\vec{R}_g) &= -\sum_\alpha \frac{\omega^2_{p\alpha}(\vec{R}_g)}{\omega^2} \int_{-\infty}^{\infty} \frac{e^{-u^2}}{\sqrt{\pi}} du \\ \sum_{p=-\infty}^{\infty} (-i\omega) &\int_{-\infty}^{t} \mathcal{G}'_p(\vec{k},\vec{k}_b) e^{-ip(\bar{\delta}'-\delta)} \sum_K \underline{\underline{\Pi}}^{(p)}_K dt' \end{aligned} \tag{4.126}$$

For simplicity we use the same symbols as in the general case, although of course they now have a different meaning. The sum here includes only the bulk isotropic and the diamagnetic contribution, and we recall that, a small term arising from the Lorentz $\vec{v} \times \vec{B}$ has been neglected. The retained terms are

a) In the 'bulk' conductivity

$$\underline{\underline{\Pi}}_M^{(p)} = 2\underline{\underline{R}}(\delta) \cdot \underline{\underline{P}}^{(p)} \cdot \underline{\underline{R}}^{-1}(\bar{\delta}') \tag{4.127}$$

with

$$\begin{aligned}
&\underline{\underline{P}}_{\psi\psi}^{(p,M)} = \frac{p^2}{\nu\bar{\nu}\,'} S^p(\nu, \bar{\nu}\,') \quad \underline{\underline{P}}_{\psi\eta}^{(p,M)} = i\sqrt{\frac{B_0'}{B_0}}\frac{p}{\nu} D^p(\nu, \bar{\nu}\,') \quad \underline{\underline{P}}_{\psi\zeta}^{(p,M)} = u'\frac{p}{\nu} S^p(\nu, \bar{\nu}\,') \\
&\underline{\underline{P}}_{\eta\psi}^{(p,M)} = -i\frac{p}{\bar{\nu}\,'} D^p(\bar{\nu}\,', \nu) \quad \underline{\underline{P}}_{\eta\eta}^{(p,M)} = \sqrt{\frac{B_0'}{B_0}} T^p(\nu, \bar{\nu}\,') \quad \underline{\underline{P}}_{\eta\zeta}^{(p,M)} = -iu' D^p(\bar{\nu}\,', \nu) \\
&\underline{\underline{P}}_{\zeta\psi}^{(p,M)} = u\frac{p}{\bar{\nu}\,'} S^p(\bar{\nu}\,', \nu) \quad \underline{\underline{P}}_{\zeta\eta}^{(p,M)} = i\sqrt{\frac{B_0'}{B_0}} u D^p(\nu, \bar{\nu}\,') \quad \underline{\underline{P}}_{\zeta\zeta}^{(p,M)} = uu' S^p(\nu, \bar{\nu}\,')
\end{aligned} \tag{4.128}$$

b) In the diamagnetic conductivity

$$\underline{\underline{\Pi}}_D^{(p)} = -\frac{a}{L_{n\alpha}}\frac{v_{th\alpha}}{2\Omega_{g\alpha}'} \underline{\underline{R}}(\delta) \cdot \underline{\underline{P}}^{(p,D)} \tag{4.129}$$

with

$$\begin{aligned}
&\underline{\underline{P}}_{\psi\psi}^{(p,D)} = 0 \quad \underline{\underline{P}}_{\psi\eta}^{(p,D} = \frac{p}{\nu} S_D^p(\nu, \bar{\nu}\,') K_{B\eta}' \quad \underline{\underline{P}}_{\psi\zeta}^{(p,D)} = \frac{p}{\nu} S_D^p(\nu, \bar{\nu}\,') K_{B\zeta}' \\
&\underline{\underline{P}}_{\eta\psi}^{(p,D)} = 0 \quad \underline{\underline{P}}_{\eta\eta}^{(p,D)} = -iD_D^p(\bar{\nu}\,', \nu) K_{B\eta}' \quad \underline{\underline{P}}_{\eta\zeta}^{(p,D)} = -iD_D^p(\bar{\nu}\,', \nu) K_{B\zeta}' \\
&\underline{\underline{P}}_{\zeta\psi}^{(p,D)} = 0 \quad \underline{\underline{P}}_{\zeta\eta}^{(p,D)} = uS_D^p(\nu, \bar{\nu}\,') K_{B\eta}' \quad \underline{\underline{P}}_{\zeta\zeta}^{(p,D)} = uS_D^p(\nu, \bar{\nu}\,') K_{B\zeta}'
\end{aligned} \tag{4.130}$$

where we have introduced the notations

$$\begin{aligned}
S_D^p(\nu, \bar{\nu}\,') &= \left[1 + \eta_T \left(u^2 - \frac{3}{2}\right)\right] S^p(\nu, \bar{\nu}\,') + \eta_T Y^p(\nu, \bar{\nu}\,') \\
D_D^p(\bar{\nu}\,', \nu) &= \left[1 + \eta_T \left(u^2 - \frac{3}{2}\right)\right] D^p(\bar{\nu}\,', \nu) + \eta_T X^p(\bar{\nu}\,', \nu)
\end{aligned} \tag{4.131}$$

If we further assume, that toroidal effects on the parallel motion can be neglected in the evaluation of the time and parallel velocity integrals, the guiding center propagator can be approximated as

$$G_p'(\vec{k}_b) = e^{i\left(k_\zeta^{mn} v_\parallel + \bar{k}_\eta v_{D\eta} - \omega + p\Omega_c\right)(t'-t)} \tag{4.132}$$

where it has already been stipulated that V_D is either negligible or independent from the particle velocity. The t' and $v_\parallel$ integrals can then be performed explicitly in terms of the Plasma Dispersion Function $Z(x_p^D)$ and its derivatives, with argument

$$x_p = \frac{\omega - p\Omega_{cg}}{k_\zeta v_{th\alpha}} \qquad x_p^D = \frac{\omega - p\Omega_{cg} - k_\eta V_D}{k_\zeta v_{th\alpha}} \tag{4.133}$$

In this limit the tensor conductivity operator reduces to the form already evaluated in the case of straight geometry. From this form, the uniform plasma dielectric tensor can again be recovered in the appropriate limit.

Chapter 5

Wave equation in the low frequency range in a torus

In this chapter we will obtain the wave equations for drift and sheared Alfvén waves in a tokamak using the mathematical formalism developed above and some approximations simplifying the calculations. The approach applied here to investigate these gradient driven instabilities is similar to that in the plain stratified plasma case (Chap. 3), but toroidal geometry of the system bring some new features to our consideration. Equations (4.108) together with (4.114)-(4.117) give us general constitutive relation, which must be simplified by taking an appropriate limit. Namely, we consider frequencies well below the ion cyclotron frequency

$$\omega \ll \Omega_{ci} \tag{5.1}$$

Again, in the low frequency range the only resonant wave-particles interactions occur at the Cerenkov resonance, $p = 0$, where p is harmonic number in the expansion in cyclotron harmonics (4.114). As the frequency approaches the diamagnetic frequency, on the other hand, these resonances can be affected by the drift motion of the guiding centers. Diamagnetic and toroidal effects, therefore, must be taken into account. As long as the integrals along the gyrocenter orbits are left unevaluated, however, the algebra needed to obtain the low frequency conductivity parallels closely that of the straight magnetic field limit. Just as in the plane stratified case, the contributions to the total plasma current in the low frequency range can classified as follows:

- The polarisation and $\vec{E} \times \vec{B}_0$ currents ($p \neq 0$);
- The resonant 'Landau' contribution ($p = 0$) to the 'bulk' conductivity;
- The current associated with the drift motion of the particles.

We will neglect third part of conductivity operator since its contribution is too small at the low frequency in comparison with other ones.

5.1 Polarisation and $\vec{E} \times \vec{B}_0$ current

Because of condition (5.1) the parallel velocity integrals can be evaluated asymptotically in the harmonics $p \geq 1$, i.e with $|x_p| \gg 1$. We again will use approximation (3.2) neglecting the v_D in the denominator here Then the contribution of the polarisation and $\vec{E} \times \vec{B}_0$ currents to the conductivity operator has the form

$$\underline{\underline{\sigma}}_{pol}(\vec{k}, \vec{k}'; \vec{R}_g) = \underline{\underline{R}}(\delta) \cdot \underline{\underline{\sigma}}_{pol}^{Smpl} \cdot \underline{\underline{R}}^{-1}(\bar{\delta}) \tag{5.2}$$

with

$$\frac{4\pi i}{\omega} \underline{\underline{\sigma}}_{pol}^{Smpl} = \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2} \begin{pmatrix} \Pi_{XX} & \Pi_{XY} & 0 \\ \Pi_{YX} & \Pi_{YY} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3}$$

The nonvanished elements are

$$\begin{aligned}
\Pi_{XX} &= 2 \sum_{p=1}^{\infty} \frac{I_p(\lambda_1) e^{-\lambda_2}}{\lambda_1} \left[\cos p(\delta - \bar{\delta}) - ip \frac{\Omega_c}{\omega} \sin p(\delta - \bar{\delta}) \right] \\
\Pi_{XY} &= 2i \sum_{p=1}^{\infty} \left[I_p'(\lambda_1) - I_p(\lambda_1) - \frac{\bar{\nu}\,' - \nu}{\nu} I_p(\lambda_1) \right] e^{-\lambda_2} \\
&\left[\frac{\Omega_c}{\omega} \cos p(\delta - \bar{\delta}) - \frac{i}{p} \sin p(\delta - \bar{\delta}) \right] \\
\Pi_{YX} &= -2i \sum_{p=1}^{\infty} \left[I_p'(\lambda_1) - I_p(\lambda_1) - \frac{\nu - \bar{\nu}\,'}{\bar{\nu}\,'} I_p(\lambda_1) \right] e^{-\lambda_2} \\
&\left[\frac{\Omega_c}{\omega} \cos p(\delta - \bar{\delta}) - \frac{i}{p} \sin p(\delta - \bar{\delta}) \right] \\
\Pi_{YY} &= \Pi_{XX} + 2 \sum_{p=1}^{\infty} \left[(\lambda_1 - \lambda_2) I_p(\lambda_1) - \lambda_2 [I_p'(\lambda_1) - I_p(\lambda_1)] \right] e^{-\lambda_2} \\
&\left[\frac{\cos p(\delta - \bar{\delta})}{p^2} - i \frac{\Omega_c}{\omega} \frac{\sin p(\delta - \bar{\delta})}{p} \right]
\end{aligned} \tag{5.4}$$

With the exception of the rapidly convergent series weighted with p^{-2} in the difference between Π_{YY} and Π_{XX}, the sums over harmonics which appear in these expressions can be evaluated using Bessel function summation formulas which can be derived from identity [44]

$$\sum_{p=-\infty}^{\infty} I_p(\lambda) e^{ipx} = e^{\lambda \cos x} \tag{5.5}$$

Taking the real part we obtain

$$2\sum_{p=1}^{\infty} I_p(\lambda)\cos px = e^{\lambda\cos x} - I_0(\lambda) \tag{5.6}$$

Deriving with respect to λ and taking the difference

$$2\sum_{n=1}^{\infty}[I_n'(\lambda) - I_n(\lambda)]\cos nx = [I_0'(\lambda) - I_0(\lambda)] - (1-\cos x)e^{\lambda\cos x} \tag{5.7}$$

Deriving with respect to x gives

$$\begin{aligned} &2\sum_{p=1}^{\infty} pI_p(\lambda)\sin px = \lambda\sin x e^{\lambda\cos x} \\ &2\sum_{p=1}^{\infty} p[I_p'(\lambda) - I_p(\lambda)]\sin px = \sin x[1+\lambda(1-\cos x)]e^{\lambda\cos x} \end{aligned} \tag{5.8}$$

The sine summations weighted with p^{-1} can be obtained by integrating over x

$$\begin{aligned} &2\sum_{p=1}^{\infty} I_p(\lambda)\frac{\sin px}{p} = \int_0^x e^{\lambda\cos x}dx - xI_0(\lambda) \\ &2\sum_{p=1}^{\infty}[I_p'(\lambda) - I_p(\lambda)]\frac{\sin px}{p} = x[I_0(\lambda) - I_0'(\lambda)] - \int_0^x e^{\lambda\cos x}(1-\cos x)dx \end{aligned} \tag{5.9}$$

The results, putting for the brevity $\Delta = \delta - \bar{\delta}$, can be written

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{pol}^{Smpl} = \sum_{\alpha}\frac{\omega_{p\alpha}^2}{\Omega_{c\alpha}^2}\left(\underline{\underline{S}} + i\frac{\Omega_{c\alpha}}{\omega}\underline{\underline{D}}\right) \tag{5.10}$$

with

$$\underline{\underline{S}} = \begin{pmatrix} S_0\cos\Delta & (S_1+\gamma s_1)\sin\Delta & 0 \\ -(S_1+\gamma^\dagger s_1)\sin\Delta & (S_0+s_0)\cos\Delta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

and

$$\underline{\underline{D}} = \begin{pmatrix} D_1\sin\Delta & (D_0+\gamma d_0)\cos\Delta & 0 \\ -(D_0+\gamma^\dagger d_0)\cos\Delta & d_1\sin\Delta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.12}$$

Here

$$
\begin{aligned}
S_0(\lambda_1, \lambda_2, \Delta) &= \frac{e^{\lambda_1 \cos\Delta} - I_0(\lambda_1)}{\lambda_1 \cos\Delta} e^{-\lambda_2} \\
S_1(\lambda_1, \lambda_2, \Delta) &= \left\{ \Delta[I_0(\lambda_1) - I_0'(\lambda_1)] - \int_0^\Delta e^{\lambda_1 \cos x}(1 - \cos x)dx \right\} \frac{e^{-\lambda_2}}{\sin\Delta} \\
s_1(\lambda_1, \lambda_2, \Delta) &= \left[\Delta I_0(\lambda_1) - \int_0^\Delta e^{\lambda_1 \cos x} dx \right] \frac{e^{-\lambda_2}}{\sin\Delta} \\
s_0(\lambda_1, \lambda_2, \Delta) &= [(\lambda_1 - \lambda_2)\tau_1(\lambda_1, \Delta) - \lambda_2 \tau_2(\lambda_1, \Delta)]\, e^{-\lambda_2}
\end{aligned}
\tag{5.13}
$$

with

$$
\begin{aligned}
\tau_1(\lambda_1, \Delta) &= 2\sum_p I_p(\lambda_1) \frac{\cos p\Delta}{p^2 \cos\Delta} \\
\tau_2(\lambda_1, \Delta) &= 2\sum_p [I_p'(\lambda_1) - I_p(\lambda_1)] \frac{\cos p\Delta}{p^2 \cos\Delta}
\end{aligned}
\tag{5.14}
$$

and

$$
\begin{aligned}
D_1(\lambda_1, \lambda_2, \Delta) &= -e^{\lambda_1 \cos\Delta - \lambda_2} \\
D_0(\lambda_1, \lambda_2, \Delta) &= \Big[[I_0(\lambda_1) - I_0'(\lambda_1)] - (1 - \cos\Delta) e^{\lambda_1 \cos\Delta} \Big] \frac{e^{-\lambda_2}}{\cos\Delta} \\
d_0(\lambda_1, \lambda_2, \Delta) &= \lambda_1 S_0(\lambda_1, \lambda_2, \Delta) \\
d_1(\lambda_1, \lambda_2, \Delta) &= \Bigg\{ (\lambda_1 - \lambda_2) \left[\Delta I_0(\lambda_1) - \int_0^\Delta e^{\lambda_1 \cos x} dx \right] \\
&\quad - \lambda_2 \left[\Delta[I_0(\lambda_1) - I_0'(\lambda_1)] - \int_0^\Delta e^{\lambda_1 \cos x}(1 - \cos x) dx \right] \Bigg\} \frac{e^{-\lambda_2}}{\sin\Delta}
\end{aligned}
\tag{5.15}
$$

All function in (5.13)-(5.15) are defined so that their leading terms are either 1 (capital letter) or λ_1 (small letter) when $\lambda_1 \to 1$. Moreover

$$
\gamma = \frac{\bar{\nu}\,' - \nu}{\nu} \qquad\qquad \gamma^\dagger = \frac{\nu - \bar{\nu}\,'}{\bar{\nu}\,'} \tag{5.16}
$$

The complete expression of $\underline{\underline{\sigma}}_{pol}$ in the laboratory frame is quite complicated. To obtain reasonably simple results, we will exploit the fact that the perpendicular wavelength of low frequency waves is always much shorter than the parallel one. In the tokamak, this imposes an approximate relation between the values of the 'poloidal' and 'toroidal' components of the wavevector

$$k_y = -k_z \cot\Theta + \delta k_y \qquad |\delta k_y| \ll |k_y| \tag{5.17}$$

where $\cot\Theta = B_{tor}/B_{pol} \gg 1$. If we again assume that the range of relevant values of the 'radial' component k_x, k_x' are also centered on some average value with a spread which is not too large, then we obtain to a good approximation

$$\bar{\delta} = \delta \qquad |\nu - \bar{\nu}'| \ll \nu, \bar{\nu}' \tag{5.18}$$

The approximations following from (5.18) must be performed only after the 'rotations' $\underline{\underline{R}}(\delta)$, $\underline{\underline{R}}(-\bar{\delta})$, in order to respect the tensor properties of the conductivity operator. With simplifications which follows from (5.18) we obtain

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{pol} = \begin{pmatrix} \hat{S}_0 - \hat{s}_0 \sin\delta \sin\bar{\delta} & -i(\hat{D}_0 + \gamma\hat{d}_0) + \hat{s}_0 \sin\delta\cos\bar{\delta} & 0 \\ i(\hat{D}_0 + \gamma^\dagger \hat{d}_0) + \hat{s}_0 \cos\delta \sin\bar{\delta} & \hat{S}_0 - \hat{s}_0 \cos\delta\cos\bar{\delta} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.19}$$

where

$$\begin{aligned} \hat{S}_0 &= \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2} \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \\ \hat{D}_0 &= -\sum_i \frac{\omega_{pi}^2}{\omega\Omega_{ci}} \left\{ [I_0(\lambda_1) - I_0'(\lambda_1)]e^{-\lambda_2} + 1 \right\} \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \hat{s}_0 &= \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2} \sum_{p=1}^{\infty} \frac{\lambda_1 [I_p'(\lambda_1) - I_p(\lambda_1)]}{p^2} e^{-\lambda_2} \\ \hat{d}_0 &= \lambda_1 \hat{S}_0 \end{aligned} \tag{5.21}$$

Because of second of (5.18), the terms proportional to $\hat{d}_0$ can be neglected compared with $\hat{D}_0$; the same is true for terms proportional to $\hat{s}_0$ in comparison to $\hat{S}_0$. In this approximation the polarisation and $\vec{E} \times \vec{B}$ conductivity in field aligned components simplifies to the familiar form

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{pol} = \begin{pmatrix} \hat{S}_0 & -i\hat{D}_0 & 0 \\ i\hat{D}_0 & \hat{S}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.22}$$

in which the directions of the wavevectors do no more play a role.

5.2 The Landau current

We denote with this name the $p = 0$ contributions to the 'bulk' conductivity, which describe Cherenkov resonances (Landau damping and magnetic pumping) of ions and electrons. According to general results (4.115), these terms are

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\theta) = -4\sum_{\alpha}\frac{\omega_{p\alpha}^2(\vec{R}_g)}{\omega^2}\int_0^\infty d\epsilon_v \bar{F}_\alpha(\epsilon_v)\int d\mu_v \frac{B_0}{v_\parallel}$$
$$\times(-i\omega)\int_{-\infty}^{t}\mathcal{G}_0'(\vec{k}_b,\omega)\underline{\underline{R}}(\delta)\cdot\underline{\underline{P}}_{Ld}^{(0)}(\vec{k},\vec{k}_b)\cdot\underline{\underline{R}}(-\delta)dt' \tag{5.23}$$

Here

$$\underline{\underline{P}}_{Ld}^{(0)} = \vec{\pi}^0 : \vec{\pi}^{0^\dagger} \tag{5.24}$$

with

$$\begin{aligned} &\pi_\xi^0(\nu w) = 0 && \pi_\xi^{0^\dagger}(\bar{\nu}\,'w) = 0 \\ &\pi_\eta^0(\nu w) = -iwJ_0'(\nu w) && \pi_\eta^{0^\dagger}(\bar{\nu}\,'w) = -iw'J_0'(\bar{\nu}\,'w) \\ &\pi_\zeta^0(\nu w) = uJ_0(\nu w) && \pi_\zeta^{0^\dagger}(\bar{\nu}\,'w) = u'J_0(\bar{\nu}\,'w) \end{aligned} \tag{5.25}$$

and

$$\mathcal{G}_0'(\vec{k}_b,\vec{R}_g,\vec{R}_g') = \exp\left\{i[\bar{\kappa}(\psi_g' - \psi_g) + \bar{m}(\theta_g' - \theta_g) + n(\varphi_g' - \varphi_g) - \omega(t' - t)]\right\} \tag{5.26}$$

is the guiding center propagator for the cyclotron harmonic $p = 0$.

The integrals in eq.(5.23) cannot be evaluated in the closed form without further approximations. Therefore we introduce the notations

$$\mathcal{I}_{i,j}^{r,\bar{r},2s}(\nu,\bar{\nu}) = \frac{2t_{r,\bar{r}}}{(s+i+j)!}\frac{2^{i+j}}{\nu^i\bar{\nu}}\int_0^\infty d\epsilon_v\frac{2e^{-2\epsilon_v}}{\sqrt{\pi}}\sum_{\sigma_v}d\mu_v\frac{B_0}{v_\parallel}$$
$$\times(-i\omega)\int_{-\infty}^{t}\mathcal{G}_0'(\vec{k}_b,\omega)w^{2s+i}w'^{j}u^r u'^{\bar{r}}J_i(\nu w)J_j(\bar{\nu}\,'w')dt' \tag{5.27}$$

for $i,j = 0,1$, $s = 0,1$ (the letter value only in the diamagnetic conductivity), $\bar{r} = 0,1$, $0 \le r+\bar{r} \le 3$. The coefficients $t_{r,\bar{r}}$ are chosen so that in the straight, zero Larmor radius limit, the integrals tend to unity when the parallel phase velocity tends to infinity:

$$\begin{aligned} &t_{0,0} = 1 \qquad t_{1,0} = 2\frac{\omega}{k_\zeta v_{th}} \qquad t_{0,1} = 2\frac{\omega}{\bar{k}_\zeta v_{th}} \\ &t_{2,0} = t_{1,1} = t_{0,2} = 2 \qquad t_{3,0} = \frac{4}{3}\frac{\omega}{k_\zeta v_{th}} \end{aligned} \tag{5.28}$$

With the notation introduced above we have

$$\underline{\underline{\sigma}}_{Ld}(\vec{k},\vec{k}_b,\psi,\theta) = \underline{\underline{R}}(\delta)\cdot\underline{\underline{\sigma}}_{Ld}^{Smpl}(\vec{k},\vec{k}_b,\psi,\theta)\cdot\underline{\underline{R}}(-\bar{\delta}) \tag{5.29}$$

where

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{Ld}^{Smpl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2n_\perp n_\perp^b \hat{\tau} & i n_\parallel^b n_\perp \hat{\xi} \\ 0 & -i n_\parallel n_\perp^b \hat{\xi}^\dagger & \hat{P} \end{pmatrix} \tag{5.30}$$

with

$$\begin{aligned}
\hat{\tau}(\vec{k},\vec{k}_b,\psi,\theta) &= \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{g\alpha}^2}\frac{v_{th\alpha}^2}{c^2}\mathcal{I}_{1,1}^{0,0,0}(\nu,\bar{\nu}\,') \\
\hat{\xi}(\vec{k},\vec{k}_b,\psi,\theta) &= -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{g\alpha}}\frac{v_{th\alpha}^2}{c^2}\mathcal{I}_{1,0}^{0,1,0}(\nu,\bar{\nu}\,') \\
\hat{\xi}^\dagger(\vec{k},\vec{k}_b,\psi,\theta) &= -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega\Omega_{g\alpha}}\frac{v_{th\alpha}^2}{c^2}\mathcal{I}_{0,1}^{1,0,0}(\nu,\bar{\nu}\,') \\
\hat{P}(\vec{k},\vec{k}_b,\psi,\theta) &= -\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\mathcal{I}_{0,0}^{1,1,0}(\nu,\bar{\nu}\,')
\end{aligned} \tag{5.31}$$

Performing the rotation explicitly, we obtain

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_{Ld} = \frac{c^2}{\omega^2}\begin{pmatrix} -2k_\eta^{mn}\bar{k}_\eta^{\bar{m}n}\hat{\tau} & 2k_\eta^{mn}\bar{k}_\psi\hat{\tau} & -ik_\eta^{mn}\bar{k}_\zeta^{\bar{m}n}\hat{\xi} \\ 2k_\psi\bar{k}_\eta^{\bar{m}n}\hat{\tau} & -2k_\psi\bar{k}_\psi\hat{\tau} & ik_\psi\bar{k}_\zeta^{\bar{m}n}\hat{\xi} \\ i\bar{k}_\eta^{\bar{m}n}k_\zeta^{mn}\hat{\xi}^\dagger & -i\bar{k}_\psi k_\zeta^{mn}\hat{\xi}^\dagger & \dfrac{w^2}{c^2}\hat{P} \end{pmatrix} \tag{5.32}$$

We see again, that the Landau contribution to the conductivity operator has the same form as in plane-stratified geometry; of course, however, in the toroidal case the orbit integrals which enter in the definitions of its elements are appreciably more complicated.

5.3 The diamagnetic current

In the terms proportional to the equilibrium gradients we can distinguish non-resonant and resonant contributions. Using the identities

$$\frac{2}{\bar{\nu}}\sum_p X(\bar{\nu}\,',\nu)\frac{\omega}{\omega-p\Omega} \rightarrow -\frac{\omega^2}{\Omega_{g\alpha}}\frac{e^{\lambda_1}-I_0(\lambda_1)}{\lambda_1}e^{-\lambda_2}$$

$$\sum_p pY^p(\nu,\bar{\nu}\,')\frac{\omega}{\omega-p\Omega} \rightarrow -\frac{\omega}{\Omega_{g\alpha}}\left\{(1+\lambda_1+\lambda_2)\left[e^{\lambda_1}-I_0(\lambda_1)\right]\right. \tag{5.33}$$

$$\left.-\lambda_1\left[I_0'(\lambda_1)-I_0(\lambda_1)\right]\right\}e^{-\lambda_2}$$

we obtain for non-resonant terms

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\theta) = 2\sum_\alpha \frac{\omega_{p\alpha}^2(\psi)}{\Omega_{g\alpha}^2}\frac{\bar{\omega}_{*\alpha}}{\Omega_{g\alpha}}\underline{\underline{R}}(\delta)\cdot\vec{P}^D : a(\vec{b}\times\vec{\nabla}\psi) \tag{5.34}$$

where the nonvanishing components of $\vec{P}^D$ are

$$P_x^D = -(1-\eta_T)\frac{e^{\lambda_1}-I_0(\lambda_1)}{\lambda_1}e^{-\lambda_2}$$

$$-\eta_T\frac{\Omega_{g\alpha}}{\omega}\left\{(1+\lambda_1-\lambda_2)\frac{e^{\lambda_1}-I_0(\lambda_1)}{\lambda_1} - \left[I_0'(\lambda_1)-I_0(\lambda_1)\right]\right\}e^{-\lambda_2} \tag{5.35}$$

$$P_y^D = -i\eta_T\frac{e^{\lambda_1}-I_0(\lambda_1)}{\lambda_1}e^{-\lambda_2}$$

and, as before,

$$\bar{\omega}_{*\alpha} = \frac{k_\perp^b v_{th\alpha}^2}{2\Omega_{g\alpha}L_n} \tag{5.36}$$

is the diamagnetic frequency of species α built with the 'barred' perpendicular wavevector. Since $P_z^D = 0$, the neoclassic contribution to nonresonant part of the diamagnetic current vanishes.

The resonant terms ($p=0$) of diamagnetic conductivity are:

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_D(\vec{k},\vec{k}_b,\psi,\theta) = 2\sum_\alpha \frac{\omega_{p\alpha}^2(\vec{R}_g)}{\omega^2}\int_0^\infty d\epsilon_v \bar{F}_\alpha(\epsilon_v)\frac{v_{th\alpha}}{2\Omega_{c\alpha}L_n}\Lambda_D(\epsilon_v)$$

$$\times\int d\mu_v\frac{B_0}{v_\parallel}\left[(-i\omega)\int_{-\infty}^t \mathcal{G}_0'(\vec{k}_b,\omega)\underline{\underline{R}}(\delta)\cdot\vec{\pi}^0(\vec{k},u,w) : \vec{\tau}_D^0(\vec{k}_b,w')dt'\right] \tag{5.37}$$

where

$$\Lambda_D = 1 + \eta_T \left(\frac{2\epsilon_v}{v_{th\alpha}^2} - \frac{3}{2} \right) \tag{5.38}$$

and

$$\vec{\tau}_D^0 = a \left[\vec{b} \times \vec{\nabla}\psi - H_{tp} \frac{RB_0 \cos\Theta}{\mathcal{F}(\psi)} \vec{b} \right] J_0(\bar{\nu}'\omega') \tag{5.39}$$

We recall that the η component of $\vec{\tau}_D^0$ is the conventional diamagnetic term, while the ζ component is a neoclassical effect, contributed only by trapped particles.

Simplifying the nonresonant terms by taking into account that $\delta \simeq \bar{\delta}'$, we can write the diamagnetic contribution in matrix form as

$$\underline{\underline{\sigma}}_D(\vec{k}, \vec{k}_b, \psi, \theta) = \underline{\underline{R}}(\delta) \cdot \underline{\underline{\sigma}}_D^{Smpl}(\vec{k}, \vec{k}_b, \psi, \theta) \tag{5.40}$$

where

$$\frac{4\pi i}{\omega} \underline{\underline{\sigma}}_D^{Smpl} = \begin{pmatrix} 0 & \mathcal{A}^D & 0 \\ 0 & -i(\mathcal{B}^D + \mathcal{H}^D) & -i \dfrac{RB_0 \cos\Theta}{\mathcal{F}} \mathcal{H}_{tp}^D \\ 0 & \dfrac{\bar{k}_\zeta^{\bar{m}n}}{k_\perp^b} \mathcal{K}^D & -\dfrac{RB_0 \cos\Theta}{\mathcal{F}} \dfrac{\bar{k}_\zeta^{\bar{m}n}}{k_\perp^b} \mathcal{K}_{tp}^D \end{pmatrix} \tag{5.41}$$

Here the non-resonant terms are

$$\begin{aligned} \mathcal{A}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2(\psi)}{\Omega_{g\alpha}^2} \frac{\bar{\omega}_{*\alpha}}{\omega} \left\{ \frac{\omega}{\Omega_{g\alpha}} \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} - \eta_T [I_0'(\lambda_1) - I_0(\lambda_1)] \right\} e^{-\lambda_2} \\ \mathcal{B}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2(\psi)}{\Omega_{g\alpha}^2} \frac{\bar{\omega}_{*\alpha}}{\Omega_{g\alpha}} \eta_T \frac{e^{\lambda_1} - I_0(\lambda_1)}{\lambda_1} e^{-\lambda_2} \end{aligned} \tag{5.42}$$

The classic resonant terms in (5.41) are

$$\begin{aligned} \mathcal{H}^D &= -\sum_\alpha \frac{\omega_{p\alpha}^2(\psi)}{\omega^2} \frac{\bar{\omega}_{*\alpha}}{\Omega_{g\alpha}} \left\{ \left(1 - \frac{3}{2}\eta_T \right) \mathcal{I}_{1,0}^{0,0,0} + \eta_T \left[\frac{1}{2} \mathcal{I}_{1,0}^{2,0,0} + 2\mathcal{I}_{1,0}^{0,0,2} \right] \right\} \\ \mathcal{K}^D &= \sum_\alpha \frac{\omega_{p\alpha}^2(\psi)}{\omega^2} \frac{\bar{\omega}_{*\alpha}}{\omega} \left\{ \left(1 - \frac{3}{2}\eta_T \right) \mathcal{I}_{0,0}^{1,0,0} + \eta_T \left[\mathcal{I}_{0,0}^{1,0,2} + \frac{3}{2} \mathcal{I}_{0,0}^{3,0,0} \right] \right\} \end{aligned} \tag{5.43}$$

while the neoclassic ones, indicated by subscript 'tp', have the same form, but with the integrals extended only to the trapped particles. Performing the rotation explicitly finally gives

$$\frac{4\pi i}{\omega}\underline{\underline{\sigma}}_D = \begin{pmatrix} 0 & \dfrac{k_\psi \mathcal{A}^D + i k_\eta^{mn}(\mathcal{B}^D + \mathcal{H}^D)}{k_\perp} & i\dfrac{RB_0 \cos\Theta}{\mathcal{F}_p}\dfrac{k_\eta^{mn}}{k_\perp}\mathcal{H}_{tp}^D \\ 0 & \dfrac{k_\eta^{mn}\mathcal{A}^D + i k_\psi(\mathcal{B}^D + \mathcal{H}^D)}{k_\perp} & -i\dfrac{RB_0 \cos\Theta}{\mathcal{F}_p}\dfrac{k_\psi}{k_\perp}\mathcal{H}_{tp}^D \\ 0 & \dfrac{\bar{k}_\zeta^{\bar{m}n}}{k_\perp^b}\mathcal{K}^D & -\dfrac{RB_0 \cos\Theta}{\mathcal{F}_p}\dfrac{\bar{k}_\zeta^{\bar{m}n}}{k_\perp^b}\mathcal{K}_{tp}^D \end{pmatrix} \tag{5.44}$$

If the neoclassical corrections in the left column are neglected, the diamagnetic conductivity has again essentially the same form as in the plane-stratified case, except for the more complicated form of the orbit integrals.

The drift conductivity term should be consider further; however its contribution is far less than contribution of the diamagnetic one. Therefore, we omit drift term in the total conductivity operator and will not calculate its contibution here.

5.4 The compressional wave

Now, having eqs.(5.22), (5.32), (5.44) we can derive the constitutive relation in the low-frequency case. The wave equation obtained by substituting the low-frequency constitutive relation into the Maxwell equations describe three main types of low-frequency waves, namely variety of drift waves, compressional and shear Alfvén waves. Again, it is useful to consider separately the compressional Alfvén waves. The compressional waves propagates across $\vec{B}_0$ with perpendicular phase velocity of the order of Alfvén speed, which in low-beta plasma is much faster then the ion thermal velocity. Except for boundary conditions, they are little affected by the plasma inhomogeneity. Because of their fast perpendicular phase velocity, moreover, the wavelengths are always much larger than the thermal ion Larmor radius:

$$\frac{k_\perp^2 v_{thi}^2}{\Omega_{gi}^2} = \frac{v_{thi}^2}{v_A^2}\frac{\omega^2}{\Omega_{gi}^2} \simeq \beta_p \frac{\omega^2}{\Omega_{gi}^2} \tag{5.45}$$

In low-beta plasmas in the low-frequency range both factors on the r.h. side are much smaller than unity.

We simplify the general low frequency constitutive relation as it is described in the sec. (3.5). eqs. (4.84)-(4.88) by means of the local approximation. To write the wave equation for compressional Alfvén waves, it is justified to expand the constitutive relation for small Larmor radius. The displacement current, moreover, can be neglected at these frequencies.

In the ion polarisation and $\vec{E} \times \vec{B}_0$ currents we have, to second order in $k_\perp \rho_i$

$$
\begin{aligned}
\hat{S}_0 &\sim \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2}\left(1+\frac{\lambda_1}{4}-\lambda_2\right) \\
\hat{D}_0 &\sim -\sum_i \frac{\omega_{pi}^2}{\omega\Omega_{ci}}\left(\frac{\lambda_1}{2}+\lambda_2\right)
\end{aligned}
\tag{5.46}
$$

The FRL expansion of the Landau current can be left in the form (5.32), except that in the $\hat{\tau}$, $\hat{\xi}$, $\hat{\xi}^\dagger$ the limit ν,$\bar{\nu} \to 0$ can be taken in the integrals , since these quantities are already of the second order in Larmor radius. The diamagnetic contribution can be neglected altogether for the compressional waves.

The system of ordinary differential equations then simplifies to

$$
\begin{aligned}
&\int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\, e^{-im\bar{\theta}} \left\{ e^{-in\varphi}\vec{\nabla} \times \left[\vec{\nabla} \times \left(\sum_{\bar{m}} \vec{E}^{\bar{m}n}(\psi) e^{i(\bar{m}\bar{\theta}+n\varphi)}\right)\right]\right\} \\
&= \frac{4\pi i}{\omega} \sum_{\bar{m}} \int_0^{2\pi} \frac{d\bar{\theta}}{2\pi}\, e^{i(\bar{m}-m)\bar{\theta}} \underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\bar{\theta}) \cdot \vec{E}^{\bar{m}n}(\psi)
\end{aligned}
\tag{5.47}
$$

where the current can be written in vector form as

$$
\begin{aligned}
&\frac{4\pi i}{\omega}\underline{\underline{\sigma}}(\vec{k},\vec{k}_b,\psi,\bar{\theta}) \cdot \vec{E}^{\bar{m}n}(\psi) = \hat{S}_0 \vec{E}_\perp^{\bar{m}n}(\psi) + i\hat{D}_0(\vec{b} \times \vec{E}_\perp^{\bar{m}n}(\psi)) + \hat{P}_0 \vec{E}_\zeta^{\bar{m}n}(\psi)\vec{b} \\
&+\frac{c^2}{\omega^2}\vec{\nabla}_\perp \times \left[\hat{\tau}\vec{\nabla}_\perp \times \vec{E}_\perp^{\bar{m}n}(\psi)\right] + i\frac{c^2}{\omega^2}\left\{\vec{\nabla}_\perp\left[\hat{\xi}(\vec{b}\cdot\vec{\nabla})\vec{E}_\zeta^{\bar{m}n}(\psi)\vec{b}\right]\right. \\
&\left. + \vec{b} \times \vec{\nabla}\left[\hat{\xi}\vec{b}\cdot(\vec{\nabla}_\perp \times \vec{E}_\perp^{\bar{m}n}(\psi))\right]\vec{b}\right\}
\end{aligned}
\tag{5.48}
$$

Here, according to (4.93), $\vec{k}$ and $\vec{k}_b$ differ only for the poloidal wavenumber, m and $\bar{m}$, respectively. The term of zeroth order in the Larmor radius are

$$
\hat{S}_0 = \sum_i \frac{\omega_{pi}^2}{\Omega_{ci}^2} \qquad\qquad \hat{P}_0 = -\frac{\omega_{pi}^2}{\omega^2}\bar{\mathcal{I}}^{1,1,0} \tag{5.49}
$$

and those of the second order

$$
\begin{aligned}
\hat{\tau}(\vec{k},\vec{k}_b,\psi,\theta) &= \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\Omega_{g\alpha}^2}\frac{v_{th\alpha}^2}{c^2}\bar{\mathcal{I}}^{0,0,0} \\
\hat{\xi}(\vec{k},\vec{k}_b,\psi,\theta) &= -\frac{1}{2}\frac{\omega_{pe}^2}{\omega\Omega_{ge}}\frac{v_{the}^2}{c^2}\bar{\mathcal{I}}^{0,1,0} \\
\hat{\xi}^\dagger(\vec{k},\vec{k}_b,\psi,\theta) &= -\frac{1}{2}\frac{\omega_{pd}^2}{\omega\Omega_{gd}}\frac{v_{thd}^2}{c^2}\bar{\mathcal{I}}^{1,0,0}
\end{aligned}
\tag{5.50}
$$

with

$$\bar{\mathcal{I}}^{r,\bar{r},2s}(\nu,\bar{\nu}) = \frac{2t_{r,\bar{r}}}{s!}\int_0^\infty d\epsilon_v \frac{2e^{-2\epsilon_v}}{\sqrt{\pi}}\sum_{\sigma_v} d\mu_v \frac{B_0}{v_\parallel}$$
$$\times(-i\omega)\int_{-\infty}^t \mathcal{G}_0'(\vec{k}_b,\omega) w^{2s} u^r u'^{\,\bar{r}} dt' \tag{5.51}$$

The local dispersion relation corresponding to eqn. (5.47) is familiar

$$n_\perp^2 = -(n_\parallel^2 - \hat{S}_0)^2(1-2\hat{\tau}) \tag{5.52}$$

and all the well-known properties of the compressional wave at low frequencies are easily recovered from this dispersion relation. In particular, this equation confirms that as long as $n_\parallel^2 \ll \hat{S}_0$, the waves propagates almost perpendicularly to the static magnetic field, with phase and group velocities close to the Alfvén velocity v_A. We note also, that $\hat{\xi}$, $\hat{\xi}^\dagger$ and $\hat{P}_0$ do not enter explicitely in the dispersion relation. They, however, determine the electric field polarisation, or, more precisely, the ratio of the parallel and perpendicular components,

$$E_\zeta \sim -\frac{n_\zeta n_\perp}{\hat{P}_0}\left(\vec{E}_\perp - i\xi\vec{b}\times\vec{E}_\perp\right) \tag{5.53}$$

and thus the intensity of electron Landau damping, which can be evaluated iteratively.

5.5 The drift and shear-Alfven waves

To investigate the much slower low-frequency waves and instabilities of the drift and shear-Alfven type, it is convenient to factorise out the compressional Alfven wave by writing

$$\vec{E} = -\vec{\nabla}\phi + i\frac{\omega}{c}\vec{A}_\zeta\vec{b}$$
$$\vec{B} = \vec{\nabla}(A_\zeta\vec{b}) \simeq \left(\vec{\nabla}A_\zeta - \frac{\vec{\nabla}B_0}{B_0}A_\zeta\right)\times\vec{b} \tag{5.54}$$

As in the case of straight geometry, in $\vec{B}$ we have neglected a small term proportional to $\vec{\nabla}\times\vec{B}_0$, which in plasmas with steady-state current, as tokamaks, represents a very small residual coupling between compressional and shear waves. Substituting the constitutive relation into Maxwell's equations, neglecting the vacuum displacement current and corrections of order $k^2\lambda_D^2$ in Poisson's equation, where λ_D^2 is the Debye length, the equations to be solved become

$$\tilde{\rho} = 0$$
$$\vec{\nabla} \times \left(\vec{\nabla} \times (A_\zeta \vec{b})\right) = \frac{4\pi}{c} j_\zeta \vec{b} \tag{5.55}$$

The first of these equation is charge neutrality, and in the second only the current parallel to the static magnetic field plays a role.

The most important terms to the kernel of the conductivity operator are contributed by the bulk and classic diamagnetic conductivity where, neglecting for simplicity the particle drift conductivity $\underline{\underline{\sigma}}_B$ and the small neoclassical term in the diamagnetic conductivity together, they have the form

$$\begin{aligned}
\mathcal{X} &= \frac{4\pi i}{\omega} \frac{\vec{k}}{k} \cdot \underline{\underline{\sigma}} \cdot \frac{\vec{k}^b}{k} = \frac{k_\psi \bar{k}_\psi + k_\eta^{mn} \bar{k}_\eta^{\bar{m}n}}{k^2} (\hat{S}_0 + \mathcal{A}^D) + \frac{k_\zeta^{mn} \bar{k}_\zeta^{\bar{m}n}}{k^2} (\hat{P} + \mathcal{K}^D) \\
\mathcal{Y} &= \frac{4\pi i}{\omega} \frac{\vec{k}}{k} \cdot \underline{\underline{\sigma}} \cdot \vec{b} = \frac{k_\zeta^{mn}}{k} \hat{P} \\
\mathcal{Y}^\dagger &= \frac{4\pi i}{\omega} \vec{b} \cdot \underline{\underline{\sigma}} \cdot \frac{\vec{k}^b}{k} = \frac{\bar{k}_\zeta^{\bar{m}n}}{k} (\hat{P} + \mathcal{K}^D) \\
\mathcal{Z} &= \frac{4\pi i}{\omega} \vec{b} \cdot \underline{\underline{\sigma}} \cdot \vec{b} = \hat{P}
\end{aligned} \tag{5.56}$$

Here $\mathcal{A}^D$ and $\mathcal{K}^D$ are defined as in eq. (5.42), but with diamagnetic frequency

$$\omega_{*\alpha} = \frac{k_\eta^{m,n} v_{th\alpha}^2}{2\Omega_{c\alpha} L_n} \tag{5.57}$$

defined in the more usual way in terms of the component of the unbarred wavevector in the diamagnetic direction. In $\mathcal{X}$ we have omitted a term $i[(\bar{k}_\psi k_\eta^{mn} - k_\psi \bar{k}_\eta^{\bar{m}n})k^2]\hat{D}$, which is negligible when the barred and unbarred wavevectors are nearly aligned.

The coefficient $\mathcal{X}$ is easily recognized as the scalar susceptibility of the plasma. Its first term is due to the polarisation current, the one is most appropriate for the ions. Its second term is mostly determined by the electrons in which usually the adiabatic response is singled out. This splitting can be performed by the appropriate integration by parts, after which we obtain $\hat{P}_0$ and $\mathcal{K}^D$ in the form

$$\begin{aligned}
\hat{P}_0 &= 2 \sum_\alpha \frac{\omega_{p\alpha}^2(\psi_g)}{k_\zeta^{mn} \bar{k}_\zeta^{\bar{m}n} v_{th\alpha}^2} \left(1 - \mathcal{I}_{0,0}^{0,0,0}\right) \\
\mathcal{K}^D &= -2 \sum_\alpha \frac{\omega_{p\alpha}^2(\psi_g)}{k_\zeta^{mn} \bar{k}_\zeta^{\bar{m}n} v_{th\alpha}^2} \frac{\omega_{*\alpha}}{\omega} \left\{ \left(1 - \frac{3}{2}\eta_T\right) \mathcal{I}_{0,0}^{0,0,0} - \frac{\eta_T}{2} \left[\mathcal{I}_{0,0}^{2,0,0} + 2\mathcal{I}_{0,0}^{0,0,2}\right] \right\}
\end{aligned} \tag{5.58}$$

where non-adiabatic part of the electron motion is determined by the integrals $\mathcal{I}$.

For completeness, we list also the neoclassic diamagnetic current which, are less important, and are mostly omitted in the literature.

$$
\begin{aligned}
\mathcal{X}_{nc} &= \frac{4\pi i}{\omega}\frac{\vec{k}}{k}\cdot\underline{\underline{\sigma}}^{nc}\cdot\frac{\vec{k}^b}{k} = \frac{RB_0\cos\Theta}{\mathcal{F}}\frac{k_\zeta^{mn}\bar{k}_\zeta^{\bar{m}n}}{k_\perp^b}\mathcal{K}_{tp}^D \\
\mathcal{Y}_{nc} &= \frac{4\pi i}{\omega}\frac{\vec{k}}{k}\cdot\underline{\underline{\sigma}}^{nc}\cdot\vec{b} = \frac{RB_0\cos\Theta}{\mathcal{F}}\frac{k_\zeta^{mn}}{k}\mathcal{K}_{tp}^D \\
\mathcal{Y}^\dagger{}_{nc} &= \frac{4\pi i}{\omega}\vec{b}\cdot\underline{\underline{\sigma}}^{nc}\cdot\frac{\vec{k}^b}{k} = \frac{RB_0\cos\Theta}{\mathcal{F}}\frac{\bar{k}_\zeta^{\bar{m}n}}{k}\mathcal{K}_{tp}^D \\
\mathcal{Z}_{nc} &= \frac{4\pi i}{\omega}\vec{b}\cdot\underline{\underline{\sigma}}^{nc}\cdot\vec{b} = \frac{RB_0\cos\Theta}{\mathcal{F}}\mathcal{K}_{tp}^D
\end{aligned}
\tag{5.59}
$$

where $\mathcal{K}_{tp}^D$ is defined by the second of (5.43) but with the integrals extended only to the trapped particles.

The perturbed charge density can in principle be evaluated from the divergence of the current. In the general form this approach is rather cumbersome, since in curvilinear geometry Fourier components are not eigenfunction of the divergence operator. Indeed, if

$$
\vec{j}(\psi,\theta,\varphi,t) = \sum_{m,n}\int d\kappa\vec{j}(\vec{k})e^{i(\kappa\psi+m\theta+n\varphi-\omega t)} \tag{5.60}
$$

then, in the local field-aligned components,

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{j} = \sum_{m,n}\int d\kappa\,\Big[&(k_\psi^{m,n}-i\nu_\psi)j_\psi(\vec{k})+(k_\eta^{m,n}-i\nu_\eta)j_\eta(\vec{k}) \\
&+(k_\zeta^{m,n}-i\nu_\zeta)\,j_\zeta(\vec{k})\Big]\,e^{i(\kappa\psi+m\theta+n\varphi-\omega t)}
\end{aligned}
\tag{5.61}
$$

where $k_\psi^{m,n}(\psi,\theta)$, $k_\eta^{m,n}(\psi,\theta)$ and $k_\zeta^{m,n}(\psi,\theta)$ are the physical components of the wavevector, and the quantities ν_i consist of the derivatives of the metric elements and coordinates.

It is to be expected, however, that for most low-frequency waves these quantities, proportional to derivatives of the metric and to the inverse shear length, are negligible compared to the components of the typical wavevectors,

$$
|\nu_j| \ll |k_j| \tag{5.62}
$$

Assuming to be the case, it is possible to perform our calculations using the equality of the total current divergency to zero. Simple calculation in the Fourier representation gives

$$
\begin{aligned}
&-\frac{4\pi}{k^2}\rho^{mn}(\kappa) = \sum_{\bar m}\int d\bar\kappa \int_0^{2\pi}\frac{d\bar\theta}{2\pi}\int_0^1 d\bar\psi e^{i[(\bar\kappa-\kappa)\bar\psi+(\bar m-m)\bar\theta]} \\
&\times\left[\mathcal{X}(\vec k,\vec k_b;\bar\psi,\bar\theta)\phi^{\bar m n}(\bar\kappa) - \frac{\omega}{ck}\mathcal{Y}(\vec k,\vec k_b;\bar\psi,\bar\theta)A_\zeta^{\bar m n}(\bar\kappa)\right] \\
&\frac{4\pi c}{\omega^2}j_\zeta^{mn}(\kappa) = \sum_{\bar m}\int d\bar\kappa \int_0^{2\pi}\frac{d\bar\theta}{2\pi}\int_0^1 d\bar\psi e^{i[(\bar\kappa-\kappa)\bar\psi+(\bar m-m)\bar\theta]} \\
&\times\left[-\frac{c\bar k}{\omega}\mathcal{Y}^\dagger(\vec k,\vec k_b;\bar\psi,\bar\theta)\phi^{\bar m n}(\bar\kappa) + \mathcal{Z}(\vec k,\vec k_b;\bar\psi,\bar\theta)A_\zeta^{\bar m n}(\bar\kappa)\right]
\end{aligned}
\tag{5.63}
$$

Taking the inverse Fourier transform of eqs.(5.55) and substituting the results into (5.55), one obtains coupled integro-differential wave equations for $\phi^{\bar m n}(\psi)$ and $A_\zeta^{\bar m n}(\psi)$. Taking advantage of the fact that drift and sheared Alfvén waves propagate preferentially in the diamagnetic direction, we can apply the technic of sec 4.9 to transform them into differential wave equations valid to second order in $k_\psi\rho_i \ll 1$. Since we expect $k_\eta \gg k_\psi$, we do not prelimeraly develop the kernel in the Larmor radius, so that the equations obtained will be valid to all order in $k_\eta\rho_i$. The equations obtained in the such way are, for each m, $-\infty < m < \infty$

$$
\begin{aligned}
0 = \sum_{\bar m}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(\bar m-m)\theta}&\Big\{\vec\nabla_\perp\cdot\Big[\Big(\hat S_0(\psi,\theta)+\mathcal{A}^D(\psi,\theta)\Big)\Big(\vec\nabla_\perp\phi^{\bar m n}(\psi)\Big)\Big] \\
+\vec\nabla\cdot\Big[\vec b\Big\{\Big(\hat P_0(\psi,\theta)+\mathcal{K}^D(\psi,\theta)\Big)&\Big(\vec b\cdot\vec\nabla\phi^{\bar m n}(\psi)\Big)+\hat P_0(\psi,\theta)A_\zeta^{\bar m n}(\psi)\Big\}\Big]\Big\}
\end{aligned}
\tag{5.64}
$$

$$
\begin{aligned}
&\sum_{\bar m}\int d\bar\kappa\int_0^{2\pi}\frac{d\bar\theta}{2\pi}e^{-im\theta}\Big\{\vec b\Big[\vec\nabla\times\Big(\vec\nabla\times A_\zeta^{\bar m n}(\psi)e^{i\bar m\theta}\vec b\Big)\Big]\Big\} = \frac{4\pi}{c}\sum_{\bar m}\int_0^{2\pi}\frac{d\theta}{2\pi} \\
&\times\Big\{\Big(\hat P_0(\psi,\theta)+\mathcal{K}^D(\psi,\theta)\Big)\Big(\vec b\cdot\nabla\phi^{\bar m n}(\psi)\Big)+\hat P_0(\psi,\theta)A_\zeta^{\bar m n}(\psi)\Big\}e^{i(\bar m-m)\theta}
\end{aligned}
$$

where the expression for $\hat S_0$, $\mathcal{A}^D$, $\hat P_0$, $\mathcal{K}^D$ are given above.

Electrostatic limit of the equations obtained is just a generalization of the Poisson equation for electrostatic waves at low frequencies, which in the cold plasma limit is

$$
\vec\nabla_\perp\cdot(\hat S_0\vec\nabla_\perp\phi)+\vec\nabla_\parallel(\hat P_0\vec\nabla_\parallel\phi)=0 \tag{5.65}
$$

This equation describes cold-plasma electrostatic waves up to the lower hibrid frequency, and, in particular, lower-hibrid resonance cones.

Chapter 6

Summary and conclusions

In this work we have obtained the constitutive relation for low frequency waves propagating in a hot plasma confined by a nonuniform magnetic field by integrating the linearised Vlasov equation. Two cases have been considered, namely a plane-stratified plasma in a sheared magnetic field, and an axisymmetric toroidal plasma of arbitrary poloidal cross-section. The present derivation is quite general: the plasma response has been evaluated under the only assumption that the drift approximation can be used to describe the unperturbed motion of charged particles in the nonuniform confining magnetic field. As a consequence, the wave equations obtained by substituting into Maxwell's equations the results of section 2.7 in the plane-stratified case, of section 4.1 in the toroidal case, are, in principle valid in a wide range of frequencies and wavelengths. This has been made possible by using a spectral representation of the wave field (eqs. (2.47)-(2.48) in the plane-stratified case, (4.64) and (4.70) in the toroidal case). As a result, the integral relation between fields and currents in the plasma is obtained in the form of a convolution between Fourier components in each coordinate which is not ignorable, i.e. in which the plasma properties are not uniform. Although the fact that the constitutive relation of a non-uniform plasma should be an integral equation is well known, explicit expressions for it were previously available only for very special cases.

An appropriate change of the integration variables, namely from the particle position to position of the guiding center[1] has been used to separate rigorously the effects on the plasma response of the gyration motion from those of the motion along magnetic field lines and of the slower perpendicular drifts across the static magnetic field. This change of the integration variables had previously been introduced in the low frequency limit, in order to average the plasma response over fast gyro-motion [28]. In the general case considered here, it allows to expand the plasma response in harmonics of the cyclotron frequency of charged particles. As a result of the spectral representation of the fields and of this expansion, the kernel of the integral constitutive relation obtained is clearly

[1] The ideal instantaneous center of the Larmor gyration motion in the static magnetic field.

related to the conductivity tensor of the uniform plasma, with additional terms and corrections which take into account the presence of density and temperature gradients, and the effects of the particle drift motion.

Taking advantage of the fact that these effects had been retained throughout the derivation, we have finally investigated in more details the wave equations obtained in the low frequency limit, where gradient driven instabilities play an important role in determining the confinement properties of fusion plasmas. These equations are given in section 3.6 for the plane-stratified case, and in section 5.5 for the toroidal case. They can be described as the equation for charge neutrality, and the parallel component of Ampere's law; together, they describe nearly electrostatic drift waves and shear Alfvén waves, which are most easily destabilised by the free energy due to gradients. Our derivation from the Vlasov equation includes in a natural way the ion polarisation current, which was missed in the early literature on drift instabilities, and only later 'rediscovered' with the gyro-kinetic approach[2] [25]. The charge density associated with the ion polarisation current is neutralised by the electron charge density due to their motion along the static magnetic field; it thus plays a fundamental role in the equation for charge neutrality. The first of equation (5.64), which describes this mechanism of charge compensation in the toroidal case, is essentially the same equation that, at higher frequencies, describes a large family of electrostatic waves in hot plasma, namely ion Bernstein waves and Lower Hybrid waves [3], [4]. All these waves can propagate without violation the charge neutrality of the plasma due to the same mechanism. Thus our derivation of the constitutive relation allows to make a clear and explicit connection between low frequency drift waves and the general theory of wave propagation at higher frequencies based on the Vlasov equation.

The low-frequency wave equation (3.69) and (5.63) differ from those of the gyro-kinetic theory because they use the spectral representation of the fields. By exploiting the characteristics of these waves (propagating predominantly in the diamagnetic direction) these complicated integro-differential equations have been simplified, and put into a form essentially equivalent to the wave equations of the gyro-kinetic literature. In our equations finite Larmor radius effects are included to all orders in the diamagnetic direction, in which the wavelengths of gradient driven instabilities can be of the order of the Larmor radius, and to second order in the radial direction, in which wavelengths are usually longer. The coefficients of these equations, moreover, take into account also the effect of the particle drift motion on collisionless Cherenkov-Landau resonances between waves and particles. All these effects (convolution form in non-ignorable coordinates, large ion Larmor radii, influence of the drift motion) have been considered in the gyro-kinetic literature, but only for special situations, and never simultaneously. It must be admitted that if the effects of drifts (and, in the toroidal case, of toroidal trapping) are

[2] I recall, that in the gyro-kinetic theory gyro-kinetic equation for distribution function of plasma particles obtained by averaging of the Vlasov equation over fast gyro-motion is considered.

retained, the explicit evaluation of the coefficients of our equations (or of those of the gyro-kinetic approach, for that matter) is a quite difficult task. Few quantitative investigations of these important effects have yet been attempted, mainly in the plane-stratified limit in which the absence of toroidal trapping simplifies the calculations [48]. Except for this difficulty, on the other hand, our equations, in spite of their generality, are in a form suitable for numerical solution with standard techniques. Discretisation in the radial variable could be best done using finite elements; the problem is a non-hermitian eigenvalue problem, whose solution, although not easy, is well familiar in plasma wave problems.

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