FRACTIONAL LÉVY PROCESSES, CARMA PROCESSES AND RELATED TOPICS

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Zusammenfassung

In der Dissertation wird eine neue Methode zur Erzeugung von Modellen mit sogenanntem “Long Memory” Verhalten entwickelt.

Hierfür wird die Klasse der fraktionalen Lévy Prozesse definiert und ihre wahrscheinlichkeitstheoretischen Eigenschaften, sowie Pfadeigenschaften untersucht. Da für bestimmte Lévymaße der entsprechende fraktionale Lévy Prozess kein Semimartingal ist, kann man in diesem Fall die klassische Itô-Integrationstheorie nicht anwenden. Eine allgemeine Integrationstheorie für fraktionale Lévy Prozesse wird definiert und schließlich verwendet, um aus “Short Memory” Modellen “Long Memory” Modelle zu erzeugen. Hierbei stehen die zeitstetigen ARMA Modelle im Vordergrund.

Bisher wurden zeitstetige ARMA Modelle nur im Eindimensionalem definiert. Im zweiten Teil der Dissertation werden diese Modelle auf den mehrdimensionalen Fall erweitert und ihre Eigenschaften eingehend untersucht. Insbesondere entwickeln wir mehrdimensionale zeitstetige ARMA Modelle mit Long Memory Verhalten.

Abstract

The thesis develops a new approach to generate long memory models by defining the class of fractional Lévy processes (FLPs) and investigates the probabilistic and sample path properties of FLPs. As for a fairly large class of Lévy measures the corresponding FLP cannot be a semimartingale, classical Itô integration theory cannot be applied. In the thesis we give a general definition of integrals with respect to FLPs.

This integration theory is then applied to continuous time moving average processes in the sense that the driving Lévy process in the moving average integral representation of short memory processes is replaced by a FLP. It turns out, that the so-constructed process exhibits long memory properties. But an even more important result is that this process coincides with the moving average process (driven by the ordinary Lévy process) which is obtained by a fractional integration of its kernel function. This is a new method to generate fractionally integrated continuous time ARMA (FICARMA) processes.

So far only univariate CARMA and FICARMA processes have been defined and investigated. In the second part of the thesis multivariate analogues of both models are developed by constructing a random orthogonal measure which allows for a spectral representation of the driving Lévy process. Furthermore, the probabilistic properties of multivariate CARMA and FICARMA models are studied. Like in the univariate case, the multivariate FICARMA process has two kernel representations which lead to the same model.

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Contents

Introduction

In modern mathematical finance continuous time models play a crucial role because they allow handling unequally spaced data and even high frequency data, which are realistic for liquid markets.

The probably most famous example is the so-called Black and Scholes model, which is built out of Brownian motion and models the logarithm of an asset price by the solution to the stochastic differential equation

$$dX(t) = [\mu + \beta\sigma^2]dt + \sigma dW_t, \qquad t \in [0, T],$$

where $\{W_t\}_{t\geq 0}$ is standard Brownian motion. Here $\mu + \beta\sigma^2$ represents the drift of the log-price while σ is the volatility. As investors usually require a risk premium for holding stochastic assets, compared to holding their wealth in a riskless interest banking account, the drift depends upon the volatility. Hence, if the volatility increases we would expect the drift also to increase.

The above asset pricing model implies that the aggregate returns over intervals of length $h > 0$,

$$Y_n := X(nh) - X((n-1)h)$$

are normal and independently distributed with a mean of $\mu h + \beta\sigma^2 h$ and a variance of $h\sigma^2$.

However, in practice for moderate too large values of h, returns are typically heavy-tailed, exhibit volatility clustering (the $|Y_n|$ are correlated) and are skew. Obviously, the Black and Scholes model lacks these so-called stylized facts. This observation resulted in an enormous effort to develop empirically reasonable models.

One approach is to replace the Brownian motion by a heavier tailed Lévy process. This will allow returns to be both heavy-tailed and skewed and take into account jumps. However, then returns will be independent and stationary, since every Lévy processes has stationary and independent increments. Hence, Lévy-driven models are also easily rejected empirically. Furthermore, in this models the volatility is a constant. However, in order to include the risk of financial markets into these models, the volatility process σ should be time-changing. Barndorff-Nielsen & Shephard (2001b) suggested the volatility process σ to change over time according to an OU process.

In these stochastic volatility models we write

$$\begin{aligned} dX(t) &= [\mu + \beta\sigma^2(t)]dt + \sigma(t)dW_t + \rho d\bar{L}(\lambda t), \qquad t \in [0,T] \\ d\sigma^2(t) &= -\lambda\sigma^2(t)dt + dL(\lambda t), \qquad \lambda > 0, \end{aligned} \tag{0.1}$$

where $\bar{L}(t) = L(t) - E[L(t)]$ is the centered version of the driving Lévy process $\{L(t)\}_{t\geq 0}$. Observe that $\bar{L}$ covers the so-called leverage effect, that is the correlation of the volatility process with the price process. The stationary solution of (0.1) is

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)}dL(\lambda s).$$

A stationary Ornstein-Uhlenbeck process was chosen because it has a non-negative kernel $g(t) = e^{-\lambda t}1_{[0,\infty)}(t)$. Hence, as the driving Lévy process $\{L(t)\}$ is usually chosen to be a subordinator, that is a Lévy process with positive increments, the process $\{\sigma^2(t)\}$ will be non-negative as is necessary if it is to represent volatility.

However, using Ornstein-Uhlenbeck processes in order to represent volatility implies that the class of volatility autocorrelation functions is restricted to functions of the form $\rho(h) = e^{-\lambda h}$ for some $\lambda > 0$. One might extend this class by using linear combinations of independent Ornstein-Uhlenbeck processes with positive coefficients, as it was suggested by Barndorff-Nielsen & Shephard (2001b). But even then the autocorrelation functions are still restricted to be monotone decreasing. Brockwell (2004) therefore suggested to replace the Ornstein-Uhlenbeck process by a non-negative Lévy-driven continuous time ARMA (CARMA) process. The virtue of this approach is that a much larger class of autocorrelations can be modeled and one can drop the monotonicity constraint. In fact, it has been shown in an econometric analysis by Todorov & Tauchen (2004) that CARMA and in particular CARMA$(2, 1)$ processes are reasonable processes to model stochastic volatility. Being the continuous time analogue of the well-known ARMA processes (see e.g. Brockwell & Davis (1991)), Lévy-driven CARMA processes, have been extensively studied over the last years (see e.g. Brockwell (2001a), Brockwell (2001b), Todorov & Tauchen (2004) and references therein).

As the autocorrelation functions of both, CARMA and OU processes, show an exponential rate of decrease, these models are short memory moving average processes. This contradicts the fact that measurements and an increasing number of statistical papers in finance, but also in so diverse fields as hydrology, turbulence, economics or telecommunications, indicate the presence of long memory in real life time series in the sense that the latter seem to require models whose autocorrelation functions decay much less quickly.

As a consequence, in the sixties, Mandelbrot used the fractional Brownian motion and its increments to generate long memory and pointed out its relevance in applications for example in economics and finance; see his recent book on the (mis)behaviour of markets (Mandelbrot & Hudson (2005)).

An alternative method to construct long memory models is a fractional integration of the kernel function of a short memory process. Using this technique, Brockwell (2004) (see also Brockwell & Marquardt (2005)) defined fractionally integrated CARMA (FICARMA) processes which exhibit long memory properties in the sense that the autocorrelations are hyperbolically decreasing. However, due to the slow decay of the fractionally integrated kernel function, simulation algorithms for FICARMA processes have been very slow and expensive.

This is the starting point of this thesis. It is organized as follows.

In Chapter 1 we present the framework of the thesis. We devote Section 1.1 to the basic properties of Lévy processes (Section 1.1.1) and consider in Section 1.1.2 the integration theory for integrals with respect to Lévy processes. Section 1.2 is devoted to continuous time ARMA (CARMA) processes, which belong to the class of short memory models. Based on the approach of Brockwell (2004) we define fractionally integrated CARMA (FICARMA) processes by a fractional integration of the CARMA kernel in Section 1.3. In particular, we show that the FICARMA process has long memory properties. A brief summary on fractional Brownian motion (FBM) is given in Chapter 1.4.

Chapter 2, where we define and discuss fractional Lévy processes (FLPs), forms the main part of the thesis.

Starting from the moving average integral representation of fractional Brownian motion the class of fractional Lévy processes (FLP) is introduced in Section 1.2 by replacing the Brownian motion by a general Lévy process. It is shown that FLPs are indeed well-defined. In the following subsections we present different methods of constructing a FLP.

Assuming that the driving Lévy process has finite second moments, we construct a FLP as an integral with respect to a Poisson random measure in Section 2.1.1. In Section 2.1.2 we obtain a continuous modification of a FLP by showing that almost surely the integral is equal to an improper Riemann integral. Furthermore, in Section 2.1.3 we derive series representations for FLPs.

In Section 2.2 the thesis focuses on the second-order and sample path properties. Provided the second moments of the driving Lévy process are finite, a FLP has the same second-order structure as a fractional Brownian motion, whereas the sample paths are less smooth. Moreover, it turns out that self-similarity, the total variation and the semimartingale property depend on the driving Lévy process. In particular, for a broad class of Lévy measures the corresponding FLP cannot be a semimartingale and hence, classical Itô integration theory cannot be applied.

Therefore, in Section 2.3, we derive a general definition of integrals with respect to FLPs, provided that the integrand is deterministic.

In Section 2.4 this integration theory is applied to moving average (MA) processes, in the way that the driving Lévy process in the moving average representation of short memory processes is replaced by a fractional Lévy process (Section 2.4.1). Considering the sample path and second-order properties of this so-constructed new process in Section 2.4.2, it turns out that this process exhibits long memory properties. But an even more important result is that this process coincides with the moving average process (driven by the ordinary Lévy process) which is obtained by a fractional integration of its kernel function. We apply these results to CARMA and FICARMA processes in Section 2.4.3. In particular, the simulation problem described above is solved by using this new approach.

So far only univariate CARMA and FICARMA processes have been defined and investigated. However, as financial risk management has to deal with portfolios of assets, multivariate models are of foremost importance.

In Chapter 3, which is based on joint work with Robert Stelzer, multivariate analogues of CARMA processes are developed. This is not straightforward since the state space representation of a univariate CARMA process relies on the ability to exchange autoregressive and moving average operators, which is only possible in one dimension.

Therefore, in Section 3.1 a random orthogonal measure is constructed which allows for a spectral representation of the driving Lévy process and enables us in Section 3.2 to define multivariate CARMA (MCARMA) processes and follow a similar line as Brockwell (2001b). Furthermore, the probabilistic properties of multivariate CARMA models are studied in Section 3.3.

Our aim in Chapter 4 is to define multivariate FICARMA processes.

A first step is to extend fractional Lévy processes to the multivariate setting. This is done in Section 4.1. We state the definition and properties of multivariate fractional Lévy processes (MFLPs) in Section 4.1.1 and the integration theory for integrals with respect to them in Section 4.1.2. In particular, in Section 4.1.3 a spectral representation of FLPs is derived and later used to obtain a spectral representation of FICARMA processes, which has not been given for (univariate) FICARMA processes, yet.

Multivariate fractional Lévy processes are used to develop the class of multivariate FICARMA processes in Section 4.2. Two subsections on the representations (Section 4.2.1) of multivariate FICARMA processes and their properties (Section 4.2.2) follow. Like in the univariate case, the multivariate FICARMA process has two kernel representations which lead to the same process.

As mentioned at the beginning of this introduction Ornstein-Uhlenbeck (OU) processes are of great importance, in particular they can serve as a model for stochastic volatility.

In Chapter 5 we consider OU processes and show that the results obtained in this thesis immediately apply to OU process in the univariate case (Section 5.1) as well as in the multivariate case (Section 5.2). In fact, our models include the OU processes as a special case.

The results of Chapter 6 are based on recent joint work with Christian Bender which is still ongoing. Therefore, we only state the basic concept and main ideas without going into further detail. We aim on an integration theory which allows for integrals with stochastic integrands with respect to FLPs in terms of the S-transform.

Precisely, we consider the Itô integral from a white noise point of view in Section 6.1. and then in Section 6.2. obtain a Skorohod integral for convoluted Lévy processes.

Finally, we would like to mention that Chapter 2, 3 and 4 are based on the papers Marquardt (2006a), Marquardt & Stelzer (2006) and Marquardt (2006b), respectively.

1 Preliminaries

In this preliminary chapter we present the framework of this thesis. After introducing the basic properties of Lévy processes and the integration theory for integrals with respect to them, we are mainly concerned with continuous time ARMA (CARMA), fractionally integrated CARMA (FICARMA) processes and fractional Brownian motion. Most results of this chapter are already known and therefore proofs are kept to a minimum or skiped. However, we state some results and proofs that appear new.

1.1 Lévy Processes

This chapter is devoted to the basic properties of Lévy processes and infinitely divisible distributions. In particular, we introduce the integration theory with respect to Lévy processes.

Lévy processes are defined as stochastically continuous processes with stationary and independent increments and can be viewed as analogues of random walks in continuous time. In particular, Lévy processes include many important processes as special cases, e.g. Brownian motion, the Poisson process, stable and self-decomposable processes and subordinators. Therefore, Lévy processes provide powerful models and are applied in various fields like econometrics, finance, telecommunications and physics.

Let us briefly introduce our setting. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. Elements of $\mathbb{R}^m$ are column m-vectors $x = [x_1, \ldots, x_m]^T$. The inner product is $\langle x, y\rangle = \sum_{j=1}^{m} x_j y_j$ and the norm is $\|x\| = \langle x, x\rangle^{1/2}$. We call $M_m(\mathbb{R})$ the space of all real $m \times m$-matrices and denote by A^T and A^* the transposed and adjoint, respectively, of the matrix A. Furthermore, $I_m \in M_m(\mathbb{R})$ is the identity matrix and $\|A\|$ is the operator norm of $A \in M_m(\mathbb{R})$ corresponding to the norm $\|x\|$ for $x \in \mathbb{R}^m$. $1_B(\cdot)$ is the indicator function of the set B and we write *a.s.* if something holds almost surely.

Finally, throughout this work we always assume as given an underlying complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

1.1.1 Basic Facts on Lévy Processes

We state the key notions and elementary properties of multivariate Lévy processes. For a more general treatment and proofs we refer to Protter (2004) and Sato (1999).

We consider a Lévy process $L = \{L(t)\}_{t\geq 0}$ in $\mathbb{R}^m$ determined by its characteristic function in the Lévy-Khintchine form $E\left[e^{i\langle u, L(t)\rangle}\right] = \exp\{t\psi_L(u)\}, \;\; t \geq 0$, where

$$\psi_L(u) = i\langle \gamma, u\rangle - \frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}^m} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle h(x))\, \nu(dx), \quad u \in \mathbb{R}^m, \tag{1.1}$$

where $\gamma \in \mathbb{R}^m$, $\sigma \in \mathbb{R}^{m\times m}$ is symmetric and positive semidefinite and ν is a measure on $\mathbb{R}^m$ that satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^m} (\|x\|^2 \wedge 1)\, \nu(dx) < \infty.$$

Moreover, $h : \mathbb{R}^m \to \mathbb{R}$ is a bounded measurable function satisfying

$$\begin{aligned} h(x) &= 1 + o(\|x\|), \quad \|x\| \to 0, \\ h(x) &= O(1/\|x\|), \quad \|x\| \to \infty. \end{aligned}$$

In this thesis we will always use

$$h(x) = 1_{\{\|x\|\leq 1\}}.$$

The measure ν is referred to as the Lévy measure of L. In fact, $\nu(A)$ is the expected number of jumps of L per unit time, whose size belong to the Borel set A. Notice that conversely, given a generating triplet (γ, σ, ν) satisfying (1.1), the corresponding Lévy process is unique in distribution.

Every Lévy process has a modification whose sample paths are right-continuous with left limits (càdlàg). We always assume that it is this modification we are working with and that $L(0) = 0$ a.s.

From now on let $\mathbb{R}^m_0 := \mathbb{R}^m \setminus \{0\}$. It is a well-known fact that to every càdlàg Lévy process L on $\mathbb{R}^m$ one can associate a random measure J on $\mathbb{R}^m_0 \times \mathbb{R}$ describing the jumps of L. For any measurable set $B \subset \mathbb{R}^m_0 \times \mathbb{R}$,

$$J(B) = \sharp\{s \geq 0 : \ (L_s - L_{s-}, s) \in B\}.$$

The jump measure J is a Poisson random measure on $\mathbb{R}^m_0 \times \mathbb{R}$ (see e.g. Definition 2.18 in Cont & Tankov (2004)) with intensity measure $n(dx, ds) = \nu(dx)\, ds$. By the Lévy-Itô decomposition there exists a Brownian motion $\{B_t\}_{t \geq 0}$ on $\mathbb{R}^m$ with covariance matrix σ such that we can rewrite L almost surely as

$$L(t) = \gamma t + B_t + \int\limits_{\|x\| \geq 1, s \in [0,t]} x\, J(dx, ds) + \lim_{\varepsilon \downarrow 0} \int\limits_{\varepsilon \leq \|x\| \leq 1,\, s \in [0,t]} x \tilde{J}(dx, ds), \quad t \geq 0. \tag{1.2}$$

Here $\tilde{J}(dx, ds) = J(dx, ds) - n(dx, ds) = J(dx, ds) - \nu(dx)\, ds$ is the compensated jump measure, the terms in (1.2) are independent and the convergence in the last term is a.s. and locally uniform in $t \geq 0$. If in (1.1), $\sigma = 0$ and hence $B_t = 0$ for all $t \geq 0$, we call L a Lévy process without Brownian component. Throughout this work, unless stated otherwise, we will assume that the Lévy process L has no Brownian part. Assuming that ν satisfies additionally

$$\int\limits_{\|x\| > 1} \|x\|^2\, \nu(dx) < \infty, \tag{1.3}$$

L has finite mean and covariance matrix Σ_L given by

$$\Sigma_L = \int\limits_{\mathbb{R}^m} x x^*\, \nu(dx). \tag{1.4}$$

Furthermore, if we suppose that $E[L(1)] = 0$, then it follows that $\gamma = -\int_{\|x\| > 1} x\, \nu(dx)$ and (1.1) can be written in the form

$$\psi_L(u) = \int\limits_{\mathbb{R}^m} (e^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle)\, \nu(dx), \quad u \in \mathbb{R}^m, \tag{1.5}$$

and (1.2) simplifies to

$$L(t) = \int\limits_{x \in \mathbb{R}_0^m,\, s \in [0,t]} x \tilde{J}(dx, ds), \quad t \geq 0. \tag{1.6}$$

In this case $L = \{L(t)\}_{t \geq 0}$ is a martingale. In the sequel we will work with a two-sided Lévy process $L = \{L(t)\}_{t \in \mathbb{R}}$, constructed by taking two independent copies $\{L_1(t)\}_{t \geq 0}$, $\{L_2(t)\}_{t \geq 0}$ of a one-sided Lévy process and setting

$$L(t) = \begin{cases} L_1(t), & \text{if} \quad t \geq 0 \\ -L_2(-t-), & \text{if} \quad t < 0. \end{cases} \tag{1.7}$$

1.1.2 Stochastic Integrals with Respect to Lévy Processes

We use the stochastic integrals of nonrandom functions with respect to infinitely divisible independently scattered random measures developed by Urbanik & Woyczynski (1967), Rajput & Rosinski (1989) and Sato (2005) to define in this section stochastic integrals with respect to Lévy processes.

We consider the stochastic process $X = \{X(t)\}_{t \in \mathbb{R}}$ given by

$$X(t) = \int\limits_{\mathbb{R}} f(t,s)\, L(ds), \qquad t \in \mathbb{R}, \tag{1.8}$$

where $f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R})$ is a measurable function and $L = \{L(t)\}_{t \in \mathbb{R}}$ is an m-dimensional two-sided Lévy process without Brownian component.

Integration of functions f with respect to L is defined first on finite intervals $[a,b]$, $a < b$, for real step functions ($t \in \mathbb{R}$ fixed)

$$f_n(t,s) = \sum_{k=0}^{n-1} A_k 1_{(s_k, s_{k+1}]}(s), \tag{1.9}$$

where $A_0, \ldots, A_{n-1} \in M_m(\mathbb{R})$, $n \in \mathbb{N}$ and $a = s_0 < s_1 < \ldots < s_n = b$. Then we define

$$\int\limits_a^b f_n(t,s)\, L(ds) = \sum_{k=1}^{n-1} A_k (L(s_k) - L(s_{k+1})).$$

In general, a measurable function $f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R})$ is said to be integrable with respect to the Lévy process L if there exists a sequence $\{f_n\}$ of simple functions as above, such that $f_n \to f$ almost everywhere (a.e.) and the sequence $\{\int_a^b f_n(t,s)\, L(ds)\}$ converges in probability as $n \to \infty$. If f is integrable with respect to L we write

$$\int\limits_a^b f(t,s)\, L(ds) = p - \lim_{n\to\infty} \int\limits_a^b f_n(t,s)\, L(ds).$$

It has been shown by Urbanik & Woyczynski (1967) that the integral $\int_a^b f(t,s)\, L(ds)$ is well-defined, i.e. it does not depend on the approximating sequence $\{f_n\}$ of simple functions. Furthermore, if f is L-integrable the law of $Y(t) := \int_a^b f(t,s)\, L(ds)$ is infinitely divisible, $\int_a^b \|\psi_L(f(t,s)^* u)\|\, ds < \infty$ and

$$E[e^{iuY(t)}] = \exp\left\{ \int\limits_a^b \psi_L(f(t,s)^* u)\, ds \right\}, \quad t, u \in \mathbb{R}, \tag{1.10}$$

where ψ_L is given in (1.1). In fact, due to Sato (2005, Proposition 3.4), for the integral $\int_a^b f(t,s)\, L(ds)$ to exist a necessary and sufficient condition is $\int_a^b \|f(t,s)\|^2\, ds < \infty$.

Definition 1.1 *Let $f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R})$ be a measurable and $f(t,\cdot)$ be a continuous function in s. If $\int_a^b f(t,s)\, L(ds)$ converges in probability as $b \to \infty$ ($a \to -\infty$), then the limit is denoted by $\int_a^\infty f(t,s)\, L(ds)$ ($\int_{-\infty}^b f(t,s)\, L(ds)$) and we say that the integral $\int_a^\infty f(t,s)\, L(ds)$ ($\int_{-\infty}^b f(t,s)\, L(ds)$) is well-defined.*

We distinguish three cases and first assume that the process L in (1.8) is an m-dimensional real-valued Lévy process without a Gaussian component satisfying $E[L(1)] = 0$ and $E[L(1)L(1)^T] = \Sigma_L < \infty$, i.e. L can be represented as in (1.6) together with (1.7). In this case it is obvious that the process X can be represented by

$$X(t) = \int\limits_{\mathbb{R}_0^m \times \mathbb{R}} f(t,s) x\, \tilde{J}(dx, ds), \qquad t \in \mathbb{R}, \tag{1.11}$$

where $\tilde{J}(dx,ds) = J(dx,ds) - \nu(dx)\, ds$ is the compensated jump measure of L. A necessary and sufficient condition for the existence of the stochastic integral

(1.11) as limit in probability of elementary integrals $\int_{\mathbb{R}} \int_{\mathbb{R}_0^m} f_n(t,s) x \tilde{J}(dx, ds)$ is that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^m} (\|f(t,s)x\|^2 \wedge \|f(t,s)x\|) \, \nu(dx) \, ds < \infty \quad \text{for all } \ t \in \mathbb{R}$$

(see Kallenberg (1997, Theorem 10.5)). Then the above conclusions continue to hold, i.e. the law of $X(t)$ is for all $t \in \mathbb{R}$ infinitely divisible with characteristic function

$$E\left[\exp\left\{i\langle u, X(t)\rangle\right\}\right] = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} \left(e^{i\langle u, f(t,s)x\rangle} - 1 - i\langle u, f(t,s)x\rangle\right) \, \nu(dx) \, ds \right\}$$

(see e.g. Rajput & Rosinski (1989) or Marcus & Rosinski (2005)).

The following proposition shows that the integral (1.8) or (1.11), respectively, may be well-defined in an L^2-sense.

Proposition 1.2 *Let $f(t,\cdot) \in L^2(M_m(\mathbb{R}))$ and $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process with $E[L(1)] = 0$ and $E[L(1)L(1)^T] = \Sigma_L < \infty$. Then the stochastic integral (1.11) and hence (1.8), exists in $L^2(\Omega, P)$ and does not depend on the choice of the approximating sequence. Moreover,*

$$E\left[X(t)X(t)^*\right] = \int_{\mathbb{R}} f(t,s)\Sigma_L f^*(t,s) \, ds, \quad t \in \mathbb{R}. \tag{1.12}$$

Proof. Applying Rajput & Rosinski (1989, Theorem 3.3) it follows that (1.8) is well-defined and $E\|\int f \, dL\|^2 < \infty$ if and only if

$$\int_{\mathbb{R}} \left[f(t,s)\gamma + \int_{\mathbb{R}_m} f(t,s)x[h(f(t,s)x) - h(x)]\nu(dx) + f(t,s)\Sigma_L f^*(t,s) \right] ds < \infty \tag{1.13}$$

Since we have $\gamma = -\int_{\|x\|>1} x \, \nu(dx)$, (1.13) is implied by

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^m} f(t,s)x 1_{\{\|f(t,s)x\|>1\}} \, \nu(dx) \, ds + \int_{\mathbb{R}} f(t,s)\Sigma_L f^*(t,s) \, ds \\ &\leq 2 \int_{\mathbb{R}} f(t,s)\Sigma_L f^*(t,s) \, ds < \infty. \end{aligned}$$

It follows from Rajput & Rosinski (1989, Theorem 3.4) that the mapping $f \to \int_{\mathbb{R}} f\, dL$ is an isomorphism between $L^2(M_m(\mathbb{R}))$ and $L^2(\Omega, P)$. To proof (1.12) we observe that for step functions as defined in (1.9)

$$E\left[\int_{\mathbb{R}} f_n(t,s)\, L(ds)\right]^2 = E\left[\int_{\mathbb{R}} f_n^2(t,s)\, d[L,L]_s\right] = \int_{\mathbb{R}} f(t,s)\Sigma_L f^*(t,s)\, ds.$$

This isometry property is preserved when we approximate $f(t,\cdot)$ by a sequence of step functions $\{f_n(t,\cdot)\}$ satisfying $f_n \xrightarrow{L^2} f$ (observe that the step functions are dense in $L^2(M_m(\mathbb{R}))$). ◻

Now, we consider a second case: If

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} (\|f(t,s)x\| \wedge 1)\, \nu(dx)\, ds < \infty \quad \text{for all } t \in \mathbb{R}, \tag{1.14}$$

the stochastic integral (1.8) exists without a compensator and we can write

$$X(t) = \int_{\mathbb{R}_0^m \times \mathbb{R}} f(t,s)x\, J(dx, ds), \qquad t \in \mathbb{R}. \tag{1.15}$$

Observe that $\int_{\mathbb{R}}\int_{\mathbb{R}^m}(\|f(t,s)x\| \wedge 1)\, \nu(dx)\, ds \le \int_{\mathbb{R}} \|f(t,s)\|\, ds \int_{\mathbb{R}^m} \|x\|\, \nu(dx)$. Hence, (1.14) holds if $f \in L^1(M_m(\mathbb{R}))$ and ν satisfies $\int_{\|x\|\le 1} \|x\|\, \nu(dx) < \infty$, which corresponds to the finite variation case.

Finally, in the general case, where condition (1.3) is not satisfied, necessary and sufficient conditions for the integral (1.8) to exist as a limit in probability of step functions approximating $f(t,\cdot)$, are (see Rajput & Rosinski (1989), Sato (2005))

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} (\|f(t,s)x\|^2 \wedge 1)\, \nu(dx)\, ds < \infty, \quad \text{for all } t \in \mathbb{R}, \tag{1.16}$$

and

$$\int_{\mathbb{R}} \left\| f(t,s)\gamma + \int_{\mathbb{R}^m} f(t,s)x\, (h(f(t,s)x) - h(x))\, \nu(dx) \right\| ds < \infty. \tag{1.17}$$

Then we represent X as

$$\begin{aligned} X(t) = &\int_{\mathbb{R}}\int_{\mathbb{R}_0^m} f(t,s)x\left[J(dx,ds) - (1 \vee \|f(t,s)x\|)^{-1}\, \nu(dx)\, ds\right] \\ &+ \int_{\mathbb{R}} f(t,s)\gamma\, ds, \;\; t \in \mathbb{R}. \end{aligned}$$

Moreover, if the integral in (1.8) is well-defined, the distribution of $X(t)$ is infinitely divisible with characteristic triplet $(\gamma_X^t, 0, \nu_X^t)$ given by

$$\begin{aligned} \gamma_X^t &= \int_{\mathbb{R}} \left[f(t,s)\gamma + \int_{\mathbb{R}^m} f(t,s)x[h(f(t,s)x) - h(x)]\,\nu(dx) \right] ds, \\ \nu_X^t(B) &= \int_{\mathbb{R}} \int_{\mathbb{R}^m} 1_B(f(t,s)x)\,\nu(dx)\,ds. \end{aligned} \tag{1.18}$$

It follows that the characteristic function of $X(t)$ can be written as

$$\begin{aligned} E\left[e^{i\langle u, X(t)\rangle}\right] &= \exp\left\{ i\langle \gamma_X^t, u\rangle + \int_{\mathbb{R}^m} [e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle h(x)]\,\nu_X^t(dx) \right\} \\ &= \exp\left\{ \int_{\mathbb{R}} \psi_L(f(t,s)^* u)\,ds \right\}, \end{aligned} \tag{1.19}$$

where ψ_L is given as in (1.1). These facts follow from Sato (2005, Proposition 5.5).

Remark 1.3 We would like to note that if the integral in (1.8) is well-defined and L is a Lévy process with characteristic triplet (γ, σ, ν), i.e. L may have a Brownian component, then the characteristic triplet $(\gamma_X^t, \sigma_X^t, \nu_X^t)$ of $X(t)$ is given by (1.18) and

$$\sigma_X^t = \int_{\mathbb{R}} f(t,s)\sigma f^*(t,s)\,ds. \tag{1.20}$$

Of particular interest is the moving average class, where the function f in (1.8) is defined and continuous on $[0, \infty)$ and depends on s and t through $t-s$ only. In this case we simply write $f(t-s)$ for $f(t,s)$. In the following section we will consider the special case of autoregressive moving average processes.

1.2 Univariate Lévy-driven CARMA(p, q) Processes

Being the continuous time analogue of the well-known autoregressive moving average (ARMA) processes (see e.g. Brockwell & Davis (1991)), continuous time ARMA (CARMA) processes, dating back to Doob (1944), have been extensively studied over the recent years (see e.g. Brockwell (2001a), Brockwell (2001b), Todorov & Tauchen (2004) and references therein) and widely used in various areas of application like engineering, finance and the natural sciences (e.g. Jones & Ackerson (1990), Mossberg & Larsson (2004), Todorov & Tauchen (2004)). Originally the driving process was restricted to Brownian motion. However, Brockwell (2001b) allowed for Lévy processes which have a finite r-th moment for some $r > 0$. So far only univariate CARMA processes have been defined and investigated. We give a short summary of the definition and properties of univariate Lévy-driven CARMA(p, q) processes, i.e. we assume $m = 1$ throughout this whole section. Moreover, in this section we discuss CARMA processes driven by general Lévy processes, i.e. the Lévy process may have a Brownian component and does not need to have finite variance, unless stated otherwise. For further details on univariate CARMA processes see Brockwell (2001a), Brockwell (2001b) and Brockwell (2004).

CARMA processes belong to the class of stationary moving average (MA) processes.

Definition 1.4 (Stationary MA Process) *A stationary continuous time moving average (MA) process is a process of the form*

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, L(du), \qquad t \in \mathbb{R}, \tag{1.21}$$

where $g : \mathbb{R} \to \mathbb{R}$, called kernel function, is measurable and the driving process $L = \{L(t)\}_{t\in\mathbb{R}}$ is a Lévy process on $\mathbb{R}$ having generating triplet $(\gamma_L, \sigma_L^2, \nu_L)$. We call L the driving Lévy process of the MA process $Y = \{Y(t)\}_{t\in\mathbb{R}}$.

The results (1.16) - (1.20) of the previous section can be directly applied to MA processes. Specifically we have (Rajput & Rosinski (1989))

Proposition 1.5 *The MA process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ given in (1.21) is well-defined and infinitely divisible if and only if the following three conditions on the driving Lévy process L and the kernel g hold:*

(i) $\int_{\mathbb{R}} \left|\gamma_L g(s) + \int_{\mathbb{R}} x g(s)[1_{\{|xg(s)|\le 1\}} - 1_{\{|x|\le 1\}}]\, \nu_L(dx)\right| ds < \infty,$

(ii) $\sigma_L^2 \int_{\mathbb{R}} g^2(s)\, ds < \infty$,

(iii) $\int_{\mathbb{R}} \int_{\mathbb{R}} (|g(s)x|^2 \wedge 1)\, \nu_L(dx)\, ds < \infty$,

where $(\gamma_L, \sigma_L^2, \nu_L)$ *is the characteristic triplet of* L*. If* Y *is well-defined, then for* $t \in \mathbb{R}$ *the characteristic function of* $Y(t)$ *can be written as*

$$E\left[e^{iuY(t)}\right] = \exp\left\{ iu\gamma_Y^t - \frac{1}{2}u^2(\sigma_Y^t)^2 + \int_{\mathbb{R}} [e^{iux} - 1 - iux1_{\{|x|\le 1\}}]\, \nu_Y^t(dx) \right\},$$

$u \in \mathbb{R}$, *where*

$$\gamma_Y^t = \int_{\mathbb{R}} \gamma_L g(t-s)\, ds + \int_{\mathbb{R}} \int_{\mathbb{R}} xg(t-s)[1_{\{|xg(t-s)|\le 1\}} - 1_{\{|x|\le 1\}}]\, \nu_L(dx)\, ds,$$

$$(\sigma_Y^t)^2 = \sigma_L^2 \int_{\mathbb{R}} g^2(t-s)\, ds,$$

$$\nu_Y^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} 1_B(g(t-s)x)\, \nu_L(dx)\, ds, \quad B \in \mathcal{B}(\mathbb{R}).$$

In contrast to (1.1) we write here σ_L^2 (and not σ) to make clear that in the case $m = 1$, σ_L^2 is the variance of the Brownian motion B in the Lévy Itô decomposition (1.2), whereas in the case $m \geq 2$, σ denotes the covariance matrix of B.

Remark 1.6 It follows by the stationarity of the increments of L and an application of the Cramér Wold device that every moving average process, which is well-defined, is a strictly stationary process (see also the proof of Proposition 2.42).

Having necessary and sufficient conditions for the existence of MA processes at hand, we are now in a position to consider CARMA processes, which constitute a special class of stationary MA processes.

Definition 1.7 (CARMA(p,q) Process) *A Lévy-driven continuous time autoregressive moving average CARMA(p,q) process* $\{Y(t)\}_{t\geq 0}$ *of order* (p,q)

with $p, q \in \mathbb{N}_0, p > q$ is defined to be the stationary solution of the formal p-th order linear differential equation,

$$p(D)Y(t) = q(D)DL(t), \quad t \geq 0, \tag{1.22}$$

where D denotes differentiation with respect to t, $\{L(t)\}_{t\geq 0}$ is a Lévy process satisfying $\int_{|x|>1} \log|x|\nu_L(dx) < \infty$,

$$p(z) := z^p + a_1 z^{p-1} + ... + a_p \ \text{ and } \ q(z) := b_0 z^q + b_1 z^{q-1} + ... + b_q, \tag{1.23}$$

where $a_p \neq 0$, $b_q \neq 0$. The polynomials $p(\cdot)$ and $q(\cdot)$ are referred to as the autoregressive and moving average polynomial, respectively.

Since in general the derivative of a Lévy process does not exist, (1.22) is interpreted as being equivalent to the observation and state equations

$$Y(t) = b^T X(t) \qquad \text{and} \tag{1.24}$$

$$dX(t) = AX(t)dt + e\,L(dt), \quad t \geq 0, \tag{1.25}$$

where $A = \left[\begin{array}{c|ccc} 0 & & I_{p-1} & \\ \hline -a_p & -a_{p-1} & \dots & -a_1 \end{array}\right]$, $\quad e^T = [0, \dots, 0, 1]$,

$b^T = \left[\; b_q, b_{q-1}, \dots, b_{q-p+1} \;\right]$ with $b_{-1} = b_{-2} = \dots = b_{q-p+1} = 0$, if $q < p-1$. Furthermore, recall that $I_{p-1} \in M_{p-1}(\mathbb{R})$ denotes the identity matrix.

Let us give a brief intuition how (1.24) and (1.25) capture the meaning of (1.22) To see this, first note that in the case $q(z) = 1$ (i.e. $q = 0$ and $b^T = [1, 0, \dots, 0]$) rewriting (1.22) as a system of first-order differential equations in the standard way gives (1.25) and (1.24) with X being the vector of derivatives $X_t^T = [Y(t), DY(t), \dots, D^{p-1}Y(t)]$. In the general case we transform (1.22) to

$$Y(t) = p(D)^{-1}q(D)DL(t) = q(D)p(D)^{-1}DL(t), \quad t \geq 0, \tag{1.26}$$

From the previous case we infer that the process in (1.25) is formed by $p(D)^{-1}DL(t)$ and the first $p-1$ derivatives of this process. Now one can immediately see that

$$Y(t) = b^T X_t = q(D)p(D)^{-1}DL(t).$$

Note that we may commute $p^{-1}(D)$ and $q(D)$ in (1.26), since the real coefficients and the operator D all commute. However, this does not hold in the multivariate case. We show in Chapter 3 how to handle this problem.

Remark 1.8 It is easy to check that the eigenvalues $\lambda_1, \ldots, \lambda_p$ of the matrix A are the same as the zeros of the autoregressive polynomial $p(z)$.

In order to define a CARMA process also for $t < 0$ we take a two-sided Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ as in (1.7).

Proposition 1.9 (Brockwell (2004, Section 2)) *If all eigenvalues $\lambda_1, \ldots, \lambda_p$ of A, i.e. the roots of $p(z)$, have negative real parts, the process $\{X(t)\}_{t\in\mathbb{R}}$ defined by $X(t) = \int_{-\infty}^{t} e^{A(t-u)}e\, L(du)$, $t \in \mathbb{R}$, is the strictly stationary solution of (1.25) for $t \in \mathbb{R}$ with corresponding CARMA process*

$$Y(t) = \int_{-\infty}^{t} b^T e^{A(t-u)} e\, L(du), \quad t \in \mathbb{R}. \tag{1.27}$$

From (1.27) it is obvious that $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is a causal moving average process, since it has the form

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, L(du), \quad t \in \mathbb{R}, \tag{1.28}$$

with kernel

$$g(t) = b^T e^{At} e 1_{[0,\infty)}(t) \tag{1.29}$$

satisfying $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Notice that we call a MA process causal, if it depends only on the past of the driving Lévy process L, i.e. on $\{L(s)\}_{s\le t}$. Obviously, this holds as $g(t-s) = 0$ for $s > t$.

Remark 1.10 Replacing e^{At} by its spectral representation, the kernel g can be expressed as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \frac{q(i\lambda)}{p(i\lambda)} d\lambda, \quad t \in \mathbb{R}, \tag{1.30}$$

(Brockwell (2004)).

Observe that the representation of $\{Y(t)\}_{t\in\mathbb{R}}$ given by (1.28) together with (1.30) defines a strictly stationary process even if there are eigenvalues of A

with strictly positive real part. However, if there are eigenvalues with positive real part, the CARMA process will be no longer causal. Henceforth, we focus on causal CARMA processes, i.e. we assume that the condition on the eigenvalues of A in Proposition 1.9 is always satisfied.

Proposition 1.11 *Let the function $g : \mathbb{R} \to \mathbb{R}$ be given by (1.29) or (1.30), respectively. Then there exist $t_0 \geq 1$ and constants $c, C > 0$ and $\tilde{c}, \tilde{C} > 0$ such that*

$$|g(t)| \leq Ce^{-ct}, \quad t \in \mathbb{R}, \tag{1.31}$$

$$|g(t)| \geq \tilde{C}e^{-\tilde{c}t}, \quad t \geq t_0. \tag{1.32}$$

Proof. Denote by $\lambda_1, \ldots, \lambda_k$, $k \leq p$ the eigenvalues of the matrix A (i.e. the roots of $p(z)$) with multiplicity $m_1, \ldots, m_k$, $\sum_{j=1}^{k} m_j = p$ and suppose

$$\mathcal{R}(\lambda_k) \leq \mathcal{R}(\lambda_{k-1}) \leq \ldots \leq \mathcal{R}(\lambda_1),$$

where $\mathcal{R}(\lambda)$ denotes the real part of λ. The Jordan decomposition of e^{At} yields

$$b^T e^{At} e = \sum_{j=1}^{k} p_j(t) e^{\lambda_j t},$$

where $p_j(t)$ are polynomials of degree m_j. If $p_1(t) = c_0 + \ldots + c_{m_1} t^{m_1}$ then

$$|b^T e^{At} e| \sim |c_{m_1} t^{m_1}| e^{\mathcal{R}(\lambda_1)t}, \quad t \to \infty. \tag{1.33}$$

Set $c = -\mathcal{R}(\lambda_1)/2 > 0$. Then there exists a constant $C > 0$ such that

$$|b^T e^{At} e| \leq Ce^{-ct}.$$

Furthermore, we can conclude from (1.33) that there exists $t_0 \geq 1$ such that

$$|b^T e^{At} e| \geq \tilde{C}e^{-\tilde{c}t}, \quad \text{for all } t \geq t_0,$$

where $\tilde{C} = |c_{m_1}|/2$ and $\tilde{c} = -\mathcal{R}(\lambda_1) > 0$. □

Proposition 1.12 *Suppose all roots $\lambda_1, \ldots, \lambda_p$ of the autoregressive polynomial $p(z)$ have negative real parts. Then the CARMA process $Y = \{Y(t)\}_{t \in \mathbb{R}}$ given in (1.28) with kernel function g given by (1.30) is well-defined if and only if*

$$\int_{|x|>1} \log |x| \, \nu_L(dx) < \infty. \tag{1.34}$$

Proof. The proof is an application of Proposition 1.5 to the kernel function g given in (1.30) and makes heavily use of Proposition 1.11. We refer the interested reader to Chojnowska-Michalik (1987, Theorem 6.7), where an analogous result is proven for the exponentially stable semigroup to show the sufficiency of (1.34). Therefore we only show that (1.34) is a necessary condition for the well-definedness of the CARMA process. It follows from Proposition 1.5 (iii) that if the CARMA process is well-defined and stationary,

$$\begin{aligned}\infty &> \int_0^\infty \int_{\mathbb{R}} 1_{\{|g(s)x|>1\}}\, \nu_L(dx)\, ds \overset{(1.32)}{\geq} \int_{t_0}^\infty \int_{\mathbb{R}} 1_{\{|x\tilde{C}e^{-\tilde{c}s}|>1\}}\, \nu_L(dx)\, ds \\ &= \int_{|x|>1/\tilde{C}} \int_{t_0}^\infty 1_{\{s<\frac{1}{\tilde{c}}\log(|x|\tilde{C})\}}\, ds\, \nu_L(dx) \\ &= \int_{|x|>1/\tilde{C}} \left(\frac{1}{\tilde{c}}\log(|x|\tilde{C}) - t_0\right)_+ \nu_L(dx) = \int_{|x|>\frac{1}{\tilde{C}}e^{t_0\tilde{c}}} \left[\frac{1}{\tilde{c}}\log(|x|\tilde{C}) - t_0\right] \nu_L(dx).\end{aligned}$$

As ν_L is a Lévy measure this shows the necessity of (1.34). □

Finally, we state the second-order properties of CARMA processes.

Proposition 1.13 (Brockwell (2004, Section 2)) *If* $E[L(1)^2] < \infty$, *the spectral density* f_Y *of* $Y = \{Y(t)\}_{t\in\mathbb{R}}$ *is given by*

$$f_Y(\lambda) = \frac{\mathrm{var}(L(1))}{2\pi} \frac{|q(i\lambda)|^2}{|p(i\lambda)|^2}, \quad \lambda \in \mathbb{R}.$$

Thus, being the Fourier transform of the spectral density f_Y, the autocovariance function γ_Y of the CARMA process Y can be expressed as

$$\gamma_Y(h) = \mathrm{cov}(Y(t+h), Y(t)) = \frac{\mathrm{var}(L(1))}{2\pi} \int_{-\infty}^{\infty} e^{ih\lambda} \left|\frac{q(i\lambda)}{p(i\lambda)}\right|^2 d\lambda, \quad h \in \mathbb{R}.$$

Remark 1.14 Suppose the CARMA process is causal. Then, provided all eigenvalues $\lambda_1, \ldots, \lambda_p$ of the matrix A are algebraically simple, an application of the residue theorem leads to

$$g(t) = \sum_{r=1}^p \frac{q(\lambda_r)}{p'(\lambda_r)} e^{\lambda_r t} 1_{[0,\infty)}(t), \quad t \in \mathbb{R}. \tag{1.35}$$

Consequently, the autocovariance function γ_Y simplifies to

$$\gamma_Y(h) = \mathrm{var}(L(1)) \sum_{r=1}^{p} \frac{q(\lambda_r)q(-\lambda_r)}{p'(\lambda_r)p(-\lambda_r)} e^{\lambda_r |h|}, \quad h \in \mathbb{R}. \tag{1.36}$$

We say that a second-order stationary process Y having autocovariance function γ_Y belongs to the class of **short memory** processes, if $\gamma_Y(h)$ decreases at an exponential rate towards zero as $h \to \infty$. Obviously, every CARMA process is a short memory moving average process.

In the following section we show how to incorporate long memory behaviour into the class of short memory CARMA processes. Before, we give an example.

Example 1.15 (An Application to Stochastic Volatility Modeling) Barndorff-Nielsen & Shephard (2001b) introduced a model for asset-pricing in which the logarithm of an asset price is the solution of the stochastic differential equation

$$dX(t) = (\mu + \beta\sigma^2(t))dt + \sigma(t)dW(t), \quad t \geq 0,$$

where $\{\sigma^2(t)\}$, the instantaneous volatility, is a non-negative Lévy-driven Ornstein-Uhlenbeck process, $\{W(t)\}$ is standard Brownian motion and μ and β are constants. With this model they were able to derive explicit expressions for quantities of fundamental interest such as the integrated volatility. A crucial feature of volatility modeling is the requirement that the volatility must be non-negative, a property achieved by the Lévy-driven Ornstein-Uhlenbeck process since its kernel is non-negative and the driving Lévy process is chosen to be non-decreasing. A limitation of the use of the Ornstein-Uhlenbeck process (and of convex combinations of independent Ornstein-Uhlenbeck processes) is the constraint that the autocovariances $\gamma_Y(h),\ h \geq 0$, necessarily decrease as the lag h increases.

Much of the analysis of Barndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a CARMA process with non-negative kernel driven by a non-decreasing Lévy process, i.e. a subordinator. This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelation functions than is possible in the Ornstein-Uhlenbeck framework. For example, the CARMA(3,2) process with

$$p(z) = (z+0.1)(z+0.5+i\pi/2)(z+0.5-i\pi/2) \quad \text{and} \quad q(z) = 2.792+5z+z^2$$

has non-negative kernel

$$g(t) = 0.8762e^{-0.1t} + \left(0.1238\cos\frac{\pi t}{2} + 2.5780\sin\frac{\pi t}{2}\right)e^{-0.5t}, \quad t \geq 0$$

and autocovariance functions

$$\gamma(h) = 5.1161e^{-0.1h} + \left(4.3860\cos\frac{\pi h}{2} + 1.4066\sin\frac{\pi h}{2}\right)e^{-0.5h}, \quad h \geq 0,$$

both of which exhibit damped oscillatory behaviour (see Figure 1.2 and Figure 1.3 of Section 1.3).

Figure 1.1 shows the corresponding sample path, when the driving Lévy process L is a gamma subordinator, i.e. at a fixed time t the process L has the gamma distribution with density

$$f(x) = \frac{\lambda^{ct}}{\Gamma(ct)}x^{ct-1}e^{-\lambda x}.$$

1.3 Univariate FICARMA(p, d, q) Processes

Since the autocorrelation functions of CARMA processes show an exponential rate of decrease, CARMA processes are short memory processes. However, observed time series often show long memory behaviour in the sense that they seem to require models, whose autocorrelation functions follow a power law and where the decay is so slow that the autocorrelations are not integrable. Historically, long range dependence or long memory, respectively, was defined in several ways. It was associated either with a particularly slow decay of correlation or with a particular pole of the spectral density at the origin. One should note that in general neither definition implies the other.

We propose the following definition of "long memory".

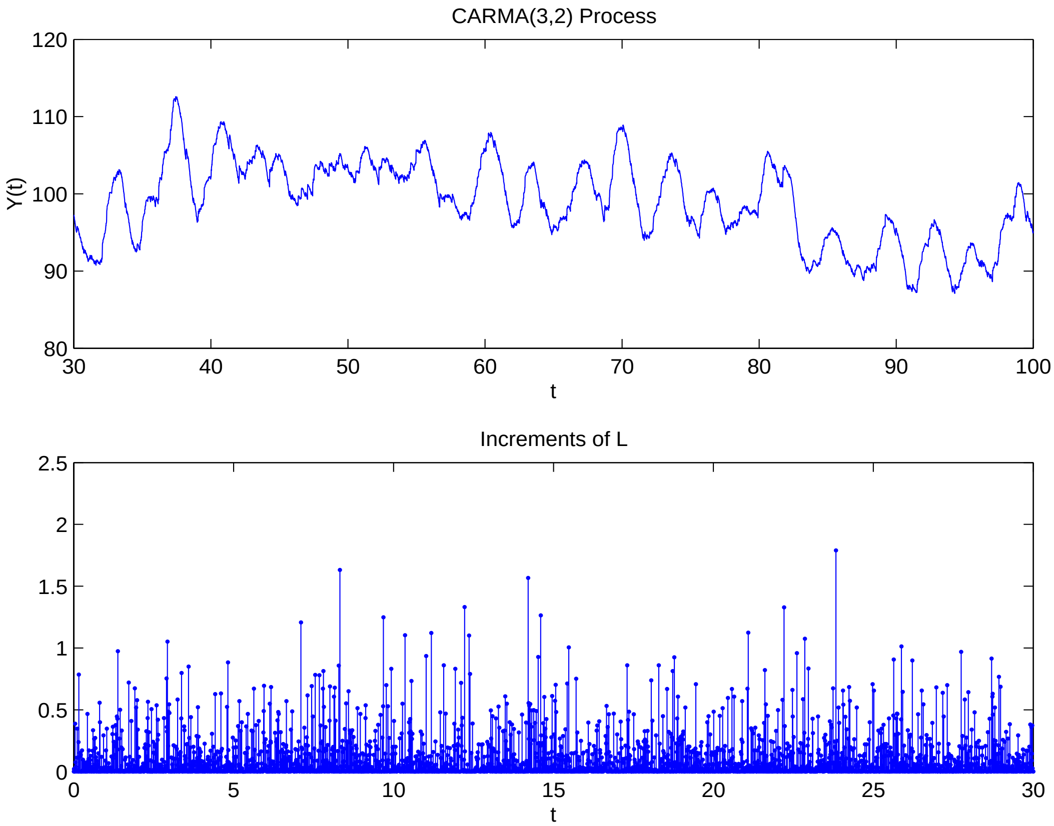

Figure 1.1: Top: The sample path of a CARMA(3,2) process which is driven by a gamma subordinator with parameters $c = 30$ and $\lambda = 3$.
Bottom: The increments of the driving gamma subordinator.

Definition 1.16 (Long Memory Process) *Let* $X = \{X_t\}_{t\in\mathbb{R}}$ *be a stationary stochastic process and* $\gamma_X(h) = \text{cov}(X_{t+h}, X_t),\ h \in \mathbb{R}$, *be its autocovariance function. If there exist* $0 < d < 0.5$ *and a constant* $c_\gamma > 0$ *such that*

$$\lim_{h\to\infty} \frac{\gamma_X(h)}{h^{2d-1}} = c_\gamma, \tag{1.37}$$

then X *is a stationary process with long memory (long range dependence).*

A generalization of the latter definition may be obtained by replacing the proportionality constant c_γ by a slowly varying function at infinity, i.e. a function $l(\cdot)$ such that for any $t > 0$,

$$\frac{l(t\lambda)}{l(\lambda)} \to 1, \qquad \text{as } \lambda \to \infty.$$

Then X is referred to as a long memory process, if

$$\gamma_X(h) \sim h^{-2d} l(h), \qquad \text{as } h \to \infty.$$

However, for our and most practical purposes this generalization is not needed. Furthermore, observe that long memory implies

$$\int_0^\infty \gamma_X(h)\, dh = \infty.$$

The subject of long range dependence has sparked considerable research interest over the last few years. An excellent survey of the present state of the art is Doukhan et al. (2003).

Aiming at long range dependent CARMA processes, using a fractional integration of the CARMA kernel, Brockwell (2004) (see also Brockwell & Marquardt (2005)) defined Lévy-driven fractionally integrated CARMA (FICARMA) processes, where the autocorrelations are hyperbolically decaying. In this section we give a summary of Lévy-driven FICARMA processes and derive the second order properties of FICARMA(p, d, q) processes. In particular, we give an explicit formula for the autocovariance function. The results of this section (and some further extensions) can also be found in Brockwell & Marquardt (2005).

First we introduce the Riemann-Liouville fractional integrals and derivatives. For details see Samko et al. (1993).

For $0 < \alpha < 1$ the Riemann-Liouville fractional integrals $I^\alpha_\pm$ are defined by

$$(I^\alpha_- f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1}\, dt, \tag{1.38}$$

$$(I^\alpha_+ f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1}\, dt, \tag{1.39}$$

if the integrals exist for almost all $x \in \mathbb{R}$. In fact, fractional integrals $I^\alpha_\pm$ are defined for functions $f \in L^p(\mathbb{R})$ if $0 < \alpha < 1$ and $1 \le p < 1/\alpha$ (Samko et al. (1993, p.94)). We refer to the integrals I^α_- and I^α_+ as right-sided and left-sided, respectively.

Fractional differentiation was introduced as the inverse operation. Let $0 < \alpha < 1$, $1 \leq p < 1/\alpha$ and denote by $I_{\pm}^{\alpha}(L^p)$ the class of functions $\phi \in L^p(\mathbb{R})$ which may be represented as an $I_{\pm}^{\alpha}$-integral of some function $f \in L^p(\mathbb{R})$. If $\phi \in I_{\pm}^{\alpha}(L^p)$, there exists a unique function $f \in L^p(\mathbb{R})$ such that $\phi = I_{\pm}^{\alpha} f$ and f agrees with the Riemann-Liouville derivative $\mathcal{D}_{\pm}^{\alpha}$ of ϕ of order α defined by

$$
\begin{aligned}
(\mathcal{D}_{-}^{\alpha}\phi)(x) &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^{\infty} \phi(t)(t-x)^{-\alpha}\,dt, \\
(\mathcal{D}_{+}^{\alpha}\phi)(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x} \phi(t)(x-t)^{-\alpha}\,dt,
\end{aligned}
$$

where the convergence of the integrals at the singularity $t = x$ holds pointwise for almost all x if $p = 1$ and in the L^p-sense if $p > 1$.

Last but not least we have the following rule for fractional integration by parts.

Proposition 1.17 (Bender (2003b, Theorem 2.6)) *Let* $0 < \alpha < 0.5$. *Then*

$$
\int_{\mathbb{R}} f(s)(I_{-}^{\alpha}g)(s)\,ds = \int_{\mathbb{R}} (I_{+}^{\alpha}f)(s)g(s)\,ds \tag{1.40}
$$

holds if $f \in L^p(\mathbb{R})$, $g \in L^r(\mathbb{R})$ *and* $p > 1$, $r > 1$, $1/p + 1/r = 1 + \alpha$.

We calculate the Riemann-Liouville fractional integral of order d of the (short memory) CARMA kernel g given in (1.30) in order to obtain the corresponding fractionally integrated kernel g_d. As in this thesis we are mainly interested in long memory processes, we always assume $0 < d < 0.5$ to be consistent with Definiton 1.16. Furthermore, we restrict ourselves to the *causal* case, i.e. $g(t) = 0$ for $t < 0$. Then

$$
\begin{aligned}
g_d(t) := (I_{+}^{d}g)(t) &= \int_0^{\infty} g(t-u)\frac{u^{d-1}}{\Gamma(d)}\,du = \int_0^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t-u)\lambda}\frac{q(i\lambda)}{p(i\lambda)}\,d\lambda\frac{u^{d-1}}{\Gamma(d)}\,du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\frac{q(i\lambda)}{p(i\lambda)}\frac{1}{\Gamma(d)}\int_0^{\infty} e^{-iu\lambda}u^{d-1}du\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}(i\lambda)^{-d}\frac{q(i\lambda)}{p(i\lambda)}\,d\lambda, \qquad (1.41)
\end{aligned}
$$

since

$$\int_0^\infty e^{-iu\lambda} u^{d-1} du = (i\lambda)^{-d}\Gamma(d).$$

Proposition 1.18 *The kernel $g_d(t)$ converges to zero at a hyperbolic rate as $t \to \infty$. In fact,*

$$g_d(t) \sim \frac{t^{d-1}}{\Gamma(d)} \cdot \frac{q(0)}{p(0)}, \qquad t \to \infty. \tag{1.42}$$

Proof. As we can rewrite the kernel g_d as

$$g_d(t) = \frac{1}{2i\pi} \int_{-i\infty}^{i\infty} e^{tz} z^{-d} \frac{q(z)}{p(z)} dz,$$

the asymptotic behaviour of $g_d(t)$ as $t \to \infty$ is a consequence of Doetsch (1974, Theorem 37.1., p.254). □

Remark 1.19 Observe that $g_d \in L^2(\mathbb{R})$ and $g_d(t) = 0$ for all $t \leq 0$.

Substituting the CARMA kernel g by the fractionally integrated kernel g_d as given in (1.41), we obtain the fractionally integrated CARMA(p, d, q) processes.

Definition 1.20 (FICARMA(p, d, q) Process) *Let $0 < d < 0.5$ and assume that all zeros of the polynomial $p(z)$ given by (1.23) have negative real parts. Then the stationary fractionally integrated CARMA(p, d, q) (FICARMA(p, d, q)) process $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ with coefficients $a_1, ..., a_p, b_0, ..., b_q$ and driven by the Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ satisfying $E[L(1)] = 0$ and $E[L(1)^2] < \infty$ is defined as*

$$Y_d(t) = \int_{-\infty}^{t} g_d(t-u)\, L(du), \qquad t \in \mathbb{R}, \tag{1.43}$$

where the kernel function g_d is given in (1.41).

Remark 1.21 As $g_d \in L^2(\mathbb{R})$, $E[L(1)] = 0$ and $E[L(1)^2] < \infty$ it follows from Proposition 1.2 and the results of Section 1.1.1 that the FICARMA process Y_d is well-defined in $L^2(\mathbb{R})$ and as a limit in probability of step functions approximating the kernel function g_d. Let $(\gamma_L, \sigma_L^2, \nu_L)$ denote the characteristic triplet of the driving Lévy process L, then the distribution of the FICARMA process $Y_d(t)$ is for all $t \in \mathbb{R}$ infinitely divisible and the stationary distribution has characteristic triplet $(\gamma_{Y_d,\infty}, \sigma^2_{Y_d,\infty}, \nu_{Y_d,\infty})$ given by

$$\begin{aligned}
\gamma_{Y_d,\infty} &= \gamma_L \int_0^\infty g_d(s)\,ds + \int_0^\infty \int_{\mathbb{R}} g_d(s)x[h(g_d(s)x) - h(x)]\,\nu_L(dx)\,ds,\\
\sigma^2_{Y_d,\infty} &= \sigma_L^2 \int_0^\infty g_d^2(s)\,ds,\\
\nu_{Y_d,\infty} &= \int_0^\infty \int_{\mathbb{R}} 1_B(g_d(s)x)\,\nu_L(dx)\,ds, \quad B \in \mathcal{B}(\mathbb{R}).
\end{aligned}$$

We turn our attention to the the second order properties of FICARMA(p, d, q) processes (see also Brockwell & Marquardt (2005)). Before, we establish two lemmata which contain important results we shall need to show the long memory property of FICARMA processes.

Lemma 1.22 *Let*

$$f(u,h) \;\sim\; F(u,h) \qquad \text{for} \quad u,h \geq 0, \quad u+h \to \infty,$$

i.e.

$$\lim_{u,h\geq 0, u+h\to\infty} \frac{f(u,h) - F(u,h)}{F(u,h)} = 0. \tag{1.44}$$

Assume that the integral

$$I_F(h) := \int_0^\infty F(u,h)\,du$$

exists for $h > 0$ and that there is a constant $C > 0$ with

$$\int_0^\infty |F(u,h)|\,du \leq C|I_F(h)| \quad \text{for all } h > M, \tag{1.45}$$

$M > 0$ large enough. Then $I_f(h) := \int_0^\infty f(u,h)\,du < \infty$ and

$$I_f(h) = \int_0^\infty f(u,h)\,du \;\sim\; I_F(h) \qquad \text{for } h \to \infty, \text{ i.e.}$$

$$\lim_{h\to\infty} \frac{I_f(h) - I_F(h)}{I_F(h)} = 0.$$

Proof. We have

$$
\begin{aligned}
|I_f(h) - I_F(h)| &\leq \int_0^\infty |f(u,h) - F(u,h)|\, du \\
&= \int_0^\infty \frac{|f(u,h) - F(u,h)|}{|F(u,h)|} |F(u,h)|\, du \\
&\leq \sup_{u \geq 0} \frac{|f(u,h) - F(u,h)|}{|F(u,h)|} \int_0^\infty |F(u+h)|\, du \\
&\overset{(1.45)}{\leq} \sup_{u \geq 0} \frac{|f(u,h) - F(u,h)|}{|F(u,h)|} C |I_F(h)|.
\end{aligned}
$$

The first factor tends to zero for $h \to \infty$ by (1.44). This shows that

$$
\lim_{h \to \infty} \frac{I_f(h) - I_F(h)}{I_F(h)} = 0.
$$

□

Lemma 1.23 *Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function on $[0, \infty)$ and*

$$
g(u) \;\sim\; C u^{d-1} =: G(u) \quad \text{for } u \to \infty, \tag{1.46}
$$

where $C > 0$ is a constant and $0 < d < 0.5$ and consider for $h \geq 0$

$$
r(h) := \int_0^\infty g(u+h) g(u)\, du
$$

and

$$
R(h) := \int_0^\infty G(u+h) G(u)\, du.
$$

Then

$$
r(h) \;\sim\; R(h) \quad \text{for } h \to \infty. \tag{1.47}
$$

Moreover,

$$
R(h) = C^2 \int_0^\infty (u+h)^{d-1} u^{d-1}\, du = h^{2d-1} \frac{\Gamma(1-2d)\Gamma(d)}{\Gamma(1-d)} C^2.
$$

Proof. We first show that

$$
\tilde{r}(h) := \int_{h^{d/2}}^\infty g(u+h) g(u)\, du \;\sim\; \int_{h^{d/2}}^\infty G(u+h) G(u)\, du =: \tilde{R}(h).
$$

To this end let

$$f(u,h) = g(u+h+h^{d/2})g(u+h^{d/2}),$$

and

$$F(u,h) = G(u+h+h^{d/2})G(u+h^{d/2}).$$

Then

$$\tilde{r}(h) = I_f(h), \qquad \tilde{R}(h) = I_F(h).$$

Moreover

$$\lim_{u,h\geq 0, u+h\to\infty} \frac{f(u,h)}{F(u,h)} = \lim_{u,h\geq 0, u+h\to\infty} \frac{g(u+h+h^{d/2})g(u+h^{d/2})}{G(u+h+h^{d/2})G(u+h^{d/2})} = 1$$

and thus

$$f(u,h) \;\sim\; F(u,h) \qquad \text{for } \; u+h\to\infty.$$

Finally,

$$|I_F(h)| = C^2 \int_{h^{d/2}}^{\infty} (u+h)^{d-1} u^{d-1}\, du = C^2 \int_0^{\infty} |F(u,h)|\, du$$

and thus (1.45) holds. Hence, Lemma 1.22 yields

$$\tilde{r}(h) \;\sim\; \tilde{R}(h) \qquad \text{for } h\to\infty.$$

Now (1.47) follows from the observation that for $h\to\infty$,

$$\frac{|r(h)-R(h)|}{|R(h)|} \leq \frac{|r(h)-\tilde{r}(h)|}{|R(h)|} + \frac{|\tilde{r}(h)-\tilde{R}(h)|}{|R(h)|} + \frac{|\tilde{R}(h)-R(h)|}{|R(h)|} \to 0.$$

In fact, $|R(h)| \geq |\tilde{R}(h)|$ and thus

$$\frac{|\tilde{r}(h)-\tilde{R}(h)|}{|R(h)|} \leq \frac{|\tilde{r}(h)-\tilde{R}(h)|}{|\tilde{R}(h)|} \to 0, \quad h\to\infty,$$

as we have just shown. Moreover, since $d < 0.5$,

$$|R(h)| = C^2 \int_0^{\infty} (u+h)^{d-1} u^{d-1}\, du \geq C^2 \int_0^{\infty} (u+h)^{2d-2}\, du = C^2 \frac{h^{2d-1}}{1-2d}.$$

On the other hand, we have by (1.46)

$$|g(u+h)| \leq 2|G(u+h)| \leq 2Ch^{d-1}$$

for all $h \geq M$, M large enough, and there is a constant $\tilde{C} > 0$ with

$$\sup_{u \geq 0} |g(u)| \leq \tilde{C}.$$

This yields

$$|r(h) - \tilde{r}(h)| = \left| \int_0^{h^{d/2}} g(u+h)g(u)\, du \right| \leq h^{d/2} 2Ch^{d-1}\tilde{C} \leq 2C\tilde{C}h^{2d-1-d/2}.$$

This gives

$$\frac{|r(h) - \tilde{r}(h)|}{|R(h)|} \leq \frac{2C\tilde{C}h^{2d-1-d/2}}{C^2 \frac{h^{2d-1}}{1-2d}} \to 0, \quad h \to \infty.$$

Similarly we obtain

$$\frac{|R(h) - \tilde{R}(h)|}{|R(h)|} \leq \frac{C\tilde{C}h^{2d-1-d/2}}{C^2 \frac{h^{2d-1}}{1-2d}} \to 0, \quad h \to \infty.$$

It remains to calculate the function $R(h)$.

$$\begin{aligned} R(h) &= C^2 \int_0^\infty (u+h)^{d-1} u^{d-1}\, du \\ &\overset{u=(\frac{1}{x}-1)h}{=} C^2 \int_0^1 x^{-2d}(1-x)^{d-1}\, dx \\ &= h^{2d-1} \frac{\Gamma(1-2d)\Gamma(d)}{\Gamma(1-d)} C^2, \end{aligned}$$

since $\int_0^1 (1-x)^{a-1} x^{b-1}\, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. □

Theorem 1.24 *The FICARMA(p,d,q) process Y_d as defined in Definition 1.20 is a long memory moving average process.*

Proof. For $h \geq 0$ we have

$$\begin{aligned} \gamma_d(h) &= \operatorname{cov}(Y_d(t+h), Y_d(t)) \\ &= \operatorname{cov}\left(\int_{-\infty}^{t+h} g_d(t+h-u)\, L(du), \int_{-\infty}^{t} g_d(t-u)\, L(du) \right) \\ &= \operatorname{cov}\left(\int_0^\infty g_d(u+h)\, L(du), \int_0^\infty g_d(u)\, L(du) \right) \\ &= E[L(1)^2] \int_0^\infty g_d(u+h) g_d(u)\, du. \end{aligned}$$

Moreover, we know from (1.42), $g_d(t) \sim \frac{t^{d-1}}{\Gamma(d)}\frac{q(0)}{p(0)}$ as $t \to \infty$. Hence, we can apply Lemma 1.23 with $C = \frac{q(0)}{\Gamma(d)p(0)}$ and obtain that the autocovariance function γ_d of the FICARMA process Y_d is hyperbolically decaying, namely

$$\gamma_d(h) \sim \frac{E[L(1)^2]\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \left[\frac{q(0)}{p(0)}\right]^2 h^{2d-1}, \quad h \to \infty. \tag{1.48}$$

Hence, Y_d satisfies the conditions of Definition 1.16. □

Proposition 1.25 *The spectral density f_d of the FICARMA(p, d, q) process equals*

$$f_d(\lambda) = \frac{E[L(1)^2]}{2\pi|\lambda|^{2d}} \left|\frac{q(i\lambda)}{p(i\lambda)}\right|^2, \quad \lambda \in \mathbb{R}. \tag{1.49}$$

Proof. We observe that

$$\gamma_d(h) = E[L(1)^2] \int_0^\infty g_d(u+h)g_d(u)\,du = E[L(1)^2] \int_{\mathbb{R}} \tilde{g}_d(h-u)g_d(u)\,du,$$

where $\tilde{g}_d(x) = g_d(-x)$. Then, using the representation (1.30) of g_d and the fact that the spectral density is the inverse Fourier transform of the autocovariance function, we find that

$$f_d(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda h}\gamma_d(h)\,dh = \frac{E[L(1)^2]}{2\pi|\lambda|^{2d}} \left|\frac{q(i\lambda)}{p(i\lambda)}\right|^2,$$

where we made use of the convolution theorem for Fourier transforms. □

Corollary 1.26

(i) The spectral density has a pole at the origin. In fact,

$$f_d(\lambda) \sim \frac{E[L(1)^2]}{2\pi} \left[\frac{q(0)}{p(0)}\right]^2 |\lambda|^{-2d}, \quad \lambda \to 0.$$

(ii) As a consequence of (1.49), the autocovariance function γ_d can be expressed as

$$\gamma_d(h) = \frac{E[L(1)^2]}{2\pi} \int_{\mathbb{R}} e^{ih\lambda} \left|\frac{q(i\lambda)}{p(i\lambda)}\right|^2 \frac{1}{|\lambda|^{2d}}\,d\lambda, \quad h \in \mathbb{R},$$

since γ_d is the Fourier transform of f_d.

Finally, we consider the special case that the roots $\lambda_1, \dots, \lambda_p$ of the autoregressive polynomial $p(\cdot)$ are distinct. Then the kernel g_d of the FICARMA process $\{Y_d(t)\}_{t\in\mathbb{R}}$ can be expressed as

$$g_d(t) = \sum_{r=1}^{p} \frac{q(\lambda_r)}{p'(\lambda_r)} \lambda_r^{-d} e^{\lambda_r t} P(\lambda_r t, d), \quad t \in \mathbb{R}, \tag{1.50}$$

and we obtain

$$\gamma_d(h) = \frac{E[L(1)^2]}{2\cos(\pi d)} \sum_{r=1}^{p} \frac{q(\lambda_r)q(-\lambda_r)}{p'(\lambda_r)p(-\lambda_r)} v(d, \lambda_r, h), \quad h \in \mathbb{R}, \tag{1.51}$$

where

$$v(d,\lambda,h) = 2(-\lambda)^{-2d}\cosh(\lambda h) + \lambda^{-2d} e^{h\lambda} P(\lambda d, 2d) - (-\lambda)^{-2d} e^{-\lambda j} P(-\lambda h, 2d)$$

(see Brockwell & Marquardt (2005) for a proof). Here $P(z,d)$ is the incomplete gamma function with complex argument z,

$$P(z,d) = \frac{1}{\Gamma(d)} \int_0^z e^{-x} x^{d-1}\, dx, \tag{1.52}$$

where integration is along the radial line in the complex plane form 0 to z. Observe that alternatively the function P can be expressed as

$$P(z,d) = \frac{z^d}{\Gamma(d+1)}{}_1F_1(d; d+1; -z),$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind.

Figure 1.2 and Figure 1.3 show the kernel and autocovariance function of the CARMA$(3,2)$ process given in Example 1.15 and of the corresponding FICARMA$(3,d,2)$ process for $d = 0.25$. We recognize the long memory property of the FICARMA process.

Due to the slow decay (1.42) of the fractionally integrated kernel g_d, simulation algorithms for FICARMA processes are very slow and expensive. Introducing the so-called fractional Lévy processes in Chapter 2, we obtain an alternative representation of FICARMA processes which allows much more efficient simulation. Fractional Lévy processes are a generalization of fractional Brownian motion. Therefore, in the following section we will give a brief summary of the definition and properties of fractional Brownian motion.

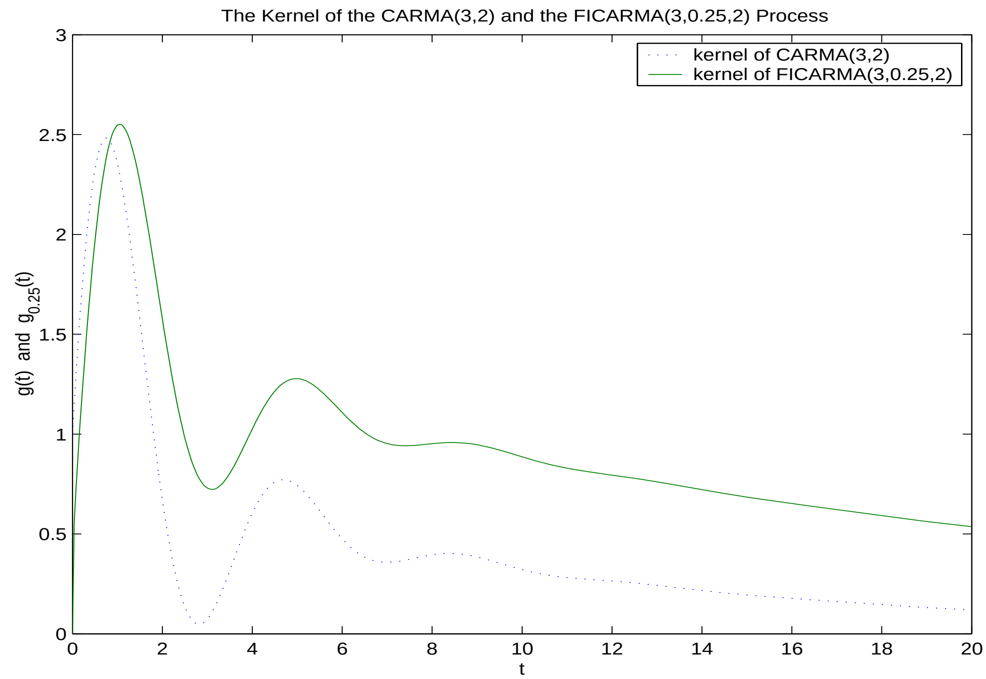

Figure 1.2: The kernel of the CARMA(3,2) and the FICARMA(3,0.25,2) process.

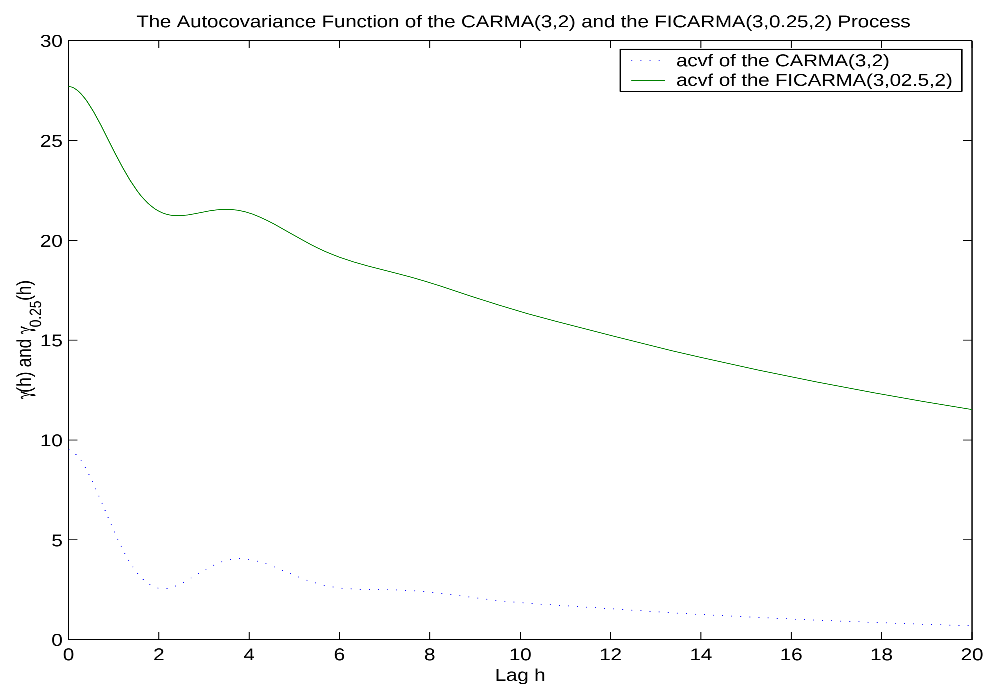

Figure 1.3: The autocovariance function of the CARMA(3,2) and the FICARMA(3,0.25,2) process.

1.4 Fractional Brownian Motion

In this thesis we consider fractional Lévy processes. The name "fractional Lévy process" already suggests that it can be regarded as a generalization of fractional Brownian motion (FBM). In the past years fractional Brownian motion has been the subject of numerous investigations and played a role in many fields of application such as economics, finance, turbulance and telecommunications. Let us recall the definition and properties of FBM (see Doukhan et al. (2003), part A or Samorodnitsky & Taqqu (1994), chapter 7.2 for proofs and further results).

Definition 1.27 *Let $0 < H < 1$. The Gaussian stochastic process $\{B_H(t)\}_{t\geq 0}$ satisfying the following three properties*

(i) $B_H(0) = 0$

(ii) $E[B_H(t)] = 0$ *for all* $t \geq 0$,

(iii) for all $s, t \geq 0$,

$$E[B_H(t)B_H(s)] = \frac{1}{2}\left(|t|^{2H} - |t-s|^{2H} + |s|^{2H}\right), \tag{1.53}$$

is called the (standard) ***fractional Brownian motion*** *with parameter* H.

The parameter H is also referred to as the Hurst coefficient. It is obvious from (1.53) that FBM has stationary increments but that for $H \neq 1/2$ the increments are not independent.
We can define a parametric family of FBMs in terms of the stochastic Weyl integral (see e.g. Samorodnitsky & Taqqu (1994), chapter 7.2).
For any $a, b \in \mathbb{R}$,

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \tag{1.54}$$
$$\left\{\int_{\mathbb{R}} \left\{a\left[(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right] + b\left[(t-s)_-^{H-\frac{1}{2}} - (-s)_-^{H-\frac{1}{2}}\right]\right\} dB(s)\right\}_{t\in\mathbb{R}},$$

where $u_+ = \max(u, 0)$, $u_- = \max(-u, 0)$ and $\{B(t)\}_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion. Notice that we can construct a two-sided standard Brownian motion as in (1.7).

If $H = 1/2$, it is clear that $\{B_{1/2}(t)\}_{t\in\mathbb{R}} = \{B(t)\}_{t\in\mathbb{R}}$ is ordinary Brownian motion. If we choose $a = \sqrt{\Gamma(2H+1)\sin(\pi H)}/\Gamma(H+1/2)$ and $b = 0$ in (1.54) then $\{B_H(t)\}_{t\in\mathbb{R}}$ is a FBM satisfying (1.53).

Many properties of FBM are given by its fractional index H. For instance H governs the self-similarity property. First, let us precise the definition of a self-similar process (we refer to Embrechts & Maejima (2002) for an excellent survey on self-similar processes).

Definition 1.28 *A real-valued stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ is self-similar with index H if for all $c > 0$,*

$$\{X(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} c^H \{X(t)\}_{t\in\mathbb{R}}. \tag{1.55}$$

Proposition 1.29 *Fractional Brownian motion is self-similar with index H. Moreover, FBM is the only self-similar Gaussian process with stationary increments.*

Remark 1.30 In higher dimension $m \geq 2$ the preceding proposition does not remain true, i.e. there exist Gaussian models which are stationary and selfsimilar (see e.g. Bonami & Estrade (2003)).

Consider now the covariance between two increments. It follows by the stationarity of the increments of B_H,

$$\begin{aligned}\rho_H(n) :&= \operatorname{cov}(B_H(k) - B_H(k-1), B_H(k+n) - B_H(k+n-1)) \\ &= \frac{1}{2}(|n+1|^{2H} - 2|n|^{2H} - |n-1|^{2H}), \quad n \in \mathbb{N}.\end{aligned} \tag{1.56}$$

Proposition 1.31
(i) If $0 < H < 1/2$, ρ_H is negative and $\sum_{n=1}^{\infty} |\rho_H(n)| < \infty$.
(ii) If $H = 1/2$, ρ_H equals 0.
(iii) If $1/2 < H < 1$, ρ_H is positive, $\sum_{n=1}^{\infty} |\rho_H(n)| = \infty$, and $\rho_H(n) \sim Cn^{2H-2}$, as $n \to \infty$.

Corollary 1.32 *For $1/2 < H < 1$ the increments of FBM exhibit long memory in the sense of Definition 1.16 (with $d = H - \frac{1}{2}$).*

Having observed these distributional properties, which make FBM a promising model in various applications, we briefly review the sample path properties.

Proposition 1.33 *The trajectories of FBM are continuous. In particular, for every $\tilde{H} < H$ there exists a modification of B_H whose sample paths are a.s. locally $\tilde{H}$-Hölder continuous on $\mathbb{R}$.*

Let us introduce the notion of p-variation:

Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be a stochastic process. Given a real number $p \geq 1$ and a point partition $a = t_0^n < t_1^n < \ldots < t_n^n = b$ of the compact interval $[a, b]$ such that $\max\limits_{1\leq k\leq n} \{|t_k^n - t_{k-1}^n|\} \to 0$ as $n \to \infty$, we define the random variable

$$Var^p_{[a,b];n}(X) = \sum_{k=1}^{n} |X(t_k^n) - X(t_{k-1}^n)|^p.$$

Then the limit in probability

$$Var^p_{[a,b]}(X) = p - \lim_{n\to\infty} Var^p_{[a,b];n}(X)$$

is called the p-variation of X over $[a, b]$. If $p = 1$ we call $Var_{[a,b]}(X) := Var^1_{[a,b]}(X)$ the total variation. For $p = 2$, $[X, X]_t := Var^2_{[0,t]}(X)$ denotes the quadratic variation of X on the interval $[0, t]$.

Proposition 1.34 *The sample paths of FBM are of finite p-variation for every $p > 1/H$ and of infinite p-variation if $p < 1/H$.*

Consequently, for $H < 1/2$ the quadratic variation is infinite. On the other hand, if $H > 1/2$ it is known that the quadratic variation of FBM is zero, whereas the total variation is infinite.

Corollary 1.35 *This shows that for $H \neq 1/2$, FBM cannot be a semimartingale.*

A proof of this well-known fact can be found in e.g. Rogers (1997) or Cheridito (2001).

Figure 1.4 shows the sample paths of FBM for various values of the Hurst parameter H.

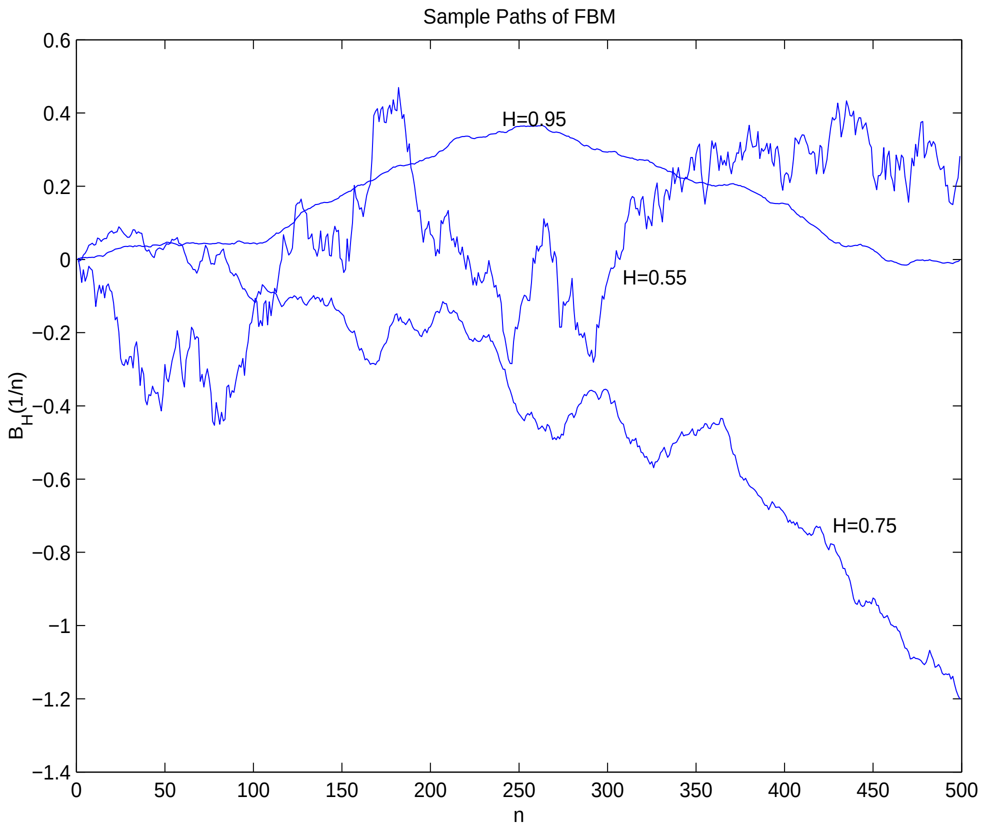

Figure 1.4: Various sample paths of FBM .

2 Fractional Lévy Processes

We define and discuss fractional Lévy processes (FLP) in this chapter. FLPs are constructed by a natural generalization of the integral representation of fractional Brownian motion (FBM) and were first introduced by (Benassi et al. (2004)). However, our approach is less restrictive, as we allow also for Lévy processes without a finite second moment.

After presenting different methods of constructing FLPs we derive the second-order and sample path properties. The remaining part of this chapter is devoted to integrals with respect to FLPs. In particular, we focus on moving average processes and show how our findings apply to CARMA and FICARMA processes.

The results of this chapter can also be found in Marquardt (2006a).

Since definitions and calculations are easier to understand in one dimension, we consider here univariate FLPs and then generalize our results to the multivariate setting in Chapter 4. We would like to stress that throughout we assume a Lévy process without Brownian component.

2.1 Construction of Univariate Fractional Lévy Processes

In this section we introduce univariate fractional Lévy processes (FLPs) as a natural counterpart to fractional Brownian motion (FBM).

In order to be consistent with the notation of the previous Sections 1.2 and 1.3 and as we are mainly interested in fractionally integrated processes, in what follows we will work with the fractional integration parameter $d := H - 1/2 \in (-0.5, 0.5)$ rather than the Hurst parameter. Furthermore, we restrict ourselves to $0 < d < 0.5$ as we are interested in long memory processes (see Definition 1.16).

The integral representation of FBM was first generalized to a fractional Lévy motion by Benassi et al. (2004), who start with the so-called "well-balanced" FBM with $a = b = 1$ in (1.54). Their approach is the basis of our definition of a FLP as, like them, we replace the Brownian motion B in the moving average representation (1.54) by a two-sided Lévy process as defined in (1.7). However, we will allow for Lévy processes with infinite second moments and also consider integrals with respect to FLPs.

Furthermore, like Mandelbrot & Van Ness (1968) for FBM, we choose $a = 1/\Gamma(H + 1/2) = 1/\Gamma(d + 1)$ and $b = 0$ in (1.54). This choice will simplify calculations when we apply our results to moving average processes.

Based on the moving average representation (1.54) of FBM we define a FLP as follows.

Definition 2.1 (Fractional Lévy Process) *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided Lévy process on $\mathbb{R}$ without Brownian component and satisfying*

$$E[L(1)^\alpha] < \infty \quad \textit{for some } \; 1 < \alpha \leq 2. \tag{2.1}$$

For fractional integration parameter $0 < d < 1 - \frac{1}{\alpha}$, a stochastic process

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d\right] \, L(ds), \quad t \in \mathbb{R}, \tag{2.2}$$

is called a ***fractional Lévy process*** *(FLP). We refer to the process L as the driving Lévy process of the FLP M_d. If $\alpha = 2$ in (2.1) we call M_d a square-integrable FLP.*

Remark 2.2 The general Lévy-Itô representation (1.2) guarantees that every Lévy process can be decomposed into a linear term, a Brownian and a jump component which is independent of the Brownian part. However, the Brownian part induces a FBM which has already been extensively studied (see e.g. Doukhan et al. (2003) or Samorodnitsky & Taqqu (1994)) and considered in Section 1.4. Therefore we assume a Lévy process without Brownian component.

Remark 2.3 Note that in general an infinitely divisible distribution with characteristic triplet (γ, σ^2, ν) has finite α-th moment, if and only if $\int_{|x|>C} |x|^\alpha \nu(dx) < \infty$ for one and hence all $C > 0$ (see Sato (1999, Corollary 25.8)). Thus, (2.1) holds if and only if the Lévy measure ν of L satisfies

$$\int\limits_{|x|>1} |x|^\alpha \, \nu(dx) < \infty. \tag{2.3}$$

Example 2.4 A prominent example for a class of Lévy processes satisfying (2.1) are the symmetric α-stable Lévy processes having Lévy measure

$$\nu(dx) = \frac{1}{|x|^{1+\alpha}} dx,$$

as for an α-stable process L, $E[|L(1)|^\gamma] < \infty$ for any $\gamma < \alpha$, whereas $E[|L(1)|^\alpha] = \infty$ (see Sato (1999, Example 25.10)). The resulting FLP is then referred to as a *linear fractional stable motion (LFSM)*. Linear fractional stable motions are of increasing interest in many fields of applications, in particular because they belong to the class of non-Gaussian self-similar processes (see e.g. Samorodnitsky & Taqqu (1994), chapter 7.4).

We would like to stress that our class of fractional Lévy processes includes the linear fractional stable motions ($1 < \alpha < 2$) as a special case.

Before making precise the meaning of the integral (2.2), we summarize the following two important properties of the kernel function

$$f_t(s) := \frac{1}{\Gamma(1+d)} [(t-s)_+^d - (-s)_+^d], \quad s \in \mathbb{R}, \tag{2.4}$$

where $0 < d < 1 - \frac{1}{\alpha}$, $1 < \alpha \leq 2$.

Proposition 2.5 *For $0 < d < 1 - \frac{1}{\alpha}$, $1 < \alpha \leq 2$ and $t \in \mathbb{R}$ the kernel function f_t as defined in (2.4) is bounded. Moreover, $f_t \in L^p(\mathbb{R})$ for $p > (1-d)^{-1}$. In particular, $f_t \in L^\alpha(\mathbb{R})$ and $f_t \in L^2(\mathbb{R})$.*

Proof. For $s \geq \max(t, 0)$ we have $f_t(s) = 0$.
Now let $s \leq \min(t, 0)$, then

$$f_t(s) = [(t-s)^d - (-s)^d]/\Gamma(d+1) = (\Gamma(d))^{-1} \int_{-s}^{t-s} u^{d-1} \, du.$$

Hence,

$$|f_t(s)| \leq \begin{cases} t\,(-s)^{d-1}/\,\Gamma(d), & \text{for } s \leq -1 \text{ and } t \geq 0, \\ |t|\,(t-s)^{d-1}/\,\Gamma(d), & \text{for } s \leq t-1 \text{ and } t < 0. \end{cases} \tag{2.5}$$

Moreover, for $t \geq 0$ and $s \in [-1, \max(t,0)] = [-1, t]$, we obtain

$$0 \leq f_t(s) \leq \frac{1}{\Gamma(d+1)}(t-s)^d \leq \frac{1}{\Gamma(d+1)}(t+1)^d < \infty,$$

and for $t < 0$ and $s \in [t-1, \max(t,0)] = [t-1, 0]$,

$$0 \geq f_t(s) \geq -\frac{1}{\Gamma(d+1)}(-s)^d \geq -\frac{1}{\Gamma(d+1)}(1-t)^d > -\infty.$$

Hence, for all $t \in \mathbb{R}$,

$$|f_t(s)| \leq \frac{1}{\Gamma(d+1)}(1+|t|)^d < \infty, \quad s \in \mathbb{R},$$

which shows that f_t is bounded. It remains to show $f_t \in L^p(\mathbb{R})$ for $p > (1-d)^{-1}$. In fact, from (2.5) we have for $t > 0$,

$$\begin{aligned} \| f_t(s) \|^p_{L^p(-\infty,-1)} &= \int_{-\infty}^{-1} |f_t(s)|^p \, ds \leq \left(\frac{t}{\Gamma(d)}\right)^p \int_{-\infty}^{-1} (-s)^{p(d-1)} \, ds \\ &= -\left(\frac{t}{\Gamma(d)}\right)^p \frac{(-s)^{p(d-1)+1}}{p(d-1)+1}\Bigg|_{-\infty}^{-1} < \infty, \quad \text{if } p > \frac{1}{1-d}. \end{aligned}$$

Analogously, for $t < 0$ and $p > (1-d)^{-1}$ it follows $\| f_t(s) \|^p_{L^p(-\infty,t-1)} < \infty$. □

Proposition 2.6 *The function $t \mapsto (t-s)_+^d - (-s)_+^d$ is locally Hölder continuous of every order $\beta \leq d$ and for an order $\beta > d$ it is not Hölder continuous on any interval containing s. Furthermore, the total variation is finite on compacts.*

Proof. Define for $t_1 > t_2$, $x := (t_1 - s)_+$ and $y := (t_2 - s)_+$. We first consider the case $\frac{x}{2} \leq y \leq x$. Hence, $x - y \leq \frac{x}{2}$ and $x^d - y^d = d\,\tilde{y}^{d-1}(x-y)$, where $\tilde{y} \in \left[\frac{x}{2}, x\right]$, i.e., $\tilde{y} \geq x - y$. Thus,

$$x^d - y^d \leq d\,(x-y)^{d-1}(x-y) = d\,(x-y)^d.$$

Now assume $y \leq \frac{x}{2}$. Then $x - y \geq \frac{x}{2}$ and

$$x^d - y^d \leq x^d = 2^d \left(\frac{x}{2}\right)^d \leq 2^d (x-y)^d.$$

Therefore,

$$\left|(t_1 - s)_+^d - (-s)_+^d - (t_2 - s)_+^d + (-s)_+^d\right| = |(t_1 - s)_+^d - (t_2 - s)_+^d| \leq 2^d\,(t_1 - t_2)^d$$

and $t \mapsto (t-s)_+^d - (-s)_+^d$ is locally Hölder continuous of every order $\beta \leq d$. On the other hand setting $s = 0$, $t_2 = 0$ and $t_1 = 1/n$ we obtain

$$\frac{(t_1 - s)_+^d - (t_2 - s)_+^d}{(t_1 - t_2)^{d+\epsilon}} = \frac{(1/n)_+^d - (0)_+^d}{(1/n)^{d+\epsilon}} = \left(\frac{1}{n}\right)^{-\epsilon} = n^\epsilon \to \infty, \quad \text{as } n \to \infty.$$

Thus the function $t \longmapsto (t-s)_+^d - (-s)_+^d$ is not Hölder continuous of any order $\beta > d$.

It remains to show that the function $t \mapsto (t-s)_+^d - (-s)_+^d$ is of finite total variation on compacts. However, for fixed $s \in \mathbb{R}$, $g(t) := (t-s)_+^d - (-s)_+^d$ is monotone increasing. Hence,

$$Var_{[a,b]}(g) = (b-s)_+^d - (a-s)_+^d,$$

which is finite. □

Figure 2.1 shows the kernel function f_t for fixed value $d = 0.25$ and different values of t, whereas in Figure 2.2, t is fixed ($t = 5$) and the fractional integration parameter d varies.

We have defined a fractional Lévy process in terms of a stochastic integral (2.2) without specifying in which sense the integration is understood. Now we make precise the meaning of (2.2).

Theorem 2.7 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[L(1)^\alpha] < \infty$ for some $1 < \alpha \leq 2$. For $t \in \mathbb{R}$ and $0 < d < 1 - \frac{1}{\alpha}$ define the kernel function f_t as in (2.4). Then for every $t \in \mathbb{R}$, the integral*

$$M_d(t) = \int_{\mathbb{R}} f_t(s)\, L(ds)$$

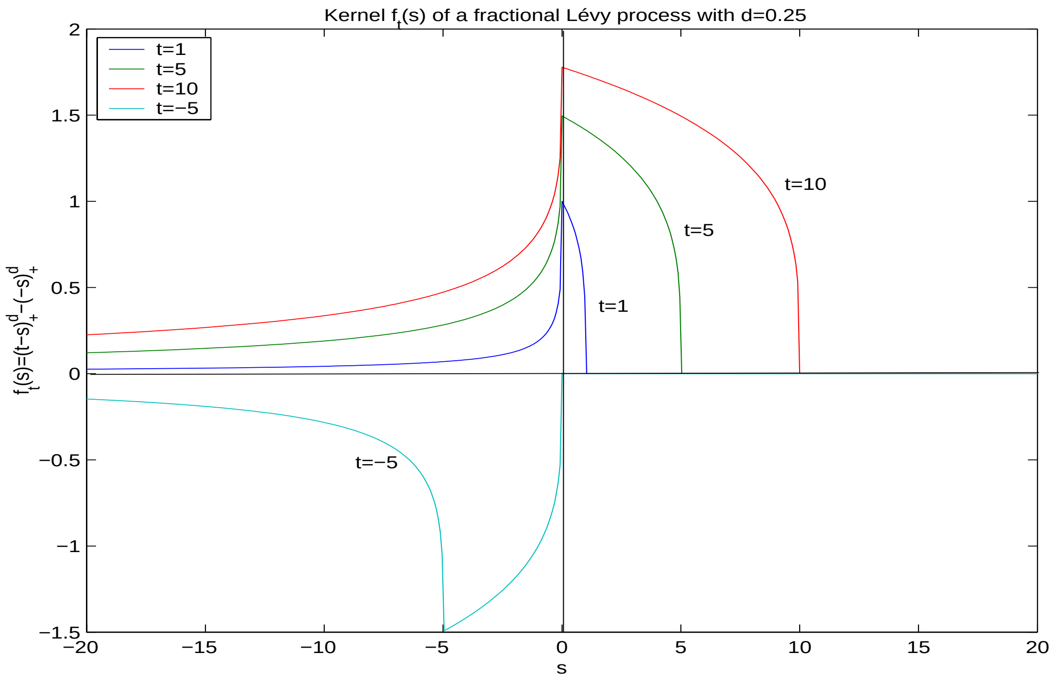

Figure 2.1: The kernel f_t of a fractional Lévy process with $d = 0.25$ for different values of t.

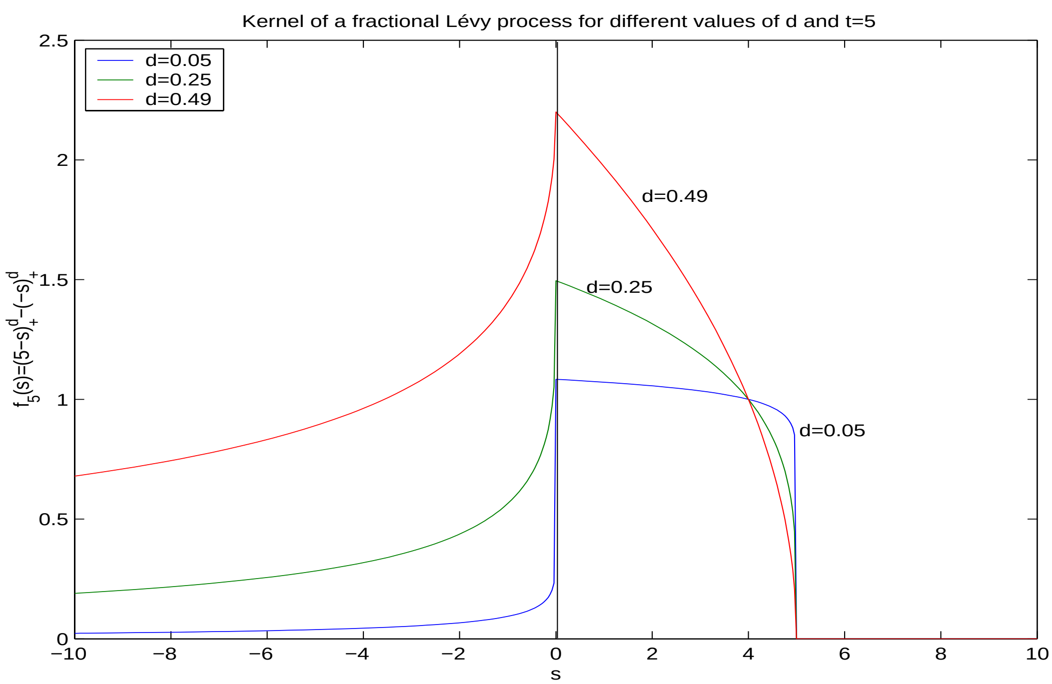

Figure 2.2: The kernel f_t of a fractional Lévy process for $t = 5$ and different values of d.

is well-defined in the sense that it exists as the limit in probability of step functions approximating f_t. Moreover, let $u_1, \dots, u_m \in \mathbb{R}$, $-\infty < t_1 < \dots < t_m < \infty$ and $m \in \mathbb{N}$. Then the finite dimensional distributions of the process M_d have the characteristic functions

$$E[\exp\{iu_1 M_d(t_1) + \dots + iu_m M_d(t_m)\}] = \exp\left\{ \int_{\mathbb{R}} \psi\left(\sum_{j=1}^{m} u_j f_{t_j}(s) \right) ds \right\}, \tag{2.6}$$

where ψ is given as in (1.5).

Proof. It follows from our findings at the end of Section 1.1.2 that (2.2) is well-defined if we verify conditions (1.16) and (1.17). We know from the proof of Proposition 2.5 that for all $t \in \mathbb{R}$,

$$|f_t(s)| \leq \frac{1}{\Gamma(d+1)}(1+|t|)^d, \quad s \in \mathbb{R}.$$

Furthermore, since $1 < \alpha \leq 2$, $E[L(1)] = 0$ and hence $\gamma = -\int_{|x|>1} x\,\nu(dx)$, (1.17) is implied by

$$\begin{aligned}
&\int_{\mathbb{R}} \left| f_t(s)\gamma + \int_{\mathbb{R}} f_t(s)x \left(1_{\{|f_t(s)x|\leq 1\}} - 1_{\{|x|\leq 1\}}\right) \nu(dx) \right| ds \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f_t(s)x 1_{\{|f_t(s)x|>1\}}\, \nu(dx) \right| ds \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f_t(s)x|^\alpha 1_{\{|x|>\Gamma(d+1)(1+|t|)^{-d}\}}\, \nu(dx)\, ds < \infty,
\end{aligned}$$

where the finiteness of the last term is a consequence of (2.3) and $f_t \in L^\alpha(\mathbb{R})$ (see Proposition 2.5).

To show (1.16) we observe that for $1 < \alpha \leq 2$,

$$(|f_t(s)x|^2 \wedge 1) \leq |f_t(s)|^\alpha |x|^\alpha.$$

In fact, if $|f_t(s)x|^2 > 1$, then $|f_t(s)x| > 1$ and hence

$$|f_t(s)x|^\alpha > 1 \geq (|f_t(s)x|^2 \wedge 1).$$

On the other hand, if $|f_t(s)x|^2 \leq 1$, then

$$|f_t(s)x|^\alpha \geq |f_t(s)x|^2 \geq (|f_t(s)x|^2 \wedge 1).$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} (|f_t(s)x|^2 \wedge 1)\, \nu(dx)\, ds \\
&= \int_{\mathbb{R}} \int_{|x|\leq 1} (|f_t(s)x|^2 \wedge 1)\, \nu(dx)\, ds + \int_{\mathbb{R}} \int_{|x|>1} (|f_t(s)x|^2 \wedge 1)\, \nu(dx)\, ds \\
&\leq \int_{\mathbb{R}} \int_{|x|\leq 1} x^2 f_t^2(s)\, \nu(dx)\, ds + \int_{\mathbb{R}} \int_{|x|>1} |x|^\alpha |f_t(s)|^\alpha\, \nu(dx)\, ds < \infty,
\end{aligned}$$

since $f_t \in L^\alpha(\mathbb{R})$, $f_t \in L^2(\mathbb{R})$, (2.3) and ν is a Lévy measure.

Finally, (2.6) is a consequence of (1.19), when we insert $\gamma = -\int_{|x|>1} x\, \nu(dx)$ and write

$$\sum_{j=1}^m u_j M_d(t_j) = \sum_{j=1}^m u_j \int_{\mathbb{R}} f_{t_j}(s)\, L(ds) = \int_{\mathbb{R}} \sum_{j=1}^m u_j f_{t_j}(s)\, L(ds).$$

□

Theorem 2.8 *Let $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ be a FLP as defined in Definition 2.1. Then the process M_d is well-defined if and only if the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ satisfies (2.1) or equivalently (2.3).*

Proof. The sufficiency of (2.1) has already been proven in Theorem 2.7. Hence, it remains to show the necessity.

If M_d is well-defined we know from (1.16) that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_{\{|xf_t(s)|>1\}}\, \nu(dx)\, ds < \infty.$$

W.l.o.g. assume $t > 0$. Then for $-\infty < s \leq 0$,

$$f_t(s) = (t-s)^d - (-s)^d = d \int_{-s}^{t-s} u^{d-1}\, du \geq dt(t-s)^{d-1} =: C(t-s)^{d-1}.$$

This yields, using Fubini's theorem,

$$\begin{aligned}
\infty > \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{\{|xf_t(s)|>1\}}\,\nu(dx)\,ds &\geq \int_{-\infty}^{0}\int_{\mathbb{R}} 1_{\{|x|C(t-s)^{d-1}>1\}}\,\nu(dx)\,ds \\
&= \int_{-\infty}^{0}\int_{\mathbb{R}} 1_{\{s>t-|x|^{\frac{1}{1-d}}C^{\frac{1}{1-d}}\}}\,\nu(dx)\,ds \\
&= \int_{\mathbb{R}} \left(-t+|x|^{\frac{1}{1-d}}C^{\frac{1}{1-d}}\right)_+\,\nu(dx) \\
&= C^{\frac{1}{1-d}} \int_{|x|>t^{1-d}/C} |x|^{\frac{1}{1-d}}\,\nu(dx) - t\int_{|x|>t^{1-d}/C} \nu(dx).
\end{aligned}$$

This shows the necessity of (2.3). The proof is complete. □

Remark 2.9 As a consequence of (2.6) the generating triplet of $M_d(t)$ is $(\gamma_M^t, 0, \nu_M^t)$, where

$$\begin{aligned}
\gamma_M^t &= -\int_{\mathbb{R}}\int_{\mathbb{R}} f_t(s)x1_{\{|f_t(s)x|>1\}}\,\nu(dx)\,ds \quad \text{and} \\
\nu_M^t(B) &= \int_{\mathbb{R}}\int_{\mathbb{R}} 1_B(f_t(s)x)\,\nu(dx)\,ds. \qquad (2.7)
\end{aligned}$$

2.1.1 The L^2-Integral based on the Poisson Representation of L

We have just defined the integral (2.2) as a limit in probability of step functions approximating the kernel function f_t. However our findings in Section 1.1.2 allow for a definition of the integral (2.2) in an L^2-sense, provided that $\alpha = 2$, i.e. the Lévy process L has finite second moments.

Theorem 2.10 (Fractional Lévy Process in L^2-sense) *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying $E[L(1)] = 0$, $E[L(1)^2] < \infty$ and $\tilde{J}(ds, du) = J(ds, du) - ds\nu(du)$ be the compensated jump measure of L. For $t \in \mathbb{R}$ and $0 < d < 0.5$ define the kernel function f_t as in (2.4). Then for every $t \in \mathbb{R}$, $M_d(t) = \int_{\mathbb{R}} f_t(s)\,L(ds)$ exists in the sense that*

$$M_d(t) = \int_{\mathbb{R}\times\mathbb{R}_0} f_t(s)u\,\tilde{J}(ds, du), \quad t \in \mathbb{R}, \qquad (2.8)$$

i.e. M_d is the limit in $L^2(\Omega, P)$ of integrals of simple functions $(\phi_k)_{k\in\mathbb{N}}$ satisfying

$$E\left[\int_{\mathbb{R}\times\mathbb{R}_0} |\phi_k(s,u) - f_t(s)u|^2 \,\nu(du)\,ds\right] \to 0, \text{ as } k \to \infty.$$

Moreover, for all $t \in \mathbb{R}$ the distribution of $M_d(t)$ is infinitely divisible and

$$E[M_d(t)]^2 = \| f_t \|^2_{L^2(\mathbb{R})} E[L(1)^2], \quad t \in \mathbb{R}. \tag{2.9}$$

Let $u_1, \ldots, u_m \in \mathbb{R}$, $-\infty < t_1 < \ldots < t_m < \infty$ and $m \in \mathbb{N}$. Then the finite dimensional distributions of the process M_d have the characteristic functions (2.6).

Proof. The assertions are direct consequences of the results of Section 1.1.2, since $f_t \in L^2(\mathbb{R})$, $E[L(1)] = 0$ and $E[L(1)^2] < \infty$. (2.6) follows from (1.19) when we write (as in the proof of Theorem 2.7),

$$\sum_{j=1}^m u_j M_d(t_j) = \sum_{j=1}^m u_j \int_{\mathbb{R}} f_{t_j}(s)\,L(ds) = \int_{\mathbb{R}} \sum_{j=1}^m u_j f_{t_j}(s)\,L(ds).$$

□

Remark 2.11 In Theorem 2.7, we have shown that M_d is well-defined as a limit in probability of step functions approximating the kernel function f_t. However, the L^2-limit and the p-limit agree, since L^2-convergence implies convergence in probability and the p-limit is unique.

We have seen that, if $\alpha = 2$, (2.2) can be understood as L^2-limit and we can now apply the Kolmogorov-Centsov Theorem to obtain a continuous modification of $\{M_d(t)\}_{t\in\mathbb{R}}$ (see Theorem 2.19 below). However, we can also show that $\{M_d(t)\}_{t\in\mathbb{R}}$ has a continuous modification by proving in the following section that $M_d(t)$ is a.s. equal to an improper Riemann integral for all $t \in \mathbb{R}$.

2.1.2 The Improper Riemann Integral

We give here a pathwise construction of a FLP as an improper Riemann integral. As in the preceding subsection we fix $\alpha = 2$.

Theorem 2.12 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[L(1)^2] < \infty$. For $t \in \mathbb{R}$ define the kernel function f_t as in (2.4). Then for all $t \in \mathbb{R}$, $M_d(t) = \int_{\mathbb{R}} f_t(s)\, L(ds)$ has a modification which is equal to the improper Riemann integral*

$$M_d(t) = \frac{1}{\Gamma(d)} \int\limits_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}]\, L(s)\, ds, \quad t \in \mathbb{R}. \tag{2.10}$$

Moreover (2.10) is continuous in t.

Proof. We assume $t > 0$. For $t \le 0$ the proof is analogous. For a Lévy process L on $\mathbb{R}$ that satisfies $E[L(1)] = 0$ and $E[L(1)^2] < \infty$ we have a generalization of the law of the iterated logarithm of random walks (Sato (1999), Proposition 48.9), that is

$$\limsup_{t\to\infty} \frac{|L(t)|}{(2t\log\log t)^{1/2}} = (E[L(1)^2])^{1/2} \;\; a.s.$$

Moreover, $(t-s)^d - (-s)^d \sim td(-s)^{d-1}$ as $s \to -\infty$ and therefore,

$$\lim_{s\to-\infty} L(s)[(t-s)^d - (-s)^d] = 0 \quad a.s.$$

If g is a continuously differentiable function on $[a,b] \subset \mathbb{R}$ it is always possible to use the integration by parts formula to define $\int_a^b g(s)\, L(ds)$ as a Riemann integral by

$$\int\limits_{[a,b]} g(s)\, L(ds) = g(b)L(b) - g(a)L(a) - \int\limits_{[a,b]} L(s)\, dg(s). \tag{2.11}$$

(see e.g. Eberlein & Raible (1999, Lemma 2.1)). Since we have,

$$M_d(t) = \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \int\limits_a^0 [(t-s)^d - (-s)^d]\, L(ds) + \lim_{\epsilon\to 0} \int\limits_\epsilon^t \frac{(t-s)^d}{\Gamma(d+1)}\, L(ds),$$

it follows by (2.11),

$$\begin{aligned} M_d(t) &= \lim_{\epsilon\to 0} \int\limits_\epsilon^t \frac{(t-s)^{d-1}}{\Gamma(d)}\, L(s)\, ds - \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \left\{L(a)[(t-a)^d - (-a)^d]\right\} \\ &\quad + \frac{1}{\Gamma(d+1)} \lim_{a\to-\infty} \left\{ d \int\limits_a^0 [(t-s)^{d-1} - (-s)^{d-1}]\, L(s)\, ds \right\} \\ &= \frac{1}{\Gamma(d)} \int\limits_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}]\, L(s)\, ds, \quad t \in \mathbb{R}. \end{aligned}$$

To show that (2.10) is continuous in t we define for $t > 0$,

$$g_t(s) = (t-s)^{d-1} L(s) 1_{[0,t]}(s), \quad s \in \mathbb{R}.$$

Then for all $T > 0$ the family $\{g_t\}_{t\in[0,T]}$ is uniformly integrable with respect to the Lebesgue measure and the continuity of $\int_0^t (t-s)^{d-1} L(s)\, ds$ follows from Shiryaev (1996, Theorem 5, Chapter II.6). Furthermore, by Lebesgue's dominated convergence theorem

$$\int_{-\infty}^{0} \left[(t-s)^{d-1} - (-s)^{d-1}\right] L(s)\, ds$$

is continuous in t. □

2.1.3 Series Representations of Fractional Lévy Processes

The results in this section are based on series representation of Lévy processes summarized in Rosinski (2001).

Theorem 2.13 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[L(1)^\alpha] < \infty$ for some $1 < \alpha \leq 2$. For $t \in \mathbb{R}$ define the kernel function f_t as in (2.4). Suppose the Lévy measure ν of L is symmetric and set $\nu^{\leftarrow}(s) = \inf\{x > 0 :\ \nu((x,\infty)) \leq s\},\ s > 0$, the right continuous inverse of $x \mapsto \nu((x,\infty))$. Let Λ be an arbitrary probability measure on $\mathbb{R}$ with nowhere vanishing density ρ. Furthermore, let $\{T_i\}_{i=1,2,\ldots}$ and $\{U_i\}_{i=1,2,\ldots}$ be independent sequences of random variables, such that $\{T_i\}_{i=1,2,\ldots}$ is a sequence of independent indentically distributed (i.i.d.) standard exponential random variables and $\{U_i\}_{i=1,2\ldots}$ is a sequence of i.i.d. random variables with distribution Λ. Put $\tau_0 = 0$ and $\tau_i = \sum_{j=1}^{i} T_j,\ i = 1,2,\ldots$. Furthermore, let $\{\varepsilon_i\}_{i=1,2\ldots}$ be an i.i.d. sequence of random variables with $P(\varepsilon_i = -1) = P(\varepsilon_i = 1) = \frac{1}{2}$. Then for every $t \in \mathbb{R}$ the series*

$$X(t) = \sum_{i=1}^{\infty} \varepsilon_i \nu^{\leftarrow}(\tau_i \rho(U_i)) f_t(U_i) \tag{2.12}$$

converges a.s. and

$$\{M_d(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t)\}_{t\in\mathbb{R}}. \tag{2.13}$$

Proof. As ν is symmetric, we have from (2.6),

$$E[e^{iuM_d(t)}] = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} \left[e^{iuxf_t(s)} - 1 - iuxf_t(s) \right] \nu(dx)\, ds \right\}$$
$$= \exp\left\{ 2 \int_{\mathbb{R}} \int_0^\infty [\cos(uxf_t(s)) - 1]\, \nu(dx)\, ds \right\}.$$

Therefore, the assertion is an immediate consequence of Rosinski (1989, Proposition 2). □

If ν is not symmetric we obtain a similar result by taking into account the left continuous inverse of ν.

Theorem 2.14 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[L(1)^\alpha] < \infty$ for some $1 < \alpha \leq 2$. Set $\nu^{\leftarrow}(s) = \inf\{x > 0 : \nu((x,\infty)) \leq s\}$, $s > 0$, and $\nu^{\rightarrow}(s) = \sup\{x < 0 : \nu((-\infty, x)) \leq s\}$, $s > 0$, the right and left continuous inverse of ν, respectively. Define Λ and the sequences $\{T_i\}, \{U_i\}$ and $\{\tau_i\}$ as in Theorem 2.13. Then for every $t \in \mathbb{R}$ the series*

$$X(t) = \sum_{i=1}^{\infty} \{[\nu^{\leftarrow}(\tau_i\rho(U_i)) + \nu^{\rightarrow}(\tau_i\rho(U_i))]\, f_t(U_i) - C_t(\tau_i)\} \tag{2.14}$$

converges a.s., where

$$C_t(\tau_i) = \int_{\mathbb{R}} \int_{\tau_{i-1}}^{\tau_i} [\nu^{\leftarrow}(\tau\rho(u)) + \nu^{\rightarrow}(\tau\rho(u))] f_t(u)\, d\tau\, \rho(u)\, du.$$

Moreover,

$$\{M_d(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t)\}_{t\in\mathbb{R}}.$$

Proof. $X(t)$ in (2.14) is a generalized shot noise series which converges a.s. if we show that

$$G^t(B) := \int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(H_t(\tau, u))\, d\tau\, \Lambda(du)$$
$$= \int_{\mathbb{R}} \int_0^\infty 1_{\{B\setminus\{0\}\}}(H_t(\tau, u))\, d\tau\, \rho(u)\, du, \quad B \in \mathcal{B}(\mathbb{R})$$

is a Lévy measure, where

$$H_t(\tau,u) = [\nu^{\leftarrow}(\tau\rho(u)) + \nu^{\rightarrow}(\tau\rho(u))]f_t(u), \quad \tau > 0,\ t,u \in \mathbb{R}$$

(see Rosinski (1990, Theorem 2.4)). Observe, that for every $x \geq 0$, $u \in \mathbb{R}$,

$$\begin{aligned} &Leb(\{\tau > 0:\ \nu^{\leftarrow}(\tau\rho(u)) > x\}) \\ &= Leb(\{\tau > 0:\ \nu^{\leftarrow}(\tau) > x\})/\rho(u) \\ &= \nu((x,\infty))/\rho(u) \end{aligned}$$

and thus

$$\int_{\mathbb{R}}\int_0^{\infty} 1_{\{B\setminus\{0\}\}}(\nu^{\leftarrow}(\tau\rho(u))f_t(u))\,d\tau\,\rho(u)\,du = \int_{\mathbb{R}}\int_0^{\infty} 1_{\{B\setminus\{0\}\}}(xf_t(u))\,\nu(dx)\,du.$$

Analogously, for every $x \leq 0$ and $u \in \mathbb{R}$,

$$Leb(\{\tau > 0:\ \nu^{\rightarrow}(\tau\rho(u)) < x\}) = \nu((-\infty,x))/\rho(u),$$

which yields

$$\int_{\mathbb{R}}\int_0^{\infty} 1_{\{B\setminus\{0\}\}}(\nu^{\rightarrow}(\tau\rho(u))f_t(u))\,d\tau\,\rho(u)\,du = \int_{\mathbb{R}}\int_{-\infty}^{0} 1_{\{B\setminus\{0\}\}}(xf_t(u))\,\nu(dx)\,du.$$

Therefore,

$$G^t(B) = \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{\{B\setminus\{0\}\}}(xf_t(u))\,\nu(dx)\,du.$$

From (2.7) follows that $G^t = \nu_M^t$ is the Lévy measure of an infinitely divisible random variable. Furthermore, it follows from Theorem 3.1(iii), Rosinski (1990) and its proof that $X(t)$ has characteristic function given by

$$E[e^{iuX(t)}] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{iuf_t(s)x} - 1 - iuf_t(s)x]\,\nu(dx)\,ds\right\},$$

i.e. $X(t) \stackrel{d}{=} M_d(t)$. Finally, repeating the same arguments for $\sum_{j=1}^m w_j H_{t_j}(\tau,u)$, where $m \in \mathbb{N}$, $t_1,\ldots,t_m \in \mathbb{R}$ and $w_1,\ldots,w_m \in \mathbb{R}$, we obtain that the finite dimensional distributions of X are identical to those of M_d.

□

The series representation (2.13) can be used for a simulation of FLPs. Of course, for practical simulations the series must be truncated. However, simulation from it is not so easy since the inverse of the tail mass of the Lévy measure is rarely known in closed form.

Recently an alternative generalized shot noise representation for fractional fields was developed by Cohen et al. (2005): Assume that ν is symmetric and $\nu(\mathbb{R}) < \infty$. Let $\{V_i\}_{1,2\ldots}$ be a sequence of random variables such that $\mathcal{L}(V_i) = \nu(dx)/\nu(\mathbb{R})$. Moreover, define $\{\tau_i\}_{i=1,2\ldots}$ and $\{\varepsilon_i\}_{i=1,2\ldots}$ as in Theorem 2.13. Then for every $t \in \mathbb{R}$, the series

$$Y(t) = \sum_{i=1}^{\infty} f_t\left(\frac{\tau_i \varepsilon_i}{\nu(\mathbb{R})}\right) V_i \tag{2.15}$$

converges a.s. and

$$\{M_d(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{Y(t)\}_{t\in\mathbb{R}}\,.$$

Cohen et al. (2005) also give a rate of convergence.

If ν is symmetric with $\nu(\mathbb{R}) = \infty$, we define $\nu^{\epsilon,1}(dx) = \nu(dx)1_{\{|x|\le 1\}}$ and $\nu^{\epsilon,2}(dx) = \nu(dx)1_{\{|x|>1\}}$. Notice that $\nu^{\epsilon,2}(\mathbb{R}) < \infty$ and $M_d = M_d^{\epsilon,1} + M_d^{\epsilon,2}$. Let $\sigma(\epsilon) = \left(\int_{|x|\le\epsilon} x^2\, \nu(dx)\right)^{1/2}$. Then for all $t \in \mathbb{R}$, if $\lim_{\epsilon\to 0+} \sigma(\epsilon)/\epsilon = \infty$,

$$M_d(t) \stackrel{d}{=} \sigma(\epsilon) B_d(t) + \sum_{i=1}^{\infty} f_t\left(\frac{\tau_i \varepsilon_i}{\nu^{\epsilon,2}(\mathbb{R})}\right) V_i, \tag{2.16}$$

where $\{B_d(t)\}_{t\in\mathbb{R}}$ is a FBM as given in (1.54) (see Cohen et al. (2005) for a proof).

2.2 Second Order and Sample Path Properties

Having defined FLPs we want to investigate their second-order and sample path properties.

We first consider the second-order properties. Therefore we assume $\alpha = 2$. Hence, $0 < d < 0.5$. Then representation (2.8) of a FLP in the L^2-sense gives us a direct way to calculate the second-order properties. It turns out that up to a constant FLPs have the same second-order structure as FBM.

Theorem 2.15 (Autocovariance Function) *For $s, t \in \mathbb{R}$ the autocovariance function of a FLP $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ is given by*

$$\mathrm{cov}(M_d(t), M_d(s)) = \frac{E[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}\left[|t|^{2d+1} - |t-s|^{2d+1} + |s|^{2d+1}\right]. \tag{2.17}$$

Proof. Notice that $M_d(0) = 0$ a.s. and $E[M_d(t)] = 0$ for all $t \in \mathbb{R}$, since $E[L(1)] = 0$. For every $t > 0$ we have from (2.9),

$$\begin{aligned}
E[M_d(t)]^2 &= \frac{E[L(1)^2]}{\Gamma(d+1)^2}\int_{-\infty}^{\infty}\left[(t-s)_+^d - (-s)_+^d\right]^2 ds \\
&= \frac{E[L(1)^2]}{\Gamma(d+1)^2}t^{2d+1}\int_{-\infty}^{\infty}\left[(1-u)_+^d - (-u)_+^d\right]^2 du \\
&= \frac{E[L(1)^2]}{\Gamma(d+1)^2}t^{2d+1}\left[\int_{-\infty}^{0}\left[(1-u)^d - (-u)^d\right]^2 du + \int_0^1 (1-u)^{2d}\,du\right] \\
&= \frac{E[L(1)^2]}{\Gamma(d+1)^2}t^{2d+1}\left[\int_0^{\infty}\left[(1+u)^d - u^d\right]^2 du + \frac{1}{2d+1}\right] \\
&= \frac{E[L(1)^2]}{\Gamma(d+1)^2}t^{2d+1}\frac{\Gamma(d+1)^2}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} \\
&= \frac{E[L(1)^2]}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}t^{2d+1}.
\end{aligned}$$

Further for any $s, t \in \mathbb{R}$,

$$\begin{aligned}
E[M_d(t) - M_d(s)]^2 &= \frac{E[L(1)^2]}{\Gamma(1+d)^2}\int_{-\infty}^{\infty}\left[(t-u)_+^d - (s-u)_+^d\right]^2 du \qquad (2.18)\\
&= \frac{E[L(1)^2]}{\Gamma(1+d)^2}\int_{-\infty}^{\infty}\left[(t-s-u)_+^d - (-u)_+^d\right]^2 du \\
&= \frac{E[L(1)^2]}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}|t-s|^{2d+1}.
\end{aligned}$$

Hence, for any $s, t \in \mathbb{R}$,

$$\begin{aligned}
E\left[M_d(t)M_d(s)\right] &= \frac{1}{2}\left\{E[M_d(t)]^2 + E[M_d(s)]^2 - E[M_d(t) - M_d(s)]^2\right\} \\
&= \frac{E[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}\left\{|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}\right\}.
\end{aligned}$$

□

As already mentioned, in this thesis we are particularly interested in processes showing long memory behaviour. In order to study the memory properties of a FLP we consider the covariance between two increments.

Theorem 2.16 (Covariance between two Increments) *Let $h > 0$ and the FLP M_d given as in (2.8). The covariance between two increments $M_d(t+h) - M_d(t)$ and $M_d(s+h) - M_d(s)$, where $s + h \leq t$ and $t - s = nh$ is*

$$\begin{aligned}\delta_d(n) &= \frac{E[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} h^{2d+1}\left[(n+1)^{2d+1} + (n-1)^{2d+1} - 2n^{2d+1}\right] \\ &= \frac{E[L(1)^2]d(2d+1)}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} h^{2d+1} n^{2d-1} + O(n^{2d-2}), \quad n \to \infty. \end{aligned} \tag{2.19}$$

Proof. We use Theorem 2.15 and the stationarity of the increments of M_d (see Theorem 2.21 below),

$$\begin{aligned}\delta_d(n) &= \operatorname{cov}(M_d(t+h) - M_d(t), M_d(s+h) - M_d(s)) \\ &= \operatorname{cov}(M_d(nh+h), M_d(h)) - \operatorname{cov}(M_d(nh), M_d(h)) \\ &= \frac{E[L(1)^2]}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} h^{2d+1}\left[(n+1)^{2d+1} + (n-1)^{2d+1} - 2n^{2d+1}\right].\end{aligned}$$

Applying a binomial expansion we have for $n \to \infty$,

$$\begin{aligned}(n+1)^{2d+1} &= n^{2d+1} + (2d+1)n^{2d} + 2d(2d+1)n^{2d-1} + O(n^{2d-2}), \\ (n-1)^{2d+1} &= n^{2d+1} - (2d+1)n^{2d} + 2d(2d+1)n^{2d-1} + O(n^{2d-2}).\end{aligned}$$

Therefore, as $n \to \infty$,

$$\delta_d(n) = \frac{E[L(1)^2]d(2d+1)}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} h^{2d+1} n^{2d-1} + O(n^{2d-2}).$$

□

Corollary 2.17 *Under the same assumptions as in Theorem 2.16,*

$$\delta_d(n) \to 0 \quad \text{as} \quad n \to \infty.$$

As a direct consequence of (2.19) we have $\delta_d(n) > 0$,

$$\sum_{n=1}^{\infty} \delta_d(n) = \infty,$$

*and the increments of a FLP exhibit **long memory** in the sense of Definition 1.16.*

It is this long memory property allowing us in Section 2.4 to construct long memory moving average processes without a fractional integration of the kernel.

Remark 2.18 Note that for a martingale X with zero expectation the covariance function must be identical zero, since

$$\begin{aligned}&\operatorname{cov}(X(h)-X(h-1),X(h+n)-X(h+n-1))\\&=E[(X(h)-X(h-1))E[X(h+n)-X(h+n-1)\,|\,\mathcal{F}_{h+n-1}]]=0.\end{aligned}$$

This shows that M_d cannot be a martingale. We will prove later that for a broad class of Lévy processes, M_d is not a semimartingale either.

Before, we derive important sample path properties of FLPs.

Theorem 2.19 (Hölder Continuity) *Let $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ be a square-integrable FLP, i.e. the driving Lévy process L satisfies $E[L(1)] = 0$ and $E[L(1)^2] < \infty$. Then for every $\beta < d$ there exists a continuous modification of M_d and there exist an a.s. positive random variable H_ϵ and a constant $\delta > 0$ such that*

$$P\left[\omega\in\Omega:\sup_{0<h<H_\epsilon(\omega)}\left(\frac{M_d(t+h,\omega)-M_d(t,\omega)}{h^\beta}\right)\le\delta\right]=1.$$

This means that the sample paths of FLPs are a.s. locally Hölder continuous of any order $\beta < d$. Moreover, for every modification of M_d and for every $\beta > d$, $P(\{\omega\in\Omega:\ M_d(\,\cdot\,,\omega)\notin C^\beta[a,b]\}) > 0$, where $C^\beta[a,b]$ is the space of Hölder continuous functions on $[a,b]$. Furthermore, if $\nu(\mathbb{R}) = \infty$ then $P(\{\omega\in\Omega:\ M_d(\,\cdot\,,\omega)\notin C^\beta[a,b]\}) = 1$.

Proof. The first assertion follows directly from (2.18) and an application of the Kolmogorov-Centsov Theorem (see e.g. Loève (1960), p.519). Furthermore, from Proposition 2.6 we know that $t\mapsto(t-s)_+^d-(-s)_+^d\notin C^\beta[a,b]$ for every $\beta > d$. Therefore, the proof of the second part is analogous to the proof of Proposition 3.3. in Benassi et al. (2004). □

If $\alpha < 2$, we have for the linear fractional stable motion (see Example 2.4) the following sample path behaviour (Samorodnitsky & Taqqu (1994, Example 12.2.3)).

Proposition 2.20 *Let* $1 < \alpha < 2$, $0 < d < 1 - \frac{1}{\alpha}$ *and* $L_\alpha = \{L_\alpha(t)\}_{t\in\mathbb{R}}$ *be* α*-stable. Then the linear fractional stable motion (LFSM)*

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-s)_+^d - (-s)_+^d\right] L_\alpha(ds)$$

has a.s. continuous sample paths.

Proposition 2.21 (Stationary Increments) *Let* $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ *be a FLP with driving Lévy process* L *satisfying* $E[L(1)] = 0$ *and* $E[L(1)^\alpha] < \infty$ *for some* $1 < \alpha \leq 2$. *Then*
(i) M_d *is a process with stationary increments.*
(ii) M_d *is symmetric, i.e.* $\{M_d(-t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{-M_d(t)\}_{t\in\mathbb{R}}$.

Proof. (i) For any $s, t \in \mathbb{R}$, $s < t$ we have

$$M_d(t) - M_d(s) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-u)_+^d - (s-u)_+^d\right] L(du)$$

$$\stackrel{d}{=} \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-s-v)_+^d - (-v)_+^d\right] L(dv) = M_d(t-s),$$

where equality in distribution follows from the stationarity of the increments of L.
(ii) $M_d(-t) = M_d(-t) - M_d(0) \stackrel{d}{=} M_d(0) - M_d(t) = -M_d(t)$. □

Recall that fractional Brownian motion is the only Gaussian stochastic process which is self-similar with stationary increments. As self-similar processes are invariant in distribution under judicious scaling of time and space (see Definition 1.28), they are of great interest in modeling in e.g. turbulence, economics and physics. For FLPs we obtain the following result.

Theorem 2.22 (Self-Similarity) *(i) A square-integrable FLP* M_d *(i.e.* $\alpha = 2$*) cannot be self-similar.*
(ii) If the driving Lévy process L *of* M_d *is* $\alpha -$ *stable with index* $1 < \alpha < 2$, *then* M_d *is self-similar with index* $H := d + \frac{1}{\alpha} \in (0.5, 1)$.

Proof. (i) Assume that M_d is self-similar with index H. Then we have for all $c > 0$,

$$\{M_d(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} c^H\{M_d(t)\}_{t\in\mathbb{R}}. \tag{2.20}$$

The generating triplet of $M_d(t)$ is $(\gamma_M^t, 0, \nu_M^t)$ (see (2.7)). Observe that,

$$\begin{aligned} E[e^{iuc^{-H}M_d(ct)}] &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{ic^{-H}uxf_{ct}(s)} - 1 - ic^{-H}uxf_{ct}(s)]\,\nu(dx)\,ds\right\} \\ &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{ic^{d-H}uxf_t(s)} - 1 - ic^{d-H}uxf_t(s)]\,c\,\nu(dx)\,ds\right\} \\ &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{iuyf_t(s)} - 1 - iuyf_t(s)]\,c\,\nu(c^{H-d}dy)\,ds.\right\} \end{aligned} \tag{2.21}$$

Define for $r > 0$ the transformation T_r of measures ν on $\mathbb{R}$ by $(T_r\nu)(B) = \nu(r^{-1}B)$, $B \in \mathcal{B}(\mathbb{R})$. Then the Lévy measure of $c^{-H}M_d(ct)$ is given by $c(T_b\nu_M^t)$ with $b = c^{d-H}$. Therefore, if M_d is self-similar, by the uniqueness of the generating triplet

$$\nu_M^t = b^{-1/(H-d)}(T_b\nu_M^t), \qquad \text{for all } b > 0.$$

Then by Sato (1999, Theorem 14.3 (ii)) and its proof it follows that $\frac{1}{H-d} < 2$ and that ν_M^t is the Lévy measure of an α-stable process with $\alpha = 1/(H-d)$. Hence, $E[M_d(t)^2] = \infty$, contradicting the square integrability of M_d.
(ii) Now, suppose that L is α-stable. Then L is self-similar with index $\frac{1}{\alpha}$. and $\nu = b^{-1/\alpha}(T_b\nu)$ for all $b > 0$ (Sato (1999, Theorem 14.3)), i.e. setting $c = b^{-1/\alpha}$, we obtain that $c\nu(c^{1/\alpha}dx) = \nu(dx)$. Then for $1 < \alpha < 2$ such that $H - d = \frac{1}{\alpha}$,

$$\begin{aligned} (2.21) &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{iuyf_t(s)} - 1 - iuyf_t(s)]\,c\,\nu(c^{1/\alpha}dy)\,ds.\right\} \\ &\stackrel{d}{=} \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} [e^{iuyf_t(s)} - 1 - iuyf_t(s)]\,\nu(dy)\,ds.\right\} = E[e^{iuM_d(t)}], \end{aligned}$$

which shows $c^{-H}M_d(ct) \stackrel{d}{=} M_d(t)$. Repeating the same arguments for $\sum_{j=1}^m u_j c^{-H} M_d(ct_j)$, where $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}$ and $u_1, \dots, u_m \in \mathbb{R}$, we obtain (2.20) for $H = d + \frac{1}{\alpha}$. □

Though square-integrable FLPs cannot be self-similar, we have the following result concerning asymptotic self-similarity (see Benassi et al. (2004, Proposition 3.1) for a proof).

Proposition 2.23 *FLPs are asymptotically self-similar with parameter* $0 < d < 0.5$*, i.e.*

$$d - \lim_{c \to \infty} \left\{ \frac{M_d(ct)}{c^d} \right\}_{t \in \mathbb{R}} \stackrel{d}{=} \{B_d(t)\}_{t \in \mathbb{R}},$$

where the limit is the distribution of a fractional Brownian motion with parameter d*,* $0 < d < 0.5$*.*

The next theorem on the local self-similarity of FLPs is crucial for our further investigations.

Theorem 2.24 *Let* M_d *be a (not necessarily square-integrable) FLP. Define for* $1 < \alpha < 2$ *and* $0 < d < 1 - \frac{1}{\alpha}$ *the parameter* $\tilde{H}$ *by* $\tilde{H} = d + \frac{1}{\alpha}$*, i.e.* $0.5 < \tilde{H} < 1$*. Assume that* $\nu(dx) = g(x)\,dx$*, where* $g : \mathbb{R} \to \mathbb{R}_+$ *is measurable and satisfies*

$$g(x) \sim |x|^{-1-\alpha}, \quad x \to 0$$

and

$$g(x) \le C|x|^{-1-\alpha} \quad \text{for all } x \in \mathbb{R},$$

with a constant $C > 0$*.*

Then M_d *is locally self-similar with parameter* $\tilde{H}$*, i.e. for every fixed* $t \in \mathbb{R}$*,*

$$d - \lim_{\epsilon \downarrow 0} \left\{ \frac{M_d(t + \epsilon x) - M_d(t)}{\epsilon^{\tilde{H}}} \right\}_{x \in \mathbb{R}} \stackrel{d}{=} \{Y_{\tilde{H}}(x)\}_{x \in \mathbb{R}}. \qquad (2.22)$$

Here $Y_{\tilde{H}}$ *is a linear fractional stable motion with representation*

$$Y_{\tilde{H}}(t) = \frac{1}{\Gamma(\tilde{H} - \frac{1}{\alpha} + 1)} \int_{\mathbb{R}} [(t-s)_+^{\tilde{H}-\frac{1}{\alpha}} - (-s)_+^{\tilde{H}-\frac{1}{\alpha}}]\, L_\alpha(ds),$$

where L_α *is a symmetric* α*-stable Lévy process (see e.g. Samorodnitsky & Taqqu (1994) and Example 2.4).*

Proof. Since M_d has stationary increments it is enough to show the convergence for $t = 0$. For $u_1, \ldots, u_n \in \mathbb{R}$, $-\infty < t_1 < \ldots < t_n < \infty$ and $n \in \mathbb{N}$, we have by (2.6)

$$\begin{aligned}
&\log E\left[\exp\left\{i\sum_{k=1}^n u_k \frac{M_d(\epsilon t_k)}{\epsilon^{\tilde{H}}}\right\}\right] \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\exp\left\{ix\sum_{k=1}^n u_k \frac{f_{\epsilon t_k}(s)}{\epsilon^{\tilde{H}}}\right\} - 1 - ix\sum_{k=1}^n u_k \frac{f_{\epsilon t_k}(s)}{\epsilon^{\tilde{H}}}\right] \nu(dx)\, ds \\
&\overset{\epsilon v = s}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\exp\left\{ix\epsilon^{d-\tilde{H}}\sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - ix\epsilon^{d-\tilde{H}}\sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon\, \nu(dx)\, dv \\
&\overset{\epsilon^{d-\tilde{H}}x = y}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - iy\sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon\, \nu(\epsilon^{\tilde{H}-d}dy)\, dv \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - iy\sum_{k=1}^n u_k f_{t_k}(v)\right] \epsilon g(\epsilon^{\tilde{H}-d}y)\epsilon^{\tilde{H}-d} dy\, dv \\
&=: \int_{\mathbb{R}}\int_{\mathbb{R}} G_\epsilon(y,v)\, dy\, dv.
\end{aligned}$$

For any $y \neq 0$ the asymptotic behavior of g yields

$$\epsilon g(\epsilon^{\tilde{H}-d}y)\epsilon^{\tilde{H}-d} \sim \epsilon^{\tilde{H}-d+1}|\epsilon^{\tilde{H}-d}y|^{-1-\alpha} = |y|^{-1-\alpha}, \quad \epsilon \to 0,$$

which is the Lévy measure of a symmetric α-stable Lévy process. This shows that

$$G_\epsilon(y,v) \to G(y,v), \quad \epsilon \to 0 \quad \text{for all } (y,v) \in \mathbb{R}^2, \quad y \neq 0$$

with

$$G(y,v) = \left[\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\} - 1 - iy\sum_{k=1}^n u_k f_{t_k}(v)\right] |y|^{-1-\alpha}.$$

We will show below that there exists $F \in L^1(\mathbb{R}^2)$ with

$$|G_\epsilon| \leq F \quad \text{for all } \epsilon > 0. \tag{2.23}$$

Then by the dominated convergence theorem

$$
\begin{aligned}
&\lim_{\epsilon\downarrow 0}\log E\left[\exp\left\{i\sum_{k=1}^n u_k\frac{M_d(\epsilon t_k)}{\epsilon^{\tilde H}}\right\}\right]\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}\left[\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\}-1-iy\sum_{k=1}^n u_k f_{t_k}(v)\right]|y|^{-1-\alpha}\,dy\,dv\\
&=\int_{\mathbb{R}}\int_0^\infty\left[2\cos\left(y\sum_{k=1}^n u_k f_{t_k}(v)\right)-2\right]|y|^{-1-\alpha}\,dy\,dv\\
&=\int_{\mathbb{R}}\int_0^\infty[2\cos(x)-2]\left|\sum_{k=1}^n u_k f_{t_k}(v)\right|^\alpha\frac{dx}{x^{1+\alpha}}\,dv=C(\alpha)\int_{\mathbb{R}}\left|\sum_{k=1}^n u_k f_{t_k}(v)\right|^\alpha dv,
\end{aligned}
$$

where $C(\alpha)=2\int_0^\infty[\cos(x)-1]\frac{dx}{x^{1+\alpha}}$. Since,

$$
\log E\left[\exp\left\{i\sum_{k=1}^n u_k Y_{\tilde H}(\epsilon t_k)\right\}\right]=C(\alpha)\int_{\mathbb{R}}\left|\sum_{k=1}^n u_k f_{t_k}(v)\right|^\alpha dv
$$

(see Samorodnitsky & Taqqu (1994, p.114)), this yields the assertion. It remains to show (2.23). By the upper bound for g we have

$$
\begin{aligned}
|G_\epsilon(y,v)|&=\left|\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\}-1-iy\sum_{k=1}^n u_k f_{t_k}(v)\right|\epsilon g(\epsilon^{\tilde H-d}y)\epsilon^{\tilde H-d}\\
&\le\left|\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\}-1-iy\sum_{k=1}^n u_k f_{t_k}(v)\right|\epsilon C|\epsilon^{\tilde H-d}y|^{-1-\alpha}\epsilon^{\tilde H-d}\\
&=\left|\exp\left\{iy\sum_{k=1}^n u_k f_{t_k}(v)\right\}-1-iy\sum_{k=1}^n u_k f_{t_k}(v)\right|C|y|^{-1-\alpha}\\
&:=F(y,v).
\end{aligned}
$$

We show finally that $F\in L^1(\mathbb{R}^2)$. Consider the function

$$
h(y,z)=\exp(iyz)-1-iyz.
$$

We have

$$
h(y,z)=iyz\int_0^1(\exp(iyzs)-1)\,ds\;=iyz\int_0^1 iyzs\int_0^1\exp(iyzsw)\,dw\,ds
$$

This yields the estimates

$$|h(y,z)| \leq 2|y||z| \tag{2.24}$$

$$|h(y,z)| \leq \frac{1}{2}|y|^2|z|^2. \tag{2.25}$$

Taking for arbitrary $\beta \in (0,1)$ both sides of the inequality (2.24) to the power β and both sides of (2.25) to the power $1-\beta$ and multiplying both inequalities yields

$$|h(y,z)| \leq \frac{1}{2^{1-\beta}} 2^\beta |y|^{2-\beta}|z|^{2-\beta} \leq 2|y|^{2-\beta}|z|^{2-\beta}. \tag{2.26}$$

Now choose $\beta \in (0,1)$ such that

$$2-\beta = p \in (\frac{1}{1-d}, \frac{1}{\tilde{H}-d}) = (\frac{1}{1-d}, \alpha).$$

Then with (2.25) and (2.26)

$$\begin{aligned} F(y,v) &= \left| \exp\left\{ iy \sum_{k=1}^n u_k f_{t_k}(v) \right\} - 1 - iy \sum_{k=1}^n u_k f_{t_k}(v) \right| C|y|^{-1-\alpha} \\ &\leq 1_{\{|y|\leq 1\}} 2|y|^2 \left| \sum_{k=1}^n u_k f_{t_k}(v) \right|^2 C|y|^{-1-\alpha} \\ &\quad + 1_{\{|y|>1\}} 2|y|^p \left| \sum_{k=1}^n u_k f_{t_k}(v) \right|^p C|y|^{-1-\alpha}. \end{aligned}$$

Since $f_{t_k} \in L^p(\mathbb{R})$ and $f_{t_k} \in L^2(\mathbb{R})$ by Proposition 2.5 we conclude that

$$\begin{aligned} \|F\|_{L^1(\mathbb{R}^2)} &\leq \int_{\mathbb{R}} \int_{|y|>1} 2|y|^p \left| \sum_{k=1}^n u_k f_{t_k}(v) \right|^p C|y|^{-1-\alpha}\, dy\, dv \\ &\quad + \int_{\mathbb{R}} \int_{|y|\leq 1} 2|y|^2 \left| \sum_{k=1}^n u_k f_{t_k}(v) \right|^2 C|y|^{-1-\alpha}\, dy\, dv \\ &= \left\| \sum_{k=1}^n u_k f_{t_k} \right\|_{L^p(\mathbb{R})}^p \int_{|y|>1} 2C|y|^{p-1-\alpha}\, dy \\ &\quad + \left\| \sum_{k=1}^n u_k f_{t_k} \right\|_{L^2(\mathbb{R})}^2 \int_{|y|\leq 1} 2C|y|^{2-1-\alpha}\, dy \end{aligned}$$

$$\leq \left(\sum_{k=1}^{n} |u_k| \|f_{t_k}\|_{L^p(\mathbb{R})} \right)^p 4C \left. \frac{y^{p-\alpha}}{p-\alpha} \right|_1^\infty$$
$$+ \left(\sum_{k=1}^{n} |u_k| \|f_{t_k}\|_{L^2(\mathbb{R})} \right)^2 4C \left. \frac{y^{2-\alpha}}{2-\alpha} \right|_0^1$$
$$\leq \left(\sum_{k=1}^{n} |u_k| \|f_{t_k}\|_{L^p(\mathbb{R})} \right)^p 4C \frac{1}{\alpha - p} + \left(\sum_{k=1}^{n} |u_k| \|f_{t_k}\|_{L^2(\mathbb{R})} \right)^2 4C \frac{1}{2-\alpha}$$
$$< \infty,$$

where we have used that $p < \alpha < 2$. Hence the upper bound F in (2.23) is in $L^1(\mathbb{R}^2)$ and the proof is complete. ☐

In the following let, as in Section 1.4, $Var_{[a,b]}(M_d)$ denote the total variation of the sample paths of M_d on the interval $[a, b] \subset \mathbb{R}$.

Theorem 2.25 (Total Variation) *If ν is given as in Theorem 2.24, the sample paths of M_d are a.s. of infinite total variation on compacts, i.e. $Var_{[a,b]}(M_d) = \infty$ a.s. If $\nu(\mathbb{R}) < \infty$, they are of finite total variation.*

Proof. We know from (2.22) that

$$d - \lim_{h \downarrow 0} \frac{M_d(t \pm h) - M_d(t)}{h^{\tilde{H}}} \stackrel{d}{=} Y_{\tilde{H}}(\pm 1)$$

Thus,

$$d - \lim_{h \downarrow 0} \frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}} \stackrel{d}{=} |Y_{\tilde{H}}(\pm 1)| > 0 \quad a.s. \tag{2.27}$$

As $|Y_{\tilde{H}}(\pm 1)| > 0$ *a.s.*, for all $\Omega' \subset \Omega$ with $P(\Omega') > 0$ it follows

$$\lim_{h \downarrow 0} E \left[1_{\Omega'} \frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}} \right] > 0. \tag{2.28}$$

In fact, let $\Omega' \subset \Omega$ with $P(\Omega') > 0$. Then $\lim_{\delta \downarrow 0} P(|Y_{\tilde{H}}(\pm 1)| \leq \delta) \to 0$. Choose $\delta > 0$ small enough such that δ is a continuity point of the distribution function of $|Y_{\tilde{H}}(\pm 1)|$ and $P(|Y_{\tilde{H}}(\pm 1)| \leq \delta) \leq \frac{P(\Omega')}{4}$. Then by (2.27)

$$\lim_{h \downarrow 0} P \left(\frac{|M_d(t \pm h) - M_d(t)|}{|h|^{\tilde{H}}} \leq \delta \right) = P(|Y_{\tilde{H}}(\pm 1)| \leq \delta) \leq \frac{P(\Omega')}{4}.$$

Hence, there exists $\epsilon_t > 0$ such that

$$P \left(\frac{|M_d(t+h) - M_d(t)|}{|h|^{\tilde{H}}} \leq \delta \right) \leq \frac{P(\Omega')}{2} \quad \text{for all } h \neq 0,\ |h| \leq \epsilon_t.$$

This yields

$$P\left(\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} \leq \delta\right\}\right) \leq \frac{P(\Omega')}{2} \quad \text{for all } h \neq 0,\ |h| \leq \epsilon_t,$$

and hence

$$P\left(\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}\right) \geq \frac{P(\Omega')}{2} \quad \text{for all } h \neq 0,\ |h| \leq \epsilon_t.$$

Therefore,

$$\begin{aligned}
&E\left[1_{\Omega'}\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&= E\left[1_{\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} \leq \delta\right\}}\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&\quad + E\left[1_{\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}}\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}}\right] \\
&\geq 0 + E\left[1_{\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}}\delta\right] = \delta\, P\left(\Omega' \cap \left\{\frac{|M_d(t+h)-M_d(t)|}{|h|^{\tilde{H}}} > \delta\right\}\right) \\
&\geq \frac{P(\Omega')}{2}\delta, \quad \text{for all } h \neq 0,\ |h| \leq \epsilon_t.
\end{aligned}$$

This shows (2.28).

Now, assume that $P(Var_{[a,b]}(M_d) < \infty) > 0$. Then there exist $\Omega' \subset \Omega,\ P(\Omega') > 0$ and $K > 0$ such that $Var_{[a,b]}(M_d) < K$ on Ω'. Hence,

$$E\left[1_{\Omega'}\, Var_{[a,b]}(M_d)\right] \leq K. \tag{2.29}$$

We lead this to a contradiction:

For any sequence $a \leq t_0 < t_1 < \ldots t_n \leq b$, we have

$$\begin{aligned}
E\left[1_{\Omega'} Var_{[a,b]}(M_d)\right] &\geq E\left[1_{\Omega'}\sum_{i=0}^{n}|M_d(t_{i+1})-M_d(t_i)|\right] \\
&= \sum_{i=0}^{n} E\left[1_{\Omega'}|M_d(t_{i+1})-M_d(t_i)|\right].
\end{aligned} \tag{2.30}$$

Fix $[a,b'] \subset [a,b]$, $a < b' < b$. We construct a sequence $a \leq t_0 < t_1 < ... < t_n \leq b' < t_{n+1} < b$ for some n with

$$E\left[1_{\Omega'}|M_d(t_{i+1})-M_d(t_i)|\right] \geq (t_{i+1}-t_i)\frac{2K}{b'-a}, \quad 0 \leq i \leq n. \tag{2.31}$$

Since $\tilde{H} < 1$, (2.28) yields

$$\lim_{h\downarrow 0} E\left[1_{\Omega'}\frac{|M_d(t\pm h)-M_d(t)|}{h}\right] = \lim_{h\downarrow 0} h^{\tilde{H}-1}E\left[1_{\Omega'}\frac{|M_d(t\pm h)-M_d(t)|}{h^{\tilde{H}}}\right] = \infty. \tag{2.32}$$

Thus, for any $t \in [a,b']$, we find $0 < \epsilon_t < b - b'$ with

$$E\left[1_{\Omega'}|M_d(t+h)-M_d(t)|\right] \geq |h|\frac{2K}{b'-a}, \quad \forall\ h, |h| \leq \epsilon_t. \tag{2.33}$$

Now, $(]t-\epsilon_t, t+\epsilon_t[)$ is an open covering of $[a,b']$ and thus we find a finite covering $(]t_{2i}-\epsilon_{t_{2i}}, t_{2i}+\epsilon_{t_{2i}}[)$, $t_0 < t_2 < \ldots < t_{2m}$, $t_{2m}+\epsilon_{t_{2m}} = t_{2m+1} > b'$. Now we choose $t_{2i+1} \in]t_{2i}, t_{2i}+\epsilon_{t_{2i}}[\cap]t_{2i+2}-\epsilon_{t_{2i+2}}, t_{2i+2}[$. Then by (2.33) in fact (2.31) holds for all i, $0 \leq i \leq 2m =: n$. Now summation of (2.33) gives together with (2.30)

$$\begin{aligned}E[1_{\Omega'}\,Var_{[a,b]}(M_d)] &\geq \sum_{i=0}^{n} E[1_{\Omega'}\,|M_d(t_{i+1})-M_d(t_i)|] \geq \sum_{i=0}^{n}|t_{i+1}-t_i|\frac{2K}{b'-a}\\ &= (t_{n+1}-t_0)\frac{2K}{b'-a} \geq 2K.\end{aligned}$$

This is a contradiction to (2.29). Consequently, $Var_{[a,b]}(M_d) = \infty$ a.s.

It remains to show $Var_{[a,b]}(M_d) < \infty$, if $\nu(\mathbb{R}) < \infty$. The proof is based on the series representation of FLPs. For simplicity we assume that the Lévy measure ν of the driving Lévy process L is symmetric. Now, consider the series representation (2.12). Since $\nu(\mathbb{R}) < \infty$, there is only a finite number $n \in \mathbb{N}$ of jumps τ_i on every interval $[a,b]$. Now, we divide the interval $[a,b]$ into subintervals $]\tau_{i-1},\tau_i[$, $i = 1,\ldots,n$. Let $(t_k^m)_{k=1,\ldots,m}$ be a refining sequence on the interval $]\tau_{i-1},\tau_i[$, i.e. $\max_k |t_k^m - t_{k-1}^m| \to 0$ as $m \to \infty$. Then

$$\begin{aligned}Var_{[\tau_{i-1},\tau_i]}(M_d) &= p-\lim_{m\to\infty}\sum_{k=1}^{m}|M_d(t_k^m)-M_d(t_{k-1}^m)|\\ &= p-\lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{j=1}^{\infty}\varepsilon_j\nu^{\leftarrow}(\tau_j\rho(U_j))f_{t_k^m}(U_j)-\sum_{j=1}^{\infty}\varepsilon_j\nu^{\leftarrow}(\tau_j\rho(U_j))f_{t_{k-1}^m}(U_j)\right|\\ &= p-\lim_{m\to\infty}\sum_{k=1}^{m}\left|\sum_{j=1}^{\infty}\varepsilon_j\nu^{\leftarrow}(\tau_j\rho(U_j))[f_{t_k^m}(U_j)-f_{t_{k-1}^m}(U_j)]\right|\end{aligned}$$

$$= p - \lim_{m\to\infty} \sum_{k=1}^{m} \left| \sum_{j=1}^{\infty} \varepsilon_j \nu^{\leftarrow}(\tau_j \rho(U_j)) \frac{1}{\Gamma(d+1)} [(t_k^m - U_j)_+^d - (t_{k-1}^m - U_j)_+^d] \right|$$

$$\leq \sum_{j=1}^{\infty} |\nu^{\leftarrow}(\tau_j \rho(U_j))| \frac{1}{\Gamma(d+1)}\, p - \lim_{m\to\infty} \sum_{k=1}^{m} \left|(t_k^m - U_j)_+^d - (t_{k-1}^m - U_j)_+^d\right|$$

$$\leq \sum_{j=1}^{\infty} |\nu^{\leftarrow}(\tau_j \rho(U_j))| Var_{[\tau_{i-1},\tau_i]}(f.(U_j)).$$

Since the total variation of the function $t \mapsto (t-s)_+^d - (-s)_+^d$ is finite on every interval $[\tau_{i-1}, \tau_i]$ (see Proposition 2.6) and since there are only finitely many τ_i, we can conclude that the sample paths of M_d have finite variation on compacts.

If ν is not symmetric the proof uses the spectral representation (2.14) and the same arguments. □

Remark 2.26 Observe that as a consequence of Theorem 2.25, M_d is a semimartingale if $\nu(\mathbb{R}) < \infty$.

Theorem 2.27 (Semimartingale I) *Let M_d be a FLP. If the Lévy measure ν of the driving Lévy process is given as in Theorem 2.24, the fractional Lévy process M_d is not a semimartingale.*

Proof. Let $0 = t_0^n < \ldots < t_n^n = t$, $n \in \mathbb{N}$ be a partition of $[0,t]$ such that $\max_{0\leq i\leq n} |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. Assume that M_d is a semimartingale. Then its quadratic variation

$$[M_d, M_d]_t = p - \lim_{n\to\infty} \sum_{i=0}^{n-1} |M_d(t_{i+1}^n) - M_d(t_i^n)|^2$$

exists for all $t \in [0,T]$, $T > 0$. Hence, there exists a refining subsequence $\{t_i^{n_k}\}$ such that as $k \to \infty$,

$$\sum_{i=0}^{n_k-1} |M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})|^2 \to [M_d, M_d]_t \quad a.s.$$

Therefore, we can apply Fatou's Lemma to obtain together with Theorem 2.15,

$$E[M_d, M_d]_t = E\left[\lim_{k\to\infty} \sum_{i=0}^{n_k-1} [M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2\right] \tag{2.34}$$

$$\leq \liminf_{k\to\infty} E\left[\sum_{i=0}^{n_k-1} [M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2\right] \tag{2.35}$$

$$= \liminf_{k\to\infty} \sum_{i=0}^{n_k-1} E[M_d(t_{i+1}^{n_k}) - M_d(t_i^{n_k})]^2$$

$$= \frac{E[L(1)^2]}{\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])} \liminf_{k\to\infty} \sum_{i=0}^{n_k-1} |t_{i+1}^{n_k} - t_i^{n_k}|^{2d+1} = 0.$$

It follows from $M_d(0) = 0$ a.s., (2.34) and Protter (2004, Theorem II.22 (ii)) that $[M_d, M_d]_t = 0$ a.s. for all $t \in [0,T]$, $T > 0$. As M_d is a continuous semimartingale it is of the form

$$M_d(t) = M_d(0) + A(t) + B(t),$$

where $M_d(0)$ is an $\mathcal{F}_0$-measurable random variable, $A(0) = B(0) = 0$, B is an a.s. continuous local martingale with respect to $\mathcal{F}$ and A is an a.s. right-continuous, $\mathcal{F}$-adapted finite variation process. It follows

$$0 = [M_d, M_d]_t = [A, A]_t = [A, B]_t = [B, A]_t = [B, B]_t \ a.s. \text{ for all } t \in [0,T].$$

Therefore, as M_d and hence B has continuous sample paths, Protter (2004, Theorem II.27) implies $B(t) = 0$ a.s. for all $t \in [0,T]$, $T > 0$. Hence, M_d is a finite variation process. This contradicts Theorem 2.25, if ν is of the form given in Theorem 2.24. □

Remark 2.28 Let the driving Lévy process L be a symmetric α-stable Lévy process with $1 < \alpha < 2$. Then the Lévy measure ν is of the form given in Theorem 2.24. Consequently, Theorem 2.27 yields that the corresponding FLP is not a semimartingale. This shows that linear fractional stable motions (see Example 2.4) cannot be semimartingales.

We conclude this section with a result concerning the lower bound for the L^1-norm of $M_d(t)$ and which will lead to a further class of Lévy processes for which the corresponding FLP cannot be a semimartingale

Proposition 2.29 *Let M_d be a FLP. Suppose the Lévy measure ν of the driving Lévy process L satisfies*

$$\int_{|x|\geq\epsilon} |x|\nu(dx) \geq C\,\epsilon^{-\beta} \tag{2.36}$$

for some $\beta \geq 1$ *and a constant* $C > 0$*. Then there exists* $0 < \delta < \frac{\beta - d(1+\beta)}{1+\beta}$*, and* $\bar{t} > 0$*, such that*

$$E|M_d(t)| \geq C t^{1-\delta}, \quad \textit{for all } \ 0 \leq t \leq \bar{t}. \tag{2.37}$$

Proof. We apply the results of Marcus & Rosinski (2001): Let l_t be a solution of the equation

$$\xi(l) := \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{|f_t(s)||x|}{l} \wedge \frac{|f_t(s)|^2 |x|^2}{l^2} \right) \nu(dx)\, ds = 1.$$

Then

$$0.25\, l_t \leq E|M_d(t)| \leq 2.125\, l_t \tag{2.38}$$

(Marcus & Rosinski (2001, Theorem 1.1)). We observe that $\xi(l_t)$ is monotone decreasing in l_t. Therefore we show that $\xi(t^{1-\delta}) > 1$ for some $0 < \delta < 1$, since then $l_t \geq t^{1-\delta}$. We have

$$\begin{aligned} \{(x,s):\ |f_t(s)||x| \geq l\} &\supset \{(x,s):\ s \in [0,t/2],\ (t-s)^d |x| \geq l\} \\ &\supset \{(x,s):\ s \in [0,t/2],\ (t/2)^d |x| \geq l\} \end{aligned}$$

Therefore,

$$\begin{aligned} \xi(l) &\geq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f_t(s)||x|}{l} 1_{\{|f_t(s)||x| \geq l\}}\, \nu(dx)\, ds \\ &\geq \int_0^{t/2} \int_{|x| \geq l(t/2)^{-d}} |x| \frac{(t-s)^d}{l}\, \nu(dx)\, ds \\ &= \int_{|x| \geq l(t/2)^{-d}} |x|\, \nu(dx) \ \left. \frac{(t-s)^{d+1}}{(d+1)l} \right|_{t/2}^{0} \\ &= \int_{|x| \geq l(t/2)^{-d}} |x|\, \nu(dx) \frac{t^{d+1} - (t/2)^{d+1}}{(d+1)l} \\ &= \int_{|x| \geq l(t/2)^{-d}} |x|\, \nu(dx) \frac{1 - 1/(2^{d+1})}{(d+1)l} t^{d+1}. \end{aligned}$$

Now it follows from (2.36) that

$$\xi(l) \geq \frac{1}{2(d+1)} \frac{C}{\left(\frac{l 2^d}{t^d}\right)^{\beta}} \frac{t^{d+1}}{l}.$$

Let $l = t^{1-\delta}$. Then

$$\begin{aligned}\xi(t^{1-\delta}) &\geq \frac{1}{2(d+1)}\frac{C}{2^{d\beta}}\frac{t^{d+\delta}}{t^{(1-\delta-d)\beta}}\\ &= \frac{1}{2(d+1)}\frac{C}{2^{d\beta}}t^{d(1+\beta)+\delta(1+\beta)-\beta}.\end{aligned}$$

Hence, for $0 < \delta < \frac{\beta-d(1+\beta)}{1+\beta}$,

$$\lim_{t\downarrow 0}\xi(t^{1-\delta}) = \infty.$$

This shows $l_t \geq t^{1-\delta}$ for $t \leq \bar{t}$, $\bar{t} > 0$ small and (2.38) yields the assertion. □

Theorem 2.30 *Let M_d be a FLP. Suppose the Lévy measure ν of the driving Lévy process L satisfies (2.36). Then the sample paths of M_d are a.s. of infinite total variation on compacts.*

Proof. Define

$$X_j = M_d((j+1)t) - M_d(jt).$$

Since M_d has stationary increments, the sequence $\{X_j\}$ is stationary and ergodic. Thus Proposition 2.29 and the ergodic theorem imply that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}|X_j| = E|X_1| \geq Ct^{1-\delta} \tag{2.39}$$

a.s and in $L^1(\Omega, P)$. We observe that

$$\frac{1}{n}\sum_{j=0}^{n-1}|X_j| \leq \frac{Var_{[0,nt]}(M_d)}{nt}t.$$

This together with (2.39) shows that for every $t > 0$, $\epsilon > 0$ there exists $n_{\epsilon,t}$ such that

$$P\left(\frac{Var_{[0,n_{\epsilon,t}t]}(M_d)}{n_{\epsilon,t}t}t \geq \frac{C}{2}t^{1-\delta}\right) \geq 1-\epsilon,$$

i.e.

$$P\left(\frac{Var_{[0,n_{\epsilon,t}t]}(M_d)}{n_{\epsilon,t}t} \geq \frac{C}{2}t^{-\delta}\right) \geq 1-\epsilon.$$

Now, let $t \downarrow 0$. Then for every $\epsilon > 0$ and $M > 0$ there exists $T > 0$ such that

$$P\left(\frac{Var_{[0,T]}(M_d)}{T} \geq M\right) \geq 1-\epsilon.$$

Consequently, for every $\epsilon > 0$, $M > 0$, there exists $s \geq 0$ such that

$$P\left(Var_{[s,s+1]}(M_d) \geq M\right) \geq 1 - \epsilon. \tag{2.40}$$

Furthermore, we have by monotone convergence

$$\begin{aligned} E[Var_{[t,t+1]}(M_d)] &= \lim_{n\to\infty} E\left[\sum_{i=1}^{2^n} |M_d(j2^{-n}) - M_d((j-1)2^{-n})|\right] \\ &= \lim_{n\to\infty} \sum_{i=1}^{2^n} E|M_d(j2^{-n}) - M_d((j-1)2^{-n})| \\ &\geq \lim_{n\to\infty} C \sum_{i=1}^{2^n} (2^{-n})^{1-\delta} \to \infty. \end{aligned}$$

Assume $P(Var_{[t,t+1]}(M_d) < \infty) > 0$. Then there exists $M > 0$ and $\Omega' \subset \Omega$ with $P(\Omega') =: 2\epsilon > 0$ such that

$$Var_{[t,t+1]}(M_d)\big|_{\Omega'} \leq \frac{M}{2},$$

i.e. $P(Var_{[t,t+1]}(M_d) \leq \frac{M}{2}) \geq P(\Omega') = 2\epsilon$, and hence,

$$P(Var_{[t,t+1]}(M_d) > \frac{M}{2}) \leq 1 - 2\epsilon. \tag{2.41}$$

Since M_d has stationary increments we have $Var_{[t,t+1]}(M_d) \stackrel{d}{=} Var_{[s,s+1]}(M_d)$. In fact, let $t = t_0 < t_1 < \ldots < t_n = t+1$ be a partition of the interval $[t, t+1]$. Then

$$\begin{aligned} \sum_{i=1}^{n} |M_d(t_i) - M_d(t_{i-1})| &\stackrel{d}{=} \sum_{i=1}^{n} |M_d(t_i - (t-s)) - M_d(t_{i-1} - (t-s))| \\ &= \sum_{i=1}^{n} |M_d(s_i) - M_d(s_{i-1})|, \end{aligned}$$

where $s_i = t_i - (t-s)$. This shows $Var_{[t,t+1]}(M_d) \stackrel{d}{=} Var_{[s,s+1]}(M_d)$, since all possible partitions of the interval $[t, t+1]$ are the same as those of $[s, s+1]$.

Now, it follows from (2.40) that there exists $s \geq 0$ with

$$P(Var_{[s,s+1]}(M_d) \geq M) \geq 1 - \epsilon, \quad i.e.$$

$$P(Var_{[t,t+1]}(M_d) \geq M) = P(Var_{[s,s+1]}(M_d) \geq M) \geq 1 - \epsilon$$

contradicting (2.41). The proof is complete.

□

Corollary 2.31 (Semimartingale II) *Let M_d be a FLP. If the Lévy measure ν of the driving Lévy process satisfies (2.36), then M_d is not a semimartingale.*

Proof. The proof is analogous to the proof of Theorem 2.27. □

Figure 2.3 and Figure 2.4 show sample paths of FLPs, where the driving Lévy process has a truncated stable Lévy measure

$$\nu(dx) = \frac{1_{\{|x|\leq 1\}}}{|x|^{1+\alpha}} dx, \tag{2.42}$$

with $1 < \alpha < 2$. Observe that (2.42) satisfies the assumptions of Theorem 2.24. Hence, the resulting FLP is not a semimartingale

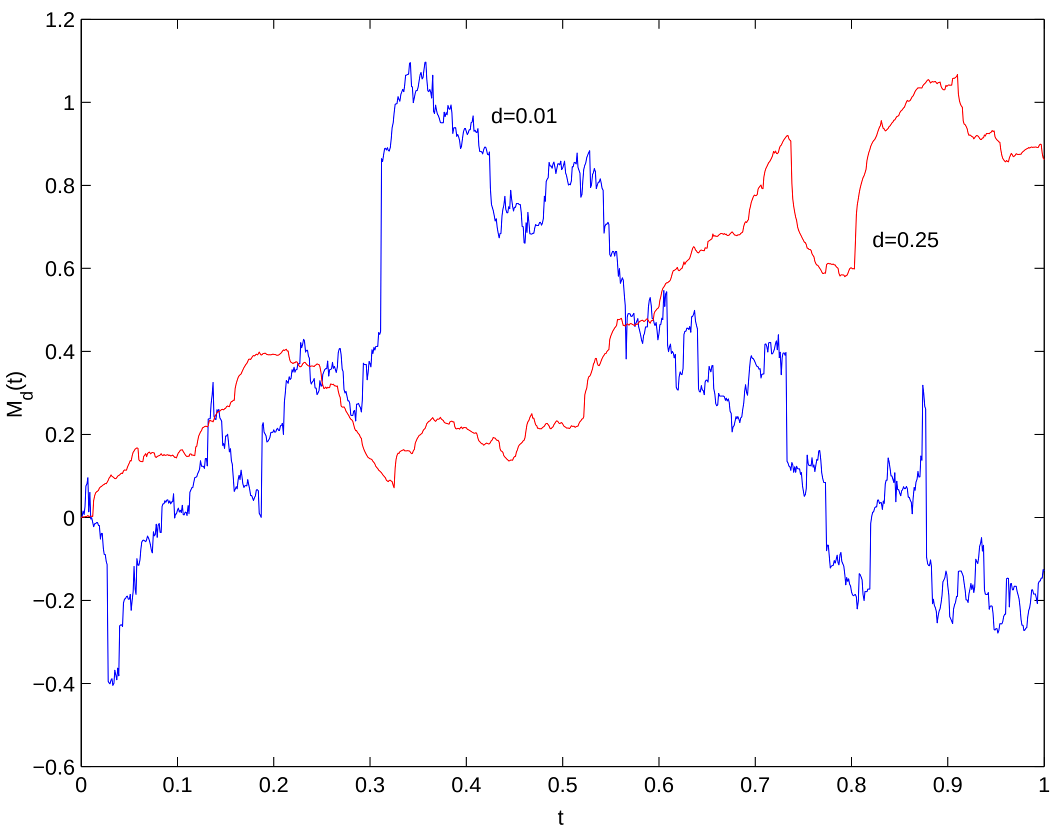

Figure 2.3: The sample paths of a FLP for different values of d, where the driving Lévy process has Lévy measure (2.42) with $\alpha = 1.5$.

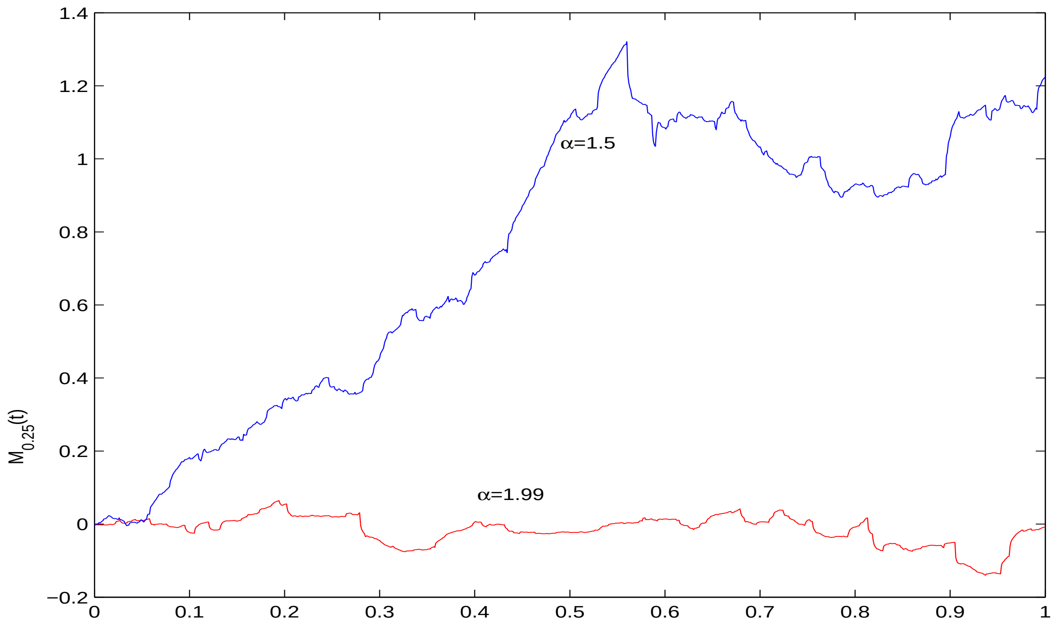

Figure 2.4: The sample paths of a FLP for $d = 0.25$, where the driving Lévy process has Lévy measure (2.42) with $\alpha = 1.5$ and $\alpha = 1.99$.

2.3 Integrals with respect to Fractional Lévy Processes

In the present section we define integrals with respect to fractional Lévy processes. As pointed out in Theorem 2.27 and Corollary 2.31 a FLP is not always a semimartingale. Therefore, classical Itô integration theory cannot be applied. Recently, integration with respect to FBMs has been studied extensively and various approaches have been made to define a stochastic integral with respect to FBM (see Nualart (2003) for a survey). For instance Zähle (1998) introduced a pathwise stochastic integral using fractional integrals and derivatives. If the integrand is β-Hölder continuous with $\beta > 1 - H$, then the integral with respect to FBM can be interpreted as a Riemann-Stieltjes integral. Other approaches use the Gaussianity and define a Wiener integral or they apply Malliavin calculus to obtain Skorohod-like integrals with respect to FBM (see e.g. Decreusefond & Üstünel (1999) and the references therein). Malliavin calculus was also used by Decreusefond & Savy (2006) to construct a stochastic calculus for filtered Poisson processes. A new integral of Itô type with zero mean defined by means of the Wick product was introduced in Duncan et al. (2000) who also give some Itô formulae (see also Bender (2003a)).

In this section we consider the special case of a deterministic integrand which is sufficient for our present purposes and turns out to be easy to handle. Furthermore, until the end of this thesis we only consider *square integrable FLPs*, i.e. we always assume $\alpha = 2$ in Definition 2.1. The reason is that, as already mentioned several times, we are interested in long memory processes. Our definition of long memory requires the autocovariances to exist (see Definition 1.16).

We give a general definition of integrals with respect to FLPs which is closely related to the integral with respect to FBM defined in Pipiras & Taqqu (2000).

Recall from Section 1.3 the definition of the Riemann-Liouville fractional integral $I^d_\pm$ and derivative $\mathcal{D}^d_\pm$ of order d ($0 < d < 0.5$). Observe that we can rewrite

$$M_d(t) = \int_{\mathbb{R}} (I^d_- 1_{(0,t)})(s)\, L(ds). \tag{2.43}$$

Now, consider for $g \in L^1(\mathbb{R})$ the right-sided Riemann-Liouville fractional integral $I^d_- g$ of order d and denote by $\tilde{H}$ the set of functions $g : \mathbb{R} \to \mathbb{R}$, $g \in L^1(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} (I^d_- g)^2(u)\, du < \infty. \tag{2.44}$$

Proposition 2.32 *If $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $g \in \tilde{H}$.*

Proof. Starting from the fact that $(I^d_- g) \in L^2(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} |h(u)(I^d_- g)(u)|\, du \le C\|h\|_{L^2} \quad \text{for all } h \in L^2(\mathbb{R}),$$

it is sufficient to show that for all $h \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} \int_0^{\infty} |h(u)s^{d-1} g(s+u)|\, ds\, du \le C\|h\|_{L^2}. \tag{2.45}$$

Now (2.45) holds if

$$I_1 = \int_{\mathbb{R}} \int_1^{\infty} |h(u)s^{d-1} g(s+u)| \, ds \, du \leq C \|h\|_{L^2}$$

and

$$I_2 = \int_{\mathbb{R}} \int_0^{1} |h(u)s^{d-1} g(s+u)| \, ds \, du \leq C \|h\|_{L^2}.$$

Applying Fubini's Theorem and the Hölder inequality we obtain for I_2,

$$\int_0^1 s^{d-1} \int_{\mathbb{R}} |h(u)g(s+u)| \, du \, ds \leq \int_0^1 s^{d-1} \|h\|_{L^2} \|g\|_{L^2} \, ds = d^{-1} \|g\|_{L^2} \|h\|_{L^2}.$$

Furthermore, setting $t = s + u$ and using again Hölder's inequality,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} |g(t)| \int_1^{\infty} |h(t-s)| s^{d-1} \, ds \, dt \leq \int_{\mathbb{R}} \|h\|_{L^2} \left(\int_1^{\infty} s^{2(d-1)} \, ds \right)^{1/2} |g(t)| \, dt \\ &= \int_{\mathbb{R}} \|h\|_{L^2} \frac{1}{\sqrt{1-2d}} |g(t)| \, dt \leq (1-2d)^{-1/2} \|g\|_{L^1} \|h\|_{L^2} \end{aligned}$$

□

We define the space H as the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\| g \|_H := \left(E[L(1)^2] \int_{\mathbb{R}} (I_-^d g)^2(u) \, du \right)^{1/2}.$$

If follows from Pipiras & Taqqu (2000, Theorem 3.2) that $\| \cdot \|_H$ defines in fact a norm. Then from the proof of Proposition 2.32 we know that for $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\| g \|_H \leq C \left[\| g \|_{L^1} + \| g \|_{L^2} \right]. \tag{2.46}$$

To construct the integral $I_{M_d}(g) := \int_{\mathbb{R}} g(s) \, M_d(ds)$ for $g \in H$ we proceed as follows. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a simple function, i.e. $\phi(s) = \sum_{i=1}^{n-1} a_i 1_{(s_i, s_{i+1}]}(s)$, where $a_i \in \mathbb{R}, \; i = 1, ..., n$ and $-\infty < s_1 < s_2 < ... < s_n < \infty$. Notice that $\phi \in H$. Define

$$I_{M_d}(\phi) = \int_{\mathbb{R}} \phi(s) \, M_d(ds) = \sum_{i=1}^{n-1} a_i [M_d(s_{i+1}) - M_d(s_i)].$$

Obviously, I_{M_d} is linear in the simple functions

Proposition 2.33 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a simple function. Then*

$$\int_{\mathbb{R}} \phi(s)\, M_d(ds) = \int_{\mathbb{R}} (I_-^d \phi)(u)\, L(du) \tag{2.47}$$

and $\phi \mapsto I_{M_d}(\phi) = \int_{\mathbb{R}} \phi(s)\, M_d(ds)$ is an isometry between H and $L^2(\Omega, P)$.

Proof. It is sufficient to show (2.47) for indicator functions $\phi(s) = 1_{[0,t]}(s)$, $t > 0$. In fact,

$$\int_{\mathbb{R}} \phi(s)\, M_d(ds) = \int_{\mathbb{R}} 1_{[0,t]}(s)\, M_d(ds) = M_d(t)$$

and for the r.h.s. of (2.47) we obtain,

$$\begin{aligned}
\int_{\mathbb{R}} (I_-^d \phi)(u)\, L(du) &= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^{\infty} (s-u)^{d-1} 1_{[0,t]}(s) ds\, L(du) \\
&= \begin{cases} \dfrac{1}{\Gamma(d+1)} \displaystyle\int_{\mathbb{R}} [(t-u)^d - (-u)^d]\, L(du), & u < 0, \\ \dfrac{1}{\Gamma(d+1)} \displaystyle\int_{\mathbb{R}} (t-u)^d\, L(du), & 0 \le u \le t, \\ 0, & u > t \end{cases} \\
&= \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} \left[(t-u)_+^d - (-u)_+^d\right]\, L(du) = M_d(t).
\end{aligned}$$

Moreover, for all simple functions ϕ it follows from (2.9),

$$\begin{aligned}
\| I_{M_d}(\phi) \|_{L^2(\Omega,P)}^2 &= E\left[\int_{\mathbb{R}} (I_-^d \phi)(u)\, L(du)\right]^2 \\
&= E[L(1)^2] \int_{\mathbb{R}} (I_-^d \phi)^2(u)\, du = \| \phi \|_H^2\, .
\end{aligned} \tag{2.48}$$

□

Theorem 2.34 *Let $M_d = \{M_d(t)\}_{t \in \mathbb{R}}$ be a fractional Lévy process and let the function $g \in H$. Then there are simple functions $\phi_k : \mathbb{R} \to \mathbb{R},\ k \in \mathbb{N}$, satisfying $\| \phi_k - g \|_H \to 0$ as $k \to \infty$ such that $I_{M_d}(\phi_k)$ converges in $L^2(\Omega, P)$ towards a limit denoted as $I_{M_d}(g) = \int_{\mathbb{R}} g(s)\, M_d(ds)$ and $I_{M_d}(g)$ is independent of the approximating sequence ϕ_k. Moreover,*

$$\| I_{M_d}(g) \|_{L^2(\Omega,P)}^2 = \| g \|_H^2\, . \tag{2.49}$$

Proof. The simple functions are dense in H. This follows from the fact that the simple functions are dense in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in H by construction and (2.46). Hence, there exists a sequence (ϕ_k) of simple functions such that

$$\| \phi_k - g \|_H = \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} [\phi_k(s) - g(s)] \, ds \right)^2 du \to 0$$

as $k \to \infty$. It follows from the isometry property (2.48) that $\int_{\mathbb{R}} \phi_k(s) \, M_d(ds)$ converges in $L^2(\Omega, P)$ towards a limit denoted as $\int_{\mathbb{R}} g(s) \, M_d(ds)$ and the isometry property is preserved in this procedure. Last but not least (2.49) implies that the integral $\int_{\mathbb{R}} g(s) \, M_d(ds)$ is the same for all sequences of simple functions converging to g. $\square$

Corollary 2.35 *If M_d is a semimartingale, then $\int_{\mathbb{R}} g(s) \, M_d(ds)$ is well-defined as a limit in probability of elementary integrals. Observe that, since the limit in probability is unique, this limit is then equal to the limit $I_{M_d}(g)$ of Theorem 2.34.*

Using (2.47) and Theorem 2.34 the next proposition is obvious.

Proposition 2.36 *Let $g \in H$. Then*

$$\int_{\mathbb{R}} g(s) \, M_d(ds) = \int_{\mathbb{R}} (I_-^d g)(u) \, L(du), \tag{2.50}$$

where the equality holds in the L^2-sense.

Remark 2.37 Notice that our conditions on the integrand g differ from those imposed in the work by Zähle (1998). In particular we do not require the function g to be Hölder continuous of order greater than $1-d$. Furthermore, if the function g is Hölder continuous and g is defined on a compact interval, then $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence, $g \in H$.

The second order properties of integrals which are driven by FLPs follow by direct calculation. As $E[L(1)] = 0$, first note that we have for $g \in H$,

$$E\left[\int_{\mathbb{R}} g(t) \, M_d(dt) \right] = E\left[\frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^\infty (s-u)^{d-1} g(s) \, ds \, L(du) \right] = 0.$$

Proposition 2.38 *Let* $|f|, |g| \in H$. *Then*

$$E\left[\int_{\mathbb{R}} f(t)\, M_d(dt) \int_{\mathbb{R}} g(u)\, M_d(du)\right] = \frac{\Gamma(1-2d)E[L(1)^2]}{\Gamma(d)\Gamma(1-d)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)|t-u|^{2d-1}\, dt\, du. \tag{2.51}$$

Proof. It is a well-known fact that (Gripenberg & Norros (1996), p.405),

$$\int_{-\infty}^{\min(u,t)} (t-s)^{d-1}(u-s)^{d-1} ds = |t-u|^{2d-1}\frac{\Gamma(d)\Gamma(1-2d)}{\Gamma(1-d)}, \quad u,t \in \mathbb{R}.$$

Hence, by the isometry (2.49),

$$\begin{aligned} E\left[\int_{\mathbb{R}} f(t)\, M_d(dt) \int_{\mathbb{R}} g(u)\, M_d(du)\right] &= \frac{E[L(1)^2]}{\Gamma^2(d)} \int_{-\infty}^{\infty}\int_{s}^{\infty}\int_{s}^{\infty} f(t)g(u)(t-s)^{d-1}(u-s)^{d-1}\, dt\, du\, ds \\ &= \frac{E[L(1)^2]}{\Gamma^2(d)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u) \int_{-\infty}^{\min(u,t)} (t-s)^{d-1}(u-s)^{d-1}\, ds\, dt\, du \\ &= \frac{\Gamma(1-2d)E[L(1)^2]}{\Gamma(d)\Gamma(1-d)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(u)|t-u|^{2d-1}\, dt\, du, \end{aligned}$$

where we have used Fubini's theorem. □

2.4 Application to Long Memory Moving Average Processes

In discrete time, moving average (MA) processes are very popular in classical time series analysis and are widely used in applications in engineering, physics and metrology.

We consider the continuous time version of a MA process. Continuous time MA processes play an important role since they are very flexible models, e.g.

MA processes can capture volatility jumps or exhibit long memory properties. Typical examples are the stochastic volatility models by Barndorff-Nielsen & Shephard (2001b) which are based on Ornstein-Uhlenbeck processes, the CARMA processes (Chapter 1.2) and FICARMA processes (Chapter 1.3) or the stable MA processes (Samorodnitsky & Taqqu (1994)).

Considering short memory MA processes, we construct a special class of MA processes, the long memory MA processes. We would like to stress that throughout this section we assume that L is a Lévy process without Brownian component satisfying $E[L(1)] = 0$ and $E[L(1)^2] < \infty$.

Stationary continuous time moving average (MA) processes have already been introduced in Definition 1.4. Moreover, Proposition 1.5 gives conditions for a MA process to be well-defined, stationary and infinitely divisible, which are formulated in terms of the kernel g and the generating triplet $(\gamma_L, \sigma_L^2, \nu_L)$ of the driving Lévy process L.

Definition 2.39 (Short Memory Causal MA Process) *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a second-order Lévy process having generating triplet $(\gamma_L, \sigma_L^2, \nu_L)$. Then we define a short memory causal moving average process by (1.21), where we assume that the kernel $g : \mathbb{R} \to \mathbb{R}$ is measurable and satisfies the following two conditions:*

(M1) $g(t) = 0$ *for all* $t < 0$ *(causality),*

(M2) $|g(t)| \leq Ce^{-ct}$ *for some constants* $C > 0$ *and* $c > 0$ *(short memory).*

From now on, if not stated otherwise, a MA process means a short memory causal MA process, i.e. g satisfies (M1) and (M2) which imply $g \in L^1(\mathbb{R})$.

Proposition 2.40 *A short memory MA process is well-defined if*

$$\int_{|x|>1} \log|x|\, \nu_L(dx) < \infty. \tag{2.52}$$

Proof. The assertion is shown by substituting (M2) in Proposition 1.5. □

We can use a short memory MA process to construct a long memory MA process.

2.4.1 Lévy-driven Long Memory MA Processes

We recall from Section 1.3 that one possibility to incorporate long memory into a class of short memory processes, is to fractionally integrate the kernel g in (1.21). Recall that the definition of the Riemann-Liouville fractional integrals has been given in Section 1.3.

We calculate the left-sided Riemann-Liouville fractional integral of the kernel g in (1.21), where we only consider functions $g \in H$. Then we obtain for $0 < d < 0.5$ the fractionally integrated kernel

$$g_d(t) := (I_+^d g)(t) = \int_0^t g(t-s) \frac{s^{d-1}}{\Gamma(d)} \, ds, \quad t \in \mathbb{R}. \tag{2.53}$$

From (M1) and (M2) follows that $g_d(t) = 0$ for $t \leq 0$ and

$$|g_d(t)| = \left| \int_0^t g(t-s) \frac{s^{d-1}}{\Gamma(d)} \, ds \right| \leq K \, t^{d-1}, \quad t > 0, \tag{2.54}$$

for some constant $K > 0$. Moreover, $g_d \in L^2(\mathbb{R})$ as $g \in H$. We can now define a fractionally integrated MA process by replacing the kernel g by the kernel g_d.

Definition 2.41 (FIMA Process) *Let $0 < d < 0.5$. Then the fractionally integrated moving average (FIMA) process $Y_d = \{Y_d(t)\}_{t \in \mathbb{R}}$ driven by the Lévy process L with $E[L(1)] = 0$ and $E[L(1)^2] < \infty$ is defined by*

$$Y_d(t) = \int_{-\infty}^t g_d(t-u) \, L(du), \qquad t \in \mathbb{R}, \tag{2.55}$$

where the fractionally integrated kernel g_d is given in (2.53).

The next proposition summarizes results on the stationarity and infinite divisibility of FIMA processes.

Proposition 2.42 (Stationarity, Infinite Divisibility) *The FIMA process (2.55) is well-defined and stationary. Moreover, for all $t \in \mathbb{R}$ the distribution* *of $Y_d(t)$ is infinitely divisible with characteristic triplet $(\gamma_Y^t, 0, \nu_Y^t)$, where*

$$\gamma_Y^t = -\int_{-\infty}^{t}\int_{\mathbb{R}} x g_d(t-s) 1_{\{|g_d(t-s)x|>1\}}\, \nu_L(dx)\, ds \quad \text{and} \tag{2.56}$$

$$\nu_Y^t(B) = \int_{-\infty}^{t}\int_{\mathbb{R}} 1_B(g_d(t-s)x)\, \nu_L(dx)\, ds, \quad B \in \mathcal{B}(\mathbb{R}). \tag{2.57}$$

Here $(\gamma_L, 0, \nu_L)$ denotes the characteristic triplet of L.

Proof. Since $g_d \in L^2(\mathbb{R})$ we can apply Proposition 1.2 to $Y_d(0)$ and obtain that Y_d is well-defined. Let $u_1, \dots, u_n \in \mathbb{R}$ and $-\infty < t_1 < \dots < t_n < \infty$, $n \in \mathbb{N}$. Then by the stationary increments of L,

$$u_1 Y_d(t_1+h) + \dots + u_n Y_d(t_n+h) = \sum_{k=1}^{n} u_k \int_{-\infty}^{t_k+h} g_d(t_k+h-s)\, L(ds)$$

$$\stackrel{d}{=} \sum_{k=1}^{n} u_k \int_{-\infty}^{t_k} g_d(t_k-s)\, L(ds) = u_1 Y_d(t_1) + \dots + u_n Y_d(t_n). \tag{2.58}$$

The characteristic functions of the left and the right hand side of (2.58) coincide. Hence, by the Cramér Wold device Y_d is stationary. □

So far we constructed a FIMA process by a fractional integration of the corresponding short memory kernel g. The next theorem states that we can also construct a long memory MA process by replacing in the short memory MA process (1.21) the driving Lévy process by the corresponding fractional Lévy process. The resulting process coincides in L^2 with the process (2.55).

Theorem 2.43 *Suppose $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ to be the FIMA process $Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(ds)$, $t \in \mathbb{R}$, with $g_d \in L^2(\mathbb{R})$ such that $g_d \in I_+^d(L^2)$. Then Y_d can be represented as*

$$Y_d(t) = \int_{-\infty}^{t} g(t-s)\, M_d(ds),\ t \in \mathbb{R}, \qquad \text{with} \tag{2.59}$$

$$g(x) = \frac{1}{\Gamma(1-d)} \frac{d}{dx} \int_{0}^{x} g_d(s)(x-s)^{-d} ds, \quad x \in \mathbb{R},$$

i.e. g is the Riemann-Liouville derivative $\mathcal{D}^d_+ g_d$ of the kernel g_d.

On the other hand, if Y_d is given by (2.59) with $g \in H$, then Y_d can be rewritten as $Y_d(t) = \int_{-\infty}^t g_d(t-s)\, L(ds)$, $t \in \mathbb{R}$, where $g_d(x) = (I^d_+ g)(x)$.

Proof. For every $t \in \mathbb{R}$ it holds a.s.

$$
\begin{aligned}
Y_d(t) &= \int_{-\infty}^{t} g(t-s)\, M_d(ds) = \frac{1}{\Gamma(d)} \int_{-\infty}^{t} \left(\int_{u}^{\infty} (s-u)^{d-1} g(t-s)\, ds \right) L(du) \\
&= \frac{1}{\Gamma(d)} \int_{-\infty}^{t} \left(\int_{0}^{\infty} s^{d-1} g(t-u-s) ds \right) L(du) = \int_{-\infty}^{t} g_d(t-u)\, L(du),
\end{aligned}
$$

where we used (2.50). □

Using representation (2.59) of a FIMA process we show that this class of processes has long memory properties.

Theorem 2.44 (Long Memory) *A FIMA process $Y_d = \{Y_d(t)\}_{t \in \mathbb{R}}$ is a long memory MA process.*

Proof. Since Y_d can be expressed as (2.59), we have from Proposition 2.38 for $h > 0$,

$$
\begin{aligned}
&\gamma_{Y_d}(h) = \operatorname{cov}(Y_d(t+h), Y_d(t)) \\
&= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} E[L(1)^2] \int_{\mathbb{R}} \int_{\mathbb{R}} g(t+h-u) g(t-v) |u-v|^{2d-1}\, du\, dv \\
&= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} E[L(1)^2] \int_{\mathbb{R}} \int_{\mathbb{R}} g(s) g(\tilde{s}) |h-s+\tilde{s}|^{2d-1}\, ds\, d\tilde{s}.
\end{aligned}
$$

It follows,

$$
\gamma_{Y_d}(h) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} E[L(1)^2] \left(\int_{\mathbb{R}} g(u)\, du \right)^2 |h|^{2d-1}, \quad \text{as } h \to \infty. \tag{2.60}
$$

Hence, γ_{Y_d} satisfies condition (1.37) and Y_d is a long memory process.

It remains to show (2.60):

$$
\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} g(s) g(\tilde{s}) \frac{|h-s+\tilde{s}|^{2d-1}}{|h|^{2d-1}}\, ds\, d\tilde{s} = \int_{\mathbb{R}} \int_{\mathbb{R}} g(s) g(\tilde{s}) \left| 1 - \frac{s}{h} + \frac{\tilde{s}}{h} \right|^{2d-1} ds\, d\tilde{s} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} g(s) g(\tilde{s})\, ds\, d\tilde{s} + I, \qquad (2.61)
\end{aligned}
$$

where

$$I = \int_{\mathbb{R}} \int_{\mathbb{R}} g(s)g(\tilde{s}) \left(\left|1 - \frac{s}{h} + \frac{\tilde{s}}{h}\right|^{2d-1} - 1 \right) ds\, d\tilde{s}.$$

Then

$$\begin{aligned} |I| \leq & \int_{\mathbb{R}} \int_{\mathbb{R}} |g(s)||g(\tilde{s})| \left| \left|1 - \frac{s}{h} + \frac{\tilde{s}}{h}\right|^{2d-1} - 1 \right| 1_{\{|\tilde{s}-s| \leq \epsilon h\}}\, ds\, d\tilde{s} \\ & + \int_{\mathbb{R}} \int_{\mathbb{R}} Ce^{-cs}Ce^{-c\tilde{s}} \left| \left|1 - \frac{s}{h} + \frac{\tilde{s}}{h}\right|^{2d-1} - 1 \right| 1_{\{|\tilde{s}-s| > \epsilon h\}}\, ds\, d\tilde{s} \end{aligned}$$

Since for $|\tilde{s} - s| \leq \epsilon h$ we have

$$\left| \left|1 - \frac{s}{h} + \frac{\tilde{s}}{h}\right|^{2d-1} - 1 \right| \leq \max\left((1-\epsilon)^{2d-1} - 1, 1 - (1+\epsilon)^{2d-1}\right) \leq (1-\epsilon)^{2d-1} - 1,$$

we obtain

$$\begin{aligned} |I| &\leq \left((1-\epsilon)^{2d-1} - 1\right) \int_{\mathbb{R}} \int_{\mathbb{R}} |g(s)||g(\tilde{s})|\, ds\, d\tilde{s} + I_2(h) \\ &= \left((1-\epsilon)^{2d-1} - 1\right) \|g\|^2_{L^1(\mathbb{R})} + I_2(h) \end{aligned}$$

where

$$I_2(h) = \int_{\mathbb{R}} \int_{\mathbb{R}} C^2 e^{-cs} e^{-c\tilde{s}} \left| \left|1 - \frac{s}{h} + \frac{\tilde{s}}{h}\right|^{2d-1} - 1 \right| 1_{\{|\tilde{s}-s| > \epsilon h\}}\, ds\, d\tilde{s}.$$

Define $(u, v) = (s + \tilde{s}, s - \tilde{s}) =: T(s, \tilde{s})$, i.e. $d(u, v) = |\mathrm{det(T'(s,\tilde{s})|d(s,\tilde{s})} = \mathrm{2d(s,\tilde{s})}$. Since $s + \tilde{s} \geq |\tilde{s} - s|$ we have

$$\begin{aligned} I_2(h) = & \int_{u \geq |v| \geq \epsilon h} 2C^2 e^{-cu} ||1 - \frac{v}{h}|^{2d-1} - 1|\, d(u, v) \\ = & \; 2 \int_{\epsilon h}^{\infty} \int_{v}^{\infty} C^2 e^{-cu}\, du\, ||1 - \frac{v}{h}|^{2d-1} - 1|\, dv \\ & + 2 \int_{-\infty}^{-\epsilon h} \int_{-v}^{\infty} C^2 e^{-cu}\, du\, ||1 - \frac{v}{h}|^{2d-1} - 1|\, dv \\ \leq & \int_{\epsilon h}^{\infty} \frac{2C^2}{c} e^{-cv} (|1 - \frac{v}{h}|^{2d-1} + 1)\, dv + \int_{-\infty}^{\epsilon h} \frac{2C^2}{c} e^{cv} (\underbrace{|1 - \frac{v}{h}|^{2d-1}}_{\leq 1} + 1)\, dv \end{aligned}$$

$$\leq \int_{\epsilon h}^{2h} \frac{2C^2}{c} e^{-cv} |1 - \frac{v}{h}|^{2d-1} \, dv + \int_{\epsilon h}^{\infty} \frac{4C^2}{c} e^{-cv} \, dv + \int_{-\infty}^{-\epsilon h} \frac{4C^2}{c} e^{cv} \, dv$$

$$\leq \frac{2C^2}{c} e^{-c\epsilon h} \frac{h}{2d} \, (1 - \frac{v}{h})^{2d} \Big|_{h}^{\epsilon h} + \frac{2C^2}{c} e^{-ch} \frac{h}{2d} \, (\frac{v}{h} - 1)^{2d} \Big|_{h}^{2h}$$

$$+ \frac{4C^2}{c} e^{-cv} \Big|_{\infty}^{\epsilon h} + \frac{4C^2}{c} e^{cv} \Big|_{-\infty}^{-\epsilon h}$$

$$= \frac{2C^2}{c} e^{-c\epsilon h} \frac{h}{2d} (1 - \epsilon)^{2d} + \frac{2C^2}{c} e^{-ch} \frac{h}{2d} + 8 \frac{C^2}{c^2} e^{-c\epsilon h}.$$

Now, we choose $\epsilon > 0$ such that

$$\big((1 - \epsilon)^{2d-1} - 1\big) \, \|g\|^2_{L^1(\mathbb{R})} \leq \frac{K}{2}$$

for some $K > 0$. As $\epsilon > 0$ is fixed, it follows that $I_2(h) \to 0$ as $h \to \infty$. This shows that $|I|$ becomes arbitrarily small as $h \to \infty$, which yields the assertion by (2.61). □

2.4.2 Second Order, Sample Path and Distributional Properties of Long Memory MA Processes

In the preceding proof we have already derived an expression for the autocovariance function of a FIMA process. However, the following representation will be needed to calculate the spectral density.

Theorem 2.45 (Autocovariance Function) *Let $0 < d < 0.5$. The autocovariance function γ_d of a FIMA process Y_d is*

$$\gamma_d(h) = E[L(1)^2] \int_{\mathbb{R}} g_d(u + h) g_d(u) \, du, \quad h \in \mathbb{R}, \tag{2.62}$$

where g_d is the fractionally integrated kernel given in (2.53).

Proof. Let $h \geq 0$. Then from representation (2.55),

$$\gamma_d(h) = \operatorname{cov}(Y_d(t + h), Y_d(t)) = \operatorname{var}(L(1)) \int_{-\infty}^{t} g_d(t + h - s) g_d(t - s) \, ds$$

$$= E[L(1)^2] \int_{0}^{\infty} g_d(u + h) g_d(u) \, du = E[L(1)^2] \int_{\mathbb{R}} g_d(u + h) g_d(u) \, du,$$

since $g_d(t) = 0$ for $t \leq 0$. □

Theorem 2.46 (Spectral Density) *The spectral density f_d of a FIMA process Y_d equals*

$$f_d(\lambda) = \frac{E[L(1)^2]}{2\pi} |G_d(\lambda)|^2, \quad \lambda \in \mathbb{R}, \tag{2.63}$$

where $G_d(\lambda) = \int_{\mathbb{R}} e^{iu\lambda} g_d(u)\, du$, $\lambda \in \mathbb{R}$, is the Fourier transform of the kernel function g_d given in (2.53).

Proof. Since the spectral density of a stationary process is the inverse Fourier transform of the autocovariance function, we obtain from (2.62) by direct calculation,

$$\begin{aligned} f_d(\lambda) &= \frac{E[L(1)^2]}{2\pi} \int_{\mathbb{R}} e^{-ih\lambda} \int_{\mathbb{R}} g_d(u+h) g_d(u)\, du\, dh \\ &= \frac{E[L(1)^2]}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\lambda(v-u)} g_d(v) g_d(u)\, du\, dv \\ &= \frac{E[L(1)^2]}{2\pi} \left(\int_{\mathbb{R}} e^{-iv\lambda} g_d(v)\, dv \right) \left(\int_{\mathbb{R}} e^{iu\lambda} g_d(u)\, du \right) \\ &= \frac{E[L(1)^2]}{2\pi} G_d(-\lambda) G_d(\lambda) = \frac{E[L(1)^2]}{2\pi} |G_d(\lambda)|^2, \quad \lambda \in \mathbb{R}. \end{aligned}$$

□

To obtain some insight into the behaviour of the sample paths of a FIMA process we exclude path properties that do *not* hold. In fact, Rosinski (1989) provides immediately verifiable necessary conditions for interesting sample path properties.

Proposition 2.47 (p-Variation) *Let $p \geq 0$. If the kernel $t \mapsto g_d(t-s)$ is of unbounded p-variation then $P(\{\omega \in \Omega : Y_d(\cdot, \omega) \notin C_p[a,b]\}) > 0$, where $C_p[a,b]$ is the space of functions of bounded p-variation on $[a,b]$.*

Proof. The assertion follows by an application of Theorem 4 of Rosinski (1989), where we use the symmetrization argument of section 5 in Rosinski (1989), if ν_L is not already symmetric. □

We noted in Theorem 2.42 that a FIMA process Y_d has infinitely divisible margins. Moreover, since $E[L(1)] = 0$, $E[L(1)^2] < \infty$ and $g_d \in L^2(\mathbb{R})$ it follows by (1.11) that we can represent Y_d as

$$Y_d(t) = \int_{-\infty}^{t} \int_{\mathbb{R}_0} x g_d(t-s)\, \tilde{J}(dx, ds).$$

Therefore we can apply the results of Marcus & Rosinski (2005) to determine the continuity of Y_d.

Proposition 2.48 (Continuity) *Let $g_d \in C_b^1(\mathbb{R})$. Then the FIMA process Y_d has a continuous version on every bounded interval I of $\mathbb{R}$.*

Proof. Applying Theorem 2.5, Marcus & Rosinski (2005), we obtain that Y_d has a continuous version on $I \subset \mathbb{R}$, if $g_d(0) = 0$ and if for some $\epsilon > 0$,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\epsilon} |g_d(u) - g_d(v)| < \infty.$$

We have $|g_d(u) - g_d(v)| \leq |g_d'(\xi)||u-v| \leq C|u-v|, \ u \leq \xi \leq v, \ \xi \in I$. Therefore,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\epsilon} |g_d(u) - g_d(v)| \leq \sup_{t \in I'} C|t|(-\log|t|)^{1/2+\epsilon} = \sup_{t \in I'} m(t),$$

where

$$m(t) = C|t|(-\log|t|)^{1/2+\epsilon} \leq C|t|(-\log|t|) \to 0 \quad \text{as } t \to 0^+.$$

Moreover, m is continuous and assumes its maximum on any compact interval. Hence,

$$\sup_{t \in I'} m(t) < \infty.$$

□

Remark 2.49 If the process L has paths of bounded variation then

$$Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(ds) = (g_d * L)(t), \quad t \in \mathbb{R},$$

is the convolution of the kernel g_d with the jumps of L, taken pathwise. In this case, as g_d is continuous, it is obvious that Y_d is continuous.

Finally, we consider the question what happens if we fractionally differentiate a long memory FIMA process and fractionally integrate a short memory MA process.

Theorem 2.50 *The left-sided Riemann Liouville fractional derivative* $\mathcal{D}^d_+ Y_d$ *of a FIMA process* Y_d *is the corresponding short memory MA process* Y.

Proof. Applying Fubini's Theorem for stochastic integrals (Theorem 65, Protter (2004)) we have,

$$\begin{aligned}(\mathcal{D}^d_+ Y_d)(t) &= \frac{1}{\Gamma(1-d)}\frac{d}{dt}\int_{-\infty}^{t}(t-s)^{-d}Y_d(s)\,ds \\ &= \frac{1}{\Gamma(1-d)}\frac{d}{dt}\int_{-\infty}^{t}\int_{u}^{t}(t-s)^{-d}g_d(s-u)\,ds\,L(du), \quad t\in\mathbb{R}.\end{aligned}$$

Now define

$$Z(t) := \int_{-\infty}^{t}(D * g_d)(t-u)\,L(du),\ t\in\mathbb{R},$$

where

$$D(t) := t^{-d}(\Gamma(1-d))^{-1}1_{[0,\infty)}(t).$$

Then

$$(\mathcal{D}^d_+ Y_d)(t) = \frac{d}{dt}Z(t)$$

and again by Fubini's theorem,

$$\begin{aligned}Z(t) &= \int_{-\infty}^{t}(D * g_d)(t-u)\,L(du) = \int_{-\infty}^{t}\int_{u}^{t}(D * g_d)'(x-u)\,dx\,L(du) \\ &= \int_{-\infty}^{t}\int_{-\infty}^{x}(D * g_d)'(x-u)\,L(du)\,dx, \quad t\in\mathbb{R}.\end{aligned}$$

Hence for $t\in\mathbb{R}$,

$$(\mathcal{D}^d_+ Y_d)(t) = \frac{d}{dt}Z(t) = \int_{-\infty}^{t}(D * g_d)'(t-u)\,L(du),$$

and since $(D * g_d)'(t) = (\mathcal{D}^d_+ g_d)(t) = g(t)$, we obtain

$$(\mathcal{D}^d_+ Y_d)(t) = \int_{-\infty}^{t} g(t-s)\,L(ds) = Y(t),\ t\in\mathbb{R}.$$

□

Theorem 2.51 *The left-sided Riemann Liouville fractional integral $I_+^d Y$ of a short memory MA process Y is the corresponding long memory FIMA process Y_d.*

Proof. Using again Fubini's Theorem (Theorem 65, Protter (2004)) we have,

$$\begin{aligned}
(I_+^d Y)(t) &= \int_{\mathbb{R}} \int_{-\infty}^{s} g(s-u)\, L(du)\, \frac{(t-s)^{d-1}}{\Gamma(d)} 1_{[0,\infty)}(t-s)\, ds \\
&= \int_{-\infty}^{t} \int_{0}^{t-u} \frac{(t-u-s)^{d-1}}{\Gamma(d)} g(s)\, ds\, L(du) \\
&= \int_{-\infty}^{t} g_d(t-u)\, L(du) = Y_d(t), \quad t \in \mathbb{R}.
\end{aligned}$$

□

The following Corollary summarizes our findings.

Corollary 2.52 *Every long memory FIMA process $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ has the following three representations:*

(i) $Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(ds), \quad t \in \mathbb{R},$

(ii) $Y_d(t) = \int_{-\infty}^{t} g(t-s)\, M_d(ds), \quad t \in \mathbb{R},$

(iii) $Y_d(t) = (I_+^d Y)(t)$, *where* $Y(t) = \int_{-\infty}^{t} g(t-s)\, L(ds),\ t \in \mathbb{R}$, *and* I_+^d *denotes the left-sided Riemann-Liouville fracional integral of order* $0 < d < 0.5$.

For $t \in \mathbb{R}$ *the corresponding short memory MA process is given by*

$$\begin{aligned}
(\mathcal{D}_+^d Y_d)(t) &= \frac{1}{\Gamma(1-d)} \frac{d}{dt} \left[\int_{-\infty}^{t} (t-s)^{-d} Y_d(s)\, ds \right] \\
&= \int_{-\infty}^{t} g(t-s)\, L(ds) = Y(t).
\end{aligned}$$

Remark 2.53 Like a FLP a FIMA process has a generalized shot noise representation, which is given by (2.12) or (2.14), respectively, with the kernel function $f_t(\,\cdot\,)$ replaced by the kernel $g_d(t-\,\cdot\,)$ given in (2.53).

We apply the above results to CARMA and FICARMA processes.

2.4.3 Application to CARMA and FICARMA Processes

Continuous time ARMA (CARMA) processes constitute a special class of short memory MA processes and have already been defined in Chapter 1.2.

To incorporate long memory into the class of causal short memory Lévy-driven CARMA processes, we can either proceed as described in Chapter 1.3, i.e. we fractionally integrate the kernel g given in (1.30) and obtain the FICARMA(p, d, q) process

$$Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(ds),$$

where $0 < d < 0.5$ and

$$g_d(t) = (I_+^d g)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} (i\lambda)^{-d} \frac{q(i\lambda)}{p(i\lambda)} d\lambda, \quad t \in \mathbb{R}.$$

Or, applying Theorem 2.43 to CARMA and FICARMA processes, we can alternatively define a FICARMA process via the fractional Lévy process M_d by

$$Y_d(t) = \int_{-\infty}^{t} g(t-s)\, M_d(ds), \quad t \in \mathbb{R}, \tag{2.64}$$

where the kernel g is the Riemann-Liouville fractional derivative of the function g_d, i.e. g is the kernel (1.30) of the short memory CARMA process (1.28). Furthermore, when we calculate the left-sided Riemann-Liouville fractional derivative of the FICARMA process, we obtain a CARMA process (see Theorem 2.50), i.e.

$$(\mathcal{D}_+^d Y_d)(t) = Y(t), \qquad t \in \mathbb{R}.$$

This one-to-one correspondence is reflected in the following corollary.

Corollary 2.54 *The process $\{Y_d(t)\}_{t\geq 0}$ satisfies the formal SDE*

$$p(D)Y_d(t) = q(D)DM_d(t), \quad t \geq 0, \tag{2.65}$$

(which is an abbreviation of (2.67) and (2.68) below) if and only if its derivative $(\mathcal{D}_+^d Y_d)(t) = Y(t)$ of order $d \in (0, 0.5)$ satisfies the formal SDE (1.22):

$$p(D)Y(t) = q(D)DL(t), \quad t \geq 0. \tag{2.66}$$

Representation (2.65) of a FICARMA process suggests a convenient way to simulate a sample of the process.

Simulation of FICARMA Processes

We define a FICARMA process by equation (2.65) and, analogously to Section 1.2, interprete this equation as being equivalent to the following observation and state equations,

$$Y_d(t) = \mathbf{b}^T \mathbf{X}(t) \qquad \text{and} \tag{2.67}$$

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + \mathbf{e}\, M_d(dt), \quad t \geq 0, \tag{2.68}$$

where the vectors $\mathbf{e}$, $\mathbf{b}$ and the matrix A are defined as in Section 1.2.

Using equations (2.67) and (2.68) the following simulation procedure generates a sample path of a FICARMA(p, d, q) process.

(i) Set $\mathbf{X}(0) := 0$

(ii) For $j = 1, \ldots, n-1$ define $T_j := t_{j+1} - t_j$, where the times $t_1, \ldots, t_n$ are not necessarily uniformly spaced.

(iii) Generate the increments $W_j := M_d(t_{j+1}) - M_d(t_j)$ for $j = 1, \ldots, n-1$, where $\{M_d(t_j)\}_{j=1,\ldots,n-1}$ is a sample of the driving fractional Lévy process.

(iv) Apply the Euler method to obtain for $j = 1, \ldots, n-1$,

$$\mathbf{X}(t_{j+1}) = \mathbf{X}(t_j) + T_j A\mathbf{X}(t_j) + W_j\, \mathbf{e}.$$

(v) Compute for $j = 1, \ldots, n$,

$$Y_d(t_j) = \mathbf{b}^T \mathbf{X}(t_j).$$

To generate a sample $\{M_d(t_j)\}_{j=1,\ldots,n-1}$ of the fractional Lévy process M_d we approximate

$$M_d(t) = \frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} [(t-s)_+^d - (-s)_+^d]\, L(ds), \quad t \geq 0$$

by the corresponding Riemann sums

$$M_d^{(n)}(t) = \frac{1}{\Gamma(d+1)} \left\{ \sum_{k=-a_n}^{0} \left(\left(t - \frac{k}{n}\right)^d - \left(-\frac{k}{n}\right)^d \right) \left[L\left(\frac{k+1}{n}\right) - L\left(\frac{k}{n}\right) \right] + \sum_{k=1}^{[nt]} \left(t - \frac{k}{n}\right)^d \left[L\left(\frac{k+1}{n}\right) - L\left(\frac{k}{n}\right) \right] \right\}, \quad t \in \mathbb{R},$$

where $\{L(k)\}_{k\in\mathbb{Z}}$ is a discrete sample of the driving Lévy process $\{L(t)\}_{t\in\mathbb{R}}$ and a_n is a sequence that satisfies $\lim_{n\to\infty} a_n = +\infty$. As the next theorem shows the performance of this discretization depends on the choice of a_n. We take $a_n = n^2$ in order to keep the computational costs reasonable. Furthermore, for simplicity we assume $E[L(1)^2] = 1$ until the end of this section.

Theorem 2.55

$$M_d^{(n)}(t) \xrightarrow{L^2} M_d(t)$$

as $n \to \infty$. Moreover, for $t \in \mathbb{R}$,

$$\| M_d^{(n)}(t) - M_d(t) \|_{L^2} = O\left((a_n/n)^{d-1/2}\right) + O\left(a_n^{1/2} n^{-3/2}\right) + O\left(n^{\frac{1+2d-2d^2}{2d-3}}\right) \tag{2.69}$$

and the optimal convergence rate is obtained by choosing $a_n = n^{\frac{2-d}{1-d}}$.

Proof. Without loss of generality let $t \geq 0$. Define

$$f_t^{(n)}(s) = \sum_k f_t(k/n) 1_{(k/n,(k+1)/n]}(s).$$

Then

$$\int_{\mathbb{R}} f_t^{(n)}(s)\, L(ds) = \sum_k f_t(k/n)[L((k+1)/n) - L(k/n)]$$

and as $n \to \infty$, $f_t^{(n)}(s) \to f_t(s)$ in $L^2(\mathbb{R})$, since f_t is continuous in $L^2(\mathbb{R})$ and $f_t(s) = 0$ for $s \geq \max(t,0)$. Hence, Proposition 1.2 yields

$$\int_{\mathbb{R}} f_t^{(n)}(s)\, L(ds) \xrightarrow{L^2(\Omega)} \int_{\mathbb{R}} f_t(s)\, L(ds), \qquad n \to \infty.$$

We have,

$$\| M_d^{(n)}(t) - M_d(t) \|_{L^2} \leq \underbrace{\left(\int_{-\infty}^{-a_n/n} f_t(s)^2\, ds \right)^{\frac{1}{2}}}_{A} + \underbrace{\left(\int_{-a_n/n}^{t} (f_t^{(n)}(s) - f_t(s))^2\, ds \right)^{\frac{1}{2}}}_{B}$$

$A^2 = (\Gamma(d+1))^{-2} \int_{-\infty}^{-a_n/n} ((t-s)^d - (-s)^d)^2 \, ds \leq \int_{-\infty}^{-a_n/n} ((-s)^{d-1} t/\Gamma(d))^2 \, ds.$ Hence,

$$A \leq a_n^{d-1/2} \, n^{1/2-d} \, t \, / (\Gamma(d)\sqrt{1-2d}).$$

Using that f_t is Hölder continuous of order d we have $|f_t^{(n)}(s) - f_t(s)| \leq C\, n^{-d}$. Now,

$$\begin{aligned} B \leq & \left(\int_{-a_n/n}^{-\epsilon-1} \Big(\sup_{\xi \in [-\frac{a_n}{n}, -\epsilon-1]} f_t'(\xi)/n \Big)^2 \, ds \right)^{\frac{1}{2}} + \left(\int_{-\epsilon-1}^{-\epsilon} \Big(\sup_{\xi \in [-\epsilon-1, -\epsilon]} f_t'(\xi)/n \Big)^2 \, ds \right)^{\frac{1}{2}} \\ & + \left(\int_{-\epsilon}^{0} (C\, n^{-d})^2 \, ds \right)^{\frac{1}{2}} + \left(\int_{0}^{t-\epsilon} \Big(\sup_{\xi \in [0, t-\epsilon]} f_t'(\xi)/n \Big)^2 \, ds \right)^{\frac{1}{2}} + \left(\int_{t-\epsilon}^{t} (C\, n^{-d})^2 \, ds \right)^{\frac{1}{2}} \end{aligned}$$

which gives

$$\begin{aligned} B \leq & \left(\int_{-a_n/n}^{-\epsilon-1} (\Gamma(d) n)^{-2} \, ds \right)^{\frac{1}{2}} + \left(\int_{-\epsilon-1}^{-\epsilon} (|\epsilon|^{d-1} / (\Gamma(d) n))^2 \, ds \right)^{\frac{1}{2}} \\ & + \left(\int_{-\epsilon}^{0} (C\, n^{-d})^2 \, ds \right)^{\frac{1}{2}} + \left(\int_{0}^{t-\epsilon} (|\epsilon|^{d-1} / (\Gamma(d) n))^2 \, ds \right)^{\frac{1}{2}} + \left(\int_{t-\epsilon}^{t} (C\, n^{-d})^2 \, ds \right)^{\frac{1}{2}} \\ = & \, a_n^{1/2} \, n^{-3/2} / \Gamma(d) + |\epsilon|^{d-1} / (\Gamma(d) n) + \sqrt{\epsilon}\, C\, n^{-d} + \sqrt{t-\epsilon}\, |\epsilon|^{d-1} / (\Gamma(d) n) \\ & + \sqrt{\epsilon}\, C\, n^{-d} \end{aligned}$$

Setting $\epsilon = n^{\frac{2-2d}{2d-3}}$ leads to the same convergence order of the last four terms in the last equation. Hence, $B \leq {a_n}^{1/2} \, n^{-3/2} \, d + O(n^{\frac{1+2d-2d^2}{2d-3}})$ and

$$\| \, M_d^{(n)}(t) - M_d(t) \, \|_{L^2} = O((a_n/n)^{d-1/2}) + O(a_n^{1/2} n^{-3/2}) + O(n^{\frac{1+2d-2d^2}{2d-3}}).$$

The optimal rate of convergence is obtained by balancing the first two terms on the r.h.s of (2.69). This leads to $a_n = n^{\frac{2-d}{1-d}}$ and

$$\| \, M_d^{(n)}(t) - M_d(t) \, \|_{L^2} = O(n^{\frac{2d-1}{2-2d}}) + O(n^{\frac{1+2d-2d^2}{2d-3}}).$$

□

Remark 2.56 An efficient method to compute $M_d^{(n)}$ is presented in Stoev & Taqqu (2004) by using the Fast Fourier Transform algorithm. Alternatively one can simulate a sample of the driving FLP following the approach (2.15) or (2.16) of Cohen et al. (2005). However, our approach uses the increments of the Lévy process L which are very easy to simulate.

Figure 2.5 shows the sample path of a FICARMA$(3, 0.25, 2)$ process which is driven by a fractional truncated stable Lévy process, i.e. ν is given by (2.42), where we fix $\alpha = 1.8$. The autoregressive and moving average polynomials are the same as those of the CARMA process in Example 1.15, i.e. they are given by

$$p(z) = (z + 0.1)(z + 0.5 - \frac{i\pi}{2})(z + 0.5 + \frac{i\pi}{2}) \text{ and } q(z) = 2.792 + 5z + z^2.$$

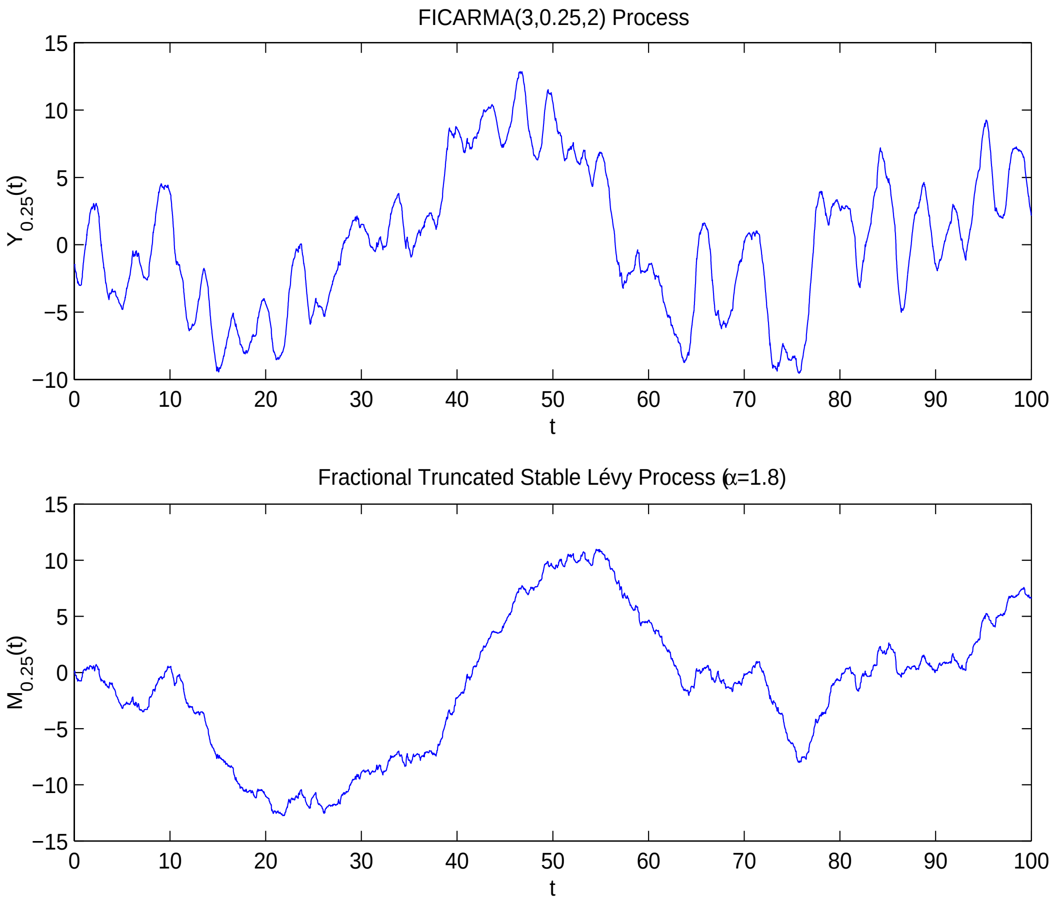

Figure 2.5: The sample path of a FICARMA$(3, 0.25, 2)$ process and the sample path of the driving fractional truncated stable Lévy process ($\alpha = 1.8$) process.

3 Multivariate CARMA Processes

CARMA processes have already been considered in the previous sections. So far only univariate CARMA processes have been defined and investigated. However, in order to model the joint behaviour of several time series (e.g. prices of various stocks) multivariate models are required. In this chapter we develop multivariate CARMA (MCARMA) processes and study their probabilistic properties.

Unfortunately, it is not straightforward to define multivariate CARMA processes analogously to the univariate ones, as the state space representation (see Chapter 1.2) relies on the ability to exchange the autoregressive and moving average operators, which is only possible in one dimension. Simply taking this approach would lead to a spectral representation which does not reflect the autoregressive moving average structure. Our approach leads to a model which can be interpreted as a solution to the formal differential equation

$$P(D)Y(t) = Q(D)DL(t),$$

where D denotes the differential operator with respect to t, L a Lévy process and P and Q the autoregressive and moving average polynomial, respectively. Moreover, it is the continuous time analogue of the multivariate ARMA model (see e.g. Brockwell & Davis (1991)).

We would like to stress that in this chapter we discuss CARMA processes driven by general Lévy processes, i.e. the Lévy processes may have a Brownian component and does not need to have finite variance, unless stated otherwise.

The results of this chapter are joint work with Robert Stelzer and can also be found in Marquardt & Stelzer (2006).

3.1 State Space Representation of Multivariate CARMA Processes

This section contains the necessary results and insights enabling us to define multivariate CARMA processes. As we shall heavily make use of spectral representations of stationary processes (see Doob (1953), Gikhman & Skorokhod (2004) or Rozanov (1967) for comprehensive treatments), let us briefly recall the notions and results we shall employ.

Definition 3.1 *Let $\mathcal{B}(\mathbb{R})$ denote the Borel-σ-algebra over $\mathbb{R}$. A family $\{\zeta(\Delta)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ of $\mathbb{C}^m$-valued random variables is called an m-dimensional random orthogonal measure, if*

(i) $\zeta(\Delta) \in L^2$ *for all bounded* $\Delta \in \mathcal{B}(\mathbb{R})$*,*

(ii) $\zeta(\emptyset) = 0$*,*

(iii) $\zeta(\Delta_1 \cup \Delta_2) = \zeta(\Delta_1) + \zeta(\Delta_2)$ *a.s., if* $\Delta_1 \cap \Delta_2 = \emptyset$ *and*

(iv) $F : \mathcal{B}(\mathbb{R}) \to M_m(\mathbb{C})$, $\Delta \mapsto E[\zeta(\Delta)\zeta(\Delta)^*]$ *defines a σ-additive positive definite matrix measure (i.e. a σ-additive set function that assumes values in the positive semi-definite matrices) and it holds that* $E[\zeta(\Delta_1)\zeta(\Delta_2)^*] = F(\Delta_1 \cap \Delta_2)$ *for all* $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$*.*

F is referred to as the spectral measure of ζ.

The definition above obviously implies $E[\zeta(\Delta_1)\zeta(\Delta_2)^*] = 0$ for disjoint Borel sets Δ_1, Δ_2.

Stochastic integrals $\int_\Delta f(t)\zeta(dt)$ of deterministic Lebesgue-measurable functions $f : \mathbb{R} \to M_m(\mathbb{C})$ with respect to a random orthogonal measure ζ are now as usually defined in an L^2-sense (see, in particular, Rozanov (1967, Chapter 1) for details). Note that the integration can be understood componentwise: Denoting the coordinates of ζ by ζ_i, i.e. $\zeta = (\zeta_1, \dots, \zeta_m)^*$, the i-th element $\left(\int_\Delta f(t)\zeta(dt)\right)_i$ of $\int_\Delta f(t)\zeta(dt)$ is given by $\sum_{k=1}^m \int_\Delta f_{ik}(t)\zeta_k(dt)$, where the integrals are standard one-dimensional stochastic integrals in an L^2-sense and

$f_{ik}(t)$ denotes the element in the i-th row and k-th column of $f(t)$. The above integral is defined whenever the integral

$$\int_\Delta f(t)F(dt)f(t)^* := \left(\sum_{k,l=1}^m \int_{\mathbb{R}} f_{ik}(t)\overline{f}_{jl}(t)F_{kl}(dt)\right)_{1\leq i,j\leq m}$$

exists. Functions satisfying this condition are said to be in $L^2(F)$. For two functions $f, g \in L^2(F)$ we have

$$E\left[\int_\Delta f(t)\zeta(dt)\left(\int_\Delta g(t)\zeta(dt)\right)^*\right] = \int_\Delta f(t)F(dt)g(t)^*. \tag{3.1}$$

In the following we will only encounter random orthogonal measures, whose associated spectral measures have constant density with respect to the Lebesgue measure λ on $\mathbb{R}$, i.e. $F(dt) = C\lambda(dt) =: C\,dt$ for some positive definite $C \in M_m(\mathbb{C})$, which simplifies the integration theory considerably. In this case it is easy to see that it is sufficient for $\int_\Delta f(t)F(dt)f(t)^*$ to exist that $\int_\Delta \|f(t)\|^2\,dt$ is finite, where $\|\cdot\|$ is some norm on $M_m(\mathbb{C})$. To ease notation we define the space of square-integrable matrix-valued functions

$$L^2(\mathbb{R}; M_m(\mathbb{C})) := \left\{f : \mathbb{R} \to M_m(\mathbb{C}),\ \int_{\mathbb{R}} \|f(t)\|^2 dt < \infty\right\}. \tag{3.2}$$

In the following we abbreviate $L^2(\mathbb{R}; M_m(\mathbb{C}))$ by $L^2(M_m(\mathbb{C}))$. This space is independent of the norm $\|\cdot\|$ on $M_m(\mathbb{C})$ used in the definition and is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_m(\mathbb{C})$ where all components f_{ij} are in the usual space $L^2(\mathbb{C})$.

$$\|f\|_{L^2(M_m(\mathbb{C}))} = \left(\int_{\mathbb{R}} \|f(t)\|^2 dt\right)^{1/2} \tag{3.3}$$

defines a norm on $L^2(M_m(\mathbb{C}))$ and again it is immaterial, which norm we use, as all norms $\|\cdot\|$ on $M_m(\mathbb{C})$ lead to equivalent norms $\|\cdot\|_{L^2(M_m(\mathbb{C}))}$. With this norm $L^2(M_m(\mathbb{C}))$ is a Banach space and even a Hilbert space, provided the original norm $\|\cdot\|$ on $M_m(\mathbb{C})$ is induced by a scalar product. Observe that as usual we do not distinguish between functions and equivalence classes in $L^2(\cdot)$. The integrals $\int_\Delta f(t)\zeta(dt)$ and $\int_\Delta g(t)\zeta(dt)$ agree (in L^2), if f and g are identical in $L^2(M_m(\mathbb{C}))$, and a sequence of integrals $\int_\Delta \|f_n(t)\|^2\,dt$ converges (in L^2) to $\int_\Delta \|f(t)\|^2\,dt$ for $n \to \infty$, if $\|f_n(t) - f(t)\|_{L^2(M_m(\mathbb{C}))} \to 0$ as $n \to \infty$. Moreover,

$$E\left[\int_\Delta f(t)\zeta(dt)\left(\int_\Delta g(t)\zeta(dt)\right)^*\right] = \int_\Delta f(t)Cg(t)^* dt. \tag{3.4}$$

Our first step in the construction of multivariate CARMA processes is the following theorem extending the well-known fact that

$$W(t) = \int_{-\infty}^{\infty} \frac{e^{i\mu t} - 1}{i\mu}\,\phi(d\mu), \qquad t \in \mathbb{R},$$

is an m-dimensional standard Wiener process, if ϕ is an m-dimensional Gaussian random orthogonal measure satisfying $E[\phi(A)] = 0$ and $E[\phi(A)\phi(A)^*] = \frac{I_m}{2\pi}\lambda(A)$ for all $A \in \mathcal{B}(\mathbb{R})$ (see e.g. Arató (1982, Section 2.1, Lemma 5)).

Theorem 3.2 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided square integrable m-dimensional Lévy process with $E[L(1)] = 0$ and $E[L(1)L(1)^*] = \Sigma_L$. Then there exists an m-dimensional random orthogonal measure Φ_L with spectral measure F_L such that $E[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ,*

$$F_L(dt) = \frac{1}{2\pi}\Sigma_L\,dt \tag{3.5}$$

and

$$L(t) = \int_{-\infty}^{\infty} \frac{e^{i\mu t} - 1}{i\mu}\,\Phi_L(d\mu). \tag{3.6}$$

The random measure Φ_L is uniquely determined by

$$\Phi_L([a,b)) = \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu}\,L(d\mu) \tag{3.7}$$

for all $-\infty < a < b < \infty$.

Proof. Observe that setting $\tilde{\Phi}([a,b)) = L(b) - L(a)$ defines a random orthogonal measure on the semi-ring of intervals $[a,b)$, with $-\infty < a < b < \infty$. Using an obvious multidimensional extension of Rozanov (1967, Theorem 2.1), we extend $\tilde{\Phi}_L$ to a random orthogonal measure on the Borel sets. It is immediate that the associated spectral measure $\tilde{F}_L$ satisfies $\tilde{F}_L(dt) = \Sigma_L\,dt$ and that integrating with respect to $\tilde{\Phi}_L$ is the same as integrating with respect to the Lévy process L.

Now define $\Phi_L([a,b))$ for $-\infty < a < b < \infty$ by (3.7) which is equivalent to

$$\Phi_L([a,b)) = \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu} \tilde{\Phi}_L(d\mu). \tag{3.8}$$

Using (3.4) we obtain for any two intervals $[a,b)$ and $[a',b')$

$$\begin{aligned} E[\Phi_L([a,b))\Phi_L([a',b'))^*] &= \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu} \Sigma_L \overline{\left(\frac{e^{-i\mu a'} - e^{-i\mu b'}}{2\pi i\mu} \right)} d\mu \qquad (3.9) \\ &= \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu} \Sigma_L^{1/2} \left(\frac{e^{-i\mu a'} - e^{-i\mu b'}}{2\pi i\mu} \Sigma_L^{1/2} \right)^* d\mu, \end{aligned}$$

where $\Sigma_L^{1/2}$ denotes the unique square root of Σ_L defined by spectral calculus. The crucial point is now to observe that the function $\hat{\phi}_{a,b}(\mu) = \frac{e^{-i\mu a} - e^{-i\mu b}}{\sqrt{2\pi} i\mu} \Sigma_L^{1/2}$ is the Fourier transform of the function $1_{[a,b)}(t)\Sigma_L^{1/2}$, i.e.

$$\hat{\phi}_{a,b}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\mu t} 1_{[a,b)}(t) \Sigma_L^{1/2} dt.$$

The standard theory of Fourier-Plancherel transforms $\mathcal{F}$ (see e.g. Chandrasekharan (1989, Chapter II) or Yosida (1965, Chapter 6)) extends immediately to the space $L^2(M_m(\mathbb{C}))$ by setting

$$\mathcal{F}_m : L^2(M_m(\mathbb{C})) \to L^2(M_m(\mathbb{C})),\ f(t) \mapsto \hat{f}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\mu t} f(t) dt$$

where $\int_{-\infty}^{\infty} e^{-i\mu t} f(t) dt$ is the limit in $L^2(M_m(\mathbb{C}))$ of $\int_{-R}^{R} e^{-i\mu t} f(t) dt$ as $R \to \infty$, because this can be interpreted as a component-wise Fourier-Plancherel transformation and, as stated before, a function f is in $L^2(M_m(\mathbb{C}))$, if and only if all components f_{ij} are in $L^2(\mathbb{C})$. In particular, $\mathcal{F}_m$ is an invertible continuous linear operator on $L^2(M_m(\mathbb{C}))$ with

$$\mathcal{F}_m^{-1} : L^2(M_m(\mathbb{C})) \to L^2(M_m(\mathbb{C})),\ \hat{f}(\mu) \mapsto f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\mu t} \hat{f}(\mu) d\mu,$$

and Plancherel's identity generalizes to:

$$\int_{\mathbb{R}} f(t)g(t)^* dt = \int_{\mathbb{R}} \hat{f}(\mu)\hat{g}(\mu)^* d\mu. \tag{3.10}$$

Combining (3.9) with (3.10) gives

$$\begin{aligned} E[\Phi_L([a,b))\Phi_L([a',b'))^*] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_{a,b}(\mu)\left(\hat{\phi}_{a',b'}(\mu)\right)^* d\mu \\ &= \frac{\Sigma_L}{2\pi}\int_{-\infty}^{\infty} 1_{[a,b)}(t)1_{[a',b')}(t)\, dt. \end{aligned}$$

This implies immediately that $E[\Phi_L([a,b))\Phi_L([a',b'))^*] = 0$, if $[a,b)$ and $[a',b')$ are disjoint,

$$E[\Phi_L([a,b))\Phi_L([a,b))^*] = \frac{\Sigma_L \lambda([a,b))}{2\pi}$$

and that Φ_L is a random orthogonal measure on the semi-ring of intervals $[a,b)$, which we extend to one on all Borel sets. Therefore, (3.8) extends to

$$\int_{-\infty}^{\infty} 1_\Delta(t)\Phi_L(dt) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\phi}_\Delta(\mu)\,\tilde{\Phi}_L(d\mu) \tag{3.11}$$

for all Borel sets Δ, where $\hat{\phi}_\Delta = \mathcal{F}_m(1_\Delta)$ is the Fourier transform of 1_Δ.

For any function $\varphi \in L^2(M_m(\mathbb{C}))$ there is a sequence of elementary functions $\varphi_k(t)$, $k \in \mathbb{N}$, (i.e. matrix-valued functions of the form $\sum_{i=1}^N C_i 1_{\Delta_i}(t)$ with appropriate $N \in \mathbb{N}, C_i \in M_m(\mathbb{C})$ and Borel sets Δ_i) which converges to φ in $L^2(M_m(\mathbb{C}))$. As the Fourier-Plancherel transform is a topological isomorphism that maps $L^2(M_m(\mathbb{C}))$ onto itself, the Fourier-Plancherel transforms $\hat{\varphi}_k(t)$ converge to the Fourier-Plancherel transform $\hat{\varphi}(t)$ in $L^2(M_m(\mathbb{C}))$, which allows us to extend (3.11), exchanging the roles of μ and t, to

$$\int_{-\infty}^{\infty} \varphi(\mu)\,\Phi_L(d\mu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\varphi}(t)\,\tilde{\Phi}_L(dt) \tag{3.12}$$

for all functions φ in $L^2(M_m(\mathbb{C}))$ and their Fourier-Plancherel transforms $\hat{\varphi}$. Now choose $\varphi(\mu) = (e^{i\mu b} - e^{i\mu a})/(i\mu)$, then $\hat{\varphi}(t) = \sqrt{2\pi}1_{[a,b]}(t)$. This shows that

$$\int_{-\infty}^{\infty} \frac{e^{i\mu b} - e^{i\mu a}}{i\mu}\Phi_L(d\mu) = L(b) - L(a)$$

and thus (3.6) is shown.

The uniqueness of Φ_L follows easily, as (3.6) implies (3.12) using arguments analogous to the above ones. ☐

Note that for one-dimensional random orthogonal measures such results can already be found in Doob (1953, Section IX.4).

Remark 3.3 If we formally differentiate (3.6), we obtain

$$\frac{dL(t)}{dt} = \int_{-\infty}^{\infty} e^{i\mu t}\Phi_L(d\mu),$$

as in the spectral representation differentiation is the transform given by

$$\int_{-\infty}^{\infty} e^{i\mu t}\Phi(d\mu) \mapsto \int_{-\infty}^{\infty} i\mu e^{i\mu t}\Phi(d\mu).$$

Thus, a univariate CARMA processes should have the representation

$$Y(t) = \int_{-\infty}^{\infty} e^{i\mu t}\frac{q(i\mu)}{p(i\mu)}\Phi_L(d\mu), \tag{3.13}$$

as this reflects the differential equation (1.22). Later, in Theorem 3.19, we will see that this is indeed the case. The square integrability necessary for (3.13) to be defined, explains why one can only consider CARMA processes with $q < p$ (c.f. Lemma 3.8).

The next lemma deals with the spectral representation of integrals of processes.

Lemma 3.4 *Let Φ be an m-dimensional random orthogonal measure with spectral measure $F(dt) = C\,dt$ for some positive definite $C \in M_m(\mathbb{C})$ and $g \in L^2(M_m(\mathbb{C}))$. Define the m-dimensional random process $G = \{G(t)\}_{t\in\mathbb{R}}$ by*

$$G(t) = \int_{-\infty}^{\infty} e^{i\mu t} g(i\mu)\,\Phi(d\mu).$$

Then G is weakly stationary,

$$\int_0^t G(s)\,ds < \infty \quad a.s. \quad \textit{for every} \quad t > 0 \qquad \textit{and}$$

$$\int_0^t G(s)\,ds = \int_{-\infty}^{\infty} \frac{e^{i\mu t}-1}{i\mu} g(i\mu)\,\Phi(d\mu), \quad t > 0.$$

Proof. Weak stationarity follows immediately from (3.4), which implies

$$E[G(t)G(s)^*] = \int_{-\infty}^{\infty} e^{i\mu(t-s)} g(i\mu) C g(i\mu)^* d\mu.$$

The weak stationarity implies that

$$\|G(s)\|_{L_2} := E[\|G(s)\|_2^2]^{1/2} = E[G(s)^* G(s)]^{1/2}$$

is finite and constant, where $\|\cdot\|_2$ denotes the Euclidean norm. Thus an elementary Fubini argument and using $\|\cdot\|_{L^1} \leq \|\cdot\|_{L^2}$ gives

$$\begin{aligned} E\left\|\int_0^t G(s)ds\right\|_2 &\leq E\left[\int_0^t \|G(s)\|_2 ds\right] = \int_0^t E\left[\|G(s)\|_2\right] ds \\ &\leq \int_0^t \|G(s)\|_{L_2} ds < \infty. \end{aligned}$$

In particular, $\int_0^t G(s)ds$ is almost surely finite. Finally, we obtain

$$\begin{aligned} \int_0^t G(s)ds &= \int_0^t \int_{-\infty}^{\infty} e^{i\mu s} g(i\mu)\,\Phi(d\mu) ds = \int_{-\infty}^{\infty} \int_0^t e^{i\mu s} ds\, g(i\mu)\,\Phi(d\mu) \\ &= \int_{-\infty}^{\infty} \frac{e^{i\mu t}-1}{i\mu} g(i\mu)\,\Phi(d\mu), \end{aligned}$$

using a stochastic version of Fubini's theorem (e.g. the obvious multidimensional extension of Gikhman & Skorokhod (2004, Section IV.4, Lemma 4)). □

Before turning to a theorem enabling us to define MCARMA processes we establish three lemmata and one corollary which contain necessary technical results relating the zeros of what is to become the autoregressive polynomial to the spectrum of a particular matrix A. The first lemma contains furthermore some additional insight into the eigenvectors of A.

Lemma 3.5 *Let* $A_1, \ldots, A_p \in M_m(\mathbb{C})$, $p \in \mathbb{N}$, *define*

$$P : \mathbb{C} \to M_m(\mathbb{C}), \quad z \mapsto I_m z^p + A_1 z^{p-1} + A_2 z^{p-2} + \ldots + A_p$$

and set

$$\mathcal{N}(P) = \{z \in \mathbb{C} : \det(P(z)) = 0\}, \tag{3.14}$$

i.e. $\mathcal{N}(P)$ *is the set of all* $z \in \mathbb{C}$ *such that* $P(z) \notin \mathcal{G}l_m(\mathbb{C})$. *Furthermore, set*

$$A = \begin{pmatrix} 0 & I_m & 0 & \ldots & 0 \\ 0 & 0 & I_m & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & I_m \\ -A_p & -A_{p-1} & \ldots & \ldots & -A_1 \end{pmatrix} \in M_{mp}(\mathbb{C}) \tag{3.15}$$

and denote the spectrum of A *by* $\sigma(A)$. *Then* $\mathcal{N}(P) = \sigma(A)$ *and* $\bar{x} \in \mathbb{C}^{mp} \setminus \{0\}$ *is an eigenvector of* A *with corresponding eigenvalue* λ, *if and only if there is an* $\tilde{x} \in \mathrm{Ker}P(\lambda) \setminus \{0\}$, *such that* $\bar{x} = (\tilde{x}^*, (\lambda\tilde{x})^*, \ldots, (\lambda^{p-1}\tilde{x})^*)^*$. *Moreover,* $0 \in \sigma(A)$, *if and only if* $0 \in \sigma(A_p)$.

Proof. It is immediate from the structure of A that A is of full rank, if and only if A_p is of full rank.

Let λ be an eigenvalue of A and $\bar{x} = (x_1^*, \ldots x_p^*)^* \in \mathbb{R}^{mp}$, $x_i \in \mathbb{R}^m$, a corresponding eigenvector, i.e. $A\bar{x} - \lambda\bar{x} = 0$ from which $\lambda x_1 = x_2$, $\lambda x_2 = x_3, \ldots, \lambda x_{p-1} = x_p$, $\lambda x_p + A_1 x_p + A_2 x_{p-1} + \ldots + A_p x_1 = 0$ follows. Hence, $x_i = \lambda^{i-1} x_1$, $i = 1, 2, \ldots, p$ and

$$\lambda^p x_1 + A_1 \lambda^{p-1} x_1 + A_2 \lambda^{p-2} x_1 + \ldots + A_p x_1 = (I_m \lambda^p + A_1 \lambda^{p-1} + \ldots + A_p) x_1 = 0. \tag{3.16}$$

As $\bar{x} \neq 0$, we have $x_1 \neq 0$ and (3.16) gives $x_1 \in \mathrm{Ker}P(\lambda)$. Hence, we can set $\tilde{x} = x_1$. Furthermore, the non-triviality of the kernel of $P(\lambda)$ implies $\det(P(\lambda)) = 0$. Thus $\mathcal{N}(P) \supseteq \sigma(A)$ has been established.

Now we turn to the converse implication. Let $\lambda \in \mathcal{N}(P)$, then $P(\lambda)$ has a non-trivial kernel. Take any $\tilde{x} \in \mathrm{Ker}P(\lambda) \setminus \{0\}$ and set $\bar{x} = (\tilde{x}^*, (\lambda\tilde{x})^*, \ldots, (\lambda^{p-1}\tilde{x})^*)^*$. Then (3.16) shows that $A\bar{x} = \lambda\bar{x}$ and thus $\lambda \in \sigma(A)$. Therefore $\mathcal{N}(P) \subseteq \sigma(A)$ and $\bar{x}$ is an eigenvector of A to the eigenvalue λ. □

Corollary 3.6 *$\sigma(A) \subseteq (-\infty, 0) + i\mathbb{R}$ if and only if $\mathcal{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$.*

Lemma 3.7 *If $\mathcal{N}(P) \subseteq \mathbb{R}\backslash\{0\} + i\mathbb{R}$, then $P(iz) \in \mathcal{G}l_m(\mathbb{C})$ for all $z \in \mathbb{R}$.*

Proof. As all zeros of $\det(P(z))$ have non-vanishing real part, all zeros of $\det(P(iz))$ must have non-vanishing imaginary part and thus $P(iz)$ is invertible for all $z \in \mathbb{R}$. □

Lemma 3.8 *Let $C_0, C_1, \ldots, C_{p-1} \in M_m(\mathbb{C})$ and $R(z) = \sum_{i=0}^{p-1} C_i z^i$. Assume that $\mathcal{N}(P) \subseteq \mathbb{R}\backslash\{0\} + i\mathbb{R}$, then*

$$\int_{-\infty}^{\infty} \|P(iz)^{-1}R(iz)\|^2 \, dz < \infty,$$

where $P(z) = I_m z^p + A_1 z^{p-1} + ... + A_p$.

Proof. As $\det(P(iz))$, $z \in \mathbb{R}$, has no zeros, $\|P(iz)^{-1}R(iz)\|$ is finite for all $z \in \mathbb{R}$, continuous and thus bounded on any compact set. Hence,

$$\int_{-K}^{K} \|P(iz)^{-1}R(iz)\|^2 \, dz$$

exists for all $K \in \mathbb{R}$. For any $x \in \mathbb{R}^m$ we have

$$\begin{aligned}\|P(z)x\| = \left\|\left(I_m z^p + \sum_{k=0}^{p-1} A_{p-k} z^k\right) x\right\| &\geq \|z^p x\| - \left\|\sum_{k=0}^{p-1} A_{p-k} z^k x\right\| \\ &\geq \left(|z|^p - \sum_{k=0}^{p-1} \|A_{p-k}\| |z^k|\right) \|x\|.\end{aligned}$$

Thus, there is $K > 0$ such that $\|P(z)x\| \geq |z|^p \|x\|/2$ for all z such that $|z| \geq K$, $x \in \mathbb{R}^m$. This implies $\|P(z)^{-1}\| \leq 2|z|^{-p}$ for all $|z| \geq K$ and thus for all $z \in \mathbb{R}$, $|z| \geq K$,

$$\|P(iz)^{-1}R(iz)\|^2 \leq \|P(iz)^{-1}\|^2 \|R(iz)\|^2 \leq \frac{4}{|z|^{2p}} \left(\sum_{i=0}^{p-1} \|C_i\| |z|^i\right)^2,$$

which gives the finiteness of $\int_{-\infty}^{-K} \|P(iz)^{-1}R(iz)\|^2 \, dz$ and $\int_{K}^{\infty} \|P(iz)^{-1}R(iz)\|^2 \, dz$. □

The following result provides the key to be able to define multivariate CARMA processes.

Theorem 3.9 *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be an m-dimensional square-integrable Lévy process with corresponding m-dimensional random orthogonal measure Φ as in Theorem 3.2 and $p, q \in \mathbb{N}_0$, $q < p$ (i.e. $p \geq 1$). Let further $A_1, A_2, \ldots, A_p$, $B_0, B_1, \ldots, B_q \in M_m(\mathbb{R})$, where $B_0 \neq 0$ and define $\beta_1 = \beta_2 = \ldots = \beta_{p-q-1} = 0$ (if $p > q+1$) and $\beta_{p-j} = -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}$ for $j = 0, 1, 2, \ldots, q$. (Alternatively, $\beta_{p-j} = -\sum_{i=1}^{p-j-1} A_i \beta_{p-j-i} + B_{q-j}$ for $j = 0, 1, \ldots, p-1$, setting $B_i = 0$ for $i < 0$.) Assume $\sigma(A) \subseteq (-\infty, 0) + i\mathbb{R}$, which implies $A_p \in \mathcal{G}l_m(\mathbb{R})$.*

Denote by $G = (G_1^*(t), \ldots, G_p^*(t))^*$ an mp-dimensional process and set $\beta^* = \left(\beta_1^*, \ldots, \beta_p^*\right)$. Then the stochastic differential equation

$$dG(t) = AG(t)dt + \beta dL_t \tag{3.17}$$

is uniquely solved by the process G given by

$$\begin{aligned} G_j(t) &= \int_{-\infty}^{\infty} e^{i\lambda t} w_j(i\lambda)\, \Phi(d\lambda), \quad j = 1, 2, \ldots, p, \quad t \in \mathbb{R}, \text{ where} \qquad (3.18) \\ w_j(z) &= \frac{1}{z}(w_{j+1}(z) + \beta_j), \quad j = 1, 2, \ldots, p-1 \quad \text{and} \end{aligned}$$

$$w_p(z) = \frac{1}{z}\left(-\sum_{k=0}^{p-1} A_{p-k} w_{k+1}(z) + \beta_p\right).$$

The strictly stationary process G can also be represented as

$$G(t) = \int_{-\infty}^{t} e^{A(t-s)}\beta\, L(ds), \quad t \in \mathbb{R}. \tag{3.19}$$

Moreover, $G(0)$ and $\{L(t)\}_{t\geq 0}$ are independent, in particular,

$$E[G_j(0)L(t)^*] = 0 \quad \text{for all} \quad t \geq 0, \quad j = 1, 2, \ldots, p.$$

Finally, it holds that

$$w_p(z) = P(z)\left(\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right), \tag{3.20}$$

$$w_1(z) = (P(z))^{-1} Q(z), \tag{3.21}$$

where

$$\begin{aligned} P(z) &= I_m z^p + A_1 z^{p-1} + \ldots + A_p, \\ Q(z) &= B_0 z^q + B_1 z^{q-1} + \ldots + B_q, \end{aligned}$$

denote the autoregressive and moving average polynomial, respectively and $\int\limits_{-\infty}^{\infty} \|w_j(i\lambda)\|^2 \, d\lambda < \infty$ *for all* $j \in \{1, 2, \ldots, p\}$.

Proof. $A_p \in \mathcal{G}l_m(\mathbb{R})$ follows from Lemma 3.5. That (3.19) is the strictly stationary solution of (3.17) is a standard result, since all elements of $\sigma(A)$ have strictly negative real part, and a simple application of Gronwall's Lemma shows that the solution of (3.17) is a.s. unique for all $t \in \mathbb{R}$ (see e.g. Ikeda & Watanabe (1989, Theorem 3.1)). Since $G(0) = \int_{-\infty}^{0} e^{-As} \beta \, L(ds)$ and the processes $\{L(t)\}_{t<0}$ and $\{L(t)\}_{t\geq 0}$ are independent according to our definition (1.7) of L, $G(0)$ and $\{L(t)\}_{t\geq 0}$ are independent.

To prove (3.20) and (3.21) we first show

$$w_j(z) = \frac{1}{z^{p-j}} \left(w_p(z) + \sum_{i=1}^{p-j} \beta_{p-i} z^{i-1} \right) \quad \text{for} \quad j = 1, \ldots, p-1. \tag{3.22}$$

In fact, for $p - j = 1$ (3.22) becomes $w_{p-1} = \frac{1}{z}(w_p(z) + \beta_{p-1})$ which proves the identity for $j = p - 1$ immediately. Assume the identity holds for $j + 1 \in \{2, 3, \ldots, p - 1\}$, then

$$\begin{aligned} w_j(z) = \frac{1}{z}(w_{j+1}(z) + \beta_j) &= \frac{1}{z} \left[\frac{1}{z^{p-j-1}} \left(w_p(z) + \sum_{i=1}^{p-j-1} \beta_{p-i} z^{i-1} \right) + \beta_j \right] \\ &= \frac{1}{z^{p-j}} \left(w_p(z) + \sum_{i=1}^{p-j-1} \beta_{p-i} z^{i-1} + \beta_{p-(p-j)} z^{p-j-1} \right) \\ &= \frac{1}{z^{p-j}} \left(w_p(z) + \sum_{i=1}^{p-j} \beta_{p-i} z^{i-1} \right), \end{aligned}$$

which proves (3.22). Now we turn to (3.20):

$$\begin{aligned} w_p(z) &= \frac{1}{z} \left(- \sum_{k=0}^{p-1} A_{p-k} w_{k+1}(z) + \beta_p \right) \\ &\overset{(3.22)}{=} \frac{1}{z} \left[- \sum_{k=0}^{p-1} A_{p-k} \left(\frac{1}{z^{p-k-1}} \left(w_p(z) + \sum_{i=1}^{p-k-1} \beta_{p-i} z^{i-1} \right) \right) \right] + \frac{\beta_p}{z}. \end{aligned}$$

It follows,

$$\begin{aligned}
\left(I_m + \sum_{k=0}^{p-1} A_{p-k}\frac{1}{z^{p-k}}\right) w_p(z) &= \beta_p z^{-1} - \sum_{k=0}^{p-1}\sum_{i=1}^{p-k-1} A_{p-k}\beta_{p-i} z^{i-1-p+k} \\
\left(I_m z^p + \sum_{k=0}^{p-1} A_{p-k} z^k\right) w_p(z) &= \beta_p z^{p-1} - \sum_{k=0}^{p-1}\sum_{i=1}^{p-k-1} A_{p-k}\beta_{p-i} z^{k+i-1}.
\end{aligned}$$

Set $j = k + i - 1$, then

$$\begin{aligned}
w_p(z) = (P(z))^{-1}&\left(\beta_p z^{p-1} - \sum_{k=0}^{p-2}\sum_{j=k}^{p-2} A_{p-k}\beta_{p+k-j-1} z^j\right) \\
&= (P(z))^{-1}\left(\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right),
\end{aligned}$$

which proves (3.20).

Let now $l \in \{1, 2, \dots, p-1\}$. Then setting $A_0 = I_m$,

$$\begin{aligned}
w_l(z) &= \frac{1}{z^{p-l}}\left(w_p(z) + \sum_{i=1}^{p-l}\beta_{p-i} z^{i-1}\right) \\
&\overset{(3.20)}{=} \frac{1}{z^{p-l}}\left[(P(z))^{-1}\left(\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right) + \sum_{i=1}^{p-l}\beta_{p-i} z^{i-1}\right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}}\left[\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right. \\
&\qquad\left. + \left(\sum_{k=0}^{p} A_{p-k} z^k\right)\left(\sum_{i=1}^{p-l}\beta_{p-i} z^{i-1}\right)\right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}}\left[\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j\right. \\
&\qquad\left. + \left(\sum_{k=0}^{p} A_{p-k} z^k\right)\left(\sum_{i=0}^{p-l-1}\beta_{p-i-1} z^{i}\right)\right] \\
&= \frac{(P(z))^{-1}}{z^{p-l}}\left[\beta_p z^{p-1} - \sum_{j=0}^{p-2}\sum_{k=0}^{j} A_{p-k}\beta_{p+k-j-1} z^j + \sum_{k=0}^{p}\sum_{i=0}^{p-l-1} A_{p-k}\beta_{p-i-1} z^{i+k}\right].
\end{aligned}$$

Setting $j = k + l$ we obtain,

$$
\begin{aligned}
w_l(z) &= \frac{(P(z))^{-1}}{z^{p-l}} \Bigg[\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j \\
&\quad + \sum_{k=0}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \Bigg] \\
&= \frac{(P(z))^{-1}}{z^{p-l}} \Bigg[- \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=0}^{p-l-1} \sum_{j=k}^{p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \\
&\quad + \beta_p z^{p-1} + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=1}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \Bigg] \\
&= \frac{(P(z))^{-1}}{z^{p-l}} \Bigg[\beta_p z^{p-1} - \sum_{j=0}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{j=0}^{p-l-1} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j \\
&\quad + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=0}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \Bigg]
\end{aligned}
$$

It follows,

$$
\begin{aligned}
w_l(z) &= \frac{(P(z))^{-1}}{z^{p-l}} \Bigg[\beta_p z^{p-1} - \sum_{j=p-l}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^j \\
&\quad + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j + \sum_{k=0}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^j \Bigg] \\
&= (P(z))^{-1} \Bigg[\beta_p z^{l-1} - \sum_{j=p-l}^{p-2} \sum_{k=0}^{j} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} \\
&\quad + \sum_{k=p-l}^{p} \sum_{j=k}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} + \sum_{k=1}^{p-l-1} \sum_{j=p-l}^{k+p-l-1} A_{p-k} \beta_{p+k-j-1} z^{j-p+l} \Bigg].
\end{aligned}
$$

The last term in the bracket appears only if $p-l-1 \geq 1$, i.e. $p-l-2 \geq 0$. Therefore, the whole term in the bracket is a polynomial of at most order $p-1$.

Fixing $l = 1$ we obtain,

$$\begin{aligned}
w_1(z) &= P(z)^{-1}\left[\beta_p + \sum_{k=p-1}^{p}\sum_{j=k}^{k+p-2} A_{p-k}\beta_{p+k-j-1}z^{j-p+1}\right.\\
&\quad \left. + \sum_{k=1}^{p-2}\sum_{j=p-1}^{k+p-2} A_{p-k}\beta_{p+k-j-1}z^{j-p+1}\right]\\
&= P(z)^{-1}\left[\beta_p + \sum_{k=p-1}^{p}\sum_{i=k-p+1}^{k-1} A_{p-k}\beta_{k-i}z^i + \sum_{k=1}^{p-2}\sum_{i=0}^{k-1} A_{p-k}\beta_{k-i}z^i\right]\\
&= P(z)^{-1}\left[\beta_p + \sum_{k=1}^{p-1}\sum_{i=0}^{k-1} A_{p-k}\beta_{k-i}z^i + A_0\sum_{i=1}^{p-1}\beta_{p-i}z^i\right]\\
&= P(z)^{-1}\left[\sum_{i=0}^{p-1}\beta_{p-i}z^i + \sum_{i=0}^{p-2}\sum_{k=i+1}^{p-1} A_{p-k}\beta_{k-i}z^i\right].
\end{aligned}$$

Using the fact that $\beta_1 = B_{q-p+1}$, we finally get

$$\begin{aligned}
w_1(z) &= (P(z))^{-1}\left[B_{q-p+1}z^{p-1} + \sum_{i=0}^{p-2}\left(\beta_{p-i} + \sum_{j=1}^{p-i-1} A_j\beta_{p-j-i}\right)z^i\right]\\
&= P(z)^{-1}\left[B_{q-p+1}z^{p-1} + \sum_{i=0}^{p-2} B_{q-i}z^i\right] = P(z)^{-1}\sum_{i=0}^{p-1} B_{q-i}z^i\\
&= P(z)^{-1}\sum_{i=0}^{q} B_{q-i}z^i = P(z)^{-1}Q(z).
\end{aligned}$$

The finiteness of $\int_{-\infty}^{\infty}\|w_j(i\lambda)\|^2\,d\lambda$ for all $j = 1, 2, \ldots, p$ is now a direct consequence of Lemmata 3.7, 3.8 and Corollary 3.6.

It remains to show that the process defined in (3.18) solves (3.17): For $j = 1, \ldots, p$ we have as a consequence of (3.18),

$$G_j(t) - G_j(0) = \int_{-\infty}^{\infty}(e^{i\lambda t} - 1)w_j(i\lambda)\,\Phi(d\lambda). \tag{3.23}$$

For $j = 1, \dots, p-1$ the recursion for w_j together with Lemma 3.4 gives

$$
\begin{aligned}
G_j(t) - G_j(0) &= \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} w_{j+1}(i\lambda)\, \Phi(d\lambda) + \beta_j \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda}\, \Phi(d\lambda) \\
&= \int_0^t \int_{-\infty}^{\infty} w_{j+1}(i\lambda) e^{i\lambda s}\, \Phi(d\lambda)\, ds + \beta_j L(t) \\
&= \int_0^t G_{j+1}(s)\, ds + \beta_j L(t).
\end{aligned}
$$

Hence,

$$
dG_j(t) = G_{j+1}(t)dt + \beta_j dL(t). \tag{3.24}
$$

Analogously we obtain for G_p,

$$
\begin{aligned}
G_p(t) - G_p(0) &= \int_{-\infty}^{\infty} (e^{i\lambda t} - 1) w_p(i\lambda)\, \Phi(d\lambda) \\
&= \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} \left(-\sum_{k=0}^{p-1} A_{p-k} w_{k+1}(i\lambda) + \beta_p \right) \Phi(d\lambda) \\
&= -\sum_{k=0}^{p-1} \int_0^t \int_{-\infty}^{\infty} e^{i\lambda s} A_{p-k} w_{k+1}(i\lambda)\, \Phi(d\lambda)\, ds + \beta_p L(t) \\
&= -\sum_{k=0}^{p-1} A_{p-k} \int_0^t G_{k+1}(s)\, ds + \beta_p L(t) \\
&= -\left(\int_0^t A_p G_1(s) + \dots + A_1 G_p(s)\, ds \right) + \beta_p L(t).
\end{aligned}
$$

Therefore,

$$
dG_p(t) = -(A_p G_1(t) + \ldots + A_1 G_p(t))dt + \beta_p dL(t).
$$

Together with (3.24) this gives that the process G defined by (3.18) solves (3.17). □

Obviously, $E[G(t)] = 0$ for the process $G = \{G(t)\}_{t \in \mathbb{R}}$ which solves (3.17). Turning to the second order properties we have the following proposition.

Proposition 3.10 *Let $G = \{G(t)\}_{t\in\mathbb{R}}$ be the process that solves (3.17). Then its autocovariance matrix function has the form*

$$\Gamma(h) = E[G(t+h)G(t)^*] = e^{Ah}\Gamma(0), \quad h \geq 0, \qquad \textit{with} \tag{3.25}$$

$$\Gamma(0) = \int_0^\infty e^{Au}\beta\Sigma_L\beta^* e^{A^*u}\, du, \tag{3.26}$$

satisfying

$$A\Gamma(0) + \Gamma(0)A^* = -\beta\Sigma_L\beta^*.$$

Proof. As the solution of (3.17) has the representation (3.19) formulae (3.25) and (3.26) are obvious. As we can write

$$G(t) = e^{At}G(0) + \int_0^t e^{A(t-s)}\beta\, L(ds)$$

and $G(0)$ and $L(t)$, $t \geq 0$, are independent, we obtain

$$\begin{aligned}
\Gamma_t &= E[G(t)G(t)^*] \\
&= E\left[\left(e^{At}G(0) + \int_0^t e^{A(t-s)}\beta\, L(ds)\right)\left(e^{At}G(0) + \int_0^t e^{A(t-s)}\beta\, L(ds)\right)^*\right] \\
&= e^{At}\, E[G(0)G(0)^*]\, e^{A^*t} + \int_0^t e^{A(t-s)}\beta\Sigma_L\beta^* e^{A^*(t-s)}\, ds \\
&= e^{At}\left[\Gamma_0 + \int_0^t e^{-As}\beta\Sigma_L\beta^* e^{-A^*s}\right] e^{A^*t}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d\Gamma_t}{dt} &= Ae^{At}\left[\Gamma_0 + \int_0^t e^{-As}\beta\Sigma_L\beta^* e^{-A^*s}\, ds\right] e^{A^*t} \\
&\qquad + e^{At}\left[\Gamma_0 + \int_0^t e^{-As}\beta\Sigma_L\beta^* e^{-A^*s}\right] e^{A^*t}A^* + e^{At}e^{-At}\beta\Sigma_L\beta^* e^{-A^*t}e^{A^*t} \\
&= A\Gamma_t + \Gamma_t A^* + \beta\Sigma_L\beta^*
\end{aligned} \tag{3.27}$$

Now let $t > s$. Then

$$\begin{aligned}\Gamma(t,s) &= E[G(t)G(s)^*] \\ &= E\left[\left(e^{At}G(0) + \int_0^t e^{A(t-u)}\beta\, L(du)\right)\left(e^{As}G(0) + \int_0^s e^{A(s-u)}\beta\, L(du)\right)^*\right] \\ &= e^{At}E[G(0)G(0)^*]e^{A^*s} + \int_0^s e^{A(t-u)}\beta\Sigma_L\beta^* e^{A^*(s-u)}\,du \\ &= e^{A(t-s)}e^{As}\left(\Gamma_0 + \int_0^s e^{-Au}\beta\Sigma_L\beta^* e^{-A^*u}\,du\right)e^{A^*s} = e^{A(t-s)}\Gamma_s.\end{aligned}$$

Therefore,

$$\Gamma(t,s) = E[G(t)G(s)^*] = \begin{cases} e^{A(t-s)}\Gamma_s, & t \geq s \\ \Gamma_t e^{A^*(s-t)}, & s \leq t. \end{cases} \tag{3.28}$$

Since the solution (3.19) is strictly stationary we have $\Gamma_t = \Gamma(0)$. Moreover, it follows from (3.27) that $\Gamma(0)$ is the solution of

$$A\Gamma(0) + \Gamma(0)A^* = -\beta\Sigma_L\beta^*. \tag{3.29}$$

□

From Chojnowska-Michalik (1987), Jurek & Mason (1993), Sato & Yamazato (1984) and Wolfe (1982) we know that (3.19) is the unique stationary solution to (3.17), whenever the Lévy measure ν of the driving process $L(t)$ satisfies

$$\int_{\|x\|>1} \log\|x\|\nu(dx) < \infty.$$

This condition is sufficient (and necessary, provided β is injective) for the stochastic integral in (3.19) to exist, as can be seen from substituting $f(t,s) = e^{A(t-s)}\beta 1_{[0,\infty)}(t-s)$ in (1.16) and (1.17). As we shall use this fact later on to define CARMA processes driven by Lévy processes with infinite second moment, we state the following two results on the process G in a general manner.

Proposition 3.11 *For any driving Lévy process $L(t)$, the process $G = \{G(t)\}_{t\in\mathbb{R}}$ solving (3.17) in Theorem 3.9 is a temporally homogeneous strong*

Markov process with an infinitely divisible transition probability $P_t(x, dy)$ having characteristic function

$$\int\limits_{\mathbb{R}^{mp}} e^{i\langle u,y\rangle}\, P_t(x, dy) = \exp\left\{ i\langle x, e^{A^*t}u\rangle + \int\limits_0^t \psi_L((e^{Av}\beta)^* u)\, dv \right\}, \quad u \in \mathbb{R}^{mp}. \tag{3.30}$$

Proof. See (Sato & Yamazato 1984, Th. 3.1) and additionally (Protter 2004, Theorem V.32) for the strong Markov property. □

Proposition 3.12 *Consider the unique solution $G = \{G(t)\}_{t\geq 0}$ of (3.17) with initial value $G(0)$ independent of $L = \{L(t)\}_{t\geq 0}$, where L is a Lévy process on $\mathbb{R}^m$ with generating triplet (γ, σ, ν) satisfying $\int_{\|x\|>1} \log \|x\|\, \nu(dx) < \infty$. Let $\mathcal{L}(G(t))$ denote the marginal distribution of the process $G = \{G(t)\}_{t\geq 0}$ at time t. Then there exists a limit distribution F such that*

$$\mathcal{L}(G(t)) \to F \qquad \text{as } t \to \infty.$$

This F is infinitely divisible with generating triplet given by $(\gamma_G^\infty, \sigma_G^\infty, \nu_G^\infty)$, where

$$\begin{aligned}
\gamma_G^\infty &= \int\limits_0^\infty e^{As}\beta\gamma\, ds + \int\limits_0^\infty \int\limits_{\mathbb{R}^m} e^{As}\beta x [1_{\{\|e^{As}\beta x\|\leq 1\}} - 1_{\{\|x\|\leq 1\}}]\, \nu(dx)\, ds, \\
\sigma_G^\infty &= \int\limits_0^\infty e^{As}\beta\, \sigma\beta^* e^{A^*s}\, ds, \\
\nu_G^\infty(B) &= \int\limits_0^\infty \int\limits_{\mathbb{R}^m} 1_B(e^{As}\beta x)\, \nu(dx)\, ds.
\end{aligned}$$

Moreover,

$$E\left[e^{i\langle u,F\rangle}\right] = \exp\left\{ \int\limits_0^\infty \psi_L((e^{As}\beta)^* u)\, ds \right\}, \quad u \in \mathbb{R}^{mp}. \tag{3.31}$$

Proof. From (3.30) the characteristic function of $G(t)$ is

$$E\left[e^{i\langle u,G(t)\rangle}\right] = \left(E\left[e^{i\langle u,e^{At}G(0)\rangle}\right]\right) \exp\left\{ \int\limits_0^t \psi_L((e^{As}\beta)^* u)\, ds \right\}, \tag{3.32}$$

where

$$\int_0^t \psi_L((e^{As}\beta)^* u)\, ds = i\langle \gamma_G^t, u\rangle - \frac{1}{2}\langle u, \sigma_G^t u\rangle + \int_{\mathbb{R}^m} [e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_{\{\|x\|\le 1\}}]\, \nu_G^t(dx), \tag{3.33}$$

$$\begin{aligned} \gamma_G^t &= \int_0^t e^{As}\beta\gamma\, ds + \int_0^t \int_{\mathbb{R}^m} e^{As}\beta x [1_{\{\|e^{As}\beta s\|\le 1\}} - 1_{\{\|x\|\le 1\}}]\, \nu(dx)\, ds, \\ \sigma_G^t &= \int_0^t e^{As}\beta\, \sigma\beta^* e^{A^* s}\, ds, \\ \nu_G^t(B) &= \int_0^t \int_{\mathbb{R}^m} 1_B(e^{As}\beta x)\, \nu(dx)\, ds, \end{aligned}$$

using (1.18) and (1.19). Therefore, as $t \to \infty$,

$$\begin{aligned} \int_{\|x\|\le 1} \|x\|^2\, \nu_G^t(dx) &= \int_0^t \int_{\mathbb{R}^m} \|e^{As}\beta x\|^2 1_{\{\|e^{As}\beta x\|\le 1\}}\, \nu(dx)\, ds \\ &\to \int_0^\infty \int_{\mathbb{R}^m} \|e^{As}\beta x\|^2 1_{\{\|e^{As}\beta x\|\le 1\}}\, \nu(dx)\, ds < \infty, \\ \int_{\|x\|>1} \nu_G^t(dx) &= \int_0^t \int_{\mathbb{R}^m} 1_{\{\|e^{As}\beta x\|>1\}}\, \nu(dx)\, ds \\ &\to \int_0^\infty \int_{\mathbb{R}^m} 1_{\{\|e^{As}\beta x\|>1\}}\, \nu(dx)\, ds < \infty, \\ \gamma_G^t \to \int_0^\infty e^{As}\beta\gamma\, ds &+ \int_0^\infty \int_{\mathbb{R}^m} e^{As}\beta x [1_{\{\|e^{As}\beta x\|\le 1\}} - 1_{\{\|x\|\le 1\}}]\, \nu(dx)\, ds < \infty, \\ \sigma_G^t &\to \int_0^\infty e^{As}\beta\, \sigma\beta^* e^{A^* s}\, ds < \infty. \end{aligned}$$

The convergences above follow from Sato & Yamazato (1984) and Sato (2005). Hence, it is shown that as $t \to \infty$, γ_G^t and σ_G^t tend to γ_G^∞ and σ_G^∞, respectively. Moreover, ν_G^t increases to the measure ν_G^∞ satisfying $\int(\|x\|^2 \wedge 1)\, \nu_G^\infty(dx) < \infty$, i.e. ν_G^∞ is a Lévy measure. By dominated convergence the right-hand side of (3.33) tends to

$$i\langle\gamma_G^\infty, u\rangle - \frac{1}{2}\langle u, \sigma_G^\infty u\rangle + \int_{\mathbb{R}^m} [e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle 1_{\{\|x\|\le 1\}}]\, \nu_G^\infty(dx)$$

for all $u \in \mathbb{R}^m$. Thus, there exists an infinitely divisible distribution F with triplet $(\gamma_G^\infty, \sigma_G^\infty, \nu_G^\infty)$ which satisfies (3.31). Convergence of $\mathcal{L}(G(t))$ to F is then a consequence of (3.31) and (3.32) and F does not depend on $G(0)$. □

Remark 3.13 Obviously F is also the marginal distribution of the stationary solution considered in Theorem 3.9.

The sample path behaviour of the process $G = \{G(t)\}_{t\in\mathbb{R}}$ is described below.

Proposition 3.14 *If the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ of the process $G = \{G(t)\}_{t\in\mathbb{R}}$ in Theorem 3.9 is Brownian motion, the sample paths of G are continuous. Otherwise the process G has a jump, whenever L has one. In particular, $\Delta G(t) = \beta\Delta L(t)$.*

3.2 Multivariate CARMA Processes

We are now in a position to define an m-dimensional CARMA (MCARMA) process by using the spectral representation for square-integrable driving Lévy processes. Then we extend this definition making use of the insight obtained in Theorem 3.9.

Definition 3.15 (MCARMA Process) *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be a two-sided square integrable m-dimensional Lévy-process with $E[L(1)] = 0$ and $E[L(1)L(1)^T] = \Sigma_L$. An m-dimensional Lévy-driven continuous time autoregressive moving average process $\{Y(t)\}_{t\in\mathbb{R}}$ of order (p, q), $p > q$ (MCARMA(p, q) process) is defined as*

$$Y(t) = \int_{-\infty}^{\infty} e^{i\lambda t} P(i\lambda)^{-1} Q(i\lambda)\, \Phi(d\lambda), \quad t \in \mathbb{R}, \qquad \textit{where} \tag{3.34}$$

$$P(z) := I_m z^p + A_1 z^{p-1} + ... + A_p, \tag{3.35}$$

$$Q(z) := B_0 z^q + B_1 z^{q-1} + ... + B_q \quad \textit{and} \tag{3.36}$$

Φ is the Lévy orthogonal random measure of Theorem 3.2 satisfying

$$E[\Phi(d\lambda)] = 0 \quad \textit{and} \quad E[\Phi(d\lambda)\Phi(d\lambda)^*] = \frac{d\lambda}{2\pi}\Sigma_L.$$

Here $A_j \in M_m(\mathbb{R})$, $j = 1, ..., p$ and $B_j \in M_m(\mathbb{R})$ are matrices satisfying $B_q \neq 0$ and

$$\mathcal{N}(P) := \{z \in \mathbb{C} : \det(\mathrm{P(z)}) = 0\} \subset (-\infty, 0) + \mathrm{i}\mathbb{R}.$$

The process G defined as in Theorem 3.9 is called the state space representation of the MCARMA process Y*.*

Remark 3.16 (i) There are several reasons why the name "multivariate continuous time ARMA process" is indeed appropriate. The same arguments as in Remark 3.3 show that an MCARMA process Y can be interpreted as a solution to the p-th order formal m-dimensional differential equation

$$P(D)Y(t) = Q(D)DL(t),$$

where D denotes the differentiation operator. Moreover, the upcoming Theorem 3.19 shows that for $m = 1$ the well-known univariate CARMA processes are obtained and finally, the spectral representation (3.34) is the obvious continuous time analogue of the spectral representation of multivariate discrete time ARMA processes (see, for instance, Brockwell & Davis (1991, Section 11.8)).

(ii) The well-definedness is ensured by Lemma 3.8. Observe also that, if $\det(\mathrm{P(z)})$ has zeros with positive real part, all assertions of Theorem 3.9 except the alternative representation (3.19) and the independence of $G(0)$ and $\{L(t)\}_{t\geq 0}$ remain still valid interpreting the stochastic differential equation as an integral equation as in the proof of the theorem. However, in this case the process is no longer causal, i.e. adapted to the natural filtration of the driving Lévy process. In the following we focus on the causal case.

(iii) Assuming $E[L(1)] = 0$ is actually no restriction.
If $E[L(1)] = \mu_L \neq 0$, one simply observes that $\tilde{L}(t) = L(t) - \mu_L t$ has zero

expectation and

$$P(D)^{-1}Q(D)DL(t) = P(D)^{-1}Q(D)D\tilde{L}(t) + P(D)^{-1}Q(D)\mu_L.$$

The first term simply is the MCARMA process driven by $\tilde{L}_t$ and the second an ordinary differential equation having the unique "stationary" solution $-A_p^{-1}B_q\mu_L$, as simple calculations show. Thus, the definition can be immediately extended to $E[L(1)] \neq 0$. Moreover, it is easy to see that the SDE representation given in Theorem 3.9 still holds and one can also extend the spectral representation by adding an atom with mass μ_L to $\Phi_{\tilde{L}}$ at 0.

(iv) Furthermore, observe that the representation of MCARMA processes by the stochastic differential equation (3.17) is a continuous time version of state space representations for (multivariate) ARMA processes as given in Brockwell & Davis (1991, Example 12.1.5) or Wei (1990, p. 387). In the univariate Gaussian case it can already be found in Arató (1982, Lemma 3, Chapter 2.2).

As already noted before, we extend the definition of MCARMA processes to driving Lévy processes L with finite logarithmic moment using Theorem 3.9. As they agree with the above defined MCARMA processes, when L is square-integrable, and are always causal, we call them causal MCARMA processes.

Definition 3.17 (Causal MCARMA Process) *Let $L = \{L(t)\}_{t\in\mathbb{R}}$ be an m-dimensional Lévy process satisfying*

$$\int_{\|x\|>1} \log \|x\| \, \nu(dx) < \infty, \tag{3.37}$$

$p, q \in \mathbb{N}_0$ with $q < p$, and further $A_1, A_2, \ldots, A_p, B_0, B_1, \ldots, B_q \in M_m(\mathbb{R})$, where $B_0 \neq 0$. Define the matrices A, β and the polynomial P as in Theorem 3.9 and assume $\sigma(A) = \mathcal{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$. Then the m-dimensional process

$$Y(t) = \left(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})}\right) G(t) \tag{3.38}$$

where G is the unique stationary solution to

$$dG(t) = AG(t)dt + \beta dL(t)$$

is called causal MCARMA(p, q) *process. Again* G *is referred to as the state space representation.*

Remark 3.18 In the following we will write "MCARMA" when referring to Definition 3.15 and "causal MCARMA" when we refer to Definition 3.17. Moreover, we write "(causal) MCARMA" when referring to both Definitions 3.15 and 3.17.

Let us now state a result extending the short memory moving average representation of univariate CARMA processes to our MCARMA processes and showing that our definition is in line with univariate CARMA processes.

Theorem 3.19 *Analogously to a one-dimensional CARMA process (see Section 1.2), the MCARMA process (3.34) can be represented as a moving average process*

$$Y(t) = \int_{-\infty}^{\infty} g(t-s)\, L(ds), \quad t \in \mathbb{R}, \tag{3.39}$$

where the kernel matrix function $g : \mathbb{R} \to M_m(\mathbb{R})$ *is given by*

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, d\mu. \tag{3.40}$$

Proof. Using the notation of the proof of Theorem 3.2 we obtain this immediately from (3.12):

$$\begin{aligned} Y(t) &= \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, \Phi(d\mu) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mu(t-s)} P(i\mu)^{-1} Q(i\mu)\, d\mu\, \tilde{\Phi}_L(ds) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mu(t-s)} P(i\mu)^{-1} Q(i\mu)\, d\mu\, L(ds) \\ &= \int_{-\infty}^{\infty} g(t-s)\, L(ds). \end{aligned}$$

□

Remark 3.20 For causal MCARMA processes with representation (3.38) an analogous result holds with the kernel function g replaced by

$$\tilde{g}(s) = (I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})e^{As}\beta 1_{[0,\infty)}(s).$$

Moreover, the function g simplifies in the square-integrable causal case as the following extension of a well-known result for univariate CARMA processes shows.

Lemma 3.21 *Assume that $\sigma(A) = \mathcal{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$. Then the function g given in (3.40) vanishes on the negative real line.*

Proof. We need the following consequence of the residue theorem from complex analysis (cf., for instance, Lang (1993, Section VI.2, Theorem 2.2)):

Let q and $p : \mathbb{C} \mapsto \mathbb{C}$ be polynomials where p is of higher degree than q. Assume that p has no zeros on the real line. Then

$$\int_{-\infty}^{\infty} \frac{q(t)}{p(t)} \exp(i\alpha t)dt \;=\; 2\pi i \sum_{z\in\mathbb{C}:\Im(z)>0, p(iz)=0} Res\,(f,z) \;\;\forall\, \alpha > 0, \quad (3.41)$$

$$\int_{-\infty}^{\infty} \frac{q(t)}{p(t)} \exp(i\alpha t)dt \;=\; -2\pi i \sum_{z\in\mathbb{C}:\Im(z)<0, p(iz)=0} Res\,(f,z) \;\;\forall\, \alpha < 0 \quad (3.42)$$

with $f : \mathbb{C} \mapsto \mathbb{C}$, $z \mapsto \frac{q(z)}{p(z)} \exp(i\alpha z)$ and $Res(f, a)$ denoting the residual of the function f at point a.

Turning to our function g, we have from elementary matrix theory that

$$P(iz)^{-1}Q(iz) = \frac{S(z)}{\det(P(iz))}$$

where $S : \mathbb{C} \mapsto M_m(\mathbb{C})$ is some matrix-valued polynomial in z. Observe that $\det(P(iz))$ is a complex-valued polynomial in z and that Lemma 3.8 applied to $R = Q$ implies that $\det(P(iz))$ is of higher degree than $S(z)$. Thus, we can apply the above stated results from complex function theory componentwise to (3.40). But as all zeros of $\det(P(z))$ are in the left half plane $(-\infty, 0) + i\mathbb{R}$, all zeros of $\det(P(iz))$ are in the upper half plane $\mathbb{R} + i(0,\infty)$ and therefore (3.42) shows that

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1}Q(i\mu)\, d\mu = 0 \text{ for all } t < 0.$$

□

Remark 3.22 The above result again reflects the causality, i.e. that the MCARMA process $Y(t)$ only depends on the past of the driving Lévy process, i.e. on $\{L(s)\}_{s\leq t}$. Similarly g vanishes on the positive half line, if $\mathcal{N}(P) \subset (0,\infty) + i\mathbb{R}$. In this case the MCARMA process $Y(t)$ depends only on the future of the driving Lévy process, i.e. on $\{L(s)\}_{s\geq t}$. In all other non-causal cases the MCARMA process depends on the driving Lévy process at all times.

Using the kernel representations, strict stationarity of MCARMA processes is obtained by applying Applebaum (2004, Theorem 4.3.16).

Proposition 3.23 *The (causal) MCARMA process is strictly stationary.*

Furthermore, we can characterize the stationary distribution by applying representation (3.39) and the results mentioned in Section 1.1.2.

Proposition 3.24 *If the driving Lévy process L has characteristic triplet (γ,σ,ν), then the distribution of the MCARMA process $Y(t)$ is infinitely divisible for all $t \in \mathbb{R}$ and the characteristic triplet of the stationary distribution is $(\gamma_Y^\infty, \sigma_Y^\infty, \nu_Y^\infty)$, where*

$$\begin{aligned}
\gamma_Y^\infty &= \int_{\mathbb{R}} g(s)\gamma\, ds + \int_{\mathbb{R}}\int_{\mathbb{R}^m} g(s)x[1_{\{\|g(s)x\|\leq 1\}} - 1_{\{\|x\|\leq 1\}}]\,\nu(dx)\,ds,\\
\sigma_Y^\infty &= \int_{\mathbb{R}} g(s)\sigma g^*(s) ds\\
\nu_Y^\infty(B) &= \int_{\mathbb{R}}\int_{\mathbb{R}^m} 1_B(g(s)x)\,\nu(dx)\,ds. \qquad (3.43)
\end{aligned}$$

For a causal MCARMA process the same result holds with g replaced by $\tilde{g}$.

3.3 Further Properties of MCARMA Processes

Having defined multivariate CARMA processes above, we analyse their probabilistic behaviour further in this section. First we turn to the second order properties.

Proposition 3.25 *Let $Y = \{Y(t)\}_{t\in\mathbb{R}}$ be the MCARMA process defined by (3.34). Then its autocovariance matrix function is given by*

$$\Gamma_Y(h) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda h} P(i\lambda)^{-1}Q(i\lambda)\Sigma_L Q(i\lambda)^*(P(i\lambda)^{-1})^*\,d\lambda, \quad h\in\mathbb{R}.$$

Proof. It follows directly from the spectral representation (3.34) that the MCARMA process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ has the spectral density

$$f_Y(\lambda) = \frac{1}{2\pi}P(i\lambda)^{-1}Q(i\lambda)\Sigma_L Q(i\lambda)^*(P(i\lambda)^{-1})^*, \quad \lambda\in\mathbb{R}. \tag{3.44}$$

The autocovariance function is the Fourier transform of (3.44). □

Remark 3.26 Note that in Proposition 3.10 we already obtained an expression for the autocovariance matrix function of the process $\{G(t)\}_{t\in\mathbb{R}}$ of Theorem 3.9: The upper left $m\times m$ block of (3.25) is also equal to Γ_Y.

Regarding the general existence of moments, it is mainly the driving Lévy process that matters.

Proposition 3.27 *Let Y be a causal MCARMA process and assume that the driving Lévy process L is in $L^r(\Omega, P)$ for some $r > 0$. Then Y and its state space representation G are in $L^r(\Omega, P)$. Provided β is injective, the converse is true as well for G.*

Proof. We use the general fact that an infinitely divisible distribution with characteristic triplet (γ, σ, ν) has finite r-th moment, if and only if

$$\int_{\|x\|>C} \|x\|^r \nu(dx) < \infty$$

for one and hence all $C > 0$ (see Sato (1999, Corollary 25.8)). Using the Kernel representation (3.39) with

$$\tilde{g}(s) = (I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})e^{As}\beta 1_{[0,\infty)}(s),$$

and the fact that there are $C, c > 0$ such that

$$\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})e^{As}\beta\| \le Ce^{-cs}$$

we obtain for the stationary distribution of Y

$$\begin{aligned}\int_{\|x\|>1} \|x\|^r \nu_Y^\infty(dx) &= \int_0^\infty \int_{\mathbb{R}^m} 1_{[1,\infty)}\left(\left\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})e^{As}\beta x\right\|\right) \\ &\quad \times \left\|(I_m, 0_{M_m(\mathbb{C})}, \ldots, 0_{M_m(\mathbb{C})})e^{As}\beta x\right\|^r \nu(dx)ds \\ &\leq \int_0^\infty \int_{\mathbb{R}^m} 1_{[1,\infty)}\left(Ce^{-cs}\|x\|\right) C^r e^{-rcs} \|x\|^r \nu(dx)ds \\ &= \int_{\|x\|>1/C} \int_0^{\frac{\log(1/(C\|x\|))}{-c}} C^r e^{-rcs}\|x\|^r ds\nu(dx) \\ &= \frac{C^r}{rc}\int_{\|x\|>1/C} \left(\|x\|^r - 1/C^r\right)\nu(dx),\end{aligned}$$

which is finite, if and only if L has a finite r-th moment.

Basically the same arguments apply to $G(t) = \int_{-\infty}^t e^{A(t-s)}\beta L(ds)$. Provided β is injective, there are $D, d > 0$ such that $\|e^{As}\beta\| \geq De^{-ds}$ and calculations analogous to the above one lead to a lower bound which establishes the necessity of $L \in L^r(\Omega, P)$ for $G \in L^r(\Omega, P)$. □

Since the characteristic function of $Y(t)$ for each t is explicitly given, we can investigate the existence of a C_b^∞ density, where C_b^∞ denotes the space of bounded continuous, infinitely often differentiable functions whose derivatives are bounded.

Proposition 3.28 *Suppose that there exists an $\alpha \in (0,2)$ and a constant $C > 0$ such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^m} |\langle u, g(t-s)x\rangle|^2 1_{\{|\langle u, g(t-s)x\rangle| \leq 1\}}\, \nu(dx)\, ds \geq C\|u\|^{2-\alpha} \tag{3.45}$$

for any vector u such that $\|u\| \geq 1$. Then the MCARMA process $Y(t)$ has a C_b^∞ density.

The same holds for a causal MCARMA $Y(t)$ process with g replaced by $\tilde{g}$.

Proof. It is sufficient to show that $\int \|u\|^k \|\Phi(u)\|\, du < \infty$ for any non-negative integer k, where Φ denotes the characteristic function of $Y(t)$. (see e.g. Picard (1996, Proposition 0.2))

The characteristic function of the (causal) MCARMA process $Y(t)$ is given by

$$\Phi(u) = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} \left[e^{i\langle u, g(t-s)x\rangle} - 1 - i\langle u, g(t-s)x\rangle 1_{\{|\langle u,g(t-s)x\rangle|\le 1\}} \right] \nu(dx)\, ds \right\},$$

where g stands for either g or $\tilde{g}$. Thus,

$$\begin{aligned}
\|\Phi(u)\| &= \left(\exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} \left[e^{i\langle u,g(t-s)x\rangle} + e^{-i\langle u,g(t-s)x\rangle} - 2 \right] \nu(dx)\, ds \right\} \right)^{1/2} \\
&= \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} (\cos\langle u, g(t-s)x\rangle - 1)\ \nu(dx)\, ds \right\} \\
&\le \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} (\cos\langle u, g(t-s)x\rangle - 1)\, 1_{\{|\langle u,g(t-s)x\rangle|\le 1\}}\, \nu(dx)\, ds \right\},
\end{aligned}$$

as $\cos\langle u, g(t-s)x\rangle - 1 \le 0$. Then, using the inequality $1 - \cos(x) \ge 2(x/\pi)^2$ for $|x| \le \pi$ and assumption (4.30) we have

$$\begin{aligned}
\|\Phi(u)\| &\le \exp\left\{ -\tilde{C} \int_{\mathbb{R}} \int_{\mathbb{R}^m} |\langle u, g(t-s)x\rangle|^2 1_{\{|\langle u,g(t-s)x\rangle|\le 1\}}\, \nu(dx)\, ds \right\} \\
&\le \exp\{-C\|u\|^{2-\alpha}\},
\end{aligned}$$

where $C, \tilde{C} > 0$ are generic constants and the proof is complete.

The inequality $1 - \cos(x) \ge 2(x/\pi)^2$ for $|x| \le \pi$ can be easily shown: Define

$$f(x) = 1 - \cos(x) - 2(x/\pi)^2.$$

Then $f(0) = f(\pi) = 0$ and there is $y \in (0, \pi)$ such that $f'(x) > 0$, $x \in [0, y)$ and $f'(x) < 0$, $x \in (y, \pi]$. Hence, $f(x) > 0$ for all $x \in (0, \pi)$. □

We summarize the sample path behaviour of the MCARMA(p, q) process $Y = \{Y(t)\}_{t\in\mathbb{R}}$, which is immediate from the state space representation (3.17) and the proof of Theorem 3.9.

Proposition 3.29 *(i) If $p > q + 1$, then the (causal) MCARMA(p, q) process $Y = \{Y(t)\}_{t\in\mathbb{R}}$ is $(p - q - 1)$-times differentiable. Using the state space representation $G = \{G(t)\}_{t\in\mathbb{R}}$ we have $\frac{d^i}{dt^i}Y(t) = G_{i+1}(t)$ for $i = 1, 2, \ldots, p - q - 1$.*

(ii) *If $p = q + 1$, then $\Delta Y(t) = \beta_1 \Delta L(t)$, i.e. Y has a jump, whenever L has one.*

(iii) *If the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ of the MCARMA(p, q) process is Brownian motion, the sample paths of Y are continuous and $(p-q-1)$-times continuously differentiable, provided $p > q + 1$.*

Ergodicity and mixing properties (see, for instance, Doukhan (1994) for a comprehensive treatment) have far reaching implications. We thus conclude the analysis of MCARMA processes with a result on their mixing behaviour. Recall the following notions:

Definition 3.30 (cf. Davydov (1973)) *A continuous time stationary stochastic process $X = \{X_t\}_{t\in\mathbb{R}}$ is called* strongly (or α-) mixing, *if*

$$\alpha_l := \sup\left\{|P(A\cap B) - P(A)P(B)| : A \in \mathcal{F}^0_{-\infty}, B \in \mathcal{F}^\infty_l\right\} \to 0$$

as $l \to \infty$, where $\mathcal{F}^0_{-\infty} := \sigma\left(\{X_t\}_{t\leq 0}\right)$ and $\mathcal{F}^\infty_l = \sigma\left(\{X_t\}_{t\geq l}\right)$.

It is said to be β- mixing (or completely regular), *if*

$$\beta_l := E\left(\sup\left\{\left|P(B|\mathcal{F}^0_{-\infty}) - P(B)\right| : B \in \mathcal{F}^\infty_l\right\}\right) \to 0$$

as $l \to \infty$.

Note that $\alpha_l \leq \beta_l$ and thus any β-mixing process is strongly mixing.

Proposition 3.31 *Let Y be a causal MCARMA process and G be its state space representation. If the driving Lévy process L satisfies*

$$\int_{\|x\|>1} \|x\|^r \nu(dx) < \infty \tag{3.46}$$

for some $r > 0$, then G is β-mixing with mixing coefficients $\beta_l = O(e^{-al})$ for some $a > 0$ and Y is strongly mixing. In particular, both G and Y are ergodic.

Proof. As $G(t) = \int_{-\infty}^t e^{A(t-s)}\beta L(ds)$ is a multidimensional Ornstein-Uhlenbeck process driven by the Lévy process βL, we may apply Masuda (2004, Theorem 4.3) noting that (3.46) together with Proposition 3.27 ensure that all conditions are satisfied. Hence, the β-mixing of G with exponentially decaying coefficients is shown. But this implies that $G = (G_1^*, G_2^*, \ldots, G_p^*)^*$ is also strongly mixing, which in turn shows the strong mixing property for Y, since Y is equal to G_1 and it is obvious from the definition of strong mixing that strong mixing of a multidimensional process implies strong mixing of its components. Note that we also obtain $\alpha_l \leq \beta_l$ for the mixing coefficients α_l of Y. Using the well-known result that mixing implies ergodicity concludes the proof. □

For a plot of the sample paths of a 2-dimensional MCARMA$(1, 0)$ we refer to Figure 5.4 in Chapter 5.

We have seen in Proposition 3.25 that the autocorrelations of MCARMA processes are exponentially decaying. Hence, MCARMA processes have short memory. In the following chapter we introduce fractionally integrated MCARMA processes which have long memory in the sense that the autocorrelations are hyperbolically decaying.

4 Multivariate FICARMA Processes

A multivariate analogue of the fractionally integrated continuous time autoregressive moving average (FICARMA) process (see Chapter 1.3, Brockwell (2004) or Brockwell & Marquardt (2005)) is introduced in this chapter (see also Marquardt (2006b)). We show that the multivariate FICARMA process has two kernel representations: as an integral over the fractionally integrated CARMA kernel with respect to a Lévy process and as an integral over the original (not fractionally integrated) CARMA kernel with respect to the corresponding fractional Lévy process (FLP). In order to obtain the latter representation we extend FLPs to the multivariate setting.

4.1 Multivariate Fractional Lévy Processes

4.1.1 Definition and Properties of MFLPs

Fractional Lévy processes have been introduced in Chapter 2 by replacing the Brownian motion in the moving average representation of fractional Brownian motion by a Lévy processes without Gaussian part. Here we extend the definition of a univariate fractional Lévy process to the multivariate setting. Since most results are similar to the univariate case, we only give a brief sketch of the proofs. Furthermore, we would like to stress that we only consider the case where the driving Lévy process L has zero mean and finite second moments, i.e. $\alpha = 2$ in Definition 2.1.

Definition 4.1 (MFLP) *For fractional integration parameter* $0 < d < 0.5$

we define a multivariate fractional Lévy process (MFLP) by

$$M_d(t) = (M_d^1(t), \dots, M_d^m(t))^T = \int_{\mathbb{R}} f_t(s)\, L(ds), \quad t \in \mathbb{R}, \tag{4.1}$$

where the kernel f_t *is defined as in (2.4) and* $L(t) = (L^1(t), \dots, L^m(t))^T$ *is a square-integrable Lévy process on* $\mathbb{R}^m$*, whose components* $L^j = \{L^j(t)\}_{t\in\mathbb{R}}$*,* $j = 1, \dots, m$ *are Lévy processes without Gaussian part on* $\mathbb{R}$ *satisfying* $E[L^j(1)] = 0$ *and* $E[L^j(1)^2] < \infty$*,* $j = 1, \dots m$*.*

The following proposition is obvious. It is a generalization of Theorem 2.10 to the multivariate setting.

Proposition 4.2 *The process* $\{M_d(t)\}_{t\in\mathbb{R}}$ *given in (4.1) is well-defined in* $L^2(\Omega, P)$*. The distribution of* $M_d(t)$ *is infinitely divisible with characteristic triplet* $(\gamma_M^t, 0, \nu_M^t)$*, where*

$$\gamma_M^t = -\int_{\mathbb{R}} \int_{\mathbb{R}^m} f_t(s) x 1_{\{\|f_t(s)x\|>1\}}\, \nu(dx)\, ds \quad \text{and} \tag{4.2}$$

$$\nu_M^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}^m} 1_B(f_t(s)x)\, \nu(dx)\, ds, \tag{4.3}$$

where ν *denotes the Lévy measure of the driving Lévy process* L*. Furthermore, for* $t \in \mathbb{R}$ *and* $z \in \mathbb{R}^m$*,*

$$E[\exp i\langle z, M_d(t)\rangle] = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^m} \left(e^{i\langle z, f_t(s)x\rangle} - 1 - i\langle z, f_t(s)x\rangle\right) \nu(dx)\, ds \right\}. \tag{4.4}$$

Remark 4.3 As M_d is well-defined in an L^2-sense, analogously to the one-dimensional case and (1.11), the process M_d can be represented as

$$M_d(t) = \int_{\mathbb{R}} \int_{\mathbb{R}_0^m} f_t(s) x\, \tilde{J}(dx, ds), \quad t \in \mathbb{R},$$

where $\tilde{J}(dx, ds) = J(dx, ds) - \nu(dx)\, ds$ is the compensated jump measure of the Lévy process L. Moreover, M_d is a.s. equal to the improper Riemann integral

$$M_d(t) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} [(t-s)_+^{d-1} - (-s)_+^{d-1}]\, L(s)\, ds, \quad t \in \mathbb{R}, \tag{4.5}$$

and (4.5) is continuous in t (see Section 2.1.2).

Furthermore, we have the isometry property

$$E[M_d(t)M_d(t)^T] = \|f_t\|^2_{L^2(\mathbb{R})}\Sigma_L. \tag{4.6}$$

and we see that the second-order properties of the MFLP $\{M_d(t)\}_{t\in\mathbb{R}}$ are specified by $E[M_d(t)] = 0$ and covariance matrices

$$\Gamma(s,t) = E[M_d(s)M_d(t)^T] = [\gamma_{ij}(s,t)]_{i,j=1}^m, \quad s,t \in \mathbb{R},$$

where for $s, t \in \mathbb{R}$,

$$\begin{aligned}\gamma_{ij}(s,t) &= E[M_d^i(s)M_d^j(t)] \\ &= \frac{\operatorname{cov}(L^i(1), L^j(1))}{2\Gamma(2d+2)\sin(\pi[d+\frac{1}{2}])}\left[|t|^{2d+1} - |t-s|^{2d+1} + |s|^{2d+1}\right].\end{aligned}$$

Recall that

$$\operatorname{cov}(L^i(1), L^j(1)) = \int_{\mathbb{R}^m} x^i x^j \,\nu(dx),$$

where $x = (x^1, \ldots, x^m)^T \in \mathbb{R}^m$. Hence, the MFLP $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ inherits its dependence structure form the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$.

To the end of this chapter we use the notation

$$\Gamma(h) = E[X(t+h)X(t)^T] = [\gamma_{ij}(h)]_{i,j=1}^m,$$

if the series $\{X(t)\}_{t\in\mathbb{R}^m}$ is stationary. We shall refer to $\Gamma(h)$ as the covariance matrix at lag h. Notice that, if $\{X(t)\}_{t\in\mathbb{R}^m}$ is stationary with covariance matrix function Γ, then for each j, $\{X^j(t)\}_{t\in\mathbb{R}}$, $j = 1, \ldots, m$ is stationary with covariance matrix function γ_{jj}. The function γ_{ij}, $i \neq j$, is called the cross-covariance function of the two series $\{X^i(t)\}_{t\in\mathbb{R}}$ and $\{X^j(t)\}_{t\in\mathbb{R}}$. It should be noted that γ_{ij} is not in general the same as γ_{ji}.

The sample path properties of a MFLP are analogous to the one-dimensional case. We therefore omit the proof of the following proposition and refer to chapter 2.

Proposition 4.4 (Sample Path Properties) *Every MFLP is a long memory process with stationary increments, which cannot be self-similar. Moreover, it is symmetric and Hölder continuous of every order less than d.*

In particular, a MFLP has less smooth sample paths than a fractional Brownian motion. Note also, that the upper bound on the Hölder exponent of the MFLP cannot be improved. In fact, if the Lévy measure $\nu(\mathbb{R}) = \infty$, the sample paths of MFLPs are not Hölder continuous with probability 1 for every order $\beta > d$.

MFLPs are not always semimartingales. In fact, it has been shown in Theorem 2.27 and Corollary 2.31 that this is the case for a fairly large class of Lévy measures ν.

4.1.2 Integration with respect to MFLPs

MFLPs are not always semimartingales and thus ordinary Itô integration theory cannot be applied. Therefore, this section contains the integration theory for stochastic integrals with respect to MFLPs. Our approach is heavily based on the integration theory with respect to a one-dimensional FLP (see Chapter 2).

Again, let the space H be the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\|g\|_H := \left(E[L(1)^2] \int_{\mathbb{R}} (I^d_- g)^2(u)\, du \right)^2, \tag{4.7}$$

where $(I^d_- g)(u)$, $u \in \mathbb{R}$, is the right-sided Riemann-Liouville fractional integral of order d of the function $g : \mathbb{R} \to \mathbb{R}$, $g \in L^1(\mathbb{R})$.

Now let $G : \mathbb{R} \to M_m(\mathbb{R})$ be a matrix function whose components $G_{jk} : \mathbb{R} \to \mathbb{R}$, $j, k = 1, \ldots, m$, are in the space H. To ease notation we write $G \in H_m$. Moreover, let $M_d = \{M_d(t)\}_{t \in \mathbb{R}}$ denote an m-dimensional FLP. Then we define the integral

$$\int_{\mathbb{R}} G(t)\, M_d(dt) \tag{4.8}$$

componentwise as the limit in $L^2(\Omega, P)$ of simple functions $\phi^n_{jk} : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ approximating each component G_{jk} of G in the sense that

$$\|\phi^n_{jk} - G_{jk}\|_H \to 0, \quad \text{as } n \to \infty.$$

Denoting the coordinates of M_d by M^j_d, the j-th element $(\int G(t)\, M_d(dt))^j$ of $\int G(t)\, M_d(dt)$ is then given by $\sum_{k=1}^m \int G_{jk}(t)\, M^k_d(dt)$, where the integrals are one-dimensional stochastic integrals as in (2.50). This reflects that the integration is understood componentwise. As a consequence of Theorem 2.34 the integral (4.8) is well-defined, whenever $G \in H_m$. In fact,

$$\int_{\mathbb{R}} G(t)\, M_d(dt) = \int_{\mathbb{R}} \begin{bmatrix} G_{11}(t) & \dots & G_{1m}(t) \\ \vdots & & \vdots \\ G_{m1}(t) & \dots & G_{mm}(t) \end{bmatrix} \begin{bmatrix} M_d^1(dt) \\ \vdots \\ M_d^m(dt) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{\mathbb{R}} G_{11}(t)\, M_d^1(dt) + \ldots + \int_{\mathbb{R}} G_{1m}(t)\, M_d^m(dt) \\ \vdots \\ \int_{\mathbb{R}} G_{m1}(t)\, M_d^1(dt) + \ldots + \int_{\mathbb{R}} G_{mm}(t)\, M_d^m(dt) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{\mathbb{R}} \frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} G_{11}(s)\, ds\, L^1(du) + \ldots + \int_{\mathbb{R}} \frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} G_{1m}(s)\, ds\, L^m(du) \\ \vdots \\ \int_{\mathbb{R}} \frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} G_{m1}(s)\, ds\, L^1(du) + \ldots + \int_{\mathbb{R}} \frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} G_{mm}(s)\, ds\, L^m(du) \end{bmatrix}$$

$$= \int_{\mathbb{R}} \frac{1}{\Gamma(d)} \begin{bmatrix} \int_u^\infty (s-u)^{d-1} G_{11}(s)\, ds & \dots & \int_u^\infty (s-u)^{d-1} G_{1m}(s)\, ds \\ \vdots & & \vdots \\ \int_u^\infty (s-u)^{d-1} G_{m1}(s)\, ds & \dots & \int_u^\infty (s-u)^{d-1} G_{mm}(s)\, ds \end{bmatrix} \mathbf{L}(du)$$

$$= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^\infty (s-u)^{d-1} \begin{bmatrix} G_{11}(s) & \dots & G_{1m}(s) \\ \vdots & & \vdots \\ G_{m1}(s) & \dots & G_{mm}(s) \end{bmatrix} ds\, \mathbf{L}(du)$$

$$= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^\infty (s-u)^{d-1} G(s)\, ds\, L(du).$$

Hence, analogous to Section 2.3 we obtain

$$\int_{\mathbb{R}} G(t)\, M_d(dt) = \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^\infty (s-u)^{d-1} G(s)\, ds\, L(du), \tag{4.9}$$

where the equality holds in the L^2-sense.

Like in the univariate case, we have the following isometry property.

Proposition 4.5 *Let $F : \mathbb{R} \to M_m(\mathbb{R})$ and $G : \mathbb{R} \to M_m(\mathbb{R})$ be matrix functions with components $F_{ij} : \mathbb{R} \to \mathbb{R}$ and $G_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j = 1, \ldots, m$ such* *that* $|F_{ij}|$, $|G_{ij}|$, $i, j = 1, \ldots, m$ *are in the space* H*. Then*

$$E\left[\left(\int_{\mathbb{R}} F(t)\, M_d(dt)\right)\left(\int_{\mathbb{R}} G(u)\, M_d(du)\right)^T\right]$$
$$= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \int_{\mathbb{R}}\int_{\mathbb{R}} |t-u|^{2d-1} F(t)\Sigma_L G(u)^T\, dt\, du. \tag{4.10}$$

Proof.
Analogous to the proof of Proposition 2.38, we have

$$E\left[\left(\int_{\mathbb{R}} F(t)\, M_d(dt)\right)\left(\int_{\mathbb{R}} G(u)\, M_d(du)\right)^T\right]$$
$$= \frac{1}{(\Gamma(d))^2}\int_{-\infty}^{\infty}\int_{s}^{\infty}\int_{s}^{\infty} F(t)\Sigma_L G(u)^T (t-s)^{d-1}(u-s)^{d-1}\, dt\, du\, ds$$
$$= \frac{1}{(\Gamma(d))^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\Sigma_L G(u)^T \int_{-\infty}^{\min(u,t)} (t-s)^{d-1}(u-s)^{d-1}\, ds\, dt\, du$$
$$= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t)\Sigma_L G(u)^T |t-u|^{2d-1}\, dt\, du,$$

where we have used Fubini's theorem. □

4.1.3 The Spectral Representation of MFLPs

We start with a brief summary of the results on random orthogonal measures obtained in Chapter 3. There it is shown that for every m-dimensional Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ with $E[L(1)] = 0$ and $E[L(1)L(1)^T] = \Sigma_L$ there exists an m-dimensional random orthogonal measure Φ_L such that $E[\Phi_L(\Delta)] = 0$ and $E[\Phi_L(\Delta)\Phi_L(\Delta)^*] = \frac{1}{2\pi}\Sigma_L\Lambda(\Delta)$ for any bounded Borel set Δ, where Λ denotes the Lebesgue measure. The random measure Φ_L is uniquely determined by

$$\Phi_L([a,b)) = \int_{\mathbb{R}} \frac{e^{-i\lambda a} - e^{-i\lambda b}}{2\pi i\lambda}\, L(d\lambda) \tag{4.11}$$

for all $-\infty < a < b < \infty$. Moreover,

$$L(t) = \int_{\mathbb{R}} \frac{e^{i\lambda t} - 1}{i\lambda}\, \Phi_L(d\lambda), \quad t \in \mathbb{R}. \tag{4.12}$$

Finally, for any function $f \in L^2(M_m(\mathbb{C}))$,

$$\int_{\mathbb{R}} f(\lambda)\,\Phi_L(d\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda t} f(\lambda)\,d\lambda\,L(dt) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(t)\,L(dt), \tag{4.13}$$

$$\int_{\mathbb{R}} \hat{f}(t)\,L(dt) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t}\hat{f}(t)\,dt\,\Phi_L(d\lambda) = \sqrt{2\pi}\int_{\mathbb{R}} f(\lambda)\,\Phi_L(d\lambda). \tag{4.14}$$

Here,

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t} f(\lambda)\,d\lambda \quad \text{and} \quad f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda t}\hat{f}(t)\,dt$$

are the Plancherel Fourier transform and the inverse Plancherel Fourier transform, respectively. We will use those results to obtain a spectral representation for MFLPs and integrals with respect to them.

Theorem 4.6 *Let $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ be an m-dimensional FLP. Then M_d has the spectral representation*

$$M_d(t) = \int_{\mathbb{R}} \frac{e^{i\lambda t}-1}{(i\lambda)^{d+1}}\,\Phi_L(d\lambda), \quad t \in \mathbb{R}, \tag{4.15}$$

where Φ_L is the random orthogonal measure defined in (4.11). Furthermore, let

$$\Phi_M([a,b]) = \int_{\mathbb{R}} \frac{1}{(i\lambda)^d} 1_{(a,b)}(\lambda)\,\Phi_L(d\lambda), \quad a < b, \tag{4.16}$$

define a random measure. Then

$$\Phi_M([a,b]) = \int_{\mathbb{R}} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\,M_d(ds). \tag{4.17}$$

Proof. Observe that (Bronstein et al. (1999, Formula 4, p. 1081))

$$\frac{1}{\Gamma(d+1)}\int_{\mathbb{R}} [(b-s)_+^d - (a-s)_+^d] e^{i\lambda s}\,ds = \frac{e^{i\lambda b}-e^{i\lambda a}}{(i\lambda)^{d+1}}. \tag{4.18}$$

Using (4.14) and (4.18) we obtain

$$\begin{aligned} M_d(b) - M_d(a) &= \frac{1}{\Gamma(d+1)}\int_{\mathbb{R}} [(b-s)_+^d - (a-s)_+^d]\,L(ds) \\ &= \frac{1}{\Gamma(d+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} [(b-s)_+^d - (a-s)_+^d]\,e^{i\lambda s}\,ds\,\Phi_L(d\lambda) \\ &= \int_{\mathbb{R}} \frac{e^{i\lambda b}-e^{i\lambda a}}{(i\lambda)^{d+1}}\,\Phi_L(d\lambda). \end{aligned}$$

It remains to prove (4.17):

$$\begin{aligned}
\Phi_M([a,b]) &= \int_{\mathbb{R}} \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, M_d(ds) \\
&= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_u^\infty (s-u)^{d-1} \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, ds\, L(du) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d-1}}{\Gamma(d)} \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, ds\, du\, \Phi_L(d\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d-1}}{\Gamma(d)}\, du\, \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, ds\, \Phi_L(d\lambda) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{i\lambda s}}{(i\lambda)^d} \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, ds\, \Phi_L(d\lambda) = \int_{\mathbb{R}} \frac{1}{(i\lambda)^d} 1_{(a,b)}(\lambda)\, \Phi_L(d\lambda).
\end{aligned}$$

□

Remark 4.7 From the proof of Theorem 4.6 follows that we can write

$$\int_{\mathbb{R}} g(t)\, M_d(dt) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda t}(i\lambda)^{-d} g(t)\, dt\, \Phi_L(d\lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda t} g(t)\, dt\, \Phi_M(d\lambda). \tag{4.19}$$

Figure 4.1 and Figure 4.2 display the sample paths of a 2-dimensional FLP, where the driving Lévy process is a 2-dimensional symmetric truncated stable Lévy process (see (2.42)) and where the dependence of the driving Lévy process is given by the Clayton Lévy copula

$$F_\theta(u,v) = (u^{-\theta} + v^{-\theta})^{-1/\theta} \tag{4.20}$$

for which the conditional distribution function takes a particularly simple form

$$F(v|u) = \frac{\partial F_\theta(u,v)}{\partial u} = \left[1 + \left(\frac{u}{v}\right)^\theta\right]^{-1-1/\theta}. \tag{4.21}$$

As (4.21) can be easily inverted, namely

$$F^{-1}(y|u) = u\left(y^{-\frac{\theta}{1+\theta}} - 1\right)^{-1/\theta},$$

we can use the state space representation (2.13) of a univariate FLP to simulate the sample paths (see Cont & Tankov (2004, Example 6.18) for further details). We simulated trajectories of a MFLP with fractional truncated α-stable margins, where we set $\alpha = 1.8$ and $d = 0.2$. Observe that the dependence increases with the value of θ.

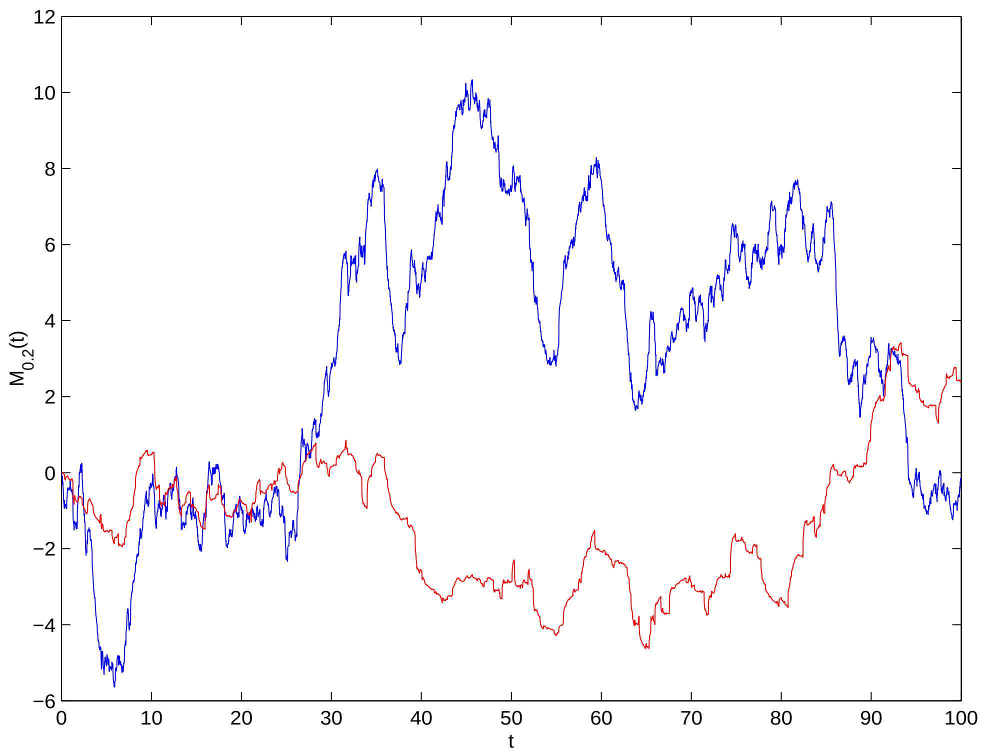

Figure 4.1: The sample path of a 2-dim FLP ($d = 0.2$) with fractional truncated 1.8-stable margins and where the dependence of the driving Lévy process is given by the Clayton Lévy copula (4.20) with $\theta = 0.3$.

4.2 Multivariate FICARMA Processes

Our aim in this section is to define a multivariate FICARMA process, since so far only univariate FICARMA processes have been defined and investigated (see Section 1.3). The advantage of continuous time multivariate modeling is that it allows handling irregularly spaced time series and high frequency data but also modeling the joint behaviour of several time series. For instance, to

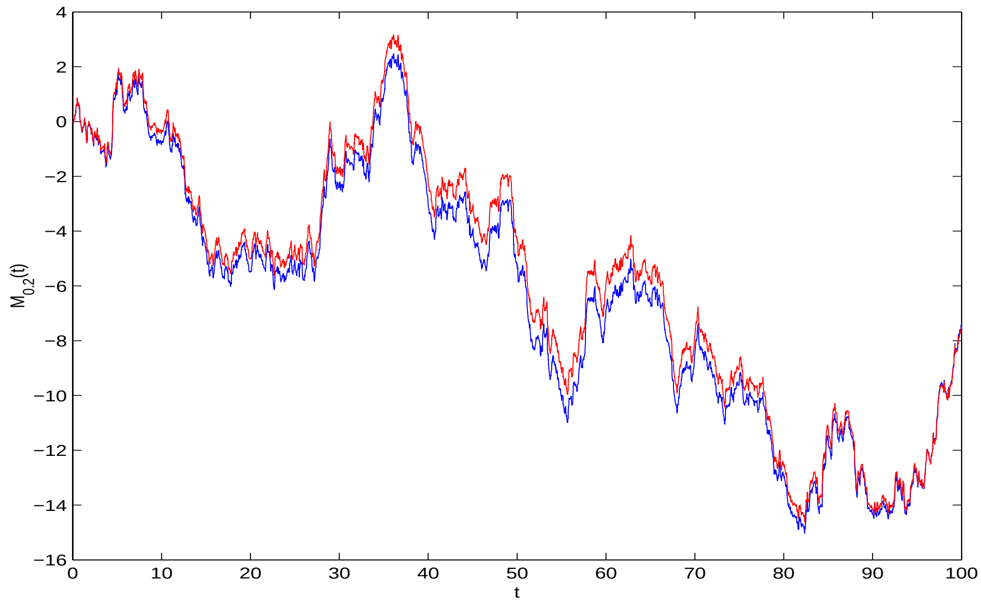

Figure 4.2: The sample path of a 2-dim FLP ($d = 0.2$) with fractional truncated 1.8-stable margins and where the dependence of the driving Lévy process is given by the Clayton Lévy copula (4.20) with $\theta = 20$.

model prices of various stocks on a tic-by-tic basis, continuous time multivariate time series models are required.

4.2.1 Representations of MFICARMA Processes

In one dimension, starting from a short memory moving average process, there are at least two possible ways to construct a long memory moving average process:

(I) a fractional integration of the kernel of the short memory process,

(II) a substitution of the driving Lévy process by the corresponding fractional Lévy process.

Both approaches lead to the same long memory L^2-process (see Theorem 2.43).

We apply approach (I) to MCARMA processes to obtain MFICARMA processes, i.e. we fractionally integrate the MCARMA kernel g as given in (3.40) (and which satisfies $g \in H_m$) and obtain for $t \in \mathbb{R}$,

$$
\begin{aligned}
g_d(t) := (I_+^d g)(t) &= \frac{1}{\Gamma(d)} \int_0^t g(t-u) u^{d-1}\, du \\
&= \frac{1}{2\pi\Gamma(d)} \int_0^t \int_{\mathbb{R}} e^{i\mu(t-u)} P(i\mu)^{-1} Q(i\mu)\, d\mu\, u^{d-1}\, du \\
&= \frac{1}{2\pi\Gamma(d)} \int_{\mathbb{R}} \int_0^t e^{i\mu(t-u)} P(i\mu)^{-1} Q(i\mu) u^{d-1}\, du\, d\mu \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\mu t} (i\mu)^{-d} P(i\mu)^{-1} Q(i\mu)\, d\mu. \qquad (4.22)
\end{aligned}
$$

Note that $g_d(t) = 0$ for all $t \leq 0$ and $g_d \in L^2(M_m(\mathbb{R}))$. Moreover, for $m = 1$ (4.22) is equivalent to (1.41). This leads to the following definition.

Definition 4.8 (MFICARMA Process I) *Let $0 < d < 0.5$. For $p > q$ the multivariate fractionally integrated CARMA(p, d, q) (MFICARMA) process driven by the m-dimensional Lévy process $L = \{L(t)\}_{t \in \mathbb{R}}$ with $E[L(1)] = 0$ and $E[L(1)L(1)^T] = \Sigma_L < \infty$ is defined by*

$$
Y_d(t) = \int_{-\infty}^t g_d(t-s)\, L(ds), \quad t \in \mathbb{R}, \qquad (4.23)
$$

where the fractionally integrated kernel g_d is given as in (4.22) and where the polynomials $P(\cdot)$ and $Q(\cdot)$ are defined as in (3.35) and (3.36), respectively.

Now, we turn our attention to approach (II) and substitute in the MCARMA representation the driving Lévy process by the corresponding MFLP.

Definition 4.9 (MFICARMA Process II) *Let $0 < d < 0.5$. For $p > q$ the multivariate fractionally integrated CARMA(p, d, q) (MFICARMA) process driven by the m-dimensional fractional Lévy process $M_d = \{M_d(t)\}_{t \in \mathbb{R}}$ is defined by*

$$
Y_d(t) = \int_{-\infty}^t g(t-s)\, M_d(ds), \quad t \in \mathbb{R}, \qquad (4.24)
$$

where the kernel g is the CARMA kernel given in (3.40).

Representation (4.24) is equal to (4.23). In fact, using (4.9), we have

$$\int_{\mathbb{R}} g(t-s)\, M_d(ds) = \int_{\mathbb{R}} \begin{bmatrix} g_{11}(t-s) & \dots & g_{1m}(t-s) \\ \vdots & & \vdots \\ g_{m1}(t-s) & \dots & g_{mm}(t-s) \end{bmatrix} \begin{bmatrix} M_d^1(ds) \\ \dots \\ M_d^m(ds) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{\mathbb{R}} g_{11}(t-s)\, M_d^1(ds) + \ldots + \int_{\mathbb{R}} g_{1m}(t-s)\, M_d^m(ds) \\ \vdots \\ \int_{\mathbb{R}} g_{1m}(t-s)\, M_d^1(ds) + \ldots + \int_{\mathbb{R}} g_{mm}(t-s)\, M_d^m(ds) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} g_{11}(t-s)\, ds \right) L^1(du) + \ldots \\ \vdots \\ \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} g_{m1}(t-s)\, ds \right) L^1(du) + \ldots \end{bmatrix}$$

$$\begin{matrix} + \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} g_{1m}(t-s)\, ds \right) L^m(du) \\ \vdots \\ + \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} g_{mm}(t-s)\, ds \right) L^m(du) \end{matrix} \Bigg]$$

$$= \begin{bmatrix} \int_{\mathbb{R}} g_{d_{11}}(t-s)\, L^1(ds) + \ldots + \int_{\mathbb{R}} g_{d_{1m}}(t-s)\, L^m(ds) \\ \vdots \\ \int_{\mathbb{R}} g_{d_{m1}}(t-s)\, L^1(ds) + \ldots + \int_{\mathbb{R}} g_{d_{mm}}(t-s)\, L^m(ds) \end{bmatrix} = \int_{\mathbb{R}} g_d(t-s)\, L(du).$$

Representation (4.23) is useful to obtain distributional and sample path properties, whereas representation (4.24) is useful in simulations (see Chapter 2 for the univariate case). In particular, we use representation (4.24) to obtain a spectral representation for MFICARMA processes.

Theorem 4.10 *The MFICARMA(p, d, q) process $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ has the spectral representation*

$$Y_d(t) = \int_{\mathbb{R}} e^{i\mu t} (i\mu)^{-d} P(i\mu)^{-1} Q(i\mu)\, \Phi_L(d\mu) \tag{4.25}$$

$$= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, \Phi_M(d\mu), \quad t \in \mathbb{R},$$

where Φ_L is the random orthogonal measure corresponding to the Lévy process L and Φ_M is the random measure defined in Theorem 4.6.

Proof. We use equality (4.19) and obtain

$$\begin{aligned}\int_{\mathbb{R}} g(t-s)\,M_d(ds) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} e^{i\mu(t-s)} P(i\mu)^{-1} Q(i\mu)\,d\mu\,M_d(ds)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} e^{i\mu(t-s)} e^{i\lambda s} (i\lambda)^{-d} P(i\mu)^{-1} Q(i\mu)\,d\mu\,ds\,\Phi_L(d\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\mu t} (i\lambda)^{-d} P(i\mu)^{-1} Q(i\mu) \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(\lambda-\mu)s}\,ds\,d\mu\,\Phi_L(d\lambda)\\ &= \int_{\mathbb{R}} e^{i\mu t} (i\mu)^{-d} P(i\mu)^{-1} Q(i\mu)\,\Phi_L(d\mu)\\ &= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\,\Phi_M(d\mu).\end{aligned}$$

□

Remark 4.11 Note that for $d = 0$ the MCARMA processes are obtained. Moreover, an MFICARMA process Y_d can be interpreted as a solution to the p-th order m-dimensional formal differential equation

$$P(D)Y_d(t) = Q(D)DM_d(t),$$

where D denotes the differentiation operator. Furthermore, the spectral representation (4.25) shows that MFICARMA processes are the continuous time analogue of the well-known discrete time multivariate fractionally integrated ARMA (ARFIMA) processes (see e.g. Brockwell & Davis (1991)).

4.2.2 Properties of MFICARMA Processes

Having defined MFICARMA processes, we consider their distributional, second-order and sample path properties. First note that, since (4.23) is a moving average process, the MFICARMA is stationary.

Theorem 4.12 (Infinite Divisibility) *The MFICARMA process as given in (4.23) is well-defined in $L^2(\Omega, P)$. For all $t \in \mathbb{R}$ the distribution of $Y_d(t)$ is infinitely divisible with characteristic triplet $(\gamma_Y^t, 0, \nu_Y^t)$, where*

$$\gamma_Y^t = -\int_{\mathbb{R}} \int_{\mathbb{R}^m} x g_d(t-s) 1_{\{\|g_d(t-s)x\|>1\}} \, \nu(dx)\, ds \quad \textit{and} \tag{4.26}$$

$$\nu_Y^t(B) = \int_{\mathbb{R}} \int_{\mathbb{R}^m} 1_B(g_d(t-s)x)\, \nu(dx)\, ds, \quad B \in \mathcal{B}(\mathbb{R}^m) \tag{4.27}$$

and $(\gamma, 0, \nu)$ is the characteristic triplet of the driving Lévy process L in (4.23).

Proof. Obviously, (4.23) is well-defined in $L^2(\Omega, P)$, since $g_d \in L^2(M_m(\mathbb{R}))$. This fact, as well as (4.26) and (4.27) follow from Proposition 1.2. □

Remark 4.13 From Theorem 4.12 we can conclude that the generating triplet of the stationary limiting distribution of $Y_d(t)$ as $t \to \infty$ is given by $(\gamma_Y^\infty, 0, \nu_Y^\infty)$, where

$$\gamma_Y^\infty = -\int_0^\infty \int_{\mathbb{R}^m} x g_d(s) 1_{\{\|g_d(s)x\|>1\}} \, \nu(dx)\, ds \quad \text{and} \tag{4.28}$$

$$\nu_Y^\infty(B) = \int_0^\infty \int_{\mathbb{R}^m} 1_B(g_d(s)x)\, \nu(dx)\, ds, \quad B \in \mathcal{B}(\mathbb{R}^m). \tag{4.29}$$

Moreover, if $g_d \in L^r(M_m(\mathbb{R}))$ and the driving Lévy process L is in $L^r(\Omega, P)$ for some $r > 0$, then the MFICARMA process Y_d is in $L^r(\Omega, P)$. This follows from the general fact that an infinitely divisible distribution with characteristic triplet (γ, σ, ν) has finite r-th moment, if and only if $\int_{\|x\|>\varepsilon} \|x\|^r \, \nu(dx) < \infty$ for some $\epsilon > 0$ (Sato (1999, Corollary 25.8.), see also Propostion 3.27).

The proofs of the following two propositions are analogous to the proofs of the corresponding results for MCARMA processes (see Chapter 3). Therefore they are omitted.

Proposition 4.14 *Suppose that there exist an $\alpha \in (0, 2)$ and a constant $C > 0$ such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^m} |\langle w, g_d(t-s)x\rangle|^2 \, 1_{\{|\langle w, g_d(t-s)x\rangle| \le 1\}} \, \nu(dx)\, ds \ge C\|w\|^{2-\alpha} \tag{4.30}$$

for any vector w such that $\|w\| \geq 1$. Then $Y_d(t)$ has a C_b^∞ density.

Proposition 4.15 (Continuity) *If $g_d \in C_b^1(\mathbb{R})$, then the MFICARMA process Y_d has a continuous version on every bounded interval I of $\mathbb{R}$.*

So far we only used representation (4.23) to derive probabilistic properties. However, having the spectral representation (4.25), we can immediately conclude that the spectral density of an MFICARMA(p, d, q) process has the form

$$f_{Y_d}(\lambda) = \frac{1}{2\pi}(i\lambda)^{-2d} P(i\lambda)^{-1} Q(i\lambda)\Sigma_L Q(i\lambda)^* (P(i\lambda)^{-1})^*, \quad \lambda \in \mathbb{R}.$$

The following proposition is therefore obvious.

Proposition 4.16 *Let $Y_d = \{Y_d(t)\}_{t\in\mathbb{R}}$ be an MFICARMA(p, d, q) process. Then it has the autocovariance matrix function*

$$\Gamma_{Y_d}(h) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda h}(i\lambda)^{-2d} P(i\lambda)^{-1} Q(i\lambda)\Sigma_L Q(i\lambda)^* (P(i\lambda)^{-1})^* \, d\lambda, \quad h \in \mathbb{R}.$$

Alternatively, we can use (4.10) together with representation (4.24) and obtain for the autocovariance matrix function of an MFICARMA process

$$\begin{aligned}\Gamma_{Y_d}(h) &= E\left[\left(\int_{-\infty}^{t+h} g(t+h-s)\, M_d(ds)\right)\left(\int_{-\infty}^{t} g(t-u)\, M_d(du)\right)^T\right] \\ &= \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \int_{-\infty}^{t+h}\int_{-\infty}^{t} |s-u|^{2d-1} g(t+h-s)\Sigma_L g(t-u)^T \, ds\, du, \quad h \geq 0,\end{aligned}$$

and $\Gamma_{Y_d}(h) = (\Gamma_{Y_d}(-h))^*$, $h < 0$. It follows

$$\Gamma_{Y_d}(h) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} |h|^{2d-1} \int_0^\infty\int_0^\infty g(s)\Sigma_L g(u)^T \, ds\, du \quad \text{as } h \to \infty.$$

Therefore an MFICARMA(p, d, q) process is a **long memory process** according to Definition 1.16.

Finally, we would like to mention that Figure 5.3 in Chapter 5 displays the sample path of a 2-dimensional MFICARMA$(1, 0.25, 0)$ which is driven by a fractional symmetric truncated α-stable Lévy process M_d, where $\alpha = 1.8$.

5 Ornstein-Uhlenbeck Processes

Lévy-driven processes of Ornstein-Uhlenbeck (OU) type have been extensively studied over the last recent years and widely used in applications, especially in the context of finance and econometrics. Several examples of univariate non-Gaussian OU processes can be found in Barndorff-Nielsen & Shephard (2001a), where OU processes are used to model stochastic volatility. Recently multidimensional non-Gaussian OU processes have been considered in Masuda (2004). Moreover, Buchmann & Klüppelberg (2006) discussed among other processes univariate fractional OU processes which were driven by a fractional Brownian motion. In this section we apply the results of the previous chapters to OU processes. In particular, we obtain a multivariate fractional OU process which shows long memory.

5.1 The Univariate (Fractionally Integrated) Ornstein-Uhlenbeck Process

We apply our findings of Sections 1.2 and 1.3 to univariate Ornstein-Uhlenbeck (OU) and univariate fractionally integrated OU processes, respectively.

Definition 5.1 *Let $c > 0$ and $L = \{L(t)\}_{t\in\mathbb{R}}$ be a Lévy process. The process*

$$Y(t) = \int_{-\infty}^{t} e^{-c(t-s)}\, L(ds), \quad t \in \mathbb{R}, \tag{5.1}$$

is called non-Gaussian ***Ornstein-Uhlenbeck*** *process.*

Remark 5.2 Note that for any fixed $s < t$,

$$Y(t) - e^{-c(t-s)}Y(s) = \int_{s}^{t} e^{-c(t-s)}\, L(ds)$$

is a random variable independent of $\sigma\{Y(u),\ u \leq s\}$, the σ-algebra generated by $\{Y(u),\ u \leq s\}$. This implies that the OU process (5.1) is a Markov process. Moreover, for all $c > 0$, $Y(t)$ is self-decomposable (see Barndorff-Nielsen & Shephard (2001a, Theorem 6.1)). Furthermore, it is well-known that a necessary and sufficient condition for the OU process Y to be well-defined is that

$$\int_{|x|>1} \log|x|\, \nu(dx) < \infty$$

(see also (1.34)).

Obviously, for the univariate non-Gaussian Ornstein-Uhlenbeck process, $p(z) = z + c$ for some $c > 0$ and $q(z) = 1$ in the CARMA representation (1.22) and in the kernel representation (1.30), respectively. Hence, the OU process is a special case of a CARMA(p, q) process, namely it is a CARMA$(1, 0)$ process. From (1.35) we obtain the familiar expression for the OU kernel,

$$g(t) = e^{-ct} 1_{[0,\infty)}(t). \tag{5.2}$$

Calculating the left-sided Riemann-Liouville fractional integral of (5.2), i.e. inserting (5.2) in (1.50) we obtain the fractionally integrated kernel,

$$g_d(t) = (-c)^{-d} e^{-ct} P(-ct, d) 1_{[0,\infty)}(t), \tag{5.3}$$

where the function $P(\cdot)$ is the incomplete gamma function as defined in (1.52). From (1.42), the asymptotic form of $g_d(t)$ in this special case is

$$g_d(t) \sim \frac{t^{d-1}}{c\Gamma(d)} \text{ as } t \to \infty. \tag{5.4}$$

If $c = 1$ and $d = 0.45$, the exact and asymptotic expressions (5.3) and (5.4) agree to within 0.1 percent for $h \geq 100$. The exact and asymptotic expressions for $g_{0.45}(100)$ are 0.0405849 and 0.040359 respectively, as compared with the much smaller value of the unintegrated kernel, $g(100) = 3.72 \times 10^{-44}$. Figure 5.1 displays these results.

Considering the second-order properties we obtain from (1.36) the well-known expression for the autocovariance function, $\gamma(h) = E[L(1)^2]e^{-c|h|}/(2c)$, and

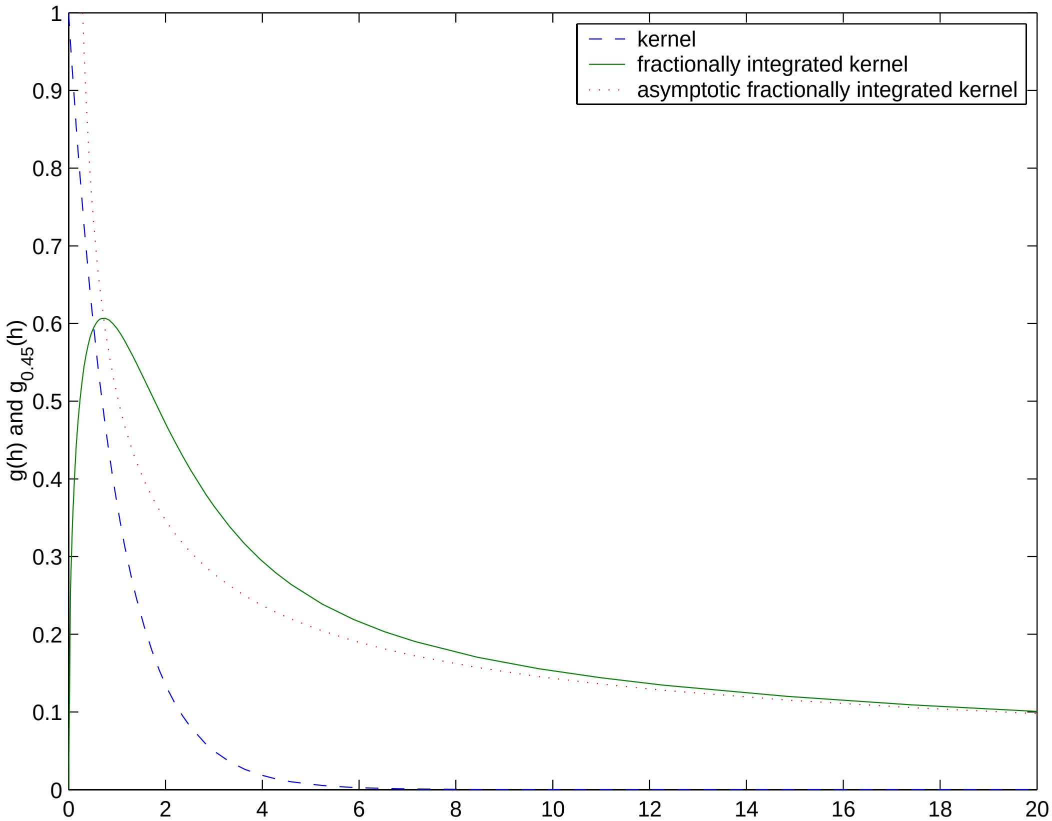

Figure 5.1: The kernel g, the fractionally integrated kernel g_d ($d = 0.45$) and the asymptotic fractionally integrated kernel of the OU process for $c = 1$.

from (1.51) we find, for the fractionally integrated process, that the variance is

$$\gamma_d(0) = \frac{E[L(1)^2]}{2c^{2d+1}\cos(\pi d)}$$

while the autocorrelation function, $\rho_d(h) = \gamma_d(h)/\gamma_d(0)$, is

$$\rho_d(h) = \cosh(ch) - \frac{e^{ch}}{2}P(ch, 2d) + \frac{e^{-ch}}{2}(-1)^{-2d}P(-ch, 2d),\ h \geq 0. \quad (5.5)$$

The autocorrelation function (5.5), interestingly, depends on c and h only through the value of ch. The following table displays the autocorrelation function for $d = .01, .05, .1, .2, .3, .4, .45, .49$ and for $ch = 0, 5, 10, 15, 20, 25, 30$.

Table 1. The autocorrelation function of the fractionally integrated OU process

$ch\backslash d$	.01	.05	.10	.20	.30	.40	.45	.49
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	.01133	.03345	.07070	.18462	.36528	.63081	.80127	.95788
10	.00221	.01354	.03514	.11450	.26901	.54338	.74421	.94393
15	.00144	.00926	.02513	.08918	.22790	.50029	.71414	.93618
20	.00108	.00712	.01990	.07490	.20289	.47209	.69373	.93077
25	.00087	.00582	.01663	.06545	.18547	.45138	.67835	.92661
30	.00072	.00493	.01436	.05864	.17237	.43516	.66606	.92323

From (1.48) we obtain the asymptotic expression for the autocorrelation function,

$$\rho_d(h) \sim (ch)^{2d-1} \frac{2\Gamma(1-2d)\cos(\pi d)}{\Gamma(d)\Gamma(1-d)} \text{ as } h \to \infty. \tag{6.8}$$

The relative error of the asymptotic approximation when $ch = 30$ is less than 0.3% across the range of d-values tabulated (see Figure 5.2).

Remark 5.3 Observe that, in order that the fractionally integrated OU process $\int_{\mathbb{R}} g_d(t-s)\, L(ds)$ with kernel g_d given by (5.3) is well-defined, the driving Lévy process must be square integrable with zero mean. This is a consequence of Remark 1.21, since the fractionally integrated OU process is a FICARMA$(1, d, 0)$ process.

5.2 The Multivariate Fractional Ornstein-Uhlenbeck Process

In this section we define multivariate fractional Ornstein-Uhlenbeck processes as a special case of the multivariate fractionally integrated CARMA processes considered in Chapter 4.

Definition 5.4 (Multivariate Fractional OU Process)
Let $A \in M_m(\mathbb{R})$ be a matrix such that all the eigenvalues of A have negative real part. Let $B \in M_m(\mathbb{R})$ be positive definite and $M_d = \{M_d(t)\}_{t\in\mathbb{R}}$ be a

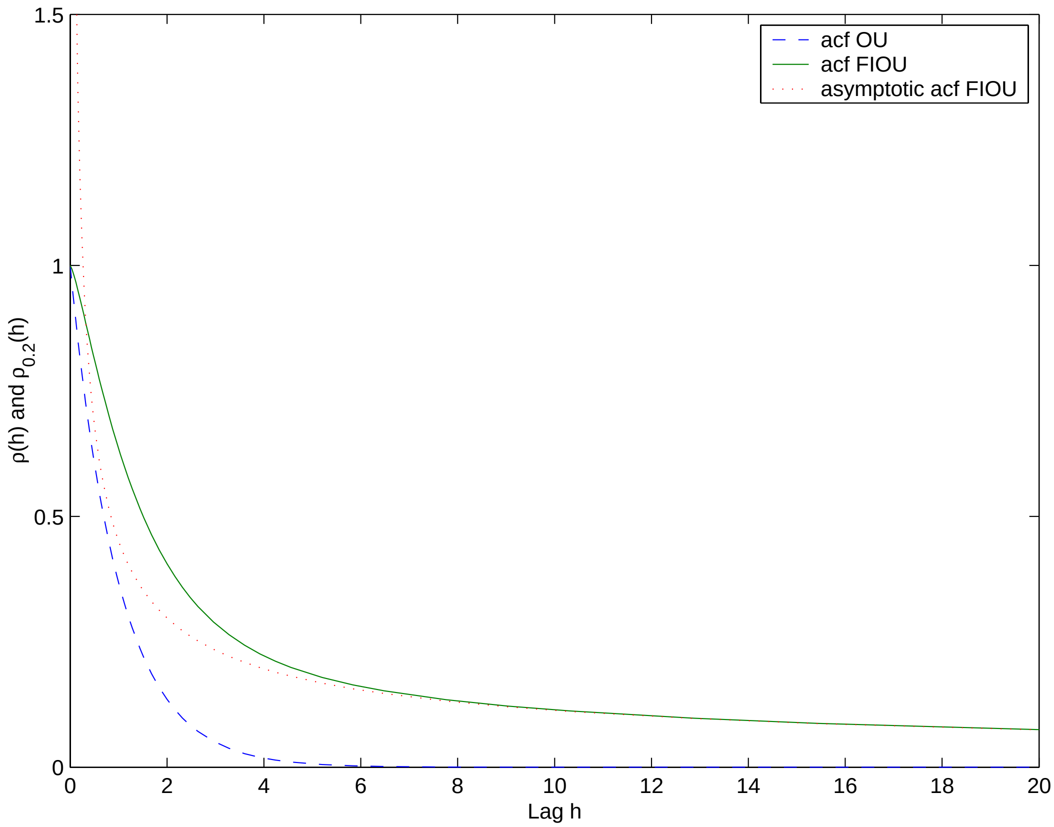

Figure 5.2: The autocorrelation function of the (fractionally integrated, $d = 0.2$) OU process ($c = 1$) and the asymptotic autocorrelation function of the fractionally integrated OU process.

square-integrable m-dimensional fractional Lévy process as defined in Section 4.1.1. We define the fractional Ornstein-Uhlenbeck process by

$$O_t^{d,A,B} = \int_{-\infty}^{t} e^{A(t-s)} B \, M_d(ds), \quad t \in \mathbb{R}. \tag{5.6}$$

Remark 5.5 Obviously (5.6) is an MFICARMA$(1, d, 0)$ process and is therefore stationary and well-defined. Moreover, it is a process with long memory (see Chapter 4).

Now, without serious loss of generality we assume that the matrix A is diagonizable. Therefore, let $U \in M_m(\mathbb{R})$ be such that $A = UDU^{-1}$, where $D = \mathrm{diag}(\lambda_i)_{i=1,\ldots,m}$ and λ_i, $i = 1, \ldots, m$, are the eigenvalues of A. Then, calculating the left-sided Riemann-Liouville fractional integral of the kernel

$$G(t-s) = e^{A(t-s)} B 1_{[0,\infty)}(t-s),$$

we obtain

$$\begin{aligned} G_d(t) :&= (I_+^d G)(t) = \frac{1}{\Gamma(d)} \int_0^\infty s^{d-1} e^{A(t-s)} B 1_{[0,\infty)}(t-s)\, ds \\ &= \frac{e^{At} U}{\Gamma(d)} \int_0^t s^{d-1} \operatorname{diag}(e^{-\lambda_i s})\, ds\, U^{-1} B \\ &= \frac{e^{At} U}{\Gamma(d)} \begin{pmatrix} \lambda_1^{-d} \int_0^{\lambda_1 t} s^{d-1} e^{-s}\, ds & & \\ & \ddots & \\ & & \lambda_m^{-d} \int_0^{\lambda_m t} s^{d-1} e^{-s}\, ds \end{pmatrix} U^{-1} B \\ &= e^{At} U \begin{pmatrix} \lambda_1^{-d} P(\lambda_1 t, d) & & \\ & \ddots & \\ & & \lambda_m^{-d} P(\lambda_m t, d) \end{pmatrix} U^{-1} B, \end{aligned}$$

where $P(x,d) = \frac{1}{\Gamma(d)} \int_0^x e^{-t} t^{d-1}\, dt$ denotes again the lower incomplete gamma function with complex argument $x \in \mathbb{C}$ (see also (1.52)). Hence, it follows from (4.9) and (4.23),

$$O_t^{d,A,B} = \int_{\mathbb{R}} G_d(t-u)\, L(du), \quad t \in \mathbb{R}. \tag{5.7}$$

We see that OU processes have the nice property that the explicit expression of the fractionally integrated kernel is easy to compute, which is unfortunately not the case for general MFICARMA processes.

Finally, we would like to mention that the usual definition of an (not fractional) OU process driven by Brownian motion is as the solution of a stochastic differential equation, the so-called Langevin equation. The next proposition shows that this is also true for multivariate fractional OU processes.

Proposition 5.6 *The process $O_t^{d,A,B}$ as given in (5.6) is the unique stationary solution of the SDE of Langevin-type*

$$dO(t) = AO(t)dt + BM_d(dt), \qquad t > 0, \tag{5.8}$$

where the matrices $A, B \in M_m(\mathbb{R})$ are defined as in Definition 5.4.

Proof. Let $t_0 < s < t$. Notice that equation (5.8) can be written in the integral form

$$O(t) - O(t_0) = \int_{t_0}^{t} AO(s)\,ds + B[M_d(t) - M_d(t_0)].$$

Therefore, using (4.9) and Fubini's theorem we obtain

$$\begin{aligned}
\int_{t_0}^{t} AO(s)\,ds &= \int_{t_0}^{t} A \int_{-\infty}^{s} e^{A(s-u)} B\, M_d(du)\,ds \\
&= \int_{t_0}^{t} A \int_{-\infty}^{t_0} e^{A(s-u)} B\, M_d(du)\,ds + \int_{t_0}^{t} A \int_{t_0}^{s} e^{A(s-u)} B\, M_d(du)\,ds \\
&= \int_{t_0}^{t} A \int_{-\infty}^{t_0} e^{A(s-t_0)} e^{A(t_0-u)} B\, M_d(du)\,ds \\
&\quad + \frac{1}{\Gamma(d)} \int_{t_0}^{t} A \int_{t_0}^{s} \int_{v}^{\infty} (u-v)^{d-1} e^{A(s-u)} B\,du\,L(dv)\,ds \\
&= \int_{t_0}^{t} Ae^{A(s-t_0)} O(t_0)\,ds + \frac{1}{\Gamma(d)} \int_{t_0}^{t} A \int_{v}^{\infty} \int_{u}^{t} (u-v)^{d-1} e^{A(s-u)} B\,ds\,du\,L(dv) \\
&= [e^{A(t-t_0)} - I_m] O(t_0) + \frac{1}{\Gamma(d)} \int_{t_0}^{t} \int_{v}^{\infty} (u-v)^{d-1} [e^{A(t-u)} - I_m] B\,du\,L(dv) \\
&= [e^{A(t-t_0)} - I_m] O(t_0) + \int_{t_0}^{t} [e^{A(t-u)} - I_m] B\, M_d(du) \\
&= O(t) - O(t_0) - B[M_d(t) - M_d(t_0)].
\end{aligned}$$

The proof of the uniqueness is a simple application of Gronwall's Lemma (see e.g. Ikeda & Watanabe (1989, Theorem 3.1)). □

Figure 5.3 and Figure 5.4 show the sample paths of a 2-dimensional OU process ($d = 0$) and a 2-dimensional fractional OU process ($d = 0.25$), respectively.

The driving Lévy process is a 2-dimensional (fractional) symmetric truncated stable Lévy process (see (2.42)) and the dependence of the driving Lévy process is given by the Clayton Lévy copula (4.20). We simulated the trajectories for $\alpha = 1.8$ and $\theta = 0.5$.

Furthermore the matrix A is given by $A = \begin{pmatrix} -1 & -2 \\ -0.3 & -1.4 \end{pmatrix}$ with eigenvalues $\lambda_1 = -0.4$ and $\lambda_2 = -2$ and we set $B = I_2$.

We observe that the roughness of the sample paths decreases when the value of d increases.

Figure 5.5 shows the 2-dimensional fractional OU process ($d = 0.25$) for $t \in [0, 100]$.

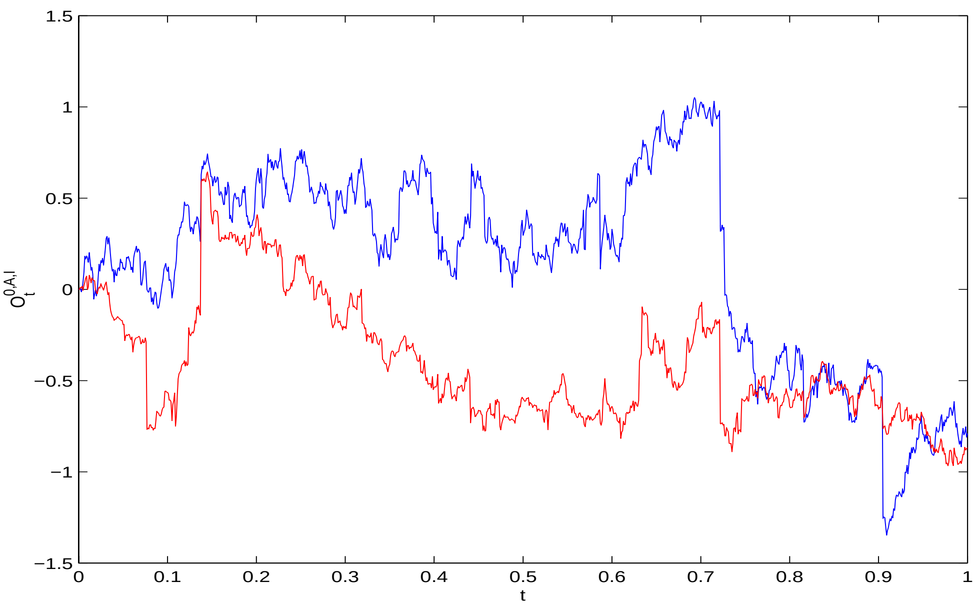

Figure 5.3: The sample path of a 2-dim OU process with truncated 1.8-stable margins.

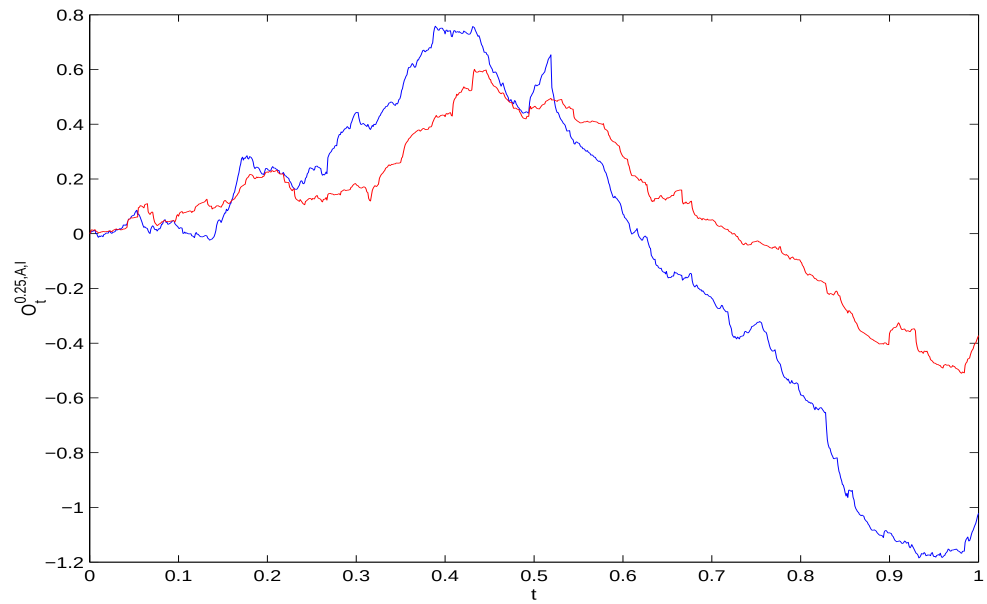

Figure 5.4: The sample path of a 2-dim fractional OU process with fractional truncated 1.8-stable margins ($d = 0.25$).

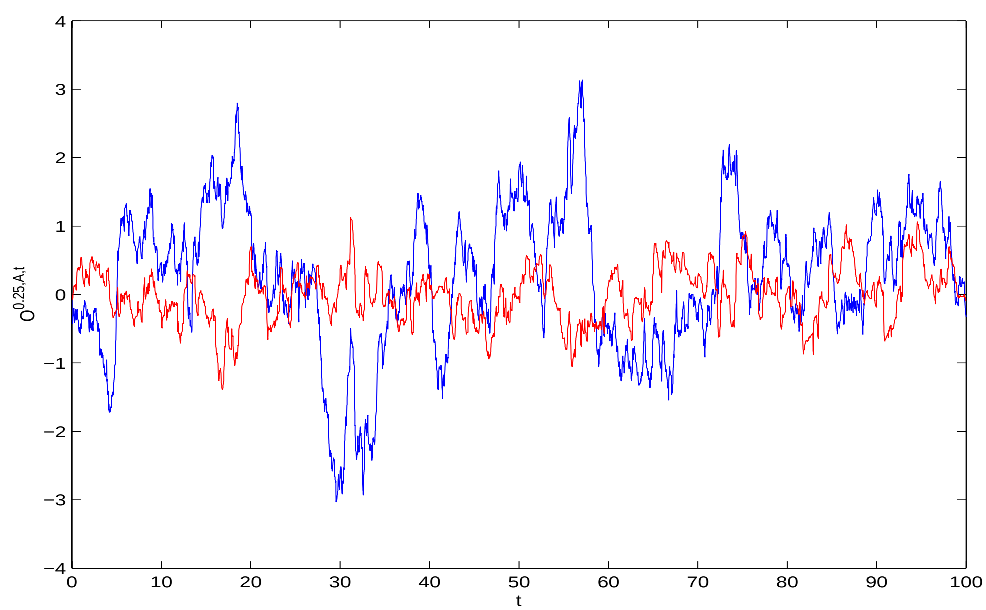

Figure 5.5: The sample path of a 2-dim fractional OU process with fractional truncated 1.8-stable margins ($d = 0.25$), where $t \in [0, 100]$.

6 Outlook: Stochastic Calculus for Convoluted Lévy Processes

We know from Chapter 2 that for a large class of Lévy processes the corresponding fractional Lévy process cannot be a semimartingale and hence ordinary Itô calculus cannot be applied. Therefore, in Section 2.3 we defined integrals with respect to fractional Lévy processes in the special case of a deterministic integrand.

Our aim in this chapter is to generalize this integral to stochastic integrands. We give an elementary definition of the (Wick-)Itô integral with respect to FLPs in terms of the S-transform. In particular, we define a stochastic calculus not only for FLPs but for general convouluted Lévy processes.

In the case of fractional Brownian motion an S-transform approach has been developed by Bender (2003a) (see also Bender (2003b)). Equivalently, the fractional Itô integral with respect to FBM can be defined in a Malliavin calculus setting with the aid of the Skorohod integral (see e.g. Alòs et al. (2001)). However, the definition based on the S-transform avoids all the technical constructions of the Malliavin and the white noise calculus.

The research on Itô integrals with respect to convoluted Lévy processes is joint work with Christian Bender and still ongoing. In this last chapter we state the basic concept and main ideas without going into further detail.

6.1 Itô's Integral from a White Noise Point of View

Our aim is to define a stochastic calculus for convoluted Lévy processes.

Definition 6.1 *We call a stochastic process* $M = \{M(t)\}_{t\in\mathbb{R}}$ *given by*

$$M(t) = \int_{\mathbb{R}} f(t,s)\, L(ds), \qquad t \in \mathbb{R}, \tag{6.1}$$

a **convoluted Lévy process** *with kernel* f*. Here,* $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *is a measurable function and* $L = \{L(t)\}_{t\in\mathbb{R}}$ *is a Lévy process without Brownian component satisfying* $E[L(1)] = 0$ *and* $E[L(1)^2] < \infty$*, i.e.* L *can be represented as in (1.6).*

Remark 6.2 We know from Section 1.1.2 that the process M can be represented by (1.11), that is

$$M(t) = \int_{\mathbb{R}_0\times\mathbb{R}} f(t,s)x\, \tilde{J}(dx,ds), \qquad t \in \mathbb{R}, \tag{6.2}$$

where J is the jump measure and $\tilde{J}(dx,ds) = J(dx,ds) - \nu(dx)\,ds$ is the compensated jump measure of L.

Important examples for convoluted Lévy processes are the fractional Lévy processes (see Chapter 2).

In this section we give a simple characterization of the classical Itô integral for Lévy processes in terms of the S-transform, which is an important tool from white noise analysis. This characterization is the starting point to define stochastic integrals with respect to convoluted Lévy proceses in the next section.

To introduce the S-transform we first require some preliminaries. For $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ we define the Wiener integral with respect to the compensated jump measure $\tilde{J}(dx,ds) = J(dx,ds) - \nu(dx)\,ds$ by

$$I(f) = I_1(f) = \int_{\mathbb{R}}\int_{\mathbb{R}_0} xf(s,x)\tilde{J}(dx,ds). \tag{6.3}$$

Observe that according to Proposition 1.2, the integral (6.3) can be defined for step functions in a straightforward manner and may then be extended by the isometry

$$E\left[I(f)^2\right] = \int_{\mathbb{R}}\int_{\mathbb{R}_0} f(s,x)^2 x^2 \nu(dx) ds. \tag{6.4}$$

To simplify notation we introduce the measures

$$
\begin{aligned}
n(dx,ds) &= \nu(dx)ds,\\
\lambda(dx,ds) &= x^2 n(dx,ds).
\end{aligned}
$$

Definition 6.3 (Wick Exponential) *Let $J_L(\omega) = \{s \in \mathbb{R} : \ \Delta L(s;\omega) \neq 0\}$, where $\Delta L(s;\omega) = L(s;\omega) - L(s-;\omega)$ are the jumps of the Lévy process L. Then for $f \in L^2(\mathbb{R}^2, \lambda)$ such that $xf(s,x)$ belongs to $L^1 \cap L^\infty(\mathbb{R} \times \mathbb{R}_0, n)$ the Wick exponential of $I(f)$ is defined as*

$$
: e^{I(f)} : = \exp\left\{ -\int_{\mathbb{R}}\int_{\mathbb{R}_0} f(s,x)x\nu(dx)ds \right\} \prod_{s\in J_L} \left[1 + \Delta L(s) f(s, \Delta L(s))\right]. \quad (6.5)
$$

Remark 6.4 (i) By Lee & Shih (2004, Theorem 3.1),

$$
: e^{I(f)} := \sum_{n=0}^{\infty} \frac{I_n(f^{\otimes n})}{n!}, \quad (6.6)
$$

where I_n denotes the multiple Wiener integral of order n with respect to the compensated Lévy measure. This representation justifies the name Wick exponential.

(ii) Since $: e^{I(f)} :$ coincides with the Doléans-Dade exponential of $I(f)$ at $t = \infty$ it is straightforward that for f, g, which satisfy the conditions of Definition 6.3, we have

$$
E[: e^{I(f)} :] = 1 \quad \text{and} \quad E[: e^{I(f)} : \cdot : e^{I(g)}] = e^{(f,g)_1},
$$

where

$$
(f,g)_1 := \int_{\mathbb{R}}\int_{\mathbb{R}_0} x^2 f(s,x) g(s,x)\, \nu(dx)\, ds.
$$

Define

$$
\Xi = \left\{ \sum_{j=1}^{n} \eta_j \otimes f_j; \ n \in \mathbb{N}, \ \eta_j \in \mathcal{S}(\mathbb{R}), \ f_j \in L^1 \cap L^\infty(\mathbb{R}_0, \nu) \right\}, \quad (6.7)
$$

where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space of smooth rapidly decreasing functions.

Lemma 6.5 *Every $f \in \Xi$ satisfies the conditions of Definition 6.3. Moreover, Ξ is a dense subset of $L^2(\mathbb{R}^2, \lambda)$.*

We can now define the S-transform

Definition 6.6 (S-transform) *For $F \in L^2(\Omega, P)$ the S-transform is defined by*

$$S(F)(\eta) = E^{Q_\eta}[F], \quad \eta \in \Xi,$$

where

$$dQ_\eta =: e^{I(\eta)} : dP.$$

Due to the following proposition every square integrable random variable is uniquely determined by its S-transform.

Proposition 6.7 *The S-transform is injective, i.e. if $S(F)(\eta) = S(G)(\eta)$ for all $\eta \in \Xi$, then $F = G$.*

Proof. In view of Lemma 6.5, the assertion follows from Remark 5.9 and Theorem 5.13 in Lee & Shih (1999). □

We shall now calculate the S-transform of an Itô integral with respect to the Lévy process L. To this end let $0 \leq a \leq b$ and $X : [a, b] \times \Omega \to \mathbb{R}$ a predictable (with respect to the filtration $\mathcal{F}_t$ generated by the Lévy process $L(s),\ 0 \leq s \leq t$) process satisfying

$$E\left[\int_a^b |X(t)|^2 \, dt\right] < \infty.$$

Then the compensated Poisson integral $\int_a^b \int_{\mathbb{R}_0} X(t) y \, \tilde{J}(dy, dt)$, which is equivalent to the integral $\int_a^b X(t) \, L(dt)$ with respect to the Lévy process L, exists and by the isometry property it is an element of $L^2(\Omega, P)$ (see Cont & Tankov (2004, Proposition 8.8)).

We start our considerations with a Girsanov theorem.

Theorem 6.8 (Girsanov) *Let $L_1(t) = L(t),\ t \geq 0,\ L_2(t) = L(-t),\ t \geq 0$, where L is a Lévy process with characteristics $(\gamma, 0, \nu)$.Moreover, let $f \in \Xi$ and define an equivalent measure Q_f to P by*

$$dQ_f =: e^{I(f)} : dP.$$

Then under Q_f

(i) For $0 \le t < \infty$,

$$\left(L_1(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds,\, L_2(t) + \int_0^t \int_{\mathbb{R}_0} x^2 f(-s,x)\,\nu(dx)\,ds\right)$$

is a two-dimensional Process with independent increments which is stochastically continuous (PII for short). Moreover, it is a Q_f*-martingale.*

(ii) $L(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds$ *is a two-sided PII. Moreover, its expectation under* Q_f *is zero.*

Proof. (i) By Girsanov's theorem for semimartingales (Jacod & Shiryaev (2003, Theorem III.3.24)) follows that

$$\tilde{L}_1(t) = L_1(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds, \quad t \ge 0,$$

is a semimartingale having characteristics $(\tilde{\gamma}_s, 0, \tilde{\nu}_s)$, where

$$\begin{aligned} \tilde{\gamma}_s &= -\int_{|x|>1} x(1 + xf(s,x))\,\nu(dx), \\ \tilde{\nu}_s(dx) &= (1 + xf(s,x))\,\nu(dx). \end{aligned}$$

Since $\tilde{L}(0) = 0$ and the characteristics are deterministic, $\tilde{L}_1$ is a PII under the measure Q_f (Jacod & Shiryaev (2003, Theorem II.4.15)). Analogously, $\tilde{L}_2$ is a PII under Q_f. The Q_f-martingale property follows from Jacod & Shiryaev (2003, Theorem III.3.8).
(ii) Obviously, it follows from (i) that

$$\begin{aligned} \tilde{L}(t) &= L(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds \\ &= \begin{cases} L_1(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds, & t \ge 0 \\ L_2(-t) - \int_0^{-t} \int_{\mathbb{R}_0} x^2 f(-s,x)\,\nu(dx)\,ds, & t < 0 \end{cases} \end{aligned}$$

is a two-sided PII. As the two-sided process can be decomposed into two one-sided Q_f-martingale and its time zero value is null, its expectation under Q_f is zero. □

Remark 6.9 From the proof of Theorem 6.8 we know the characteristics of

$$\tilde{L}(t) = L(t) - \int_0^t \int_{\mathbb{R}_0} x^2 f(s,x)\,\nu(dx)\,ds, \quad t \geq 0,$$

explicitly. Therefore, it is straightforward that under the measure Q_f for all $t \geq 0$ the characteristic function of the distribution of $\tilde{L}(t)$ is given by

$$E^{Q_f}\left[e^{iu\tilde{L}(t)}\right] = \exp(\psi_t(u)),$$

where

$$\psi_t(u) = \int_0^t iu\tilde{\gamma}_s + \int_{\mathbb{R}_0} [e^{iux} - 1 - iux1_{\{|x|\leq 1\}}][1 + xf(s,x)]\,\nu(dx)\,ds \tag{6.8}$$

$$= \int_0^t \int_{\mathbb{R}_0} [e^{iux} - 1 - iux][1 + xf(s,x)]\,\nu(dx)\,ds. \tag{6.9}$$

Hence,

$$E^{Q_f}[\tilde{L}(t)] = 0, \tag{6.10}$$

$$E^{Q_f}[\tilde{L}(t)^2] = \int_0^t \int_{\mathbb{R}_0} x^2(1 + xf(s,x))\,\nu(dx)\,ds. \tag{6.11}$$

Theorem 6.10 *Let $0 \leq a \leq b$ and $X : [a,b] \times \Omega \to \mathbb{R}$ be a predictable process such that $E[\int_a^b |X(t)|^2\,dt] < \infty$. Then $\int_a^b X(s)\,L(ds)$, is the unique square integrable random variable with S-transform given by*

$$\int_a^b \int_{\mathbb{R}_0} S(X(s))(\eta)\,y^2\eta(s,y)\,\nu(dy)\,ds, \quad \eta \in \Xi.$$

Proof. By the Girsanov theorem we know that $L(t) - \int_0^t \int_{\mathbb{R}_0} x^2\eta(s,x)\,\nu(dx)\,ds$ is a two-sided PII and a Q_η-martingale with zero expectation.

Consequently, $\int_a^t X(s)\,\tilde{L}(ds)$, $a \le t \le b$ is a zero mean Q_η-martingale. Hence, by Fubini's theorem we obtain

$$E^{Q_\eta}\left[\int_a^b X(s)\,L(ds)\right] = E^{Q_\eta}\left[\int_a^b X(s)\,\tilde{L}(ds) + \int_a^b \int_{\mathbb{R}_0} X(s)y^2\,\eta(s,y)\,\nu(dy)\,ds\right]$$

$$= \int_a^b \int_{\mathbb{R}_0} E^{Q_\eta}[X(s)]\,y^2\eta(s,y)\,\nu(dy)\,ds.$$

□

From Theorem 6.8 (ii), we also obtain that

$$S(L(t))(\eta) = \int_0^t \int_{\mathbb{R}_0} y^2\eta(s,y)\,\nu(dy)\,ds.$$

Hence, in the context of Theorem 6.10,

$$S\left(\int_a^b X(s)\,L(ds)\right)(\eta) = \int_a^b S(X(s))(\eta)\frac{d}{ds}S(L(s))(\eta)ds.$$

The latter identity is the starting point for defining an integral for convoluted Lévy processes.

6.2 A Skorohod Integral for Convoluted Lévy Processes

For the rest of this section we assume:

Standing Assumption: $M = \{M(t)\}_{t\in\mathbb{R}}$ is a convoluted Lévy process on the real line, such that for every $\eta \in \Xi$

$$t \mapsto S(M(t))(\eta)$$

is differentiable.

Definition 6.11 *Let $B \in \mathcal{B}(\mathbb{R})$ and $X : B \to L^2(\Omega, P)$. Then X is said to have Skorohod integral with respect to M if*

$$S(X(\cdot))(\eta)\frac{d}{dt}S(M(\cdot))(\eta) \in L^1(B) \quad \textit{for any } \eta \in \Xi$$

and there is a $\Phi \in L^2(\Omega, P)$ *such that for all* $\eta \in \Xi$,

$$S(\Phi)(\eta) = \int_B S(X(t))(\eta) \frac{d}{dt} S(M(t))(\eta)\, dt.$$

In that case Φ is uniquely determined by the injectivity of the S-transform and we denote

$$\Phi = \int_B X(t)\, M^\diamond(dt).$$

Remark 6.12 (i) The definition of the Skorohod integral does not require measurability conditions such as predictability or progressive measurability. Hence, it also generalizes the Itô integral with respect to the underlying Lévy process to anticipative integrands.

(ii) Since the Lévy process itself is stochastically continuous, the S-transform cannot distinguish between $L(t)$ and $L(t-)$ for fixed t. Consequently, we obtain e.g.

$$\int_0^t L(s)\, L^\diamond(ds) = \int_0^t L(s-)\, L^\diamond(ds) = \int_0^t L(s-)\, L(ds),$$

where the last integral is the classical Itô integral.

The following properties of the Skorohod integral are an obvious consequence of the definition:

Proposition 6.13 *(i) For all* $a < b \in \mathbb{R}$,

$$M(a) - M(b) = \int_a^b M^\diamond(dt).$$

(ii) Let $X : B \to L^2(\Omega, P)$ *be Skorohod integrable. Then*

$$\int_B X(t)\, M^\diamond(dt) = \int_{\mathbb{R}} 1_B(t) X(t)\, M^\diamond(dt).$$

(iii) Let $X : B \to L^2(\Omega, P)$ *be Skorohod integrable. Then*

$$E\left[\int_B X(t)\, M^\diamond(dt)\right] = 0.$$

Because of (ii) there is no loss of generality in proving the majority of results for $B = \mathbb{R}$, only. Note that (iii) holds since the expectation coincides with the S-transform at $\eta = 0$.

We shall now specialize from a convoluted Lévy process to a fractional one (see Chapter 2). Similar considerations as in Theorem 6.10 yield together with (2.43)

$$S(M_d(t))(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} (I_-^d 1_{(0,t)})(\eta)\, y^2 \eta(s,y)\, \nu(dy)\, ds,$$

where I_-^d is the right-sided Riemann-Liouville fractional integral of order d.

Recall, $\eta = \sum_j \eta_j \otimes f_j$. Hence, by Fubini's theorem and fractional integration by parts (see 1.40),

$$\begin{aligned} S(M_d(t))(\eta) &= \sum_j \int_{\mathbb{R}_0} f_j(y) y^2 \int_0^t (I_+^d \eta_j)(s)\, ds\, \nu(dy) \\ &= \int_0^t \int_{\mathbb{R}_0} \left(\sum_j f_j(y)(I_+^d \eta_j)(s) \right) y^2\, \nu(dy)\, ds \end{aligned}$$

Consequently,

Proposition 6.14 *Suppose M_d is a fractional Lévy process with $0 < d < 0.5$. Then for all $\eta \in \Xi$,*

$$\frac{d}{dt} S(M_d(t))(\eta) = \int_{\mathbb{R}_0} (I_+^d \eta)(t,y) y^2 \nu(dy),$$

where, by convention, fractional integral and differential operators are applied only to the time variable t.

We now state a theorem which is well-known in the case of fractional Brownian motion. For the proof we refer to Bender (2003b, Theorem 3.4).

Theorem 6.15 *Let $0 < d < 0.5$. Suppose that $X \in L^p(\mathbb{R}; L^2(\Omega, P))$ with $1 \leq p < 1/d$. Then*

$$\int_{\mathbb{R}} X(t)\, M_d(dt) = \int_{\mathbb{R}} (I_-^d X)(t)\, L(dt)$$

in the sense that if one of the integrals exists then so does the other and both coincide.

Remark 6.16 Observe that Theorem 6.15 coincides with (2.50): If the integrand X is deterministic, the integral defined in Definition 6.11 and the integral of Theorem 2.34 are equal.

We now define the Wick product:

Definition 6.17 *Let $F, G \in L^2(\Omega, P)$ and assume that there is an element $F \diamond G \in L^2(\Omega, P)$ such that*

$$S(F \diamond G)(\eta) = S(F)(\eta)S(G)(\eta), \quad \textit{for all} \quad \eta \in \Xi.$$

*Then $F \diamond G$ is referred to as the **Wick product** of F and G.*

Example 6.18 Let $f, g \in L^2(\mathbb{R})$. Then

$$: e^{I(f)} : \diamond : e^{I(g)} : = : e^{I(f+g)} : \ .$$

Theorem 6.19 *Let $X : \mathbb{R} \to L^2(\Omega, P)$ and $Y \in L^2(\Omega, P)$. Then*

$$Y \diamond \int_{\mathbb{R}} X(s)\, M_d(ds) = \int_{\mathbb{R}} Y \diamond X(s)\, M_d(ds),$$

in the sense that if one side is well-defined then so is the other and both coincide.

Proof. The assertion follows by calculating the S-transform of both sides. □

Example 6.20 We want to calculate the Wick product $L(T) \diamond L(T)$. On the one hand we have

$$
\begin{aligned}
S\left(\int_0^T L(t)^{\diamond} L(dt)\right)(\eta) &= \int_0^T \int_{\mathbb{R}_0} S(L(t))(\eta) y^2\, \eta(t,y)\, \nu(dy)\, dt \\
&= \int_0^T \left(\int_0^t \int_{\mathbb{R}_0} x^2\, \eta(s,x)\, \nu(dx)\, ds\right) \underbrace{\int_{\mathbb{R}_0} y^2\, \eta(t,y) \nu(dy)}_{=:a(t)}\; dt \\
&= \int_0^T \left(\int_0^t a(s)\, ds\right) a(t)\, dt = \frac{1}{2}\left(\int_0^T a(t)\, dt\right)^2 \\
&= \frac{1}{2}\left(\int_0^T \int_{\mathbb{R}_0} y^2\, \eta(t,y)\, \nu(dy)\, dt\right)^2 \\
&= \frac{1}{2}\left[S(L(T))(\eta)\right]^2 = \frac{1}{2} S(L(T))(\eta) S(L(T))(\eta) \\
&= \frac{1}{2} S(L(T) \diamond L(T))(\eta).
\end{aligned}
$$

On the other hand we know by Itô's formula (e.g. Protter (2004, Theorem II.32))

$$
2\int_0^T L(t)\, L(dt) = (L(T))^2 - [L,L]_T = (L(T))^2 - \sum_{s \le T} (\Delta L(s))^2.
$$

Hence,

$$
L(T) \diamond L(T) = (L(T))^2 - [L,L]_T.
$$

In the general case, applying Lee & Shih (2004, Theorem 3.6), we obtain the following result

Proposition 6.21 *Let $f, g \in L^2(\mathbb{R})$. Then*

$$
I(f) \diamond I(g) = I(f) \cdot I(g) - \int_{\mathbb{R}} \int_{\mathbb{R}_0} x^2 f^2(x,s)\, J(dx, ds).
$$

Example 6.22 For a fractional Lévy process $M_d(t) = \int_{\mathbb{R}} (I_-^d 1_{(0,t)})(s)\, L(ds)$, we obtain from proposition (6.21)

$$
\begin{aligned}
M_d(t) \diamond M_d(t) &= (M_d(t))^2 - \int_{\mathbb{R}} \int_{\mathbb{R}_0} x^2 (I_-^d 1_{(0,t)})^2(s)\, J(dx, ds) \\
&= (M_d(t))^2 - \frac{1}{\Gamma^2(d+1)} \sum_{s \in J_L} [(t-s)_+^d - (-s)_+^d]^2 (\Delta L(s))^2.
\end{aligned}
$$

Proposition 6.21 gives rise to a guess on the form of an Itô formula for functionals of stochastic integrals with respect to convoluted Lévy processes. This will be the topic of future research.

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Notation

Abbreviations

a.e.	almost everywhere
a.s.	almost surely
AR	autoregressive
ARMA	autoregressive moving average
ARFIMA	autoregressive fractionally integrated moving average
CARMA	continuous time autoregressive moving average
FBM	fractional Brownian motion
FICARMA	fractionally integrated CARMA
FLP	fractional Lévy process
i.i.d.	independent identically distributed
l.h.s., r.h.s.	left hand side, right hand side
MA	moving average
MCARMA	multivariate CARMA
MFLP	multivariate fractional Lévy process
MFICARMA	multivariate FICARMA
OU	Ornstein Uhlenbeck
SDE	stochastic differential equation
SV	stochastic volatility
w.l.o.g.	without loss of generality

Symbols

$\mathbb{R}, \mathbb{R}^+, \mathbb{R}_0^m$	$(-\infty, \infty),\ [0, \infty),\ \mathbb{R}^m \setminus \{0\}$
$\mathbb{N}, \mathbb{Z}, \mathbb{N}_0$	$\{1, 2, \ldots\},\ \{\ldots, -2, -1, 0, 1, 2, \ldots\},\ \mathbb{N} \cup \{0\}$
$\mathbb{C}$	complex numbers
$\mathcal{B}(\mathbb{R})$	Borel σ-algebra over $\mathbb{R}$
$M_m(\mathbb{R}), M_m(\mathbb{C})$	space of all real, complex $m \times m$-matrices
$Gl_m(\mathbb{C})$	space of all invertible $m \times m$-matrices
$\mathcal{R}(z), \mathcal{I}(z)$	real and imaginary part of $z \in \mathbb{C}$
$a \wedge b, a \vee b$	minimum, maximum of $a, b \in \mathbb{R}$
a_+	$0 \vee a$
a_-	$0 \vee -a$
$f', f'', f^{(n)}$	first, second, n-fold derivative of f
$f(t) \sim g(t)$	$f(t)/g(t) \to 1,\ \ t \to \infty$
$\log, \exp$	natural logarithm, exponential function
P, E	probability, expectation
$\mathrm{var}, \mathrm{cov}$	variance, covariance
1_B	indicator function of the set B
I_m	identity matrix, $I_m \in M_m(\mathbb{C})$
A^T	transposed of the matrix A
A^*	adjoint of the matrix A
$\mathrm{Ker}A$	kernel of the matrix A
det(A)	determinant of the matrix A
$\lvert x \rvert$	absolute value of $x \in \mathbb{C}$
$\lVert x \rVert$	norm of $x \in \mathbb{C}^m$
$\lVert A \rVert$	operator norm corresponding to the norm $\lVert x \rVert$, $x \in \mathbb{C}^m$
C_b^∞	space of bounded continuous, infinitely often differentiable functions with bounded derivatives
L^p	space of p-integrable functions
$L^p(M_m(\mathbb{R}))$	$\left\{ f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R}),\ \int_{\mathbb{R}} \lVert f(t,s) \rVert^p \, ds < \infty \right\},\ p > 0$
H	Hurst coefficient
d	fractional integration parameter
D	differential operator
$\{B_t\}_{t \in \mathbb{R}}$	ordinary Brownian motion
$\{B_H(t)\}_{t \in \mathbb{R}}$	fractional Brownian motion with parameter H

$\{L(t)\}_{t\in\mathbb{R}}$	Lévy process
$\{M_d(t)\}_{t\in\mathbb{R}}$	fractional Lévy process
$\mathcal{L}(X)$	distribution of the random variable X
(γ, σ, ν)	generating triplet of a Lévy process
$\stackrel{d}{\to}, \stackrel{d}{=}$	convergence, equality in (all finite dimensional) distribution(s)
$\stackrel{L^2}{\to}, \stackrel{L^2}{=}$	convergence, equality in L^2
$\stackrel{P}{\to}$, p-lim	convergence in probability
d-lim	convergence of the finite dimensional margins