TECHNISCHE UNIVERSITÄT MÜNCHEN
ZENTRUM MATHEMATIK

# On the control of nonlinear dynamical systems

Tanja Vocke

MÜNCHEN
2001

ZENTRUM MATHEMATIK DER TECHNISCHEN UNIVERSITÄT MÜNCHEN
LEHRSTUHL UNIV.-PROF. DR. J. SCHEURLE

# On the control of nonlinear dynamical systems

Tanja Vocke

Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines

**Doktors der Naturwissenschaften (Dr. rer. nat.)**

genehmigten Dissertation.

| | | |
|---|---|---|
| Vorsitzende: | | Univ.-Prof. Dr. Heike Faßbender |
| Prüfer der Dissertation: | 1. | Univ.-Prof. Dr. Jürgen Scheurle |
| | 2. | Univ.-Prof. Dr. Fritz Colonius, Universität Augsburg |

Die Dissertation wurde am 27.04.2001 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 20.07.2001 angenommen.

# Contents

# Preface

Control theory deals with the design and analysis of control systems. Its aim is the computation of a control law such that a particular state of the control system is driven to a predefined state, e.g., a fixed point. Such controls are widespread. For example, water storage tanks are control systems where the float inside the tank restricts the inlet flow of the water as the water level rises. The water tank with the float and the in- and outlet are the control system. The position of the float is the control, because it adjusts the inlet flow. It either restricts it as explained above or it keeps the water height in the tank if water is taken out. Another example of a control system is the so-called rocket car. This is a car that runs on rails and is equipped with two rocket engines. The goal is to move this car from some location to a predefined place. The system is then given by the car and its track, and the state is the position of the car and its velocity. Moreover, the control is represented by the firing of one of the engines. Depending on which engine is fired, the car moves to the left or right on the track, cf. [MS82].

The classical approaches to control theory deal with linear time-invariant control systems, which are well understood today. Not until the middle of the 20th century, control theory for nonlinear control systems has emerged and thus, the restriction to linear systems has at least partly been overcome. There exist several approaches to nonlinear control theory, e.g., nonlinear controllability that uses Lie-algebraic methods and nonlinear stabilization, which can be achieved by Lyapunov functions. For a detailed introduction to nonlinear control systems, cf. [Son98, Isi89] and [NS90]. In the present work, we are particularly interested in time-discrete control systems given by $x_{n+1} = f(x_n, F(x_n))$, where $f\colon X \times U \to X$, and $F$ is a feedback control. That is a function $F\colon X \to U$, where $X$ is the state space and $U$ the control space. The problem is to find a feedback $F$ so that the controlled system is asymptotically stable near some periodic orbit.

Our approach is the stabilization of such a system from the point of view of dynamical system theory. We can interpret a physical system as a control system, when we use one of its system parameters $u$ as time dependent feedback control $F(x_n) = u_n$. Thus, let us consider a nonlinear discrete dynamical system governed by the evolution equation

$$x_{n+1} = f(x_n, u_n),$$

where $f(\cdot, u_n): X \to X$ and $u_n \in U$ is a system parameter. The control $u_n$ is a nonlinear feedback control, which means that $u_n$ can be calculated in terms of the current state $x_n$. Assume that the dynamical system possesses a hyperbolic unstable periodic orbit $O$ with a stable manifold of dimension one at least. Since we approach the stabilization problem from the point of view of dynamical system theory, we will make use of the dynamics of this system. Especially, we take into account the stable manifold of the periodic orbit $O$ and the fact that this manifold is invariant under the map $f$. Stabilization of the nonlinear discrete dynamical system at the unstable periodic orbit is achieved in the following manner: An orbit, that is not on the stable manifold of $O$ but within some neighborhood of it, usually will move away from the periodic orbit due to its instability. To prevent this from happening, we compute the feedback control $u_n$ such that this orbit is forced onto the stable manifold of $O$. After the application of the feedback law $u_n$ at time $n$, control is switched off again and by invariance of the stable manifold under $f$, the orbit is attracted to the hyperbolic unstable periodic orbit $O$. Thus, orbits that usually move away from the periodic orbit, stay close to it and hence, the system is stabilized at $O$. This kind of nonlinear stabilization has a background in dynamical system theory and uses feedback control as is done in control theory. In contrast to our approach, nonlinear stabilization in terms of Lyapunov functions uses a feedback $F$ to construct a control so that one obtains a descent in the energy levels of the Lyapunov function in order to reach its minimum, i.e., the periodic orbit $O$.

Let us introduce our nonlinear stabilization method in more detail. For the sake of simplicity, we first consider the stabilization at hyperbolic saddles in two dimensions, cf. Figure 1. Note that fixed points are just a special case of periodic orbits, which we will deal with later. Let the nonlinear discrete dynamical system be given by

$$x_{n+1} = f(x_n, u_n),$$

where $f(\cdot, u_n): \mathbb{R}^2 \to \mathbb{R}^2$ is a $C^r$-map, $r \geq 1$, and $u_n \in \mathbb{R}$. Assume that for $u_n = u_\star$, the origin is a hyperbolic saddle fixed point of the uncontrolled system $x_{n+1} = f(x_n, u_\star)$, i.e., $f(0, u_\star) = 0$. Then the eigenvalues of $\mathrm{D}f(0, u_\star)$ are given by $\lambda_s$, $\lambda_u \in \mathbb{R}$ with $|\lambda_s| < 1$ and $|\lambda_u| > 1$. Moreover, we assume that the corresponding eigenvectors are unit vectors in $\mathbb{R}^2$. Hence, the linear subspaces of the fixed point are given by $E^s(0) = \mathrm{span}\{(1 \quad 0)^t\}$ and $E^u(0) = \mathrm{span}\{(0 \quad 1)^t\}$. The Stable Manifold Theorem guarantees the existence of the local stable manifold $W^s(0)$ with $\dim W^s(0) = \dim E^s(0) = 1$ in an open neighborhood $U$ of $0$. Moreover, this theorem tells us that the local stable manifold $W^s_{loc}(0)$ is represented by the graph of a $C^r$-function $\psi: E^s(0) \cap U \to E^u(0) \cap U$ such that

$$W^s_{loc}(0) = \{(x^{(1)}, \psi(x^{(1)})) : x^{(1)} \in E^s(0) \cap U\}.$$

Let $x_0$ be a given initial condition that generates an orbit $O(x_0)$ which is not on the stable manifold $W^s(0)$. Then we compute the feedback law $u_n$ at some time $n$ so that $x_{n+1} = f(x_n, u_n)$ is a point on the stable manifold of the origin. Since

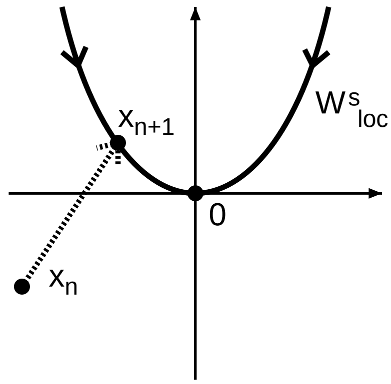


Figure 1: The nonlinear stabilization method in two dimensions.

$W^s(0)$ is positively invariant under $f$, the controlled orbit $O(x_{n+1})$ approaches the saddle fixed point. Thus, the nonlinear system is stabilized at the origin. Note that one can derive an implicit equation for the computation of the feedback law $u_n$. One of the main results of this thesis is the following local existence and uniqueness theorem for such a nonlinear feedback control.

**Theorem 1**
*Consider $x_{n+1} = f(x_n, u_n)$ as above with $f(0, u_\star) = 0$. The system is stabilizable at the hyperbolic saddle fixed point $0$ provided that*

$$\partial_u \left[\psi(f_1(0,0,u)) - f_2(0,0,u)\right]\Big|_{u=u_\star} \neq 0.$$

*Let $U \in \mathbb{R}^2$ be an appropriate open neighborhood of the fixed point and $V \subset \mathbb{R}$ be an appropriate open neighborhood of $u_\star$. If $x_n \in U$, then the control $u_n$ at time $n$ is given by the unique solution $u_n \in V$ of*

$$\psi\left(f_1(x_n, u_n)\right) = f_2\left(x_n, u_n\right),$$

*where $f = (f_1, f_2)^t$.*

Later in the thesis, we do not restrict ourselves to two dimensions and fixed points, but generalize the above theorem to arbitrary dimension $m \in \mathbb{N}$ and hyperbolic periodic orbits of period $k \geq 1$, which have a global stable manifold with dimension $s$, where $1 \leq s < m$. The idea is the same as in case of the fixed point control. Let $\{p_0, p_1, \ldots, p_{k-1}\}$ be a hyperbolic periodic orbit of period $k$ of $x_{n+1} = f(x_n, u_n)$ for $u_n = u_\star$. At some time $n$, we compute the feedback $u_n$ such that $x_{n+1} = f(x_n, u_n)$ is a point of the stable manifold of the following periodic point $p_{i+1}$, i.e., $x_{n+1} \in W^s(p_{i+1})$. We obtain the following local existence and uniqueness result:

**Theorem 2**

*Let $x_{n+1} = f(x_n, u_n)$ define a discrete dynamical system in $\mathbb{R}^m$, where $f(\cdot, u_n)\colon \mathbb{R}^m \to \mathbb{R}^m$ is a $C^r$-diffeomorphism and $u \in \mathbb{R}^{m-s}$ is a system paramter. Assume that $f(x_n, \cdot)\colon \mathbb{R}^{m-s} \to \mathbb{R}^m$ is at least $C^1$ and that $f$ possesses a hyperbolic periodic orbit $\{p_0, \ldots, p_{k-1}\}$ of period $k \geq 1$ for $u_n = u_\star$. Assume that $\mathrm{D}f^k(p_i)$ has $s$ stable eigenvalues, $m > s \geq 1$, and $m - s$ unstable ones with corresponding generalized eigenvectors $v_1^i, v_2^i, \ldots, v_s^i$ and $w_1^i, \ldots, w_{m-s}^i$, where $i = 0, \ldots, k-1$. Let $U_i \subset \mathbb{R}^m$ be an appropriate open neighborhood of $p_i$ for each $i \in \{0, \ldots, k-1\}$ and $D \subset \mathbb{R}^{m-s}$ be an appropriate open neighborhood of $u_\star$. If $x_n \in U_i$ at time $n$ for one $i \in \{0, \ldots, k-1\}$, then there exists a unique control $u_n \in D$ near $u_\star$ that stabilizes the dynamical system at $O(p_0)$ provided that*

$$\pi_{E^u(p_{i+1})}\Big[\partial_u(f(p_i, u_\star))\Big] \tag{1}$$

*is invertible. $\pi_{E^u(p_{i+1})}\colon \mathbb{R}^m \to E^u(p_{i+1})$ denotes the projection onto the linear unstable subspace $E^u(p_{i+1})$.*

Note that the feedback control $u_n$ is determined by an implicit equation similar to

$$\psi\left(f_1(x_n, u_n)\right) = f_2\left(x_n, u_n\right)$$

in Theorem 1. In the proof of the Theorem 2, we apply the Implicit Function Theorem to this equation. We obtain $\frac{\partial f}{\partial u}(p_i, u_\star) \in \mathbb{R}^{m \times m-s}$ and the projection onto $E^u(p_{i+1})$ leads to a matrix of dimension $\mathbb{R}^{(m-s)\times(m-s)}$ which is invertible by the assumption in the theorem above. Thus, local existence and uniqueness of $u_n$ can be proven.

Summarizing, we will establish a stabilization method for nonlinear dynamical systems that does not use typical control theory methods like Lyapunov functions. One possible application of this stabilization procedure is so-called chaos control, which was also a motivation for the derivation of the above results. One considers a nonlinear dynamical system that possesses a chaotic attractor, in which unstable periodic orbits are typically dense. Thus, there exists a large number of periods and the system can be stabilized in many different hyperbolic periodic orbits. Various controls for chaotic system have been developed, e.g., [HL98] or [OGY90a]. A good overview of present research on chaos control is given in [JMTV97, Sch99].

One of these chaos control techniques has been introduced in 1990 by Ott, Grebogi and Yorke, cf. [OGY90a, OGY90b]. They present a simple geometric approach of how to compute the control $u_n$ at a given time $n$ for the nonlinear system $x_{n+1} = f(x_n, u_n)$ which contains a chaotic attractor. Ott et al. linearize the system at the saddle fixed point at which the system should be stabilized. The feedback law is computed for the linear system so that with respect to the stable subspace, stabilization can be achieved. But since the original nonlinear system is iterated and the controlled orbit is only forced onto the stable subspace, it might not reach the saddle fixed point but wanders off again and undergoes a chaotic transient. In this case, control has to be applied again. This so-called OGY-method makes use of the chaotic dynamics of the system, which ensures that an orbit comes eventually close enough to the chosen hyperbolic saddle fixed point or periodic orbit. Since this approach relies on the linearization of the original system $x_{n+1} = f(x_n, u_n)$, stabilization of the system can only be achieved within the small neighborhood of the fixed point in which the approximation of the nonlinear system is still valid. The nonlinear stabilization method introduced above does not have this disadvantage, because it uses the local stable manifold instead of the stable subspace. Here, orbits which are further away from the saddle point can be controlled. As a result, a much larger set of initial conditions is stabilizable compared to the OGY-method. Accordingly, the nonlinear stabilization procedure is more global than the linear one. Moreover, if one considers hyperbolic periodic orbits with periods greater than one, it turns out that the nonlinear method is more uniform and needs less control steps.

Stabilization procedures such as the OGY-method have been applied to physical systems such as the driven pendulum and the driven bronze ribbon [HDM94, $S^+97$]. Moreover, in [ND92], the Duffing equation is controlled. A magneto-elastic ribbon is stabilized in [DRS90] and [SGOY93] and a model of a laser given by the Ikeda map in [SO95]. Recently, some experiments in medicine such as in [CC96, $S^+94$] have been started. In [WJ96], the OGY-method is applied to a nonlinear one-dimensional map that represents the relationship between action potential duration and heart rate. This relation plays an important role in lethal heart rhythm disorders. The authors point out that the chaos control algorithm might have applications for the prevention of cardiac rhythm disturbances. An overview about applications of chaos control is presented in [Sch99] and [JMTV97]. The very first illustration of the OGY-method was an application to the Hénon map, cf. [OGY90a, OGY90b].

Let us return to the results established in Theorems 1 and 2. We point out that, in theory, it would be sufficient to apply the feedback control $u_n$ once in order to achieve stabilization at a periodic orbit. The problem is that we have to find a representation of the stable manifold or the periodic orbit. In order to obtain such a representation locally, we use the graph of the function $\psi\colon E^s(p_i) \cap U \to E^u(p_i) \cap U$. In the stabilization algorithm, this function is approximated by a Taylor series and thus, in the implementation of the results, we work with this approximation and can not expect that stabilization is successful with only one application of the control. Furthermore, depending on the degree of nonlinearity of the system, the feedback law has to be approximated as well. So in general, the controlled orbit is not exactly on the stable manifold and thus, it leaves a certain neighborhood of the manifold after a number of iterations. In conclusion, the system is not stabilized. To overcome this, we check whether the controlled orbit is still within a neighborhood of the local stable manifold. If it leaves this neighborhood, the feedback control is applied again to stabilize the system. Note that in this regard, the OGY-method is just a special case of our stabilization procedure. Ott et al. control the orbit so that it is on the linear stable subspace $E^u$ instead of the local stable manifold $W^s$.

The subspace $E^u$ is the $0$-th order approximation of the stable manifold. Therefore, the OGY-method is the same as our nonlinear stabilization method using this approximation.

In order to demonstrate the performance of our nonlinear stabilization algorithm, we have chosen to stabilize the Hénon map and the Ikeda map from all the different systems the OGY-method has been applied to. To compare our results with those obtained by the OGY-method, we implement the OGY algorithm and use mainly the Hénon map to point out its shortcomings. As one expects, our implemented nonlinear stabilization method acts more global, since it uses a higher order approximation of the local stable manifold instead of the linear stable subspace. Therefore, stabilization can be achieved even when an orbit is still far from the periodic orbit. In contrast, the linear method from [OGY90a] can only be applied for orbits within a small strip at the periodic orbit. Therefore, the globality of the nonlinear stabilization is an enormous advantage. It is able to stabilize more orbits than the OGY-method. Especially for periodic orbits, less control steps with smaller values than in the linear setting are used. The price for controlling the fully nonlinear system is that one has to put more computational effort into the algorithm. Therefore, our method is more costly than the OGY-method.

The thesis is organized as follows: In the first chapter, an introduction to dynamical systems and chaos is given. The purpose of this chapter is to provide a brief overview. Since the main results are applied to nonlinear discrete systems we restrict ourselves to dynamical systems with discrete time. We present basic notations and results in Section 1.1, in particular for steady state solutions and their stability as well as hyperbolic periodic orbits. Further on, we state the Hartman Grobman Theorem and introduce stable and unstable manifolds of hyperbolic fixed points. This allows us to introduce the Stable Manifold Theorem in the second section, which plays a central role later on. In Section 1.3, we define invariant sets and attractors and finish the chapter with the basic definitions for chaotic systems such as sensitive dependence on initial conditions, strange attractors and Lyapunov exponents. These definitions will be of use in Chapters 3 and 5, where we apply our nonlinear stabilization method to the Hénon map and the Ikeda map, which are both chaotic.

Chapter 2 is concerned with the introduction of our stabilization method. We restrict ourselves to the stabilization at a hyperbolic fixed point $p_\star$ that has a one-dimensional stable and a one-dimensional unstable manifold. Section 2.1 introduces the nonlinear stabilization method and in Section 2.2, we prove Theorem 1 using the Stable Manifold Theorem and the Implicit Function Theorem. The last section of Chapter 2 introduces the corresponding algorithm for the nonlinear stabilization method.

Chapter 3 gives some computational results and aspects of the algorithm, which has been developed in the previous chapter. Our goal is to stabilize the Hénon map at a hyperbolic saddle point, which is embedded in the strange attractor $\mathcal{A}$ of the map. We present the results for our method and the OGY-method, which we have also implemented. Both algorithms were tested for various initial conditions and with different bounds on the control parameter and on the neighborhood of the local stable manifold.

Chapter 4 discusses the nonlinear stabilization method for periodic orbits of period $k$ in $m$ dimensions. The first section introduces basic facts from linear algebra as well as properties of the local stable manifolds of periodic orbits. Section 4.2 introduces local coordinate transformations $\phi_i$ and further relevant details. Then we prove the existence and uniqueness Theorem 2. In Section 4.4, the algorithm for stabilizing systems at periodic orbits of period greater than one is given. Finally, we apply this algorithm to the Hénon map and its hyperbolic period-2 orbit. Again, we compare the results to those of the OGY-method.

In Chapter 5, we proceed with the application to a dynamical system that has been derived from a realistic model. We have chosen the Ikeda map, which represents a two-dimensional laser system. First, we introduce the system dynamics, and then the stabilization at a hyperbolic periodic orbit of period 3 is shown. Such a laser system is useful in many different areas of application, e.g., medicine, computer science or in biotechnologies. As it has been shown, lasers are very sensitive to small perturbations, cf. [HJM85, Ott93, Sch99] and thus, it is desirable to stabilize laser systems at a periodic behavior.

This work was supported by the Deutsche Forschungsgemeinschaft within the Graduiertenkolleg Angewandte Algorithmische Mathematik at the Technische Universität München. My advisor, Professor Jürgen Scheurle, encouraged me to stay with the subject of nonlinear stabilization and chaos control. During the past three years, he kept giving me constant advice and support. I would like to thank him as well as Professor James A. Yorke, who first introduced me to this interesting topic. Special thanks to my colleagues Dr. Peter Giesl, Dr. Matthias Rumberger and Dr. Hans-Peter Kruse for numerous fruitful discussions and for proof-reading my thesis. I am most grateful to my parents for encouraging me. Their constant support made it a lot easier to focus on this work. Last, but not least, I am deeply indebted to Oliver Knopf. He always gave me steadying support, never lost faith in me and kept patient during all the difficult moments.

# Chapter 1

# Preliminaries

This chapter introduces some standard results from discrete dynamical system theory. We provide notations, definitions and fundamental theorems which will be used throughout this thesis. In the first section, we define discrete autonomous dynamical systems as well as orbits of general and specific solutions such as fixed points. The question of how to determine stability of periodic solutions is treated as well as hyperbolicity. We close this section with the Hartman-Grobman Theorem. Section 2 introduces invariant eigenspaces for hyperbolic fixed points. We define stable and unstable manifolds and present the Stable Manifold Theorem. The third section deals with attractors. The chapter finishes with some aspects of chaotic dynamics. The results, which are presented in this chapter, are taken from [Wig88, Wig90, GH83, Rob95, Dev86, KH97, Ott93, ASY97, Hal88].

## 1.1 Discrete dynamical systems and special solutions

The emphasis of this work lies on autonomous discrete dynamical systems which we define as follows. Let $\phi^n : X \to X$ be given, where $(X, d)$ is a metrix space. We call $X$ the *phase space* and $n$ the discrete time, i.e., $n \in \mathbb{Z}$. A dynamical system is characterized by the property that given any initial state $x_0 \in X$ at initial time $0$, i.e., $x_0 = x(0)$, the system assigns to any future time a unique state. In other words, the dynamical system is given by

$$\phi^n : x_0 = x(0) \mapsto x(n) = \phi^n(x_0) \in X$$

such that $\phi^n$ fulfills the so-called flow property, i.e.,

$$\phi^{s+n}(x_0) = \phi^s(\phi^n(x_0)).$$

Throughout this thesis, we consider discrete dynamical systems given by an autonomous difference equation

$$x_{n+1} = f(x_n), \tag{1.1}$$

where $f\colon X \to X$, $X \subset \mathbb{R}^m$ nonempty and open, is a $C^r$-diffeomorphism, $r \geq 1$. If the map $f$ is linear, the system (1.1) is called a *linear dynamical system*, otherwise, the system is *nonlinear*.

Consider the initial value problem

$$x_{n+1} = f(x_n), \quad x_0 \in X.$$

If we apply $f$ to the initial condition $x_0$, then $f(x_0) \in X$, $f(f(x_0)) \in X$ and so on. Clearly, the initial value problem has a unique solution given by the sequence $\{x_n\}_{n\in\mathbb{N}}$. Since $f$ is $C^r$, for each fixed $n$, $x_n$ depends $C^r$ on $x_{n-1}$.

For an arbitrary initial condition $x_0 \in X$, we define the $n$-th iterate of $x_0$ as follows:

$$f(f(\ldots(f(x_0)\ldots))) = \underbrace{f \circ f \circ \ldots f \circ f}_{n \text{ times}}(x_0) := f^n(x_0).$$

The *orbit* of $x_0$ under $f$ is a bi-infinite sequence

$$O(x_0) := \{\ldots, f^{-n}(x_0), \ldots, f^{-1}(x_0), x_0, f(x_0), \ldots, f^n(x_0), \ldots\}.$$

Sometimes, one distinguishes between a *forward* and *backward orbit*, i.e.,

$$O^+(x_0) := \{x_0, f(x_0), \ldots, f^n(x_0), \ldots\}$$

and

$$O^-(x_0) := \{\ldots, f^{-n}(x_0), \ldots, f^{-1}(x_0), x_0\}.$$

Note that the orbits of discrete dynamical systems differ from those which are generated by an ordinary differential equation. Orbits of continuous systems are curves, whereas orbits of maps are discrete sets of points.

Studying specific types of solutions of (1.1) turns out to be useful when one is interested in the qualitative behavior of a dynamical system. Therefore, one of the very first steps in the analysis is to seek special solutions such as fixed points or periodic orbits of the system (1.1). Furthermore, a characterization of the behavior of solutions nearby a specific solution is helpful, especially for nonlinear systems. We call $p \in X$ a *fixed point* or *equilibrium* for the difference equation $x_{n+1} = f(x_n)$, if $f(p) = p$. From a geometrical point of view, a fixed point is the point of intersection of the graph of $f$ and the diagonal $g(x) = x$. We also refer to a fixed point as a *period*-1 *orbit*. Periodic orbits with period greater than one are fixed points of $f^k$ where $k$ is the corresponding period $k$.

**Definition 1.1.1** (Periodic Orbit)
*A point $p \in X$ is called a periodic point of period $k$ for the map $f$ if $f^k(p) = p$. Here $k$ is the smallest such positive integer. The orbit $O(p)$ is called a periodic orbit of period $k$ or a period-$k$ orbit and consists of $k$ points.*

From now on, let $p$ be a fixed point of $f$. Let us introduce the following definition of stability for $p$, cf. [Wig90].

**Definition 1.1.2** (Lyapunov Stability)
*The fixed point $p \in X$ of a dynamical system given by (1.1) is said to be Lyapunov stable or stable, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if for $x_0 \in X$ satisfying*

$$d(x_0 - p) < \delta,$$

*then*

$$d(f^n(x_0) - p) < \epsilon$$

*for $n > 0$.*

If the fixed point $p$ is not stable, then it is called *unstable*.

**Definition 1.1.3** (Asymptotic Stability)
*Let $x_{n+1} = f(x_n)$ be a discrete dynamical system given as in Definition 1.1.2. The fixed point $p \in X$ is asymptotically stable, if it is stable, and in addition, it is attractive, i.e., there exists a $\delta > 0$ such that*

$$d(x_0 - p) < \delta \quad \Rightarrow \quad \lim_{n\to\infty} f^n(x_0) = p.$$

**Remark**
In order to describe the stability of periodic orbits of period $k > 1$, we just have to replace $f$ by $g := f^k$.

We are not only interested in the stability of $p$, but also in the behavior of solutions nearby $p$. Hence, we consider the linearization of the nonlinear dynamical system (1.1) at $p$ which is given by

$$y_{n+1} = \mathrm{D}f(p)\, y_n, \tag{1.2}$$

where $\mathrm{D}f(p) \in \mathbb{R}^{m\times m}$ denotes the Jacobian matrix of $f$ at $p$. The linearization (1.2) of (1.1) has a fixed point at $0$. It is a well-known fact that if all eigenvalues of $\mathrm{D}f(p)$ lie inside the unit disk, then $0$ is asymptotically stable. If one of the eigenvalues lies outside the unit disk, then the fixed point is unstable.

Throughout this chapter, we will consider the nonlinear system $x_{n+1} = f(x_n)$ and assume that it possesses a fixed point $p$. In Chapter 2, we will discuss the stabilization of a so-called hyperbolic saddle fixed point $p$.

**Definition 1.1.4** (Hyperbolic Saddle Fixed Point)
*Consider the discrete linear dynamical system (1.2). The fixed point $0$ of (1.2) is called a hyperbolic fixed point, if none of the eigenvalues of $\mathrm{D}f(p)$ has absolute value one. Moreover, if at least one eigenvalue of $\mathrm{D}f(p)$ has absolute value less than one and at least one eigenvalue has absolute value greater than one, $0$ is called a hyperbolic saddle fixed point.*

Now suppose that $p$ is a fixed point of the nonlinear system (1.1). It is hyperbolic, if none of the eigenvalues of $\mathrm{D}f(p)$ have absolute value equal to one. The aim is to relate the stability of $0$ of (1.2) to the stability of $p$ from the original nonlinear system. The following theorem states that the nonlinear system (1.1) near $p$ is topologically conjugated to its linearization (1.2).

**Theorem 1.1.5** (HARTMAN GROBMAN)
*Let $p$ be a hyperbolic fixed point of $x_{n+1} = f(x_n)$, where $f$ is a diffeomorphism. The stability of the hyperbolic fixed point $0$ of $y \mapsto \mathrm{D}f(p)\,y$ corresponds to stability of the hyperbolic fixed point $p$.*

A proof the theorem can be found, for example, in [KH97, Rob95]. This theorem states that, in the hyperbolic case, instead of considering the original system, we can determine the stability of a fixed point via the linearized system. As has been said before, the stability of a fixed point is determined by the Jacobian matrix $\mathrm{D}f(p)$:

**Theorem 1.1.6**
*Let $f\colon X \to X$ be a diffeomorphism and let $p \in X$ be a fixed point of $x_{n+1} = f(x_n)$.*

*(i) If the absolute value of each eigenvalue of $\mathrm{D}f(p)$ is strictly less than $1$, then $p$ is asymptotically stable.*

*(ii) If the absolute value of at least one of the eigenvalues of $\mathrm{D}f(p)$ is greater than $1$, then $p$ is unstable.*

A proof of this theorem can be found in [Dev86].

**Remark**
If we consider a period-$k$ orbit of the map $f$, we just replace $f$ by $g := f^k$. In this case, the linearization of (1.1) is

$$y_{n+1} = \mathrm{D}f^k(p)\,y_n$$

instead of (1.2). Provided that all eigenvalues of $\mathrm{D}f^k(p)$ lie inside the unit disk, the periodic orbit

$$\{p, f(p), \dots, f^{k-1}(p)\}$$

is asymptotically stable.

## 1.2 Stable and unstable manifolds of hyperbolic fixed points

Throughout this section, let $p \in X$ be a hyperbolic fixed point of $x_{n+1} = f(x_n)$. The previous section told us how to determine the stability of p. Now we characterize the behavior of orbits near $p$ in detail. We start with the simplest discrete dynamical system, namely the *linear system*

$$x_{n+1} = A\,x_n, \quad x_n \in \mathbb{R}^m, \tag{1.3}$$

where $A \in \mathbb{R}^{m\times m}$. The only fixed point of (1.3) is the origin. Assume that $0$ is hyperbolic. Theorem 1.1.6 tells us that the Jacobian matrix $\mathrm{D}f(0)$ determines the stability of the fixed point.

The eigenvectors corresponding to the eigenvalues of $A$ define, depending on the modulus of the eigenvalue, subspaces as follows: Suppose $A$ has $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$. and suppose there are $n_s$ eigenvalues $\lambda_i$ which have absolute value less than one. These are the so called *stable eigenvalues*, since orbits lying in the eigenspaces of $\lambda_i$ are attracted to $p$ with the rate $|\lambda_i| < 1$ for $i = 1, \ldots, n_s$. The space spanned by the corresponding generalized eigenvectors $v_1, \ldots, v_{n_s}$ is the so called *linear stable subspace* which we denote $E^s$:

$$E^s(0) := \mathrm{span}\{v_1, \ldots, v_{n_s}\}$$

The analogue is true for the *unstable eigenvalues* $\lambda_j$ with corresponding generalized eigenvectors $u_1, \ldots, u_{n_u}$. Here, $|\lambda_j| > 1$ for $j = 1, \ldots, n_u$ and we define the *linear unstable subspace*

$$E^u(0) := \mathrm{span}\{u_1, \ldots, u_{n_u}\}.$$

Note that the orbits in $E^s(0)$ and $E^u(0)$ are characterized by contraction and expansion, respectively, cf. also Figure 1.1.

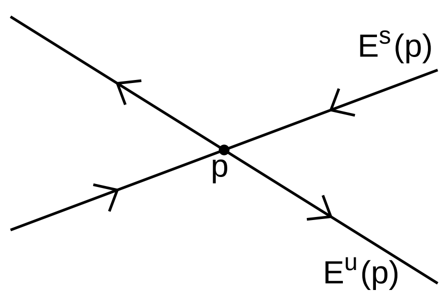


Figure 1.1: Stable and unstable eigenspaces of a hyperbolic fixed point $p$ in $\mathbb{R}^2$.

**Remarks**

(i) Since $0$ is a hyperbolic fixed point, then we have $n_s + n_u = dim(\mathbb{R}^m) = m$ and in particular, $\mathbb{R}^m = E^s(p) \oplus E^u(p)$. Due to this *splitting* of the phase space, we identify $\mathbb{R}^m$ with $E^s(p) \oplus E^u(p)$ such that we write $x \in \mathbb{R}^m$ as $x = (x^s, x^u)$, where $x^s \in E^s(p), x^u \in E^u(p)$.

(ii) For the linear map (1.3) given by $A \in \mathbb{R}^{m\times m}$, we define the *spectral radius* of A by the maximal absolute value of an eigenvalue of $A$. We denote the spectral radius by $r(A)$. Given any norm on $\mathbb{R}^n$, we define

$$\|A\| := \sup_{\|v\|=1} \|A\, v\|.$$

One can show that for every $\delta > 0$ there exists a norm in $\mathbb{R}^n$ such that

$$\|A\| \leq r(A) + \delta,$$

cf. [KH97]. Furthermore, we can define the largest contraction and smallest expansion rate of $A$ as follows:

$$\begin{aligned} \lambda(A) &:= r\left(A\Big|_{E^s(0)}\right) \\ \mu(A) &:= \left(r\left(\left(A\Big|_{E^u(0)}\right)^{-1}\right)\right)^{-1} \end{aligned}$$

The above result about $\|A\|$ leads to $\|A|_{E^s(0)}\| \leq \lambda(A) + \delta < 1$ and $\|A|_{E^u(0)}\| \geq \mu(A) - \delta > 1$.

(iii) In case, that $p$ is not a hyperbolic fixed point, i.e., at least one eigenvalue has modulus equal to one, the so-called *linear center subspace* is given by

$$\begin{aligned} E^c(p) := \operatorname{span}\{&n_c \text{ generalized eigenvectors} \\ &\text{whose eigenvalues have modulus } = 1\}, \end{aligned}$$

where $n_s + n_u + n_c = m$.

For the remaining part of this section, let $p$ be a hyperbolic saddle fixed point of the nonlinear $C^r$-diffeomorphism $f$ that defines the dynamical system $x_{n+1} = f(x_n)$. We consider the linearization of the form (1.3) with $A = \mathrm{D}f(p)$. Note that the stability behavior is completely determined by $\mathrm{D}f(p)$. The Hartman-Grobman Theorem 1.1.5 tells us that $0$ is a fixed point for (1.2) with the same stability properties as those of $p$ for (1.1). Furthermore, in some neighborhood $U \subset X \subset \mathbb{R}^m$ of $p$, the system $x \mapsto f(x)$ is topologically conjugated to $y \mapsto \mathrm{D}f(p)\, y$. Due to the considerations above, we can classify the stability of $p$. We define the stable and unstable manifold of a hyperbolic fixed point for a nonlinear discrete dynamical system as follows:

**Definition 1.2.1** (Local Manifolds)
*Let $p$ be a hyperbolic fixed point of the local $C^r$-diffeomorphism $f$ and let $U \subset X$ be a neighborhood of $p$. The local stable manifold of $p$ is defined as*

$$W^s_{loc}(p) := \{x \in U : f^n(x) \overset{n\to\infty}{\longrightarrow} p \text{ and } f^n(x) \in U \quad \forall n \geq 0\}.$$

*Analogously, the local unstable manifold of $p$ is defined as*

$$W^u_{loc}(p) := \{x \in U : f^{-n}(x) \overset{n\to\infty}{\longrightarrow} p \text{ and } f^{-n}(x) \in U \quad \forall n \geq 0\}.$$

The corresponding global manifolds are given by the union of all preiterates, respectively iterates, of the local manifolds:

$$W^s(p) := \bigcup_{n\geq 0} f^n(W^s_{loc}(p))$$

and

$$W^u(p) := \bigcup_{n\geq 0} f^{-n}(W^u_{loc}(p)).$$

One interesting property of the stable and unstable manifold of a hyperbolic fixed point is invariance with respect to the system (1.1). In the following chapters, where the control procedure is introduced, we will especially make use of the fact that the local stable manifold is positively invariant under $f$.

**Definition 1.2.2** (Invariant Set)
*A set $A \subset X$ is called positively invariant under $f$, if $f(x) \in A$ for all $x \in A$. The set $A$ is negatively invariant, if $f^{-1}(x) \in A$ for all $x \in A$. Finally, $A$ is said to be invariant provided that $f(A) = A$.*

Note that fixed points and periodic orbits are always invariant sets under $f$. Let us return to the local manifolds of $p$. The nonlinear system $x_{n+1} = f(x_n)$ defined on $X \subset \mathbb{R}^m$ can locally be transformed around $p$ so that $p$ is translated to the origin and the coordinates are chosen such that the unit vectors $e_1, \dots, e_s$ span $E^s(0)$ and $e_{s+1}, \dots, e_m$ span $E^u(0)$, cf. [Wig90]. This transformed system then reads

$$\begin{aligned} y_{n+1} &= A^s y_n + P(y_n, z_n) \\ z_{n+1} &= A^u z_n + Q(y_n, z_n), \end{aligned}$$

where $y \in \mathbb{R}^s, z \in \mathbb{R}^u, s + u = m$. Let $r \geq 1$, $V \subset \mathbb{R}^s \times \mathbb{R}^u$ be an open neighborhood of $(0,0)$, $P \in C^r(V, \mathbb{R}^s)\,, Q \in C^r(V, \mathbb{R}^u)$ such that $P(0,0) = Q(0,0) = 0$, $\mathrm{D}P(0,0) = \mathrm{D}Q(0,0) = 0$. The matrices $A^s \in \mathbb{R}^{s\times s}, A^u \in \mathbb{R}^{u\times u}$ have only eigenvalues with absolute value smaller or greater than one, respectively. This implies that $(0,0)$ is a hyperbolic fixed point. Since we assume $f$ to be $C^r$ with $r \geq 1$, the transformed system is also $C^r$. The following theorem states that $E^s(0,0)$ is the 0-th order approximation of $W^s_{loc}(0,0)$.

**Theorem 1.2.3** (STABLE MANIFOLD THEOREM)
*Let* $(0,0)$ *be a hyperbolic fixed point of the system*

$$\begin{aligned} y_{n+1} &= A^s\, y_n + P(y_n, z_n) \\ z_{n+1} &= A^u\, z_n + Q(y_n, z_n), \end{aligned}$$

*with the assumptions made above. There exists unique local stable and local unstable* $C^r$*-manifolds*

$$W^s_{loc}(0,0) = \{(y,z) \in U \subset V : z = \psi(y)\}$$

*and*

$$W^u_{loc}(0,0) = \{(y,z) \in U \subset V : y = \theta(z)\},$$

*where* $U$ *is an open neighborhood of* $(0,0)$ *and* $\psi(0) = \theta(0) = 0$, $\mathrm{D}\psi(0) = \mathrm{D}\theta(0) = 0$, *i.e.,* $W^s_{loc}(0,0)$ *and* $W^u_{loc}(0,0)$ *are tangent to* $E^s$ *and* $E^u$ *at* $(0,0)$*, respectively. Moreover,* $W^s_{loc}(0,0)$ *is positively invariant with respect to the system and* $W^u_{loc}(0,0)$ *is negatively invariant with respect to the system.*

A proof of the theorem can be found in, e.g., [KH97].

**Remark**
The dimensions of the local stable and unstable manifold correspond to the dimensions of the stable and unstable subspace, respectively. Furthermore, $\psi$ and $\theta$ are as smooth as $f$. The theorem allows us to represent the local manifolds as graphs of functions $\psi$ and $\theta$, respectively, where $\psi\colon E^s(0,0) \cap U \to E^u(0,0) \cap U$ and $\theta\colon E^u(0,0) \cap U \to E^s(0,0) \cap U$.

The Stable Manifold Theorem gives us the following picture of the local dynamics of $x_{n+1} = f(x_n)$ near the fixed point $p$, cf. Figure 1.2. Every point that is not on $W^s_{loc}(p)$ leaves $U$ under forward iteration. Points on the local stable manifold converge to $p$ at an exponential rate given by the bound on the stable spectrum, cf. Figure 1.2 and the remark on page 6.

## 1.3 Attractors

In Section 1, we have seen that fixed points and periodic orbits can be attracting. Besides those attractors, there exist other sets with attracting properties. We introduce some basic definitions in order to define attractors in a general way. In Chapters 3 and 5, we are going to apply the control methods from Chapters 2 and 4 to two different dynamical systems, the Hénon and the Ikeda map. Both systems possess a local attractor $\mathcal{A}$. Thus, we give a defintion of local attractor that suits the set-up in the corresponding chapters. As before, we consider a discrete dynamical system as in (1.1), where $X \subset \mathbb{R}^m$, $r \geq 1$ and $f$ is a nonlinear local $C^r$-diffeomorphism.

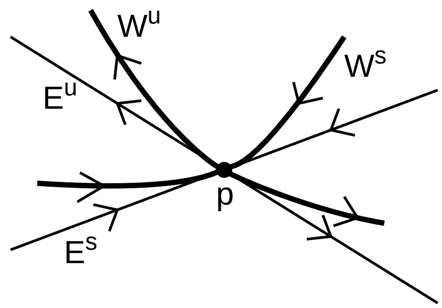


Figure 1.2: Local stable and unstable manifolds of a hyperbolic fixed point $p$ in $\mathbb{R}^2$ with corresponding eigenspaces.

**Definition 1.3.1** (Attracting Set)
*Let $A, B \neq \emptyset$ and $A \subset B \subset X$. The set $A$ is said to attract $B$ under $f$ if for all $x \in B$*

$$\lim_{n\to\infty} \operatorname{dist}(f^n(x), A) = 0.$$

Note that we assume that $A \subset B$ because this is the case in the examples in Chapters 3 and 5. One could also define an attracting set without assuming that $B$ is a set that contains $A$. In Definition 1.2.2, we have already defined invariance. Together with the attracting property of a set $A$, we are able to define local attractors according to [Hal88].

**Definition 1.3.2** (Local Attractor)
*A set $\mathcal{A} \subset X$ is called a local attractor if $\mathcal{A}$ is compact and invariant with respect to the system (1.1) and if there exists a bounded neighborhood $B$ of $\mathcal{A}$ such that $\mathcal{A}$ attracts $B$.*

There exists no generally accepted definition of an attractor. As it is pointed out in [Mil85, Rob95], there exists several other definitions. For example, Milnor introduced a definition that requires $\mathcal{A}$ to attract a set of positive measure. Instead, we prefer the Definition 1.4.2, where points in a whole neighborhood of $\mathcal{A}$ have to approach $\mathcal{A}$. Furthermore, one could also define a global attractor of a system. For example, if we consider the Hénon map, which will be introduced in Chapter 3, and restrict the phase space from $\mathbb{R}^2$ to a certain rectangle $R$ (cf. Section 3.1), then $R$ corresponds to the bounded neighborhood $B$ and there exists a global attractor $\mathcal{A}$ for the Hénon map. But if one chooses $X = \mathbb{R}^2$, then $\mathcal{A}$ is only a local attractor.

The set $B$ in Definition 1.3.1 that is attracted by $A$ is also called a trapping region, cf. [Wig90, GH83].

**Definition 1.3.3** (Trapping Region)
*A closed connected set $B \subset X$ is a trapping region if $f^n(B) \subset B$ for all $n \geq 0$.*

In case that there exists such a trapping region, one can define the associated attracting set by

$$A = \bigcap_{n \geq 0} f^n(B).$$

## 1.4 Chaotic dynamics

In this section, we give a definition of chaotic dynamical systems. Since the research of chaos theory started comparatively recently, there exists no terminology that is generally agreed upon. For some notions like chaotic attractors, there even exist several definitions. Thus, we emphasize that there exists no standardized definition of chaos. In Chapters 3 and 5 of this thesis, we consider systems which contain a single chaotic attractor. Accordingly, we define chaotic behavior in terms of chaotic attractors, i.e., a system is chaotic if the dynamics of $f$ on a local attractor $\mathcal{A}$ is chaotic.

**Definition 1.4.1** (Sensitive Dependence on Initial Conditions)
*A map $f\colon X \subset \mathbb{R}^m \to X$ is said to have sensitive dependence on initial conditions, if there exists $R \in \mathbb{R}_+$ such that, for every $x \in X$ and for each $\varepsilon > 0$, there is a point $y \in X$ with $d(x, y) < \varepsilon$ and an $n \geq 0$ so that $d(f^n(x), f^n(y)) \geq R$.*

This definition is given in [Dev86, Rob95]. So far we have restricted our considerations to local attractors that are not necessarily undergoing sensitive dependence on intial conditions. Now, we define chaotic dynamics on such an attractor as follows, cf. [Wig90].

**Definition 1.4.2** (Chaotic System)
*Let $x_{n+1} = f(x_n)$ be a discrete dynamical system with phase space $(X, d)$ and local attractor $\mathcal{A} \subset X$. If the system displays sensitive dependence on initial conditions on $\mathcal{A}$, then the system is called chaotic.*

In Chapter 3, we treat the Hénon map as an example for the control mechanism developed in Chapter 2. It will be pointed out that there exists a rectangle $R \subset \mathbb{R}^2$ and a compact invariant set $\mathcal{A} \subset R$ that attracts $R$. Thus, $\mathcal{A}$ is a local attractor for the Hénon map and one can show that the map undergoes sensitive dependence on intial conditions an $\mathcal{A}$. Hence, by Definition 1.4.2, the Hénon map is a chaotic dynamical system. An analogous result can be shown for the Ikeda map, cf. Chapter 5.

Another interesting classification of a dynamical system is given by the Lyapunov exponents of the system. These exponents characterize the stretching and the contracting characteristics of attractors.

**Definition 1.4.3** (Lyapunov Exponent)
*Let $f\colon X \to X$ be a diffeomorphism on the metric space $(X, d)$ and $x_0 \in X$ be an initial condition with corresponding orbit $O(x_0)$. Consider an infinitesimal displacement from $x_0$ in the direction of a tangent vector $y_0$. We define the Lyapunov exponent for $x_0$ and initial orientation of the infinitesimal displacement given by $u_0 = y_0/|y_0|$ by*

$$L(x_0, u_0) = \lim_{n\to\infty} \frac{1}{n} \ln |\mathrm{D}f^n(x_0)\, u_0|.$$

Depending on the orientation of $u_0$, there are several possible values of the Lyapunov exponents. In general, there will be $dim(X) = m$ or less distinct Lyapunov exponents for one given initial value. For a more detailed discussion of Lyapunov exponents, the reader is referred to the literature cited within the beginning of this chapter. [ASY97, Ott93] introduce the concept of a Lyapunov exponent of an attractor $\mathcal{A}$. The authors call $\mathcal{A}$ chaotic, if the largest Lyapunov exponent of this attractor is positive. With respect to this definition, the Hénon map possesses a chaotic attractor, cf. also Chapter 3.

Later on, when we consider the Hénon map and the Ikeda map in Chapters 3 and 5, we will see that the local chaotic attractors $\mathcal{A}$ of these maps have a noninteger dimension. The dimension of an attractor can be defined in many different ways. Here, we use the box counting dimension, which is defined as follows.

**Definition 1.4.4** (Box Counting Dimension)
*Let $A \subset X$ be a compact set. The box counting dimension of $A$ is defined by*

$$\dim_b(A) := \liminf_{\varepsilon\to 0} \frac{\log(N(\varepsilon, A))}{\log(\varepsilon^{-1})},$$

*where $N(\varepsilon, A)$ is the minimal number of closed cubes with length $\varepsilon$ that cover $A$.*

Provided that an attractor $\mathcal{A}$ has a box counting dimension that is not an integer, we call it a *strange attractor*.

# Chapter 2

# The stabilization at hyperbolic saddle fixed points in $\mathbb{R}^2$

This chapter introduces the main part of the thesis, the nonlinear stabilization procedure from the point of view of dynamical systems theory. For a better illustration of this particular kind of stabilization, we first restrict our considerations to a two-dimensional dynamical system and stabilize it at a hyperbolic saddle fixed point In Chapter 4, we will generalize the stabilization procedure to $m \geq 2$ dimensions and to hyperbolic periodic orbits of general period $k \geq 1$. By stabilizing a nonlinear dynamical system at a hyperbolic saddle fixed point, we mean that an orbit with an aperiodic behavior is forced onto the the stable manifold of the fixed point. By invariance of the stable manifold under the systems evolution equation, the controlled orbit is attracted to the unstable fixed point. Thus, the irregular movement of the orbit is stabilized.

Section 1 is concerned with the introduction of the stabilization procedure for the special case described above. We introduce all relevant details such that in the second section, we prove the local existence and uniqueness result of the feedback control by which stabilization is achieved. In Section 3, our nonlinear stabilization method is implemented. Moreover, we mention that the OGY-method, cf. [OGY90a, OGY90b], is nothing but a special case of our method.

## 2.1 The nonlinear stabilization at a saddle in two dimensions

The goal of this section is to set up all details needed for the stabilization of a two-dimensional nonlinear autonomous discrete dynamical system. We consider a system together with one of its system parameters $u$. This parameter is taken to be the feedback control such that it stabilizes the system at a hyperbolic saddle fixed point $p_\star$. Note that in the two-dimensional case, $p_\star$ possesses a one-dimensional stable and a one-dimensional unstable manifold. Since we approach the stabilization problem from the point of view of dynamical system theory, we will make use of the dynamics of this system. Especially, we take into account the stable manifold of the saddle and the fact that this manifold is invariant under the evolution equation. Stabilization is achieved in the following manner: An orbit, that is not on the stable manifold of $p_\star$ but within some neighborhood of it, usually will move away from the saddle due to its instability. To prevent this, we compute the feedback control $u_n$ at time $n$ so that this orbit is forced onto the stable manifold of $p_\star$. After the application of the feedback law $u_n$, control is switched off again and by invariance of the stable manifold, the orbit is attracted to $p_\star$. Thus, orbits that usually move away from the saddle fixed point, stay close to it and hence, the system is stabilized at $p_\star$. Schematically, the stabilization procedure can be viewed as in Figure 2.1.

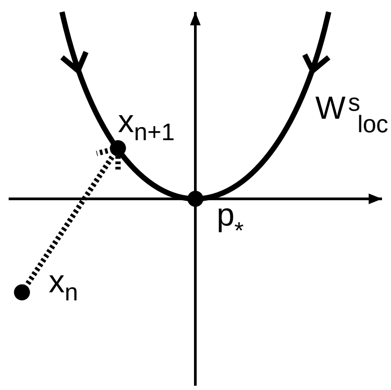


Figure 2.1: The feedback control takes the iterate of a point $x_n$, which is in a neighborhood of the fixed point $p_\star$, onto the local stable manifold $W^s_{loc}$ of $p_\star$. Then, $x_{n+1}$ is a point on $W^s_{loc}$, and by invariance of the manifold, the orbit $O(x_{n+1})$ approaches $p_\star$. Thus, the system is stabilized at $p_\star$.

Let us carry out the stabilization method in detail. We consider a nonlinear discrete dynamical system given by

$$x_{n+1} = f(x_n, u_n) \tag{2.1}$$

where $f\colon X \times \mathbb{R} \to X$, $X \subset \mathbb{R}^2$, is a $C^r$-map, $r \geq 1$, with respect to $x \in X$ and $f$ is at least $C^1$ with respect to $u_n \in \mathbb{R}$. The parameter $u \in \mathbb{R}$ is an adjustable system parameter, which we use as the time dependent feedback control. We assume that for $u_n = u_\star$, there exists a hyperbolic saddle fixed point $p_\star$ of (2.1), i.e., $f(p_\star, u_\star) = p_\star$. The so-called uncontrolled system is given by

$$x_{n+1} = f(x_n, u_\star). \tag{2.2}$$

Since $p_\star$ is a hyperbolic saddle in $X \subset \mathbb{R}^2$, the Jacobian of $f$ at $p_\star$ has two eigenvalues, $\lambda_s, \lambda_u \in \mathbb{R}$ with $|\lambda_s| < 1$ and $|\lambda_u| > 1$. Let $v_s \in \mathbb{R}^2$ and $v_u \in \mathbb{R}^2$ be the corresponding eigenvectors. Then the linear stable and unstable subspace are given by $E^s(p_\star) = \mathrm{span}\{v_s\}$ and $E^u(p_\star) = \mathrm{span}\{v_u\}$, respectively.

Our goal is to force an orbit $O(x_0)$ onto the stable manifold $W^s(p_\star)$. Thus, the main tasks are to find a suitable description of the stable manifold $W^s(p_\star)$ and to determine an equation for the computation of the necessary control $u_n$. The first task is answered by the Stable Manifold Theorem 1.2.3. For $u_n = u_\star$, the theorem tells us that there exists $W^s(p_\star)$ tangent to $E^s(p_\star)$ at $p_\star$ with the same dimension as $E^s$. In this case, $\dim E^s(p_\star) = 1$. Therefore, we need to find a representation of the one-dimensional local stable manifold $W^s(p_\star)$. The easiest way is to represent the manifold locally as a graph of a function $\psi\colon E^s(p_\star) \cap U \to E^u(p_\star) \cap U$ where $U \in X$ is an open neighborhood of $p_\star$. Theorem 1.2.3 leads to the existence of the local stable manifold $W^s_{loc}(p_\star)$ such that

$$W^s_{loc}(p_\star) = \{(x^{(1)}, \psi(x^{(1)})) : x^{(1)} \in E^s(p_\star) \cap U\}$$

for fixed $u_\star$. The remaining task is the computation of the feedback control $u_n$ at time $n$. Furthermore, we have to ensure that such a control exists at all. This will be done in the next section. For now, we concentrate on the problem of how to compute $u_n$.

First, for the sake of a simpler illustration, we shift the hyperbolic saddle $p_\star$ to the origin while we keep the parameter $u$ fixed at $u_\star$. Moreover, we transform the linear subspaces $E^s(p_\star)$ and $E^u(p_\star)$ such that they are equal to the new axes of the transformed coordinate system. For the moment, we fix $u_n$ at $u_\star$ since we consider the local transformation of the uncontrolled system.

Recall that the spectrum of the Jacobian matrix of $f$ at $p_\star$ consists of

$$\lambda_s, \lambda_u \in \mathbb{R} \text{ with } |\lambda_s| < 1, |\lambda_u| > 1$$

with corresponding eigenvectors $v_s \in \mathbb{R}^2$ and $v_u \in \mathbb{R}^2$. The local coordinate transformation $\phi$ in Figure 2.2 is defined as follows:

**Definition 2.1.1** (Local Coordinate Transformation)
*Let $f\colon X \to X \subset \mathbb{R}^2$ be a $C^r$-map with $r \geq 1$ that defines the nonlinear dynamical system (2.1). We define $\phi : X \to Y \subset \mathbb{R}^2$ as follows:*

$$z = \phi(x) := T^{-1}(x - p_\star), \tag{2.3}$$

*where $T \in Gl_2(R)$ consists of the two eigenvectors $v_s, v_u$, i.e.,*

$$T = \begin{pmatrix} v_s^{(1)} & v_u^{(1)} \\ v_s^{(2)} & v_u^{(2)} \end{pmatrix}. \tag{2.4}$$

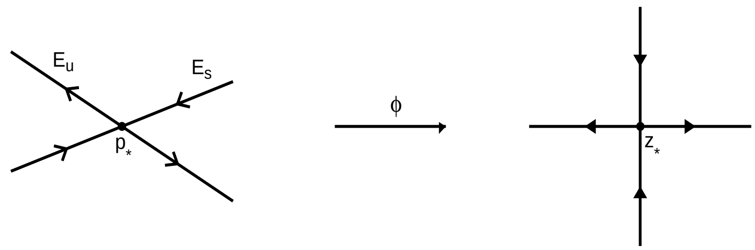


Figure 2.2: Local coordinate transformation $\phi$ at the hyperbolic saddle $p_\star$ of the uncontrolled nonlinear dynamical system $x_{n+1} = f(x_n, u_\star)$.

The inverse transformation is given by

$$x = \phi^{-1}(z) = T\,z + p_\star \tag{2.5}$$

where

$$T^{-1} = \frac{1}{\det T} \begin{pmatrix} v_u^{(2)} & -v_u^{(1)} \\ -v_s^{(2)} & v_s^{(1)} \end{pmatrix}.$$

Using Definition 2.1.1, we can define a transformed discrete system such that the origin is the hyperbolic saddle fixed point, cf. Figure 2.2. In the new coordinate system, the eigenspaces coincide with the axes of the transformed system, and the evolution equation for the transformed system reads

$$\begin{aligned} z_{n+1} &= \phi(x_{n+1}) \\ &= T^{-1}\,(x_{n+1} + p_\star) \\ &= T^{-1}\,(f(x_n, u_\star) - p_\star) \\ &= T^{-1}\,(f(T\,z_n + p_\star, u_\star) - p_\star). \end{aligned}$$

Now let us come back to the system with general parameter $u_n$. Instead of $x_{n+1} = f(x_n, u_n)$, we consider the discrete dynamical system

$$z_{n+1} = h(z_n, u_n), \tag{2.6}$$

where the $C^r$-map $h\colon Y \to Y$ is defined by

$$h(z, u) = T^{-1}\,(f(T\,z + p_\star, u) - p_\star).$$

If we use (2.3) with $z_n = (z_n^{(1)}, z_n^{(2)})^t, p_\star = (p_\star^{(1)}, p_\star^{(2)})^t, f = (f_1, f_2)^t$, then (2.6) is equivalent to

$$\left\{ \begin{array}{rcl} z_{n+1}^{(1)} &=& h_1(z_n, u_n) \\ z_{n+1}^{(2)} &=& h_2(z_n, u_n), \end{array} \right.$$

where $(z_n^{(1)}, z_n^{(2)})^t$ denotes the the transposed. In detail, we obtain

$$\begin{aligned}
z_{n+1}^{(1)} &= \frac{1}{\det T}\Bigg[v_u^{(2)}\left(f_1(v_s^{(1)}\,z_n^{(1)} + v_u^{(1)}\,z_n^{(2)} + p_\star^{(1)},\right.\\
&\qquad \left. v_s^{(2)}\,z_n^{(1)} + v_u^{(2)}\,z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(1)}\right)\\
&\qquad -v_u^{(1)}\left(f_2(v_s^{(1)}\,z_n^{(1)} + v_u^{(1)}\,z_n^{(2)} + p_\star^{(1)},\right.\\
&\qquad \left. v_s^{(2)}\,z_n^{(1)} + v_u^{(2)}\,z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(2)}\right) - p_\star^{(1)}\Bigg]\\
z_{n+1}^{(2)} &= \frac{1}{\det T}\Bigg[-v_s^{(2)}\left(f_1(v_s^{(1)}\,z_n^{(1)} + v_u^{(1)}\,z_n^{(2)} + p_\star^{(1)},\right.\\
&\qquad \left. v_s^{(2)}\,z_n^{(1)} + v_u^{(2)}\,z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(1)}\right)\\
&\qquad +v_s^{(1)}\left(f_2(v_s^{(1)}\,z_n^{(1)} + v_u^{(1)}\,z_n^{(2)} + p_\star^{(1)},\right.\\
&\qquad \left. v_s^{(2)}\,z_n^{(1)} + v_u^{(2)}\,z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(2)}\right) - p_\star^{(2)}\Bigg].
\end{aligned}$$

**Corollary 2.1.2**
*The point* $z_\star = 0 \in \mathbb{R}^2$ *is a fixed point for* $h$ *for* $u_n = u_\star$*, with eigenvalues* $\lambda_s$ *and* $\lambda_u$ *and corresponding eigenvectors* $e_1 = \binom{1}{0}$ *and* $e_2 = \binom{0}{1}$*, respectively.*

**Proof**
It is easy to see that

$$\mathrm{D}h(z_n, u_\star) = \mathrm{D}(T^{-1}\,(f(T\,z_n + p_\star, u_\star) - p_\star)) = T^{-1}\,\mathrm{D}(f(T\,z_n + p_\star, u_\star))\,T,$$

which implies

$$\mathrm{D}h(0, u_\star) = T^{-1}\,\mathrm{D}f(p_\star, u_\star)\,T.$$

Since $T$ consists of the eigenvectors of $\mathrm{D}f(p_\star)$, it follows from linear algebra that

$$\mathrm{D}h(0, u_\star) = \begin{pmatrix} \lambda_s & 0 \\ 0 & \lambda_u \end{pmatrix},$$

and the corresponding eigenvectors are $e_1$ and $e_2$. ■

The set-up with the transformed coordinate system allows us to establish an implicit equation from which one can compute $u_n$, so that $z_{n+1} = h(z_n, u_n)$ is a point on $W_{loc}^s(0)$. As already has been mentioned, the local stable manifold $W_{loc}^s(0)$ can be represented as a graph over the linear stable subspace. Due to the coordinate change, we consider $W_{loc}^s(0)$ locally as a graph over $E^s(0) = \mathrm{span}\{e_1\}$ which is the $z^{(1)}$ axis, cf. also Figure 2.3. Using the Stable Manifold Theorem 1.2.3, we conclude that $\psi\colon E^s(0) \cap U \to E^u(0) \cap U$ exists with $\psi(0) = 0$ and $\mathrm{D}\psi(0) = 0$ such that

$$W^s_{loc}(0) = \{(z^{(1)}, z^{(2)}) \in U : z^{(2)}\psi(z^{(1)})\},$$

where $U \subset Y$ is an open neighborhood of $0$. Since $f$ is a $C^r$-map, so is $h$, and by Theorem 1.2.3, $\psi$ is also $C^r$, where $r \geq 1$.

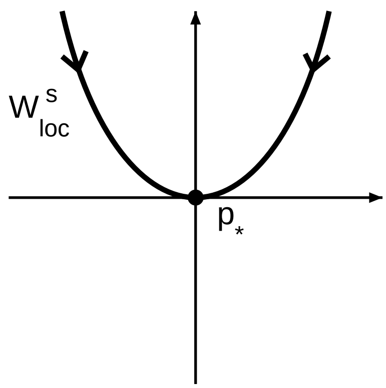


Figure 2.3: The local stable manifold $W^s_{loc}(0)$ for the uncontrolled system $z_{n+1} = h(z_n, u_\star)$ is given by the graph of $\psi$.

Finally, we can introduce our stabilization procedure. We consider the system (2.6), which possesses a hyperbolic saddle fixed point at the origin for $u_n = u_\star$. The linear subspaces of the fixed point are given by the spans of the unit vectors in $\mathbb{R}^2$. Moreover, the local stable manifold $W^s_{loc}(0)$ is represented as graph of $\psi$ in a neighborhood $U$ of $0$.

Our goal is to stabilize the system (2.6) at the origin by forcing an orbit onto $W^s(0)$. Let $z_0$ be a given initial condition that generates an orbit $O(z_0)$, which is not on the stable manifold $W^s_{loc}(0)$. Assume that at some time $n \in \mathbb{N}$, the state $z_n \in O(z_0)$ is in the given neighborhood $U$ of the origin. In this case, we compute $u_n$ such that the next iterate $z_{n+1}$ of the orbit $O(z_0)$ is on the stable manifold $W^s(0)$, or equivalently, $z_n = \left(z_n^{(1)}, z_n^{(2)}\right)$ is a point on $W^s(0)$. More precisely, we want

$$z^{(2)}_{n+1} = \psi\left(z^{(1)}_{n+1}\right).$$

We need to find $u_n$ so that $\left(z^{(1)}_{n+1}, z^{(2)}_{n+1}\right)$ is on $W^s(0)$. In other words,

$$\left\{ \begin{array}{rcl} z^{(1)}_{n+1} & = & h_1(z^{(1)}_n, z^{(2)}_n, u_n) \\ z^{(2)}_{n+1} & = & h_2(z^{(1)}_n, z^{(2)}_n, u_n) \end{array} \right. \tag{2.7}$$

should be a point on $W^s(0)$, which is equivalent to

$$\begin{aligned} \psi\left(z_{n+1}^{(1)}\right) &= z_{n+1}^{(2)} \\ \Leftrightarrow \psi\left(h_1(z_n^{(1)}, z_n^{(2)}, u_n)\right) &= h_2\left(z_n^{(1)}, z_n^{(2)}, u_n\right). \end{aligned} \tag{2.8}$$

As a consequence, $\left(z_{n+1}^{(1)}, z_{n+1}^{(2)}\right)$ is automatically a point on the stable manifold given for $u_\star$. Now we switch $u_k$ back to $u_\star$ for all $k \geq n+1$, i.e., we turn the control off. Due to the invariance of $W^s(0)$ under $h$, all the succeeding iterates of $z_{n+1}$ lie on the stable manifold and the controlled orbit $O(z_{n+1})$ is attracted to $0$. Hence, the system is stabilized. Consequently, the nonlinear control law can be computed from equation (2.8), since all parameters and functions are known except for $u_n$. Note that there may exist different solutions $u_n$ of this equation or no solution at all. Thus, we prove an existence theorem in the following section.

**Remarks**

(i) In case that there exists more than one possible solution of (2.8), one has to choose the optimal solution, for example, the $u_n$ with $|u_n - u_\star|$ having the smallest absolute value.

(ii) Note that in the derivation of the implicit equation (2.8) for $u_n$, we have used the fact that the graph of $\psi$ represents the local stable manifold for $u_n = u_\star$. Therefore, this equation determines $u_n$ only locally. As a consequence, we have to consider an open neighborhood $U$ of $p_\star$ and assume that $x_n \in U$ at time $n$.

## 2.2 An existence and uniqueness theorem

So far, we have shown that it is indeed possible to find an implicit equation that determines the feedback control which stabilizes the nonlinear system (2.1) at a given saddle fixed point. Naturally, the question arises whether one can always find such a control and if so, under what conditions.

Let us consider the nonlinear discrete dynamical system

$$z_{n+1} = h(z_n, u_n)$$

given as in (2.6) and assume that for $u_n = u_\star$, the system possesses a hyperbolic saddle fixed point at the origin. As we have seen in the previous section, we can derive the system given by $z_{n+1} = h(z_n, u_n)$ from the original system $x_{n+1} = f(x_n, u_n)$, which has a hyperbolic fixed point $p_\star$ for $u_n = u_\star$. Therefore, we can either consider the map $f$ and $p_\star$ or the map $g$ and the origin with the same stability properties as $p_\star$.

The following theorem states the local existence and uniqueness of a control $u_n$ for the transformed system such that (2.8) is satisfied.

**Theorem 2.2.1**
*Consider* $z_{n+1} = h(z_n, u_n)$ *with the assumptions made above. The system is stabilizable at the hyperbolic saddle fixed point* $0$ *provided that*

$$\partial_u \left[\psi(h_1(0,0,u)) - h_2(0,0,u)\right]\Big|_{u=u_\star} \neq 0. \tag{2.9}$$

*Let* $U \subset Y$ *be an appropriate open neighborhood of* $0 \in Y$ *and* $V \subset \mathbb{R}$ *be an appropriate open neighborhood of* $u_\star$*. Assume that* $z_n = (z_n^{(1)}, z_n^{(2)}) \in U$ *for some time* $n \in \mathbb{N}$*. Then the local feedback control near* $u_\star$ *is given by the unique solution* $u_n \in V$ *of*

$$\psi\left(h_1(z_n^{(1)}, z_n^{(2)}, u_n)\right) = h_2\left(z_n^{(1)}, z_n^{(2)}, u_n\right)$$

*for* $(z_n^{(1)}, z_n^{(2)}) \in U$.

**Proof**
The proof of the theorem is based on the Implicit Function Theorem. Let $F : Y \times \mathbb{R} \subset \mathbb{R}^3 \to \mathbb{R}$ be a $C^r$-function defined by

$$F\left(z^{(1)}, z^{(2)}, u\right) := \psi\left(h_1(z^{(1)}, z^{(2)}, u)\right) - h_2\left(z^{(1)}, z^{(2)}, u\right).$$

Obviously, $F$ has a zero at $(0, 0, u_\star)$. The goal is to solve $F$ for $u$ in a neighborhood of $(0, 0, u_\star)$. A necessary and sufficient condition for this is that $\partial_u F(0, 0, u_\star)$ does not vanish. Hence, we compute

$$\frac{\partial F}{\partial u}(z^{(1)}, z^{(2)}, u) = \partial_u[\psi(h_1(z^{(1)}, z^{(2)}, u)) - h_2(z^{(1)}, z^{(2)}, u)]$$

and in particular, using assumption (2.9),

$$\frac{\partial F}{\partial u}(0, 0, u_\star) \neq 0.$$

Also by assumption, $z_n \in U$. The Implicit Function Theorem implies that there exists a unique $C^1$-mapping $G : U \to V$ with appropriate neighborhoods $U, V$ as stated in the theorem, such that

$$F\left(z_n^{(1)}, z_n^{(2)}, u_n\right) = 0 \quad \forall \quad \left(z_n^{(1)}, z_n^{(2)}, u_n\right) \in U \times V$$

if and only if

$$u_n = G\left(z_n^{(1)}, z_n^{(2)}\right) \quad \forall \quad \left(z_n^{(1)}, z_n^{(2)}\right) \in U.$$

Therefore, the existence of a unique control $u_n \in V$ is implied by the Implicit Function Theorem.

Let us return to the stabilization procedure. We iterate the uncontrolled system

$$\begin{cases} z_{n+1}^{(1)} &= h_1\left(z_n^{(1)}, z_n^{(2)}, u_\star\right) \\ z_{n+1}^{(2)} &= h_2\left(z_n^{(1)}, z_n^{(2)}, u_\star\right) \end{cases}$$

for $n = 0, 1, \ldots$ until $\left(z_n^{(1)}, z_n^{(2)}\right) \in U$ for some time $n$. This conditions is satisfied by assumption. In this case, we set

$$u_n := G\left(z_n^{(1)}, z_n^{(2)}\right) \tag{2.10}$$

and by construction it follows that $F\left(z_n^{(1)}, z_n^{(2)}, u_n\right) = 0$. With $u_n$ given by (2.10), we compute

$$\begin{aligned} z_{n+1}^{(1)} &= h_1\left(z_n^{(1)}, z_n^{(2)}, u_n\right) \\ z_{n+1}^{(2)} &= h_2\left(z_n^{(1)}, z_n^{(2)}, u_n\right) \end{aligned}$$

Therefore, by definition of $F$, we obtain that $\psi\left(z_{n+1}^{(1)}\right) = z_{n+1}^{(2)}$, i.e., $z_{n+1}$ is a point on $W_{loc}^s(0)$. Setting $u_K := u_\star$ for $K > n$, we iterate the uncontrolled system further. Due to the invariance of the stable manifold under $h$ fro $u_\star$, the orbit is attracted to the origin. Thus, the system is stabilized at the origin. ■

If we switch back to the original system $x_{n+1} = f(x_n, u_n)$ using the inverse transformation $\phi^{-1}$, we have reached our original goal: The successful stabilization of a nonlinear system at a hyperbolic fixed point $p_\star$. Since $\phi$ and its inverse are affine linear transformations, one can switch back and forth between the two systems $x_{n+1} = f(x_n, u_n)$ and $z_{n+1} = h(z_n, u_n)$. Thus, the original problem is equivalent to stabilizing $z_{n+1} = h(z_n, u_n)$ at the origin. Note that in the transformed system, it is easier to find a representation of the local stable manifold of the fixed point. Therefore, we establish the control algorithm within this system and then go back to the original one.

## 2.3 The nonlinear control algorithm

So far, we have introduced the theory of stabilization at hyperbolic saddle fixed points in two-dimensional nonlinear dynamical systems. In order to be able to implement our nonlinear stabilization method, we need to establish a corresponding algorithm. In Chapters 3, 4 and 5, we will use this algorithm.

From the theoretical point of view, it suffices to compute the feedback control $u_n$ once because of the invariance of the stable manifold. The orbit $O(x_{n+1})$, where $x_{n+1} = f(x_n, u_n)$, stays on the manifold, when $u_K$ is set back to $u_\star$ for

$K > n$. Hence, the system is stabilized by a single control step. However, we do not obtain the stable manifold $W^s(p_\star)$ exactly. Instead, we use a Taylor approximation to determine the map $\psi$ that represents only the local stable manifold. Furthermore, the control $u_n$ is obtained numerically in most cases, depending on the degree of nonlinearity of $f$. Thus, it is not sufficient to control the system once in order to stabilize it. From this point of view, we introduce the algorithm.

Our approach is as follows: First, we fix the hyperbolic saddle fixed point $p_\star$ and the nominal value $u_\star$. The computation of the Jacobian $\mathrm{D}f(p_\star)$ leads to the eigenvalues $\lambda_s, \lambda_u$ and corresponding eigenvectors $v_s, v_u$, respectively. This leads to the definition of the transformation $\phi$. Altogether, we obtain the transformed evolution equation $z_{n+1} = h(z_n, u_n)$ with $h = \binom{h_1}{h_2}$. As explained in Section 1, we obtain

$$\begin{aligned} h_1(z_n,u_n) \;=\; \frac{1}{\det T}\Bigg[ & v_u^{(2)}\Big(f_1(v_s^{(1)}\, z_n^{(1)} + v_u^{(1)}\, z_n^{(2)} + p_\star^{(1)}, \\ & v_s^{(2)}\, z_n^{(1)} + v_u^{(2)}\, z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(1)}\Big) \\ & -v_u^{(1)}\Big(f_2(v_s^{(1)}\, z_n^{(1)} + v_u^{(1)}\, z_n^{(2)} + p_\star^{(1)}, \\ & v_s^{(2)}\, z_n^{(1)} + v_u^{(2)}\, z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(2)}\Big) - p_\star^{(1)}\Bigg] \end{aligned} \tag{2.11}$$

$$\begin{aligned} h_2(z_n,u_n) \;=\; \frac{1}{\det T}\Bigg[ & - v_s^{(2)}\Big(f_1(v_s^{(1)}\, z_n^{(1)} + v_u^{(1)}\, z_n^{(2)} + p_\star^{(1)}, \\ & v_s^{(2)}\, z_n^{(1)} + v_u^{(2)}\, z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(1)}\Big) \\ & +v_s^{(1)}\Big(f_2(v_s^{(1)}\, z_n^{(1)} + v_u^{(1)}\, z_n^{(2)} + p_\star^{(1)}, \\ & v_s^{(2)}\, z_n^{(1)} + v_u^{(2)}\, z_n^{(2)} + p_\star^{(2)}, u_\star) - p_\star^{(2)}\Big) - p_\star^{(2)}\Bigg]. \end{aligned} \tag{2.12}$$

Next, we determine $W^s_{loc}(0)$ as graph of $\psi : E^s(0) \to E^u(0)$ with $\psi(0) = 0, \mathrm{D}\psi(0) = 0$. To obtain an approximation of $\psi$, we expand $\psi$ in a Taylor series at the origin. Assume that $\psi$ is given by

$$\psi(z^{(1)}) = s_0 + s_1\,(z^{(1)}) + s_2\,(z^{(1)})^2 + s_3\,(z^{(1)})^3 + \ldots + s_N\,(z^{(1)})^N + \mathcal{O}((z^{(1)})^{N+1})$$

and that we expand $\psi$ up to order $N$. Since $\psi(0) = 0, \mathrm{D}\psi(0) = 0$ has to be true because of Theorem 1.2.3, we conclude that

$$s_0 = s_1 = 0,$$

i.e.,

$$\psi(z^{(1)}) \approx s_2\,(z^{(1)})^2 + s_3\,(z^{(1)})^3 + \ldots + s_N((z^{(1)})^N. \tag{2.13}$$

We need to determine the remaining coefficients $s_i, i = 2, \ldots, N$ of $\psi$ by comparison. We make use of the fact that the local stable manifold $W^s_{loc}(0)$, i.e., the graph of $\psi$, is invariant under the map $h$. If $z_n$ is a point on $W^s_{loc}(0)$, then $h(z_n, u_\star) = z_{n+1}$ is also a point on $W^s_{loc}(0)$ and thus,

$$z^{(2)}_{n+1} = \psi\left(z^{(1)}_{n+1}\right) \Leftrightarrow h_2\left(z^{(1)}_n, \psi(z^{(1)}_n), u_\star\right) = \psi\left(h_1(z^{(1)}_n, \psi(z^{(1)}_n), u_\star)\right)$$

must be true. If we now replace $\psi$ by (2.13), use the evolution equations (2.11), and (2.12) and $z^{(2)}_n = \psi(z^{(1)}_n)$, we obtain two polynomials with unknown coefficients $s_i, i = 2, \ldots, N$. We can calculate the coefficients of $\psi$ by comparison. The graph of $\psi(z^{(1)}) = s_2\,(z^{(1)})^2 + s_3\,(z^{(1)})^3 + \ldots + s_N\,(z^{(1)})^N$ gives us an approximation of the local stable manifold $W^s_{loc}(0)$. Once the function $\psi$ is determined, one can write down the control equation (2.8). In special cases, depending on the degree of nonlinearity of $f$, this equation can be solved explicitely for $u_n$. Otherwise, we use Newton's method to obtain $u_n$ near $u_\star$.

Now we are able to formulate the control algorithm. We transform points $x_n$ to $z_n = \phi(x_n)$ as defined in Definition 2.1.1. Then we check whether some point $z_n$ lies within a neighborhood $U$ of the saddle $z_\star = 0$. In this case, we solve the equation (2.8) for $u_n$. In order to do so, we need the evolution equation $z_{n+1} = h(z_n, u_n)$ with $h = \binom{h_1}{h_2}$ and the coefficients of $\psi$. The solution $u_n$ is plugged into $z_{n+1} = h(z_n, u_n)$. Now we switch off the control, i.e., $u_K = u_\star$ for $K > n$, and iterate the uncontrolled system. Due to the computational errors made within the approximation of $\psi$ and $u_n$, we might have to control again if points $z_K, K > n$, leave an $\varepsilon$-strip around the local stable manifold.

Consequently, the new control algorithm is as follows.

```
% Set the initial condition
% and initialize all necessary values
% such as the position of the saddle (P1,P2)
n=1;
while (n<k)
  if (x(n),y(n)) is not in U(P1,P2)
    Iterate the uncontrolled system;
  else
    if (x(n),y(n)) is in U(P1,P2)
      if (x(n),y(n)) is within nbhd.of W^s(P1,P2)
        Iterate the uncontrolled system;
      else
        % Orbit is close enough to (P1,P2)
        % and the control is switched on
        Compute local coordinates of (x(n),y(n));
        Solve the control equation for u(n);
        Apply u(n) to local coordinates;
        Compute (x(n+1),y(n+1));
```

```
          if abs(u*-u(n))>u(max)
            % Control is too large !
            u(n)=u*;
            Compute (x(n+1),y(n+1)) with u(n)=u*;
          end
        end
      end
    end
    n=n+1;
  end
```

Summarizing, a nonlinear autonomous discrete dynamical system

$$x_{n+1} = f(x_n, u_n)$$

in $\mathbb{R}^2$ can be stabilized at a hyperbolic saddle fixed point $p_\star$ via the feedback control $u_n \in \mathbb{R}$. Due to the necessary approximations of $W^s_{loc}(p_\star)$ and the control, stabilization is possible when one corrects the controlled orbit whenever it leaves a neighborhood of the local stable manifold. Results of this algorithm are shown in the next chapters.

**Remark**

Note that one motivation for our stabilization method was so-called chaos control. Here, a nonlinear dynamical system that possesses a chaotic attractor $\mathcal{A}$ is considered. Within such a chaotic system, unstable periodic orbits are typically dense, cf. [GOY88]. Thus, there exists a large number of periods and the system can be stabilized in many different hyperbolic periodic orbits. The most well-known method for chaos control has been established by Ott, Grebogi and Yorke in 1990, cf. [OGY90a, OGY90b]. As has been pointed in [Voc98], many interesting applications and succeeding results of the so-called OGY-method have been obtained during the last decade. For example, Nitsche and Dressler [ND92] improved the control method and applied it to the Duffing oscillator. The articles [PMT94, RGOD92] also present the OGY-method, and in [RGOD92], the control is applied to the kicked double rotor. Another example of a useful application to a laser, which is represented by the Ikeda map, is given in [SO95], cf. also Chapter 5. Furthermore, [SGOY93] provides a good overview of chaos control and in [JMTV97] and [Sch99], a variety of results on this research topic is given.

We point out that the OGY-method is only a special case of our nonlinear stabilization. Ott et al. present a simple geometric approach of how to compute the feedback control $u_n$ at a given time $n$ for the nonlinear system $x_{n+1} = f(x_n, u_n)$ which contains a single chaotic attractor $\mathcal{A}$. The system is linearized at the saddle fixed point $p_\star$, that is embedded in $\mathcal{A}$ for $u_n = u_\star$. The system should be stabilized at $p_\star$. The corresponding feedback law is computed for the linear system so that, with respect to the stable subspace, stabilization can be achieved. Hence, Ott et al. work with the $0$-th order approximation of $W^s_{loc}(p_\star)$, namely, the linear stable subspace $E^s(p_\star)$.

Let us review the OGY-method, because we will use it in the applications in Chapter 3 to compare it with our stabilization algorithm. Without loss of generality, Ott, Grebogi and Yorke set $u_\star := 0$ and $p_\star := 0$, which can be achieved by a simple transformation. Then $p_\star = 0$ is a hyperbolic saddle for the uncontrolled system (2.2). The linearization of the system at $p_\star$ is given by

$$x_{n+1} - p(u_n) = x_{n+1} - u_n\, g = A\,(x_n - u_n\, g),$$

where

$$g := \frac{\partial p(u)}{\partial u}\Big|_{u=u_\star} = \lim_{u_n \to 0} \frac{p(u_\star + u_n) - p_\star}{u_n} \approx \frac{1}{u_n}\, p(u_n). \tag{2.14}$$

Here, $p(u_n)$ denotes the position of the saddle fixed point, when $u_n \neq u_\star$. The matrix $A \in \mathbb{R}^{2\times 2}$ is given by

$$A := \mathrm{D}f(p_\star, u_\star)$$

and its eigenvalues are $\lambda_u$, $|\lambda_u| > 1$, and $\lambda_s$, $|\lambda_s| < 1$, with $\lambda_u,\, \lambda_s \in \mathbb{R}$. Let $v_u,\, v_s \in \mathbb{R}^{2\times 1}$ be corresponding right eigenvectors and $w_u, w_s \in \mathbb{R}^{1\times 2}$ left eigenvectors so that

$$\begin{aligned} <v_s, w_s> &= <v_u, w_u> &= 1 \\ <v_s, w_u> &= <v_u, w_s> &= 0. \end{aligned}$$

In [OGY90a, OGY90b], the formula for finding an appropriate control $u_n$ is given, but not proven. The following theorem introduces the OGY formula, which is proven in detail in [Voc98].

**Theorem 2.3.1**
*We consider the nonlinear two-dimensional dynamical system* $x_{n+1} = f(x_n, u_n)$ *and its linearization* $x_{n+1} - u_n\, g = A\,(x_n - u_n\, g)$ *with all the assumptions made above. The system can be stabilized at its hyperbolic saddle fixed point* $p_\star = 0$ *with*

$$u_n = \frac{\lambda_u}{\lambda_u - 1}\, \frac{<x_n, w_u^t>}{<g, w_u^t>}, \tag{2.15}$$

*provided that* $x_n \in U$ *at some time* $n$*, where* $U \subset \mathbb{R}^2$ *is an open neighborhood of the hyperbolic saddle fixed point* $0$*.*

Here, $<\cdot,\cdot>$ denotes the standard inner product in $\mathbb{R}^2$ and $w_u^t$ is the transposed of $w_u$. Note that the original nonlinear system is iterated and the controlled orbit is only forced onto the stable subspace $E^s(p_\star)$.

The OGY-method makes use of the chaotic dynamics of the system, which ensures that an orbit comes eventually close enough to the chosen hyperbolic saddle fixed point or periodic orbit. Since this approach relies on the linearization of the

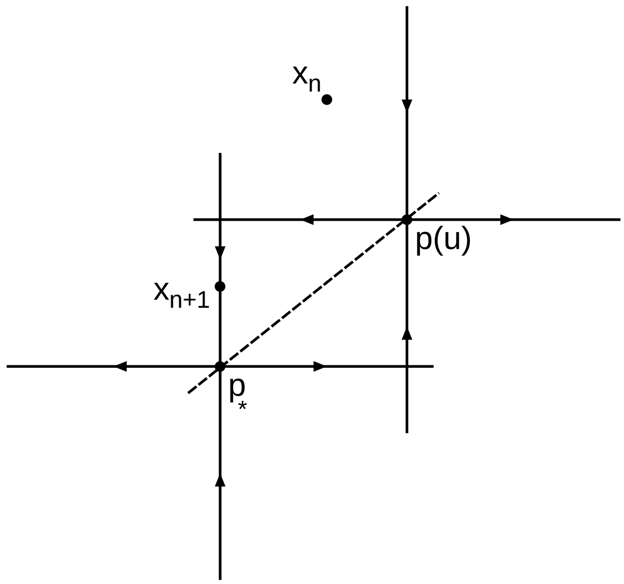


Figure 2.4: Geometric interpretation of the OGY-method: The saddle fixed point $p_\star$ is shifted along $g$ (dashed line) as $u$ changes to $u_n$. Choose $u_n \neq u_\star$ so that the point $x_{n+1}$ lies on the stable linear subspace $E^s(p_\star)$.

original system $x_{n+1} = f(x_n, u_n)$, stabilization of the system can only be achieved within the small strip of $p_\star = 0$ given by

$$x_{max} := u_{max} \left| \frac{\lambda_u - 1}{\lambda_u} < g, w_u^t > \right|, \tag{2.16}$$

with the assumption that $|u_n - u_\star| = |u_n| < u_{max}$ for $u_n$ given by (2.15), Therefore, we activate the control $u_n$ only for $x_n$ being in

$$|<x_n, w_u^t>| < x_{max}.$$

For small $u_{max}$, a typical initial condition will execute a chaotic orbit, unchanged from the uncontrolled case, until $x_n$ is in the strip. Because of the nonlinearity not included in the feedback law, the control at time $n$ may not be able to keep the controlled orbit near the fixed point. In this case, the orbit leaves the strip again and wanders around chaotically as before, despite the activated control. This is called a *chaotic transient*. Ott et al. derived a formula for the length of such a chaotic transient in [OGY90a, OGY90b]. They show that after some finite amount of time, the orbit will come back into the strip, since by assumption almost all trajectories are dense in the attractor $\mathcal{A}$. Thus, if the orbit is again within the strip, then control is achieved. So we are finally able to stabilize the orbit which is preceded by a chaotic transient, where the orbit is similar to orbits on the uncontrolled attractor.

**Remark**
Note that the OGY-method for controlling chaos can be seen from a control theoretical point of view. It was shown in [Voc98] that the OGY-method is quite similar to a common control theoretical approach. In this case, we consider the control system

$$x_{n+1} = A\, x_n + B\, u_n,$$

or equivalently,

$$x_{n+1} = (A - B\, F)\, x_n,$$

with $u_n = -F\, x_n$. We can compute the so-called feedback gain matrix from the Pole Shifting Theorem [Son98]. It turns out that Ott et al. do just the same, although from a geometrical point of view. In fact, the control law (2.15) is one possibility to compute the feedback gain matrix $F$. Both the control theoretical approach and the OGY-method do not at all depend on the dynamics of the system. No matter whether the dynamical system is chaotic or not, the system can be stabilized near the saddle.

In conclusion, we have seen in Chapter 2 that in a neighborhood $U$ of the hyperbolic saddle fixed point $p_\star$ with $u_n = u_\star$, there exists, at least locally, a unique control of the nonlinear system $x_{n+1} = f(x_n, u_n)$. We expect that our stabilization method works better than the OGY-method, which is based on the linearization of the system. In the OGY scenario, only a small number of initial conditions can be controlled, because they use the $0$-th order approximation of $W^s_{loc}(p_\star)$. On the other hand, our nonlinear stabilization technique is valid for a wider range of initial conditions, since this algorithm works with a higher order approximation of $W^s_{loc}(p_\star)$. As has been pointed out before, the OGY-method is just a special case of our stabilization method. In Chapter 3, we give an example for such a stabilization. Both methods are applied to the Hénon map and it turns out that the our algorithm indeed works more globally than the one from Ott, Grebogi and Yorke.

# Chapter 3

# An example: The Hénon map

In this chapter, we present an application of the algorithm for stabilizing nonlinear dynamical systems which has been developed in Chapter 2. We want to compare the new algorithm with the OGY-method. For this reason, we consider the Hénon map, which has already been taken as an example in the work by Ott, Grebogi and Yorke [OGY90a, OGY90b]. The Hénon map is a two-dimensional quadratic map that was introduced by Hénon in 1976, cf. [Hén76]. It is a model problem of a simple two-dimensional map that exhibits the same essential properties as the Lorenz system, cf. [Lor63]. The first section begins with an overview of the dynamics of the map. Section 2 introduces all necessary computations in order to implement the nonlinear algorithm from Section 2.3. In the last section, we actually stabilize the map at an unstable fixed point which lies on the strange attractor. We numerically illustrate both stabilization methods, the one by Ott, Grebogi and Yorke and our nonlinear one and compare the results.

## 3.1 Dynamics of the Hénon map

Consider the Hénon map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is usually given by

$$f(x, y, a) = (a - x^2 + b\,y, x) \tag{3.1}$$

or equivalently, by defining the two components of $f$ by $f_1, f_2$, i.e.,

$$\begin{cases} x_{n+1} &= a_n - x_n^2 + b\,y_n =: f_1(x_n, y_n, a_n) \\ y_{n+1} &= x_n =: f_2(x_n, y_n, a_n). \end{cases}$$

The Hénon map has two parameters $a, b \in \mathbb{R}$ with $a > 0$ and $|b| < 1$. Throughout this chapter, we fix $b$ at $b = 0.3$, whereas $a$ is the feedback control with nominal value $a_\star = 1.4$, cf. [OGY90a, OGY90b]. Consequently, the uncontrolled Hénon map is given by

$$f(x, y, 1.4) = (1.4 - x^2 + 0.3\,y, x).$$

By varying $a$ over $a_\star$, we can stabilize the system as will be shown in Section 3.

Let us have a look at the properties of the Hénon map. Obviously, $f$ is a $C^\infty$-map. Since $b \neq 0$, $f$ is invertible and its inverse reads

$$f^{-1}(x, y, a) = (y, \frac{1}{b}\,(x - a + y^2)).$$

Thus, the Hénon map is one-to-one. Since $f^{-1}$ is also a $C^\infty$-map, the map $f$ is a $C^\infty$-diffeomorphism with respect to the state variables $x$ and $y$.

Another property of the Hénon map is that the determinant of the Jacobian matrix is constant. Let us consider the Jacobian of $f$ at every point $(x, y) \in \mathbb{R}^2$, i.e.,

$$\mathrm{D}f(x, y) = \begin{pmatrix} -2\,x & b \\ 1 & 0 \end{pmatrix}.$$

Note that

$$\det \mathrm{D}f(x, y) = -b \quad \forall\,(x, y) \in \mathbb{R}^2.$$

Thus, $|\det \mathrm{D}f(x, y)| = |b| < 1$ by assumption. It follows that the Hénon map is *area contracting*, or, as one can also say, *dissipative*.

As it has been pointed out by Hénon in 1976, the dynamical system given by (3.1) possesses a strange attractor $\mathcal{A}$ for certain parameter values $a$ and $b$. In our case, $b = 0.3$ and $a$ varies about $a_\star = 1.4$. It can be shown that for $b = 0.3$ and $a_\star = 1.4$, there exists a quadrilateral $R \in \mathbb{R}^2$ which is mapped inside itself. The set $R$ is compact and $f^k(R) \subset R$ for $k > 0$. In Figure 3.1, the set $\mathcal{A} \subset R$ is shown. $R$ is attracted by the invariant set $\mathcal{A}$. By Definition 1.3.3, $R$ is the trapping region for the attractor $\mathcal{A}$. The attracting set $\mathcal{A}$ is a strange attractor, since

$$\dim_b(\mathcal{A}) \approx 1.27.$$

Thus, the box counting dimension of $\mathcal{A}$ is noninteger, see e.g. [ASY97]. It can be shown that orbits $O(x_0, y_0)$ either diverge to minus infinity or tend to the strange attractor $\mathcal{A}$, if $(x_0, y_0) \in R$.

The attracting set $\mathcal{A}$ has not only a non-integer dimension, but with respect to Lyapunov exponents, it is also a chaotic attractor. In Chapter 1, we called an attractor $\mathcal{A}$ of a dynamical system chaotic, if the largest Lyapunov exponent with respect to $\mathcal{A}$ is positive. A computation of the Lyapunov exponents for $\mathcal{A}$ for the Hénon map leads to the approximate values $L_1 = 0.42$ and $L_2 = -1.2$, cf. [ASY97]. Therefore, the largest Lyapunov exponent is greater than zero. By definition, $\mathcal{A}$ is a chaotic attractor. A picture of the attractor $\mathcal{A}$ of the system (3.1) is shown below. We start with some initial condition $(x_0, y_0) = (0, 0)$ in $R$ and iterate the system (3.1) $10^4$ times with $b = 0.3$, $a = 1.4$. The first 20 iterates are not plotted. Thus, we obtain the following picture of the strange attractor $\mathcal{A}$.

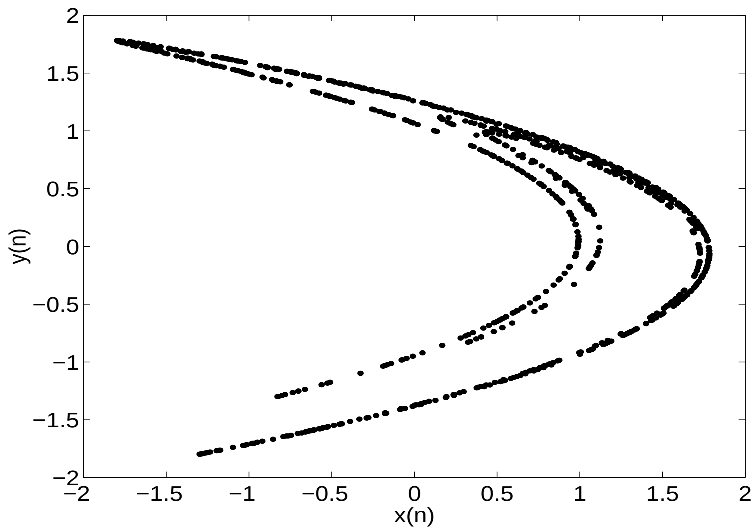


Figure 3.1: The strange attractor of the Hénon map with $a = 1.4$ and $b = 0.3$.

## 3.2 Necessary computations

We want to stabilize the Hénon map at a hyperbolic saddle fixed point. So let us determine, for general parameters $a$ and $b$, the fixed points of $f$. They are obtained by solving $f(x, y, a_\star) = (x, y, a_\star)$ for $x$ and $y$. The second evolution equation $f_2(x, y, a_\star) = x$ yields $y = x$. Thus,

$$0 = x^2 + x\,(1 - b) - a,$$

i.e.,

$$x_{1,2} = \frac{1}{2}\,(b - 1 \pm \sqrt{(b-1)^2 + 4\,a}).$$

Accordingly, fixed points of the Hénon map exist as long as

$$-4\,a < (b - 1)^2,$$

which is true for $a = a_\star = 1.4$ and $b = 0.3$. Due to the fact that $f_2(x, y, a_\star) = x$, the fixed points lie on the line $x = y$. Let us check if one of the fixed points is embedded in the strange attractor $\mathcal{A}$ and let us determine their stability. Using the specific parameter values $a = a_\star = 1.4$ and $b = 0.3$, we obtain approximately

$$(x_F, y_F) = (0.8839, 0.8839) \text{ and } (\tilde{x}_F, \tilde{y}_F) = (-1.5839, -1.5839).$$

The second fixed point $(\tilde{x}_F, \tilde{y}_F)$ does not lie within the strange attractor $\mathcal{A}$, as can be seen in Figure 3.2. However, the fixed point $(x_F, y_F)$ is contained in $\mathcal{A}$. Thus, this is our fixed point of interest, at which the system is stabilized, provided

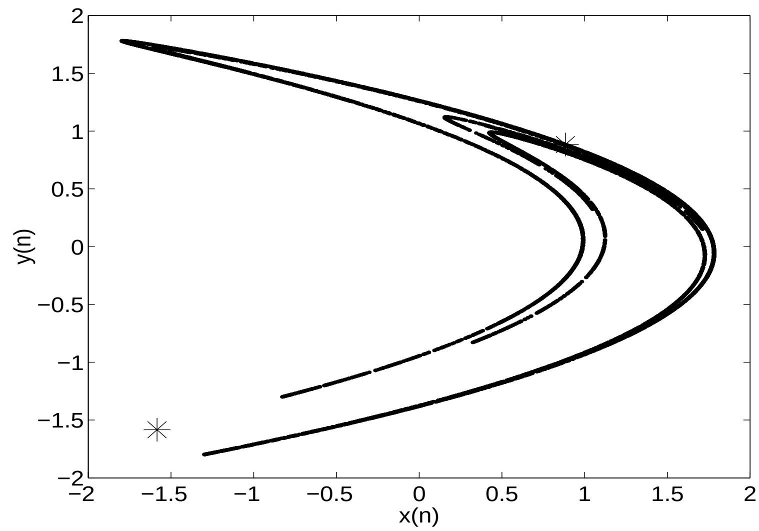


Figure 3.2: The strange attractor and the two fixed points of the Hénon map.

$(x_F, y_F)$ is a hyperbolic saddle. In order to determine the stability of the fixed point, we evaluate the Jacobian of $f$ at $(x_F, y_F)$ and compute the corresponding eigenvalues and eigenvectors. We obtain

$$\mathrm{D}f(x_F, y_F) = \begin{pmatrix} -1.7678 & 0.3 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_s = 0.1559 \quad \text{and} \quad \lambda_u = -1.9237 \tag{3.2}$$

and corresponding eigenvectors

$$v_s = \begin{pmatrix} -0.1541 \\ -0.9881 \end{pmatrix} \quad \text{and} \quad v_u = \begin{pmatrix} -0.8873 \\ 0.4612 \end{pmatrix}.$$

Thus, by Theorem 1.1.6, the fixed point is a hyperbolic saddle point with a one-dimensional stable and one-dimensional unstable subspace.

The goal is to apply our stabilization method developed in the previous chapter and stabilize the system at the hyperbolic saddle $(x_F, y_F)$. A given orbit should be stabilized by forcing it onto the stable manifold $W^s_{loc}(x_F, y_F)$. To achieve this, we use the system parameter $a$ as a feedback control and vary $a$ over $a_\star = 1.4$. Since we want to apply the theory from Chapter 2, we need to shift the fixed point to the origin and transform the coordinates such that the stable linear subspace $E^s(0,0)$ is equal to the x-axis and $E^u(0,0)$ is perpendicular to $E^s(0,0)$. Moreover, a Taylor approximation for the local stable manifold has to be done to obtain the local stable manifold as a graph over the new $x$-axis. We consider

$$f(x, y, a_\star) = (a_\star - x^2 + by, x)$$

and its hyperbolic fixed point $(x_F, y_F)$ in $\mathcal{A}$. We shift $(x_F, y_F)$ to $(0,0)$ and transform the linear subspaces by $\phi$ given as in Definition 2.1.1

$$\begin{pmatrix} u \\ v \end{pmatrix} := \phi \begin{pmatrix} x \\ y \end{pmatrix} := T^{-1} \left( \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_F \\ y_F \end{pmatrix} \right),$$

where

$$T := (v_s \;\; v_u) = \left( \begin{array}{cc} v_s^1 & v_u^1 \\ v_s^2 & v_u^2 \end{array} \right) = \left( \begin{array}{cc} -0.1541 & -0.8873 \\ -0.9881 & 0.4612 \end{array} \right).$$

The inverse matrix is

$$T^{-1} := \frac{1}{\det T} \left( \begin{array}{cc} v_u^2 & -v_u^1 \\ -v_s^2 & v_s^1 \end{array} \right)$$

with $\det T = -1$. These computations can be carried out either by Matlab or Maple. $\phi$ transforms $v_s$ and $v_u$ such that the stable eigenvector of the hyperbolic fixed point $\phi(x_F, y_F) = (0,0)$ is $e_1$ and the unstable one is $e_2$, where $e_1, e_2$ denote the unit vectors in $\mathbb{R}^2$. Using the transformation $\phi$ and its inverse

$$\begin{pmatrix} x \\ y \end{pmatrix} = \phi^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} x_F \\ y_F \end{pmatrix},$$

we obtain the transformed Hénon map $h(u, v)$. Consider

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \phi^{-1} \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = T \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} + \begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} v_s^1 \, u_{n+1} + v_u^1 \, v_{n+1} + x_F \\ v_s^2 \, u_{n+1} + v_u^2 \, v_{n+1} + y_F \end{pmatrix}$$

and

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_\star - x_n^2 + b \, y_n \\ x_n \end{pmatrix}.$$

Write $x_n$ and $y_n$ in terms of $u_n$ and $v_n$ by the inverse transformation and use the fact that $x_F = y_F$. Then set the two equations equal and obtain two equations

$$\begin{aligned} v_s^1 \, u_{n+1} + v_u^1 \, v_{n+1} &= a_\star - (v_s^1 \, u_n + v_u^1 \, v_n + x_F)^2 + b \, (v_s^2 \, u_n + v_u^2 \, v_n + x_F) - x_F \\ v_s^2 \, u_{n+1} + v_u^2 \, v_{n+1} &= v_s^1 \, u_n + v_u^1 \, v_n \end{aligned}$$

with two unknowns, namely $u_{n+1}$ and $v_{n+1}$. We solve this set of equations using Maple and obtain the transformed Hénon map

$$(u_{n+1}, v_{n+1}) = h(u_n, v_n, a_n)$$

with its components

$$\begin{aligned} h_1(u_n, v_n, a_n) \;\; = \;\; & \frac{1}{v_s^2} \Big( - v_u^2 \left( v_s^2 \, (a_n - (v_s^1 \, u_n + v_u^1 \, v_n + x_F)^2 \right. \\ & + b \, (v_s^2 \, u_n + v_u^2 \, v_n) + x_F \, (b-1)) - v_s^1 \, (v_s^1 \, u_n + v_u^1 \, v_n) \Big) \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad +v_s^1\, u_n + v_u^1\, v_n \Big) \\
h_2(u_n, v_n, a_n) &= v_s^2 \Big( a_n - (v_s^1\, u_n + v_u^1\, v_n + x_F)^2 \\
&\quad + b\,(v_s^2\, u_n + v_u^2\, v_n) + x_F\,(b-1) \Big) - v_s^1\,(v_s^1\, u_n + v_u^1\, v_n).
\end{aligned}
$$

The map $h$ possesses a hyperbolic fixed point at $(0,0)$ with $E^s(0,0) = \mathrm{span}\{e_1\}$ and $E^u(0,0) = \mathrm{span}\{e_2\}$. Note that the original strange attractor $\mathcal{A}$ is just undergoing a coordinate transformation $\phi$ such that its properties remain the same.

The local stable manifold $W^s_{loc}(0,0)$ is obtained by a Taylor approximation as described in Section 2.3. The coefficients for the function $\psi\colon \mathbb{R} \to \mathbb{R}$ were computed by Maple. The function $\psi$ reads

$$\psi(u) = 0.01414\, u^2 + 0.00218\, u^3 + \mathcal{O}(u^4),$$

where the values for the coefficients have been rounded.

In order to perform the desired control, we have to verify the assumption of Theorem 2.2.1, which is necessary and sufficient for the existence of a control $a_n \neq 1.4$, i.e., we need to check that

$$\partial_a \left[\psi(h_1(u,v,a)) - h_2(u,v,a)\right] \Big|_{a=a_\star, u=v=0} \neq 0$$

In this case we obtain

$$
\begin{aligned}
&\partial_a \Big[ s_2 \Big( \tfrac{1}{v_s^2} \left(-v_u^2 \left(v_s^2 \left(a_n - (v_s^1\, u_n + v_u^1\, v_n + x_F)^2 + b\,(v_s^2\, u_n + v_u^2\, v_n)\right.\right.\right. \\
&\left.\left.\left. + x_F\,(b-1)\right) - v_s^1\,(v_s^1\, u_n + v_u^1\, v_n)\right) + v_s^1\, u_n + v_u^1\, v_n\right) \Big)^2 \\
&+ s_3 \Big( \tfrac{1}{v_s^2} \left(-v_u^2 \left(v_s^2 \left(a_n - (v_s^1\, u_n + v_u^1\, v_n + x_F)^2 + b\,(v_s^2\, u_n + v_u^2\, v_n)\right.\right.\right. \\
&\left.\left.\left. + x_F\,(b-1)\right) - v_s^1\,(v_s^1\, u_n + v_u^1\, v_n)\right) + v_s^1\, u_n + v_u^1\, v_n\right) \Big)^3 \\
&- v_s^2\,(a_n - (v_s^1\, u_n + v_u^1\, v_n + x_F)^2 \\
&- b\,(v_s^2\, u_n + v_u^2\, v_n) - x_F\,(b-1)) - v_s^1\,(v_s^1\, u_n + v_u^1\, v_n) \Big] \Big|_{a=a_\star, u=v=0} \\
&\approx 0.4866 \neq 0
\end{aligned}
$$

## 3.3 Controlling a saddle fixed point

The implicit equation (2.7), from which $a_n$ is computed, is rather complicated. This is the reason why we do not write it down here. We have implemented both the nonlinear stabilization algorithm and the OGY-method in Matlab. Our nonlinear method is programmed as described in Section 2.3 and the OGY-method is programmed according to [OGY90a] with $a_n$ given by (2.15) and $x_{max}$ given by (2.16). All necessary symbolic computations have been done in Maple. The nonlinear implicit equation (2.7) that determines $a_n$ was obtained by Maple as well as the transformed Hénon map. We use these results in the main Matlab program. The following pictures illustrate the results of the two algorithms.

We consider an initial condition $(x_0, y_0)$ within $[-2, 2]^2$. Note that $\mathcal{A}$ lies within this square. Let $(x_0, y_0) = (0.75, -0.5)$ be an exemplary initial condition, which lies on the inner right arc of $\mathcal{A}$.

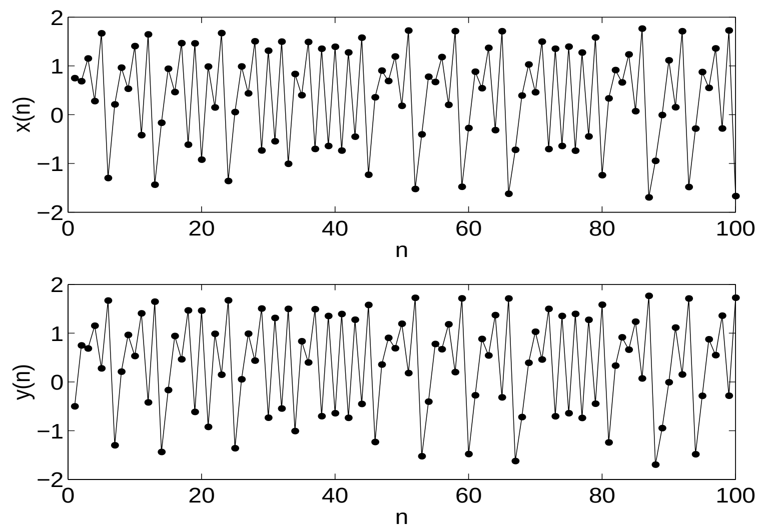


Figure 3.3: The uncontrolled Hénon map with $(x_0, y_0) = (0.75, -0.5)$.

Figure 3.3 shows the first hundred iterations of $(x_0, y_0)$ under the original uncontrolled map $f$ in order to show what happens without control. The two components $x$ and $y$ of the map are shown separately so that one is able to see the evolution of points dependent on time. The first row shows the points $x_n$ on the $y$ axis and the second row shows $y_n$ on the $y$ axis. In all figures, time is on the $x$ axis.

As one can see in Figure 3.3, $(x_0, y_0) = (0.75, -0.5)$ generates an orbit which behaves irregular. We apply both stabilization methods, our nonlinear one and the OGY-method, to the orbit $O(x_0, y_0)$. As we will see, both methods are able to control this orbit and thus stabilize the system behavior at the saddle. The first step is to apply our algorithm to this initial condition. Note that it is implemented such that if the orbit leaves an $\varepsilon$-neighborhood of the local stable manifold, then the control is switched on again. Here we take $\varepsilon = 0.01$.

More precisely, we start with the initial condition and iterate the uncontrolled system as long as $\phi^{-1}(x_n, y_n) = (z_n^{(1)}, z_n^{(2)}) \in \mathrm{B}_r(0,0)$ where $\mathrm{B}_r(0,0)$ is the ball centered at the fixed point $(0,0)$ with radius $r$. Here, we take $r = 1$. In this case, we compute $a_n$ and the controlled point $(z_{n+1}^{(1)}, z_{n+1}^{(2)})$. As described in Section 2.3, the orbit $O(z_{n+1}^{(1)}, z_{n+1}^{(2)})$ eventually leaves an $\varepsilon$- neighborhood of the local stable manifold. Thus, after each iteration of the uncontrolled system, we check whether $(z_K^{(1)}, z_K^{(2)})$ for $K > n$ is in the $\varepsilon$-neighborhood of $W^s_{loc}(0,0)$. Equivalently, take $(z_K^{(1)}, z_K^{(2)})$ and compute $(\bar{z}_1, \bar{z}_2)$ such that $\psi(\bar{z}_1) = \bar{z}_2$ is a point on the local stable manifold so that

$$\left( \begin{pmatrix} z_K^{(1)} \\ z_K^{(2)} \end{pmatrix} - \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \right) \perp \begin{pmatrix} 1 \\ \mathrm{D}\psi(\bar{z}_1) \end{pmatrix}.$$

If $|z_K^{(1)} - \bar{z}_1|^2 + |z_K^{(2)} - \bar{z}_2|^2 > \varepsilon^2$, then $(z_K^{(1)}, z_K^{(2)})$ has left the $\varepsilon$-neighborhood and we have to control again, i.e., compute $a_K$ and the controlled point $(z_{K+1}^{(1)}, z_{K+1}^{(2)})$. To prevent the vertical component $z_K^{(2)}$ from being too far away from the fixed point, we also ensure that $|z_n^{(2)}| < \delta$ where $\delta = 0.1$. Moreover, we set a bound on the control $a_n$. Let $a_{max} := 0.3$, then $|a_n - 1.4| < 0.3$.

To check whether $z_K$, $K > n$, is still within the $\varepsilon$-neighborhood of $W^s_{loc}(0)$, we use the following routine.

```
% Suppose that we have already controlled
% at time n and let (X_k,Y_k) be some
% state at time k>n.
% We now compute (X,Y) which fulfills the
% above requirement, i.e., (X_k-X,Y_k-Y) is
% perpendicular to the stable manifold
% s(x)=0.01414*xˆ2+0.00218*xˆ3 and
% s'(x) is its derivative.

X=fsolve('(X_k-x)+(Y_k-s(x))*s'(x)',X_k);
Y=s(X);
while (((abs(X_k-X))ˆ2+(abs(Y_k-Y))ˆ2)<epsilonˆ2)
  & (abs(Y_k)<delta)
  & (k<=max. number of iterations)
      compute (X_(k+1),Y_(k+1)) with a*;
      k=k+1;
....
```

**Remark**

Note that this subroutine is very costly and that we use it only to test the algorithms, cf. also the following Figures, where we apply our control algorithm to the hyperbolic fixed point of the Hénon map. In practical situations, it will not be wise to use such an expensive Newton's method like 'fsolve' in Matlab. Instead, one could think of the following implementation. Regularly, after a fixed amount of time $T$, one switches the control on in order to achieve stabilization. The time $T$ has to be found by trial and error.

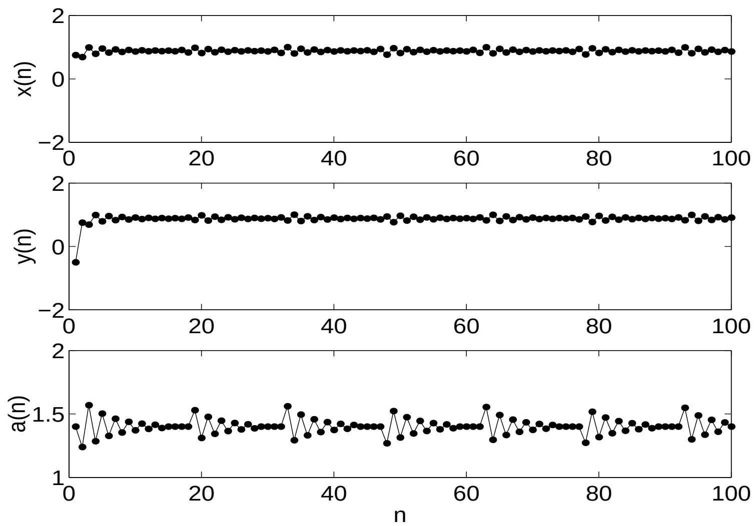


Figure 3.4: The nonlinearly controlled Hénon map with $(x_0, y_0) = (0.75, -0.5)$.

As one can see in Figure 3.4, our nonlinear method forces the orbit $O(0.75, -0.5)$ onto the local stable manifold of the hyperbolic fixed point $(x_F, y_F)$. The orbit stays close to the hyperbolic fixed point $(x_F, y_F)$. Thus, the control is successful. Note that the control $a_n$ has to be activated several times since the orbit leaves the $\varepsilon$-neighborhood of the stable manifold. One of our further investigations will be to determine the behavior of the control algorithm, if $\varepsilon$ is varied.

Now we apply the OGY-method to the same initial condition $(x_0, y_0) = (0.75, -0.5)$. According to the control law (2.15) given in Theorem 2.3.1, $a_n$ is computed using the right eigenvectors $v_s$ and $v_u$ as given above, and the left eigenvectors $w_s$ and $w_u$ and the vector $g$. Recall that the left and right eigenvectors have to fulfill the following conditions.

$$\begin{aligned} <v_s, w_s> &= <v_u, w_u> &= 1 \\ <v_s, w_u> &= <v_u, w_s> &= 0. \end{aligned}$$

We obtain

$$w_s = (0.4867\ 0.9362), w_u = (-1.0425\ 0.1626) \text{ and } g = \begin{pmatrix} 0.4052 \\ 0.4052 \end{pmatrix}.$$

Since we have chosen $a_{max} := 0.3$, we compute the width of the strip $|<x_n, w_u^t>| < x_{max} = 0.1621$. If $x_n$ is in this strip, then the control is activated according to (2.15). This happens at $n = 91$, cf. Figure 3.5.

Let us compare the results of the two algorithms, which are shown in Figures 3.4 and 3.5. It is obvious, that with our stabilization method, the control can be activated much earlier than with the OGY-method. In the OGY set-up, we have to wait

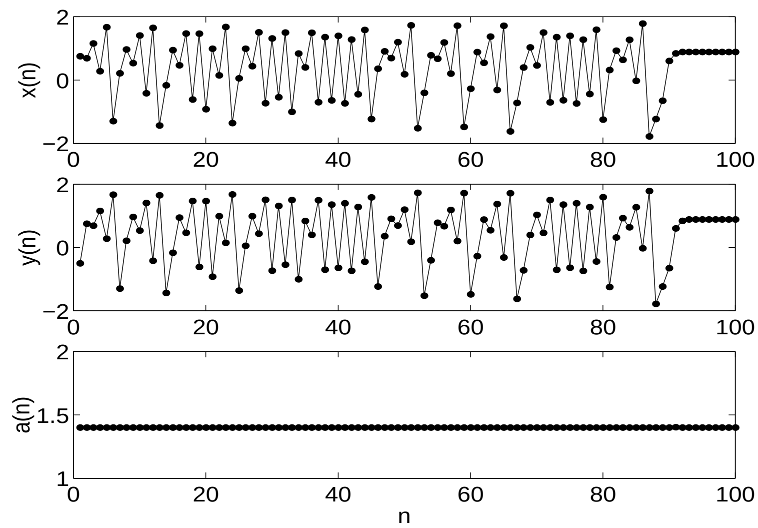


Figure 3.5: The OGY-controlled Hénon map with $(x_0, y_0) = (0.75, -0.5)$.

91 iterations before we can activate the control. This is due to the fact that control can only be activated if $x_n$ is in the strip defined by $|{<}x_n, w_u^t{>}| < x_{max}$. In contrast, in the nonlinear case, control is switched on when $(z_n^{(1)}, z_n^{(2)}) \in \mathrm{B}_r(0,0)$ and the control is activated within the first iteration. Thus, the globality of new algorithm, in the sense that one uses $W^s_{loc}(0,0)$ instead of $E^s(0,0)$, is a big advantage. Nevertheless, our method is more costly, since we have to approximate $W^s_{loc}(0,0)$. Due to this approximation and computational errors, the controlled orbit leaves the $\varepsilon$-neighborhood of $W^s_{loc}(0,0)$ and we have to adjust the control value. The last row in the figures show the control values $a_n$ depending on time $n$. The differences $|a_n - 1.4|$ for all $n$, where $a_n \neq 1.4$, in Figure 3.4 are larger than the difference $|a_{91} - 1.4| = 0.0026$ of the single control step used in Figure 3.5. Thus, in this case, the OGY-method is better than our method, since it only requires one control step and no further control steps are needed as in the nonlinear set-up. Nevertheless, in general, our nonlinear stabilization acts more global than the OGY-method.

Let us take a different initial condition, e.g., $(x_0, y_0) = (-1, 1.5)$. Note that this initial condition lies in the upper left part of $\mathcal{A}$. The parameters $\varepsilon, \delta$ and $a_{max}$ are the same as before, cf. Figures 3.6 and 3.7.

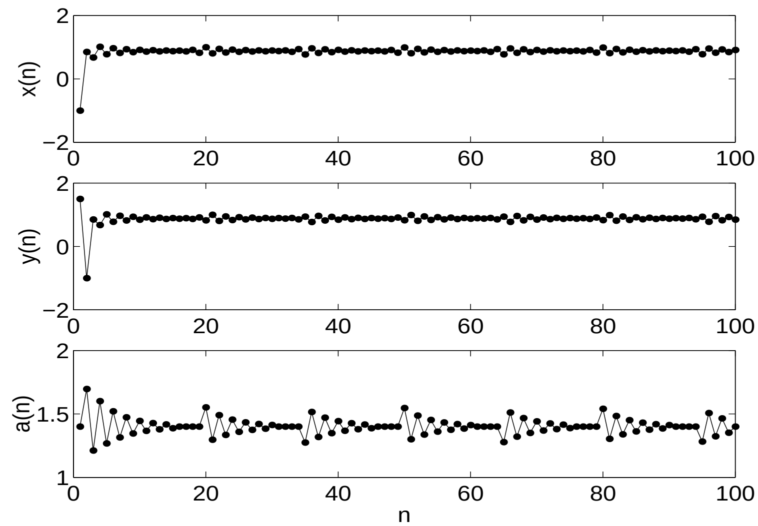


Figure 3.6: The nonlinearly controlled Hénon map with $(x_0, y_0) = (-1, 1.5)$.

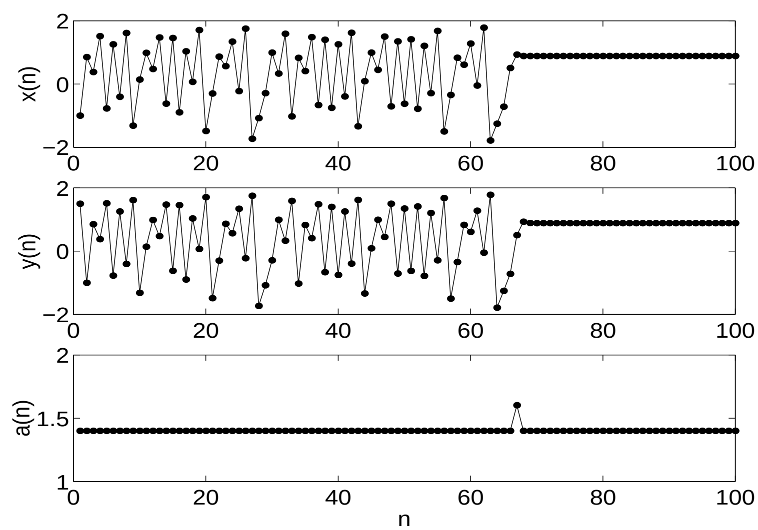


Figure 3.7: The OGY-controlled Hénon map with $(x_0, y_0) = (-1, 1.5)$.

The following two figures in Figure 3.8 show the controllable initial conditions in the square $[-3, 3]^2$ with unbounded control. We observe that controlling initial conditions, which are further away from $(x_F, y_F)$, is possible with our algorithm, whereas the OGY-method fails to control these orbits. The reason for this failure is that those initial conditions are not in a small vicinity of the fixed point. But this is a necessary condition for the OGY control in order to stabilize the system. In

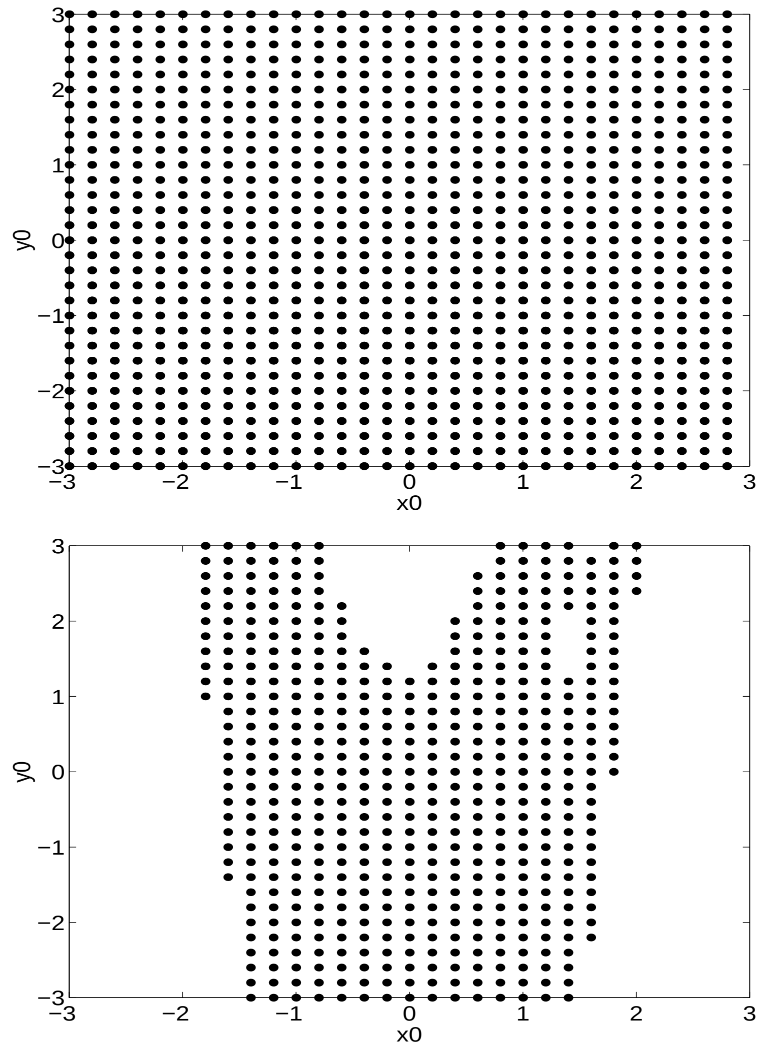


Figure 3.8: The nonlinearly controlled Hénon map (upper figure) and the OGY-controlled Hénon map (lower figure) with $(x_0, y_0) \in [-3, 3]^2$.

contrast, the local stable manifold given by the approximation of $\psi$ is a much better approximation of the stable manifold and thus, our algorithm is able to stabilize the system, even if initial conditions are further apart from the hyperbolic fixed point.

Now we pose the following question: Is our stabilization better, when the approximation of the local stable manifold has a higher order? So far, we have used the graph of the function

$$\psi(u) = 0.01414\,u^2 + 0.00218\,u^3.$$

It is possible to obtain coefficients for higher order terms.

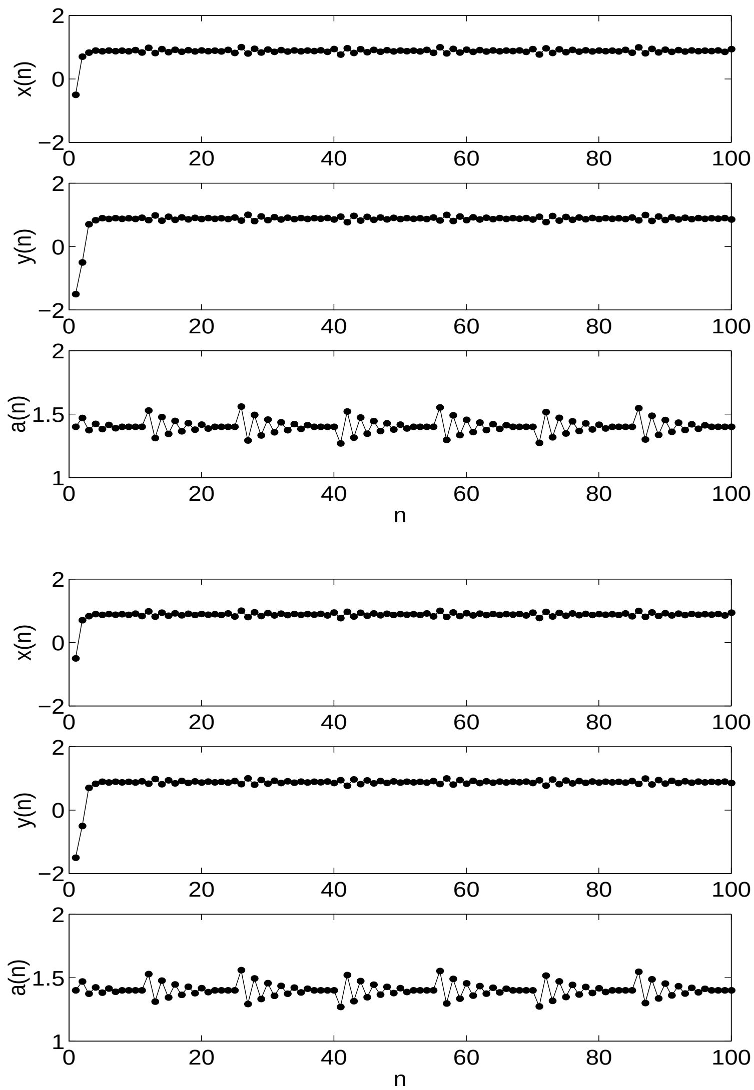


Figure 3.9: The nonlinear stabilization algorithm applied to $(x_0, y_0) = (-0.5, -1.5)$ with $\varepsilon = 0.2, \delta = 0.1$. The upper figure uses the approximation $\psi(u) = 0.01414\, u^2$ and the lower one $\psi(u) = 0.01414\, u^2 + 0.00218\, u^3 + 0.00042\, u^4$.

But as our computational experiments show, an approximation with higher orders of $W^s_{loc}(0,0)$ leads not necessarily to better results. In Figure 3.9, we apply the nonlinear stabilization method to the initial condition $(x_0, y_0) = (-0.5, -1.5)$ with $\varepsilon = 0.2$ and $\delta = 0.1$. First, we use the lowest approximation of the local stable manifold, i.e.,

$$\psi(u) = 0.01414\, u^2 + \mathcal{O}(u^3).$$

The result is shown in the upper figure. The lower figure of Figure 3.9 shows the same algorithm applied to the same initial data, but with the local stable manifold given as graph of

$$\psi(u) = 0.01414\,u^2 + 0.00218\,u^3 + 0.00042\,u^4 + \mathcal{O}(u^5).$$

Note that there is no difference in the stabilization that is achieved. The reason might be that the computational errors are adding up and that an approximation with higher order is no more accurate than a lower one. Furthermore, if we compute $s_4$, we obtain $s_4 \approx 0.0004195$, which is of order $10^{-4}$. Such a small value might not make much difference in the control procedure and it only adds up to the computational errors. Thus, the order of the approximation plays no role in the implementation of the stabilization procedure, as long as a higher order than in the OGY case is used.

The last observation we want to make is how the nonlinear control algorithm depends on the choice of $\varepsilon$. We take the initial condition to be $(x_0, y_0) = (0.75, -0.5)$. Figure 3.10 shows results for different values of $\varepsilon$. In Figure 3.4, we had $\varepsilon = 0.01$. In Figure 3.10, $\varepsilon$ is ten times larger than in Figure 3.4, i.e., $\varepsilon = 0.1$. If one compares the last row of Figure 3.4 to that of 3.10, then one notes that in the latter figure, more control steps are needed and in particular, those are larger than in the first figure. This is due to the fact that in Figure 3.10, we have an $\varepsilon$-neighborhood around $W^s_{loc}(x_F, y_F)$ with $\varepsilon$ ten times bigger than before.

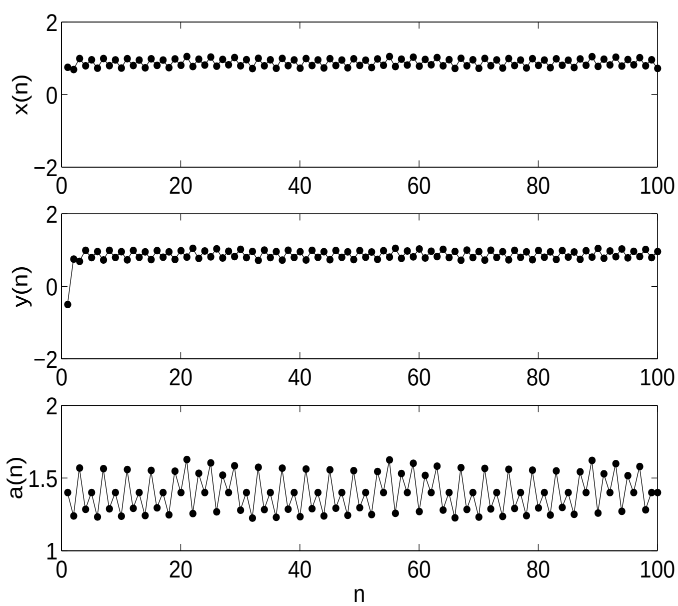


Figure 3.10: The nonlinear stabilization algorithm applied to $(x_0, y_0) = (0.75, -0.5)$ with $\varepsilon = 0.1$.

Consequently, the controlled orbit needs a longer time until it leaves this neighborhood and we have to apply larger controls in order to get back to the local stable manifold.

Summarizing, our stabilization of the Hénon map is successful. The nonlinear method is able to control a much larger set of initial conditions compared to the OGY-method. Furthermore, the smaller the $\varepsilon$-neighborhood of $W^s_{loc}(x_F, y_F)$ is, the better the stabilization. There is no difference in the result of nonlinear stabilization, when a higher order approximation of $W^s_{loc}(x_F, y_F)$ is used.

# Chapter 4

# The stabilization at hyperbolic periodic orbits in $\mathbb{R}^m$

In Chapter 2, our nonlinear method for stabilizing nonlinear autonomous dynamical systems has been introduced. The corresponding algorithm for the stabilization of a system at a hyperbolic fixed point in two dimensions has successfully been applied in Chapter 3. So far, we have restricted our considerations to a two-dimensional phase space. We now get rid of this simplification and introduce the analogue stabilization method for higher dimensions. In applications it might be necessary to force trajectories onto a more complicated behavior than it is represented by a hyperbolic fixed point. Thus, we show how to stabilize systems at hyperbolic periodic orbits of period greater than one. As before, this method will be implemented. The resulting algorithm is based on the algorithm, which has already been introduced in Section 2.3.

The first section establishes necessary facts about local stable manifolds at periodic points. Section 2 is concerned with local coordinate transformations that will be used to compute the local stable manifolds and the control parameter. Thereafter, we are able to introduce the complete stabilization procedure. In Section 4, we prove a local existence and uniqueness result for the feedback control. The corresponding algorithm, which is based on the one from Section 2.3, is introduced in Section 5 as well as an illustration of a stabilization at a period-2 orbit of the Hénon map.

## 4.1 Preliminaries

In the following let $X$ be the phase space, where $X \subset \mathbb{R}^m$ is an open subset. We consider the nonlinear discrete dynamical system given by

$$x_{n+1} = f(x_n, u_n), \tag{4.1}$$

where $f(\cdot, u_n)\colon X \to X$ is a $C^r$-map with $r \geq 1$. The parameter $u_n \in \mathbb{R}^{m-s}$ is a system parameter, which represents the feedback control, and $s \geq 1$ is the dimension of the linear stable subspace of the periodic orbit at which we are planning to stabilize the system (4.1). We assume that $f$ is $C^1$ with respect to $u_n$. Note that we do not write $f : X \times \mathbb{R}^{m-s} \to X$, but instead, we think of a family of maps $f_{u_n} : X \to X$. We assume that the control parameter $u_n$ can be varied over $\mathbb{R}^{m-s}$. As in Chapter 2, the control is bounded by $|u_n - u_\star| < u_{max}$, where $u_\star$ is the nominal value for the uncontrolled system

$$x_{n+1} = f(x_n, u_\star). \tag{4.2}$$

Throughout this chapter, we assume that for $u_n = u_\star$, the system (4.2) possesses a hyperbolic periodic point $p_0$. Let $p_0$ generate a hyperbolic periodic orbit of period $k \geq 1$, i.e.,

$$O(p_0) = \{p_0, f(p_0), f^2(p_0), \ldots, f^{k-1}(p_0)\}$$

and $f^k(p_0) = p_0$. We abbreviate $f^i(p_0) =: p_i$, where $i \in \{0, \ldots, k-1\}$, i.e., $p_k = p_0,\ p_{k+1} = p_1,\ p_{2\,k} = p_0$ and so on. We assume that the $C^r$-map $f$ is a local diffeomorphism at the periodic orbit $O(p_0)$.

Within this section, we consider the uncontrolled system with $u_n = u_\star$ for all $n$. First, we determine the stability behavior at each point of the periodic orbit. In the following, we introduce a series of lemmas, which will be of use in the later sections. We assume that $p_0$ is a saddle fixed point of $f^k$ with $s \geq 1$ stable directions and $m - s$ unstable ones, where $s < m$. As we have seen in Theorem 1.1.6, the stability of the periodic orbit $\{p_0, p_1, p_2, \ldots, p_{k-1}\}$ is determined by the Jacobian matrix $\mathrm{D}f^k(p_0)$. Using the chain rule and the fact that $f^i(p_0) = p_i$ yields

**Lemma 4.1.1**
*Consider the periodic orbit $\{p_0, p_1, \ldots, p_{k-1}\}$ of the system (4.2). Then the following holds:*

$$\mathrm{D}f^k(p_0) = \mathrm{D}f(p_{k-1})\, \mathrm{D}f(p_{k-2}) \cdots \mathrm{D}f(p_1)\, \mathrm{D}f(p_0) \tag{4.3}$$

Thus, instead of $\mathrm{D}f^k(p_0)$, we can consider the product of Jacobians of $f$. As we will show, it is sufficient to compute the eigenvalues of just one of the Jacobian matrices $\mathrm{D}f^k(p_i)$. The following lemmas are basic facts from linear algebra, so we omit the proofs.

**Lemma 4.1.2**
*Let $A \in \mathbb{R}^{m\times m}, B \in Gl_m(R)$ and let $\lambda$ be an eigenvalue of the matrix product $A\,B$ with eigenvector $v \in \mathbb{R}^m$. Then $\lambda$ is also an eigenvalue of $B\,A$ with corresponding eigenvector $w = B\,v \in \mathbb{R}^m$.*

The result of Lemma 4.1.2 can be generalized to arbitrary finite products of matrices.

**Lemma 4.1.3**
*Let $A_1, \ldots, A_n \in \mathbb{R}^{m\times m}$. Assume that for $1 \leq l \leq n$, the matrix product $A_{l+1} \cdots A_n$ is invertible. Then $A_1 \cdots A_n$ and the cyclic permutation $A_{l+1} \cdots A_n\, A_1 \cdots A_l$ have the same set of eigenvalues, where $1 \leq l \leq n$.*

The assertion follows from setting $A := A_1 \ldots A_l$ and $B := A_{l+1} \ldots A_n$ in Lemma 4.1.2

**Remark**
The assumption in Lemma 4.1.3 that $A_{l+1} \cdots A_n$ is invertible, is equivalent to $A_{l+1}, \ldots, A_n$ each being in $\mathrm{Gl}_m(R)$. In our case, the matrices $A_l$ are the Jacobians $\mathrm{D}f(p_l)$ of $f$ at $p_l$. We consider $\mathrm{D}f^k(p_i)$, which is the product of the $\mathrm{D}f(p_l)$, cf. Lemma 4.1.1. Due to Lemma 4.1.3, we need to assume that all Jacobians $\mathrm{D}f(p_l)$ are invertible, i.e., $\mathrm{D}f(p_l) \in \mathrm{Gl}_m(R)$, which yields $\mathrm{D}f^k(p_i) \in \mathrm{Gl}_m(R)$. Equivalently, we note that none of the eigenvalues of $\mathrm{D}f^k(p_i)$ is zero. Hence, $\mathrm{D}f^k(p_i)$ is an isomorphism for all $i$ and thus, $f$ is a local diffeomorphism.

Due to the lemmas above, it is possible to determine the stability of each point of the periodic orbit. By assumption, $p_0$ is hyperbolic, thus all eigenvalues of $\mathrm{D}(f^k(p_0)) : \mathbb{R}^m \to \mathbb{R}^m$ have modulus different from one. We call the eigenvalues $\lambda_j$ and $\mu_l$, where

$$\lambda_j, \mu_l \in \mathbb{C} \text{ and } |\lambda_j| < 1, |\mu_l| > 1 \text{ with } j = 1, \ldots, s,\ l = 1, \ldots, m - s. \tag{4.4}$$

The corresponding generalized eigenvectors are given by $v_1^0, \ldots, v_s^0$ for the stable eigenvectors $\lambda_j$, and $w_1^0, \ldots, w_{m-s}^0$ for the unstable eigenvalues $\mu_l$. Lemma 4.1.3 and Lemma 4.1.1 yield that $\lambda_j$ and $\mu_l$ are also the eigenvalues of $\mathrm{D}f^k(p_i)$ for all $i \in \{1, \ldots, k-1\}$. Hence, the saddle point structure is preserved at each point of the periodic orbit. Also, the dimensions of the linear stable and unstable subspaces are the same at each $p_i$. Only the eigenvectors differ for each $\mathrm{D}f^k(p_i)$. They are determined by the product of Jacobians in the following way.

**Lemma 4.1.4**
*Let $\{p_0, p_1, \ldots, p_{k-1}\}$ be a periodic orbit of period $k$ of the system (4.2). Let $\lambda_j$ and $\mu_l$ be given as in (4.4), and let $v_j^0 \neq 0$ and $w_l^0 \neq 0$ be the corresponding eigenvectors for $\mathrm{D}f^k(p_0)$. Then the eigenvectors of $\mathrm{D}f^k(p_q)$ for $1 \leq q \leq k-1$ are determined by*

$$v_j^q = \mathrm{D}f(p_{q-1}) \ldots \mathrm{D}f(p_0)\, v_j^0, \quad j = 1, \ldots, s \tag{4.5}$$

*and*

$$w_l^q = \mathrm{D}f(p_{q-1}) \ldots \mathrm{D}f(p_0)\, w_l^0, \quad l = 1 \ldots, m - s,$$

*respectively.*

It is sufficient to prove the lemma just for one of the eigenvectors. If one uses

$$\mathrm{D}f^k(p_0)\, v_j^0 = \lambda_j\, v_j^0$$

and applies Lemma 4.1.1 to the left hand side, the multiplication of $\mathrm{D}f(p_{q-1})\,\mathrm{D}f(p_{q-2})\ldots\mathrm{D}f(p_0)$ from the left leads to the desired result. Thus, all stable eigenvectors $v_j^q$ for $1 \le q \le k-1$ are determined by the Jacobian matrices $\mathrm{D}f(p_l)$ and the eigenvectors $v_j^0$.

Next, we show that each local stable manifold $W^s_{loc}(p_i)$ is invariant under $f^k$. Let $M := W^s_{loc}(p_0)$ be the local stable manifold at $p_0$ with respect to $f^k$. By Definition 1.2.1, it follows that $M$ is invariant under $f^k$. Since the whole periodic orbit is to be stabilized, we need to know more about the structure at each periodic point $p_i$. The fact that $M$ is positively invariant under $f^k$ implies that the $f$-images of the local stable manifold $M$ are also positively invariant:

**Lemma 4.1.5**
*For all $i \in \{1,\ldots,k-1\}$, the images of $M$, $f^i(M)$, are positively invariant under $f^k$.*

**Proof**
Due to the invariance of $M$, we know that for all $x \in M$ it follows that $f^k(x) \in M$. Now let $y \in f^i(M)$ for some $i \in \{1,\ldots,k-1\}$. There exists $x \in M$ such that $f^i(x) = y$. Hence,

$$f^k(y) = f^k(f^i(x)) = f^{k+i}(x) = f^i(f^k(x)) \in f^i(M),$$

which holds for all $y \in f^i(M)$ and $i \in \{1,\ldots,k-1\}$. Thus, $f^i(M)$ is invariant with respect to $f^k$ for all $i = 1,\ldots,r-1$. ■

Following the definition of a local stable manifold, we know that for all $x \in M$ we have $(f^k)^n(x) \to p_0$ as $n \to \infty$. One immediately deduces

**Lemma 4.1.6**
*The $f^i(M)$ are local stable manifolds with respect to $f^k$ at $p_i$, i.e.,*

$$f^i(M) = W^s_{loc}(p_i),$$

*where $i \in \{0,\ldots,k-1\}$.*

**Proof**
By definition, for all $x \in M$ the iterates $(f^k)^n(x)$ tend to $p_0$ as $n \to \infty$. The continuity of $f$ leads to

$$\forall\ y \in f^i(M) : (f^k)^n(y) \longrightarrow p_i \quad \text{as } n \to \infty$$

for all $i \in \{0,\ldots,k-1\}$. ■

Moreover, the local stable manifold of the hyperbolic periodic orbit $\{p_0, p_1, \ldots, p_{k-1}\}$ is given by the union of all $k$ local stable manifolds $\bigcup_{i=0}^{k-1} W^s_{loc}(p_i)$ with respect to $f^k$. The existence of the local stable manifolds at each periodic point $p_i$ is guaranteed by the Stable Manifold Theorem 1.2.3.

**Remark**
The same is true for the local unstable manifolds of each periodic point. The proofs are the same except that one has to change $n \to \infty$ to $n \to -\infty$. We are only interested in the stable manifolds, because the control algorithms forces points of an orbit onto the local stable manifold of the periodic orbit.

## 4.2 Local coordinate systems at each periodic point

Our goal is to stabilize a dynamical system at the hyperbolic period orbit $\{p_0, \ldots, p_{k-1}\}$ using the system parameter $u_n \in \mathbb{R}^{m-s}$. The strategy will be the same as in Section 2.1. In the case of periodic orbits, we have to be careful of how to define the stabilization procedure itself. As already has been pointed out in [RGOD92, Voc98], control of a periodic orbit can be achieved by taking the $k$-th iterate of $f$ and use $f^k$ to control a hyperbolic saddle fixed point $p_i$ of $f^k$. In this case, the control takes place only at one of the periodic points $p_i$. We could then use the stabilization method from Chapter 2. However, taking $f^k$ is overly sensitive to noise, especially, when large periods are involved. Moreover, the neighborhood in which control could be achieved would be very small. Therefore, we introduce an alternative method, where we are able to stabilize the system at the whole periodic orbit and not only at one periodic point. This has the advantage of a more uniform stabilization. By a uniform stabilization we mean that we can stabilize at each $p_i$. In case that the controlled orbit leaves the local stable manifold of the periodic orbit, we are able to adjust the control whenever necessary. If one uses $f^k$, then this can only be done every $k$-th iteration and we need to wait until the orbit comes into the neighborhood of that particular $p_i$. Hence, our goal is to control an orbit $O(x_0)$ in the following manner. If $x_n$ of $O(x_0)$ is within a neighborhood of some $p_i$, then force $x_{n+1} = f(x_n, u_n)$ onto the local stable manifold of the next periodic point $p_{i+1}$.

In this section, we introduce local coordinate transformations which makes the dealing with the local stable manifolds more manageable. As we will see, one can establish a stabilization method for the original system given by $f$. However, it is easier to obtain an explicit representation for the local stable manifolds of the transformed system. We wish to achieve the diagram shown in Figure 4.1.

In the following, we first fix $u_n$ at $u_\star$ and establish the necessary transformations as in Chapter 2. Later on, we will come back to variable $u_n$, since we then consider the transformed system with control $u_n$. Using Theorem 1.2.3, we can introduce a coordinate chart near $p_i$ for all $i \in \{0, \ldots, k-1\}$, mapping $p_i$ onto the origin so that the stable and unstable linear subspaces $E^s(p_i)$ and $E^u(p_i)$

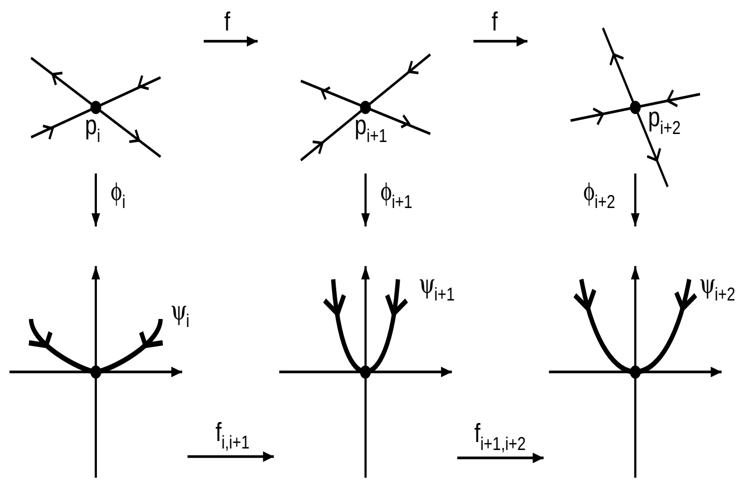


Figure 4.1: Local charts $\phi_i$, the local stable manifolds given as graph of $\psi_i$ and the induced maps $f_{i,i+1}$.

are tangent to $\mathbb{R}^s \times \{0\}$ and $\{0\} \times \mathbb{R}^{m-s}$ at $p_i$, correspondingly. Therefore, we obtain smooth adapted coordinates such that $W^s_{loc}(p_i)$ is the graph of a function $\psi_i : \mathbb{R}^s \to \mathbb{R}^{m-s}$. Later in the implementation of the algorithm, we use this fact to approximate the local stable manifolds at each $p_i$. Let us define transformations $\phi_i$ in a neighborhood of $p_i$:

**Definition 4.2.1** (Local Charts)
*Let $x_{n+1} = f(x_n, u_\star)$ be given as in Section 1. Suppose that the hyperbolic periodic orbit $\{p_0, \ldots, p_{k-1}\}$ of the uncontrolled system has $s$ stable directions and $m - s$ unstable ones. The linear stable and unstable subspace are given by*

$$E^s(p_i) = \text{span}\{v^i_1, \ldots, v^i_s\}$$

*and*

$$E^u(p_i) = \text{span}\{w^i_1, \ldots, w^i_{m-s}\}$$

*for $i \in \{0, \ldots, k-1\}$, respectively. Define a local coordinate transformation $\phi_i : U_i \to V_i$ by*

$$z = \phi_i(x) := T_i^{-1}(x - p_i), \quad \forall\, i \in \{0, \ldots, k-1\}. \tag{4.6}$$

*where*

$$T_i := [v^i_1 \; v^i_2 \ldots v^i_s \; w^i_1 \ldots w^i_{m-s}] \tag{4.7}$$

*and $U_i \subset X, V_i \subset \mathbb{R}^m$ are neighborhoods of $p_i$ and $\phi_i(p_i) = 0$, respectively.*

The charts $\phi_i$ are exactly those charts introduced in Figure 4.1. Note that each $\phi_i$ is an isomorphism. The transformation matrices $T_i$ consist of generalized eigenvectors corresponding to the stable and unstable eigenvalues $\lambda_j, \mu_l$ and their multiplicities.

**Remark**

Since we will make use of the real Jordan normal form of the matrices $\mathrm{D}f^k(p_i)$, we recall some basic facts. Depending on the eigenvalues $\lambda_j$ and $\mu_l$, the Jordan normal form of $\mathrm{D}f^k(p_i)$ consists of real Jordan blocks $J_{\lambda_j}$ and $J_{\mu_l}$. These Jordan blocks can be different for each eigenvalue. They are of one of the following types:

(i) If all eigenvalues are real and distinct with multiplicities $M_1, \dots, M_m$, then

$$\mathrm{D}f^k(p_i) = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \mu_{m-s-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \mu_{m-s} \end{pmatrix} \in \mathbb{R}^{m \times m}.$$

(ii) If an eigenvalue $\alpha$ is real with multiplicity $m_\alpha > 1$, then the corresponding Jordan block $J_\alpha$ has the form

$$J_\alpha = \begin{pmatrix} \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & \dots & 0 & 0 \\ 0 & 0 & \alpha & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & \alpha & 1 \\ 0 & 0 & 0 & \dots & 0 & \alpha \end{pmatrix} \in \mathbb{R}^{m_\alpha \times m_\alpha}.$$

(iii) If $\lambda = a + i\,b$ is a complex eigenvalue with multiplicity one, then

$$J_\lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

(iv) In case that such a complex eigenvalue occurs with multiplicity $m_\lambda > 1$, the Jordan block consists of $J_\lambda$ as in (iii) and the identity matrix $I_2 \in \mathbb{R}^{2 \times 2}$.

$$J_\lambda = \begin{pmatrix} J_\lambda & I_2 & \dots & 0 & 0 \\ 0 & J_\lambda & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & J_\lambda & I_2 \\ 0 & 0 & \dots & 0 & J_\lambda \end{pmatrix}$$

Now we define a new map $h_i$ that represents $f^k$ in the local coordinate system at $p_i$. The map $h_i$ maps points from $V_i$ to $\tilde{V}_i$, where $V_i \cap \tilde{V}_i \neq \emptyset$ such that $V_i \cap \tilde{V}_i$ is still a whole neighborhood of the transformed periodic point $0$.

**Definition 4.2.2**
*Under the assumptions made above, we define a map $h_i\colon V_i \to \tilde{V}_i$ in the local coordinate system of each $p_i$ for fixed $u_n = u_\star$, where*

$$h_i(z) = \phi_i(f^k(\phi_i^{-1}(z), u_\star)) = T_i^{-1}\left(f^k(T_i\, z + p_i, u_\star) - p_i\right)$$

*for each $i \in \{0, \ldots, k-1\}$.*

Obviously, $h_i(0) = 0$ for each $i$, i.e., the origin $\phi_i(p_i) = 0$ of each local coordinate system is a fixed point of $h_i$. We will show that the Jacobian of $h_i$ evaluated at this fixed point has the same set of eigenvalues as $\mathrm{D}f^k(p_0)$.

**Lemma 4.2.3**
*Let $h_i$ be defined as in Definition 4.2.2. Then*

$$\mathrm{D}h_i(0) = T_i^{-1}\,\mathrm{D}f^k(p_i)\,T_i$$

*and thus, the eigenvalues of $\mathrm{D}h_i(0)$ are identical to those of $\mathrm{D}f^k(p_i)$.*

**Proof**
Clearly,

$$\begin{aligned}
\mathrm{D}h_i(z) &= \mathrm{D}[T_i^{-1}\left(f^k(T_i\, z + p_i, u_\star) - p_i\right)] \\
&= T_i^{-1}\,\mathrm{D}(f^k(T_i\, z + p_i, u_\star)) \\
&= T_i^{-1}\,\mathrm{D}f^k((T_i\, z + p_i, u_\star))\,T_i.
\end{aligned}$$

Thus, $\mathrm{D}h_i(0) = T_i^{-1}\,\mathrm{D}f^k(p_i)\,T_i$. Due to the definition of $T_i$, $\mathrm{D}h_i(0)$ is the Jordan normal form of $\mathrm{D}f^k(p_i)$ and the eigenvalues are the same. ■

The fact that $T_i$ is the coordinate transformation matrix which transforms $E^s(p_i)$ to $\mathbb{R}^s \times \{0\}$ implies that the linear stable subspaces at the fixed points $\phi_i(p_i) = 0$ are given by $E_i^s(0) = \mathrm{span}\{e_1, \ldots, e_s\}$. Analogously, the linear unstable subspaces are $E_i^u(0) = \mathrm{span}\{e_{s+1}, \ldots, e_m\}$. Actually, there exists a so-called *splitting* of $h_i$ into its stable and unstable component.

**Corollary 4.2.4**
*The map $h_i$ can be split into two components, namely*

$$h_i(z) = \begin{pmatrix} h_i^{(1)}(z) \\ h_i^{(2)}(z) \end{pmatrix} = \begin{pmatrix} J_s\, z^{(1)} + H_1(z^{(1)}, z^{(2)}) \\ J_{m-s}\, z^{(2)} + H_2(z^{(1)}, z^{(2)}) \end{pmatrix},$$

*where $z^{(1)} \in E_i^s(0), z^{(2)} \in E_i^u(0), H_\alpha(0,0) = 0, \mathrm{D}H_\alpha(0,0) = 0$ for $\alpha = 1, 2$ and $J_s \in \mathbb{R}^{s\times s}, J_{m-sl} \in \mathbb{R}^{(m-s)\times(m-s)}$ are the block matrices that consist of the Jordan forms corresponding to the eigenvalues $\lambda_j$ and $\mu_l$, respectively.*

**Proof**
As we have seen above, the Jacobian $\mathrm{D}h_i(0)$ is the Jordan normal form of $\mathrm{D}f^k(p_i)$. By Lemma 4.2.3, we obtain

$$\mathrm{D}h_i(0) = \begin{pmatrix} \square & | & 0 \\ ----- & & ----- \\ 0 & | & \square \end{pmatrix},$$

where the upper left block is of dimension $s \times s$ and the lower right block is of dimension $(m-s) \times (m-s)$. All the Jordan blocks of the stable eigenvalues $\lambda_j, j = 1, \ldots, s$, together form a Jordan block $J_s$ of dimension $s \times s$ and the Jordan blocks of the unstable ones form $J_{\mu_l}$ of dimension $(m-s) \times (m-s)$ where $l = 1, \ldots, m-s$. Therefore, $\mathrm{D}h_i(0)$ leaves $E_i^s(0)$ and $E_i^u(0)$ invariant. Since $\mathrm{D}h_i(0)$ is the derivative of $h_i$ at zero and $h_i(0) = 0$ holds, we conclude that the linear part of $h_i$ has to be of the form $(J_{\lambda_j}\, z^{(1)}, J_{\mu_l}\, z^{(2)})^t$, where $z^{(1)} \in \mathbb{R}^s, z^{(2)} \in \mathbb{R}^{m-s}$. Due to $H_\alpha(0,0) = 0$ and $\mathrm{D}H_\alpha(0,0) = 0$, there is no linear part in $H_\alpha$ with respect to $z = (z^{(1)}, z^{(2)})^t$ for $\alpha = 1.2$. Thus, the nonlinear part of $h_i$ is given by $H_\alpha(z^{(1)}, z^{(2)}), \alpha = 1, 2$, of corresponding dimensions and we obtain the desired splitting of $h_i$. ■

Now we can apply the Stable Manifold Theorem 1.2.3 to $h_i$ with $0$ as hyperbolic saddle fixed point for $u_n = u_\star$ fixed. It tells us that there exists, for each $i$, the local stable $C^r$-manifold $W^s_{loc}(0)$ which is tangent to $E^s(0)$ at $0$. Moreover, $W^s_{loc}(0)$ is the graph of a $C^r$-function $\psi_i\colon E_i^s(0) \cap W_i \subset \mathbb{R}^s \to E_i^u \cap W_i \subset \mathbb{R}^{m-s}$ with $\psi_i(0) = 0$ and $\mathrm{D}\psi_i(0) = 0$ such that

$$W^s_{loc}(0) = \{(v, \psi_i(v)) : v \in E_i^s(0) \cap W_i\},$$

where $W_i \subset \mathbb{R}^m$ is an open neighborhood of $0$. There exist $k$ such local stable manifolds, one for each periodic point $p_i$. By Definition 1.2.1, all these manifolds are positively invariant with respect to the corresponding $h_i$.

So far we defined the coordinate charts $\phi_i$, which have been introduced in Figure 4.1. In order to be able to present the complete stabilization method for hyperbolic periodic orbits with a saddle structure, we still need to establish the induced maps $f_{i,i+1}$ from Figure 4.1 for arbitrary $u_n$. In order to compute the feedback control $u_n$, we set up the local coordinates, cf. Definition 4.2.1 and Figure 4.1, and a new map $f_{i,i+1}$, which maps points from the local coordinate system at $p_i$ to the one of the following periodic point $p_{i+1}$. Let us define $f_{i,i+1}$ as follows.

**Definition 4.2.5** (Induced Maps)
*Let $\phi_i\colon U_i \to V_i$ be given as in (4.6), where $U_i$ is an open neighborhood of $p_i$ for each $i$. Then define $f_{i,i+1}\colon V_i \times \mathbb{R}^{m-s} \to V_{i+1}$ for general $u_n$ by*

$$f_{i,i+1}(z,u) = \phi_{i+1} \circ f \circ \phi_i^{-1}(z,u) = T_{i+1}^{-1}\left(f(T_i\, z + p_i, u) - p_{i+1}\right). \tag{4.8}$$

For $u_n = u_\star$, the map $f_{i,i+1}$ can be split into an $s$- and an $(m-s)$-dimensional part since

$$\mathrm{D}f_{i,i+1}\colon \begin{cases} E_i^s(0) & \to & E_{i+1}^s(0) \\ E_i^u(0) & \to & E_{i+1}^u(0) \end{cases}$$

is a bijective mapping. The proof is the same as the one of Corollary 4.2.4, except that $h_i$ contains $f^k$ whereas $f_{i,i+1}$ contains only $f$. We denote

$$f_{i,i+1}(z,u) = \begin{pmatrix} f_{i,i+1}^{(1)}(z,u) \\ \\ f_{i,i+1}^{(2)}(z,u) \end{pmatrix},$$

where $f_{i,i+1}^{(1)} \in \mathbb{R}^s$ and $f_{i,i+1}^{(2)} \in \mathbb{R}^{m-s}$.

**Lemma 4.2.6**
*Each $f_{i,i+1}$ maps $W^s_{loc}(\phi_i(p_i))$ in one local coordinate system to the next, i.e., to $W^s_{loc}(\phi_{i+1}(p_{i+1}))$.*

**Proof**
The assertion follows from the set-up in the diagram introduced in Figure 4.1. If we take a part of the diagram consisting of two successive periodic points together with the corresponding transformations, then this part of the diagram commutes. So let us consider two periodic points $p_i$ and $p_{i+1}$, where $i \in \{0,\dots,k-1\}$. We have seen in the global coordinates that $f^k(W^s_{loc}(p_i)) \subset W^s_{loc}(p_i)$ and $f^k(W^s_{loc}(p_{i+1})) \subset W^s_{loc}(p_{i+1})$, i.e., the local stable manifolds are positively invariant under $f^k$, compare Lemma 4.1.5 and Theorem 1.2.3. In the local coordinate systems, we obtain analogously $h_i(W^s_{loc}(\phi_i(p_i))) \subset W^s_{loc}(\phi_i(p_i))$ and $h_{i+1}(W^s_{loc}(\phi_{i+1}(p_{i+1}))) \subset W^s_{loc}(\phi_{i+1}(p_{i+1}))$. It follows by Definition 1.2.1, that for $w_1 \in W^s_{loc}\phi_i((p_i))$ and $w_2 \in W^s_{loc}(\phi_{i+1}(p_{i+1}))$ we have

$$\lim_{n\to\infty} h_i^n(w_1) = \phi_i(p_i) = 0$$

and

$$\lim_{n\to\infty} h_{i+1}^n(w_2) = \phi_{i+1}(p_{i+1}) = 0,$$

respectively. Using Definitions 4.2.2 and 4.2.5, we conclude

$$\begin{aligned} & \lim\nolimits_{n\to\infty} f_{i,i+1} \circ h_i^n(w_1) \\ = \; & \lim\nolimits_{n\to\infty} \phi_{i+1} \circ f \circ \phi_i^{-1} \circ \phi_i \circ (f^k)^n \circ \phi_i^{-1}(w_1) \\ = \; & \lim\nolimits_{n\to\infty} \phi_{i+1} \circ f^{(k\,n+1)} \circ \phi_i^{-1}(w_1) \\ = \; & \lim\nolimits_{n\to\infty} \phi_{i+1} \circ (f^k)^n \circ \phi_{i+1}^{-1} \circ \phi_{i+1} \circ f \circ \phi_i^{-1}(w_1) \\ = \; & \lim\nolimits_{n\to\infty} h_{i+1}^n \circ f_{i,i+1}(w_1) \\ = \; & \lim\nolimits_{n\to\infty} h_{i+1}^n(w_2) \\ = \; & \phi_{i+1}(p_{i+1}) = 0, \end{aligned}$$

which completes the proof. ■

## 4.3 An existence and uniqueness theorem

The aim is to activate the nonlinear feedback control $u_n \in \mathbb{R}^{m-s}$ within one iteration of $f$. Suppose we start with some point $x_n$, which is in the vicinity of $p_i$ for some $i$. We define $U_i$ to be an open neighborhood of the periodic point $p_i$ for each $i \in \{0, \ldots, k-1\}$. If one computes the following iterate $x_{n+1} = f(x_n, u_n)$ with $u_n = u_\star$, then $x_{n+1} \in U_{i+1}$. Controlling the orbit $O(x_n)$ means to find $u_n$ such that $x_{n+1} = f(x_n, u_n)$ is a point on the local stable manifold of $p_{i+1}$.

After introducing all necessary details shown in Figure 4.1, we are able to derive an implicit equation for the control $u_n$. Assume that $x_n \in U_i$ for one $i = 0, \ldots, k-1$ and $u_{n-1} = u_\star$. First, one switches to the local coordinate system, using $\phi_i(x_n) = z_n$. Then we wish to obtain a point $z_{n+1} = f_{i,i+1}(z_n, u_n)$ on $W^s_{loc}(\phi_{i+1}(p_{i+1}))$ for suitable $u_n$. The goal is to determine $u_n$. Since $f_{i,i+1}$ can be split into the stable and the unstable component $f^{(1)}_{i,i+1}(z_n, u_n) \in \mathbb{R}^s, f^{(2)}_{i,i+1}(z_n, u_n) \in \mathbb{R}^{m-s}$, respectively, we can split $z_{n+1}$ into its two components as well. Explicitely, $z_{n+1} = (z^{(1)}_{n+1}, z^{(2)}_{n+1})^t$. Now we require $z_{n+1}$ to be a point on the local stable manifold $W^s_{loc}(\phi_{i+1}(p_{i+1}))$. Taking into account that the manifold is given as the graph of $\psi_{i+1}$, one deduces that

$$z^{(2)}_{n+1} = \psi_{i+1}\left(z^{(1)}_{n+1}\right). \tag{4.9}$$

Using $z_{n+1} = f_{i,i+1}(z_n, u_n)$, one immediately concludes that

$$f^{(2)}_{i,i+1}(z_n, u_n) = \psi_{i+1}\left(f^{(1)}_{i,i+1}(z_n, u_n)\right). \tag{4.10}$$

The nonlinear equation (4.10) now determines the control value $u_n$, because all other variables and maps are known.

The question that arises now is whether one can solve equation (4.10) for $u_n$. Indeed, we can show that under certain assumptions such a control exists locally. The proof of the following existence theorem is mainly based on the previous sections, where we have introduced all the relevant notations and results, together with an application of the Implicit Function Theorem, cf. also Theorem 2 in the Introduction of the thesis.

**Theorem 4.3.1** (EXISTENCE AND UNIQUENESS THEOREM)
*Let $x_{n+1} = f(x_n, u_n)$ be a discrete dynamical system given as in (4.1), which possesses a hyperbolic periodic orbit $\{p_0, \ldots, p_{k-1}\}$ of period $k \geq 1$ for $u_n = u_\star$. Assume that $\mathrm{D}f^k(p_i)$ has $s$ stable eigenvalues, $m > s \geq 1$, and $m - s$ unstable ones with corresponding generalized eigenvectors $v^i_1, v^i_2, \ldots, v^i_s$ and $w^i_1, \ldots, w^i_{m-s}$, where $i = 0, \ldots, k-1$. Define $\phi_i, f_{i,i+1}$ as in Definitions 4.2.1 and 4.2.5. Let $U_i \subset X$ be an appropriate open neighborhood of $p_i$ for each $i \in \{0, \ldots, k-1\}$ and $D \subset \mathbb{R}^{m-s}$ be an appropriate open neighborhood of $u_\star$. We denote $V_i = \phi_i(U_i)$, which is an open neighborhood of $\phi_i(p_i) = 0$. If $x_n \in U_i$ at time $n$ for one*

$i \in \{0,\ldots,k-1\}$, *then there exists a unique control* $u_n \in D$ *near* $u_\star$ *that stabilizes the dynamical system at* $O(p_0)$ *provided that*

$$\pi_{E^u(p_{i+1})}\Big[\partial_u(f(p_i,u_\star))\Big] \tag{4.11}$$

*is invertible.* $\pi_{E^u(p_{i+1})}$ *denotes the projection onto the linear unstable subspace* $E^u(p_{i+1})$.

**Proof**

In order to control the system, we demand $x_{n+1}$, or equivalently, $z_{n+1} = \phi_{i+1}(x_{n+1})$, to be on the local stable manifold $W^s_{loc}(p_{i+1})$ for $i \in \{0,\ldots,k-1\}$, respectively on $W^s_{loc}(0)$, i.e.,

$$\begin{array}{rrcl} & z^{(2)}_{n+1} & = & \psi_{i+1}(z^{(1)}_{n+1}) \\ \Leftrightarrow & f^{(2)}_{i,i+1}(z_n,u_n) & = & \psi_{i+1}\left(f^{(1)}_{i,i+1}(z_n,u_n)\right). \end{array}$$

We define $F_i\colon \mathbb{R}^m \times \mathbb{R}^{m-s} \to \mathbb{R}^{m-s}$ by

$$F_i(z,u) = f^{(2)}_{i,i+1}(z,u) - \psi_{i+1}\left(f^{(1)}_{i,i+1}(z,u)\right).$$

Then

$$f^{(2)}_{i,i+1}(z,u) = \psi_{i+1}(f^{(1)}_{i,i+1}(z,u)) \quad \Leftrightarrow \quad F_i(z,u) = 0.$$

In particular, for $z = \phi_i(p_i) = 0$ and $u = u_\star$,

$$F(0,u_\star) = 0$$

holds true with $0 \in V_i, u_\star \in D$, where $V_i := \phi_i(U_i)$ is determined by an appropriate $U_i$ as stated in the theorem. Let us consider

$$\partial_u F_i(z,u)\Big|_{(0,u_\star)} \in \mathbb{R}^{(m-s)\times(m-s)}$$

where $|_{(0,u_\star)}$ means that one takes $z = 0$ and $u = u_\star$. If this matrix is invertible, then we are able to apply the Implicit Function Theorem in order to obtain the existence of the control $u_n$ which stabilizes the system. We compute

$$\begin{array}{rl} & \partial_u F_i(z,u)\Big|_{(0,u_\star)} \\ = & \partial_u\left(f^{(2)}_{i,i+1}(z,u) - \psi_{i+1}(f^{(1)}_{i,i+1}(z,u))\right)\Big|_{(0,u_\star)} \\ = & \partial_u\left(f^{(2)}_{i,i+1}(z,u)\right)\Big|_{(0,u_\star)} - \partial_u\left(\psi_{i+1}(f^{(1)}_{i,i+1}(z,u))\right)\Big|_{(0,u_\star)} \end{array}$$

Recall that $\psi_{i+1}(0) = 0$ and, in particular, $\mathrm{D}\psi_{i+1}(0) = 0$. We conclude

$$\begin{array}{rl} & \partial_u\left(\psi_{i+1}(f^{(1)}_{i,i+1}(z,u))\right)\Big|_{(0,u_\star)} \\ \overset{y:=f^{(1)}_{i,i+1}(z,u)}{=} & \partial_y\ \psi_{i+1}(y)\Big|_{y=0}\partial_u\left(f^{(1)}_{i,i+1}(z,u)\right)\Big|_{(0,u_\star)} \\ = & \mathrm{D}\,\psi_{i+1}(y)\Big|_{y=0}\partial_u\left(f^{(1)}_{i,i+1}(z,u)\right)\Big|_{(0,u_\star)} \\ = & 0 \end{array}$$

Therefore,

$$\partial_u F_i(z,u)\Big|_{(0,u_\star)} = \partial_u \left( f^{(2)}_{i,i+1}(z,u) \right) \Big|_{(0,u_\star)}.$$

Our goal is now to rewrite the derivative of $f^{(2)}_{i,i+1}$ with respect to $u$ in terms of the original map $f$ and the local coordinate transformations $\phi_i$. Let us recall the Definitions 4.2.1 and 4.2.5 in order to obtain the following.

$$\begin{aligned} & \partial_u \left( f^{(2)}_{i,i+1}(z,u) \right) \Big|_{(0,u_\star)} \\ = \quad & \partial_u \left[ \pi_{E^u(p_{i+1})} \left( (\phi_{i+1}(f(\phi_i^{-1}(z),u))) \Big|_{(0,u_\star)} \right) \right] \end{aligned}$$

Here, $\pi_{E^u(p_{i+1})} \colon \mathbb{R}^m \to E^u(p_{i+1})$ denotes the projection onto the $(m-s)$-dimensional linear unstable subspace $E^u(p_{i+1})$. The projection can be exchanged with the derviative such that we obtain

$$\begin{aligned} & \pi_{E^u(p_{i+1})} \left[ \partial_u \left( \phi_{i+1}(f(\phi_i^{-1}(z),u)) \right) \Big|_{(0,u_\star)} \right] \\ = \quad & \pi_{E^u(p_{i+1})} \left[ \partial_u \left( T^{-1}_{i+1} \left( f(T_i\, z + p_i, u) - p_{i+1} \right) \right) \Big|_{(0,u_\star)} \right] \\ = \quad & \pi_{E^u(p_{i+1})} \left[ \partial_u \left( T^{-1}_{i+1} \left( f(T_i\, z + p_i, u) \right) - T^{-1}_{i+1}\, p_{i+1} \right) \Big|_{(0,u_\star)} \right] \\ = \quad & \pi_{E^u(p_{i+1})} \left[ \partial_u \left( T^{-1}_{i+1} \left( f(T_i\, z + p_i, u) \right) \right) \Big|_{(0,u_\star)} \right. \\ & \left. - \partial_u \left( T^{-1}_{i+1}\, p_{i+1} \right) \Big|_{(0,u_\star)} \right] \\ \overset{\partial_u(T^{-1}_{i+1}\, p_{i+1})=0}{=} \quad & \pi_{E^u(p_{i+1})} \left[ \partial_u \left( T^{-1}_{i+1} \left( f(T_i\, z + p_i, u) \right) \right) \Big|_{(0,u_\star)} \right] \\ = \quad & \pi_{E^u(p_{i+1})} \left[ T^{-1}_{i+1} \left( \partial_u (f(T_i\, z + p_i, u)) \right) \Big|_{(0,u_\star)} \right]. \end{aligned}$$

Finally, we have shown that

$$\partial_u F_i(z,u)\Big|_{(0,u_\star)} = \pi_{E^u(p_{i+1})} \left[ T^{-1}_{i+1} \left( \partial_u (f(T_i\, z + p_i, u)) \right) \Big|_{(0,u_\star)} \right].$$

By assumption (4.11),

$$\pi_{E^u(p_{i+1})} \Big[ \partial_u (f(p_i, u_\star)) \Big]$$

is invertible. Thus, we conclude that $\partial_u F_i(z,u)\Big|_{(0,u_\star)}$ is invertible, which allows us to apply the Implicit Function Theorem. There exist open neighborhoods $V_i$ of $0 = \phi_i(p_i)$ and $D \subset \mathbb{R}^{m-s}$ of $u_\star$ and a unique mapping $G_i \colon V_i \to D$ such that

$$F_i(z, G_i(z)) = 0 \quad \forall\ z \in V_i.$$

Is $(z,u)$ a point with $F_i(z,u) = 0$, it follows that $u = G_i(z)$. Hence, we can find open neighborhoods $V_i$ and $D$ so that equation (4.10) has a unique solution $u_n$. ■

**Remark**

The proof of the above theorem relies on the Implicit Function Theorem and the Stable Manifold Theorem 1.2.3, which have also been used to prove the main theorem in Chapter 2. Since we use $W^s_{loc}$ instead of $W^s$ of the periodic orbit, this result is only a local one.

## 4.4 An application to the Hénon map

In Section 3.3, we have already demonstrated the stabilization strategy for hyperbolic saddle fixed points of the Hénon map. Since we are now able to stabilize systems at hyperbolic periodic orbits of period greater than one, we apply our method from the previous sections to a period-2 orbit of the uncontrolled Hénon map

$$f(x, y, a_\star) = (1.4 - x^2 + 0.3\, y, x).$$

Recall that the fixed points of the map are given by

$$x_{1,2} = \frac{1}{2}\,(b - 1 \pm \sqrt{(b-1)^2 + 4\,a}).$$

Now we compute the period-2 orbits for $f$. Here we use the knowledge about the fixed points in order to simplify the equation. We determine $(x, y)$ such that $f^2(x, y, a_\star) = (x, y)$, i.e.,

$$\begin{aligned} f^2(x, y, a_\star) &= f(a_\star - x^2 + b\,y, x) = (x, y) \\ \Leftrightarrow (a_\star - (a_\star - x^2 + b\,y)^2 + b\,x, a_\star - x^2 + b\,y) &= (x, y) \end{aligned}$$

Solving the second equation for $y$, we get

$$y = \frac{a_\star - x^2}{1 - b}.$$

Now we solve the first equation for $x$ and obtain

$$\begin{aligned} x &= a_\star - (a_\star - x^2 + \frac{b}{1-b}\,(a_\star - x^2))^2 + b\,x \\ \Leftrightarrow \quad 0 &= (a_\star - x^2)^2 + x\,(1-b)^3 - a_\star\,(1-b)^2 \\ \overset{\text{fixed pt. eq.}}{\Leftrightarrow} \quad 0 &= (x^2 - (1-b)\,x - a_\star + (1-b)^2)\,(x^2 + x\,(1-b) - a_\star) \end{aligned}$$

The term $(x^2 + x\,(1-b) - a_\star) = 0$ is the part which comes from solving the fixed point equation. Therefore, we only need to solve

$$0 = x^2 - (1-b)\,x - a_\star + (1-b)^2$$

for $x$ to obtain the periodic orbits of period 2. There exists exactly one period-2 orbit for the Hénon map as long as $4\,a > 3\,(1-b)^2$, which is true for $a = a_\star = 1.4$ and $b = 0.3$. The $x$-coordinates of the period-2 orbit are given by

$$x_{P1,P2} = \frac{1}{2}\,(1 - b \pm \sqrt{4\,a_\star - 3\,(1-b)^2}).$$

If we denote the periodic orbit by $\{(x_{P1}, y_{P1}), (x_{P2}, y_{P2})\}$, it follows from the evolution equation that the orbit lies along the line $x + y = 1 - b$ and moreover, $x_{P1} + y_{P1} = x_{P2} + y_{P2} = x_{P1} + x_{P2} = y_{P1} + y_{P2} = 1 - b$. Thus, we can easily write down the period-2 orbit:

$$\Big\{(\frac{1}{2}\,(1-b+\sqrt{4\,a_\star-3\,(1-b)^2}),\frac{1}{2}\,(1-b-\sqrt{4\,a_\star-3\,(1-b)^2})),$$
$$(\frac{1}{2}\,(1-b-\sqrt{4\,a_\star-3\,(1-b)^2}),\frac{1}{2}\,(1-b+\sqrt{4\,a_\star-3\,(1-b)^2}))\Big\}.$$

We check that the periodic orbit we have found, is a hyperbolic saddle for $f^2$ and embedded in the strange attractor $\mathcal{A}$, cf. Figure 4.2.

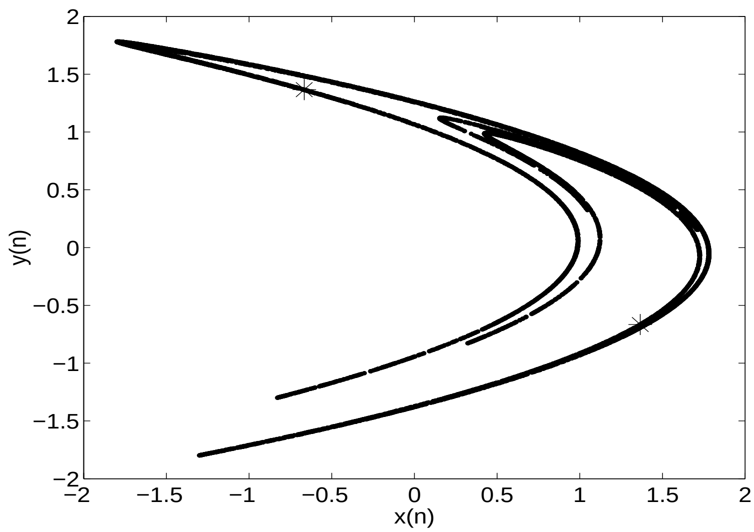


Figure 4.2: The strange attractor $\mathcal{A}$ and the hyperbolic saddle period-2 orbit of the Hénon map.

Using the specific parameter values $a = a_\star = 1.4$ and $b = 0.3$, we obtain approximately

$$(x_{P1}, y_{P1}) = (1.3661, -0.6661)$$

and

$$(x_{P2}, y_{P2}) = (-0.6661, 1.3661).$$

Since we already introduced all the extensive computations for the fixed point case of the Hénon map in Section 3.2, we will not go into detail here. We shift $(x_{Pi}, y_{Pi})$ for $i = 1, 2$ to $(0, 0)$ by defining the coordinate charts $\phi_i$. The eigenvalues of $\mathrm{D}f^2(x_{P1}, y_{P1})$ are approximately $\lambda_s = 0.1758$ and $\lambda_u = -3.2158$. The two transformation matrices $T_1$ and $T_2$ are given by the corresponding eigenvectors

$$v_s^1 = \begin{pmatrix} -0.0454 \\ -0.9990 \end{pmatrix}, v_u^1 = \begin{pmatrix} -0.7896 \\ -0.6136 \end{pmatrix}, v_s^2 = \begin{pmatrix} 0.0929 \\ -0.9957 \end{pmatrix}, v_u^2 = \begin{pmatrix} -0.9351 \\ 0.3543 \end{pmatrix},$$

where $v_s^1, v_u^1$ are the eigenvectors of $\mathrm{D}f^2(x_{P1}, y_{P1})$ and $v_s^2, v_u^2$ are those of $\mathrm{D}f^2(x_{P2}, y_{P2})$. Thus, we can write down the transformations $\phi_1$ and $\phi_2$ as well as the functions $f_{1,2}$ and $f_{2,1}$ according to Sections 2 and 3. The coefficients for the two local stable manifolds can be computed by Maple. Approximately, they are given as follows:

$$s_{1_2} = 0.01840966191 \text{ and } s_{1_3} = 0.003068471051$$

for the manifold at the first periodic point $(x_{P1}, y_{P1})$ and

$$s_{2_2} = 0.04312614006 \text{ and } s_{2_3} = -0.01655869774$$

for $W^s_{loc}(x_{P2}, y_{P2})$. Note that in the periodic set-up, we use the same algorithm in Maple as for the fixed point case. We just replace the map $f$ with $f^2$.

Now the control algorithm can be implemented as it was described in Section 2.3. Let $U_1$ be an open neighborhood of $(x_{P1}, y_{P1})$ and $U_2$ be an open neighborhood of $(x_{P2}, y_{P2})$ such that $U_1 \cap U_2 \neq \emptyset$. For example, one can compute

$$d := \mathrm{dist}((x_{P1}, y_{p1}), (x_{P2}, y_{P2}))$$

and take $r := d - \delta$ for some small positive $\delta$. Then $U_1 := \mathrm{B}_r(x_{P1}, y_{P1})$ and $U_2 := \mathrm{B}_r(x_{P2}, y_{P2})$ are balls with radius $r$ centered at the corresponding periodic point such that $U_1 \cap U_2 \neq \emptyset$. A given orbit $O(x_0, y_0)$ with either $(x_n, y_n) \in U_1$ or $(x_n, y_n) \in U_2$ at time $n$ should be controlled by forcing it onto one of the local stable manifolds $W^s_{loc}(x_{Pi}, y_{Pi})$, $i = 1, 2$. We need to make a distinction of the two cases where $(x_n, y_n) \in U_1$ or $(x_n, y_n) \in U_2$. Then, depending on the fact in which neighborhood $(x_n, y_n)$ lies, this point is transformed into local coordinates by the corresponding transformation $\phi_i$. Afterwards, the control value $a_n$ is computed according to (4.10). In Section 3.3, using Maple, we were able to compute the control explicitly, although the control law was rather complicated. Unfortunately, it is not possible to derive such an explicit formula for the period-2 case. We have to use Newton's method ('fsolve' in Matlab) in order to obtain the control value $a_n$. Using $a_n$, we compute the next iterate $(x_{n+1}, y_{n+1})$. Then the control is switched off again, i.e., $a_K = a_\star$ for all $K > n$, until the orbit leaves an $\varepsilon$-neighborhood of $W^s_{loc}(x_{Pi}, y_{Pi})$, $i = 1, 2$.

The difference between the control algorithm in Section 2.3 and the one presented here is that we have to distinguish between the two neighborhoods $U_1$ and $U_2$ and choose the corresponding coordinate charts $\phi_i$. The stabilization algorithm for the period-2 case is successfully implemented as we will illustrate now.

We consider the initial condition $(x_0, y_0) = (0, 0)$ within the trapping region $R$ of the strange attractor $\mathcal{A}$.

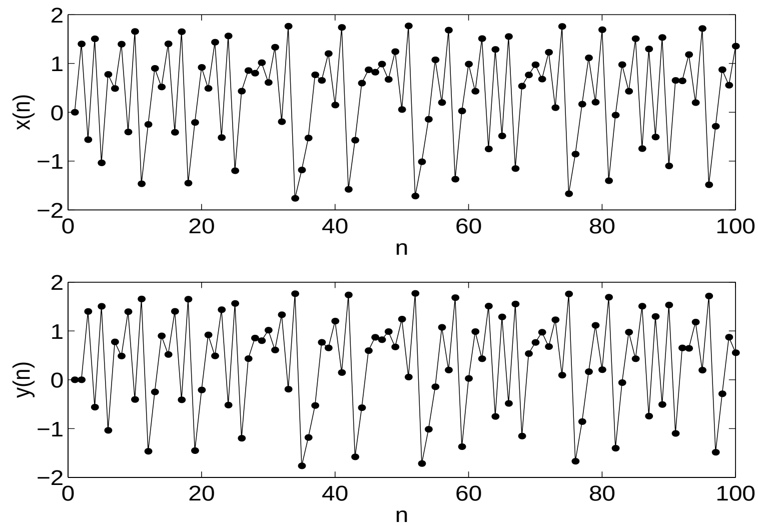


Figure 4.3: The uncontrolled Hénon map with $(x_0, y_0) = (0, 0)$.

Figure 4.3 shows the iterations of $(x_0, y_0) = (0, 0)$ under the original uncontrolled map. Our goal is to compare our nonlinear method with the OGY-method for periodic orbits. Note that the algorithm for the OGY-method was implemented using the procedure described in [RGOD92]. The next two figures show both algorithms applied to the same initial condition $(x_0, y_0) = (0, 0)$.

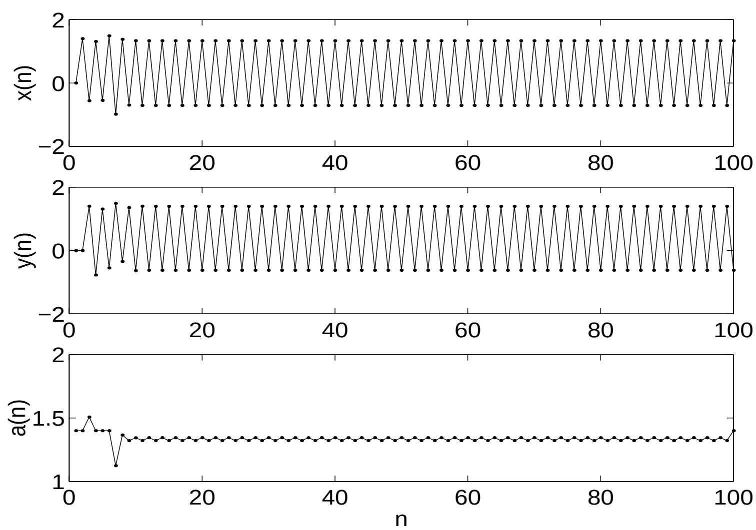


Figure 4.4: The nonlinearly controlled Hénon map with $(x_0, y_0) = (0, 0)$.

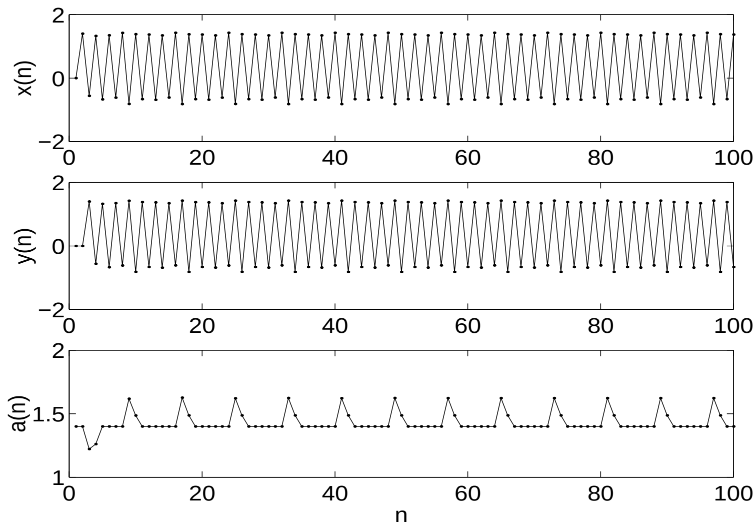


Figure 4.5: The OGY-controlled Hénon map with $(x_0, y_0) = (0, 0)$.

Again, we have divided each Figure in three parts, showing the two state variables and the control. As can be seen in Figures 4.4 and 4.5, both methods work successfully. The $x$-coordinates are stabilized onto $x_{P1}$ and $x_{P2}$ and the $y$-coordinates are stabilized onto $y_{P1}$ and $y_{P2}$. Nevertheless, there are some differences in the way the stabilization is achieved. On a first glance, both methods force the orbit $O(0, 0)$ on the periodic orbit. But in comparison, our method is more precise in the following sense: If one compares the distances $|x_n - x_{Pi}|$ and $|y_n - y_{Pi}|$ for $i = 1, 2$, then one realizes the following. The nonlinearly controlled orbit $O(x_0, y_0)$ has almost no difference in the coordinates to the period-2 orbit. In contrast, the OGY controlled orbit differs more from the coordinates of the periodic orbit. Thus, the nonlinear stabilization is more precise and even. Furthermore, the orbit controlled with the OGY-method leaves the coordinates of the periodic orbit quite often, cf. Figure 4.5. As a result, the OGY-method has to control much more often with larger controls than our method. Considering the last row of the Figures 4.4 and 4.5, one can see the values of the control $a_n$. In the OGY case, there are large peaks within about every five iterations whereas in the nonlinear control case, the control is basically applied at the beginning and then differs only in tiny amounts from $a_\star = 1.4$. Thus, only slight control steps are necessary to keep the orbit near the period-2 orbit if one uses our stabilization procedure.

In conclusion, we emphasize that our stabilization method is more global and effective in case of periodic orbits than the OGY-method. The stabilization is uniform and less control steps are needed. However, the new method demands more computational effort, and thus, it is more costly than the OGY-method.

# Chapter 5

# A second application: The Ikeda map

In this chapter, we illustrate our nonlinear stabilization method, which has been developed in Chapter 4. So far, we applied the feedback control to the Hénon map, that is somewhat artificial. Therefore, our remaining task is to apply the stabilization procedure to a dynamical system that has been derived from a realistic model. We choose the so-called Ikeda map which represents a nonlinear optical ring cavity. In the first section, we give an overview of the map and its dynamics. Section 2 is a numerical illustration of the control method. Here, we stabilize the system at a hyperbolic periodic orbit of period 3.

## 5.1 Dynamics of the Ikeda map

Generally speaking, a laser is an optical oscillator where coherent radiation is generated by stimulated emission of radiation from an atomic medium contained in the sample cell. In our case, the atomic medium (electrons, atoms or molecules) is a two-level atom, e.g. $CO_2$. It has been shown in numerous examples, that at high input power, the output of a laser system can behave in an irregular manner, cf. [Mil91]. Numerical and theoretical examination have shown that this irregular behavior illustrates aspects of chaos theory such as period doubling bifurcation and strange attractors, cf. for example [HJM85, Ott93, Sch99].

We consider the one-dimensional complex map $f\colon \mathbb{C} \to \mathbb{C}$, which is given by

$$f(z) = a + R\,z\,\exp\left(i\,(\varphi - \frac{\rho}{1+|z|^2})\right). \tag{5.1}$$

The map $f$ defines a discrete dynamical system

$$z_{n+1} = f(z_n) \tag{5.2}$$

that describes the dynamics of a simplified laser. It was introduced by K. Ikeda in 1979 [Ike79]. In the setting of [Ike79], equation (5.2) describes a nonlinear optical

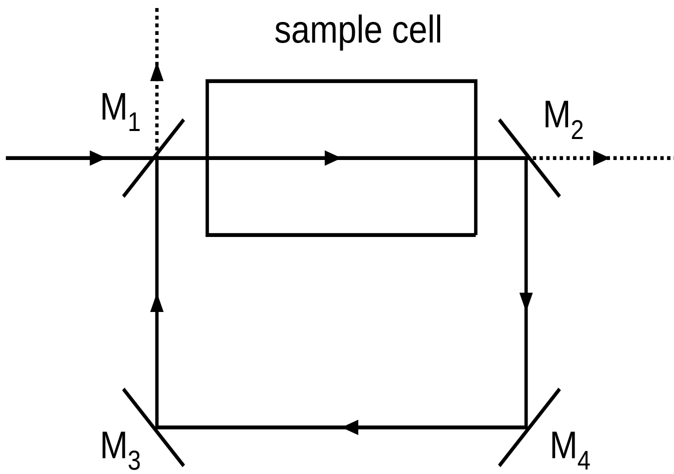


Figure 5.1: The nonlinear ring cavity which is described by the Ikeda map (5.1).

ring cavity. The schematic diagram in Figure 5.1 depicts situation.
The Ikeda map represents the evolution of the electric field (the light) inside the ring cavity, i.e., the laser. The ring cavity itself consists of four mirrors $M_1$, $M_2$, $M_3$ and $M_4$, where $M_3$ and $M_4$ have reflectivity 1, whereas $M_1$ and $M_2$ have reflectivity $R$. Moreover, the sample cell inside the cavity contains the atomic medium that has two levels. One can view the system given by (5.1) as follows: A string of light pulses with amplitude $a$ enters at $M_1$ and is partially transmitted. The light now enters the sample cell where stimulated emission and absorption take place. The transmitted light hits $M_2$, and is then partially transmitted to $M_3$ and $M_4$. Thus, we obtain a ring resonator. Let the state $z_n \in \mathbb{C}$ be given. Then $|z_n|$ is the amplitude and the angle of $z_n$ is the phase of the $n$-th light pulse just to the right of $M_1$. The amplitude $a$ is the amplitude of the so-called pumping light on the left of the ring cavity. The parameter $\varphi$ is the phase shift experienced by the pulse in the vacuum region and the term $-\rho/(1+|z_n|^2)$ is the phase shift in the nonlinear medium, which is caused by the stimulated emission and absorption in the two-level atom. Thus, the complex amplitude of $z_{n+1}$ of the electric field at the $(n+1)$st cavity pass can be understood as function of the electric field amplitude at the $n$-th cavity pass.

**Remark**
The one-dimensional complex system given by (5.1) can be written as a two-dimensional real system which reads

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a + R\,(x_n\,\cos(\theta) - y_n\,\sin(\theta)) \\ R\,(x_n\,\sin(\theta) + y_n\,\cos(\theta)) \end{pmatrix}, \tag{5.3}$$

where $\theta := \varphi - \rho/(1 + x_n^2 + y_n^2)$.

In [Ike79], a detailed derivation of the system (5.2) is given and it is stated that the transmitted light exhibits chaotic behavior. Hammel et al. [HJM85] put the emphasis on a detailed description of the dynamics of the map. They point out that the map $f\colon \mathbb{C} \to \mathbb{C}$ is invertible and the inverse map is

$$f^{-1} = \frac{1}{R}\,(z-a)\,\exp\left(-i\,\left(\varphi - \frac{\rho\,R}{R^2 + |z-a|^2}\right)\right).$$

Moreover, the Ikeda map is area contracting since $\det(\mathrm{D}f(x,y)) = R^2 < 1$ for all $(x,y)$. The Jacobian $\mathrm{D}f(x,y)$ is given by

$$\mathrm{D}f(x,y) = \frac{R}{(1+x^2+y^2)^2}\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned}
A &:= (1+x^2+y^2)^2\,\cos(\theta) - 2\,\rho\,x^2\,\sin(\theta) - 2\,\rho\,x\,y\,\cos(\theta),\\
B &:= -(1+x^2+y^2)^2\,\sin(\theta) - 2\,\rho\,y^2\,\cos(\theta) - 2\,\rho\,x\,y\,\sin(\theta),\\
C &:= (1+x^2+y^2)^2\,\sin(\theta) + 2\,\rho\,x^2\,\cos(\theta) - 2\,\rho\,x\,y\,\sin(\theta),\\
D &:= (1+x^2+y^2)^2\,\cos(\theta) - 2\,\rho\,y^2\,\sin(\theta) + 2\,\rho\,x\,y\,\cos(\theta).
\end{aligned}$$

Another important observation in [HJM85] is the existence of a positively invariant disk $\mathrm{B}_r(a,0)$ in the complex plane centered at $(a,0)$ with radius $r := a\,R/(1-R)$ such that all points in the complex plane are mapped into $\mathrm{B}_r(a,0)$. Hence, by Definition 1.3.3, the disk $\mathrm{B}_r(a,0)$ is a trapping region and it turns out that there exists an attracting set $\mathcal{A}$ defined by

$$\mathcal{A} := \bigcap_{i=1}^{\infty} f^i(\mathrm{B}_r(a,0)).$$

Because the map $f$ is area contracting, the area of the image under $f$ of the disk is $R^2$ times the area of the disk itself. Since $R < 1$, the attracting set $\mathcal{A}$ has a fractal dimension less than two. A figure of the strange attracting set $\mathcal{A}$ can be seen in Figure 5.2.

## 5.2 Stabilization in a period-3 orbit

The stabilization of the Ikeda map is achieved by changes of the amplitude of the light pulses entering the optical ring cavity. Thus, we use $a$ as our time dependent feedback control parameter. The nominal value is $a_n = a_\star = 1$. Moreover, let $R = 0.9, \varphi = 0.4$ and $\rho = 6$, cf. [SO95]. The aim is to stabilize the Ikeda map at a higher periodic orbit that is hyperbolic with a saddle structure.

We choose to find periodic orbits of period 3. Due to the nonlinearities in the system given by (5.1), we can not expect to find periodic orbits of period 3

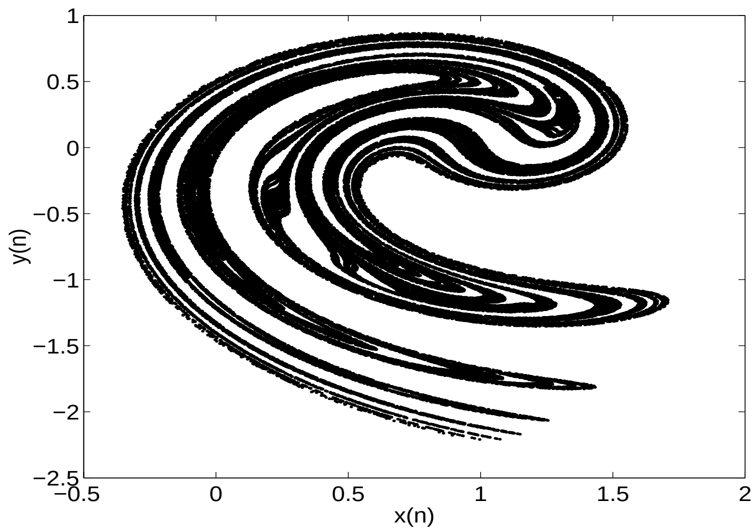


Figure 5.2: The strange attractor $\mathcal{A}$ of the Ikeda map with $a = 1, R = 0.9, \varphi = 0.4$ and $\rho = 6$.

analytically. Thus, one could use one of the numerous dynamical system programs such as Dynamics [NY94]. Another possibility is to write a Matlab program in the following manner, which is what we have done.

```
% size of the grid is d
d=0.1;

for j=-2:d:2
  for i=-3:d:3
    x=[i;j];
    fsolve('f^3(x,y)-(x,y)',[x;y])
  end
end
```

It turns out that a hyperbolic periodic orbit of period $k = 3$ is approximately given by

$$\begin{aligned} & \{(x_{P1}, y_{P1}), (x_{P2}, y_{P2}), (x_{P3}, y_{P3})\} \\ = \quad & \{(0.085797, -0.88323), (0.77797, 0.76717), (1.014, -0.98324)\}, \end{aligned}$$

cf. also [SO95]. The period-3 orbit is not stable. The strange attractor in Figure 5.2 has been obtained in the following way. We compute the first $10^5$ iterations of the Ikeda map with initial condition $(x_0, y_0) = (x_{P1}, y_{P1})$. After about 13 iterations, the orbit $O(x_{P1}, y_{P1})$ leaves the hyperbolic periodic orbit of period $3$ which is embedded in the strange attracting set $\mathcal{A}$, that is shown in Figure 5.2. Our goal is the stabilization of the Ikeda map at this periodic orbit.

As in Chapter 4, we compute the Jacobians $\mathrm{D}f^3(x_{Pi}, y_{Pi})$ for $i = 1, 2, 3$ with the corresponding eigenvalues and generalized eigenvectors. We obtain the eigenvalues $\lambda_s = -0.0479$ and $\lambda_u = -11.0979$. The transformation matrices $T_i$ are given by

$$T_1 := \begin{pmatrix} 0.8440 & -0.4312 \\ 0.5364 & 0.9022 \end{pmatrix}, \; T_2 := \begin{pmatrix} 0.8723 & -0.9999 \\ -0.4890 & 0.0107 \end{pmatrix}$$

and

$$T_3 := \begin{pmatrix} -0.4215 & -0.8980 \\ -0.9068 & 0.4401 \end{pmatrix}.$$

Again we use the Maple program to compute the approximations of the stable manifolds. We obtain

$$\psi_1(u) = 0.5437\,u^2 + 0.2358\,u^3, \; \psi_2(u) = 0.9117\,u^2 - 1.5458\,u^3$$

and

$$\psi_3(u) = 0.3277\,u^2 - 0.0347\,u^3.$$

**Remark**

This time, the coefficients of $\psi_1$, $\psi_2$ and $\psi_3$ could not be obtained as directly as in Section 4.4. Instead, we have to expand the equations from which the coefficients are computed into a Taylor series. This is due to the fact that we do not have a polynomial evolution equation, but one with sine and cosine terms. These terms have to be approximated by series in order to make a comparison of two polynomials to find the coefficients.

We start with the control of the orbit which is generated by the initial condition $(x_0, y_0) = (0, 0)$ where $\varepsilon = 0.01$ and $a_{max} = 1$. The uncontrolled orbit $O(0, 0)$ can be seen in Figure 5.3. The successful stabilization of the orbit with initial condition $(x_0, y_0) = (0, 0)$ is shown in Figure 5.4. It takes quite a long time for the control to reach its goal, but at time $n = 46$, we can control. If we take a smaller bound on the control, e.g., $a_{max} = 0.05$, then stabilization is impossible. In this case, the values of the feedback control $a_n$ are such that $|a_n - 1| > 0.05$. The controls that would be needed for stabilization are too large. Therefore, we can not stabilize the Ikeda map, because we have chosen the bound on the control too small.

Now we choose an initial condition which is close to one of the periodic points, e.g., $(x_0, y_0) = (0.08, -0.9) \in U(x_{P1}, y_{P1})$. As we have seen in Figure 5.2, the period-3 orbit generates the picture of the strange attracting set. It is shown below that it takes only small changes in the control $a_n$ in order to stabilize the system. In contrast to the control in Figure 5.4, where stabilization was achieved with quite large control values, the nonlinear stabilization is much easier achieved here, since

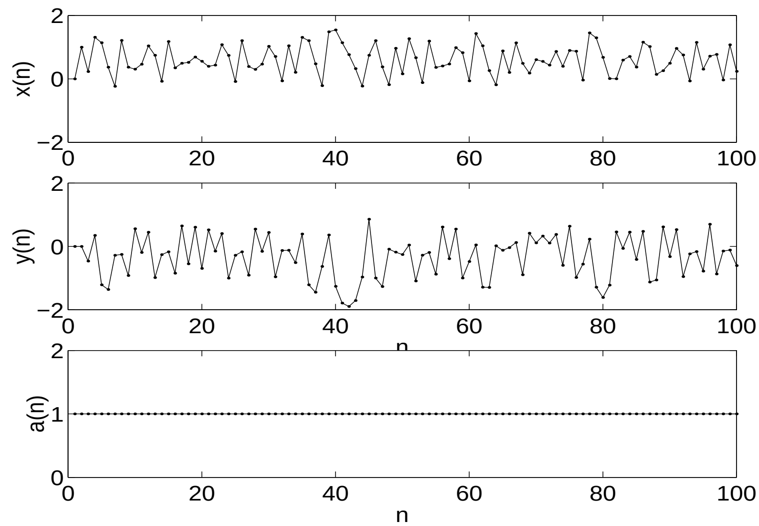


Figure 5.3: The uncontrolled Ikeda map with $(x_0, y_0) = (0, 0)$.

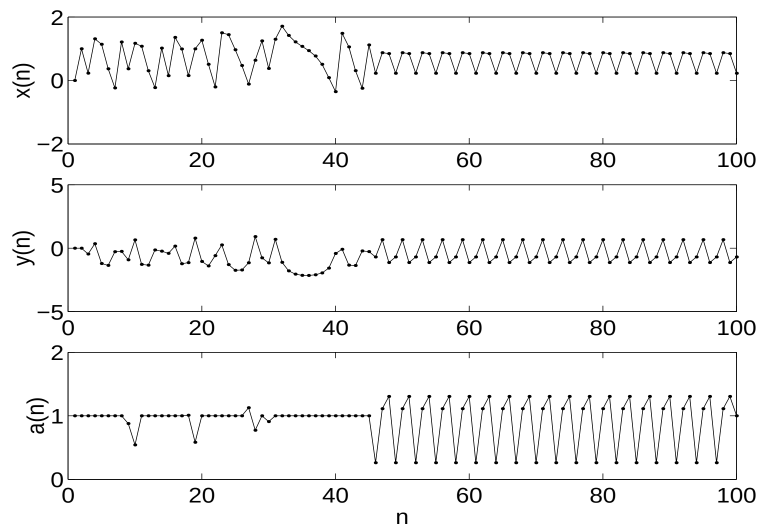


Figure 5.4: The nonlinearly controlled Ikeda map with $(x_0, y_0) = (0, 0)$ and $\varepsilon = 0.01,\ a_{max} = 1$.

the initial condition is close to one of the periodic points.
This orbit is stabilizable and can be kept close to the period-3 orbit, cf. Figure 5.5. If one reduces the maximal control value to $a_{max} = 0.05$, then control is no more possible, cf. Figure 5.6. The reason for this is that larger control values are needed than those admitted by the inequality $|a_n - 1| < 0.05$. But since we do not allow

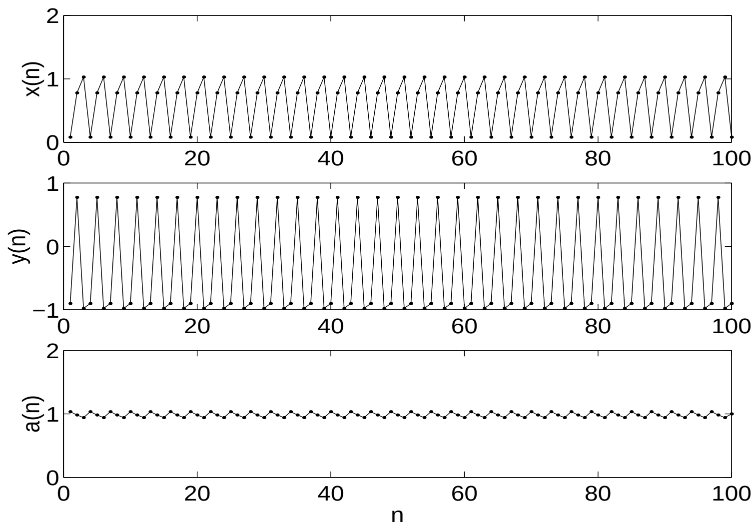


Figure 5.5: The nonlinearly controlled Ikeda map with $(x_0, y_0) = (0.08, -0.9)$ and $\varepsilon = 0.01,\ a_{max} = 0.06$.

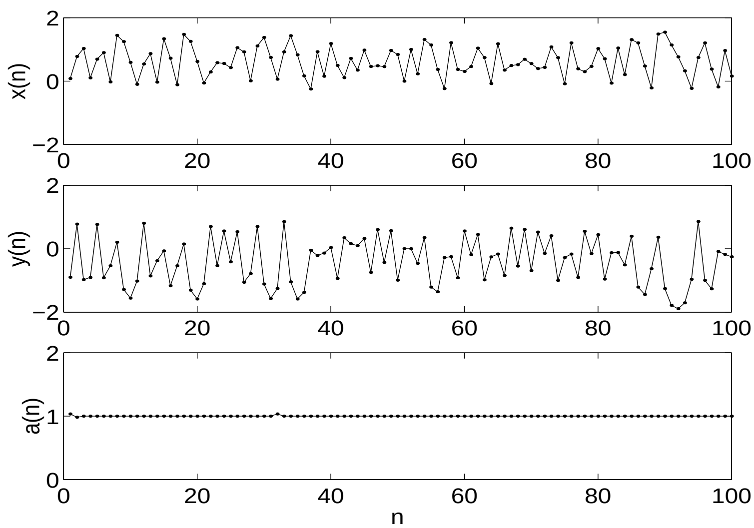


Figure 5.6: The nonlinearly controlled Ikeda map with $(x_0, y_0) = (0.08, -0.9)$ and $\varepsilon = 0.01,\ a_{max} = 0.05$.

the control to leave this bound, the stabilization of the system fails if $a_{max}$ is too small. Nevertheless, for $a_{max} = 0.06$, nonlinear stabilization is possible.

In summary, our method is able to stabilize the Ikeda map.

# Bibliography


[ASY97] K. T. Alligood, T. D. Sauer, and J. A. Yorke. *Chaos - An Introduction to Dynamical Systems*. Springer Verlag, New York, 1997.

[CC96] D. J. Christini and J. J. Collins. Using chaos control and tracking to suppress a pathological nonchaotic rythm in a cardiac model. *Physical Review E*, **53**:49–52, 1996.

[Dev86] R. Devaney. *An Introduction to Chaotic Dynamical Systems*. The Benjamin/Cummings Publ. Co. Inc., Menlo Park, 1986.

[DRS90] W. L. Ditto, S. N. Rauseo, and M. L. Spano. Experimental control of chaos. *Physical Review Letters*, **65**:3211–3214, 1990.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*. Springer Verlag, New York, 1983.

[GOY88] C. Grebogi, E. Ott, and J. A. Yorke. Unstable periodic orbits and the dimensions of multifractal chaotic attractors. *Physical Review A*, **37**:1711–1724, 1988.

[Hal88] J. K. Hale. *Asymptotic behavior of dissipative systems*. AMS, Providence, 1988.

[HDM94] B. Hübinger, R. Doerner, and W. Martienssen. Controlling chaos experimentally in systems exhibiting large effective lyapunov exponents. *Physical Review E*, **50**:932–948, 1994.

[Hén76] M. Hénon. A two-dimensional mapping with a strange attractor. *Communications in Mathematical Physics*, **50**:69–77, 1976.

[HJM85] S. M. Hammel, C. K. R. T. Jones, and J. V. Moloney. Global dynamical behavior of the optical field in a ring cavity. *Journal of Optical Society Am. B*, **2**:552–564, 1985.

[HL98] A. Hübler and E. Lüscher. Resonant stimulation and control of nonlinear oscillators. *Naturwissenschaften*, **76**:67–69, 1998.

[Ike79] K. Ikeda. Multiple-valued stationary state and its instability of the transmitted light by a ring cavity system. *Optics Communications*, **30**:257–261, 1979.

[Isi89] A. Isidori. *Nonlinear Control Systems*. Springer Verlag, New York, 1989.

[JMTV97] K. Judd, A. Mees, K. L. Teo, and T. L. Vincent. *Control and Chaos*. Birkhäuser Verlag, Boston, 1997.

[KH97] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1997.

[Lor63] E. N. Lorenz. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, **20**:130–141, 1963.

[Mil85] J. Milnor. On the concept of attractor. *Communications in Mathematical Physics*, **99**:177–195, 1985.

[Mil91] D. L. Mills. *Nonlinear Optics: Basic Concepts*. Springer Verlag, New York, 1991.

[MS82] J. Macki and A. Strauss. *Introduction to Optimal Control Theory*. Springer Verlag, New York, 1982.

[ND92] G. Nitsche and U. Dressler. Controlling chaotic dynamical systems using time delay coordinates. *Physica D*, **58**:153–164, 1992.

[NS90] H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, New York, 1990.

[NY94] H. E. Nusse and J. A. Yorke. *Dynamics: Numerical Explorations*. Springer Verlag, New York, 1994.

[OGY90a] E. Ott, C. Grebogi, and J. A. Yorke. Controlling chaos. *Physical Review Letters*, **64**:1196–1199, 1990.

[OGY90b] E. Ott, C. Grebogi, and J. A. Yorke. Controlling chaotic dynamical systems. *CHAOS-Soviet American perspectives on nonlinear science*, **64**:153–172, 1990. ed. D. K. Campbell.

[Ott93] E. Ott. *Chaos in Dynamical Systems*. Cambridge University Press, Cambridge, 1993.

[PMT94] M. Paskota, A. I. Mees, and K. L. Theo. Stabilizing higher periodic orbits. *International Journal of Bifurcation and Chaos*, **4**:457–460, 1994.

[RGOD92] F. J. Romeiras, C. Grebogi, E. Ott, and W. P. Dayawansa. Controlling chaotic dynamical systems. *Physica D*, **58**:165–192, 1992.

[Rob95] C. Robinson. *Dynamical Systems: Stability, Symbolic Dynamics and Chaos*. CRC Press, Boca Raton, 1995.

[S+94] S. J. Schiff et al. Controlling chaos in the brain. *Nature*, **370**:615–620, 1994.

[S+97] A. Schenck zu Schweinsberg et al. Quasicontinuous control of a bronze ribbon experiment using time-delay coordinates. *Physical Review E*, **55**:2145–2158, 1997.

[Sch99] H. G. Schuster. *Handbook of Chaos Control*. Wiley-VCH, Weinheim, 1999.

[SGOY93] T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke. Using small perturbations to control chaos. *Nature*, **363**:411–417, 1993.

[SO95] P. So and E. Ott. Controlling chaos using time delay coordinates. *Physical Review E*, **51**:2955–2962, 1995.

[Son98] E. Sontag. *Mathematical Control Theory*. Springer Verlag, New York, 2nd edition, 1998.

[Voc98] T. Vocke. *Control of Chaos*. Diplomarbeit, Bremen, 1998.

[Wig88] S. Wiggins. *Global Bifurcations and Chaos: Analytical Methods*. Springer Verlag, New York, 1988.

[Wig90] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer Verlag, New York, 1990.

[WJ96] M. Watanabe and R. F. Gilmour Jr. Strategy for control of complex low-dimensional dynamics in cardiac tissue. *Journal of Mathematical Biology*, **35**:73–87, 1996.

# Index