Nonlinear Analysis: Hybrid Systems 44 (2022) 101173

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Compositional construction of abstractions for infinite networks of discrete-time switched systems ☆

Maryam Sharifi [a,*], Abdalla Swikir [b], Navid Noroozi [c], Majid Zamani [c,d]

[a] School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Stockholm, Sweden
[b] Department of Electrical and Computer Engineering, Technical University of Munich, Germany
[c] Institute for Informatics, LMU Munich, Germany
[d] Computer Science Department, University of Colorado Boulder, USA

ARTICLE INFO

Article history:
Received 20 January 2021
Received in revised form 30 October 2021
Accepted 28 January 2022
Available online 8 February 2022

Keywords:
Compositionality
Continuous abstractions
Infinite networks
Small-gain theorem
Switched systems

ABSTRACT

In this paper, we develop a compositional scheme for the construction of continuous abstractions for networks of *infinitely* many discrete-time switched systems. In particular, the constructed abstractions are themselves also continuous-space systems with potentially lower dimensions, which can be used as replacements of the original (also known as concrete) systems in the controller design process. Having designed a controller for the abstract system, it is refined to a more detailed one for the concrete system. We use the notion of so-called simulation functions to quantify the mismatch between the original system and its approximation. Each subsystem in the concrete network and its corresponding one in the abstract network are related through a notion of *local* simulation functions. We show that if the local simulation functions satisfy a spectral small-gain condition, then the aggregation of the individual simulation functions provides an overall simulation function quantifying the error between the overall abstract network and the concrete one. In addition, we show that our methodology results in a *scale-free* compositional approach for any finite-but-arbitrarily large networks obtained from truncation of an infinite network. We provide a systematic approach to construct local abstractions and simulation functions for networks of linear switched systems. In this case, the conditions are expressed in terms of linear matrix inequalities that can be efficiently computed. We illustrate the effectiveness of our approach through an application to AC islanded microgrids.

1. Introduction

Recent technological advances in sensing, computation, and data management have enabled us to develop smart networked systems providing more *autonomy* and *flexibility*. Smart grids, swarm robotics, connected automated vehicles and smart manufacturing are just a few examples of such emerging smart networked systems, in which a large number of dispersed agents interact and communicate with each other to achieve a common objective. The size and the structure of such networks can be arbitrarily large, time-varying or even unknown, and agents can be constantly plugged into and out

☆ The work of N. Noroozi is supported by the DFG, Germany through the grant WI 1458/16-1. The work of M. Zamani is supported in part by the H2020 ERC starting grant AutoCPS (grant agreement No. 804639) and the DFG, Germany through the grant ZA 873/4-1.
* Corresponding author.
E-mail addresses: msharifi@kth.se (M. Sharifi), abdalla.swikir@tum.de (A. Swikir), navid.noroozi@lmu.de (N. Noroozi), majid.zamani@colorado.edu (M. Zamani).

https://doi.org/10.1016/j.nahs.2022.101173
1751-570X/

from the network. Emerging control networks necessitate also *sophisticated* control objectives, which go beyond standard goals pursued in classical control theory. For instance, a sophisticated objective is to control connected autonomous vehicles merging at a traffic intersection while ensuring safety and fuel economy constraints.

The complexity of control objectives, the large number of participating agents, as well as safety concerns call for automated and provably correct techniques to verify or synthesize controllers for the emerging applications of control systems. A promising methodology to address the above issues is achieved by a careful integration of concepts from control theory (e.g. Lyapunov methods and small-gain theory) and those of computer science (e.g. formal methods and assume-guarantee rules) [1,2]. Discrete abstractions (a.k.a. symbolic models) is one particular technique to provide automated synthesis of correct-by-design controllers for concrete systems. In this approach, controller synthesis problems can be algorithmically solved over finite abstractions of concrete systems by resorting to automata-theoretic approaches [3]. Then, the constructed controllers can be refined back to the original systems based on some behavioral relations between original systems and their finite abstractions such as approximate alternating simulation relations [4] or feedback refinement relations [5].

The computational complexity of constructing finite abstractions of the concrete systems makes the practical applicability of these methods considerably challenging. Hence, applying such approaches to large-scale systems is not feasible at all. An appropriate technique to overcome this challenge is to introduce a pre-processing step by constructing so-called *continuous* abstractions. In that way, a continuous-space system, but possibly with a *lower* dimension, is obtained as a substitute of the concrete system [6–9]. We note that the applicability of continuous abstractions is *not* limited to the context of symbolic controllers. In fact, they can be used in other hierarchical control approaches in the lower layers, where a simplified model of the system is used for controller design purposes.

For large-scale networks, it is often more useful to maintain the structure (i.e. topology) of the network while abstractions are constructed. In that way, corresponding to each participating subsystem of the network, a continuous abstraction is constructed individually. Therefore, the complexity of synthesizing continuous abstractions of large-scale systems is managed in an efficient way. The methodology by which an abstraction for the overall network is achieved via the interconnection of the individual abstractions is called a compositional approach [10–12]. In order to guarantee that the aggregation of the individual abstractions provides an abstraction for the overall network, the interaction between subsystems should be weak enough, which can be technically described by a small-gain condition [10–15].

Small-gain type conditions are intrinsically dependent on the size of the network. Hence, one can show that the satisfaction of compositionality conditions dramatically degrades as the number of subsystems increases and may not be valid anymore, see [11, Remark 6.1]. Inspired by recent advances in the literature on the stability analysis of dynamical networks, e.g. [16–21], we address the scalability issue using an over-approximation of a finite-but-large network with a network composed of *infinitely* many subsystems. We call such an aggregated system an infinite network. This treatment leads to an infinite-dimensional system and calls for a more rigorous and detailed setting. It is widely accepted that an infinite network captures the essence of its corresponding finite network; see e.g. a vehicle platooning application in [22].

We adapt the notion of simulation functions [6] to the case of *infinite*-dimensional switched systems. The existence of a simulation function ensures that the error between the output trajectory of the abstract system and that of the concrete system is bounded in a certain sense (cf. Definition 3). By exploiting a compositionality approach, we assign an individual simulation function to each subsystem and construct the corresponding local abstraction accordingly. Then we aggregate them to construct an abstraction for the overall network. We show that if a certain small-gain condition, recently developed in [17], is satisfied, then the aggregation yields a continuous abstraction for the overall concrete network. Particularly, for linear networks, our conditions are expressed in terms of linear matrix inequalities, where we *explicitly* construct the individual abstractions as well as the controller refinement formulation.

Motivated by the scale-dependency issue in the classic compositionality methods, in this paper, a *scale-free* compositional approach for the construction of continuous abstractions for *arbitrarily* large-scale networks of discrete-time switched systems is provided. We elucidate the scale-free property of our approach by truncating the infinite network to a finite-but-arbitrary large network and show that the compositional abstraction results are preserved under any truncation. To the best of our knowledge, our work is the first one providing a scale-free compositional approach for construction of continuous abstractions. In addition to the scalability issue, in a large number of applications, the structure of the network is time-varying in the sense that the communication links between subsystems change over time. In power networks, for instance, there exist line switches and the agents are constantly plugged into and out. This calls for considering switched dynamics describing the time dependency of the network structure. Our setting, therefore, considers an infinite network of switched systems. To validate the effectiveness of our approach, we apply our results to AC microgrids operating in an islanded mode. In particular, we show through simulations that the behavior of the network remains *independent* of the size of the network, while the network size dramatically increases.

This paper expands on the conference paper [23], where uniformity conditions with respect to the switching modes were made. The present work provides a completely non-uniform structure for the simulation functions with respect to the switching signals in the network. Moreover, the scale-free property of the result is established, which leads to constructing compositional abstractions for any finite-but-arbitrarily large network. Therefore, the current setting allows us to consider more general and realistic scenarios, including the new AC microgrid case study.

The rest of the paper is organized as follows. Section 2 provides the system description. In Section 3, we first introduce the notion of simulation functions for switched systems, and then show the importance of the existence of such functions in the construction of abstractions. Section 4 contains the main result of the paper, that is the compositional construction of continuous abstractions via small-gain theory. In Section 5, we focus on linear systems and provide easier-to-check conditions for the construction of continuous abstractions. In Section 6, we apply our results to a network of AC islanded microgrids.

2. Preliminaries and system description

2.1. Notation

We write $\mathbb{N}_0(\mathbb{N})$ for the set of nonnegative (positive) integers. For vector norms on *finite*-dimensional vector spaces, we write $|\cdot|$. By ℓ^p, $p \in [1,\infty)$, we denote the Banach space of all real sequences $x = (x_i)_{i\in\mathbb{N}}$ with finite ℓ^p-norm $|x|_p < \infty$, where $|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $p < \infty$. If X is a Banach space, we write $r(T)$ for the spectral radius of a bounded linear operator $T : X \to X$. The identity function is denoted by id. Throughout this work, we will consider $\mathcal{K}$ and $\mathcal{K}_\infty$ comparison functions; see [24, Chapter 4.4] for definitions.

We consider discrete-time switched subsystems Σ_i, defined below. The arbitrary switching signals are defined as $\sigma_i : \mathbb{N}_0 \to S_i$ for each subsystem Σ_i, $i \in \mathbb{N}$, and $S_i = \{1, 2, \ldots, r_i\}$ is a finite index set with $r_i \in \mathbb{N}$. The set of such switching signals are denoted by $\mathcal{S}_i$.

2.2. Infinite networks

First, we define discrete-time switched subsystems which are interconnected to form an infinite network consisting of countably infinite number of control subsystems.

Definition 1. A discrete-time switched system Σ_i, $i \in \mathbb{N}$, is defined by the tuple

$$\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, \mathcal{U}_i, \mathbb{Y}_i, h_{i,s_i}, f_{i,s_i}, S_i), \tag{1}$$

where $\mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, $\mathbb{W}_i \subseteq \mathbb{R}^{N_i}$, $\mathbb{U}_i \subseteq \mathbb{R}^{m_i}$, and $\mathbb{Y}_i \subseteq \mathbb{R}^{q_i}$ are the state set, internal input set, external input set, and output set, respectively. Moreover, $N_i := \sum_j n_j$, where j corresponds to subsystems Σ_j influencing Σ_i. We use symbol $\mathcal{U}_i$ to denote the set of functions $u_i : \mathbb{N}_0 \to \mathbb{U}_i$. Functions $f_{i,s_i} : \mathbb{X}_i \times \mathbb{W}_i \times \mathbb{U}_i \to \mathbb{X}_i$ are the transition functions for $s_i \in S_i$. Moreover, $h_{i,s_i} : \mathbb{X}_i \to \mathbb{Y}_i$ are the output maps.

The discrete-time switched subsystems Σ_i, $i \in \mathbb{N}$, are represented by the difference equation of the form

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = f_{i,\sigma_i(k)}(\mathbf{x}_i(k), \mathbf{w}_i(k), \mathbf{u}_i(k)), \\ \mathbf{y}_i(k) = h_{i,\sigma_i(k)}(\mathbf{x}_i(k)), \end{cases} \tag{2}$$

where $\mathbf{x}_i : \mathbb{N}_0 \to \mathbb{X}_i$, $\mathbf{w}_i : \mathbb{N}_0 \to \mathbb{W}_i$, $\mathbf{u}_i : \mathbb{N}_0 \to \mathbb{U}_i$, and $\mathbf{y}_i : \mathbb{N}_0 \to \mathbb{Y}_i$ are the state signal, internal input signal, external input signal, and output signal, respectively.

The finite set $I^{\text{in}}_{i,\sigma_i(k)} \subset \mathbb{N}\backslash\{i\}$ collects *mode-dependent in-neighbors* of Σ_i, i.e. systems $\Sigma_j, j \in I^{\text{in}}_{i,\sigma_i(k)}$, *directly* influencing Σ_i. On the other hand, the finite set $I^{\text{out}}_{i,\sigma_i(k)} \subset \mathbb{N}$, collects *mode-dependent out-neighbors* of Σ_i, i.e. $\Sigma_j, j \in I^{\text{out}}_{i,\sigma_i(k)}$, influenced by Σ_i. Note that we assume $i \notin I^{\text{in}}_{i,\sigma_i(k)} \cup I^{\text{out}}_{i,\sigma_i(k)}$, $\forall i \in \mathbb{N}$. The input–output structure of each subsystem Σ_i, $i \in \mathbb{N}$, is given by

$$\mathbf{w}_i(k) = \big(\mathbf{w}_{ij}(k)\big)_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \in \mathbb{W}_i := \prod_{j\in(I^{\text{in}}_{i,\sigma_i(k)})} \mathbb{W}_{ij}, \tag{3a}$$

$$\mathbf{y}_i(k) = \big(\mathbf{y}_{ij}(k)\big)_{j\in(i\cup I^{\text{out}}_{i,\sigma_i(k)})} \in \mathbb{Y}_i := \prod_{j\in(i\cup I^{\text{out}}_{i,\sigma_i(k)})} \mathbb{Y}_{ij}, \tag{3b}$$

$$h_{i,\sigma_i(k)}(\mathbf{x}_i(k)) = \big(h_{ij,\sigma_i(k)}(\mathbf{x}_i(k))\big)_{j\in(i\cup I^{\text{out}}_{i,\sigma_i(k)})}, \tag{3c}$$

where $\mathbf{w}_i(k) \in \mathbb{W}_i \subseteq \mathbb{R}^{N_i}$, $N_i := \sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} n_j$, denotes the internal inputs describing the interconnections among subsystems. The outputs $\mathbf{y}_{ij}(k)$, $j \in I^{\text{out}}_{i,\sigma_i(k)}$, are considered as internal outputs which are used to construct interconnections between subsystems, whereas $\mathbf{y}_{ii}(k) \in \mathbb{Y}_{ii}$ are denoted as external outputs. Note that $\mathbf{w}_i(k)$ and $\mathbf{y}_i(k)$ are partitioned into sub-vectors and we interconnect all the subsystems Σ_i through the interconnection constraints given by $\mathbf{w}_{ij}(k) = \mathbf{y}_{ji}(k)$ for all $i \in \mathbb{N}$ and for all $j \in I^{\text{in}}_{i,\sigma_i(k)}$.

To model the state (resp. input) space of the overall network, we introduce a Banach space of sequences $x = (x_i)_{i\in\mathbb{N}}$ (resp. $u = (u_i)_{i\in\mathbb{N}}$). The most natural choice is the ℓ^p-space, precisely, defined as follows: we first fix a norm on each $\mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ (that might not only depend on the dimension n_i but also on the index i). For brevity, we omit the index in our notation and simply write $|\cdot|$ for each of these norms. Then, for every $p \in [1,\infty)$, we put

$$\ell^p(\mathbb{N}, (n_i)) := \Big\{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{X}_i,\ \sum_{i\in\mathbb{N}} |x_i|^p < \infty\Big\},$$

and equip this space with the norm $|x|_p := (\sum_{i\in\mathbb{N}} |x_i|^p)^{1/p}$. Now, we provide a formal definition of the infinite network.

Definition 2. Consider subsystems $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, \mathcal{U}_i, \mathbb{Y}_i, h_{i,s_i}, f_{i,s_i}, S_i)$, $i \in \mathbb{N}$, with the input–output structure as in (3). A discrete-time infinite network Σ is defined by the tuple $\Sigma = (\mathbb{X}, \mathbb{U}, \mathcal{U}, \mathbb{Y}, h_s, f_s, S)$, where $\mathbb{X} = \ell^p(\mathbb{N}, (n_i)) \subset \prod_{i\in\mathbb{N}} \mathbb{X}_i$ with a fixed $p \in [0, \infty)$ and $\mathbb{U} = \ell^q(\mathbb{N}, (n_i)) \subset \prod_{i\in\mathbb{N}} \mathbb{U}_i$ with a fixed $q \in [0, \infty)$. The space of admissible *external input functions* $\mathbf{u}$ is defined by $\mathcal{U} := \{\mathbf{u} : \mathbb{N}_0 \to \mathbb{U}\}$. Moreover, $h_s(x) = (h_{ii,s_i}(x_i))_{i\in\mathbb{N}}$, $s \in S, S = \prod_{i\in\mathbb{N}} S_i$ denotes the output function, where $h_s : \mathbb{X} \to \mathbb{Y}$, $\mathbb{Y} \subset \prod_{i\in\mathbb{N}} \mathbb{Y}_{ii}$. In addition, we restrict $f_s(x, u) = (f_{i,s_i}(x_i, w_i, u_i))_{i\in\mathbb{N}}$ to $f_s : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$.

In that way, the interconnection of subsystems Σ_i, $i \in \mathbb{N}$, is described by

$$\Sigma : \quad \begin{cases} \mathbf{x}(k+1) = f_{\sigma(k)}(\mathbf{x}(k), \mathbf{u}(k)), \\ \mathbf{y}(k) = h_{\sigma(k)}(\mathbf{x}(k)), \end{cases} \tag{4}$$

where $\mathbf{x}(k)=(\mathbf{x}_i(k))_{i\in\mathbb{N}}$, $\mathbf{u}(k)=(\mathbf{u}_i(k))_{i\in\mathbb{N}}$, $\mathbf{y}(k) = (\mathbf{y}_{ii}(k))_{i\in\mathbb{N}}$, $\sigma(k)=(\sigma_i(k))_{i\in\mathbb{N}}$, $f_{\sigma(k)}(\mathbf{x}(k), \mathbf{u}(k)) = \big(f_{i,\sigma_i(k)}(\mathbf{x}_i(k), \mathbf{w}_i(k), \mathbf{u}_i(k))\big)_{i\in\mathbb{N}}$, and $h_{\sigma(k)}(\mathbf{x}(k)) := \big(h_{ii,\sigma_i(k)}(\mathbf{x}_i(k))\big)_{i\in\mathbb{N}}$. We call the overall system (4) an *infinite* network and denote the corresponding solutions (resp. output trajectory) by $\mathbf{x}(k, x, \sigma, \mathbf{u})$ (resp. $\mathbf{y}(k, x, \sigma, \mathbf{u})$) for any $k \in \mathbb{N}_0$, any initial value $x \in \mathbb{X}$, any switching signal $\sigma : \mathbb{N}_0 \to \mathcal{S}$, $\mathcal{S} := \{\sigma : \mathbb{N}_0 \to S\}$, and any control input $\mathbf{u} \in \mathcal{U}$.

We refer to system (4) as the *concrete* system, which is often hard to control or analyze. To simplify the controller design process, we, instead, use a simpler and less precise system called an *abstract* system.

3. Abstractions for discrete-time switched systems

In this section, we introduce a notion of simulation functions for discrete-time switched systems. A simulation function quantifies a relation between the concrete system and its abstraction in the sense that the mismatch between their output trajectories remains bounded (cf. Proposition 4). A simulation function is formally defined as follows.

Definition 3. Consider two systems $\Sigma = (\mathbb{X}, \mathbb{U}, \mathcal{U}, \mathbb{Y}, h_s, f_s, S)$ and $\hat{\Sigma} = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathcal{U}}, \hat{\mathbb{Y}}, \hat{h}_s, \hat{f}_s, S)$ with the same output space dimensions. Let $p, q \in [1, \infty)$ be given. Let $V_s : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_+$, $s \in S$, be a family of functions. Assume that there exist constants $\alpha, b > 0$ such that for all $s \in S$, all $x \in \mathbb{X}$ and all $\hat{x} \in \hat{\mathbb{X}}$,

$$\alpha \left| h_s(x) - \hat{h}_s(\hat{x}) \right|_p^b \le V_s(x, \hat{x}), \tag{5}$$

and there exist a function $\rho_{\mathrm{ext}} \in \mathcal{K}$ and a constant $0 < \lambda < 1$, such that for all $s', s \in S$ and all $x \in \mathbb{X}$, $\hat{x} \in \hat{\mathbb{X}}$ and $\hat{u} \in \hat{\mathbb{U}}$, there exists $u \in \mathbb{U}$ so that we have

$$\begin{aligned} &V_{s'}(f_s(x, u), \hat{f}_s(\hat{x}, \hat{u})) - V_s(x, \hat{x}) \\ &\le -\lambda V_s(x, \hat{x}) + \rho_{\mathrm{ext}}(|\hat{u}|_q). \end{aligned} \tag{6}$$

Functions V_s satisfying (5) and (6) are called simulation functions from $\hat{\Sigma}$ to Σ and $\hat{\Sigma}$ is called an abstraction of Σ.

Now we show that the existence of a simulation function ensures that the output trajectories of the abstract and concrete systems remain within a bounded distance from each other.

Proposition 4. *Consider systems $\Sigma = (\mathbb{X}, \mathbb{U}, \mathcal{U}, \mathbb{Y}, h_s, f_s, S)$ and $\hat{\Sigma} = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathcal{U}}, \hat{\mathbb{Y}}, \hat{h}_s, \hat{f}_s, S)$ with the same output space dimensions. Let a set of simulation functions V_s, $s \in S$, from $\hat{\Sigma}$ to Σ and $p, q \in [1, \infty)$ be given. Then there exist a function $\gamma_{\mathrm{ext}} \in \mathcal{K}$ and positive constants ϑ and $\beta < 1$, such that for any $\sigma \in \mathcal{S}$, $x \in \mathbb{X}$, $\hat{x} \in \hat{\mathbb{X}}$, $\hat{\mathbf{u}} \in \hat{\mathcal{U}}$, $k \in \mathbb{N}_0$, there exists $\mathbf{u} \in \mathcal{U}$ so that we have*

$$\begin{aligned} &\left| \mathbf{y}(k, x, \sigma, \mathbf{u}) - \hat{\mathbf{y}}(k, \hat{x}, \sigma, \hat{\mathbf{u}}) \right|_p \\ &\le \vartheta \beta^k (V_{\sigma(0)}(x, \hat{x}))^{\frac{1}{b}} + \gamma_{\mathrm{ext}}(|\hat{\mathbf{u}}|_{q,\infty}), \end{aligned} \tag{7}$$

where $|\hat{\mathbf{u}}|_{q,\infty} := \sup_{k\in\mathbb{N}_0} |\hat{\mathbf{u}}(k)|_q$ and b as in (5).

Proof. Pick any $x \in \mathbb{X}, \hat{x} \in \hat{\mathbb{X}}$, an input $\hat{\mathbf{u}} \in \hat{\mathcal{U}}$, and $\sigma \in \mathcal{S}$ and let $c := \rho_{\mathrm{ext}}(|\hat{\mathbf{u}}|_{q,\infty})$ and $\mathbf{z}(k) := V_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u}))$. Then, from (6) we have

$$\begin{aligned} &\mathbf{z}(k+1) - \mathbf{z}(k) \\ &= V_{\sigma(k+1)}(\mathbf{x}(k+1, x, \sigma, \mathbf{u}), \hat{\mathbf{x}}(k+1, \hat{x}, \sigma, \hat{\mathbf{u}})) \\ &- \ V_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u}), \hat{\mathbf{x}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})) \\ &= V_{\sigma(k+1)}(f_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u})), \hat{f}_{\sigma(k)}(\hat{\mathbf{x}}(k, \hat{x}, \sigma, \hat{\mathbf{u}}))) \\ &- \ V_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u}), \hat{\mathbf{x}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})) \\ &\le -\lambda V_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u}), \hat{\mathbf{x}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})) + c = -\lambda \mathbf{z}(k) + c, \end{aligned}$$

for all $k \in \mathbb{N}_0$. By incorporating standard comparison arguments, one can obtain

$$\mathbf{z}(k) \leq (1-\lambda)^k \mathbf{z}(0) + \sum_{i=0}^{k-1} (1-\lambda)^i c, \tag{8}$$

for all $k \in \mathbb{N}_0$. In other words, we obtain

$$\begin{aligned} &V_{\sigma(k)}(\mathbf{x}(k, x, \sigma, \mathbf{u}), \hat{\mathbf{x}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})) \\ &\leq (1-\lambda)^k V_{\sigma(0)}(x, \hat{x}) + \sum_{i=0}^{k-1} (1-\lambda)^i \rho_{\text{ext}}(|\hat{\mathbf{u}}|_{q,\infty}). \end{aligned} \tag{9}$$

It follows from (5) and (9) that

$$\begin{aligned} &\alpha \left|\mathbf{y}(k, x, \sigma, \mathbf{u}) - \hat{\mathbf{y}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})\right|_p^b \\ &\leq (1-\lambda)^k V_{\sigma(0)}(x, \hat{x}) + \sum_{i=0}^{k-1} (1-\lambda)^i \rho_{\text{ext}}(|\hat{\mathbf{u}}|_{q,\infty}), \end{aligned}$$

for all $k \in \mathbb{N}_0$. From $a_1 + a_2 \leq \max\{2a_1, 2a_2\}$ with $a_1, a_2 \geq 0$, we get

$$\begin{aligned} &\left|\mathbf{y}(k, x, \sigma, \mathbf{u}) - \hat{\mathbf{y}}(k, \hat{x}, \sigma, \hat{\mathbf{u}})\right|_p \\ &\leq \vartheta \beta^k (V_{\sigma(0)}(x, \hat{x}))^{\frac{1}{b}} + \gamma_{\text{ext}}(|\hat{\mathbf{u}}|_{q,\infty}), \end{aligned}$$

for all $k \in \mathbb{N}_0$, with $\vartheta = (2\frac{1}{\alpha})^{\frac{1}{b}}$, $\beta = (1-\lambda)^{\frac{1}{b}}$, $\gamma_{\text{ext}}(\cdot) = (2\frac{\sum_{i=0}^{k-1}(1-\lambda)^i \rho_{\text{ext}}(\cdot)}{\alpha})^{\frac{1}{b}}$. This completes the proof. □

Remark 5. Suppose that we are given an interface function ν, which maps every $x, \hat{x}, \hat{u}$, and s to an input $u = \nu(x, \hat{x}, \hat{u}, s)$ so that (6) is satisfied. Then, the input $\mathbf{u}$ that realizes (7) is readily given by $\mathbf{u}(k) = \nu(\mathbf{x}(k), \hat{\mathbf{x}}(k), \hat{\mathbf{u}}(k), \sigma(k))$; see [25, Theorem 1].

Due to the size of the systems, a simulation function from $\hat{\Sigma}$ to Σ is quite hard to be *directly* computed. To address this complexity, we follow a compositional approach and define local simulation functions for each finite-dimensional subsystem (cf. Definition 6). This enables us to verify (5) and (6) in a bottom-up way. The next section develops this strategy with the use of small-gain theory for infinite networks.

4. Compositional construction of abstractions and simulation functions

In the following, we provide a method for compositional construction of simulation functions between the infinite networks Σ and $\hat{\Sigma}$. We assume that each subsystem $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, \mathcal{U}_i, \mathbb{Y}_i, h_{i,s_i}, f_{i,s_i}, S_i)$ and $\hat{\Sigma}_i = (\hat{\mathbb{X}}_i, \hat{\mathbb{W}}_i, \hat{\mathbb{U}}_i, \hat{\mathcal{U}}_i, \hat{\mathbb{Y}}_i, \hat{h}_{i,s_i}, \hat{f}_{i,s_i}, S_i)$ admits a local simulation function as defined below.

Definition 6. Consider subsystems $\Sigma_i = (\mathbb{X}_i, \mathbb{W}_i, \mathbb{U}_i, \mathcal{U}_i, \mathbb{Y}_i, h_{i,s_i}, f_{i,s_i}, S_i)$ and $\hat{\Sigma}_i = (\hat{\mathbb{X}}_i, \hat{\mathbb{W}}_i, \hat{\mathbb{U}}_i, \hat{\mathcal{U}}_i, \hat{\mathbb{Y}}_i, \hat{h}_{i,s_i}, \hat{f}_{i,s_i}, S_i)$, $i \in \mathbb{N}$. Let $p, q \in [1, \infty)$ be given. Assume that there exist functions $V_{i,s_i} : \mathbb{X}_i \times \hat{\mathbb{X}}_i \to \mathbb{R}_+$, $s_i \in S_i$, satisfying the following properties

- There are constants $\alpha_i > 0$ so that for all $x_i \in \mathbb{X}_i$ and all $\hat{x}_i \in \hat{\mathbb{X}}_i$

$$\alpha_i \left|h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)\right|^p \leq V_{i,s_i}(x_i, \hat{x}_i). \tag{10}$$

- There are positive constants $\lambda_i < 1$, $\rho_{\text{i,int}}$, $\rho_{\text{i,ext}}$ such that for all $s_i', s_i \in S_i$, all $x_i \in \mathbb{X}_i$, all $\hat{x}_i \in \hat{\mathbb{X}}_i$, all $\hat{u}_i \in \hat{\mathbb{U}}_i$, there exists $u_i \in \mathbb{U}_i$ so that the following holds for all $w_i \in \mathbb{W}_i$ and all $\hat{w}_i \in \hat{\mathbb{W}}_i$

$$\begin{aligned} &V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \\ &-V_{i,s_i}\left(x_i, \hat{x}_i\right) \leq -\lambda_i V_{i,s_i}(x_i, \hat{x}_i) \\ &+\rho_{\text{i,int}} \left|w_i - \hat{w}_i\right|^p + \rho_{\text{i,ext}} |\hat{u}_i|^q. \end{aligned} \tag{11}$$

Then functions V_{i,s_i} are called local simulation functions from $\hat{\Sigma}_i$ to Σ_i and $\hat{\Sigma}_i$ are called abstractions of Σ_i for each $i \in \mathbb{N}$.

Remark 7. In view of inequality (11), we only consider linear external gains $\rho_{i,ext}$ for each subsystem Σ_i, while Definition 3 allows for a nonlinear one (ρ_{ext}) for the infinite network Σ. This restriction is due to the choice of the input space of the overall network as well as the use of a sum formulation of small-gain theorems. We construct the overall external gain function $\rho_{\text{ext}}(|\hat{u}|_q)$ from a summation over all individual external gains $\rho_{\text{i,ext}}|\hat{u}_i|^q$ (see the chain of inequality (17) below). In that way, the well-definedness of the resulting external gain function is guaranteed by the fact that $\hat{u}$ belongs to ℓ^q

space. To show this, we assume that the external gain functions are linear. This condition clearly is not needed for finite networks (e.g. those in [10]).

Assume that each Σ_i admits an abstraction $\hat{\Sigma}_i$, $\forall i \in \mathbb{N}$, given as in Definition 6. We establish a compositional approach for the construction of continuous abstractions of infinite networks (4) by aggregating individual continuous abstractions $\hat{\Sigma}_i$. To do so, we need interaction between subsystems to be sufficiently weak, which is quantitatively described by a small-gain condition, see Assumption 10 below.

To employ the small-gain theorem, the following conditions are required. The first one makes uniformity conditions on the constants given by Definition 6.

Assumption 8. There are constants $\underline{\alpha}, \underline{\lambda}, \overline{\rho}_{\text{ext}} > 0$ so that for all $i \in \mathbb{N}$, we have $\underline{\alpha} \leq \alpha_i$, $\underline{\lambda} \leq \lambda_i$, $\rho_{\text{i,ext}} \leq \overline{\rho}_{\text{ext}}$.

We collect the coefficients from (10) and (11) to define

$$\gamma_{ij} := \begin{cases} \rho_{i,\text{int}} \bar{N}_i \frac{1}{\alpha_j}, & j \in I^{\text{in}}_{i,s_i}, \\ 0, & j \notin I^{\text{in}}_{i,s_i}, \end{cases} \tag{12}$$

where $\bar{N}_i$ denotes the cardinality of the set I^{in}_{i,s_i}.

We additionally introduce the following matrices.

$$\Lambda := \text{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \Gamma := (\gamma_{ij})_{i,j\in\mathbb{N}}. \tag{13}$$

Now, we define the following matrix by which we express our *small-gain* condition

$$\Psi := \Lambda^{-1}\Gamma := (\psi_{ij})_{i,j\in\mathbb{N}}, \quad \psi_{ij} = \gamma_{ij}/\lambda_i. \tag{14}$$

We also make an assumption on the boundedness of the operator Γ.

Assumption 9. The operator $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{N}}$ satisfies $\sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} \gamma_{ij} < \infty$.

Note that Assumption 9 always holds if each subsystem is interconnected to finitely many subsystems and *no* global communication is used.

The following spectral radius condition provides a *quantitative* bound on the strength of couplings between the subsystems. This is, in fact, the small-gain condition that is required to guarantee that the aggregation of $\hat{\Sigma}_i$ gives a continuous abstraction for network Σ.

Assumption 10. The spectral radius $r(\Psi) < 1$.

The following theorem gives the *main result* of the paper, which is a compositional approach to construct the abstractions of infinite interconnected switched control systems and their corresponding simulation functions.

Theorem 11. *Consider infinite networks $\Sigma = (\mathbb{X}, \mathbb{U}, \mathcal{U}, \mathbb{Y}, h_s, f_s, S)$ and $\hat{\Sigma} = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathcal{U}}, \hat{\mathbb{Y}}, \hat{h}_s, \hat{f}_s, S)$. Let $p, q \in [1, \infty)$ be given. Let local simulation functions $V_{i,s_i} : \mathbb{X}_i \times \hat{\mathbb{X}}_i \to \mathbb{R}_+, s_i \in S_i$, satisfy Assumptions 8–10. Then there exists a vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ with constants $\underline{\mu}, \overline{\mu} > 0$ such that the following is satisfied*

$$\frac{[\mu^\top(-\Lambda + \Gamma)]_i}{\mu_i} \leq -\lambda_\infty, \quad \forall i \in \mathbb{N}, \tag{15}$$

for a constant $\lambda_\infty \in (0, 1)$. Moreover, the following family of functions $V_s : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_+, s \in S$, with $S = \prod_{i\in\mathbb{N}} S_i$,

$$V_s(x, \hat{x}) = \sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i, \hat{x}_i), \quad V_s : X \times \hat{X} \to \mathbb{R}_+, \tag{16}$$

are simulation functions from $\hat{\Sigma}$ to Σ with $b = p, \alpha = \underline{\mu}\underline{\alpha}$ as in (5) and $\lambda = \lambda_\infty$ and $\rho_{\text{ext}} : t \mapsto \overline{\mu}\, \overline{\rho}_{\text{ext}} t^q$ as in (6).

Proof. From [17, Lemma V.10], Assumption 10 (i.e. $r(\Psi) < 1$) implies that there exists a vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ such that (15) holds.

Now we show that V in (16) satisfies (5) with $\alpha = \underline{\mu}\underline{\alpha}$. For any $s \in S$, $s_i \in S_i$, $x \in X$ and $\hat{x} \in \hat{X}$ and taking $b = p$, it follows from (10) and Assumption 8 that

$$\begin{aligned}\sum_{i=1}^{\infty} \mu_i V_{i,s_i}(x_i, \hat{x}_i) &\geq \sum_{i=1}^{\infty} \mu_i \alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p \\ &\geq \underline{\mu}\underline{\alpha} \sum_{i=1}^{\infty} |h_{ii,s_i}(x_i) - \hat{h}_{ii,s_i}(\hat{x}_i)|^p \\ &\geq \underline{\mu}\underline{\alpha} |h_s(x) - \hat{h}_s(\hat{x})|_p^p.\end{aligned}$$

Next we show the inequality (6) holds as well. Considering (11) and (16), we obtain the chain of inequality in (17) for all $s_i', s_i \in S_i$, $s_j \in S_j$, $s', s \in S$, $i \in \mathbb{N}$.

$$
\begin{aligned}
&V_{s'}\left(f_s(x,u), \hat{f}_s(\hat{x},\hat{u})\right) - V_s(x,\hat{x}) = \sum_{i=1}^{\infty} \mu_i \left[V_{i,s_i'}\left(f_{i,s_i}(x_i,w_i,u_i), \hat{f}_{i,s_i}(\hat{x}_i,\hat{w}_i,\hat{u}_i)\right) - V_{i,s_i}(x_i,\hat{x}_i)\right] \\
&\le \sum_{i=1}^{\infty} \mu_i(-\lambda_i V_{i,s_i}(x_i,\hat{x}_i) + \rho_{i,\mathrm{int}} \left|w_i - \hat{w}_i\right|^p + \rho_{i,\mathrm{ext}}|\hat{u}_i|^q) \\
&\le \sum_{i=1}^{\infty} \mu_i(-\lambda_i V_{i,s_i}(x_i,\hat{x}_i) + \sum_{j\in I^{\mathrm{in}}_{i,s_i}} \rho_{i,\mathrm{int}}\bar{N}_i \left|w_{ij} - \hat{w}_{ij}\right|^p + \rho_{i,\mathrm{ext}}|\hat{u}_i|^q) \\
&\le \sum_{i=1}^{\infty} \mu_i(-\lambda_i V_{i,s_i}(x_i,\hat{x}_i) + \sum_{j\in I^{\mathrm{in}}_{i,s_i}} \rho_{i,\mathrm{int}}\bar{N}_i \left|h_{j,s_j}(x_j) - \hat{h}_{j,s_j}(\hat{x}_j)\right|^p + \rho_{i,\mathrm{ext}}|\hat{u}_i|^q) \\
&\le \sum_{i=1}^{\infty} \mu_i(-\lambda_i V_{i,s_i}(x_i,\hat{x}_i) + \sum_{j\in I^{\mathrm{in}}_{i,s_i}} \rho_{i,\mathrm{int}}\bar{N}_i \frac{1}{\alpha_j} V_{j,s_j}(x_j,\hat{x}_j) + \rho_{i,\mathrm{ext}}|\hat{u}_i|^q) \\
&\overset{(12)}{\le} \sum_{i=1}^{\infty} \mu_i \left(-\lambda_i V_{i,s_i}(x_i,\hat{x}_i) + \sum_{j\in I^{\mathrm{in}}_{i,s_i}} \gamma_{ij} V_{j,s_j}(x_j,\hat{x}_j) + \rho_{i,\mathrm{ext}}|\hat{u}_i|^q\right).
\end{aligned}
\tag{17}
$$

Letting $V_{s_{vec}}(x,\hat{x}) := \left(V_{i,s_i}(x_i,\hat{x}_i)\right)_{i\in\mathbb{N}}$ and using (17) and (15), we have that

$$
\begin{aligned}
&V_{s'}(f_s(x,u), \hat{f}_s(\hat{x},\hat{u})) - V_s(x,\hat{x}) \\
&\le \left[\mu^\top(-\Lambda + \Gamma)V_{s_{vec}}(x,\hat{x}) + \sum_{i=1}^{\infty} \mu_i \rho_{i,\mathrm{ext}}|\hat{u}_i|^q\right] \\
&\le -\lambda_\infty V_s(x,\hat{x}) + \rho_{\mathrm{ext}}(|\hat{u}|_q),
\end{aligned}
$$

where $\rho_{\mathrm{ext}}(t) = \overline{\mu}\ \overline{\rho}_{\mathrm{ext}} t^q$ for all $t \ge 0$. □

Remark 12. The significance of Assumptions 8 and 9 in Theorem 11 has been discussed in [17]. Specifically, the small-gain condition $r(\Psi) < 1$ is tight and cannot be relaxed. This condition, however, has to be checked globally. In view of Gelfand's formula, the spectral small-gain condition is equivalent to the existence of $k \in \mathbb{N}$ such that $\|\Psi^k\| < 1$. For networks with some special structure, e.g. (quasi) spatially invariant systems [17], one can easily check Assumption 10 with the use of Gelfand's formula. Thanks to the (quasi) periodicity in the network structure, the small-gain can be checked based on the information of a finite number of subsystems; see Section 6 for more details.

Remark 13. Note that the computational complexity of constructing individual abstractions increases linearly with the number of switching modes. Hence, our approach could become infeasible in the presence of infinitely many different network topologies. Thus, we only allow finitely many switching modes for each subsystem in this work.

4.1. From infinite to finite networks

The main purpose of dealing with infinite networks is to develop scale-free tools for the analysis and design of finite, but arbitrarily large networks. In this section we truncate the infinite network Σ and keep only the first n subsystems of the network. Roughly speaking, we show that if conditions required by Theorem 11 hold, then for any truncation of infinite network Σ and accordingly that of $\hat{\Sigma}$, the same conclusion as in Theorem 11 is obtained for truncated networks.

Consider the first $n \in \mathbb{N}$ subsystems of Σ and denote the truncated system by $\Sigma^{\langle n\rangle} = (\mathbb{X}^{\langle n\rangle}, \mathbb{X}^l, \mathbb{U}^{\langle n\rangle}, \mathcal{U}^{\langle n\rangle}, \mathbb{Y}^{\langle n\rangle}, h^{\langle n\rangle}_{s^{\langle n\rangle}}, f^{\langle n\rangle}_{s^{\langle n\rangle}}, S^{\langle n\rangle})$ whose dynamics is described by

$$
\Sigma^{\langle n\rangle} : \begin{cases} \mathbf{x}^{\langle n\rangle}(k+1) = f^{\langle n\rangle}_{\sigma^{\langle n\rangle}(k)}(\mathbf{x}^{\langle n\rangle}(k), \tilde{\mathbf{x}}(k), \mathbf{u}^{\langle n\rangle}(k)), \\ \mathbf{y}^{\langle n\rangle}(k) = h^{\langle n\rangle}_{\sigma^{\langle n\rangle}(k)}(\mathbf{x}^{\langle n\rangle}(k)), \end{cases}
\tag{18}
$$

where $\mathbf{x}^{\langle n\rangle}(k) = (\mathbf{x}_i(k))_{1\le i\le n}$ are elements of $\mathbb{X}^{\langle n\rangle} \subseteq \mathbb{R}^N$, $N := \sum_{i=1}^n n_i$, $\mathbf{u}^{\langle n\rangle}(k) = (\mathbf{u}_i(k))_{1\le i\le n}$ are elements of $\mathbb{U}^{\langle n\rangle} \subseteq \mathbb{R}^M$ and $M := \sum_{i=1}^n m_i$. Moreover, we denote by $I^{in^{\langle n\rangle}}_{\sigma^{\langle n\rangle}(k)} = \bigcup_{i=1}^n I^{in}_{i,\sigma_i(k)} \setminus \{1,\ldots,n\}$, the finite set of neighbors of the first n subsystems. Then, $\tilde{\mathbf{x}}(k) = (\mathbf{x}_j(k))_{j\in I^{in^{\langle n\rangle}}_{\sigma^{\langle n\rangle}(k)}} \in \mathbb{X}^l \subseteq \mathbb{R}^L$, $L := \sum_{j\in I^{in^{\langle n\rangle}}_{\sigma^{\langle n\rangle}(k)}} n_j$, is considered as the additional input vector. Note that we do not neglect subsystems Σ_i, $i > n$, instead we consider them as additional external inputs $\tilde{\mathbf{x}}(k)$ to the network

$\Sigma^{\langle n\rangle}$. Clearly, the case in which subsystems Σ_i, $i > n$, are entirely removed from the network is covered by our setting by taking $\tilde{x} \equiv 0$. We denote the set of input functions of the truncated network as $\mathcal{U}^{\langle n\rangle}$ and the output maps are viewed as $h^{\langle n\rangle}_{s^{\langle n\rangle}} : \mathbb{X}^{\langle n\rangle} \to \mathbb{Y}^{\langle n\rangle}$ with $S^{\langle n\rangle} = \prod_{1\leq i\leq n} S_i$. Moreover, functions $f^{\langle n\rangle}_{s^{\langle n\rangle}} : \mathbb{X}^{\langle n\rangle} \times \mathbb{X}^l \times \mathbb{U}^{\langle n\rangle} \to \mathbb{X}^{\langle n\rangle}$ are defined accordingly.

In the following, we construct the compositional construction of abstractions for the network $\Sigma^{\langle n\rangle}$ under the assumption of Theorem 11.

Theorem 14. *Consider the truncated networks $\Sigma^{\langle n\rangle} = (\mathbb{X}^{\langle n\rangle}, \mathbb{X}^l, \mathbb{U}^{\langle n\rangle}, \mathcal{U}^{\langle n\rangle}, \mathbb{Y}^{\langle n\rangle}, h^{\langle n\rangle}_{s^{\langle n\rangle}}, f^{\langle n\rangle}_{s^{\langle n\rangle}}, S^{\langle n\rangle})$ and $\hat{\Sigma}^{\langle n\rangle} = (\hat{\mathbb{X}}^{\langle n\rangle}, \hat{\mathbb{X}}^l, \hat{\mathbb{U}}^{\langle n\rangle}, \hat{\mathcal{U}}^{\langle n\rangle}, \hat{\mathbb{Y}}^{\langle n\rangle}, \hat{h}^{\langle n\rangle}_{s^{\langle n\rangle}}, \hat{f}^{\langle n\rangle}_{s^{\langle n\rangle}}, S^{\langle n\rangle})$. Let $p, q \in [1, \infty)$ be given. Consider local simulation functions $V_{i,s_i} : \mathbb{X}_i \times \hat{\mathbb{X}}_i \to \mathbb{R}_+$, $s_i \in S_i$, and suppose that Assumptions 8–10 hold. Assume that there exists a vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$, $\underline{\mu} \leq \mu_i \leq \overline{\mu}$, with some constants $\underline{\mu}, \overline{\mu} > 0$ satisfying (15). Then, the family of functions $V_{s^{\langle n\rangle}} : \mathbb{X}^{\langle n\rangle} \times \hat{\mathbb{X}}^{\langle n\rangle} \to \mathbb{R}_+$, $s^{\langle n\rangle} \in S^{\langle n\rangle}$, where*

$$V_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) = \sum_{i=1}^{n} \mu_i V_{i,s_i}(x_i, \hat{x}_i),$$

are simulation functions from $\hat{\Sigma}^{\langle n\rangle}$ to $\Sigma^{\langle n\rangle}$ with $b = p$, $\alpha = \underline{\mu}\,\underline{\alpha}$ as in (5) and satisfy the following

$$\begin{aligned} &V_{s^{\langle n\rangle'}}(f^{\langle n\rangle}_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \tilde{x}, u^{\langle n\rangle}), \hat{f}^{\langle n\rangle}_{s^{\langle n\rangle}}(\hat{x}^{\langle n\rangle}, \hat{\tilde{x}}, \hat{u}^{\langle n\rangle})) \\ &-V_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) \\ &\leq -\lambda V_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) + \rho_{\text{ext}}(|\hat{u}^{\langle n\rangle}|_q) + \rho_{\text{ext}}(|\hat{\tilde{x}}^{\langle n\rangle}|_q), \end{aligned} \tag{19}$$

for all $s^{\langle n\rangle'}, s^{\langle n\rangle} \in S^{\langle n\rangle}$, where $\lambda = \lambda_\infty$ and $\rho_{\text{ext}} : t \mapsto \overline{\mu}\,\overline{\rho}_{\text{ext}} t^q$.

Proof. By following similar arguments as in Theorem 11, one can obtain

$$\begin{aligned} \sum_{i=1}^{n} \mu_i V_{i,s_i}(x_i, \hat{x}_i) &\geq \sum_{i=1}^{n} \mu_i \alpha_i |h_{i,s_i}(x_i) - \hat{h}_{i,s_i}(\hat{x}_i)|^p \\ &\geq \underline{\mu}\,\underline{\alpha} \sum_{i=1}^{n} |h_{ii,s_i}(x_i) - \hat{h}_{ii,s_i}(\hat{x}_i)|^p \\ &\geq \underline{\mu}\,\underline{\alpha} |h^{\langle n\rangle}_{s^{\langle n\rangle}}(x^{\langle n\rangle}) - \hat{h}^{\langle n\rangle}_{s^{\langle n\rangle}}(\hat{x}^{\langle n\rangle})|_p^p. \end{aligned}$$

Moreover, by letting $V_{s^{\langle n\rangle}_{vec}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) := \big(V_{i,s_i}(x_i, \hat{x}_i)\big)_{1\leq i\leq n}$, using the chain of inequalities in (17) for all $s_i', s_i \in S_i$, $s_j \in S_j$, $s^{\langle n\rangle'}, s^{\langle n\rangle} \in S^{\langle n\rangle}$, $1 \leq i \leq n$, and (15), we have

$$\begin{aligned} &V_{s^{\langle n\rangle'}}(f^{\langle n\rangle}_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \tilde{x}, u^{\langle n\rangle}), \hat{f}^{\langle n\rangle}_{s^{\langle n\rangle}}(\hat{x}^{\langle n\rangle}, \hat{\tilde{x}}, \hat{u}^{\langle n\rangle})) \\ &-\ V_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) \leq \\ &\Big\{\Big[\mu^\top(-\Lambda + \Gamma)\Big]_{1\leq i\leq n} V_{s^{\langle n\rangle}_{vec}}(x, \hat{x}) + \sum_{i=1}^{n} \mu_i \rho_{i,\text{ext}} |\hat{\tilde{x}}_i|^q \\ &+\ \sum_{i=1}^{n} \mu_i \rho_{i,\text{ext}} |\hat{u}_i|^q\Big\} \\ &\leq -\lambda_\infty V_{s^{\langle n\rangle}}(x^{\langle n\rangle}, \hat{x}^{\langle n\rangle}) + \overline{\mu}\,\overline{\rho}_{\text{ext}} |\hat{\tilde{x}}^{\langle n\rangle}|_q^q \\ &\quad + \overline{\mu}\,\overline{\rho}_{\text{ext}} |\hat{u}^{\langle n\rangle}|_q^q. \quad \square \end{aligned}$$

As can be seen from Theorem 14, the decay rate λ_∞ as well as the gain function due to external input $\hat{u}$ are preserved under truncation. Thus, the indices of the proposed compositional method are *independent* of the network size.

5. Construction of abstractions for linear systems

In this section, we *explicitly* construct local abstractions and corresponding simulation functions for linear switched subsystems.

We make the following assumption on the simulation functions, which is an incremental version of a similar assumption used to achieve the input-to-state stability of switched systems under constrained switching conditions [26].

Assumption 15. There exist uniformly bounded constants $\tau_i \geq 1$, $i \in \mathbb{N}$, such that for all $x_i \in \mathbb{X}_i$, all $\hat{x}_i \in \hat{\mathbb{X}}_i$ and every $s_i, s_i' \in S_i$

$$V_{i,s_i}(x_i, \hat{x}_i) \leq \tau_i V_{i,s_i'}(x_i, \hat{x}_i).$$

Consider the following class of linear switched subsystems

$$\Sigma_i : \begin{cases} \mathbf{x}_i(k+1) = A_{i,\sigma_i(k)}\mathbf{x}_i(k) + D_{i,\sigma_i(k)}\mathbf{w}_i(k) \\ \qquad\qquad +B_{i,\sigma_i(k)}\mathbf{u}_i(k), \\ \mathbf{y}_i(k) = C_{i,\sigma_i(k)}\mathbf{x}_i(k), \end{cases} \tag{20}$$

where $\sigma_i \in \mathcal{S}_i$, $A_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times n_i}$, $B_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times m_i}$, $C_{i,\sigma_i(k)} \in \mathbb{R}^{q_i \times n_i}$ and $D_{i,\sigma_i(k)} \in \mathbb{R}^{n_i \times p_i}$, for $i \in \mathbb{N}$.

Choose $\mathbb{X} = \ell^2(\mathbb{N}, (n_i))$ and $\mathbb{U} = \ell^2(\mathbb{N}, (m_i))$ for the overall infinite network. By slight abuse of notation, we use the tuple $\Sigma_i = (A_{i,s_i}, B_{i,s_i}, C_{i,s_i}, D_{i,s_i})$ to refer to switched subsystem with transition and output functions of the form (20) with the specified matrices dimensions.

Assume that there exist a family of matrices K_{i,s_i}, positive definite matrices M_{i,s_i}, real numbers $\epsilon_i > 0$, and $0 < \kappa_i < 1$ such that the following matrix inequalities hold for all $s_i \in S_i$, $i \in \mathbb{N}$

$$C_{i,s_i}^\top C_{i,s_i} \preceq M_{i,s_i}, \tag{21a}$$

$$(1 + \frac{1}{\epsilon_i} + \epsilon_i)(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top M_{i,s_i}(A_{i,s_i} + B_{i,s_i}K_{i,s_i}) \preceq \kappa_i M_{i,s_i}. \tag{21b}$$

Remark 16. Given κ_i and ϵ_i, inequality (21b) is not jointly convex on the decision variables M_{i,s_i} and K_{i,s_i}. Then, this inequality is not amenable to existing semidefinite tools for linear matrix inequalities (LMI). By using the Schur complement lemma, (21b) could be transformed to the following LMI over decision variables Q_{i,s_i} and Z_{i,s_i}:

$$\begin{bmatrix} -\kappa_i Q_{i,s_i} & Q_{i,s_i}A_{i,s_i}^\top + Z_{i,s_i}^\top B_{i,s_i}^\top \\ A_{i,s_i}Q_{i,s_i} + B_{i,s_i}Z_{i,s_i} & -(1 + \frac{1}{\epsilon_i} + \epsilon_i)Q_{i,s_i} \end{bmatrix} \preceq 0,$$
$$Q_{i,s_i} \succ 0,$$

where $Q_{i,s_i} = M_{i,s_i}^{-1}$ and $Z_{i,s_i} = K_{i,s_i}Q_{i,s_i}$, $i \in \mathbb{N}$.

Consider the simulation function candidates $V_{i,s_i} : \mathbb{R}^{n_i} \times \mathbb{R}^{\hat{n}_i} \to \mathbb{R}_+$, $s_i \in S_i$, $i \in \mathbb{N}$, as

$$V_{i,s_i}(x_i, \hat{x}_i) = (x_i - P_{i,s_i}\hat{x}_i)^\top M_{i,s_i}(x_i - P_{i,s_i}\hat{x}_i). \tag{22}$$

The control inputs of the concrete subsystems are given by the interface functions ν_i as follows.

$$u_i = \nu_i(x_i, \hat{x}_i, \hat{u}_i, \hat{w}_i, s_i) = K_{i,s_i}(x_i - P_{i,s_i}\hat{x}_i) + Q_{i,s_i}\hat{x}_i + R_{i,s_i}\hat{u}_i + T_{i,s_i}\hat{w}_i, \tag{23}$$

where P_{i,s_i}, $i \in \mathbb{N}$, are some matrices of appropriate dimensions. Assume that the following inequalities hold for some matrices of appropriate dimensions Q_{i,s_i}, T_{i,s_i}.

$$A_{i,s_i}P_{i,s_i} = P_{i,s_i}\hat{A}_{i,s_i} - B_{i,s_i}Q_{i,s_i}, \tag{24a}$$

$$D_{i,s_i} = P_{i,s_i}\hat{D}_{i,s_i} - B_{i,s_i}T_{i,s_i}, \tag{24b}$$

$$C_{i,s_i}P_{i,s_i} = \hat{C}_{i,s_i}. \tag{24c}$$

Next theorem shows that functions V_{i,s_i}, $s_i \in S_i$, defined in (22), are simulation functions from $\hat{\Sigma}_i$ to Σ_i.

Theorem 17. *Consider systems $\Sigma_i = (A_{i,s_i}, B_{i,s_i}, C_{i,s_i}, D_{i,s_i})$ and $\hat{\Sigma}_i = (\hat{A}_{i,s_i}, \hat{B}_{i,s_i}, \hat{C}_{i,s_i}, \hat{D}_{i,s_i})$ for $i \in \mathbb{N}$. Suppose that for all $s_i \in S_i$, there exist appropriate matrices M_{i,s_i}, P_{i,s_i}, K_{i,s_i}, Q_{i,s_i} and T_{i,s_i} satisfying* (21) *and* (24)*. Moreover, assume that $\tau_i\kappa_i < 1$. Then, functions in* (22) *are simulation functions from $\hat{\Sigma}_i$ to Σ_i with concrete inputs given by* (23)*.*

Proof. According to (24c), we have

$$|C_{i,s_i}x_i - \hat{C}_{i,s_i}\hat{x}_i| = \left((x_i - P_{i,s_i}\hat{C}_{i,s_i})^\top C_{i,s_i}^\top C_{i,s_i}(x_i - P_{i,s_i}\hat{C}_{i,s_i})\right)^{\frac{1}{2}}.$$

Using (21a), it is clear that $|C_{i,s_i}x_i - \hat{C}_{i,s_i}\hat{x}_i|^2 \le V_{i,s_i}(x_i, \hat{x}_i)$ holds for all $x_i \in \mathbb{X}_i$, $\hat{x}_i \in \hat{\mathbb{X}}_i$. Then, (10) is satisfied with $\alpha_i = 1, i \in \mathbb{N}, p = 2$.

Now, we proceed to show that (11) is satisfied, too.

Using Assumption 15, one gets the following inequality for all switchings $s_i', s_i \in S_i$

$$V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \leq \tau_i V_{i,s_i}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right). \tag{25}$$

Using the system dynamics (20) and the candidate simulation function in (22), the inequality (25) can be written as

$$\begin{aligned} &V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \leq \\ &\tau_i[A_{i,s_i}x_i + B_{i,s_i}u_i + D_{i,s_i}w_i \\ &- P_{i,s_i}(\hat{A}_{i,s_i}\hat{x}_i + \hat{B}_{i,s_i}\hat{u}_i + \hat{D}_{i,s_i}\hat{w}_i)]^\top M_{i,s_i} \\ &\times [A_{i,s_i}x_i + B_{i,s_i}u_i + D_{i,s_i}w_i \\ &- P_{i,s_i}(\hat{A}_{i,s_i}\hat{x}_i + \hat{B}_{i,s_i}\hat{u}_i + \hat{D}_{i,s_i}\hat{w}_i)]. \end{aligned} \tag{26}$$

Substituting u_i from (23) and employing (24a) to (24b) yield

$$\begin{aligned} &V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \leq \\ &\tau_i[(A_{i,s_i} + B_{i,s_i}K_{i,s_i})(x_i - P_{i,s_i}\hat{x}_i) + D_{i,s_i}(w_i - \hat{w}_i) \\ &+ (B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i]^\top M_{i,s_i} \\ &\times [(A_{i,s_i} + B_{i,s_i}K_{i,s_i})(x_i - P_{i,s_i}\hat{x}_i) + D_{i,s_i}(w_i - \hat{w}_i) \\ &+ (B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i]. \end{aligned}$$

Applying Young's inequality as $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$ for any $a, b \geq 0$ and any $\epsilon > 0$, we have (27).

$$\begin{aligned} &V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \\ &\leq \tau_i\Bigg((x_i - P_{i,s_i}\hat{x}_i)^\top[(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top M_{i,s_i}(A_{i,s_i} + B_{i,s_i}K_{i,s_i})](x_i - P_{i,s_i}\hat{x}_i) \\ &\quad + [2(x_i - P_{i,s_i}\hat{x}_i)^\top(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top]M_{i,s_i}[D_{i,s_i}(w_i - \hat{w}_i)] \\ &\quad + [2(x_i - P_{i,s_i}\hat{x}_i)^\top(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top]M_{i,s_i}[(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i] \\ &\quad + [2(w_i - \hat{w}_i)^\top D_{i,s_i}{}^\top]M_{i,s_i}[(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i] \\ &\quad + |\sqrt{M_{i,s_i}}D_{i,s_i}(w_i - \hat{w}_i)|^2 + |\sqrt{M_{i,s_i}}(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i|^2\Bigg) \\ &\leq \tau_i\Bigg((x_i - P_{i,s_i}\hat{x}_i)^\top[(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top M_{i,s_i}(A_{i,s_i} + B_{i,s_i}K_{i,s_i})](x_i - P_{i,s_i}\hat{x}_i) \\ &\quad + \epsilon_i(x_i - P_{i,s_i}\hat{x}_i)^\top[(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top M_{i,s_i}(A_{i,s_i} + B_{i,s_i}K_{i,s_i})](x_i - P_{i,s_i}\hat{x}_i) + \frac{1}{\epsilon_i}|\sqrt{M_{i,s_i}}D_{i,s_i}(w_i - \hat{w}_i)|^2 \\ &\quad + \frac{1}{\epsilon_i}(x_i - P_{i,s_i}\hat{x}_i)^\top[(A_{i,s_i} + B_{i,s_i}K_{i,s_i})^\top M_{i,s_i}(A_{i,s_i} + B_{i,s_i}K_{i,s_i})](x_i - P_{i,s_i}\hat{x}_i) \\ &\quad + \epsilon_i|\sqrt{M_{i,s_i}}(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i|^2 + \frac{1}{\epsilon_i}|\sqrt{M_{i,s_i}}(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\hat{u}_i|^2 + \epsilon_i|\sqrt{M_{i,s_i}}D_{i,s_i}(w_i - \hat{w}_i)|^2\Bigg). \end{aligned} \tag{27}$$

By employing (21b), one has

$$\begin{aligned} &V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) \leq \\ &\tau_i\Bigg(\kappa_i V_{i,s_i}(x_i, \hat{x}_i) \\ &+ (1 + \frac{1}{\epsilon_i} + \epsilon_i)|\sqrt{M_{i,s_i}}D_{i,s_i}|^2|w_i - \hat{w}_i|^2 \\ &+ (1 + \frac{1}{\epsilon_i} + \epsilon_i)\left|\sqrt{M_{i,s_i}}(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\right|^2|\hat{u}_i|^2\Bigg). \end{aligned}$$

Since $\tau_i\kappa_i < 1$ by assumption, one can define $\hat{\kappa}_i = 1 - \tau_i\kappa_i$, and rewrite the previous inequality as follows

$$V_{i,s_i'}\left(f_{i,s_i}(x_i, w_i, u_i), \hat{f}_{i,s_i}(\hat{x}_i, \hat{w}_i, \hat{u}_i)\right) - V_{i,s_i}(x_i, \hat{x}_i) \leq$$

$$
\begin{aligned}
&- \hat{\kappa}_i V_{i,s_i}(x_i, \hat{x}_i) \\
&+ \tau_i\Big(1 + \frac{1}{\epsilon_i} + \epsilon_i\Big)\big|\sqrt{M_{i,s_i}} D_{i,s_i}\big|^2 |w_i - \hat{w}_i|^2 \\
&+ \tau_i\Big(1 + \frac{1}{\epsilon_i} + \epsilon_i\Big)\Big|\sqrt{M_{i,s_i}}(B_{i,s_i} R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})\Big|^2 |\hat{u}_i|^2.
\end{aligned}
$$

Thus, (11) holds with $p = q = 2$, $\lambda_i = \hat{\kappa}_i$, $\rho_{i,\text{ext}} = \tau_i(1 + \frac{1}{\epsilon_i} + \epsilon_i)\max_{s_i}\{|\sqrt{M_{i,s_i}}(B_{i,s_i}R_{i,s_i} - P_{i,s_i}\hat{B}_{i,s_i})|^2\}$ and $\rho_{i,\text{int}} = \tau_i(1 + \frac{1}{\epsilon_i} + \epsilon_i)\max_{s_i}\{|\sqrt{M_{i,s_i}}D_{i,s_i}|^2\}$.

Therefore, the candidate functions in (22) are simulation functions from $\hat{\Sigma}_i$ to Σ_i, for all $i \in \mathbb{N}$. $\square$

6. Example

To verify the effectiveness of our results, we apply them to a voltage regulation problem in AC islanded microgrids.

Islanded microgrids are self-sufficient small-scale power grids composed of several Distributed Generation Units (DGUs). They are designed to operate safely and reliably in the absence of connection to the main grid [27]. When the microgrids are working in connected mode, voltage and frequency are set by the main grid. However, in the islanded mode, they must be controlled by DGUs. Therefore, their connection should be robust against line faults or variations in the topology of DGUs' connections. Treating time-varying communication topologies is beneficial to evaluate the system performance in the presence of the line switches or plug-and-play operations.

We consider a switched AC islanded microgrid network modeled by an interconnection of fourth-order DGUs as underlying subsystems. In particular, we consider two *circular* topologies as shown in Figs. 1 and 2 and assume that the network topology switches between these two configurations at certain times. Let $\sigma_i(k)$ be the switching signal which takes values in the set $\{1, 2\}$, where $\sigma_i(k) = 1$ corresponds to the topology shown in Fig. 1 and $\sigma_i(k) = 2$ pertains to that in Fig. 2.

The discrete-time dynamics of each DGU in the microgrid with sampling time t_s is described by (28), adapted from [27].

$$
\Sigma_i : \left\{
\begin{aligned}
&\underbrace{\begin{bmatrix} \mathbf{V}_{i,d}(k+1) \\ \mathbf{V}_{i,q}(k+1) \\ \mathbf{I}_{ti,d}(k+1) \\ \mathbf{I}_{ti,q}(k+1) \\ \nu_{i,d}(k+1) \\ \nu_{i,q}(k+1) \end{bmatrix}}_{=:\mathbf{x}_i(k+1)} = \underbrace{\begin{bmatrix}
-\frac{t_s}{C_{ti}}\big(\sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \frac{R_{ij}}{Z_{ij}^2}\big) + 1 & t_s\omega_0 - \frac{t_s}{C_{ti}}\big(\sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \frac{X_{ij}}{Z_{ij}^2}\big) & \frac{t_s k_i}{C_{ti}} & 0 & 0 & 0 \\
-t_s\omega_0 + \frac{t_s}{C_{ti}}\big(\sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \frac{X_{ij}}{Z_{ij}^2}\big) & -\frac{t_s}{C_{ti}}\big(\sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \frac{R_{ij}}{Z_{ij}^2}\big) + 1 & 0 & \frac{t_s k_i}{C_{ti}} & 0 & 0 \\
-\frac{t_s k_i}{L_{ti}} & 0 & -\frac{t_s R_{ti}}{L_{ti}} + 1 & t_s\omega_0 & 0 & 0 \\
0 & -\frac{t_s k_i}{L_{ti}} & -t_s\omega_0 & -\frac{t_s R_{ti}}{L_{ti}} + 1 & 0 & 0 \\
-t_s & 0 & 0 & 0 & 1 & 0 \\
0 & -t_s & 0 & 0 & 0 & 1
\end{bmatrix}}_{=:A_{i,\sigma_i(k)}} \\
&\times \underbrace{\begin{bmatrix} \mathbf{V}_{i,d}(k) \\ \mathbf{V}_{i,q}(k) \\ \mathbf{I}_{ti,d}(k) \\ \mathbf{I}_{ti,q}(k) \\ \nu_{i,d}(k) \\ \nu_{i,q}(k) \end{bmatrix}}_{=:\mathbf{x}_i(k)} + \underbrace{\frac{t_s}{C_{ti}} \sum_{j\in I^{\text{in}}_{i,\sigma_i(k)}} \begin{bmatrix} \frac{R_{ij}}{Z_{ij}^2} & \frac{X_{ij}}{Z_{ij}^2} \\ -\frac{X_{ij}}{Z_{ij}^2} & \frac{R_{ij}}{Z_{ij}^2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{=:D_{i,\sigma_i(k)}} \underbrace{\begin{bmatrix} \mathbf{V}_{j,d}(k) \\ \mathbf{V}_{j,q}(k) \end{bmatrix}}_{=:\mathbf{w}_i(k)} \\
&+ \underbrace{\begin{bmatrix} -\frac{1}{C_{ti}} & 0 & 0 & 0 \\ 0 & -\frac{1}{C_{ti}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t_s & 0 \\ 0 & 0 & 0 & t_s \end{bmatrix}}_{=:H_i} \underbrace{\begin{bmatrix} \mathbf{I}_{Li,d} \\ \mathbf{I}_{Li,q} \\ \mathbf{y}_{i,d,ref} \\ \mathbf{y}_{i,q,ref} \end{bmatrix}}_{=:\mathbf{d}_i} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{t_s}{L_{ti}} & 0 \\ 0 & \frac{t_s}{L_{ti}} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{=:B_i} \mathbf{u}_i(k), \\
&\mathbf{y}_i(k) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{=:C_i} \begin{bmatrix} \mathbf{V}_{i,d}(k) \\ \mathbf{V}_{i,q}(k) \\ \mathbf{I}_{ti,d}(k) \\ \mathbf{I}_{ti,q}(k) \\ \nu_{i,d}(k) \\ \nu_{i,q}(k) \end{bmatrix}.
\end{aligned}
\right. \tag{28}
$$

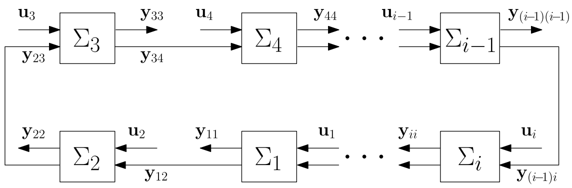


Fig. 1. The interconnected system Σ for $s_i = 1$.

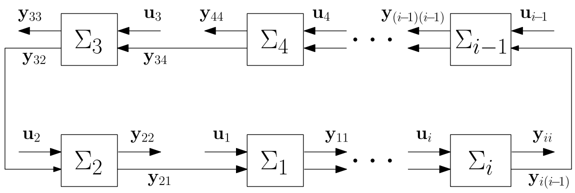


Fig. 2. The interconnected system Σ for $s_i = 2$.

In (28), $\mathbf{V}_{i,d}$ (resp. $\mathbf{V}_{i,q}$) are the d (resp. q) components of the load voltage. Similarly, $\mathbf{I}_{ti,d}$ (resp. $\mathbf{I}_{ti,q}$) denote the d (resp. q) components of the current of DGU Σ_i. In addition, the integrators $\nu_{i,d}$, $\nu_{i,q}$ are added for disturbance rejection reasons [27]. The control inputs (the voltage of corresponding voltage source converter (VSC)) and outputs are denoted by $u_i(k) = \left[\mathbf{V}_{ti,d}(k), \mathbf{V}_{ti,q}(k)\right]^\top$ and $\mathbf{y}_i(k) = \left[\mathbf{V}_{i,d}(k), \mathbf{V}_{i,q}(k)\right]^\top$, respectively. Furthermore, $D_{i,\sigma_i(k)}\mathbf{w}_i(k)$ models the coupling of DGU Σ_i with its neighbors $\Sigma_j, j \in I^{\text{in}}_{i,\sigma_i(k)}$, corresponding to each switching mode. In addition, $H_i\mathbf{d}_i$ represents the collection of load currents $I_{Li,d}$ and $I_{Li,q}$ which are considered as constant exogenous inputs acting as a disturbance and tracking references $\mathbf{y}_{i,ref} = \left[\mathbf{y}_{i,d,ref}, \mathbf{y}_{i,q,ref}\right]^\top$.

The parameters R_{ti}, L_{ti} are the resistance and inductance, respectively, corresponding to DGU Σ_i and R_{ij}, L_{ij} are those of the line between DGU Σ_i and DGU Σ_j which are connected through a three-phase line. In addition, $X_{ij} = \omega_0 L_{ij}$ and $Z_{ij} = |R_{ij} + jX_{ij}|$ with the rotation speed ω_0. Moreover, k_i is the transformer ratio which connects DGU Σ_i to the remainder of the network. The other transformer parameters are included in R_{ti} and L_{ti}. A shunt capacitance C_{ti} is used for attenuating the impact of high-frequency harmonics of the load voltage.

The interconnection structure switches between two *circular* topologies shown in Figs. 1 and 2. In these topologies, each subsystem Σ_i is fed by subsystems Σ_{i-1} for $\sigma_i(k) = 1$ ($I^{\text{in}}_{i,1} = \{i-1\}$) and Σ_{i+1} for $\sigma_i(k) = 2$ ($I^{\text{in}}_{i,2} = \{i+1\}$), respectively.

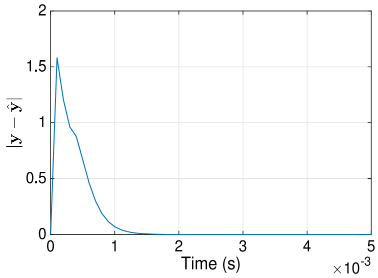
We denote Σ as the augmented infinite network consisting of infinite subsystems Σ_i. To construct an overall abstraction for Σ, we construct abstractions of subsystems Σ_i, $i \in \mathbb{N}$, with dimensions $\hat{n}_i$ for both $s_i = 1, 2$. Necessary and sufficient conditions on the geometrical properties of the involved matrices P_{i,s_i}, D_{i,s_i}, A_{i,s_i}, B_{i,s_i} in (24) are provided in [10, Sec. 4.3], which determine the lowest possible state dimension for $\hat{\Sigma}_i$, $i \in \mathbb{N}$, as $\hat{n}_i = 3$.

Now we compute the abstraction matrices satisfying (24). Considering (24a) and (24b) and taking $\hat{D}_{i,s_i} = t_s \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^\top$ and $T_{i,s_i} = 0$, we get

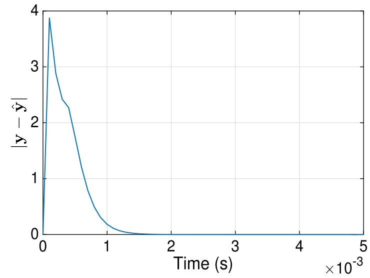
$$P_{i,s_i} = \frac{1}{2C_{ti}} \sum_{j \in I_{i,s_i}} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ \frac{X_{ij}}{Z_{ij}^2} & \frac{R_{ij}}{Z_{ij}^2} & 0 & 0 & 0 & 0 \\ \frac{R_{ij}}{Z_{ij}^2} & -\frac{X_{ij}}{Z_{ij}^2} & 0 & 0 & -1 & -1 \end{bmatrix}^\top,$$

$$Q_{i,s_i} = \frac{t_s k_i}{2C_{ti}} \sum_{j \in I_{i,s_i}} \begin{bmatrix} 0 & \frac{-X_{ij}}{Z_{ij}^2} & \frac{-R_{ij}}{Z_{ij}^2} \\ 0 & \frac{-R_{ij}}{Z_{ij}^2} & \frac{X_{ij}}{Z_{ij}^2} \end{bmatrix}, \; s_i = 1, 2.$$

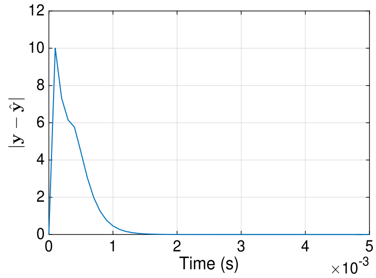
Furthermore, $\hat{A}_{i,s_i}$ is obtained by solving $\hat{n}_i \times \hat{n}_i$ equations provided that matrix $\sum_{j \in N_{i,s_i}} \begin{bmatrix} \frac{R_{ij}}{Z_{ij}^2} & \frac{-X_{ij}}{Z_{ij}^2} \\ \frac{X_{ij}}{Z_{ij}^2} & \frac{R_{ij}}{Z_{ij}^2} \end{bmatrix}$ is invertible.



(a) $i = 1, \dots, 10^2$



(b) $i = 1, \dots, 10^3$



(c) $i = 1, \dots, 10^4$

Fig. 3. The error norm between the output trajectories of Σ and $\hat{\Sigma}$ in per unit system.

In addition, $\hat{C}_{i,s_i} = \frac{1}{2C_{ti}} \sum_{j\in N_{i,s_i}} \begin{bmatrix} 0 & \frac{X_{ij}}{Z_{ij}^2} & \frac{R_{ij}}{Z_{ij}^2} \\ 0 & \frac{R_{ij}}{Z_{ij}^2} & -\frac{X_{ij}}{Z_{ij}^2} \end{bmatrix}$. By considering the computed matrices $\hat{A}_{i,s_i}$ and taking $\hat{B}_{i,s_i} = I_{\hat{n}_i}$, we choose appropriate matrices $\hat{K}_{i,s_i}$ for local controllers $\hat{u}_i = -\hat{K}_{i,s_i}\hat{x}_i$, which stabilize abstract subsystems $\hat{\Sigma}_i$ at the origin.

We also choose $R_{i,s_i} = (B_{i,s_i}^{\top} M_{i,s_i} B_{i,s_i})^{-1} B_{i,s_i}^{\top} M_{i,s_i} P_{i,s_i} \hat{B}_{i,s_i}$ to minimize $\rho_{\mathrm{i,ext}}$ as suggested in [6].

We illustrate the *scale-free* property of our approach with respect to the size of network via simulations. Following Theorem 14, we consider three truncated networks of microgrids shown in Figs. 1 and 2, respectively, consisting of 10^2, 10^3 and 10^4 subsystems. The parameters are set as $R_{ti} = 1.5$ mΩ, $L_{ti} = 300$ μH, $C_{ti} = 460$ μF, $k_i = 1$ for all subsystems Σ_i. Additionally, we choose $R_{ij} = 1$ mΩ, $L_{ij} = 10$ mH for all subsystems Σ_i, Σ_j with $s_i = 1$ and $R_{ij} = 1.2$ mΩ, $L_{ij} = 8$ mH for all subsystems Σ_i, Σ_j with $s_i = 2$. The microgrids frequency and the sampling time are set as $f_0 = 60$ Hz and $t_s = 10^{-4}$ s, respectively. The switchings between $s_i = 1$ and $s_i = 2$ occur at time steps $k = 4n$, $n \in \mathbb{N}$. We choose $\kappa_i = 0.01$ and take matrices K_{i,s_i} such that the eigenvalues of pairs (A_{i,s_i}, B_{i,s_i}) in closed loop are $[0.3; 0.15; 0.6; 0.2; 0.4; 0.5]$ for both $s_i = 1, 2$. Then, we compute $M_{i,s_i} = \begin{bmatrix} 12.291 & -0.473 & 11.082 & -1.081 & 7.809 & -1.665 \\ * & 23.041 & 0.535 & 17.258 & 1.780 & -0.585 \\ * & * & 37.993 & 26.610 & -0.607 & -0.458 \\ * & * & * & 47.840 & 0.536 & -0.358 \\ * & * & * & * & 20.913 & 2.330 \\ * & * & * & * & * & 23.559 \end{bmatrix}$, $s_i = 1$, and

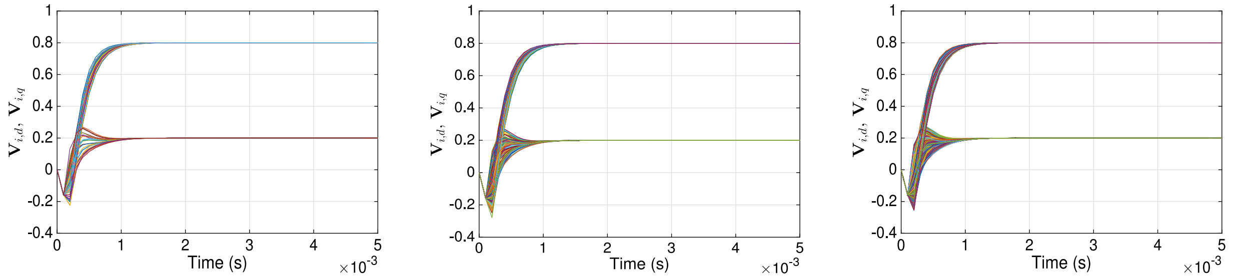
$$M_{i,s_i} = \begin{bmatrix} 13.013 & -0.801 & 10.235 & -2.132 & 6.296 & -1.819 \\ * & 25.612 & 0.669 & 15.228 & 1.171 & -1.091 \\ * & * & 38.619 & 24.581 & -1.167 & -0.915 \\ * & * & * & 49.145 & 0.251 & -0.958 \\ * & * & * & * & 22.323 & 3.244 \\ * & * & * & * & * & 25.158 \end{bmatrix}, \quad s_i = 2, \text{ satisfying (21).}$$

With the choice of (22) for V_{i,s_i}, we get $\tau_i \leq \max\{\frac{\lambda_{\max}(M_{i,s_i})}{\lambda_{\min}(M_{i,s_i})}\}$ for $s_i = 1, 2$. Thus, $1 \leq \tau_i \leq 67.61$. Therefore, the parameters in Definition 6 satisfying Assumption 8 are as $\alpha_i = 1$, $\lambda_i = \hat{\kappa}_i \in [0.3239, 0.99]$, $\epsilon_i = 1$, $\rho_{\mathrm{i,int}} \leq 0.321$, and $\rho_{\mathrm{i,ext}} \leq 512.312$. Recalling the circular interconnection topologies, each subsystem is directly fed by one other subsystem at each time instant. Thus, (12) gives $\gamma_{ij} = \tau_i(1 + \epsilon_i + \frac{1}{\epsilon_i}) \max_{s_i}\{|\sqrt{M_{i,s_i}} D_{i,s_i}|^2\} \bar{N}_i \frac{1}{\alpha_j}$ for $j \in I_{i.s_i}^{\mathrm{in}}$ and $\gamma_{ij} = 0$ for $j \notin I_{i.s_i}^{\mathrm{in}}$. Then, we get

$$\begin{aligned} r(\Psi) &< \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} \psi_{ij} < (1 + \epsilon_i + \frac{1}{\epsilon_i}) \max_{s_i}\{|\sqrt{M_{i,s_i}} D_{i,s_i}|^2\} \frac{\tau_i}{\hat{\kappa}_i} \\ &\leq 0.991, \end{aligned}$$

which implies the satisfaction of Assumption 10 on the spectral radius condition. Therefore, all the hypotheses of Theorem 11 are satisfied.

The norm of the overall error between the output trajectories of the *abstract* and *concrete* systems for three different sizes of networks are shown in Figs. 3. From the choice of $\hat{u}$ and stabilizability of $\hat{\Sigma}$ at the origin, $\lim_{k\to\infty} |\hat{\mathbf{u}}(k)|_2 \to 0$. This together with (7) implies that the mismatch between output trajectories converges to zero, illustrated by Fig. 3. The reference signals of DGUs are set as $\mathbf{y}_{i,ref} = [0.8, 0.2]^{\top}$, $i \in \mathbb{N}$. The closed-loop output trajectories of the *concrete* subsystems in a set-point tracking scenario are depicted by Fig. 4 in per unit system. From Fig. 4, one can see that the overall behavior of the network remains almost identical, though the network size grows dramatically. This admits that performances indices are independent of the network size.



(a) $\imath = 1, \ldots, 10^2$ (b) $i = 1, \ldots, 10^3$ (c) $i = 1, \ldots, 10^4$

Fig. 4. The external outputs $\mathbf{V}_{i,d}$, $\mathbf{V}_{i,q}$ in per unit system.

7. Conclusions

We proposed a compositional approach for the construction of continuous abstractions for infinite networks of switched discrete-time systems. To do this, we extended the notion of simulation functions to infinite-dimensional systems (networks of infinitely many finite-dimensional switched systems). Following the compositionality approach, we assigned to each subsystem an individual simulation function and constructed its local abstraction accordingly. Finally, we composed local abstractions to provide an abstraction of the overall network. We showed that the aggregation yields a continuous abstraction of the overall concrete network if a small-gain condition, expressed in terms of a spectral radius criterion, is satisfied. We also established that our result leads to scale-free compositional method for any finite-but-arbitrarily large networks. For linear systems, our conditions for constructing local abstractions boil down to linear matrix inequalities which can be computed efficiently. We applied our results to AC islanded microgrids under switched topologies and showed the scale-freeness of our proposed approach.

CRediT authorship contribution statement

Maryam Sharifi: Concept, Design, Analysis, Writing – review & editing. **Abdalla Swikir:** Concept, Design, Analysis, Writing – review & editing. **Navid Noroozi:** Concept, Design, Analysis, Writing – review & editing. **Majid Zamani:** Concept, Design, Analysis, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] C. Belta, B. Yordanov, E.A. Gol, Formal Methods for Discrete-Time Dynamical Systems, Vol. 89, Springer, 2017.
[2] T.A. Henzinger, S. Qadeer, S.K. Rajamani, You assume, we guarantee: Methodology and case studies, in: Int. Conf. Computer Aided Verification, Springer, 1998, pp. 440–451.
[3] R. Bloem, B. Jobstmann, N. Piterman, A. Pnueli, Y. Saar, Synthesis of reactive(1) designs, J. Comput. Syst. Sci. Int. 78 (3) (2012) 911–938.
[4] G. Pola, P. Tabuada, Symbolic models for nonlinear control systems: Alternating approximate bisimulations, SIAM J. Control Optim. 48 (2) (2009) 719–733.
[5] G. Reissig, A. Weber, M. Rungger, Feedback refinement relations for the synthesis of symbolic controllers, IEEE Trans. Automat. Control 62 (4) (2016) 1781–1796.
[6] A. Girard, G.J. Pappas, Hierarchical control system design using approximate simulation, Automatica 45 (2) (2009) 566–571.
[7] A. Lavaei, S.E.Z. Soudjani, R. Majumdar, M. Zamani, Compositional abstractions of interconnected discrete-time stochastic control systems, in: 56th Conf. Decision Control, Melbourne, 2017, pp. 3551–3556.
[8] S.W. Smith, M. Arcak, M. Zamani, Hierarchical control via an approximate aggregate manifold, in: Amer. Control Conf., 2018, pp. 2378–2383.
[9] S.W. Smith, H. Yin, M. Arcak, Continuous abstraction of nonlinear systems using sum-of-squares programming, 2019, arXiv e-prints.
[10] M. Rungger, M. Zamani, Compositional construction of approximate abstractions of interconnected control systems, IEEE Trans. Control Netw. Syst. 5 (1) (2016) 116–127.
[11] M. Zamani, M. Arcak, Compositional abstraction for networks of control systems: A dissipativity approach, IEEE Trans. Control Netw. Syst. 5 (3) (2017) 1003–1015.
[12] N. Noroozi, F.R. Wirth, M. Zamani, Compositional construction of abstractions via relaxed small-gain conditions Part I: continuous case, in: Euro. Control Conf., Limassol, 2018, pp. 76–81.
[13] K. Mallik, A.-K. Schmuck, S. Soudjani, R. Majumdar, Compositional synthesis of finite-state abstractions, IEEE Trans. Automat. Control 64 (6) (2018) 2629–2636.
[14] A. Swikir, M. Zamani, Compositional abstractions of interconnected discrete-time switched systems, in: Euro. Control Conf., 2019, pp. 1251–1256.
[15] A. Swikir, M. Zamani, Compositional synthesis of finite abstractions for networks of systems: A small-gain approach, Automatica 107 (2019) 551–561.

[16] N. Noroozi, A. Mironchenko, C. Kawan, M. Zamani, Set stability of infinite networks: ISS small-gain theory and its applications, IFAC-PapersOnLine 54 (9) (2021) 72–77, 24th Int. Symp. Math. Theory Netw. Syst..
[17] C. Kawan, A. Mironchenko, A. Swikir, N. Noroozi, M. Zamani, A Lyapunov-based small-gain theorem for infinite networks, IEEE Trans. Automat. Control (2020) http://dx.doi.org/10.1109/TAC.2020.3042410.
[18] S. Dashkovskiy, S. Pavlichkov, Stability conditions for infinite networks of nonlinear systems and their application for stabilization, Automatica 112 (2020) 108643.
[19] B. Bamieh, M.R. Jovanovic, P. Mitra, S. Patterson, Coherence in large-scale networks: Dimension-dependent limitations of local feedback, IEEE Trans. Automat. Control 57 (9) (2012) 2235–2249.
[20] P. Barooah, P.G. Mehta, J.P. Hespanha, Mistuning-based control design to improve closed-loop stability margin of vehicular platoons, IEEE Trans. Automat. Control 54 (9) (2009) 2100–2113.
[21] B. Bamieh, F. Paganini, M.A. Dahleh, Distributed control of spatially invariant systems, IEEE Trans. Automat. Control 47 (7) (2002) 1091–1107.
[22] M.R. Jovanović, B. Bamieh, On the ill-posedness of certain vehicular platoon control problems, IEEE Trans. Automat. Control 50 (9) (2005) 1307–1321.
[23] M. Sharifi, A. Swikir, N. Noroozi, M. Zamani, Compositional construction of abstractions for infinite networks of switched systems, in: 59th IEEE Conf. Decision Control, Jeju Island, 2020, pp. 476–481.
[24] H.K. Khalil, Nonlinear Systems, third ed., Prentice-Hall, Englewood Cliffs, NJ, 2002.
[25] M. Rungger, M. Zamani, Compositional construction of approximate abstractions, in: 18th Int. Conf. Hybrid Syst. Computation Control, New York, 2015, pp. 68–77.
[26] L. Vu, D. Chatterjee, D. Liberzon, Input-to-state stability of switched systems and switching adaptive control, Automatica 43 (2007) 639–646.
[27] S. Riverso, F. Sarzo, G. Ferrari-Trecate, Plug-and-play voltage and frequency control of islanded microgrids with meshed topology, IEEE Trans. Smart Grid 6 (3) (2015) 1176–1184.