Rigid toric matrix Schubert varieties

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Abstract

Fulton proves that the matrix Schubert variety $\overline{X_\pi} \cong Y_\pi \times \mathbb{C}^q$ can be defined via certain rank conditions encoded in the Rothe diagram of $\pi \in S_N$. In the case where $Y_\pi := \mathrm{TV}(\sigma_\pi)$ is toric (with respect to a $(\mathbb{C}^*)^{2N-1}$ action), we show that it can be described as a toric (edge) ideal of a bipartite graph G^π. We characterize the lower dimensional faces of the associated so-called edge cone σ_π explicitly in terms of subgraphs of G^π and present a combinatorial study for the first-order deformations of Y_π. We prove that Y_π is rigid if and only if the three-dimensional faces of σ_π are all simplicial. Moreover, we reformulate this result in terms of the Rothe diagram of π.

1 Introduction

In this paper, we are studying the *matrix Schubert varieties* $\overline{X_\pi} \cong Y_\pi \times \mathbb{C}^q$ associated with a permutation $\pi \in S_N$, where q is maximal possible. These varieties first appear during Fulton's study of the degeneracy loci of flagged vector bundles in [7]. Knutson and Miller [11] show that Schubert polynomials are multidegrees of matrix Schubert varieties. Moreover, the matrix Schubert variety is in fact related to the Schubert variety in the full flag manifold via the isomorphism in [10, Lemma A.4]. On the other hand, $\overline{X_\pi}$ is the closure of the preimage of the natural projection map $\mathrm{GL}_N \to \mathrm{GL}_N/B_+$ inside the space of $N \times N$ matrices. The matrix Schubert varieties are normal and one can define them by certain rank conditions encoded in the *Rothe diagram*. Our goal is to investigate the natural restricted torus action on these varieties. Escobar and Mészáros [6] study the *toric matrix Schubert varieties* via understanding

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their moment polytope. We present a reformulation of their classification in terms of *bipartite graphs*. The significance of this restatement is that it allows one to study the *first-order deformations* of the matrix Schubert variety in terms of graphs by [13]. The toric varieties arising from bipartite graphs have been studied in various papers [4, 9, 12, 14] in different perspectives. We will review a few aspects of this theory and bring our attention to the classification of the *rigid* toric matrix Schubert varieties. The toric varieties arising from graphs enable us to produce many interesting examples of rigid varieties. In fact, the first example of a rigid singularity in [8] is the cone over the Segre embedding $\mathbb{P}^r \times \mathbb{P}^1 \to \mathbb{P}^{2r+1}$ which is the affine toric variety associated with the complete bipartite graph $K_{r+1,2}$. Following the techniques in [1, 13] for the study of deformations of toric varieties, we classify rigid toric varieties Y_π in terms of bipartite graphs and Rothe diagram.

The organization of the paper is as follows. In preliminaries, we present some basic facts on matrix Schubert varieties and give a brief exposition of toric varieties arising from bipartite graphs. In Sect. 3, we reformulate the question of classification of toric matrix Schubert varieties to bipartite graphs. We then indicate how graphs may be used to investigate the complexity of the torus action in the sense of T-varieties [2, 3]. Section 4 starts with a discussion of deformation theory of toric varieties. Furthermore, it provides a detailed exposition of the faces of the moment (edge) cone of Y_π. In Lemma 4.5, we characterize the extremal rays of the edge cone. In Proposition 4.8 and Proposition 4.11, we present conditions for extremal rays to span a two-dimensional face and a three-dimensional face, respectively. Finally, we conclude with the following result:

Theorem (Theorem 4.12) *The toric variety Y_π is rigid if and only if the three-dimensional faces of its moment (edge) cone are all simplicial.*

We translate this result to the Rothe diagram of π and restate the classification in terms of certain shapes on the diagram, in particular solely depending on the pattern of the *essential set* (Definition 2.4) of π.

Corollary (Corollary 4.13) *Let* $\mathrm{Ess}(\pi) = \{(x_i, y_i) \mid x_{k+1} < \cdots < x_1 \text{ and } y_1 < \cdots < y_{k+1}\}$ *with $k \geq 3$ be the essential set of the Rothe diagram of $\pi \in S_N$. Then, the toric variety Y_π is rigid if and only if*

- $(x_1, y_1) \neq (m, 2)$ *and* $(x_{k+1}, y_{k+1}) \neq (2, n)$ *or*
- *for any $i \in [k]$, $(x_i, y_i) \neq (x_{i+1} + 1, y_{i+1} - 1)$.*

where m is the length and n is the width of the smallest rectangle containing $L(\pi)$ from Definition 2.4.

2 Preliminaries

2.1 Matrix Schubert varieties

In this section, we adopt the conventions from [6] for matrix Schubert varieties. We are mainly interested in matrix Schubert varieties for their effective torus actions and deformations. The statements presented in this section can be found in [7, 11].

Fig. 1 The Rothe Diagram of $[2143] \in S_4$
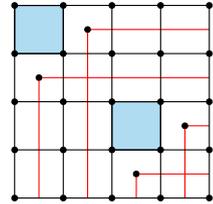
Let $\pi \in S_N$ be a permutation. We denote its *permutation matrix* by $\pi \in \mathbb{C}^{N \times N}$ as well and define it as follows:

$$\pi_{ij} = \begin{cases} 1, & \text{if } \pi(j) = i \\ 0, & \text{otherwise.} \end{cases}$$

Let B_- denote the group of invertible lower triangular matrices and B_+ denote the group of invertible upper triangular $N \times N$ matrices. The product $B_- \times B_+$ acts from left on $\mathbb{C}^{N \times N}$ and its action defined as:

$$(B_- \times B_+) \times \mathbb{C}^{N \times N} \longrightarrow \mathbb{C}^{N \times N}$$
$$((M_-, M_+), \mathcal{M}) \mapsto M_- \mathcal{M} M_+^{-1}$$

Definition 2.1 Let $\mathcal{M}_{(a,b)} \in \mathbb{C}^{a \times b}$ be the $a \times b$ matrix located at the upper left corner of $\mathcal{M} \in \mathbb{C}^{N \times N}$, where $1 \leq a \leq N$ and $1 \leq b \leq N$. The rank function of $\mathcal{M}$ is defined as $r_{\mathcal{M}}(a, b) := \text{rank}(\mathcal{M}_{(a,b)})$.

Note that the multiplication of a matrix $\mathcal{M} \in \mathbb{C}^{N \times N}$ on the left with M_- corresponds to the downward row operations and multiplication of $\mathcal{M}$ on the right with M_+ corresponds to the rightward column operations. Hence, one observes that $\mathcal{M} \in B_- \pi B_+$ if and only if $r_{\mathcal{M}}(a, b) = r_{\pi}(a, b)$ for all $(a, b) \in [N] \times [N]$. In this paper, $\mathcal{M}$ also appears as a matrix of indeterminates; however, the context clarifies which one is meant.

Definition 2.2 The Zariski closure of the orbit $\overline{X_\pi} := \overline{B_- \pi B_+} \subseteq \mathbb{C}^{N \times N}$ is called *the matrix Schubert variety* of π.

Rothe presented a combinatorial technique for visualizing inversions of the permutation π.

Definition 2.3 The *Rothe diagram* of π is defined as
$D(\pi) = \{(\pi(j), i) : i < j, \pi(i) > \pi(j)\}$.

One can draw the Rothe diagram $D(\pi)$ in the following way: Consider the permutation matrix π in an $N \times N$ grid. Cross out each box containing 1 and all the other boxes to the south and east of each box containing 1.

Definition 2.4 The connected part containing the box $(1, 1)$ in the diagram is called the dominant piece and is denoted by $\text{dom}(\pi)$. The set consisting of the southeast corners of each connected component of $D(\pi)$ is called the essential set and denoted
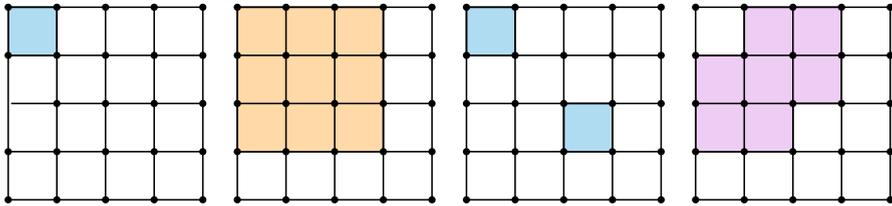
Fig. 2 The representations of $\mathrm{dom}(\pi)$, $\mathrm{NW}(\pi)$, $\mathrm{Ess}(\pi)$, and $L'(\pi)$

as $\mathrm{Ess}(\pi)$. Let $\mathrm{NW}(\pi)$ denote the union of the boxes located to the northwest of each box in $D(\pi)$. Finally, let $L(\pi) := \mathrm{NW}(\pi) \backslash \mathrm{dom}(\pi)$ and $L'(\pi) := L(\pi) \backslash D(\pi)$.

In Fig. 2, one can visualize these sets in the Rothe diagram of $[2143] \in S_4$.

Theorem 2.5 [7, Proposition 3.3, Lemma 3.10] *The matrix Schubert variety $\overline{X_\pi}$ is an affine variety of dimension $N^2 - |D(\pi)|$. It can be defined as a scheme by the equations $r_{\mathcal{M}}(a, b) \le r_\pi(a, b)$ for all $(a, b) \in \mathrm{Ess}(\pi)$.*

Remark 1 By the previous theorem, we observe that there exist no rank conditions imposed on the boxes which are not in $\mathrm{NW}(\pi)$. Thus these boxes are free in $\overline{X_\pi}$. Let $V_\pi \cong \mathbb{C}^{N^2 - |\mathrm{NW}(\pi)|}$ be the projection of the matrix Schubert variety $\overline{X_\pi} \subseteq \mathbb{C}^{N \times N}$ onto these free boxes. Also, we define Y_π as the projection onto the boxes of $L(\pi)$. Note that one obtains $(a, b) \in \mathrm{dom}(\pi)$ if and only if $r_\pi(a, b) = 0$. Hence, $\overline{X_\pi} = Y_\pi \times V_\pi$ holds. In particular, by Theorem 2.5,

$$\dim(Y_\pi) = (N^2 - |D(\pi)|) - (N^2 - |\mathrm{NW}(\pi)|) = |\mathrm{NW}(\pi)| - |D(\pi)| = |L'(\pi)|.$$

Example 1 The essential set for the permutation $\pi = [2143] \in S_4$ consists of the boxes $(1, 1)$ and $(3, 3)$. Let $\mathcal{M} = (m_{ij}) \in \mathbb{C}^{4 \times 4}$. First we note that $m_{11} = 0$ since $(1, 1) \in \mathrm{dom}(\pi)$. For the boxes in $L(\pi)$ one obtains the following inequality by Theorem 2.5:

$$r_{\mathcal{M}}(3, 3) = \mathrm{rank}(\mathcal{M}_{(3,3)}) = \mathrm{rank}\left(\begin{bmatrix} 0 & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\right) \le 2.$$

One obtains the ideal as generated by $I := (m_{11}, \det(\mathcal{M}_{(3,3)}))$. In particular $\overline{X_\pi} \cong \mathbb{V}(I) \times \mathbb{C}^7$ and $\dim(Y_\pi) = |L'(\pi)| = 7$.

2.2 Edge cones of bipartite graphs

In this section, we briefly introduce the construction of the toric varieties related to bipartite graphs as in [9, 13]. We refer the reader to [5] for details on the toric varieties and the notations. In particular $N \cong \mathbb{Z}^n$ stands for a lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is its dual lattice. We denote their associated vector spaces as $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $G \subseteq K_{m,n}$ be a bipartite graph with edge set $E(G)$ and vertex set $V(G)$. One defines the edge ring as

$$\mathrm{Edr}(G) := \mathbb{C}[t_i t_j \mid (i, j) \in E(G)].$$

Consider the following morphism:

$$\varphi_G : \mathbb{C}[x_1, \ldots, x_{|E(G)|}] \longrightarrow \mathrm{Edr}(G)$$
$$x_e \mapsto t_i t_j \quad \text{with} \quad e = (i, j).$$

The kernel of this map is a toric ideal, and it is called the *toric edge ideal* of G. The affine normal *toric variety associated with bipartite graph G* is

$$\mathrm{TV}(G) := \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_{|E(G)|}] / \ker \varphi_G).$$

Let e_i denote a canonical basis element of $\mathbb{Z}^m \times 0$ for $i = 1, \ldots, m$ and f_j denote a canonical basis element of $0 \times \mathbb{Z}^n$ for $j = 1, \ldots, n$. Set the lattices for the associated cones of the toric variety $\mathrm{TV}(G)$ as

$$N := \mathbb{Z}^{m+n} / \overline{(1, -1)} \quad \text{and} \quad M := \mathbb{Z}^{m+n} \cap \overline{(1, -1)}^{\perp}$$

where $\overline{(1, -1)} := \left\langle \sum_{i=1}^m e_i - \sum_{j=1}^n f_j \right\rangle$. We denote their associated vector spaces as $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$. In order to distinguish the elements of these vector spaces, we denote the canonical basis elements as $e_i \in N_{\mathbb{Q}}$ and $e^i \in M_{\mathbb{Q}}$.

Hence, we obtain that the (dual) edge cone associated with $\mathrm{TV}(G)$ is

$$\sigma_G^{\vee} = \mathrm{Cone}(e^i + f^j \mid (i, j) \in E(G)) \subseteq M_{\mathbb{Q}},$$

i.e., we have that

$$\mathrm{TV}(G) = \mathrm{Spec}(\mathbb{C}[\sigma_G^{\vee} \cap M]).$$

We observe in Sect. 3 that the dual edge cone $\sigma_G^{\vee}$ is in fact isomorphic to the moment cone of a matrix Schubert variety. We use this fact in order to determine the complexity of the torus action on a matrix Schubert variety.

Proposition 2.6 [13, Proposition 2.1, Lemma 2.17] *Let $G \subseteq K_{m,n}$ be a bipartite graph with k connected components and $m + n$ vertices. Then, the dimension of $\sigma_G^{\vee} \subset M_{\mathbb{R}}$ is $m + n - k$.*

Our aim is to study the first-order deformations $T^1_{\mathrm{TV}(G)}$ of the affine toric variety $\mathrm{TV}(G)$ by using the techniques from [1]. One can describe $T^1_{\mathrm{TV}(G)}$ via understanding the two and three-dimensional faces of the edge cone $\sigma_G \subseteq N_{\mathbb{Q}}$. We explain this briefly in Sect. 4.1. We now introduce terminology and notation from graph theory to describe the rays and faces of σ_G in terms of subgraphs of G. In Sect. 4.2, we will describe the rigidity of $\mathrm{TV}(G)$ in terms of graphs.

Remark 2 Note that if the bipartite graph G is the disjoint union of two connected bipartite graphs $G = G_1 \sqcup G_2$, then we have $\mathrm{TV}(G) = \mathrm{TV}(G_1) \times \mathrm{TV}(G_2)$. Thus, for the remainder of this section, we assume that $G \subseteq K_{m,n}$ is connected.

Definition 2.7 A non-empty subset A of $V(G)$ is called an *independent set* if it contains no adjacent vertices. An independent set $A \subsetneq V(G)$ is called a *maximal independent set* if there is no other independent set containing it. Let G be a bipartite graph and let U_1 and U_2 be the disjoint sets of $V(G)$. We say that an independent set is *one-sided* if it is contained either in U_1 or U_2. In a similar way, $A = A_1 \sqcup A_2$ is called a *two-sided independent set* if $\emptyset \neq A_1 \subsetneq U_1$ and $\emptyset \neq A_2 \subsetneq U_2$.

Definition 2.8 The *neighbor set* of $A \subseteq V(G)$ is defined as

$$N(A) := \{v \in V(G) \mid v \text{ is adjacent to some vertex in } A\}.$$

The *supporting hyperplane* of the dual edge cone $\sigma_G^\vee \subseteq M_{\mathbb{Q}}$ associated with an independent set $\emptyset \neq A$ is defined as

$$\mathcal{H}_A := \left\{ x \in M_{\mathbb{Q}} \mid \sum_{i \in A} x_i = \sum_{i \in N(A)} x_i \right\}.$$

Note that since no pair of vertices of an independent set A is adjacent, we obtain that $A \cap N(A) = \emptyset$.

Definition 2.9 (1) A subgraph of G with the same vertex set as G is called a *spanning subgraph* of G.

(2) Let $A \subseteq V(G)$ be a subset of the vertex set of G. The *induced subgraph* of A is defined as the subgraph of G formed from the vertices of A and all of the edges connecting pairs of these vertices. We denote it as $G[A]$ and we adopt the convention $G[\emptyset] = \emptyset$.

Now, we characterize the independent sets resulting in a facet of $\sigma_G^\vee$.

Definition 2.10 Let $G[[A]]$ denote the subgraph of G associated with the independent set A and defined as

$$\begin{cases} G[A \sqcup N(A)] \sqcup G[(U_1 \backslash A) \sqcup (U_2 \backslash N(A))], & \text{if } A \subseteq U_1 \text{ is one-sided.} \\ G[A \sqcup N(A)] \sqcup G[(U_2 \backslash A) \sqcup (U_1 \backslash N(A))], & \text{if } A \subseteq U_2 \text{ is one-sided.} \\ G[A_1 \sqcup N(A_1)] \sqcup G[A_2 \sqcup N(A_2)], & \text{if } A = A_1 \sqcup A_2 \text{ is two-sided.} \end{cases}$$

We define the *associated bipartite subgraph* $G\{A\} \subseteq G$ to the independent set A as the spanning subgraph $G[[A]] \sqcup \left(V(G) \backslash V(G[[A]]) \right)$.

Finally, we define the *first independent sets* $\mathcal{I}_G^{(1)}$ of G as

$$\mathcal{I}_G^{(1)} := \left\{ \begin{array}{l} \text{Two-sided maximal independent sets and one-sided independent sets } U_i \backslash \{\bullet\} \\ \text{where their associated bipartite subgraph has two connected components.} \end{array} \right\}$$

Note that Definition 2.10 becomes less technical for the first independent sets by [13, Proposition 2.9, Lemma 2.10]. Namely, we obtain:

> $G\{U_i\setminus\{\bullet\}\} = G[U_i\setminus\{\bullet\}\sqcup U_j]\sqcup\{\bullet\}$, for a one-sided first independent set with $i\neq j$ and
>
> $G\{A\} = G[A_1\sqcup N(A_1)]\sqcup G[A_2\sqcup N(A_2)]$, for a two-sided first independent set $A = A_1\sqcup A_2$.

Denote the set of extremal ray generators (i.e., 1-dimensional faces) of σ_G by $\sigma_G^{(1)}$. Recall that there is a bijective inclusion-reversing correspondence between the faces of σ_G and the faces of $\sigma_G^\vee$. Given a face $\tau\preceq\sigma_G^\vee$, we define the dual face τ^* of τ as $\{x\in\sigma_G^\vee\mid\langle x,u\rangle = 0\text{ for all }u\in\tau\}$. In particular, the facets of $\sigma_G^\vee$ are in bijection with the extremal rays of σ_G.

Theorem 2.11 [13, Theorem 2.8] *There is a one-to-one correspondence between the set of extremal generators $\sigma_G^{(1)}$ and the first independent set $\mathcal{I}_G^{(1)}$. In particular, the map is given as*

$$\Pi\colon \mathcal{I}_G^{(1)}\longrightarrow\sigma_G^{(1)}$$
$$A\mapsto\mathfrak{a}:=(\mathcal{H}_{A_i}\cap\sigma_G^\vee)^*$$

for a fixed $i\in\{1,2\}$ with $A_i\neq\emptyset$.

Example 2 We consider the bipartite graph $G\subset K_{2,2}$ obtained by removing one edge from the complete bipartite graph. The first independent set $\mathcal{I}_G^{(1)}$ for the graph G is colored in green. The sets $\{1\}$ and $\{3\}$ are not in $\mathcal{I}_G^{(1)}$ since they are contained in the two-sided maximal independent set $\{1,3\}$ and thus their associated subgraph has three connected components. The cone $\sigma_G\subseteq N_\mathbb{Q}$ is generated by e_1, f_1, and $e_2 - f_1$ corresponding, respectively, to the associated spanning subgraphs seen in the following figure.
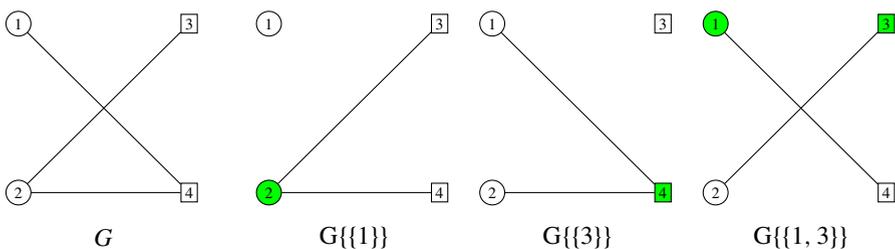
The next result classifies d-dimensional faces of σ_G via intersecting associated subgraphs related to first independent sets.

Theorem 2.12 [13, Theorem 2.18] *Let $S\subseteq\mathcal{I}_G^{(1)}$ be a subset of d first independent sets and let Π be the bijection from Theorem 2.11. The extremal ray generators $\Pi(S)$*

span a face of dimension d if and only if the dimension of the dual edge cone of the spanning subgraph $G[S] := \bigcap_{A \in S} G\{A\}$ *is* $m + n - d - 1$*, i.e.,* $G[S]$ *has* $d + 1$ *connected components. In particular, the face is equal to* $H_{Val_S} \cap \sigma_G$ *where* Val_S *is the degree sequence of the graph* $G[S]$ *and* $H_{Val_S} = \{x \in N_{\mathbb{Q}} \mid \langle Val_S, x \rangle = 0\}$ *is the usual supporting hyperplane in* $N_{\mathbb{Q}}$.

Example 3 All pairs of extremal rays of σ_G generate a two-dimensional face of σ_G since the intersection of all pairs of associated subgraphs has three connected components. In particular, the two-dimensional face generated by $(1, 0, 0, 0)$ and $(0, 1, -1, 0)$, i.e., the edge cone of $G\{\{1\}\} \cap G\{\{1, 3\}\}$, is equal to $H_{[0,1,1,0]} \cap \sigma_G$.

3 Torus action on matrix Schubert varieties in terms of graphs

The torus action on Y_π has been first studied by Escobar and Mészáros [6] where they characterize all toric varieties Y_π. We reformulate this classification in terms of graphs. Moreover, we approach the question of determining the dimension of the torus acting on Y_π from a perspective of T-varieties. These are normal varieties with effective torus action not necessarily having a dense torus orbit. They can be considered as the generalization of toric varieties with respect to the dimension of their torus action. For more details about T-varieties, we refer to [2, 3].

Definition 3.1 An affine normal variety X is called a *T-variety* of complexity-*d* if it admits an effective T torus action with $\dim(X) - \dim(T) = d$.

The matrix Schubert varieties are normal varieties (see [11, Theorem 2.4.3]). The action of $B_- \times B_+$ on $\overline{X_\pi}$ restricts to the action of $T^N \times T^N$, where $T^N \cong (\mathbb{C}^*)^N$ is a diagonal matrix of size $N \times N$. Since this action is not effective (since $(aI_N, aI_N) \cdot \mathcal{M} = \mathcal{M}$), we consider the stabilizer $\text{Stab}\left((\mathbb{C}^*)^{2N}\right)$ of this torus action and the action of the quotient $T := (\mathbb{C}^*)^{2N} / \text{Stab}\left((\mathbb{C}^*)^{2N}\right)$ on the matrix Schubert variety $\overline{X_\pi}$.

Let p be a general point in Y_π which have 1 in all boxes of $L(\pi)$ and 0 in others. Then, the closure of the torus orbit $\overline{(\mathbb{C}^*)^{2N} \cdot p}$ is the affine toric variety associated with the so-called $(\mathbb{C}^*)^{2N}$-moment cone (or weight cone) of Y_π, denoted by $\Phi(Y_\pi)$. One obtains that $\dim(\Phi(Y_\pi)) = \dim(\overline{(\mathbb{C}^*)^{2N} \cdot p})$. Since $\overline{(\mathbb{C}^*)^{2N} \cdot p}$ and Y_π are both irreducible, it suffices to examine their dimension for the complexity of the torus action on Y_π. Recall that the convex polyhedral cone generated by all weights of the torus action on Y_π in $M_{\mathbb{R}}$ (vector space spanned by the character lattice of the considered torus) is called the weight cone. Here, the weight cone of the action can be expressed as

$$\Phi(Y_\pi) = \text{Cone}(e_i - f_j \mid (i, j) \in L(w)),$$

where e_i denotes the canonical basis for $\mathbb{R}^m \times 0$ and f_j denotes the canonical basis for $0 \times \mathbb{R}^n$. Note that this cone is linearly isomorphic to a dual edge cone associated with a bipartite graph. Hence, one can define a bipartite graph $G^\pi \subseteq K_{m,n}$ from a

Rothe diagram $D(\pi)$ via the following bijection:

$$L(\pi) \longrightarrow E(G^\pi)$$
$$(a, b) \mapsto (a, b)$$

where for $(a, b) \in E(G^\pi)$, $a \in U_1$ and $b \in U_2$. Hence, we obtain also the vertex set $V(G^\pi)$. We denote the associated edge cone by σ_π. By Remark 1, we conclude the following:

Proposition 3.2 Y_π *is a T-variety of complexity-d with respect to the torus action T if and only if* $\dim(\sigma_\pi^\vee) = L'(\pi) - d$.

Example 4 Let us consider the matrix Schubert variety $\overline{X_\pi} \cong Y_\pi \times \mathbb{C}^7$ for $\pi = [2143]$ from Example 1. The second figure represents $L(\pi)$, and the third figure represents the bipartite graph G^π. For each box $(a, b) \in L(\pi)$, we construct an edge $(a, b) \in E(G^\pi)$ with vertices $a \in U_1$ and $b \in U_2$. The dimension of the associated dual edge cone $\sigma_\pi^\vee$ is 5 and $|L'(\pi)| = 7$. Hence, Y_π is a T-variety of complexity-2 with respect to the effective torus action of $T \cong (\mathbb{C}^*)^5$ and with a moment cone linearly equivalent to $\sigma_\pi^\vee$.
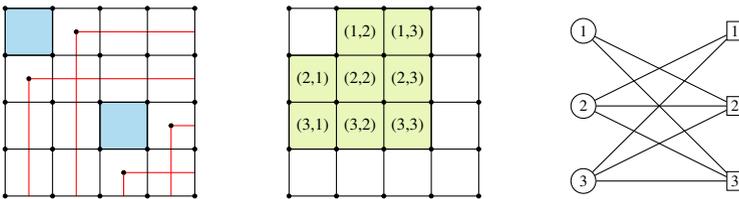

For the complexity zero case, i.e., toric case, we present an alternative proof with edge cones. A *hook* with corner (i, j) consists of boxes (i', j') such that $j = j'$ and $i' \geq i$ or $i = i'$ and $j \geq j'$.

Theorem 3.3 [6, Theorem 3.4] Y_π *is a toric variety if and only if $L'(\pi)$ consists of disjoint hooks not sharing a row or a column.*

Proof By Proposition 3.2, we aim to characterize the case when $\dim(\sigma_\pi^\vee) = L'(\pi)$. Note that $L(\pi)$ has a skew shape. Assume that $L(\pi)$ consists of k connected components with m_i rows and n_i columns for each $i \in [k]$. This means that we investigate the bipartite graph $G^\pi \subseteq K_{m,n}$ with k connected bipartite graph components $G_i^\pi \subseteq K_{m_i,n_i}$. By Proposition 2.6, the dimension of the cone $\dim(\sigma_\pi)$ is $m + n - k$. Since $L(\pi)$ has k connected components, the components of $L'(\pi)$ for each $i \in [k]$ do not share a row or a column. Therefore, we are left with proving the statement for a connected component $L_i(\pi)$ of $L(\pi)$. The dimension of the dual edge cone of G_i^π is equal to $|L_i'(\pi)|$ if and only if $L_i'(\pi)$ has a hook shape. □

Example 5 Let $\pi = [2413] \in S_4$. The first figure illustrates the Rothe diagram $D(\pi)$. The green colored boxes are $L(\pi)$ and the purple colored boxes are $L'(\pi)$. The dimension of the associated bipartite graph and $|L'(\pi)|$ is three. Also, as seen in the last figure, $L'(\pi)$ has a hook shape. Thus, $Y_{[2413]}$ is a toric variety with respect to the effective torus action of $T \cong (\mathbb{C}^*)^3$, in particular the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1$.
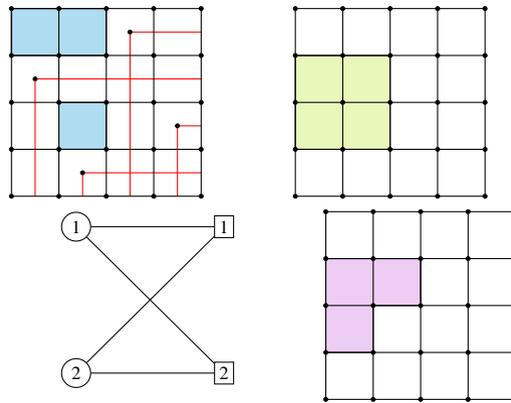
Theorem 3.3 and its alternative proof give us the opportunity to study the first-order deformations of Y_π in terms of edge cone σ_π and Rothe diagram $D(\pi)$ in Sect. 4.

4 Rigidity of toric matrix Schubert varieties

This section is devoted to the study of the detailed structure of the edge cone σ_π for matrix Schubert varieties $\overline{X_\pi}$ where $Y_\pi = \mathrm{TV}(G^\pi)$ is toric. Note that these matrix Schubert varieties are called toric matrix Schubert varieties in [6] and we adopt this convention. First, we explain briefly the combinatorial techniques for the first-order deformations of toric varieties. By studying the first independent sets of the bipartite graph G^π and the two and three-dimensional faces of σ_π, we present the necessary and sufficient condition for the rigidity of toric matrix Schubert varieties. By Remark 2 and since we investigate rigidity, we can assume that $L(\pi)$ is connected. Throughout this section, the connected bipartite graph $G^\pi \subseteq K_{m,n}$ denotes the associated bipartite graph of $L(\pi)$ which was constructed in Sect. 3.

4.1 Deformations of toric varieties

A deformation of an affine algebraic variety X_0 is a flat morphism $\pi : \mathcal{X} \longrightarrow S$ with $0 \in S$ such that $\pi^{-1}(0) = X_0$, i.e., we have the following commutative diagram.

$$
\begin{array}{ccc}
X_0 & \hookrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
0 & \hookrightarrow & S
\end{array}
$$

The variety $\mathcal{X}$ is called the total space and S is called the base space of the deformation. Let $\pi : \mathcal{X} \longrightarrow S$ and $\pi' : \mathcal{X}' \longrightarrow S$ be two deformations of X_0. We say that two deformations are isomorphic if there exists a map $\phi : \mathcal{X} \longrightarrow \mathcal{X}'$ over S inducing the identity on X_0. Let A be an Artin ring and let $S = \mathrm{Spec}(A)$. One has a contravariant functor Def_{X_0} such that $\mathrm{Def}_{X_0}(A)$ is the set of deformations of X_0 over S modulo isomorphisms.

Definition 4.1 The map π is called a first-order deformation of X_0 if $S = \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$. We set $T^1_{X_0} := \mathrm{Def}_{X_0}(\mathbb{C}[\epsilon]/(\epsilon^2))$.

The variety X_0 is called *rigid* if $T^1_{X_0} = 0$. This implies that a rigid variety X_0 has no non-trivial infinitesimal deformations. This means that every deformation $\pi \in \mathrm{Def}_{X_0}(A)$ over $S = \mathrm{Spec}(A)$ is trivial, i.e., isomorphic to the trivial deformation $X_0 \times S \longrightarrow S$.

For the case where $X_0 = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ is an affine normal toric variety, we introduce the techniques which are developed in [1] in order to investigate the $\mathbb{C}$-vector space $T^1_{X_0}$. The deformation space $T^1_{X_0}$ is multigraded by the lattice elements of M, i.e., $T^1_{X_0} = \bigoplus_{R \in M} T^1_{X_0}(-R)$. We first recall some definitions from [1, Section 2.4] in order to describe the homogeneous part $T^1_{X_0}(-R)$.

Definition 4.2 Let us call $R \in M$ a *deformation degree* and let $\sigma \subseteq N_{\mathbb{Q}}$ be generated by the extremal ray generators $a_1, \ldots, a_k$. We define the following affine space

$$[R = 1] := \{a \in N_{\mathbb{Q}} \mid \langle R, a \rangle = 1\} \subseteq N_{\mathbb{Q}}.$$

The assigned vector space is $[R = 0] := \{a \in N_{\mathbb{Q}} \mid \langle R, a \rangle = 0\} \subseteq N_{\mathbb{Q}}$. The cross-cut of σ in degree R is the polyhedron $Q(R) := \sigma \cap [R = 1]$.

The cross-cut $Q(R)$ has the cone of unbounded directions $Q(R)^\infty = \sigma \cap [R = 0]$. The compact part $Q(R)^c$ is generated by the vertices $\overline{a_i} = a_i/\langle R, a_i \rangle$ where $\langle R, a_i \rangle \geq 1$. Note that $\overline{a_i}$ is a lattice vertex in $Q(R)$ if $\langle R, a_i \rangle = 1$.

Definition 4.3 (i) Let $d^1, \ldots, d^\alpha \in R^\perp \subset N_{\mathbb{Q}}$ be the compact edges of $Q(R)$. The vector $\bar{\epsilon} \in \{0, \pm 1\}^\alpha$ is called a sign vector assigned to each two-dimensional compact face ϵ of $Q(R)$ defined as

$$\overline{\epsilon_i} = \begin{cases} \pm 1, & \text{if } d^i \text{ is an edge of } \epsilon \\ 0 \end{cases}$$

such that $\sum_{i \in [\alpha]} \overline{\epsilon_i} d^i = 0$, i.e., the oriented edges $\overline{\epsilon_i} d^i$ form a cycle along the edges of ϵ. We choose one of both possibilities for the sign of $\overline{\epsilon}$.

(ii) For every deformation degree $R \in M$, the related vector space is defined as

$$V(R) = \left\{ \bar{t} = (t_1, \ldots, t_\alpha) \in \mathbb{C}^\alpha : \sum_{i \in [\alpha]} t_i \overline{\epsilon_i} d^i = 0, \text{ for every compact 2-face } \epsilon \preceq Q(R) \right\}.$$

The toric variety $\mathrm{TV}(G)$ associated with a bipartite graph $G \subseteq K_{m,n}$ is smooth in codimension 2 [13, Theorem 4.5]. Hence, we introduce the result for this special case:

Theorem 4.4 [1, Corollary 2.7] *If the affine normal toric variety X_0 is smooth in codimension 2, then $T^1_{X_0}(-R)$ is contained in $V(R)/\mathbb{C}(1, \ldots, 1)$. Moreover, it is built by those $\bar{t}$'s satisfying $t_{ij} = t_{jk}$ where $\overline{a_j}$ is a non-lattice common vertex in $Q(R)$ of the edges $d^{ij} = \overline{\overline{a_i}\, \overline{a_j}}$ and $d^{jk} = \overline{\overline{a_j}\, \overline{a_k}}$.*

Fig. 3 Compact 2-faces sharing an edge or a non-lattice vertex in $Q(R)$
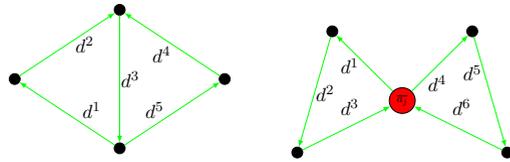
Remark 3 The following two cases of $Q(R)$ in Fig. 3 will appear often while we study $T^1_{\mathrm{TV}(G)}(-R)$. Let us interpret these cases with the previous result.

- Let $\epsilon^1, \epsilon^2 \preceq Q(R)$ be the compact 2-faces sharing the edge d^3. We choose the sign vectors $\overline{\epsilon^1} = (1, 1, 1, 0, 0)$ and $\overline{\epsilon^2} = (0, 0, 1, 1, 1)$. Suppose that $\overline{t} = (t_1, t_2, t_3, t_4, t_5) \in V(R)$. We observe that $t_1 = t_2 = t_3$ for 2-face ϵ^1 and $t_3 = t_4 = t_5$ for 2-face ϵ^2.

- Let $\epsilon^1, \epsilon^2 \preceq Q(R)$ be the compact 2-faces connected by the vertex $\overline{a_j}$. As in the previous case, we obtain that $t_1 = t_2 = t_3$ and $t_4 = t_5 = t_6$. By Theorem 4.4, if a_j is a non-lattice vertex, then we obtain $t_3 = t_4$.

We will refer to these two cases by "t is transferred by an edge or a vertex" during the investigation of $Q(R)$ in Theorem 4.12. Note that we observe certain pairs of vertices of $Q(R)$ where their convex hull is not contained in $Q(R)$. This means that the corresponding pair of extremal rays of the cone does not form a two-dimensional face (non-2-faces). In addition to non-3-faces, these are critical cases to investigate during the application of Theorem 4.4.

### 4.2 Faces of the edge cone σ_π of toric variety Y_π

In order to study the rigidity of $Y_\pi = \mathrm{TV}(\sigma_\pi)$ with Theorem 4.4 and Remark 3, we investigate the face structure of the edge cone σ_π more closely. By definition, we consider three types of first independent sets with the following notations: the one-sided first independent sets $A = U_1 \backslash \{\bullet\}$, $B = U_2 \backslash \{\bullet\}$ and two-sided (maximal) first independent sets $C = C_1 \sqcup C_2$. We label the boxes of the essential set from the bottom of the diagram starting with (x_1, y_1) to the top ending with (x_{k+1}, y_{k+1}), i.e., we have $x_{k+1} < \cdots < x_1$ and $y_1 < \cdots < y_{k+1}$.

Lemma 4.5 *For any permutation $\pi \in S_N$,*

(1) *The one-sided first independent sets of G^π are $U_i \backslash \{u_i\}$ for all $u_i \in U_i$ and for $i = 1, 2$.*

(2) *The two-sided first independent sets are the maximal two-sided independent sets of G^π.*

Proof By Theorem 3.3, $L'(\pi)$ is a hook. The boxes of $L(\pi)$ form a shape of a Ferrer diagram, i.e., we have $\lambda_1 \geq \cdots \geq \lambda_t$ where λ_i denotes the number of boxes at ith row of $L(\pi)$. Consider the smallest rectangle containing $L(\pi)$ of length m and of width n. The removed edges of the bipartite graph $G^\pi \subseteq K_{m,n}$ are linked with the free boxes of $\overline{X_\pi}$ in the rectangle. Let $(x_i, y_i) \in \mathrm{Ess}(\pi)$, equivalently let $(x_i, y_i) \in E(G^\pi)$. Then, one
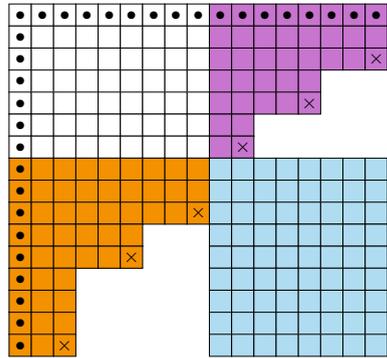
obtains naturally that there exists a two-sided maximal independent set $C = C_1 \sqcup C_2 = \{x_i + 1, \ldots, m\} \sqcup \{y_{i-1} + 1, \ldots, n\}$ where $(x_{i-1}, y_{i-1}) \in \mathrm{Ess}(\pi)$ with $x_{i-1} > x_i$ and $y_{i-1} < y_i$. Then, the neighbor sets are $N(C_1) = U_2 \backslash C_2 = \{1, \ldots, y_{i-1}\}$ and $N(C_2) = U_1 \backslash C_1 = \{1, \ldots, x_i\}$. Therefore, the boxes for the induced subgraphs $G[C_1 \sqcup N(C_1)]$ and $G[C_2 \sqcup N(C_2)]$ also form a shape of a Ferrer diagram and G{C} has two connected components. In particular, $U_i \backslash \{u_i\}$ cannot be contained in a two-sided independent set. Suppose that $G\{U_i \backslash \{u_i\}\}$ has more than three components. Then, as in [13, Proposition 2.13], there exist two-sided first independent sets $C^i \in \mathcal{I}_G^{(1)}$ such that $\bigsqcup C_1^i = U_i \backslash \{u_i\}$ which is not possible. □

Lemma 4.6 *There exist k two-sided first independent sets of G^{π} with $|\mathrm{Ess}(\pi)| = k + 1$. Moreover, if $k \geq 2$ and, C and C' are two-sided first independent sets of G^{π}, then $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$.*

Proof Consider again the smallest rectangle containing $L(\pi)$ of a length m and of a width n. If there exists only one essential set of π, then $G^{\pi} = K_{m,n}$. Assume that there is more than one essential box. Let (x_j, y_j) and (x_i, y_i) be two essential boxes with $j < i, x_j > x_i$ and $y_j < y_i$. By Lemma 4.5, we obtain two first independent sets $C = \{x_i + 1, \ldots, m\} \sqcup \{y_{i-1} + 1, \ldots, n\}$ and $C' = \{x_j + 1, \ldots, m\} \sqcup \{y_{j-1} + 1, \ldots, n\}$ of G^{π}. We infer that $C_1 \subsetneq C_1'$ and $C_2' \subsetneq C_2$. □

Example 6 The boxes of $L(\pi)$ for the toric variety Y_{π} are presented in Fig. 4. The blue boxes are removed edges between some vertex sets $C_1 \subset U_1$ and $C_2 \subset U_2$. We observe that $C := C_1 \sqcup C_2$ is a maximal independent set. In particular, the orange color represents the edges of the induced subgraph $G[C_1 \sqcup N(C_1)]$ and the purple color represents the edges of the induced subgraph $G[C_2 \sqcup N(C_2)]$. The crossed boxes are the boxes of the essential set $\mathrm{Ess}(\pi)$. The boxes with a dot form the shape of a hook and these are the boxes of $L'(\pi)$.

Let us first identify the cases where there are one or two essential boxes.

Lemma 4.7 *Let $G^{\pi} \subseteq K_{m,n}$ be the associated connected bipartite graph to the toric variety Y_{π}.*

(1) *If* $|\operatorname{Ess}(\pi)| = 1$, *then the toric variety* Y_π *is isomorphic to* $\operatorname{TV}(K_{m,n})$, *i.e., the cone over the Segre variety* $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$. *In particular,* Y_π *is rigid if* $m \neq 2$ *and* $n \neq 2$.

(2) *If* $|\operatorname{Ess}(\pi)| = 2$, *then the toric variety* Y_π *is rigid if and only if* $|C_1| \neq 1$ *and* $|C_2| \neq n - 2$ *or* $|C_1| \neq m - 2$ *and* $|C_2| \neq 1$.

Proof It follows from [13, Theorems 4.3, 4.6].   □

From now on, we assume that $|\operatorname{Ess}(\pi)| \geq 3$. This means that we consider the associated connected bipartite graph $G^\pi \subsetneq K_{m,n}$ with $m, n \geq 4$. We denote by $\mathcal{I}_{G^\pi}^{(d)}$ the set of tuples of first independent sets forming a d-dimensional face of σ_G. Let $\sigma_{G^\pi}^{(d)}$ be the set of d-dimensional faces of σ_π. Recall the classification of d-dimensional faces of an edge cone in Theorem 2.12 for a subset $S = \{A^{(1)}, \ldots, A^{(d)}\} \subseteq \mathcal{I}_G^{(1)}$ of d first independent sets. Let Π be the isomorphism from Theorem 2.11. Then, we have

$$\Pi^d : \mathcal{I}_{G^\pi}^{(d)} \longrightarrow \sigma_{G^\pi}^{(d)}$$
$$(A^{(1)}, \ldots, A^{(d)}) \mapsto (\Pi(A^{(1)}), \ldots, \Pi(A^{(d)})) = H_{\operatorname{Val}_S} \cap \sigma_{G^\pi}$$

if and only if $G[S] = \bigcap_{A \in S} G\{A\}$ has $d + 1$ connected components.

Next, we analyze the pairs of extremal rays of σ_π to determine if they form a face. This is one of the crucial results for the proof of Theorem 4.12.

Proposition 4.8 *Let* $A = U_1 \backslash \{i\}$, $B = U_2 \backslash \{j\}$, $C = C_1 \sqcup C_2$ *be three types of first independent sets of the bipartite graph* G^π.

(1) *For any* $A, B \in \mathcal{I}_{G^\pi}^{(1)}$, $(A, B) \in \mathcal{I}_{G^\pi}^{(2)}$.

(2) *For any* $C, C' \in \mathcal{I}_{G^\pi}^{(1)}$, $(C, C') \in \mathcal{I}_{G^\pi}^{(2)}$.

(3) $(A, A') \notin \mathcal{I}_{G^\pi}^{(2)}$ *if and only if there exists a first independent set* $U_1 \backslash \{i, i'\} \sqcup C_2$ *where* $C_2 \subsetneq U_2$ *is some vertex set with* $|C_2| \leq n - 2$.

(4) $(A, C) \notin \mathcal{I}_{G^\pi}^{(2)}$ *if and only* $C_1 = \{i\}$ *or there exists* $C' \in \mathcal{I}_{G^\pi}^{(1)}$ *with* $C_1 \backslash C_1' = \{i\}$.

Proof (1) Suppose that there exist a pair $(A, B) \notin \mathcal{I}_{G^\pi}^{(2)}$. Consider the intersection subgraph $G\{A\} \cap G\{B\}$ and assume that it has isolated vertices other than $\{i, j\}$. Consider the isolated vertices in $U_1 \backslash \{i\}$. This means that there exists a two-sided independent set consisting of these isolated vertices and B, which is impossible since $B \in \mathcal{I}_{G^\pi}^{(1)}$. Now assume that $G\{A\} \cap G\{B\}$ consists of the isolated vertices $\{i, j\}$ and $k \geq 2$ connected bipartite graphs G_i. Let the vertex set of G_i consist of $V_i \subsetneq U_1$ and $W_i \subsetneq U_2$. Since $B \in \mathcal{I}_{G^\pi}^{(1)}$, there exist an edge $(i, w_i) \in E(G^\pi)$ for each $i \in [k]$, where $w_i \in W_i$. Symmetrically, since $A \in \mathcal{I}_{G^\pi}^{(1)}$, there exist an edge $(j, v_i) \in E(G^\pi)$ for each $i \in [k]$ where $v_i \in V_i$. However, then for $I \subsetneq [k]$, we obtain the two-sided maximal independent sets of form $\bigsqcup_{i \in I} V_i \sqcup (B \backslash (\bigsqcup_{i \in I} W_i)$ which contradicts the construction of G^π.

(2) Let (x_j, y_j) and (x_i, y_i) be two essential boxes with $x_j > x_i$ and $y_j < y_i$, associated with two first independent sets C and C' in $\mathcal{I}_{G^\pi}^{(1)}$. It is enough to check if $G[C_1' \sqcup N(C_1')] \cap G[C_2 \cap N(C_2)]$ is connected. We observe that the edges of this

graph are represented by the square with vertices $(x_i + 1, y_{j-1} + 1)$, $(x_i + 1, y_{i-1})$, $(x_j, y_{j-1} + 1)$, and (x_j, y_{i-1}), intersected with the diagram $D(\pi)$. This intersection is also a Ferrer diagram and connected.

(3) Consider the intersection subgraph $G\{A\} \cap G\{A'\}$. Assume that it has only $\{i, i'\} \subsetneq U_1$ as isolated vertices and k connected bipartite graphs. Then, as in case (1), there exist first independent sets C, C' with $C_1 \cap C_1' = \emptyset$, which is impossible by Lemma 4.6. Assume that it has the isolated vertices $\{i, i'\} \subsetneq U_1$ and $C_2 \subsetneq U_2$ with $|C_2| \le n - 2$. Then, $C := U_1 \backslash \{i, i'\} \sqcup C_2$ is maximal and thus a first independent set.

(4) Suppose that $i \in C_1$ and $(A, C) \notin \mathcal{I}_{G^\pi}^{(2)}$. Consider the intersection subgraph $G\{A\} \cap G\{C\}$. Similarly to the last investigations, we conclude $G[C_1 \sqcup N(C_1)]$ cannot admit $\{i\}$ as its only isolated vertex. If $C_1 = \{i\}$, then the intersection subgraph admits of $|N(C_1)| + 1$ isolated vertices and $G[C_2 \sqcup N(C_2)]$. Assume that the intersection subgraph consists of the isolated vertex $\{i\} \subsetneq C_1$ and some vertex set $C_2' \subsetneq N(C_1)$. This means that $C' := C_1 \backslash \{i\} \sqcup C_2' \sqcup C_2$ is a maximal two-sided independent set. Hence, $C' \in \mathcal{I}_{G^\pi}^{(1)}$.   □

In order to eliminate the non-rigid cases of Y_π, we introduce the following result. This allows us to focus only on simplicial three-dimensional faces of σ_π.

Theorem 4.9 [13, Theorem 3.18] *Let $G \subseteq K_{m,n}$ be a connected bipartite graph. Assume that the edge cone σ_G admits a three-dimensional non-simplicial face. Then, TV(G) is not rigid.*
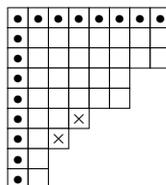
Lemma 4.10 *Assume that $|\mathrm{Ess}(\pi)| \ge 3$.*

(1) *Let C, $C' \in \mathcal{I}_{G^\pi}^{(1)}$ with $C_1' \subsetneq C_1$ and $C_2 \subsetneq C_2'$. If $|C_1| - |C_1'| = 1$ and $|C_2'| - |C_2| = 1$, then Y_π is not rigid.*

(2) *If there exists a first independent set $C \in \mathcal{I}_{G^\pi}^{(1)}$ with $|C_1| = 1$ and $|C_2| = n - 2$ or $|C_1| = m - 2$ and $|C_2| = 1$, then Y_π is not rigid.*

Proof We refer again to [13]. These are the cases from Lemma 3.10 (2)(i) and Lemma 2.11 (2). By Theorem 4.9, we conclude that Y_π is not rigid in these cases.   □

We denote the image of the first independent sets $A = U_1 \backslash \{i\}$, $B = U_2 \backslash \{j\}$, and C under the map Π of Theorem 4.9 by $\mathfrak{a} = e_i$, $\mathfrak{b} = f_j$, and $\mathfrak{c}$.

Example 7 Let $\pi = [1, 10, 8, 7, 6, 9, 4, 5, 2, 3] \in S_{10}$ and let us consider the diagram $L(\pi)$. The dotted boxes $L'(\pi)$ form a hook and therefore Y_π is toric. Consider the first independent sets $C = \{8, 9\} \sqcup \{3, 4, 5, 6, 7, 8\}$ and $C' = \{7, 8, 9\} \sqcup \{4, 5, 6, 7, 8\}$ of the associated connected bipartite graph $G^\pi \subsetneq K_{9,8}$. By Lemma 4.10, $\langle \mathfrak{c}, \mathfrak{c}', e_7, f_4 \rangle$ spans a three-dimensional face of σ_π and hence Y_π is not rigid.

The cases in Lemma 4.10 are the only cases where σ_π has non-simplicial three-dimensional faces. We conclude this by examining the non-2-face pairs from Proposition 4.8 (3) and (4).

From now on, we may assume that all three-dimensional faces of G^π are simplicial. In the next proposition, we examine the triples which do not form a three-dimensional face of σ_π. After this result, we will be ready to prove Theorem 4.12.

Proposition 4.11 *Let I be a triple of the first independent sets of G^π not forming a three-dimensional face. Assume that any pair of first independent sets of I forms a two-dimensional face. Then, the triple I is*

(1) $(A, A', A'') \notin \mathcal{I}^{(3)}_{G^\pi}$ *if and only if there exists* $C \in \mathcal{I}^{(1)}_{G^\pi}$ *with* $C_1 = U_2 \backslash \{i, i', i''\}$.
(2) $(A, A', C) \notin \mathcal{I}^{(3)}_{G^\pi}$ *if and only if* $C_1 = \{i, i'\}$ *or there exists* $C' \in \mathcal{I}^{(1)}_{G^\pi}$ *with* $C_1 \backslash C'_1 = \{i, i'\}$.

Proof The first case follows analogously as in the proof of Proposition 4.8 (3). Consider a triple of form (C, C', C'') with $C_1 \subsetneq C'_1 \subsetneq C''_1$ and $C''_2 \subsetneq C'_2 \subsetneq C_2$. Any such triple forms a 3-face, since the intersection graph $G\{C\} \cap G\{C'\} \cap G\{C''\}$ is equal to

$$G[C_1 \sqcup N(C_1)] \sqcup G[(C'_1 \backslash C_1) \sqcup (C_2 \backslash C'_2)] \sqcup G[(C''_1 \backslash C') \sqcup (C'_2 \backslash C''_2)] \sqcup G[C''_2 \sqcup N(C''_2)]$$

and has 4 connected components since any pair of two-sided first independent sets forms a 2-face. For such triples containing both A and B, similar to the arguments in the proof of Proposition 4.8 (1), we conclude that they form 3-faces. Finally, consider the triple (A, A', C). Since $(A, A') \in \mathcal{I}^{(2)}_{G^\pi}$, i and i' cannot be both in $N(C_2)$. Assume that $i \in C_1$ and $i' \in N(C_2)$. Since (A, C) and (A', C) form 2-faces, the triple (A, A', C) forms a 3-face. Hence, we have that $\{i, i'\} \subseteq C_1$. The statement follows by an analysis similar to that in the proof of Proposition 4.8 (4). □

Remark 4 In addition to the triple in Proposition 4.11, the triples of first independent sets of G^π, containing the pairs in Proposition 4.8 (3) and (4) do not form a three-dimensional face of σ_π.

4.3 Classification of rigid toric varieties Y_π

The following two results classify the rigid toric matrix Schubert varieties in terms of edge cone σ_π and in terms of its Rothe diagram $D(\pi)$.

Theorem 4.12 *The toric variety $Y_\pi = \mathrm{TV}(\sigma_\pi)$ is rigid if and only if the three-dimensional faces of σ_π are all simplicial.*

Proof We have proven the statement for $|\operatorname{Ess}(\pi)| = 1, 2$. We prove it now for $|\operatorname{Ess}(\pi)| \geq 3$. We examine the non-2-faces pairs from Proposition 4.8 (case (iii) and case (iv)) and non-3-face triples from Proposition 4.11 (case (i) and case (ii)). By Theorem 4.4 and Remark 3, we compute $T^1_{X_0}(-R)$, for all deformation degrees $R \in M \cong \mathbb{Z}^{m+n}/\overline{(1, -1)}$.

(i) Suppose that (e_1, e_2, e_3) does not span a 3-face and (e_1, e_2), (e_1, e_3) and (e_2, e_3) do span 2-faces. By Proposition 4.11 (1), there exists a first independent set $C \in \mathcal{I}_{G^\pi}^{(1)}$ with $C_1 = U_1 \backslash \{1, 2, 3\}$ and $|C_2| \leq n - 2$. Assume that $\overline{e_1}, \overline{e_2}$, and $\overline{e_3}$ are vertices in $Q(R)$ for some deformation degree $R \in M$. Let $\mathfrak{a} \in \sigma_\pi^{(1)}$ be an extremal ray. Since $(\mathfrak{a}, e_i, e_j)$ spans a 3-face of σ_π for every $i, j \in [3]$ and $i \neq j$, we are left with showing that there exists no such $\overline{\mathfrak{a}} \in Q(R)$. However, even though this is the case, i.e., $R_i \leq 0$, for every $i \in [m+n] \backslash \{1, 2, 3\}$, then $\overline{\mathfrak{c}} \in Q(R)$. By Proposition 4.11 (2), $(e_i, e_j, \mathfrak{c})$ spans a 3-face for all $i, j \in [3]$ with $i \neq j$.

(ii) Suppose that $(e_1, e_2, \mathfrak{c})$ does not span a 3-face and (e_1, e_2), $(e_1, \mathfrak{c})$ and $(e_2, \mathfrak{c})$ do span 2-faces. By Proposition 4.11, $|C_1| = \{1, 2\}$ or there exists $C' \in \mathcal{I}_{G^\pi}^{(1)}$ such that $C_1 \backslash C_1' = \{1, 2\}$. Assume that $\overline{e_1}, \overline{e_2}$, and $\overline{\mathfrak{c}}$ are vertices in $Q(R)$ for some deformation degree $R \in M$. If $|C_1| = \{1, 2\}$, then there exists $b \in N(C_1)$ such that $B = U_2 \backslash \{b\} \in \mathcal{I}_{G^\pi}^{(1)}$ and $\overline{b} \in Q(R)$ is not a lattice vertex or there exist at least three vertices $b_i \in N(C_1)$ such that $B_i = U_2 \backslash \{b_i\} \in \mathcal{I}_{G^\pi}^{(1)}$ and $\overline{b_i}$ is a lattice vertex in $Q(R)$. If $C_1 \backslash C_1' = \{1, 2\}$, then either $\overline{\mathfrak{c}'} \in Q(R)$ or $\overline{b} \in Q(R)$ for $b \in C_2' \backslash C_2$ and $B = U_2 \backslash \{b\} \in \mathcal{I}_{G^\pi}^{(1)}$. By Proposition 4.11, any triple containing a pair of $\{e_1, e_2, \mathfrak{c}\}$ and a ray generator of type $\mathfrak{b}, \mathfrak{b}_i, \mathfrak{c}'$ defined as before forms a 3-face.

(iii) In the previous two cases, we showed that t is transferred by an edge for non-3-faces as explained in Remark 3. We next continue the proof for non-2-faces.

Suppose that (e_1, e_2) does not span a 2-face and $\overline{e_1}$ and $\overline{e_2}$ are in $Q(R)$ for some deformation degree $R \in M$. Then, by Proposition 4.8 (3), there exists a first independent set $C = C_1 \sqcup C_2 \in \mathcal{I}_{G^\pi}^{(1)}$ with $C_1 = U_1 \backslash \{1, 2\}$ and $|C_2| \leq n - 2$. Assume that there exist k vertices $\overline{f_j}$ in $Q(R)$ where $j \in [k] \subseteq [n]$. If $k = 0$, then $\mathfrak{c}$ is a non-lattice vertex in $Q(R)$, and t is transferred by $\mathfrak{c}$. If $k = 1$, then $\overline{f_1}$ is a non-lattice vertex in $Q(R)$ and t is transferred by $\overline{f_1}$. If $k \geq 3$, there can be at most one non-2-face pair by Proposition 4.8 (3) and by Lemma 4.6, say (f_1, f_2). However, the other triples of type (A, B, B') not containing both $U_2 \backslash \{1\}$ and $U_2 \backslash \{2\}$ form 3-faces by Proposition 4.11.

Suppose now that there exists a lattice vertex $\overline{\mathfrak{c}'}$ in $Q(R)$. Then, by Lemma 4.6, we have that $C_1' \subset C_1$. We can assume that there exists only one such extremal ray $\mathfrak{c}'$, since any triple of type $(C, C', U_1 \backslash \{1\})$ and $(C, C', U_1 \backslash \{2\})$ form 3-faces. By Proposition 4.8 (4), there exists at most one f_j such that $(f_j, \mathfrak{c}')$ does not span a two-dimensional face. Thus t is transferred by edges of 2-faces in $Q(R)$. It leaves us to check the case where $k = 2$: if the pair $\{f_1, f_2\}$ do not span a 2-face σ_π, then there exists a first independent set $C'' = C_1'' \sqcup C_2'' \in \mathcal{I}_{G^\pi}$ with $C_2'' = U_2 \backslash \{1, 2\}$ and $|C_1''| \leq m - 2$. Then, $\overline{\mathfrak{c}} \in Q(R)$ and it is not a lattice vertex. Furthermore, $(e_i, f_j, \mathfrak{c})$ spans three-dimensional faces of σ_π for $i \in [2]$ and $j \in [2]$. Last, assume that (f_1, f_2) spans a 2-face of σ_{G^π}. Again, it is enough to check the cases for only one vertex $\overline{\mathfrak{c}'}$ in $Q(R)$. There exists at most one non-2-face pair, say $(f_1, \mathfrak{c}')$. But then $(\mathfrak{c}', f_2, e_1)$ is a 3-face of σ_{G^π} and t is transferred by the edges of $Q(R)$. In particular, if there exist $\overline{e_j} \in Q(R)$ for $j \neq 1, 2$, by Proposition 4.8 (3), there can be at most one non-2-face pair and analogous arguments follow.

Lastly, suppose that $\{\mathfrak{c}, e_i\}$ does not span a 2-face and $\overline{\mathfrak{c}}$ and $\overline{e_i}$ are in $Q(R)$ for some deformation degree $R \in M$. Remark here that we excluded the cases where there exist non-simplicial three-dimensional faces. This means $\mathfrak{c}$ and e_i forms 2-faces with

each extremal ray of σ_π. Assume that there exist more than three vertices in $Q(R)$ other than $\bar{\mathfrak{c}}$ and $\overline{e_i}$. We examined the cases where non-3-face (e_1, e_2, e_3) appears and where non-2-face (e_1, e_2) appears in $Q(R)$. Therefore, we assume that there exists another non-2-face pair, say $(\mathfrak{c}^*, e_j)$. But, since $\mathfrak{c}^*$ and e_j also forms 2-faces with each extremal ray of σ_π, it is enough to check the cases where there exist less than five vertices in $Q(R)$.

Let us first consider the case where there exist exactly two more vertices in $Q(R)$ other than $\bar{\mathfrak{c}}$ and $\overline{e_i}$. By Proposition 4.8 (4), if (A, C) is a non-2-face pair, then $C_1 = \{i\}$ or there exists $C' \in \mathcal{I}_{G^\pi}^{(1)}$ with $C_1 \backslash C_1' = \{i\}$. We first start with $C_1 = \{i\}$. Then, there exists a non-lattice vertex $\overline{f_j} \in Q(R)$ where $j \in U_2 \backslash C_2$. By Lemma 4.6, there exists no other first independent set $C' \in \mathcal{I}_{G^\pi}^{(1)}$ such that $C_2 \subsetneq C_2'$, since $C = \{i\}$. Therefore, it is impossible that there exists another non-2-face pair containing $\overline{\mathfrak{c}'}$ in $Q(R)$. Secondly, suppose that there exists $C' \in \mathcal{I}_{G^\pi}^{(1)}$ with $C_1 \backslash C_1' = \{i\}$. The vertex $\overline{\mathfrak{c}'}$ is in $Q(R)$, unless there exists $\overline{f_j} \in Q(R)$ where $j \in C_2' \backslash C_2$. This vertex cannot be $\overline{f_j}$ with $\{j\} = C_2' \backslash C_2$, because then $(\mathfrak{c}, \mathfrak{c}', e_i, f_j)$ spans a 3-face. Hence, $\overline{\mathfrak{c}'}$ is one of these two vertices. It remains to check the case where other vertex is $\overline{e_{i-1}}$. Then, there exists a first independent set $C'' \in \mathcal{I}_{G^\pi}^{(1)}$. We have that $\overline{\mathfrak{c}''} \notin Q(R)$ if and only if there exists $\overline{f_{j'}'}$ with $j' \in C_2'' \backslash C_2'$, by the same reasoning as before. Lastly, assume that there exists only one lattice vertex in $Q(R)$ other than $\bar{\mathfrak{c}}$ and $\overline{e_i}$. We observe that $\overline{\mathfrak{c}'}$ is a lattice vertex of $Q(R)$ if there exist some $\overline{f_j} \in Q(R)$ where $j \in C_2' \backslash C_2$. Therefore, we assume that this lattice vertex is $\overline{f_j}$ for some $j \in [n]$. In order to obtain $\langle R, \mathfrak{c}' \rangle = 0$, we must have $\{j\} = C_2' \backslash C_2$, but this implies that $(\mathfrak{c}, \mathfrak{c}', e_i, f_j)$ is a 3-face of σ_π. □

We also interpret the rigidity of Y_π by giving certain conditions on the Rothe diagram.

Corollary 4.13 *Let* $\mathrm{Ess}(\pi) = \{(x_i, y_i) \mid x_{k+1} < \cdots < x_1 \text{ and } y_1 < \cdots < y_{k+1}\}$ *with* $k \geq 3$. *Then, the toric variety* Y_π *is rigid if and only if*

- $(x_1, y_1) \neq (m, 2)$ *and* $(x_{k+1}, y_{k+1}) \neq (2, n)$ *or*
- *for any* $i \in [k]$, $(x_i, y_i) \neq (x_{i+1} + 1, y_{i+1} - 1)$.

Proof This follows by Lemma 4.10 which characterizes the non-simplicial three-dimensional faces and by Theorem 4.12 which classifies rigid toric varieties Y_π. □

Example 8 In the figure of Example 7, consider the essential boxes (x_2, y_2) and (x_3, y_3) which are associated with the first independent sets C' and C. We obtain that $(x_2, y_2) = (7, 3) = (x_3 + 1, y_3 - 1)$ and also $(x_1, y_1) = (9, 2)$. Therefore, Y_π is not rigid. On the other hand, the toric variety in Example 6 is rigid.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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