



Fractal hypersurfaces, affine Weyl groups, and wavelet sets

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Abstract

In this expository paper, we present some fundamental connections between iterated function systems, in particular those whose attractors are the graphs of multivariate real-valued fractal functions, and foldable figures, affine Weyl groups, and wavelet sets.

Keywords Iterated function system · Fractal surface · Affine Weyl group · Wavelet set

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1 Introduction

This expository paper deals with some fundamental connections between iterated function systems, in particular those whose attractors are multivariate real-valued fractal functions, root systems and affine Weyl groups, and wavelet sets. After a first superficial glance, these areas seem to be too different and distinct to contain commonalities. However, the common multiscale structure that appears both in the construction of fractal sets and wavelets points the way to a deeper connection. For instance, it was first shown in [19] and [27] that a class of wavelets may be constructed by piecing fractal functions together, and then later it was proved in [30] that every compactly supported refinable function, i.e., every compactly supported scaling function or wavelet, is a piecewise fractal function. The investigation into the multiscale structure of fractals and wavelets was carried out in [46] and led to the insight that the classical wavelet set concept, which is built on dilation and translation groups, may be adapted to dilation and reflection groups.

A first construction of this new type of wavelet set appeared in [38] and then some additional insights were reported in [39]. These two investigations were based on earlier results in [25, 26] which had connected the known concepts of multiresolution

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analysis and affine fractal surface construction to foldable figures. Foldable figures were shown to be in one-to-one correspondence with the fundamental domains of affine Weyl groups [32].

Here, we will revisit some of the theoretical background and present several of the main ideas that underlie the construction of dilation-reflection wavelet sets. In order to keep the presentation as self-contained as possible, we first present an updated view of iterated function systems and give a construction of (affine) fractal hypersurfaces that is based on later requirements. These themes make up the contents of Sects. 2, 3, and 4. Then, we describe root systems, affine reflections, the associated affine Weyl groups, and the concept of foldable figure in Sect. 5. There, we also show that based on the results in this section, one can construct orthonormal bases of $L^2(\mathbb{R}^n)$ consisting of affinely generated multivariate fractal functions. In Sect. 6, we introduce and revisit the classical wavelet sets. This is done first in the one-dimensional setting and then generalized to \mathbb{R}^n . Finally, we define wavelet sets based on dilation groups and affine Weyl groups coming from a foldable figure, and prove their existence for all expansive dilation matrices and all affine Weyl groups.

2 Iterated function systems

In this section, we introduce the concept of iterated function system (IFS) and highlight some of its fundamental properties. For more details and proofs, we refer the reader to, for instance, [3, 4, 6, 36] and the references given therein.

Throughout this paper, we use the following notation. The set of positive integers is denoted by $\mathbb{N} := \{1, 2, 3, \dots\}$ and the ring of integers by \mathbb{Z} . For a $1 < N \in \mathbb{N}$, we set $\mathbb{N}_N := \{1, \dots, N\}$. The pair $(\mathbb{X}, d_{\mathbb{X}})$ always denotes a complete metric space with metric $d_{\mathbb{X}}$. On occasion we simply write \mathbb{X} when the metric is understood.

Definition 1 Let $N \in \mathbb{N}$. If $f_n : \mathbb{X} \rightarrow \mathbb{X}$, $n \in \mathbb{N}_N$, are continuous mappings, then $\mathcal{F} := (\mathbb{X}; f_1, f_2, \dots, f_N)$ is called an iterated function system (IFS) on \mathbb{X} .

By a slight abuse of terminology and notation, we use the same symbol, namely \mathcal{F} , for the IFS, the set of functions in the IFS, and for the following set-valued mapping also referred to as the Barnsley–Hutchinson operator. This operator $\mathcal{F} : 2^{\mathbb{X}} \rightarrow 2^{\mathbb{X}}$ is defined by

$$\mathcal{F}(B) := \bigcup_{f \in \mathcal{F}} f(B)$$

for all $B \in 2^{\mathbb{X}}$, where $2^{\mathbb{X}}$ denotes the class of subsets of \mathbb{X} .

Let $\mathbb{H} = \mathbb{H}(\mathbb{X}) \subset 2^{\mathbb{X}}$ be the class of all nonempty compact subsets of \mathbb{X} . $(\mathbb{H}, d_{\mathbb{H}})$ becomes a metric space when endowed with the Hausdorff–Pompeiu metric $d_{\mathbb{H}}$ (cf. [21, 55]):

$$d_{\mathbb{H}}(A, B) := \max\left\{ \max_{a \in A} \min_{b \in B} d_{\mathbb{X}}(a, b), \max_{b \in B} \min_{a \in A} d_{\mathbb{X}}(a, b) \right\}, \quad A, B \in \mathbb{H}.$$

It is known that the completeness of $(\mathbb{X}, d_{\mathbb{X}})$ implies the completeness of $(\mathbb{H}, d_{\mathbb{H}})$; cf. for instance [55, Proposition 3.2].

As $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat \mathcal{F} as a mapping $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{H}$. When $U \subset \mathbb{X}$ is nonempty, we write $\mathbb{H}(U) := \mathbb{H}(\mathbb{X}) \cap 2^U$. We denote by $|\mathcal{F}|$ the number of distinct mappings in \mathcal{F} .

A metric space \mathbb{X} is termed locally compact if for every compact $C \subset \mathbb{X}$ and every positive $r \in \mathbb{R}$ the set $\text{cl}(C + r)$ is again compact. Here, the notation $\text{cl}(C + r)$ means the closure of the union of balls of radius $r > 0$, one centered on each point of C .

The Lipschitz constant of a mapping $f : \mathbb{X} \rightarrow \mathbb{X}$ is defined by

$$\text{Lip}f := \sup_{x,y \in \mathbb{X}, x \neq y} \frac{d_{\mathbb{X}}(f(x), f(y))}{d_{\mathbb{X}}(x, y)}.$$

Functions f with $\text{Lip}f < 1$ are called *contractions* on \mathbb{X} .

The following information is foundational. A proof of it can be found in [5].

Theorem 1 *The following statements are valid:*

1. If $(\mathbb{X}, d_{\mathbb{X}})$ is compact then $(\mathbb{H}, d_{\mathbb{H}})$ is compact.
2. If $(\mathbb{X}, d_{\mathbb{X}})$ is locally compact then $(\mathbb{H}, d_{\mathbb{H}})$ is locally compact.
3. If \mathbb{X} is locally compact, or if each $f \in \mathcal{F}$ is uniformly continuous, then $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{H}$ is continuous.
4. If $f : \mathbb{X} \rightarrow \mathbb{X}$ is a contraction mapping for each $f \in \mathcal{F}$, then $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{H}$ is also a contraction mapping. In this case, the Lipschitz constant of \mathcal{F} is given by $\max\{\text{Lip}f : f \in \mathcal{F}\}$.

For $B \subset \mathbb{X}$, let $\mathcal{F}^k(B)$ denote the k -fold composition of \mathcal{F} , i.e., the union of $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(B)$ over all finite words $i_1 i_2 \cdots i_k$ of length k . Define $\mathcal{F}^0(B) := B$.

Definition 2 An element $A \in \mathbb{H}(\mathbb{X})$ is said to be an attractor of the IFS \mathcal{F} if

- (i) $\mathcal{F}(A) = A$ and
- (ii) there exists an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k \rightarrow \infty} \mathcal{F}^k(B) = A$, for all $B \in \mathbb{H}(U)$, where the limit is taken with respect to the Hausdorff-Pompeiu metric.

The largest open set U such that (ii) is true is called the basin of attraction (for the attractor A of the IFS \mathcal{F}).

Remark 1 Note that if U_1 and U_2 satisfy condition (ii) in Definition 2 for the same attractor A then so does $U_1 \cup U_2$.

Remark 2 The invariance condition (i) is not needed; it follows from (ii) with $B := A$.

The following observation [40, Proposition 3 (vii)], [20, p.68, Proposition 2.4.7] is used in the proof of Theorem 2 below:

Lemma 1 Let $\{B_k\}_{k=1}^\infty \subseteq \mathbb{H}$ be a sequence of nonempty compact sets such that $B_{k+1} \subset B_k$, for all $k \in \mathbb{N}$. Then $\bigcap_{k \in \mathbb{N}} B_k = \lim_{k \rightarrow \infty} B_k$ where the convergence is with respect to the Hausdorff-Pompeiu metric $d_{\mathbb{H}}$.

The next result shows how one may obtain the attractor A of an IFS. For the proof, we refer the reader to [5]. Note that we do not assume that the functions in the IFS \mathcal{F} are contractive.

Theorem 2 Let \mathcal{F} be an IFS with attractor A and basin of attraction U . If the mapping $\mathcal{F} : \mathbb{H}(U) \rightarrow \mathbb{H}(U)$ is continuous then

$$A = \bigcap_{m \geq 1} \text{cl} \left(\bigcup_{k \geq m} \mathcal{F}^k(B) \right), \quad \text{for all } B \subset U \text{ such that } \text{cl} B \in \mathbb{H}(U).$$

The quantity on the right-hand side here is sometimes called the topological upper limit of the sequence $\{\mathcal{F}^k(B) : k \in \mathbb{N}\}$. (See, for instance, [21].)

A subclass of IFSs is obtained by imposing additional conditions on the functions that comprise the IFS. The definition below introduces this subclass.

Definition 3 An IFS $\mathcal{F} = (\mathbb{X}; f_1, f_2, \dots, f_N)$ is called contractive if each $f \in \mathcal{F}$ is a contraction (with respect to the metric $d_{\mathbb{X}}$), i.e., if there exists a constant $c \in [0, 1)$ such that

$$d_{\mathbb{X}}(f(x_1), f(x_2)) \leq c d_{\mathbb{X}}(x_1, x_2),$$

for all $x_1, x_2 \in \mathbb{X}$.

By item 4. in Theorem 1, the mapping $\mathcal{F} : \mathbb{H} \rightarrow \mathbb{H}$ is then also contractive on the complete metric space $(\mathbb{H}, d_{\mathbb{H}})$ and thus – by the Banach Fixed Point Theorem – possesses a unique fixed point or attractor A . This attractor satisfies the self-referential equation

$$A = \mathcal{F}(A) = \bigcup_{f \in \mathcal{F}} f(A). \tag{1}$$

Note that in the case of a contractive IFS, the definition of attractor given in Definition 2 coincides with the fixed point A of \mathcal{F} .

Equation (1) expresses the fact that the attractor is a finite copy of images of itself and therefore inductively a finite union of sets of the form $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(A)$, for any finite word $i_1 i_2 \dots i_k$ of length k . This observation demonstrates the immense complexity of a non-trivial attractor which for that reason is also called a fractal (set).

In the case of a contractive IFS, the basin of attraction for A is \mathbb{X} and the attractor can be computed via the following procedure: Suppose K_0 is any set in $\mathbb{H}(\mathbb{X})$. Consider the sequence of iterates

$$K_m := \mathcal{F}(K_{m-1}) = \mathcal{F}^m(K_0), \quad m \in \mathbb{N}.$$

Then K_m converges in the Hausdorff-Pompeiu metric to the attractor A as $m \rightarrow \infty$, i. e., $d_{\mathbb{H}}(K_m, A) \rightarrow 0$ as $m \rightarrow \infty$.

There exist weaker conditions for the existence of an attractor of an IFS. We refer the interested reader to, for instance, [41].

For the remainder of this paper, we deal exclusively with contractive IFSs as defined above. We will see that the self-referential Eq.(1) plays a fundamental role in the construction of fractals sets, i.e., the attractors of IFSs, and in the determination of their geometric and analytic properties.

3 Fractal hypersurfaces in \mathbb{R}^{n+1}

In this section, we construct a class of special attractors of IFSs, namely attractors that are the graphs of bounded functions $\mathfrak{f} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$, where $\Omega \in \mathbb{H}(\mathbb{R}^n)$, $n \in \mathbb{N}$. Suppose that $\{u_i : \Omega \rightarrow \Omega : i \in \mathbb{N}_N\}$ is a family of injective mappings with the property that

$$(P) \quad \{u_i(\Omega) : i \in \mathbb{N}_N\} \text{ is a set-theoretic partition of } \Omega \text{ in the sense that } \Omega = \bigcup_{i=1}^N u_i(\Omega) \text{ and } u_i(\Omega) \cap u_j(\Omega) = \emptyset, \text{ for all } i \neq j.$$

We remark that property (P) can be relaxed somewhat; for details, we refer the interested reader to [53].

We introduce the set $B(\Omega) := B(\Omega, \mathbb{R}) := \{f : \Omega \rightarrow \mathbb{R} : f \text{ is bounded}\}$ and endow it with the metric

$$d(f, g) := \sup\{|f(x) - g(x)| : x \in \Omega\}.$$

It is straight-forward to show that $(B(\Omega), d)$ is a complete metric space. Indeed it is even a complete metric linear space. Recall that a metric linear space is a vector space endowed with a metric under which the operations of vector addition and scalar multiplication are continuous. (See, for instance, [51].)

For $i \in \mathbb{N}_N$, let $v_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a mapping that is uniformly contractive in the second variable, i.e., there exists an $\ell \in [0, 1)$ so that for all $y_1, y_2 \in \mathbb{R}$

$$|v_i(x, y_1) - v_i(x, y_2)| \leq \ell |y_1 - y_2|, \quad \forall x \in \Omega. \quad (2)$$

Define a Read-Bajactarević (RB) operator $\Phi : B(\Omega) \rightarrow \mathbb{R}^\Omega$ by

$$\Phi f(x) := \sum_{i=1}^N v_i(u_i^{-1}(x), f \circ u_i^{-1}(x)) \chi_{u_i(\Omega)}(x), \tag{3}$$

where

$$\chi_M(x) := \begin{cases} 1, & x \in M \\ 0, & x \notin M \end{cases}$$

denotes the characteristic function of a set M . Note that Φ is well-defined and since f is bounded and each v_i contractive in the second variable, Φf is again an element of $B(\Omega)$.

Moreover, by (2), we obtain for all $f, g \in B(\Omega)$ the following inequality:

$$\begin{aligned} d(\Phi f, \Phi g) &= \sup_{x \in \Omega} |\Phi f(x) - \Phi g(x)| \\ &= \sup_{x \in \Omega} |v(u_i^{-1}(x), f(u_i^{-1}(x))) - v(u_i^{-1}(x), g(u_i^{-1}(x)))| \\ &\leq \ell \sup_{x \in \Omega} |f \circ u_i^{-1}(x) - g \circ u_i^{-1}(x)| \leq \ell d(f, g). \end{aligned} \tag{4}$$

To simplify notation, we set $v(x, y) := \sum_{i=1}^N v_i(x, y) \chi_{u_i(\Omega)}(x)$ in the above equation.

In other words, Φ is a contraction on the complete metric linear space $B(\Omega)$ and, by the Banach Fixed Point Theorem, has therefore a unique fixed point \bar{f} in $B(\Omega)$. This unique fixed point will be called a multivariate real-valued fractal function (for short, fractal function) and its graph a fractal hypersurface of \mathbb{R}^{n+1} .

Next, we would like to consider a special choice for the mappings v_i . To this end, define $v_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$v_i(x, y) := \lambda_i(x) + S_i(x)y, \quad i \in \mathbb{N}_N, \tag{5}$$

where $\lambda_i \in B(\Omega)$ and $S_i : \Omega \rightarrow \mathbb{R}$ is a function. Then, v_i given by (5) satisfies condition (2) provided that the functions S_i are bounded on Ω with bounds in $[0, 1)$. For then

$$\begin{aligned} |v_i(x, y_1) - v_i(x, y_2)| &= |S_i(x)y_1 - S_i(x)y_2| = |S_i(x)| \cdot |y_1 - y_2| \\ &\leq \|S_i\|_{\infty, \Omega} |y_1 - y_2| \leq s |y_1 - y_2|. \end{aligned}$$

Here, we denoted by $\|\cdot\|_{\infty, \Omega}$ the supremum norm on Ω and defined

$$s := \max\{\|S_i\|_{\infty, \Omega} : i \in \mathbb{N}_N\}.$$

Thus, for a fixed set of functions $\{\lambda_i : i \in \mathbb{N}_N\}$ and $\{S_i : i \in \mathbb{N}_N\}$, the associated RB operator (3) has now the form

$$\Phi f = \sum_{i=1}^N \lambda_i \circ u_i^{-1} \chi_{u_i(\Omega)} + \sum_{i=1}^N (S_i \circ u_i^{-1}) \cdot (f \circ u_i^{-1}) \chi_{u_i(\Omega)}, \tag{6}$$

or, equivalently,

$$\Phi f \circ u_i = \lambda_i + S_i \cdot f, \quad \text{on } \Omega \text{ and } i \in \mathbb{N}_N.$$

Thus, we have arrived at the following result.

Theorem 3 *Let $\Omega \in \mathbb{H}(\mathbb{R}^n)$ and suppose that $\{u_i : \Omega \rightarrow \Omega : i \in \mathbb{N}_N\}$ is a family of injective mappings satisfying property (P). Further suppose that the vectors of functions $\lambda := (\lambda_1, \dots, \lambda_N)$ and $\mathcal{S} := (S_1, \dots, S_N)$ are elements of $\prod_{i=1}^N B(\Omega)$.*

Define a mapping $\Phi : \left(\prod_{i=1}^N B(\Omega)\right) \times \left(\prod_{i=1}^N B(\Omega)\right) \times B(\Omega) \rightarrow B(\Omega)$ by

$$\Phi(\lambda)(\mathcal{S})f = \sum_{i=1}^N \lambda_i \circ u_i^{-1} \chi_{u_i(\Omega)} + \sum_{i=1}^N (S_i \circ u_i^{-1}) \cdot (f \circ u_i^{-1}) \chi_{u_i(\Omega)}. \tag{7}$$

If $s = \max\{\|S_i\|_{\infty, \Omega} : i \in \mathbb{N}_N\} < 1$ then the operator $\Phi(\lambda)(\mathcal{S})$ is a contraction on the complete metric linear space $B(\Omega)$ and its unique fixed point $\tilde{f} = \tilde{f}(\lambda)(\mathcal{S})$ satisfies the self-referential equation

$$\tilde{f} = \sum_{i=1}^N \lambda_i \circ u_i^{-1} \chi_{u_i(\Omega)} + \sum_{i=1}^N (S_i \circ u_i^{-1}) \cdot (\tilde{f} \circ u_i^{-1}) \chi_{u_i(\Omega)}, \tag{8}$$

or, equivalently,

$$\tilde{f} \circ u_i = \lambda_i + S_i \cdot \tilde{f}, \quad \text{on } \Omega \text{ and } i \in \mathbb{N}_N. \tag{9}$$

Remark 3 Note that the fractal function $\tilde{f} : \Omega \rightarrow \mathbb{R}$ generated by the RB operator defined by (7) does depend on the two N -tuples of bounded functions $\lambda, \mathcal{S} \in \prod_{i=1}^N B(\Omega)$ and that (8) therefore defines a two-parameter family of functions. The fixed point \tilde{f} should therefore be written more precisely as $\tilde{f}(\lambda)(\mathcal{S})$. However, for the sake of notational simplicity, we usually suppress this dependence for both \tilde{f} and Φ . Below, we will see that imposing continuity conditions on the fixed point \tilde{f} will determine the N -tuple of functions λ and \tilde{f} will then depend only on \mathcal{S} .

Now, assume that the vector of functions \mathcal{S} is fixed. Then, the following result found in [27] and in more general form in [45] describes the relationship between the vector of functions λ and the fixed point $\tilde{f}(\lambda)$ generated by the RB operator $\Phi(\lambda)$.

Theorem 4 *The mapping $\lambda \mapsto \tilde{f}(\lambda)$ is a linear isomorphism from $\prod_{i=1}^N B(\Omega)$ to $B(\Omega)$.*

Proof Let $\alpha, \beta \in \mathbb{R}$ and let $\lambda, \mu \in \prod_{i=1}^N B(\Omega)$. Injectivity follows immediately from the fixed point Eq.(8) and the uniqueness of the fixed point: $\lambda = \mu \iff \tilde{f}(\lambda) = \tilde{f}(\mu)$.

Linearity follows from (8), the uniqueness of the fixed point and injectivity:

$$\begin{aligned}\tilde{f}(\alpha\lambda + \beta\mu) &= \sum_{i=1}^N (\alpha\lambda_i + \beta\mu_i) \circ u_i^{-1} \chi_{u_i(\Omega)} \\ &\quad + \sum_{i=1}^N (S_i \circ u_i^{-1}) \cdot (\tilde{f}(\alpha\lambda + \beta\mu) \circ u_i^{-1}) \chi_{u_i(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\alpha\tilde{f}(\lambda) + \beta\tilde{f}(\mu) &= \sum_{i=1}^N (\alpha\lambda_i + \beta\mu_i) \circ u_i^{-1} \chi_{u_i(\Omega)} \\ &\quad + \sum_{i=1}^N (S_i \circ u_i^{-1}) \cdot (\alpha\tilde{f}(\lambda) + \beta\tilde{f}(\mu) \circ u_i^{-1}) \chi_{u_i(\Omega)}.\end{aligned}$$

Hence, $\tilde{f}(\alpha\lambda + \beta\mu) = \alpha\tilde{f}(\lambda) + \beta\tilde{f}(\mu)$.

For surjectivity, we define $\lambda_i := \tilde{f} \circ u_i - S_i \cdot \tilde{f}$, $i \in \mathbb{N}_N$. Since $\tilde{f} \in B(\Omega)$, we have $\lambda \in \prod_{i=1}^N B(\Omega)$. Thus, $\tilde{f}(\lambda) = \tilde{f}$.

We will see below that this theorem allows us to obtain bases for fractal functions. For more details, see also [31, 44, 47].

4 Affinely generated fractal surfaces

In this section, we specialize our construction of fractal functions even further. It is our goal to obtain continuous fractal hypersurfaces that are generated by *affine* mappings $\lambda_i : \Omega \rightarrow \mathbb{R}$ on specially chosen domains $\Omega \subset \mathbb{R}^n$, namely n -simplices, and by *constant* functions $S_i := s_i \in (-1, 1)$.

This type of fractal surface was first systematically introduced in [43] and generalized in [24]. Further generalizations were presented in [25, 26, 33]. All these constructions are based on using for Ω certain types of simplicial regions and affine mappings. Later, different types of fractal surface constructions not necessarily based on simplicial regions and affine mappings were published. A short and albeit incomplete list of them is [8–10, 13, 14, 23, 42, 50, 52].

In order to set up the connection with wavelet sets, we need to follow the construction that originated in [25, 26] and was used in [38, 47]. To this end, we first need to choose as our domain $\Omega \subset \mathbb{R}^n$ an n -simplex.

Definition 4 Let $\{p_0, p_1, \dots, p_n\}$ be a set of affinely independent points in \mathbb{R}^n . A regular n -simplex in \mathbb{R}^n is defined as the point set

$$\Delta^n := \left\{ x \in \mathbb{R}^n : x = \sum_{k=0}^n t_k p_k; t_k \geq 0; \sum_{k=0}^n t_k = 1 \right\}.$$

Over the n -simplex Δ^n , we consider the following set of functions:

$$C(\Delta^n) := C(\Delta^n, \mathbb{R}) := \{f : \Delta^n \rightarrow \mathbb{R} : f \text{ is continuous on } \Delta^n\}.$$

It is easy to verify that $(C(\Delta^n), \|\cdot\|_{\infty, \Delta^n})$ is a complete metric linear space.

Now let $1 < N \in \mathbb{N}$ and suppose that $\{\Delta_i^n : i \in \mathbb{N}_N\}$ is a family of nonempty compact subsets of Δ^n with the properties that:

- (P1) $\Delta^n = \bigcup_{i=1}^N \Delta_i^n;$
- (P2) Δ_i^n is similar to Δ^n , for all $i \in \mathbb{N}_N;$
- (P3) Δ_i^n is congruent to Δ_j^n , for all $i, j \in \mathbb{N}_N.$

Note that conditions (P1), (P2), and (P3) imply the existence of N unique contractive similitudes $u_i : \Delta^n \rightarrow \Delta_i^n$ given by

$$u_i = a O_i + \tau_i, \quad i = 1, \dots, N, \tag{10}$$

where $0 < a < 1$ is the similarity constant or the similarity ratio for Δ_i^n with respect to Δ^n , $O_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ an orthogonal transformation, and $\tau_i \in \mathbb{R}^n$ a translation.

Let V be the set of vertices of Δ^n . We denote the set of all distinct vertices of the subsimplices Δ_i^n by V_i . Suppose there exists a labelling map $\ell : \cup V_i := \bigcup_{i=1}^N V_i \rightarrow V$ such that

$$\forall i \in \mathbb{N}_N \forall v \in V_i : u_i(\ell(v)) = v. \tag{11}$$

Now, suppose that $\{\lambda_i : \Delta^n \rightarrow \mathbb{R} : i \in \mathbb{N}_N\}$ is a collection of affine functions and $\{s_i : i \in \mathbb{N}_N\}$ a set of real numbers. As in the previous section, we set $\lambda := (\lambda_1, \dots, \lambda_N)$ and $s := (s_1, \dots, s_N)$. Let us denote by $\mathbb{A}_n := \text{Aff}(\mathbb{R}^n, \mathbb{R})$ the vector space of all affine mappings $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$. We like to define an RB operator

$$\Phi(\lambda)(s)f := \sum_{i \in \mathbb{N}_N} (\lambda_i \circ u_i^{-1})\chi_{\Delta_i^n} + \sum_{i \in \mathbb{N}_N} (s_i f \circ u_i^{-1})\chi_{\Delta_i^n} \tag{12}$$

(points on common boundaries are only counted once) on a subspace $C_0(\Delta^n)$ of $C(\Delta^n)$ so that

$$\Phi : \left(\prod_{i=1}^N \mathbb{A}_n \right) \times \mathbb{R}^N \times C_0(\Delta^n) \rightarrow C_0(\Delta^n).$$

The affine mappings λ_i are usually determined by interpolation conditions. Thus, consider the interpolation set

$$Z := \{(v, z_v) \in \Delta^n \times \mathbb{R} : v \in \cup V_i\} \tag{13}$$

Then, for all $i \in \mathbb{N}_N$, the affine mappings λ_i are uniquely determined by the interpolation conditions

$$\lambda_i(\ell(v)) + s_i z_{\ell(v)} = z_v, \quad \forall v \in V_i. \tag{14}$$

In order for Φ^f to be well-defined on and continuous across adjacent triangles Δ_i^n and Δ_j^n , one needs to impose the following join-up conditions:

$$\lambda_i \circ u_i^{-1}(x, y) + s_i f \circ u_i^{-1}(x, y) = \lambda_j \circ u_j^{-1}(x, y) + s_j f \circ u_j^{-1}(x, y), \tag{15}$$

for all $(x, y) \in e_{ij} := \Delta_i^n \cap \Delta_j^n, i, j \in \mathbb{N}_N$ with $i \neq j$. Here, e_{ij} is called a common edge of Δ_i^n and Δ_j^n .

The next result, which is essentially Theorem 6 in [25] gives conditions for (15) to be satisfied.

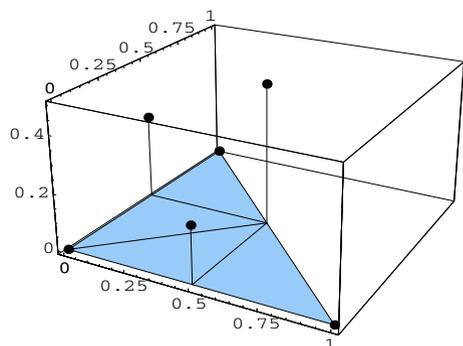
Theorem 5 *Let Δ^n be a n -simplex and $\{\Delta_i^n : i \in \mathbb{N}_N\}$ a family of nonempty compact subsets of Δ^n satisfying conditions (P1), (P2), and (P3). Let Z be an interpolation set of the form (13) and let $C_0(\Delta^n) := \{f \in C(\Delta^n) : f(v) = z_v, v \in \cup V_i\}$. Suppose that there exists a labelling map ℓ as defined in (11). Further suppose that $s := (s, \dots, s)$, with $|s| < 1$. Then, the RB operator $\Phi(\lambda)(s)$ defined by (12) maps $C_0(\Delta^n)$ into itself, is well-defined, and contractive on the complete metric subspace $(C_0(\Delta^n), \|\cdot\|_{\infty, \Delta})$ of $(C(\Delta^n), \|\cdot\|_{\infty, \Delta^n})$.*

The unique fixed point of the RB operator in Theorem 5 is called a multivariate real-valued affine fractal interpolation function and its graph an affinely generated fractal hypersurface or an affine fractal hypersurface.

Example 1 Let $n := 2$. Suppose we are given the 2-simplex Δ^2 and its associated partition as depicted in Fig. 1 below.

A simple computation yields for the four mappings u_i the following expressions:

Fig. 1 A 2-simplex and its associated partition



$$\begin{aligned}
 u_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, & u_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \\
 u_3 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, & u_4 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.
 \end{aligned}$$

The affine functions λ_i are then given by

$$\begin{aligned}
 \lambda_1(x, y) &= \lambda_2(x, y) = -z_1x + (z_2 - z_1)y + z_1, \\
 \lambda_3(x, y) &= \lambda_4(x, y) = (z_2 - z_3)x - z_3y + z_3,
 \end{aligned}$$

where $z_1, z_2,$, and z_3 are the nonzero z values. (See the figure above.) For $z_1 := \frac{1}{5}, z_2 := \frac{1}{2}, z_3 := \frac{3}{10}$, and $s := -\frac{3}{5}$, the sequence of graphs in Fig. 2 shows the generation of the fractal hypersurface in \mathbb{R}^3 .

Remark 4 In the above construction of an affine fractal hypersurface, one needs to choose all scaling factors s_i equal to a common value s in order to guarantee continuity of f . There is a related construction where one can choose different scaling factors for the maps but in this case one is forced to consider coplanar boundary conditions. The interest reader is referred to [47] for more details.

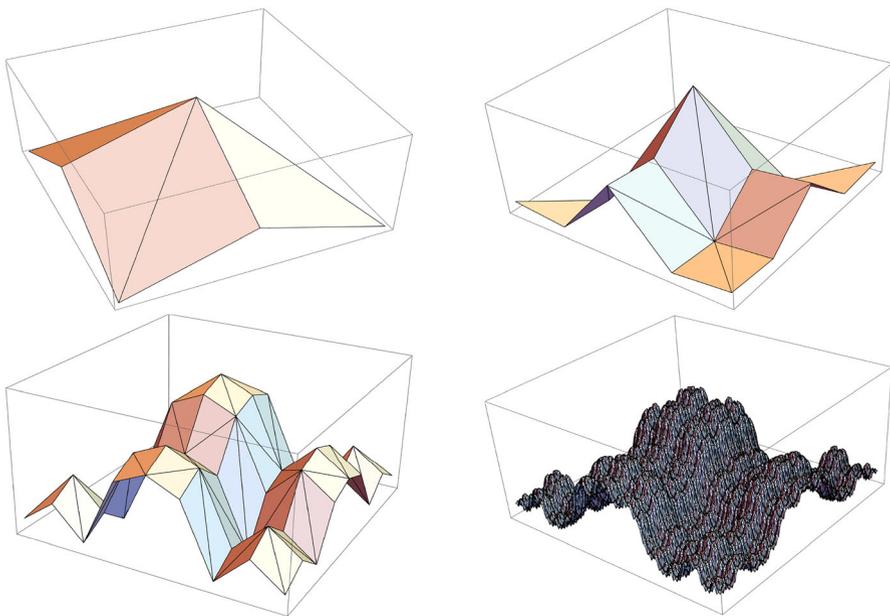


Fig. 2 The generation of an affine fractal hypersurface in \mathbb{R}^3

Using Theorem 5 we can construct a basis for fractal hypersurfaces. For this purpose, we denote by $\mathcal{S}(\Delta^n; \mathbb{A}_n)$ the set of all fractal functions generated by the RB operator (12) subject to the conditions stated in Theorem 5. The set $\mathcal{S}(\Delta^n; \mathbb{A}_n)$ becomes for fixed s a complete metric linear space inheriting its metric from $C(\Delta^n)$. Note that $\dim \mathcal{S}(\Delta^n; \mathbb{A}_n) = nN - \text{card}(\cup V_i)$; n free parameter for each of the N affine functions λ_i and $\text{card}(\cup V_i)$ many interpolation conditions at the vertices. One may interpret $\mathcal{S}(\Delta^n; \mathbb{A}_n)$ as a generalized spline space.

This observation suggests the construction of an n -dimensional system of Lagrange interpolants for each affine fractal function \mathfrak{f} of the form

$$\mathfrak{B} := \{b_v : \Delta^n \rightarrow \mathbb{R} : v \in \cup V_i\}, \tag{16}$$

where each b_v is the unique affine fractal hypersurface interpolating the set

$$Z_v := \{(v', \delta_{vv'}) : v' \in \cup V_i\}, \quad v \in \cup V_i,$$

where $\delta_{vv} := 1$ and $\delta_{vv'} := 0$, if $v \neq v'$. We also refer to the set of these affine fractal functions as an affine fractal (hypersurface) basis. Thus, we have the following result.

Theorem 6 *Let \mathfrak{f} be an affine fractal function with associated interpolation set Z as defined in (13). Then there exists an affine fractal basis of Lagrange-type (16) of cardinality $\text{card}(\cup V_i)$ such that*

$$\mathfrak{f} = \sum_{v \in \cup V_i} z_v b_v.$$

Example 2 In Fig. 3, three of the six graphs of the affine fractal basis functions for the affine fractal surface constructed in Example 1 are displayed.

As $\mathcal{S}(\Delta^n; \mathbb{A}_n)$ is finite-dimensional, we can apply the Gram–Schmidt Orthonormalization procedure to obtain an *orthonormal basis* for $\mathcal{S}(\Delta^n; \mathbb{A}_n)$ consisting of affine fractal basis functions. (See also [25, 26, 44, 47].) This orthonormal basis will play an important role in the connection with wavelet sets and affine Weyl groups.

5 Root systems and affine Weyl groups

In the current section, we introduce root systems, reflections about affine hyperplane, and the associated affine Weyl groups. We only present those concepts that are of importance for later developments and refer the interested reader to, for instance, [11, 12, 15, 28, 29, 35, 37] for a more in-depth presentation of reflection groups and root systems.

For the remainder of this section, we denote by $\mathbb{E}^n := (\mathbb{E}^n, \langle \cdot, \cdot \rangle)$ n -dimensional Euclidean space endowed with the canonical Euclidean inner product $\langle \cdot, \cdot \rangle$.

Let $H \subset \mathbb{E}^n$ be a linear hyperplane, i.e., a codimension one linear subspace of \mathbb{E}^n .

Definition 5 A linear transformation $r : \mathbb{E}^n \rightarrow \mathbb{E}^n$ is called a reflection about H if

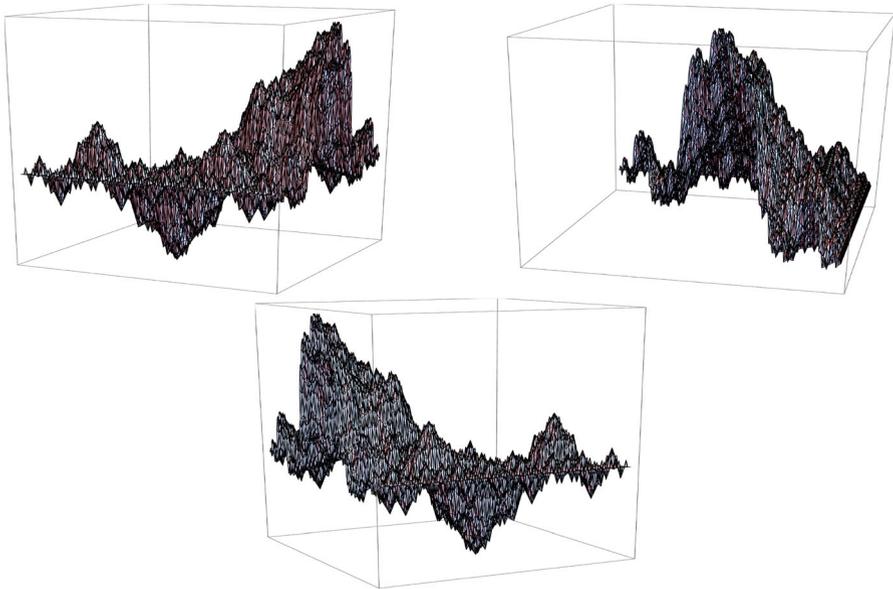


Fig. 3 Some affine fractal basis functions. Upper left: $z_i = \delta_{1i}$, upper right: $z_i = \delta_{2i}$, and lower middle: $z_i = \delta_{3i}$, $i = 1, 2, 3$

1. $r(H) = H$;
2. $r(x) = -x$, for all $x \in H^\perp$. Here, H^\perp denotes the orthogonal complement of H with respect to the Euclidean inner product $\langle \cdot, \cdot \rangle$.

A straight-forward computation yields an explicit representation of r , namely,

$$r_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad \text{for fixed } 0 \neq \alpha \in H^\perp. \tag{17}$$

Recall that a discrete group is a subgroup of the general linear group $GL(n, \mathbb{R})$ whose subspace topology in $GL(n, \mathbb{R})$ is the discrete topology.

Definition 6 A discrete group generated by a set of reflections in \mathbb{E}^n is called a Euclidean reflection group.

An abstract group that has a representation in terms of reflections is termed a Coxeter group. More precisely, a Coxeter group is a discrete group \mathcal{C} with a finite set of generators $\{\alpha_i : i = 1, \dots, k\}$ satisfying

$$\mathcal{C} := \langle \alpha_1, \dots, \alpha_k : (\alpha_i \alpha_j)^{m_{ij}} = 1, 1 \leq i, j \leq k \rangle$$

where $m_{ii} = 1$, for all i , and $m_{ij} \geq 2$, for all $i \neq j$. ($m_{ij} = \infty$ is used to indicate that no relation exists.) It can be shown that finite Coxeter groups are isomorphic to finite Euclidean reflection groups. [15]

Example 3 Klein’s Four-Group V or the dihedral group D_4 of order 4. This group is generated by two elements α and β as follows:

$$V = D_4 = \langle \alpha, \beta : \alpha^2 = \beta^2 = (\alpha\beta)^2 = 1 \rangle.$$

In geometric terms, V is the symmetry group of the unit square $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ centered at the origin.

Definition 7 A root system R consists of a finite set of vectors $\alpha_1, \dots, \alpha_k \in \mathbb{R}^n \setminus \{0\}$ having the following properties.

1. $\mathbb{R}^n = \text{span} \{ \alpha_1, \dots, \alpha_k \}$;
2. $\alpha, t\alpha \in R$ if and only if $t = \pm 1$;
3. $\forall \alpha \in R: r_\alpha(R) = R$, where r_α is the reflection through the hyperplane orthogonal to α ;
4. $\forall \alpha, \beta \in R: \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$, where $\alpha^\vee := 2\alpha / \langle \alpha, \alpha \rangle$.

The dimension of \mathbb{R}^n , i.e., n , is called the rank of the root system.

We note that condition 4. in Definition 7 restricts the possible angles between roots: Let α, β be two roots and let

$$n(\beta, \alpha) := \langle \beta, \alpha^\vee \rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

Denote by $|\alpha| := \langle \alpha, \alpha \rangle^{1/2}$ the length of α and by θ the angle between α and β . Then $\langle \alpha, \beta \rangle = |\alpha| |\beta| \cos \theta$ and thus

$$n(\beta, \alpha) = 2 \frac{|\beta|}{|\alpha|} \cos \theta. \tag{18}$$

It follows from (18) that

$$n(\beta, \alpha) n(\alpha, \beta) = 4 \cos^2 \theta.$$

Hence,

$$n(\beta, \alpha) \in \mathbb{Z} \implies 4 \cos^2 \theta \in \{0, 1, 2, 3, 4\}.$$

The following table shows the possible angles θ and thus the relation between the two roots α and β .

$n(\beta, \alpha) = 0,$	$n(\alpha, \beta) = 0,$	$\theta = \frac{\pi}{2}.$	
$n(\beta, \alpha) = 1,$	$n(\alpha, \beta) = 1,$	$\theta = \frac{\pi}{3},$	$ \beta = \alpha .$
$n(\beta, \alpha) = -1,$	$n(\alpha, \beta) = -1,$	$\theta = \frac{2\pi}{3},$	$ \beta = \alpha .$
$n(\beta, \alpha) = 1,$	$n(\alpha, \beta) = 2,$	$\theta = \frac{\pi}{4},$	$ \beta = \sqrt{2} \alpha .$
$n(\beta, \alpha) = -1,$	$n(\alpha, \beta) = -2,$	$\theta = \frac{3\pi}{4},$	$ \beta = \sqrt{2} \alpha .$
$n(\beta, \alpha) = 1,$	$n(\alpha, \beta) = 3,$	$\theta = \frac{\pi}{6},$	$ \beta = \sqrt{3} \alpha .$
$n(\beta, \alpha) = -1,$	$n(\alpha, \beta) = -3,$	$\theta = \frac{5\pi}{6},$	$ \beta = \sqrt{3} \alpha .$

Note that the value of θ determines both $n(\alpha, \beta)$ as well as $\{|\alpha|/|\beta|, |\beta|/|\alpha|\}$.

Example 4 The following are three examples of root systems in \mathbb{E}^2 . See Fig. 4.

Roots may be divided into two classes as follows.

Definition 8 Choose $v \in \mathbb{R}^n$ so that $\langle v, \alpha \rangle \neq 0$, for all roots $\alpha \in R$. The set of positive roots is defined as

$$R^+ := \{\alpha \in R : \langle v, \alpha \rangle > 0\}$$

and the set of negative roots as

$$R^- := \{\alpha \in R : \langle v, \alpha \rangle < 0\}.$$

Clearly, $R = R^+ \amalg R^-$, where \amalg denotes the disjoint union.

A root in R^+ is called simple if it cannot be written as the sum of two elements of R^+ . The set $\Delta \subset R^+$ of simple roots forms a basis of \mathbb{R}^n with the property that every $\alpha \in R$ can be written in the form

$$\alpha = \sum k_i \delta_i, \quad k_i \in \mathbb{Z}, \quad \delta_i \in \Delta,$$

where all $k_i > 0$ or all $k_i < 0$.

We summarize some properties of roots and root systems in the proposition below.

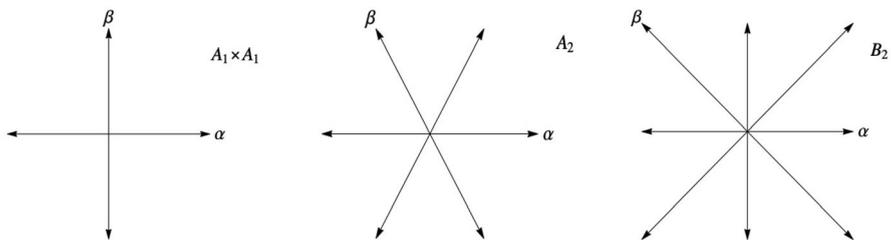


Fig. 4 Three root systems of rank 2

Proposition 1 *Let R be a root system and R^+ the set of positive roots.*

1. R^+ is that subset of R for which the following two conditions hold:
 - (i) Given $\alpha \in R$, then either $\alpha \in R^+$ or $-\alpha \in R^+$;
 - (ii) For all pairs $(\alpha, \beta) \in R^+ \times R^+$ with $\alpha \neq \beta$, such that $\alpha + \beta \in R$, $\alpha + \beta \in R^+$.
2. For $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$, $\langle \alpha, \beta \rangle \leq 0$. (I.e., the angle between two simple roots is always obtuse.)
3. Assume that $\alpha, \beta \in R$, α and β are not multiples of each other, and $\langle \alpha, \beta \rangle > 0$. Then $\alpha - \beta \in R$.

Now suppose that $\alpha, \beta \in R$. Let H be the hyperplane orthogonal to α , i.e., $H = \alpha^\perp$. We denote by $\alpha^* \in (\mathbb{E}^n)^*$ the unique element in the algebraic dual of \mathbb{E}^n such that

$$\alpha^*(H) = 0, \quad \text{and} \quad \alpha^*(\alpha) = 2.$$

Then we may rewrite (17) in the form

$$r_\alpha(\beta) = \beta - \alpha^*(\beta)\alpha = (1 - \alpha^* \otimes \alpha)(\beta).$$

Since \mathbb{E}^n and $(\mathbb{E}^n)^*$ are isomorphic via the Euclidean inner product $\langle \cdot, \cdot \rangle$, there exists a unique element $\alpha^\vee \in \mathbb{E}^n$ so that $\alpha^* = \langle \alpha^\vee, \cdot \rangle$. α^\vee is called the coroot of α and is identical to the α^\vee in Definition (7), item 4. The lattice in \mathbb{E}^n spanned by the roots R , respectively, coroots R^\vee is called the root, respectively, coroot lattice.

The finite family of hyperplanes $\{H_\alpha : \alpha \in R\}$, where H_α is the hyperplane orthogonal to α , partitions \mathbb{E}^n into finitely many regions. The connected components of $\mathbb{E}^n \setminus \bigcup_{\alpha \in R} H_\alpha$ are called the (open) Weyl chambers of \mathbb{E}^n .

The fundamental Weyl chamber C relative to Δ is defined by

$$C := \bigcap_{\delta \in \Delta} \{v \in \mathbb{E}^n : \langle v, \delta \rangle > 0\}.$$

It should be clear that C is a simplicial cone, hence convex and connected (Fig. 5).

Definition 9 The subgroup of the isometry group of a root system that is generated by the reflections through the hyperplanes orthogonal to the roots is called the Weyl group \mathcal{W} of the root system R .

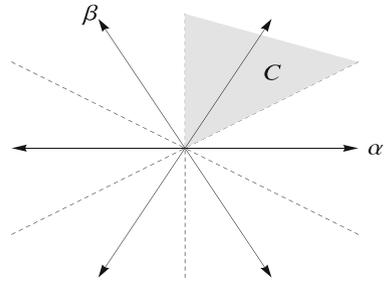
As the root system is finite, \mathcal{W} is a finite reflection group. Moreover, \mathcal{W} acts simply transitive on the Weyl chambers, i.e., if C_1 and C_2 are two Weyl chambers and $x_j \in C_j, j = 1, 2$, then there exists a unique $r \in \mathcal{W}$ so that $r(x_1) = x_2$.

An affine hyperplane with respect to a root system R is defined by

$$H_{\alpha,k} := \{x \in \mathbb{R}^n : \langle x, \alpha \rangle = k\}, \quad \alpha \in R, \quad k \in \mathbb{Z}.$$

We can also consider reflections $r_{\alpha,k}$ about affine hyperplanes. Employing conditions

Fig. 5 A root system, its Weyl chambers (regions between dashed lines) and the fundamental Weyl chamber C



(i) and (ii) in Definition 5 applied now to affine hyperplanes, we obtain the following expression for such reflections:

$$r_{\alpha,k}(x) = x - \frac{2(\langle x, \alpha \rangle - k)}{\langle \alpha, \alpha \rangle} \alpha = r_{\alpha}(x) + k \alpha^{\vee}. \tag{19}$$

Definition 10 Let R be a root system and $\{H_{\alpha,k} : \alpha \in R, k \in \mathbb{Z}\}$ its system of affine hyperplanes. The affine Weyl group $\widetilde{\mathcal{W}}$ for R is the infinite group generated by the reflections $r_{\alpha,k}$ about the affine hyperplanes $H_{\alpha,k}$:

$$\widetilde{\mathcal{W}} := \langle r_{\alpha,k} : \alpha \in R, k \in \mathbb{Z} \rangle$$

The next result characterizes the affine Weyl group of a root system and relates it to the finite Weyl group and the lattice generated by the coroots.

In reference to the theorem below, we recall the notion of semi-direct product of groups. Let (G, \cdot) be a group with identity element e . Suppose H is a subgroup of G and N a normal subgroup of G . Further suppose that $G = HN$ and $H \cap N = \{e\}$. Then G is called the semi-direct product of H and N .

Theorem 7 (Bourbaki) *The affine Weyl group $\widetilde{\mathcal{W}}$ of a root system R is the semi-direct product $\mathcal{W} \rtimes \Gamma$, where Γ is the abelian group generated by the coroots α^{\vee} . Moreover, Γ is the subgroup of translations of $\widetilde{\mathcal{W}}$ and \mathcal{W} the isotropy group (stabilizer) of the origin. The group \mathcal{W} is finite and Γ infinite.*

In this context, also note the particular form of an affine reflection (19); it is the sum of a reflection r_{α} across a linear hyperplane plus a translation along the lattice spanned by the coroot α^{\vee} .

It can be shown that $\widetilde{\mathcal{W}}$ has a fundamental domain $C \subset \mathbb{E}^n$ in the sense that no $r \in \widetilde{\mathcal{W}}$ maps a point of C to another point of C , and for all $x \in \mathbb{E}^n$ there exists an $r \in \widetilde{\mathcal{W}}$ such that $r(x) \in C$. Furthermore, C is a compact and convex simplex.

All affine Weyl groups (and therefore their fundamental domains) are classified. For a given dimension $n \in \mathbb{N}$ there exists only a finite number of possible groups and thus fundamental domains. The classification of so-called irreducible root systems (root systems that cannot be written as the union of two root systems R_1 and R_2 such

that $\langle \alpha_1, \alpha_2 \rangle = 0$, for $\alpha_i \in R_i, i = 1, 2$) follows from the representation theory of simple Lie algebras.

The classification yields four infinite families $A_n (n \geq 1), B_n (n \geq 2), C_n (n \geq 3)$, and $D_n (n \geq 4)$ and five exceptional cases E_6, E_7, E_8, F_4 , and G_2 . (The subscript indicates the rank of the root system.) For the cardinality of these families and the explicit construction of their root system as well as the geometric description of the fundamental domains, we refer the interested reader to [11, 35].

Figure 6 shows the three irreducible root systems of rank 2, their fundamental domains C and coroot lattices. Note that the fundamental domains consist of an equiangular triangle (root system A_2), a $90^\circ-45^\circ-45^\circ$ triangle (B_2), and a $90^\circ-30^\circ-60^\circ$ triangle (G_2).

The final concept we need to introduce to set up the connection between affine fractal surfaces, affine Weyl groups and wavelet sets is that of foldable figure.

Definition 11 A compact connected subset F of \mathbb{E}^n is called a foldable figure if there exists a finite set S of affine hyperplanes that cuts F into finitely many congruent subfigures F_1, \dots, F_m , each similar to F , so that reflection in any of the cutting hyperplanes in S bounding F_k takes it into some F_ℓ .

Two examples of foldable figures are given below in Fig. 7.

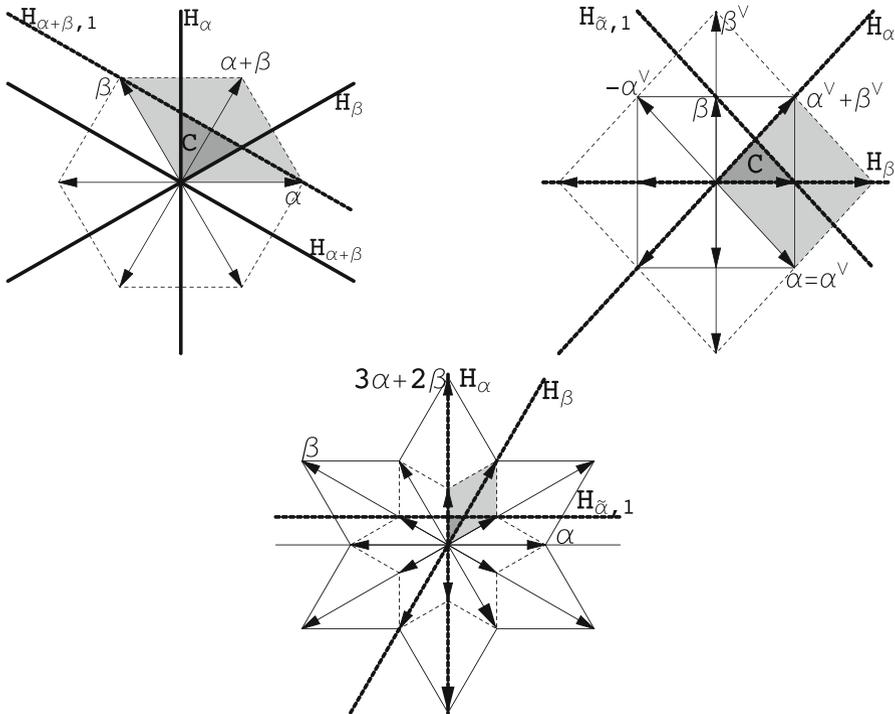


Fig. 6 The root systems A_2, B_2 , and G_2

The next theorem summarizes the connections between affine Weyl groups and their fundamental domains, and foldable figures. For proofs, see [11, 32].

Theorem 8

1. Let \mathcal{G} be a reflection group and \mathcal{O}_n the group of linear isometries of \mathbb{E}^n . Then there exists a homomorphism $\phi : \mathcal{G} \rightarrow \mathcal{O}_n$ given by $\phi(g)(x) = g(x) - -g(0)$, $g \in \mathcal{G}$, $x \in \mathbb{E}^n$. The group \mathcal{G} is called essential if $\phi(\mathcal{G})$ only fixes $0 \in \mathbb{E}^n$. The elements of $\ker \phi$ are called translations. If \mathcal{G} is essential and without fixed points then \mathcal{G} has a compact fundamental domain.
2. The reflection group generated by the reflections about the bounding hyperplanes of a foldable figure F is the affine Weyl group $\widetilde{\mathcal{W}}$ of some root system. Moreover, $\widetilde{\mathcal{W}}$ has F as a fundamental domain.
3. There exists a one-to-one correspondence between foldable figures and reflection groups that are essential and without fixed points.

Note that in view of this theorem, our construction of affine fractal hypersurfaces took place over a foldable figure, namely Δ^n together with its collection of subfigures $\{\Delta_i^n : i \in \mathbb{N}_N\}$. As the foldable figure and hence every subfigure of it is the fundamental domain of an associated affine Weyl group $\widetilde{\mathcal{W}}$, the existence of a labelling map ℓ is guaranteed. In addition – and this will become important in the next section – we also constructed an orthonormal basis on this foldable figure consisting of a finite number of basic affine fractal hypersurfaces.

In the next section it will become necessary to consider dilates of the fundamental domain of an affine Weyl group, i.e., a foldable figure. To this end, let $F \subset \mathbb{R}^n$ be a foldable figure with $0 \in \mathbb{R}^n$ as one of its vertices. Denote by \mathcal{H} be the set of hyperplanes associated with F and by Σ be the tessellation of F induced by \mathcal{H} . The affine Weyl group of the foldable figure F , $\widetilde{\mathcal{W}}$, is then the group generated by the affine reflections r_H , where $H \in \mathcal{H}$.

Now, fix $1 < \kappa \in \mathbb{N}$ and define $\Delta := \kappa F$. Then Δ is also a foldable figure, whose $N := \kappa^n$ subfigures $\Delta_i \in \Sigma$. Assume w.l.o.g that $\Delta_1 = F$. The tessellation and set of hyperplanes for Δ are $\kappa\Sigma$ and $\kappa\mathcal{H}$, respectively. Moreover, the affine reflection group generated by $\kappa\mathcal{H}$ is an isomorphic subgroup of $\widetilde{\mathcal{W}}$. Note that the similarity ratio $a = 1/\kappa$. By simple transitivity of $\widetilde{\mathcal{W}}$, define similitudes $u_i : \Delta \rightarrow \Delta_i$ by:

$$u_1 := \frac{1}{\kappa}(\cdot) \quad \text{and} \quad \forall j = 2, \dots, N : \quad u_j := r_{j,1} \circ u_1,$$

where $r_{j,1}$ is the composition of reflections from Δ_1 to Δ_j . Now the construction

Fig. 7 Foldable figures corresponding to the reducible root system $A_1 \times A_1$ (left) and the irreducible root system B_2 (right)



proceeds as in Sect. 4. This yields a fractal function \tilde{f} defined over Δ and an orthonormal basis of the form (16) whose elements are defined over the foldable figure $\varkappa F$.

Given a fractal function \tilde{f} defined over a foldable figure F , we can extend \tilde{f} to all of \mathbb{R}^n by using the fact that the foldable figure F is a fundamental domain for its associated affine Weyl group and that it tessellates \mathbb{R}^n by reflections in its bounding hyperplanes, i.e., under the action of $\widetilde{\mathcal{W}}$. For any foldable figure $F' \in \Sigma$ there exists a unique $r \in \widetilde{\mathcal{W}}$ so that $F' = r(F)$. Define an extension $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ of \tilde{f} by

$$\tilde{f}|_{F'} := \tilde{f} \circ r^{-1}.$$

Similarly, we define an orthonormal affine fractal basis \mathfrak{B}' over F' by setting $\mathfrak{B}' := \{\mathfrak{b} \circ r^{-1} : \mathfrak{b} \in \mathfrak{B}\}$. Hence, we obtain

$$\bigcup_{r \in \widetilde{\mathcal{W}}} \{\mathfrak{b} \circ r^{-1} : \mathfrak{b} \in \mathfrak{B}\}$$

as an orthonormal affine fractal basis for $L^2(\mathbb{R}^n)$.

6 Wavelet Sets

In this section, we establish a connection between the various concepts introduced in the previous sections and the multiscale structure of wavelets. Our emphasis will be entirely on wavelet sets in \mathbb{R}^n and most of the material found here is taken from [16–18, 38]. The reader interested in a more operator-theoretic formulation of wavelet sets is referred to [17]. For motivational purposes, we present first the one-dimensional setting, i.e., wavelet sets in \mathbb{R} , and then proceed to the higher-dimensional scenario.

To this end, we need to first define wavelets.

Definition 12 A dyadic orthonormal wavelet on \mathbb{R} is a unit vector $\psi \in L^2(\mathbb{R}) := L^2(\mathbb{R}, m)$, where m denotes Lebesgue measure, with the property that the set

$$\{2^{\frac{n}{2}}\psi(2^n t - \ell) : n, \ell \in \mathbb{Z}\} \tag{20}$$

of all integral translates of ψ followed by dilations by arbitrary integral powers of 2, is an orthonormal basis for $L^2(\mathbb{R})$.

We remark that this is not the most general definition of wavelet but for our purposes it is sufficient. For more details along this line, the interested reader is referred to the enormous literature on wavelets.

For later developments, we require the following two operators.

Definition 13 Let $B(L^2(\mathbb{R}))$ denote the Banach space of bounded linear operators from $L^2(\mathbb{R})$ to itself. The unitary translation operator $T \in B(L^2(\mathbb{R}))$ and the unitary dilation operator $D \in B(L^2(\mathbb{R}))$ are defined by

$$(Tf)(t) := f(t - 1) \quad \text{and} \quad (Df)(t) := \sqrt{2}f(2t), \quad f \in L^2(\mathbb{R}), \quad t \in \mathbb{R},$$

respectively.

With these two operators, we may write (20) more succinctly as

$$2^{\frac{n}{2}}\psi(2^n t - \ell) = (D^n T^\ell \psi)(t)$$

for all $n, \ell \in \mathbb{Z}, t \in \mathbb{R}$. Note that an easy computation shows $TD = DT^2$.

Next, we introduce the Fourier transform \mathcal{F} and its inverse \mathcal{F}^{-1} on functions $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$(\mathcal{F}f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t) dt =: \widehat{f}(s), \tag{21}$$

$$(\mathcal{F}^{-1}g)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} g(s) ds = \widehat{g}(-s) =: g^\vee(t). \tag{22}$$

Note that this particular form of the Fourier-Plancherel transform defines a unitary operator on $L^2(\mathbb{R})$. In particular, we remark that

$$(\mathcal{F}T)(f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t - 1) dt = e^{-is} (\mathcal{F}f)(s) =: e^{-is} \widehat{f}(s).$$

Hence, $\mathcal{F}T\mathcal{F}^{-1}\widehat{f} = e^{-is}\widehat{f} =: M_{e^{-is}}\widehat{f}$. If we define $\widehat{T} := \mathcal{F}T\mathcal{F}^{-1}$ then $\widehat{T} = M_{e^{-is}}$. Similarly, we obtain for any $n \in \mathbb{Z}$,

$$\begin{aligned} (\mathcal{F}D^n f)(s) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} (\sqrt{2})^n f(2^n t) dt = (\sqrt{2})^{-n} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i2^{-n}st} f(t) dt \\ &= (\sqrt{2})^{-n} (\mathcal{F}f)(2^{-n}s) = (D^{-n} \mathcal{F}f)(s). \end{aligned}$$

As above, if we set $\widehat{D}^n := \mathcal{F}D^n\mathcal{F}^{-1}$ then $\widehat{D}^n = D^{-n}$ and therefore $\widehat{D} = D^{-1}$.

Wavelet sets belong to the theory of wavelets via the Fourier transform.

Definition 14 A wavelet set in \mathbb{R} is a Lebesgue measurable subset E of \mathbb{R} for which $\frac{1}{\sqrt{2\pi}} \chi_E$ is the Fourier transform of a wavelet.

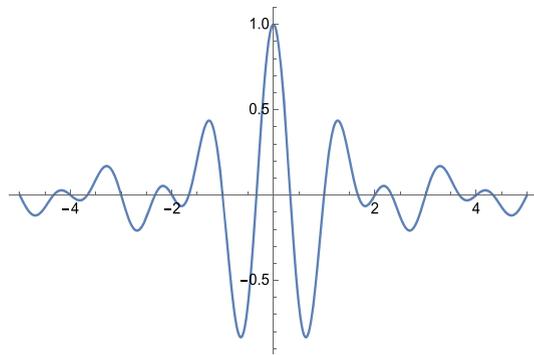
The wavelet $\widehat{\psi}_E := \frac{1}{\sqrt{2\pi}} \chi_E$ is sometimes called s -elementary. The class of wavelet sets was also investigated in [22, 34]. In their theory, the corresponding wavelets are called MSF (minimally supported frequency) wavelets (Fig. 8).

Example 5 The prototype of a wavelet set is the Shannon or Paley-Wiener set given by constructed in Section

$$E_S = [-2\pi, -\pi) \cup [\pi, 2\pi). \tag{23}$$

The orthonormal wavelet for E_S is then given by $\psi_S(t) = \text{sinc}(2t) - \text{sinc } t$, where

Fig. 8 The Shannon or Paley-Wiener wavelet



$$\text{sinc}(t) := \frac{\sin(\pi t)}{\pi t}.$$

To prove that ψ_S is indeed an orthonormal wavelet, note that the set of exponentials

$$\{e^{i\ell s} : \ell \in \mathbb{Z}\}$$

restricted to $[0, 2\pi]$ and normalized by $\frac{1}{\sqrt{2\pi}}$ is an orthonormal basis for $L^2[0, 2\pi]$. Write $E_S = E_- \cup E_+$ where $E_- = [-2\pi, -\pi)$, $E_+ = [\pi, 2\pi)$. Since $\{E_- + 2\pi, E_+\}$ is a partition of $[0, 2\pi)$ and since the exponentials $e^{i\ell s}$ are invariant under translation by 2π , it follows that

$$\left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_S} : \ell \in \mathbb{Z} \right\}$$

is an orthonormal basis for $L^2(E_S)$. Since $\widehat{T} = M_{e^{-is}}$, this set can be written as

$$\{\widehat{T}^\ell \widehat{\psi}_S : \ell \in \mathbb{Z}\}.$$

Next, note that any ‘‘dyadic interval’’ of the form $J = [b, 2b)$, for some $b > 0$ has the property that $\{2^n J : n \in \mathbb{Z}\}$, is a partition of $(0, \infty)$. Similarly, any set of the form

$$\mathcal{K} = [-2a, -a) \cup [b, 2b)$$

for $a, b > 0$, has the property that

$$\{2^n \mathcal{K} : n \in \mathbb{Z}\}$$

is a partition of $\mathbb{R} \setminus \{0\}$.

To complete the proof, we need to introduce one more item.

Definition 15 Let U be a given unitary operator on a Hilbert space \mathcal{H} . A nonempty subspace \mathcal{K} of \mathcal{H} is called a wandering subspace for U if $U^m(\mathcal{K}) \perp U^n(\mathcal{K})$, for all $m \neq n \in \mathbb{N}$. If, in addition, $\mathcal{H} = \bigoplus_{n \in \mathbb{Z}} U^n(\mathcal{K})$, then we say that \mathcal{K} is a complete wandering subspace of \mathcal{H} for U .

It follows that the space $L^2(\mathcal{K})$, considered as a subspace of $L^2(\mathbb{R})$, is a complete wandering subspace for the unitary dilation operator $(Df)(s) = \sqrt{2} f(2s)$. For each $n \in \mathbb{Z}$,

$$D^n(L^2(\mathcal{K})) = L^2(2^{-n}\mathcal{K}).$$

So $\bigoplus_{n \in \mathbb{Z}} D^n(L^2(\mathcal{K}))$ is a direct sum decomposition of $L^2(\mathbb{R})$. In particular E_S has this property. Thus,

$$D^n \left\{ \frac{e^{i\ell s}}{\sqrt{2\pi}} \Big|_{E_S} : \ell \in \mathbb{Z} \right\} = \left\{ \frac{e^{2^n i\ell s}}{\sqrt{2\pi}} \Big|_{2^{-n}E_S} : \ell \in \mathbb{Z} \right\}$$

is a basis for $L^2(2^{-n}E_S)$ for each n . Hence

$$\{D^n \widehat{T}^\ell \widehat{\psi}_S : n, \ell \in \mathbb{Z}\} = \{\widehat{D}^n \widehat{T}^\ell \widehat{\psi}_S : n, \ell \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, as required.

The above arguments yield the following two *sufficient* conditions for E to be a wavelet set:

1. The set of normalized exponentials $\left\{ \frac{1}{\sqrt{2\pi}} e^{i\ell s} : \ell \in \mathbb{Z} \right\}$, when restricted to E , constitutes an orthonormal basis for $L^2(E)$.
2. The family $\{2^n E : n \in \mathbb{Z}\}$ of dilates of E by integral powers of 2 should constitute a measurable partition (i.e., a partition modulo null sets) of \mathbb{R} .

These observations now motivate the next definitions.

Definition 16 Two measurable sets $E, F \subseteq \mathbb{R}$ are called translation congruent modulo 2π

- if there exists a measurable bijection $\phi : E \rightarrow F$ such that $\phi(s) - s = n(s) 2\pi$, for each $s \in E$, and a unique $n(s) \in \mathbb{Z}$,

or, equivalently,

- if there is a measurable partition $\{E_n : n \in \mathbb{Z}\}$ of E such that

$$\{E_n + n 2\pi : n \in \mathbb{Z}\}$$

is a measurable partition of F .

Two measurable sets $G, H \subseteq \mathbb{R}$ are called dilation congruent modulo 2

- if there is a measurable bijection $\tau : G \rightarrow H$ such that for each $s \in G$ there exists an $n = n(s) \in \mathbb{Z}$ with $\tau(s) = 2^n s$,

or equivalently,

- if there is a measurable partition $\{G_n : n \in \mathbb{Z}\}$ of G such that

$$\{2^n G_n : n \in \mathbb{Z}\}$$

is a measurable partition of H .

Example 6 $E := [0, 2\pi]$ and $F := [-2\pi, -\pi] \cup [\pi, 2\pi]$ are translation congruent modulo 2π .

The following lemma is proved in [16].

Lemma 2 Let $f \in L^2(\mathbb{R})$, and let $E = \text{supp}(f)$. Then f has the property that

$$\{e^{i\ell s} f : \ell \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(E)$ if and only if

1. E is congruent to $[0, 2\pi]$ modulo 2π , and
2. $|f(s)| = \frac{1}{\sqrt{2\pi}}$ a.e. on E .

Observe that if E is 2π -translation congruent to $[0, 2\pi)$, then since

$$\{[0, 2\pi) + 2\pi n : n \in \mathbb{Z}\}$$

is a measurable partition of \mathbb{R} , so is

$$\{E + 2\pi n : n \in \mathbb{Z}\}.$$

Similarly, if F is 2-dilation congruent to the Shannon wavelet set $E_S = [-2\pi, -\pi) \cup [\pi, 2\pi)$, then since $\{2^n E_S : n \in \mathbb{Z}\}$ is a measurable partition of \mathbb{R} , so is $\{2^n F : n \in \mathbb{Z}\}$.

A measurable subset $G \subseteq \mathbb{R}$ is a 2-dilation generator of a partition of \mathbb{R} if the sets

$$2^n G := \{2^n s : s \in G\}, \quad n \in \mathbb{Z}$$

are disjoint and $\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} 2^n G$ is a null set.

Analogously, $E \subseteq \mathbb{R}$ is a 2π -translation generator of a partition of \mathbb{R} if the sets

$$E + 2n\pi := \{s + 2n\pi : s \in E\}, \quad n \in \mathbb{Z},$$

are disjoint and $\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} (E + 2n\pi)$ is a null set.

The next theorem gives necessary and sufficient conditions for a measurable set $E \subseteq \mathbb{R}$ to be a wavelet set. Before we state it, we need a definition.

Definition 17 A fundamental domain for a group \mathcal{G} of (measurable) transformations on a measure space (X, \mathcal{A}, μ) is a measurable set $C \in \mathcal{A}$ with the property that $\{g(C) : g \in \mathcal{G}\}$ is a measurable partition (tessellation) of X ; that is, $X \setminus \left(\bigcup_{g \in \mathcal{G}} g(C) \right)$ is a μ -null set and $g_1(C) \cap g_2(C)$ is a μ -null set for $g_1 \neq g_2$.

Theorem 9 Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set if and only if one of the following equivalent conditions holds.

1. E is both a 2-dilation generator of a partition (modulo null sets) of \mathbb{R} and a 2π -translation generator of a partition (modulo null sets) of \mathbb{R} .
2. E is both translation congruent to $[0, 2\pi) \bmod 2\pi$ and dilation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi) \bmod 2$.
3. E is a fundamental domain for the dilation group $\langle D^n : n \in \mathbb{Z} \rangle$ and at the same time a fundamental domain for the translation group $\langle T_{2\pi}^k : k \in \mathbb{Z} \rangle$. Here $T_{2\pi}$ is translation by 2π along the real axis.)

Proof We refer to [16].

Now, we like to extend the above concepts and definitions to \mathbb{R}^n . We will do this in a slightly more general setting. To this end, let \mathbb{X} be a metric space and μ a σ -finite non-atomic Borel measure on \mathbb{X} for which the measure of every open set is positive and for which bounded sets have finite measure.

Let \mathcal{T} and \mathcal{D} be countable groups of homeomorphisms of \mathbb{X} that map bounded sets to bounded sets and which are absolutely continuously in the sense that they map μ -null sets to μ -null sets. Furthermore, let \mathcal{G} be a countable group of absolutely continuous Borel measurable bijections of \mathbb{X} . Denote by $\mathcal{B} := \mathcal{B}(\mathbb{X})$ the family of Borel sets of \mathbb{X} .

The following definition completely generalizes the definitions of 2π -translation congruence and 2-dilation congruence given above.

Definition 18 Let $E, F \in \mathcal{B}$. We call E and F \mathcal{G} -congruent and write $E \sim_{\mathcal{G}} F$, if there exist measurable partitions $\{E_g : g \in \mathcal{G}\}$ and $\{F_g : g \in \mathcal{G}\}$ of E , respectively, F such that $F_g = g(E_g)$, for all $g \in \mathcal{G}$, modulo μ -null sets.

This definition immediately entails the next two results.

Proposition 2

1. \mathcal{G} -congruence is an equivalence relation on the family of m -measurable sets.
2. If E is a fundamental domains for \mathcal{G} , then F is a fundamental domain for \mathcal{G} iff $F \sim_{\mathcal{G}} E$.

Proof We refer again to [16].

Definition 19 We call $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair if

1. for each bounded set E and each open set F there exist elements $\delta \in \mathcal{D}$ and $\tau \in \mathcal{T}$ such that $\tau(F) \subset \delta(E)$;

- there exists a fixed point $\theta \in \mathbb{X}$ for \mathcal{D} with the property that if N is any neighborhood of θ and E any bounded set, there is an element $\delta \in \mathcal{D}$ such that $\delta(E) \subset N$.

The following result and its proof can be found in [17].

Theorem 10 *Let $\mathbb{X}, \mathcal{B}, \mu, \mathcal{D}$, and \mathcal{T} as above. Let $(\mathcal{D}, \mathcal{T})$ be an abstract dilation–translation pair with θ being the fixed point of \mathcal{D} . Assume that E and F are bounded measurable sets in \mathbb{X} such that E contains a neighborhood of θ , and F has non-empty interior and is bounded away from θ . Then there exists a measurable set $G \subset X$, contained in $\bigcup_{\delta \in \mathcal{D}} \delta(F)$, which is both \mathcal{D} –congruent to F and \mathcal{T} –congruent to E .*

The next result is a consequence of Proposition 2 and Theorem 10 and is the key to constructing wavelet sets.

Corollary 1 *With the terminology of Theorem 10, if in addition F is a fundamental domain for \mathcal{D} and E is a fundamental domain for \mathcal{T} , then there exists a set G which is a common fundamental domain for both \mathcal{D} and \mathcal{T} .*

In order to apply the above result to wavelet sets in \mathbb{R}^n , we require the following two definitions.

Definition 20 Let $A \in M_n(\mathbb{R})$ be an $(n \times n)$ –matrix with real coefficients. An orthonormal (D_A, T) –wavelet is a function $\psi \in L^2(\mathbb{R}^n)$ such that

$$\left\{ |\det(A)|^{\frac{n}{2}} \psi(A^n t - \ell) : n \in \mathbb{Z}, \ell \in \mathbb{Z}^n, i = 1, \dots, n \right\}, \tag{24}$$

where $\ell = (\ell_1, \ell_2, \dots, \ell_n)^T$ (T transpose), is an orthonormal basis for $L^2(\mathbb{R}^n, m)$. (Here m is product Lebesgue measure.)

A matrix $A \in M_n(\mathbb{R})$ is called expansive, if all its eigenvalues are strictly greater than 1.

If $A \in M_n(\mathbb{R})$ is invertible (so in particular if A is expansive), then the operator defined by

$$(D_A f)(t) := |\det A|^{\frac{n}{2}} f(At)$$

for $f \in L^2(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, is unitary.

For $1 \leq i \leq n$, let T_i be the unitary operator defined by translation by 1 in the i^{th} coordinate direction. Then, the set (24) above can be written as

$$\{D_A^k T^\ell \psi : k, \ell \in \mathbb{Z}^n\},$$

with $T^\ell := T_1^{\ell_1} \dots T_n^{\ell_n}$.

Definition 21 A (D_A, T) –wavelet set is a Lebesgue measurable subset E of \mathbb{R}^n for which the inverse Fourier transform of $\frac{1}{m(E)^{n/2}} \chi_E$ is an orthonormal (D_A, T) –wavelet.

Two measurable subsets H and K of \mathbb{R}^n are called A -dilation congruent, in symbols $H \sim_{\delta_A} K$, if there exist measurable partitions $\{H_\ell : \ell \in \mathbb{Z}\}$ of H and $\{K_\ell : \ell \in \mathbb{Z}\}$ of K such that $K_\ell = A^\ell H_\ell$ modulo Lebesgue null sets. Moreover, two measurable sets E and F of \mathbb{R}^n are called 2π -translation congruent, written $E \sim_{\tau_{2\pi}} F$, if there exist measurable partitions $\{E_\ell : \ell \in \mathbb{Z}^n\}$ of E and $\{F_\ell : \ell \in \mathbb{Z}^n\}$ of F such that $F_\ell = E_\ell + 2\pi\ell$ modulo Lebesgue null sets.

We remark that this generalizes to \mathbb{R}^n the previous definition of 2π -translation congruence for subsets of \mathbb{R} . Observe that A -dilation by an expansive matrix together with 2π -translation congruence is a special case of an abstract dilation-translation pair as introduced in Definition 19.

Let $\mathcal{D} := \langle A^k : k \in \mathbb{Z} \rangle$ be the dilation group generated by powers of A and let $\mathcal{T} := \langle T_{2\pi}^\ell : \ell \in \mathbb{Z}^n \rangle$ be the group of translations generated by the translations $T_{2\pi}$ along the coordinate directions. Let E be any bounded set and let F be any open set that is bounded away from 0.

Let $r > 0$ be such that $E \subseteq B_r(0)$. Since A is expansive there is an $\ell \in \mathbb{N}$ such that $A^\ell F$ contains a ball B of radius large enough so that B contains some lattice point $2k\pi$ together with the ball $B_R(2k\pi)$ of radius $R > 0$ centered at the lattice point. Then $E + 2k\pi \subseteq A^\ell F$. That is, the $2k\pi$ -translate of E is contained in the A^ℓ -dilate of F , as required in (1) of Definition 19.

For (2) of Definition 19, let $\theta = 0$, and let N be a neighborhood of 0, and let E be any bounded set. As above, choose $r > 0$ with $E \subseteq B_r(0)$. Let $\ell \in \mathbb{N}$ be such that $A^\ell N$ contains $B_r(0)$. Then $A^{-\ell}$ is the required dilation such that $A^{-\ell} E \subseteq N$.

Note that if W is a measurable subset of \mathbb{R}^n that is 2π -translation congruent to the n -cube $E := \prod_{i=1}^n [-\pi, \pi]$, it follows from the exponential form of \widehat{T}_j that $\left\{ \widehat{T}_1^{\ell_1} \widehat{T}_2^{\ell_2} \dots \widehat{T}_n^{\ell_n} (m(W))^{-n/2} \chi_W : \ell = (\ell_1, \ell_2, \dots, \ell_n)^\top \in \mathbb{Z}^n \right\}$ is an orthonormal basis for $L^2(W)$.

Furthermore, if A is an expansive matrix and B the unit ball of \mathbb{R}^n then with $F_A := A(B) \setminus B$ the collection $\{A^k F_A : k \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n \setminus \{0\}$. Consequently, $L^2(F_A)$, considered as a subspace of $L^2(\mathbb{R}^n)$, is a complete wandering subspace for D_A . Hence, $L^2(\mathbb{R}^n)$ is a direct sum decomposition of the subspaces $\{D_A^k L^2(F_A) : k \in \mathbb{Z}\}$. Clearly, any other measurable set $F' \sim_{\delta_A} F_A$ has this same property.

The following theorem gives the existence of wavelet sets in \mathbb{R}^n . For the proof, we refer to [18].

Theorem 11 *Let $n \in \mathbb{N}$ and let A be an expansive $n \times n$ matrix. Then there exist (D_A, T) -wavelet sets.*

Example 7 The following is an example of a (fractal) wavelet set in \mathbb{R}^2 taken from [16] and shown below.

To this end, let $A := 2I$, where I denotes the identity matrix in \mathbb{R}^2 . For $n \in \mathbb{N}$, define vectors $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^2$ by

$$\vec{\alpha}_n := \frac{1}{2^{2n-2}} \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{pmatrix};$$

$$\vec{\beta}_0 := 0, \quad \vec{\beta}_n := \sum_{k=1}^n \vec{\alpha}_k.$$

Set

$$G_0 := \left[0, \frac{\pi}{2}\right];$$

$$G_n := \frac{1}{2^{2n}} G_0 + \vec{\beta}_n;$$

$$E_1 := \bigcup_{k=1}^{\infty} G_k \subset 2G_0 \setminus G_0;$$

$$C_1 := G_0 \cup E_1 + \begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix};$$

$$B_1 := 2G_0 \setminus (G_0 \cup E_1).$$

Finally, let

$$A_1 := B_1 \cup C_1;$$

$$A_2 := \{(-x, y) : (x, y) \in A_1\};$$

$$A_3 := \{(-x, -y) : (x, y) \in A_1\};$$

$$A_4 := \{(x, -y) : (x, y) \in A_1\};$$

$$W := A_1 \cup A_2 \cup A_3 \cup A_4.$$

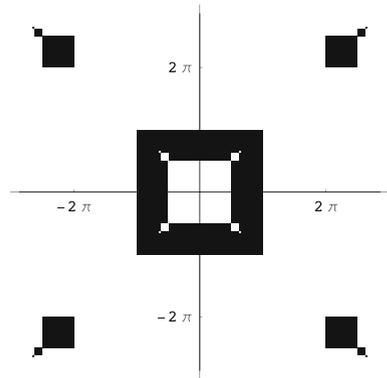
It is not hard to verify that W is 2π -translation-congruent to $C = [-\pi, \pi) \times [-\pi, \pi)$ and a 2-dilation generator of a measurable partition for the two-dimensional plane $\mathbb{R}^2 \setminus \{0\}$. The wavelet set W is graphically displayed in Fig. 9 below.

For more examples and constructions of wavelet sets, we encourage the reader to consult, for instance, [1, 7, 18, 48, 49, 54].

Finally, we put the concepts of fractal hypersurface, foldable figure, affine Weyl group and wavelet set together and introduce a new type of wavelet set called a dilation-reflection wavelet set. The idea is to adapt Definition 19, replacing the group of translations \mathcal{T} in the traditional wavelet theory by an affine Weyl group $\widetilde{\mathcal{W}}$ whose fundamental domain is a foldable figure C , and then use the orthonormal basis of affine fractal hypersurfaces constructed in Sect. 4.

In Definition 19, we take $\mathbb{X} := \mathbb{R}^n$ endowed with the Euclidean affine structure and distance. For the abstract translation group \mathcal{T} we take the affine Weyl group $\widetilde{\mathcal{W}}$

Fig. 9 The two-dimensional wavelet set W from Example 7



generated by a group of affine reflections arising from a locally finite collection of affine hyperplanes of \mathbb{R}^n .

Let C denote a fundamental domain for $\widetilde{\mathcal{W}}$ which is also a foldable figure. Recall that C is a simplex, i.e., a convex connected polytope, which tessellates \mathbb{R}^n by reflections about its bounding hyperplanes. Let θ be any fixed interior point of C . Let A be any real expansive matrix in $M_n(\mathbb{R})$ acting as a linear transformation on \mathbb{R}^n . In the case where θ is the origin 0 in \mathbb{R}^n we simply take D_A to be the usual dilation by A and the abstract dilation group to be $\mathcal{D} = \{D_A^k : k \in \mathbb{Z}\}$. For a general θ , define $D_{A,\theta}$ to be the affine mapping $D_{A,\theta}(x) := A(x - \theta) + \theta, x \in \mathbb{R}^n$, and $\mathcal{D}_\theta = \{D_{A,\theta}^k : k \in \mathbb{Z}\}$.

Proposition 3 $(\mathcal{D}_\theta, \widetilde{\mathcal{W}})$ is an abstract dilation-translation pair in the sense of Definition 19.

Proof The proof is given in [38] but for the sake of completeness, we repeat it here.

By the definition of D, θ is a fixed point for \mathcal{D}_θ . By a change of coordinates we may assume without loss of generality that $\theta = 0$ and consequently that D is multiplication by A on \mathbb{R}^n .

Let $B_r(0)$ be an open ball centered at 0 with radius $r > 0$ containing both E and C . Since F is open and A is expansive, there exists a $k \in \mathbb{N}$ sufficiently large so that $D^k F$ contains an open ball $B_{3r}(p)$ of radius $3r$ and with some center p . Since C tiles \mathbb{R}^n under the action of $\widetilde{\mathcal{W}}$, there exists a word $w \in \widetilde{\mathcal{W}}$ (i.e., a concatenation of reflections) such that $w(C) \cap B_r(p)$ has positive measure. (Note here that $B_r(p)$ is the ball with the same center p but with smaller radius r .) Then $w(B_r(0)) \cap B_r(p) \neq \emptyset$. Since reflections (and hence words in $\widetilde{\mathcal{W}}$) preserve diameters of sets in \mathbb{R}^n , it follows that $w(B_r(0))$ is contained in $B_{3r}(p)$. Hence $w(E)$ is contained in $D^k(F)$, as required.

This establishes part (1) of Definition 19. Part (2) follows from the fact that $\theta = 0$ and D is multiplication by an expansive matrix in $M_n(\mathbb{R})$. \square

Now we extend the definition of (D_A, T) -wavelet set in \mathbb{R}^n to this new setting.

Definition 22 Assume that an affine Weyl group $\widetilde{\mathcal{W}}$ acting on \mathbb{R}^n is given with a foldable figure C as its fundamental domain. Further assume that $\theta \in C$ is a

designated interior point and A an expansive matrix on \mathbb{R}^n . A $(D_{A,\theta}, \widetilde{\mathcal{W}})$ -wavelet set is a Lebesgue measurable subset E of \mathbb{R}^n satisfying the following two conditions:

1. E is congruent to C (in the sense of Definition 2.4) under the action of $\widetilde{\mathcal{W}}$, and
2. $\widetilde{\mathcal{W}}$ generates a measurable partition of \mathbb{R}^n under the action of the affine mapping $D(x) := A(x - \theta) + \theta$.

In the case where $\theta = 0$, we abbreviate $(D_{A,\theta}, \widetilde{\mathcal{W}})$ to $(D_A, \widetilde{\mathcal{W}})$.

The next result establishes the existence of $(D_{A,\theta}, \widetilde{\mathcal{W}})$ -wavelet sets. The proof is a direct application of Theorem 10 and can be found in [38].

Theorem 12 *There exist $(D_{A,\theta}, \widetilde{\mathcal{W}})$ -wavelet sets for every choice of $\widetilde{\mathcal{W}}$, A , and θ .*

In the case of a dilation–translation wavelet set W , the two systems of unitary operators are $\mathcal{D} := \{D_A^k : k \in \mathbb{Z}\}$, where $A \in M_n(\mathbb{R})$ is an expansive matrix, and $\mathcal{T} := \{T^\ell : \ell \in \mathbb{Z}^n\}$. An orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ is then obtained by setting $\widehat{\psi}_W := (m(W))^{-n/2} \chi_W$ and taking

$$\left\{ \widehat{D}_A^k \widehat{T}^\ell \widehat{\psi}_W : k \in \mathbb{Z}, \ell \in \mathbb{Z}^n \right\}. \tag{25}$$

For the systems of unitary operators $\mathcal{D} := \{D_A^k : k \in \mathbb{Z}\}$ and $\widetilde{\mathcal{W}}$, the affine Weyl group associated with a foldable figure C , one obtains as an orthonormal basis for $L^2(\mathbb{R}^n)$

$$\left\{ D_{\mathcal{A}I}^k \mathfrak{B}_r : k \in \mathbb{Z}, r \in \widetilde{\mathcal{W}} \right\}, \tag{26}$$

where $\mathfrak{B}_r = \{\mathfrak{b} \circ r : \mathfrak{b} \in \mathfrak{B}\}$ is an affine fractal hypersurface basis as constructed in the previous section and $I \in \mathbb{R}^{n \times n}$ denotes the unit matrix.

It was shown in [25, 26] that one can even construct multiresolution analyses based on affinely generated fractal functions as constructed in Sects. 3 and 4. The resulting sets of scaling vectors and multiwavelets are piecewise affine fractal functions and they generate orthonormal bases for the underlying approximation, respectively, wavelet spaces. If one uses these orthonormal multiwavelet bases in (26) to obtain orthonormal bases for $L^2(\mathbb{R}^n)$ then the analogy to the classical case as exemplified by (25) is complete. For more details, we refer to [38].

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