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Differentiability results and sensitivity calculation for optimal control of incompressible two-phase Navier-Stokes equations with surface tension

Elisabeth Diehl[1] · Johannes Haubner[2] · Michael Ulbrich[2] · Stefan Ulbrich[1]

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Abstract

We analyze optimal control problems for two-phase Navier-Stokes equations with surface tension. Based on L_p-maximal regularity of the underlying linear problem and recent well-posedness results of the problem for sufficiently small data we show the differentiability of the solution with respect to initial and distributed controls for appropriate spaces resulting from the L_p-maximal regularity setting. We consider first a formulation where the interface is transformed to a hyperplane. Then we deduce differentiability results for the solution in the physical coordinates. Finally, we state an equivalent Volume-of-Fluid type formulation and use the obtained differentiability results to derive rigorosly the corresponding sensitivity equations of the Volume-of-Fluid type formulation. For objective functionals involving the velocity field or the discontinuous pressure or phase indciator field we derive differentiability results with respect to controls and state formulas for the derivative. The results of the paper form an analytical foundation for stating optimality conditions, justifying the application of derivative based optimization methods and for studying the convergence of discrete sensitivity schemes based on Volume-of-Fluid discretizations for optimal control of two-phase Navier-Stokes equations.

Keywords Two-phase flow · Surface tension · Sharp interface · Navier-Stokes equations · Volume of fluid · Differentiability · Optimal control

Mathematics Subject Classification 49M41 · 76D55 · 35Q35 · 76D45 · 65K10

✉ Stefan Ulbrich
stefan.ulbrich@tu-darmstadt.de

Extended author information available on the last page of the article

1 Introduction

We consider the incompressible sharp interface two-phase Navier-Stokes equations. To this end, let the hypersurface (interface) $\Gamma(t)$ divide $\mathbb{R}^{n+1}$ into two open domains $\Omega_1(t)$ and $\Omega_2(t) = \mathbb{R}^{n+1} \setminus \overline{\Omega_1}(t)$, $i = 1, 2$, occupied by two viscous incompressible immiscible capillary Newtonian fluids with constant densities $\rho_i > 0$ and constant viscosities $\mu_i > 0, i = 1, 2$. We set

$$\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$$

and with the indicator functions 1_{Ω_i}

$$\rho = \rho_1 1_{\Omega_1} + \rho_2 1_{\Omega_2}, \quad \mu = \mu_1 1_{\Omega_1} + \mu_2 1_{\Omega_2}.$$

Moreover, we denote by $\nu(t, \cdot)$ the normal field on $\Gamma(t)$ pointing form $\Omega_1(t)$ to $\Omega_2(t)$, by $V(t, \cdot)$ the normal velocity of the interface $\Gamma(t)$ and by $\kappa(t, \cdot)$ the mean curvature of $\Gamma(t)$ with respect to $\nu(t, \cdot)$. Then $\kappa(t, x)$ is negative when $\Omega_1(t)$ is convex close to $x \in \Gamma(t)$ and is for sufficiently smooth $\Gamma(t)$ given by

$$\kappa(t, \cdot) = -\operatorname{div}_{\Gamma} \nu(t, \cdot)$$

(note that this coincides with $-\operatorname{div} \nu(t, \cdot)$ if $\nu(t, \cdot)$ admits a differentiable extension to a neighborhood of $\Gamma(t)$). Finally, if v is defined and admits boundary traces on both domains $\Omega_i(t)$ then

$$[v] = (v|_{\Omega_2(t)} - v|_{\Omega_1(t)})|_{\Gamma(t)}$$

denotes the jump of v accross $\Gamma(t)$. The two-phase Navier-Stokes equations with surface tension then read

$$\begin{aligned}
\rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla q &= c && \text{in } \Omega(t),\\
\operatorname{div} u &= 0 && \text{in } \Omega(t),\\
-[S(u, q; \mu)\nu] &= \sigma \kappa \nu && \text{on } \Gamma(t),\\
[u] &= 0 && \text{on } \Gamma(t),\\
V &= u^{\top} \nu && \text{on } \Gamma(t),\\
u(0) &= u_0 && \text{on } \Omega(0),\\
\Gamma(0) &= \Gamma_0,
\end{aligned} \tag{1}$$

with the velocity u, the pressure q, the stress tensor $S(u, q; \mu) = -qI + \mu(\nabla u + \nabla u^{\top})$ and the surface tension coefficient $\sigma > 0$. Here, c denotes some control.

The conditions on the interface ensure the balance between surface tension and the jump of the normal stress on the interface, the continuity of the velocity across the interface and the transport of the interface by the fluid velocity.

We note that the first four equations can be written in weak form on the whole domain by

$$\int_{\mathbb{R}^{n+1}} \left(\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - c\right)^\top \varphi + S(u, q; \mu) : \nabla\varphi\right) dx = \int_{\Gamma(t)} \sigma\kappa\nu^\top \varphi \, dS(x)$$
$$\forall\, \varphi \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}), \tag{2}$$

$$\int_{\mathbb{R}^{n+1}} \operatorname{div}(u)\, \psi \, dx = 0 \quad \forall\, \psi \in C_c^1(\mathbb{R}^{n+1}). \tag{3}$$

Our aim is to study the differentiability properties of local solutions with respect to u_0 and c. To this end, we will work in an L_p-maximal regularity setting proposed in [22], see also [20, 23].

There exist several papers on the existence and uniqueness of local solutions for (1). In [8, 9, 24, 25] Lagrangian coordinates are used to obtain local well-posedness. Since this approach makes it difficult to establish smoothing of the unknown interface, [20, 22, 23] use a transformation to a fixed domain and are then able to show local well-posedness in an L_p maximum regularity setting for the case $c = 0$ [20, 22] or for the case of gravitation [23]. Moreover, they prove that the interface as well as the solution become instantaneously real analytic. Since we are considering a distributed control c of limited regularity, the instant analyticity is in general lost.

While optimal control problems for the Navier-Stokes equations have been studied by many researchers, see for example [12, 15, 19, 26], there are only a few contributions in the context of two-phase Navier-Stokes equations, mainly for phase-field formulations with semidiscretization in time. In [18] optimal boundary control of a time-discrete Cahn-Hilliard-Navier-Stokes system with matched densities is studied. By using regularization techniques, existence of optimal solutions and optimality conditions are derived. Analogous results for distributed optimal control with unmatched densities for the diffuse interface model of [1] have been obtained in [17]. Using the same model, [14] derive based on the stable time discretization proposed in [13] necessary optimality conditions for the time-discrete and the fully discrete optimal control problem. Moreover, the differentiability of the control-to-state mapping for the semidiscrete problem is shown. Optimal control of a binary fluid described by its density distribution, but without surface tension, is studied in [4]. Different numerical approaches for the optimal control of two-phase flows are discussed in [5].

In this paper we derive differentiability results of the solution of the two-phase Navier-Stokes equations (1) with respect to controls. The results can be used to state optimality conditions and to justify the application of derivative based optimization methods. To the best of our knowledge, this is the first work providing differentiability properties of control-to-state mappings for sharp interface models of two-phase Navier-Stokes flow. The analysis is involved, since the moving interface renders a variational analysis difficult. Therefore it is beneficial, to first consider a transformed problem with fixed interface. However, since most numerical approaches are working in physical coordinates, we derive also differentiability results for the original problem. Since the normal derivative of the velocity is in general discontinuous at the interface, the sensitivities of the velocity are discontinuous across the interface. Moreover, the pressure is in general discontinuous at the interface and thus differentiability properties with respect to controls in strong spaces hold only away from the interface while at the interface differentiability properties can only be expected in the weak topology of measures. The same applies to phase indicators which are often used in Volume-of-Fluid (VoF)-type approaches. In order to obtain a PDE-formulation for the sensitivity equations, we work with a Volume-of-Fluid (VoF)-type formulation based on a discontinuous phase indicator and derive carefully a corresponding sensitivity equation.

We build on the quite recent existence and uniqueness results obtained for sufficiently small data by [22], see also [20, 23]. We consider first a formulation, where the interface is transformed to a hyperplane. By using L_p-maximal regularity of a linear system and applying a refined version of a fixed point theorem, we show differentiability of the transformed state with respect to controls in the maximum regularity spaces. A similar technique was recently used in [16] to show differentiability properties for shape optimization of fluid-structure interaction, but the analysis of the fixed point iteration is very different from two-phase flows considered here. In fact, the main difficulties in fluid-structure interaction arise from the coupling of a hyperbolic equation for the solid with the Navier-Stokes equations for the fluid while in two phase flows the moving interface and the surface tension are the main challenge. In a second step we deduce differentiability results for the control-to-state map in the physical coordinates. Finally, we derive an equivalent Volume-of-Fluid (VoF)-type formulation based on a discontinuous phase indicator that is governed by a multidimensional transport equation. By using the obtained differentiability results, we are able to justify a sensitivity system for the VoF-type formulation, which invokes measure-valued solutions of the linearized transport equation. This can be used as an analytical foundation to study the convergence of discrete sensitivity schemes for VoF-type methods. Moreover, we obtain the differentiability of objective functionals invoking the velocity field or the discontinuous pressure or phase indicator field and state formulas for the derivative.

The paper is organized as follows. In Sect. 2, the transformed problem is formulated. In Sect. 3, existence, uniqueness and differentiability of the control-to-state mapping is shown. The analysis starts in 3.1 for the transformed problem with flat interface. In 3.2 differentiability results for the original problem in physical coordinates are derived. In 3.3 the VoF-type formulation and its sensitivity equation are justified. In Sect. 4 we derive some analytical settings for the application of optimization methods. In 4.1 we consider objective functionals involving the velocity field and state differentiability results. Subsequently, we discuss in 4.2 objective functionals involving the pressure field or the phase indicator, obtain their differentiability with respect to controls as well as a formula for the derivative.

2 Transformation to a flat interface

In this paper, we consider as in Prüss and Simonett [22] the problem in $n+1$ dimensions, where Γ_0 is the graph of a sufficiently smooth function $h_0 : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\Gamma_0 = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,:\, y = h_0(x)\big\},$$
$$\Omega_1(0) = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,:\, y < h_0(x)\big\},$$
$$\Omega_2(0) = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,:\, y > h_0(x)\big\}.$$

The interface has then the form

$$\Gamma(t) = \{(x, h(t,x)) \,:\, x \in \mathbb{R}^n\},$$

where $h : [0,t_0] \times \mathbb{R}^n \to \mathbb{R}$ with $h(0,\cdot) = h_0$ and $t_0 > 0$ is some final time. We note that the case of bounded fluid domains is considered in [20]. The analysis of this paper should also extend to this setting, but the presentation would be more technical.

If $h(t,\cdot)$ has second derivatives then normal and curvature of the interface $\Gamma(t)$ are given by

$$\begin{aligned}\hat{\nu}(t,x) &= \nu(t,x,h(t,x)) = \frac{1}{\sqrt{1+|\nabla h(t,x)|^2}}\begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix},\\ \hat{\kappa}(t,x) &= \kappa(t,x,h(t,x)) = \operatorname{div}_x\left(\frac{\nabla h(t,x)}{\sqrt{1+|\nabla h(t,x)|^2}}\right) = \Delta h - G_\kappa(h),\end{aligned} \tag{4}$$

where ∇h and Δh denote the gradient and Laplacian of h with respect to x and

$$G_\kappa(h) = \frac{|\nabla h|^2 \Delta h}{\left(1+\sqrt{1+|\nabla h|^2}\right)\sqrt{1+|\nabla h|^2}} + \frac{\nabla h^\top \nabla^2 h \nabla h}{\left(1+|\nabla h|^2\right)^{3/2}}. \tag{5}$$

Following [22], we now transform the problem to $\dot{\mathbb{R}}^{n+1} = \{(x,y) \in \mathbb{R}^{n+1} \,:\, y \neq 0\}$ with a flat interface at $y = 0$ by using the transformation

$$\hat{u}(t,x,y) = \begin{pmatrix} v(t,x,y) \\ w(t,x,y) \end{pmatrix} := u(t,x,h(t,x)+y), \pi(t,x,y) := q(t,x,h(t,x)+y). \tag{6}$$

Analogously, let with $\mathbb{R}^{n+1}_\pm = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,:\, \pm y > 0\}$

$$\hat{\rho}(t,x,y) = \rho(t,x,h(t,x)+y) = \chi_{\mathbb{R}^{n+1}_-}(x,y)\rho_1 + \chi_{\mathbb{R}^{n+1}_+}(x,y)\rho_2,$$
$$\hat{\mu}(t,x,y) = \mu(t,x,h(t,x)+y) = \chi_{\mathbb{R}^{n+1}_-}(x,y)\mu_1 + \chi_{\mathbb{R}^{n+1}_+}(x,y)\mu_2.$$

As in [22], we work with the following function spaces. Let $\Omega \subset \mathbb{R}^m$ be open and X be a Banach space. $L_p(\Omega;X)$, $H^s_p(\Omega;X)$, $1 \le p \le \infty$, $s \in \mathbb{R}$, denote the X-valued Lebesgue and Bessel potential spaces of order s, respectively. We note that $H^k_p(\Omega;X) = W^k_p(\Omega;X)$ for $k \in \mathbb{N}_0$, $1 < p < \infty$ with the Sobolev-Slobodetskiĭ spaces W^k_p. Moreover, we will use the fractional Sobolev-Slobodetskiĭ spaces $W^s_p(\Omega;X)$, $1 \le p < \infty$, $s \in (0,\infty) \setminus \mathbb{N}$, with norm

$$\|g\|_{W^s_p(\Omega;X)} = \|g\|_{W^{[s]}_p(\Omega;X)} + \sum_{|\alpha|=[s]} \left(\int_\Omega \int_\Omega \frac{\|\partial^\alpha g(x) - \partial^\alpha g(y)\|^p_X}{|x-y|^{m+(s-[s])p}} \, dx \, dy \right)^{1/p}$$

We recall that $W^s_p(\Omega;X) = B^s_{pp}(\Omega;X)$ for $s \in (0,\infty) \setminus \mathbb{N}$ with the Besov space B^s_{pp}. Finally, the homogeneous Sobolev space $\dot{H}^1_p(\Omega)$ is defined by

$$\dot{H}^1_p(\Omega) := \left(\left\{ g \in L_{1,loc}(\Omega) : \|\nabla g\|_{L_p(\Omega)} < \infty \right\}, \|\cdot\|_{\dot{H}^1_p(\Omega)} \right), \quad \|g\|_{\dot{H}^1_p(\Omega)} := \|\nabla g\|_{L_p(\Omega;\mathbb{R}^m)}.$$

Then $\dot{H}^1_p(\Omega)$ is for connected Ω a Banach space if we factor out the constant functions and equip the resulting space with the corresponding quotient norm.

Finally, for $\Omega \subset \mathbb{R}^m$ open or closed we denote by $BUC(\Omega;X)$ and $BC(\Omega;X)$ the space of bounded uniformly continuous and the space of bounded continuous functions equipped with the supremum norm, respectively. Analogously, $BUC^k(\Omega;X)$ and $BC^k(\Omega;X)$, $k \in \mathbb{N}_0$, are defined for k-times continuously differentiable functions with bounded uniformly continuous or bounded continuous derivatives up to order k. If Ω is compact, we may briefly write $C^k(\Omega;X)$, since boundedness und uniform continuity are automatically satisfied.

To state the transformed problem, we follow [22] and we use a fixed point formulation consisting of a linearized Stokes problem with nonlinear right hand side. In fact, define with $J = (0, t_0)$ and $p > n+3$ the space $\mathbb{E}(t_0)$ by

$$\begin{aligned}
\mathbb{E}_1(t_0) &= \left\{ \hat{u} \in H^1_p\big(J;L_p\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\big) \cap L_p\Big(J;H^2_p\big(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}\big)\Big) : [\hat{u}] = 0 \right\}, \\
\mathbb{E}_2(t_0) &= L_p\Big(J;\dot{H}^1_p\big(\dot{\mathbb{R}}^{n+1}\big)\Big), \\
\mathbb{E}_3(t_0) &= W^{1/2-1/(2p)}_p\big(J;L_p(\mathbb{R}^n)\big) \cap L_p\Big(J;W^{1-1/p}_p(\mathbb{R}^n)\Big), \\
\mathbb{E}_4(t_0) &= W^{2-1/(2p)}_p\big(J;L_p(\mathbb{R}^n)\big) \cap H^1_p\Big(J;W^{2-1/p}_p(\mathbb{R}^n)\Big) \\
&\quad \cap W^{1/2-1/(2p)}_p\Big(J;H^2_p(\mathbb{R}^n)\Big) \cap L_p\Big(J;W^{3-1/p}_p(\mathbb{R}^n)\Big), \\
\mathbb{E}(t_0) &= \left\{ (\hat{u}, \pi, r, h) \in \mathbb{E}_1(t_0) \times \mathbb{E}_2(t_0) \times \mathbb{E}_3(t_0) \times \mathbb{E}_4(t_0) : [\pi] = r \right\},
\end{aligned} \tag{7}$$

and denote by

$$L(\hat{u}, \pi, r, h) = (f, f_d, g_v, g_w, g_h), \ (\hat{u}(0), h(0)) = (\hat{u}_0, h_0), \ (\hat{u}, \pi, r, h) \in \mathbb{E}(t_0) \tag{8}$$

(i.e., $r = [\pi]$ by the definition of $\mathbb{E}(t_0)$) the Stokes problem with free boundary

$$
\begin{aligned}
\hat\rho\partial_t\hat u - \hat\mu\Delta\hat u + \nabla\pi &= f && \text{in } \dot{\mathbb{R}}^{n+1},\\
\operatorname{div}\hat u &= f_d && \text{in } \dot{\mathbb{R}}^{n+1},\\
-[\hat\mu\partial_y v] - [\hat\mu\nabla_x w] &= g_v && \text{on } \mathbb{R}^n,\\
-2[\hat\mu\partial_y w] + [\pi] - \sigma\Delta h &= g_w && \text{on } \mathbb{R}^n,\\
[\hat u] &= 0 && \text{on } \mathbb{R}^n,\\
\partial_t h - \gamma w &= g_h && \text{on } \mathbb{R}^n,\\
\hat u(0) = \hat u_0, \quad h(0) &= h_0
\end{aligned}
\tag{9}
$$

for $t>0$. Here, $[\hat u]$ denotes the jump across the transformed interface $y=0$ and $\gamma w(x) = w(x,0)$ denotes the trace of a function $w : \dot{\mathbb{R}}^{n+1} \to \mathbb{R}$ at $y=0$ satisfying $[w]=0$. Then it is shown in [22] that the transformation (6) leads to the following problem for $\hat u = (v,w), \pi, h$

$$
\begin{aligned}
&L(\hat u,\pi,[\pi],h) = \big(\hat c + F(\hat u,\pi,h), F_d(\hat u,h), G_v(\hat u,[\pi],h), G_w(\hat u,h), H(\hat u,h)\big),\\
&(\hat u(0),h(0)) = (\hat u_0,h_0),
\end{aligned}
\tag{10}
$$

where the right hand sides are given by $F=(F_v,F_w)$ and

$$
\begin{aligned}
F_v(v,w,\pi,h) &= \hat\mu\Big(-2(\nabla h\cdot\nabla_x)\partial_y v + |\nabla h|^2\partial_y^2 v - \Delta h\partial_y v\Big) + \partial_y\pi\nabla h\\
&\quad + \hat\rho\big(-(v\cdot\nabla_x)v + (\nabla h^\top v)\partial_y v - w\partial_y v\big) + \hat\rho\partial_t h\partial_y v,\\
F_w(v,w,h) &= \hat\mu\Big(-2(\nabla h\cdot\nabla_x)\partial_y w + |\nabla h|^2\partial_y^2 w - \Delta h\partial_y w\Big)\\
&\quad + \hat\rho\big(-(v\cdot\nabla_x)w + (\nabla h^\top v)\partial_y w - w\partial_y w\big) + \hat\rho\partial_t h\partial_y w,\\
F_d(v,h) &= \nabla h^\top\partial_y v,\\
G_v(v,w,[\pi],h) &= -\big[\hat\mu(\nabla_x v + (\nabla_x v)^\top)\big]\nabla h + |\nabla h|^2[\hat\mu\partial_y v] + \big(\nabla h^\top[\hat\mu\partial_y v]\big)\nabla h\\
&\quad - [\hat\mu\partial_y w]\nabla h + \big([\pi] - \sigma(\Delta h - G_\kappa(h))\big)\nabla h,\\
G_w(v,w,h) &= -\nabla h^\top[\hat\mu\partial_y v] - \nabla h^\top[\hat\mu\nabla_x w] + |\nabla h|^2[\hat\mu\partial_y w] - \sigma G_\kappa(h),\\
H(v,w,h) &= -(\gamma v)^\top\nabla h.
\end{aligned}
\tag{11}
$$

Note that all terms except $G_\kappa(h)$ are polynomials in $(v,w,\pi,[\pi],h)$ and derivatives of (v,w,π,h). Moreover, all terms are linear with respect to second derivatives and $G_\kappa(h)$ is the pointwise superposition of a smooth function with ∇h and $\nabla^2 h$.

Remark 1 The transformed version of the deformation tensor $D(u) = \nabla u + \nabla u^\top$ is given by $\mathcal{D}(\hat u,h) = \mathcal{D}(v,w,h)$, where

$$
\mathcal{D}(\hat u,h) = \nabla\hat u + \nabla\hat u^\top - \begin{pmatrix}\nabla h\partial_y\hat u^\top\\ 0\end{pmatrix} - \begin{pmatrix}\nabla h\partial_y\hat u^\top\\ 0\end{pmatrix}^\top.
$$

Then the compatibility condition (14) can with $\hat{\nu}(0,x) = \frac{1}{\sqrt{1+|\nabla h_0(x)|^2}} \begin{pmatrix} -\nabla h_0(x) \\ 1 \end{pmatrix}$ equivalently be written as

$$\begin{aligned} &[\hat{\mu}\mathcal{D}(\hat{u}_0, h_0)\hat{\nu}(0) - \hat{\mu}(\hat{\nu}(0)^\top \mathcal{D}(\hat{u}_0, h_0)\hat{\nu}(0))\hat{\nu}(0)] = 0, \\ &\operatorname{div} \hat{u}_0 = F_d(\hat{u}_0, h_0), \quad [\hat{u}_0] = 0. \end{aligned} \tag{12}$$

3 Well-posedness and differentiability with respect to controls

3.1 Results for the transformed problem

By applying a fixed point theorem to (10), the following result is shown in [22] for $\hat{c} = 0$.

Theorem 2 *Let $p > n + 3$ and consider the case $c = 0$, i.e. $\hat{c} = 0$. Let*

$$\mathbb{U}_{\hat{u}} := W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}), \quad \mathbb{U}_h := W_p^{3-2/p}(\mathbb{R}^n) \tag{13}$$

and let with $J = (0, t_0)$ the space $\mathbb{E}(t_0)$ be defined by (7).

Then for any $t_0 > 0$ there exists $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(t_0) > 0$ such that for all initial values

$$(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h$$

satisfying, with $u_0(x, h_0(x) + y) = \hat{u}_0(x, y)$, the compatibility conditions

$$[\mu D(u_0)\nu(0) - \mu(\nu(0)^\top D(u_0)\nu(0))\nu(0)] = 0, \quad \operatorname{div} u_0 = 0, \quad [u_0] = 0, \tag{14}$$

as well as the smallness condition

$$\|\hat{u}_0\|_{\mathbb{U}_{\hat{u}}} + \|h_0\|_{\mathbb{U}_h} \le \hat{\varepsilon}_0$$

there exists a unique solution of the transformed problem (10) with

$$(\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0).$$

Moreover, $(\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$ depends continuously on $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h$ satisfying (14).

Proof See [22, Thm. 6.3]. □

Our first aim is to study the differentiability properties of the control-to-state map $(\hat{u}_0, \hat{c}) \mapsto (\hat{u}, \pi, [\pi], h)$. Note that we consider also the case $\hat{c} \neq 0$. The proof is carried out by an appropriate extension of the fixed point argument for (10) based on Theorem 7.

To apply the fixed point argument, the following L_p-maximum regularity result of [22] for the linearized problem (9) will be essential.

Theorem 3 *Let* $1 < p < \infty$ *be fixed,* $p \neq 3/2, 3$ *and assume that* ρ_i, μ_i *are positive constants. For arbitrary* $t_0 > 0$ *let* $J = (0, t_0)$ *and let* $\mathbb{E}_1(t_0), \dots, \mathbb{E}_4(t_0), \mathbb{U}_{\hat{u}}, \mathbb{U}_h$ *be defined by* (7), (26). *Set*

$$\begin{aligned}
\mathbb{F}_1(t_0) &= L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})),\\
\mathbb{F}_2(t_0) &= H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{F}_3(t_0) &= W^{1/2-1/(2p)}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1})),\\
\mathbb{F}_4(t_0) &= W^{1-1/(2p)}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),\\
\mathbb{F}(t_0) &= \mathbb{F}_1(t_0) \times \mathbb{F}_2(t_0) \times \mathbb{F}_3(t_0) \times \mathbb{F}_4(t_0).
\end{aligned} \tag{15}$$

Then for all initial values $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h$ *and* $(f, f_d, g, g_h) \in \mathbb{F}(t_0)$ *satisfying the compatibility conditions*

$$\operatorname{div} \hat{u}_0 = f_d(0) \quad \text{on } \dot{\mathbb{R}}^{n+1}, \qquad [\hat{u}_0] = 0 \quad \text{on } \mathbb{R}^n \quad \text{if } p > 3/2, \tag{16}$$

$$\text{and in addition} \qquad [-\hat{\mu}\partial_y v_0] - [\hat{\mu}\nabla_x w_0] = g_v(0) \quad \text{on } \mathbb{R}^n \quad \text{if } p > 3, \tag{17}$$

there exists a unique solution $(\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$ *of* (9) *and the solution map*

$$(f, f_d, g, g_h, \hat{u}_0, h_0) \in \mathbb{F}(t_0) \times \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto (\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$$

is continuous.

Proof This follows from [22, Thm. 5.1] and [22, Lem. 6.1, (e)]. □

For homogeneous initial data we obtain immediately

Corollary 4 *Let* $p > 3$ *and define in addition to* $\mathbb{E}(t_0)$ *and* $\mathbb{F}(t_0)$ *the spaces*

$$\begin{aligned}
{}_0\mathbb{E}(t_0) &:= \{(\hat{u}, \pi, r, h) \in \mathbb{E}(t_0) : \ \hat{u}(0) = 0, \ \ r(0) = 0, \ \ h(0) = 0\},\\
{}_0\mathbb{F}(t_0) &:= \{(f, f_d, g, g_h) \in \mathbb{F}(t_0) : \ f_d(0) = 0, \ \ g(0) = 0, \ \ g_h(0) = 0\}
\end{aligned}$$

with initial value 0 *for all components that admit a trace at* $t = 0$. *Then* (9) *has a unique and continuous solution map*

$$(f, f_d, g, g_h, 0, 0) \in {}_0\mathbb{F}(t_0) \times \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto (\hat{u}, \pi, [\pi], h) \in {}_0\mathbb{E}(t_0).$$

The fixed point argument relies on the following properties of the right hand sides (11) of (10).

Lemma 5 *Let* $p > n + 3$ *and set for* $(\hat{u}, \pi, r, h) \in \mathbb{E}(t_0)$

$$N(\hat{u}, \pi, r, h) := \big(F(\hat{u}, \pi, h), F_d(\hat{u}, h), G(\hat{u}, r, h), H(\hat{u}, h)\big), \tag{18}$$

with $F = (F_v, F_w)$, $G = (G_v, G_w)$, F_d *and* H *defined in* (11). *Then the mapping* $N : \mathbb{E}(t_0) \to \mathbb{F}(t_0)$ *is well defined and real analytic, more precisely,*

$$N \in C^\omega\big(\mathbb{E}(t_0), \mathbb{F}(t_0)\big), \quad N(0) = 0, \quad DN(0) = 0.$$

Moreover,

$$DN(\hat{u}, \pi, r, h) \in \mathcal{L}\big({}_0\mathbb{E}(t_0), {}_0\mathbb{F}(t_0)\big) \quad \forall\, (\hat{u}, \pi, r, h) \in \mathbb{E}(t_0).$$

Proof See [22, Prop. 6.2] □

Moreover, we will need the following analogue for the spaces of the initial values.

Lemma 6 *Let* $p > n + 3$, $\mathbb{U}_{\hat{u}}, \mathbb{U}_h$ *be as in* (13) *and set*

$$\mathbb{U}_{\hat{u},c} := \{\hat{u}_0 = (v_0, w_0) \in \mathbb{U}_{\hat{u}} : [\hat{u}_0] = 0\}.$$

Then with $G = (G_v, G_w)$ *and* H *defined in* (11) *the mappings*

$$(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto v_0^\top \nabla h_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}), \tag{19}$$

$$(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u},c} \times \mathbb{U}_h \mapsto H(v_0, h_0) \in W_p^{2-3/p}(\mathbb{R}^n), \tag{20}$$

$$(\hat{u}_0, r_0, h_0) \in \mathbb{U}_{\hat{u}} \times W_p^{1-2/p}(\mathbb{R}^n) \times \mathbb{U}_h \mapsto G(\hat{u}_0, r_0, h_0) \in W_p^{1-2/p}(\mathbb{R}^n) \tag{21}$$

are real analytic and the first derivatives vanish in $(\hat{u}_0, r_0, h_0) = 0$.

Proof Since $p > n + 3$ we have $W_p^{1-2/p}(\dot{\mathbb{R}}^{n+1}) \hookrightarrow BUC(\dot{\mathbb{R}}^{n+1})$ and thus $W_p^s(\dot{\mathbb{R}}^{n+1})$ is a multiplication algebra, i.e. a Banach algebra under the operation of multiplication, for all $s \geq 1 - 2/p$, see e.g. [6, Lem. 4.1, Rem. 6.4]. As a consequence, (19) is a continuous bilinear form and thus in $C^\omega(\mathbb{U}_{\hat{u}} \times \mathbb{U}_h, W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}))$.

Similarly, $W_p^s(\mathbb{R}^n)$ is a multiplication algebra for all $s \geq 1 - 2/p$. Since the trace operator $\hat{u}_0 \in \mathbb{U}_{\hat{u},c} \mapsto \gamma v_0 \in W_p^{2-3/p}(\mathbb{R}^n)$ is continuous, (20) is a continuous bilinear form and thus real analytic.

Finally $G(\hat{u}_0, r_0, h_0)$ in (21) is a polynomial in $W_p^{1-2/p}(\mathbb{R}^n)$-functions and in functions of the form $\nabla h_0/(a + (1 + \nabla h_0^\top \nabla h_0)^{k/2})$ with $a \geq 0$ and $k \in \{1, 2\}$. The function $\Psi : v \in \mathbb{R}^n \mapsto v/(a + (1 + v^\top v)^{k/2})$ is smooth with bounded derivatives and $\Psi(0) = 0$. Since $2 - 2/p > n/p$ implies $h_0 \in \mathbb{U}_h \mapsto \nabla h_0 \in W_p^{2-2/p}(\mathbb{R}^n) \hookrightarrow W^1_{(2-2/p)p}(\mathbb{R}^n) \cap BUC(\mathbb{R}^n)$, it is well known that

$$h_0 \in \mathbb{U}_h \mapsto \Psi(\nabla h_0) \in W_p^{2-2/p}(\mathbb{R}^n) \tag{22}$$

is well defined and continuous, see [6, Thm. 1.1]. It is also differentiable. In fact, for $d \in \mathbb{U}_h$

$$\Psi(\nabla h_0 + \nabla d) - \Psi(\nabla h_0) - \Psi'(\nabla h_0)\nabla d = \int_0^1 \big(\Psi'(\nabla h_0 + \tau \nabla d) - \Psi'(\nabla h_0)\big)\nabla d \, d\tau,$$

where the integrand is in $BUC([0,1] \times \mathbb{R}^n)$. Moreover, since $v \mapsto \Psi'(\nabla v) - \Psi'(0)$ is smooth with bounded derivatives and vanishes at 0, the integrand is continuous from $[0,1] \to W_p^{2-2/p}(\mathbb{R}^n)$ again by [6, Thm. 1.1]. Hence the integral is also a Bochner integral and thus by using the multiplication algebra property there is $C > 0$ with

$$\begin{aligned}&\|\Psi(\nabla h_0 + \nabla d) - \Psi(\nabla h_0) - \Psi'(\nabla h_0)\nabla d\|_{W_p^{2-2/p}(\mathbb{R}^n)} \\ &\le C \int_0^1 \|\Psi'(\nabla h_0 + \tau\nabla d) - \Psi'(\nabla h_0)\|_{W_p^{2-2/p}(\mathbb{R}^n)} \, d\tau \|\nabla d\|_{W_p^{2-2/p}(\mathbb{R}^n)} = o(\|d\|_{\mathbb{U}_h}),\end{aligned}$$

since $d \in \mathbb{U}_h \mapsto \Psi'(\nabla h_0 + \nabla d) - \Psi'(0) \in W_p^{2-2/p}(\mathbb{R}^n;\mathbb{R}^n)$ is continuous at $d = 0$ by [6, Thm. 1.1]. Now we can iteratively show that (22) is real analytic. In fact, we can write $d \in \mathbb{U}_h \mapsto \Psi'(\nabla h_0)\nabla d = (\Psi'(\nabla h_0) - \Psi'(0))\nabla d + \Psi'(0)\nabla d \in W_p^{2-2/p}(\mathbb{R}^n)$. The second term is a constant mapping in $\mathcal{L}(\mathbb{U}_h, W_p^{2-2/p}(\mathbb{R}^n))$. Moreover, as before $h_0 \in \mathbb{U}_h \mapsto \Psi'(\nabla h_0) - \Psi'(0) \in W_p^{2-2/p}(\mathbb{R}^n;\mathbb{R}^n)$ is well defined and continuous [6, Thm. 1.1] and by the same arguments as above also differentiable. Iterating the argument shows that (22) is real analytic.

We conlude that (21) is a polynomial in $W_p^{1-2/p}(\mathbb{R}^n)$-functions and in real analytic functions of the form (22). Since $W_p^{1-2/p}(\mathbb{R}^n)$ is a multiplication algebra, we conclude that (21) is real analytic.

By the product structure of (19)–(21) the first derivatives vanish in 0. □

We will work with the following extension of Banach's fixed point theorem.

Theorem 7

a) *Let U, W, Z be real Banach spaces, let $A \in \mathcal{L}(Z, W)$ be an isomorphism and set $M := \|A^{-1}\|_{\mathcal{L}(W,Z)}$. Let $B_Z \subset Z$ be a nonempty closed convex set and $B_U \subset U$ be a nonempty set. Moreover, let $K : B_Z \times B_U \to W$ be Lipschitz continuous with*

$$\|K(z,u) - K(\tilde{z},\tilde{u})\|_W \le L_z\|z - \tilde{z}\|_Z + L_u\|u - \tilde{u}\|_U \quad \forall (z,u), (\tilde{z},\tilde{u}) \in B_Z \times B_U$$

and assume that

$$A^{-1}K(z,u) \in B_Z \quad \forall (z,u) \in B_Z \times B_U \quad \text{and} \quad ML_z < 1. \tag{23}$$

Then for all $u \in B_U$ *the equation*

$$Az = K(z, u)$$

has a unique solution $z = z(u) \in B_Z$ *and*

$$\|z(u) - z(\tilde{u})\|_Z \le \frac{L_u M}{1 - ML_z} \|u - \tilde{u}\|_U \quad \forall\, u, \tilde{u} \in B_U. \tag{24}$$

b) *Assume in addition that* B_U *is a relatively open convex subset of* $u^* + U_L \subset U$, *where* U_L *is a closed linear subspace of* U ($U_L = U$ *is admitted, then* $B_U \subset U$ *is convex and open*), *and that* $K : B_Z \times B_U \to W$ *is Fréchet differentiable. Then* $u \in B_U \mapsto z(u) \in Z$ *is Fréchet differentiable, where* $\delta z_d := Dz(u)d$ *is for any* $d \in U_L$ *the unique solution of the problem*

$$A\delta z_d = D_z K(z(u), u)\delta z_d + D_u K(z(u), u)d. \tag{25}$$

If $DK : B_Z \times B_U \to \mathcal{L}(Z \times U_L, W)$ *is Lipschitz continuous then also* $Dz : B_U \to \mathcal{L}(U_L, Z)$ *is Lipschitz continuous. If* $K : B_Z \times B_U \to W$ *is k-times Fréchet differentiable then* $u \in B_U \mapsto z(u) \in Z$ *is k-times Fréchet differentiable and if* $D^k K$ *is Lipschitz continuous on* $B_Z \times B_U$ *then* $D^k z$ *is Lipschitz continuous on* B_U.

Proof a): By assumption the mapping $T : (z, u) \in B_Z \times B_U \mapsto A^{-1}K(z, u) \in B_Z$ is well defined and Lipschitz continuous with Lipschitz constants $ML_z < 1$ with respect to z and ML_u with respect to u. Hence, for all $u \in B_U$ there exists a unique fixed point $z = z(u)$ with $z = T(z, u)$ by Banach's fixed point theorem.

For $u, \tilde{u} \in B_U$ we obtain

$$\begin{aligned} \|z(u) - z(\tilde{u})\|_Z &= \|T(z(u), u) - T(z(\tilde{u}), \tilde{u})\|_W \\ &\le ML_z \|z(u) - z(\tilde{u})\|_Z + ML_u \|u - \tilde{u}\|_U \end{aligned}$$

and thus (24).

b): Now let in addition B_U be relatively open in the closed affine subspace $u^* + U_L$. Moreover, let $K : B_Z \times B_U \to W$ be Fréchet differentiable and let $u \in B_U$ be arbitrary. Then $\|D_z K(z(u), u)\|_{\mathcal{L}(Z,W)} \le L_z$ and $\|D_u K(z(u), u)\|_{\mathcal{L}(U_L,W)} \le L_u$ and thus for any $d \in U_L$ the linear problem (25) has by Banach's fixed point theorem a unique solution $\delta z_d \in Z$.

Since B_U is relatively open in $u^* + U_L$, we find $\delta > 0$ such that $u + d \in B_U$ for all $d \in U_L$ with $\|d\|_U < \delta$. Then

$$
\begin{aligned}
&A(z(u+d)-z(u)-\delta z_d)\\
&=K(z(u+d),u+d)-K(z(u),u)-D_zK(z(u),u)\delta z_d-D_uK(z(u),u)d\\
&=D_zK(z(u),u)(z(u+d)-z(u)-\delta z_d)+o_W(\|z(u+d)-z(u)\|_Z+\|d\|_U).
\end{aligned}
$$

Here, the Landau symbol o_W indicates, that the term is considered in the space W. By using (24) we conclude that for $d\in U_L, \|d\|_U\to 0$

$$
\|z(u+d)-z(u)-\delta z_d\|_Z\le \frac{ML_u}{1-ML_z}o_Z(\|d\|_U)=o_Z(\|d\|_U).
$$

If $DK : B_Z\times B_U\to \mathcal{L}(Z\times U_L, W)$ is Lipschitz continuous then (25) can be written as

$$
A\delta z_d = K^{(1)}(\delta z_d, u;d),
$$

where $K^{(1)}(\cdot,\cdot;d) : Z\times B_U\to W$ has for all $d\in U_L, \|d\|_U\le 1$ the Lipschitz constant L_z with respect to δz_d and a uniform Lipschitz constant with respect to u. Applying the first part of the theorem again yields that $Dz : B_U\to\mathcal{L}(U_L, Z)$ is Lipschitz continuous.

Repeating the argument for higher derivatives concludes the proof. □

By applying this result to (8)–(10), we obtain the following extension of Theorem 2.

Theorem 8 *Let $p>n+3$ and consider any $t_0>0$. Let $\mathbb{E}(t_0), \mathbb{F}(t_0)$ be defined as in (7) and (15) and set with $J=(0,t_0)$*

$$
\begin{aligned}
&\mathbb{U}_{\hat u} := W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+1}),\quad \mathbb{U}_h := W_p^{3-2/p}(\mathbb{R}^n),\\
&\mathbb{U}_{\hat c}(t_0) := \mathbb{F}_1(t_0)=L_p(J;L_p(\mathbb{R}^{n+1},\mathbb{R}^{n+1})).
\end{aligned}
\tag{26}
$$

Then for any $t_0>0$ there exists $\hat\varepsilon_0=\hat\varepsilon_0(t_0)>0$ such that for all data

$$
(\hat u_0, h_0, \hat c)\in \mathbb{U}_{\hat u}\times\mathbb{U}_h\times\mathbb{U}_{\hat c}(t_0)
$$

satisfying the compatibility condition (12) (or equivalently (14) with $u_0(x,h_0(x)+y)=\hat u_0(x,y)$) as well as the smallness condition

$$
\|\hat u_0\|_{\mathbb{U}_{\hat u}}+\|h_0\|_{\mathbb{U}_h}+\|\hat c\|_{\mathbb{U}_{\hat c}(t_0)}<\hat\varepsilon_0 \tag{27}
$$

there exists a unique solution of the transformed problem (10) with

$$
(\hat u,\pi,[\pi],h)\in\mathbb{E}(t_0).
$$

Moreover, the mapping

$$
\{(\hat u_0,h_0,\hat c)\in \mathbb{U}_{\hat u}\times\mathbb{U}_h\times\mathbb{U}_{\hat c}(t_0) : (\hat u_0,h_0,\hat c)\text{ satisfy (12), (27) }\}\mapsto(\hat u,\pi,[\pi],h)\in\mathbb{E}(t_0)
$$

is continuous and infinitely many times differentiable with respect to $(\hat u_0,\hat c)$.

Proof We extend the arguments in [22] and apply Theorem 7 to the transformed formulation (10).

Let $z = (\hat{u}, \pi, r, h) \in \mathbb{E}(t_0)$ and write (10)

$$Lz = N(z) + (\hat{c}, 0), \quad (\hat{u}(0), h(0)) = (\hat{u}_0, h_0), \tag{28}$$

where N is defined in (18).

Let $(\hat{u}_0, h_0)$ satisfy (12) and (27), where $\hat{\varepsilon}_0$ will be adjusted later.

Following [22], we first construct $z^* = z^*(\hat{u}_0, h_0) \in \mathbb{E}(t_0)$ that satisfies the equation

$$Lz^* = \left(0, f_d^*, g^*, g_h^*\right), \quad (\hat{u}^*(0), h^*(0)) = (\hat{u}_0, h_0), \tag{29}$$

where $(0, f_d^*, g^*, g_h^*) \in \mathbb{F}(t_0)$ resolves the compatibility conditions (16), (17). Then we can write (28) equivalently as

$$L\tilde{z} = N\left(\tilde{z} + z^*(\hat{u}_0, h_0)\right) + (\hat{c}, 0) - Lz^*(\hat{u}_0, h_0) =: K(\tilde{z}; \hat{u}_0, h_0, \hat{c}), \ \ \tilde{z} \in {}_0\mathbb{E}(t_0). \tag{30}$$

The construction of z^* can be accomplished as in [22]. Set

$$r_0(\hat{u}_0, h_0) = [\pi_0] := \left[\hat{\mu}\hat{v}(0)^\top \mathcal{D}(\hat{u}_0, h_0)\hat{v}(0)\right] + \sigma(\Delta h_0 - G_\kappa(h_0)).$$

The right hand side consists of several terms of $G(\hat{u}_0, 0, h_0)$ in (21) and thus Lemma 6 yields that the above mapping $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto [\pi_0] = r_0(\hat{u}_0, h_0) \in W_p^{1-2/p}(\mathbb{R}^n)$ is real analytic. Moreover, it is easy to check that the compatibility conditions hold

$$\begin{aligned} -[\hat{\mu}\partial_y v_0] - [\hat{\mu}\nabla_x w_0] &= G_v(\hat{u}_0, [\pi_0], h_0) \quad && \text{on } \mathbb{R}^n, \\ -2[\hat{\mu}\partial_y w_0] + [\pi_0] - \sigma\Delta h_0 &= G_w(\hat{u}_0, h_0) && \text{on } \mathbb{R}^n. \end{aligned} \tag{31}$$

Now let $D_n = -\Delta$ be the Laplacian in $L_p(\mathbb{R}^n)$ with domain $H_p^2(\mathbb{R}^n)$ and set

$$g^*(t) := e^{-tD_n} G(\hat{u}_0, r_0(\hat{u}_0, h_0), h_0), \quad g_h^*(t) := e^{-tD_n} H(\hat{u}_0, h_0).$$

By the real analyticity of $r_0(\hat{u}_0, h_0)$ and Lemma 6 the mappings

$$\begin{aligned} &(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto G(\hat{u}_0, r_0(\hat{u}_0, h_0), h_0) \in W_p^{1-2/p}(\mathbb{R}^n), \\ &(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto H(\hat{u}_0, h_0) \in W_p^{2-3/p}(\mathbb{R}^n) \end{aligned}$$

are real analytic. Now maximal L_p-regularity for D_n yields, see e.g. [11, Lem. 8.2]

$$\begin{aligned} g^* &\in H_p^1\left(J; W_p^{-1-1/p}(\mathbb{R}^n)\right) \cap L_p\left(J; W_p^{1-1/p}(\mathbb{R}^n)\right) \hookrightarrow \mathbb{F}_3(t_0), \\ g_h^* &\in H_p^1\left(J; W_p^{-1/p}(\mathbb{R}^n)\right) \cap L_p\left(J; W_p^{2-1/p}(\mathbb{R}^n)\right) \hookrightarrow \mathbb{F}_4(t_0), \end{aligned}$$

where the imbeddings follow by real interpolation and g^*, g_h^* are real analytic in $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h$. (31) ensures that (17) holds for g^*. Next, let

$$c_d^*(t) = \begin{cases} \mathcal{R}_+ e^{-tD_{n+1}} \mathcal{E}_+ v_0^\top \nabla h_0 & \text{in } \mathbb{R}_+^{n+1}, \\ \mathcal{R}_- e^{-tD_{n+1}} \mathcal{E}_- v_0^\top \nabla h_0 & \text{in } \mathbb{R}_-^{n+1}, \end{cases}$$

where $\mathcal{E}_\pm \in \mathcal{L}(W_p^{2-2/p}(\mathbb{R}_\pm^{n+1}), W_p^{2-2/p}(\mathbb{R}^{n+1})))$ are extension operators and $\mathcal{R}_\pm$ are the restrictions to $\mathbb{R}_\pm^{n+1}$. Now $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto v_0^\top \nabla h_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})$ is by Lemma 6 real analytic. By L_p-regularity for D_{n+1} $c_d^* \in H_p^1(J; L_p(\mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}))$ and thus

$$f_d^* := \partial_y c_d^* \in \mathbb{F}_2(t_0) \quad \text{with} \quad f_d^*(0) = F_d(v_0, h_0)$$

is real analytic with respect to $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h$. Hence, also (16) holds for f_d^* and we conclude that $R^* := (0, f_d^*, g^*, g_h^*) \in \mathbb{F}(t_0)$ satisfies the compatibility conditions (16), (17) and by construction $(\hat{u}_0, h_0) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \mapsto R^* \in \mathbb{F}(t_0)$ is real analytic. Hence, by Theorem 3 the linear problem (29) has a unique solution $z^* = z^*(\hat{u}_0, h_0)$ that is real analytic and by Lemma 6 the first derivative vanishes in 0, i.e., $Dz^*(0,0) = 0$.

Now consider (30). By construction of z^* the right hand side of (30) is in ${}_0\mathbb{F}(t_0)$. Denote by $L_0 \in \mathcal{L}({}_0\mathbb{E}(t_0), {}_0\mathbb{F}(t_0))$ the restriction of L which is an isomorphism by Corollary 4. Hence, (30) can be written as

$$L_0 \tilde{z} = N(\tilde{z} + z^*(\hat{u}_0, h_0)) + (\hat{c}, 0) - Lz^*(\hat{u}_0, h_0) =: K(\tilde{z}; \hat{u}_0, h_0, \hat{c}), \;\; \tilde{z} \in {}_0\mathbb{E}(t_0). \tag{32}$$

To apply Theorem 7 we set now with suitable $\hat{\varepsilon}_0 > 0$ and $\delta > 0$

$$\begin{aligned} B_U(\hat{\varepsilon}_0) &:= \left\{ (\hat{u}_0, h_0, \hat{c}) \in \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \times \mathbb{U}_{\hat{c}}(t_0) : \; (\hat{u}_0, h_0, \hat{c}) \text{ satisfy (12), (27)} \right\}, \\ B_Z(\delta) &:= \left\{ \tilde{z} \in {}_0\mathbb{E}(t_0) : \; \|\tilde{z}\|_{{}_0\mathbb{E}(t_0)} \le \delta \right\}, \end{aligned}$$

where $\hat{\varepsilon}_0, \delta > 0$ will be adjusted later.

Let $M = \|L_0^{-1}\|_{\mathcal{L}({}_0\mathbb{F}(t_0), {}_0\mathbb{E}(t_0))}$. We know by Lemma 5 and the properties of z^* that the right hand side

$$(\tilde{z}, \hat{u}_0, h_0, \hat{c}) \in {}_0\mathbb{E}(t_0) \times \mathbb{U}_{\hat{u}} \times \mathbb{U}_h \times \mathbb{U}_{\hat{c}}(t_0) \mapsto K(\tilde{z}; \hat{u}_0, h_0, \hat{c}) \in \mathbb{F}(t_0) \tag{33}$$

is real analytic with

$$K(0) = 0, \quad D_{(\tilde{z}, \hat{u}_0, h_0)} K(0) = 0.$$

Hence, the Lipschitz constant L_z of K with respect to $\tilde{z}$ is arbitrary small close to 0 and the Lipschitz constant of K with respect to $(\hat{u}_0, h_0, \hat{c})$ is $L_u = 2$ close enough to 0 (note that the Lipschitz constant with respect to $\hat{c}$ is 1). Hence, if we set $\delta = 4M\hat{\varepsilon}_0$ then for $\hat{\varepsilon}_0$ small enough K has the Lipschitz constants $L_z = 1/(2M)$ and $L_u = 2$ on $B_Z(\delta) \times B_U(\hat{\varepsilon}_0)$. Hence, for all $(\tilde{z}, \hat{u}_0, h_0, \hat{c}) \in B_Z(\delta) \times B_U(\hat{\varepsilon}_0)$

$$\begin{aligned} &\|L_0^{-1}K(\tilde z;\hat u_0,h_0,\hat c)\|_{{}_0\mathbb{E}(t_0)} \\ &\le ML_z\|\tilde z\|_{{}_0\mathbb{E}(t_0)} + ML_u(\|\hat u_0\|_{\mathbb{U}_{\hat u}} + \|h_0\|_{\mathbb{U}_h} + \|\hat c\|_{\mathbb{U}_{\hat c}(t_0)}) < \frac{1}{2}\delta + 2M\hat\varepsilon_0 = \delta. \end{aligned} \tag{34}$$

Thus, (23) is satisfied and (32) has by Theorem 7 for all $(\hat u_0, h_0, \hat c) \in B_U(\hat\varepsilon_0)$ a unique solution $\tilde z = \tilde z(\hat u_0, h_0, \hat c) \in B_Z(\delta)$ satisfying the Lipschitz stability (24). Since also the real analytic operator $z^*(\hat u_0, h_0) \in \mathbb{E}(t_0)$ is Lipschitz continuous on $B_U(\hat\varepsilon_0)$, the solution $z(\hat u_0, h_0, \hat c) = \tilde z + z^* \in \mathbb{E}(t_0)$ is unique and Lipschitz continuous on $B_U(\hat\varepsilon_0)$.

Now let $(\hat u_0^*, h_0^*, \hat c^*) \in B_U(\hat\varepsilon_0)$ be arbitrary. Then $\{(\hat u_0, h_0^*, \hat c) \in B_U(\hat\varepsilon_0)\}$ is a relatively open subset of an affine subspace of $\mathbb{U}_{\hat u} \times \mathbb{U}_h \times \mathbb{U}_{\hat c}(t_0)$. Since (33) is real analytic, it follows from Theorem 7, b) that $\tilde z(\hat u_0, h_0^*, \hat c) \in {}_0\mathbb{E}(t_0)$ is infinitely many times differentiable with respect to $(\hat u_0, \hat c)$ and the same holds for $z(\hat u_0, h_0^*, \hat c) = \tilde z + z^* \in \mathbb{E}(t_0)$. □

3.2 Results for the original problem

We transfer now the results of Theorem 8 for the transformed problem (10) to the original problem (1). To this end, we define for $h_0 \in \mathbb{U}_h$ the spaces

$$\mathbb{U}_u(h_0) := W_p^{2-2/p}\big(\mathbb{R}^{n+1} \setminus \Gamma(0), \mathbb{R}^{n+1}\big), \quad \mathbb{U}_c(t_0) := L_p\Big(J;H_p^1\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\Big). \tag{35}$$

Since the pressure $\pi(t,\cdot)$ is only determined up to a constant, we select from now on without restriction the unique representative satisfying (note that $[\pi]$ is uniquely determined in (10))

$$\pi \in \mathbb{E}_1(t_0), \qquad \int_{[-1,1]^n} \pi(t,x,0-)\,dx = 0 \quad \text{for a.a. } t \in (0,t_0). \tag{36}$$

Then with the convention (36) and by the trace theorem, we find a Poincaré constant $C_P > 0$ with

$$\|\pi(t,\cdot)\|_{H_p^1(\dot{\mathbb{R}}^{n+1})} \le C_P\Big(\|\pi(t,\cdot)\|_{\dot H_p^1(\dot{\mathbb{R}}^{n+1})} + \|[\pi]\|_{W_p^{1-1/p}(\mathbb{R}^n)}\Big). \tag{37}$$

The following imbeddings will be useful.

Lemma 9 *Let $p > n+3$. Then the following imbeddings hold with $J = (0,t_0)$, $t_0 > 0$.*

$$\mathbb{E}_1(t_0) \hookrightarrow C\big(\bar J;BUC^1\big(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}\big)\big) \cap C\big(\bar J;BUC\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\big), \tag{38}$$

$$\mathbb{E}_1(t_0) \hookrightarrow H_p^1\big(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big) \cap C\Big(\bar J;H_p^1\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\Big), \tag{39}$$

$$\mathbb{E}_1(t_0) \hookrightarrow H^1_p\big(J\times\mathbb{R}^{n+1},\mathbb{R}^{n+1}\big)\cap C\Big(\bar J;H^1_{\tilde p}\big(\mathbb{R}^{n+1},\mathbb{R}^{n+1}\big)\Big)\quad \forall\,\tilde p\in[p,\infty), \tag{40}$$

$$\mathbb{E}_4(t_0) \hookrightarrow C^1\big(\bar J;BC^1(\mathbb{R}^n)\big)\cap C\big(\bar J;BC^2(\mathbb{R}^n)\big). \tag{41}$$

Proof For the imbeddings (38), (41) see [22, Lem. 6.1]. Moreover, it is obvious that

$$\mathbb{E}_1(t_0) \hookrightarrow H^1_p(J\times\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+1})$$

and also $\mathbb{E}_1(t_0)\hookrightarrow C(\bar J;W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+1}))$ holds, see [2, Theorem III.4.10.2]. Since the functions $\hat u\in\mathbb{E}_1(t_0)$ are continuous by (38) and thus $[\hat u]=0$, this implies the imbedding (39). Now (40) follows from interpolation between (38) and (39). ☐

Theorem 10 *Let* $(\hat u,\pi,[\pi],h)\in\mathbb{E}(t_0)$, $h_0\in\mathbb{U}_h$, $u_0\in\mathbb{U}_u(h_0)$, *and consider, see* (6),

$$\begin{aligned}&u(t,x,y)=\hat u(t,x,y-h(t,x)),\ \ q(t,x,y)=\pi(t,x,y-h(t,x)),\\&u_0(x,y)=\hat u_0\big(x,y-h_0(x)\big).\end{aligned}\tag{42}$$

Then there exist constants $C(\|h\|_{\mathbb{E}_4(t_0)})>0$ *and* $C(\|h_0\|_{\mathbb{U}_h})$ *such that*

$$\begin{aligned}&\|u\|_{W^1_p(J\times\mathbb{R}^{n+1},\mathbb{R}^{n+1})}+\left(\int_J\|u(t)\|^p_{H^2_p(\mathbb{R}^{n+1}\setminus\Gamma(t),\mathbb{R}^{n+1})}\,dt\right)^{1/p}\le C(\|h\|_{\mathbb{E}_4(t_0)})\|\hat u\|_{\mathbb{E}_1(t_0)},\\&\left(\int_J\|q(t)\|^p_{\dot H^1_p(\mathbb{R}^{n+1}\setminus\Gamma(t),\mathbb{R}^{n+1})}\,dt\right)^{1/p}\le C(\|h\|_{\mathbb{E}_4(t_0)})\|\pi\|_{\mathbb{E}_2(t_0)},\\&\left(\int_J\|[q(t)]\|^p_{W^{1-1/p}_p(\Gamma(t))}\,dt\right)^{1/p}\le C(\|h\|_{\mathbb{E}_4(t_0)})\|[\pi]\|_{L_p(J;W^{1-1/p}_p(\mathbb{R}^n))},\\&\|\hat u_0\|_{\mathbb{U}_{\hat u}}\le C(\|h_0\|_{\mathbb{U}_h})\|u_0\|_{\mathbb{U}_u(h_0)}.\end{aligned}\tag{43}$$

Proof Let $(\hat u,\pi,[\pi],h)\in\mathbb{E}(t_0)$ and consider, see (6),

$$u(t,x,y)=\hat u(t,x,y-h(t,x)).$$

By (41) the mapping $T_{h(t)}:(x,y)\mapsto(x,y-h(t,x))$ is for all $t\in[0,t_0]$ a C^2-diffeomorphism with $T^{-1}_{h(t)}(x,y)=(x,y+h(t,x))$ and $\det(DT_{h(t)}(x,y))=1$. By (39) the chain rule for Sobolev functions can be applied and yields $u\in H^1_p(J\times\mathbb{R}^{n+1},\mathbb{R}^{n+1})$ with

$$\begin{aligned}\partial_t u(t,x,y)&=\partial_t\hat u\big(t,T_{h(t)}(x,y)\big)-\partial_y\hat u\big(t,T_{h(t)}(x,y)\big)\partial_t h(t,x),\\\partial_{(x,y)}u(t,x,y)&=\partial_{(x,y)}\hat u\big(t,T_{h(t)}(x,y)\big)DT_{h(t)}(x,y).\end{aligned}$$

Moreover, again by (41) and $\nabla\hat u\in L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+1,n+1}))$ we have

$$\|u\|_{H^1_p(J\times\mathbb{R}^{n+1},\mathbb{R}^{n+1})} + \left(\int_J \|u(t)\|^p_{H^2_p(\mathbb{R}^{n+1}\setminus\Gamma(t),\mathbb{R}^{n+1})}\,dt\right)^{1/p} \le C\big(\|h\|_{\mathbb{E}_4(t_0)}\big)\|\hat{u}\|_{\mathbb{E}_1(t_0)}.$$

Completely analogous one obtains

$$\left(\int_J \|q(t)\|^p_{\dot{H}^1_p(\mathbb{R}^{n+1}\setminus\Gamma(t),\mathbb{R}^{n+1})}\,dt\right)^{1/p} \le C\big(\|h\|_{\mathbb{E}_4(t_0)}\big)\|\pi\|_{\mathbb{E}_2(t_0)}.$$

Now consider $\hat{r} = [\pi]$ and $r = [q]$ then $r(t,x,h(t,x)) = \hat{r}(t,x)$ and

$$\begin{aligned}\|r(t,\cdot)\|^p_{W^{1-1/p}_p(\Gamma(t))} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|r(t,x,h(t,x)) - r(t,\tilde{x},h(t,\tilde{x}))|^p}{\left(\sqrt{|x-\tilde{x}|^2 + |h(t,x)-h(t,\tilde{x})|^2}\right)^{n+p-1}}\\ &\qquad\cdot\sqrt{1+|\nabla h(t,x)|^2}\sqrt{1+|\nabla h(t,\tilde{x})|^2}\,dx\,d\tilde{x}\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\hat{r}(t,x)-\hat{r}(t,\tilde{x})|^p}{|x-\tilde{x}|^{n+p-1}}\,dx\,d\tilde{x}(1+|h(t,\cdot)|_{BC^1(\mathbb{R}^n)})^2\\ &\le \|\hat{r}(t,\cdot)\|^p_{W^{1-1/p}_p(\mathbb{R}^n)}C\big(\|h\|_{\mathbb{E}_4(t_0)}\big)^p.\end{aligned}$$

Similarly, one obtains also the estimate for $\|\hat{u}_0\|_{\mathbb{U}_{\hat{u}}}$, see [22, Proof Thm 1.1]. □

Lemma 11 *Consider the transformation* (42), *where we choose for* $\pi \in \mathbb{E}_2(t_0), [\pi] \in \mathbb{E}_3(t_0)$ *the unique representative* π *satisfying* (36).

Then for all $\tilde{p} \in [p,\infty)$ *the mapping*

$$(\hat{u},\pi,[\pi],h) \in \mathbb{E}(t_0) \mapsto u \in C\big(\bar{J};L_{\tilde{p}}\big(\mathbb{R}^{n+1},\mathbb{R}^{n+1}\big)\big) \tag{44}$$

is continuously differentiable with derivative

$$\begin{aligned}(\delta\hat{u},\delta\pi,[\delta\pi],\delta h) \in \mathbb{E}(t_0) &\mapsto \delta u(t,x,y)\\ &= \delta\hat{u}(t,x,y-h(t,x)) - \partial_y\hat{u}(t,x,y-h(t,x))\delta h(t,x).\end{aligned}$$

Let $\mathcal{E}_\pm \in \mathcal{L}(H^l_p(\mathbb{R}^{n+1}_\pm), H^l_p(\mathbb{R}^{n+1}))$ *be extension operators for* $l = 1, 2$ *and set*

$$\begin{aligned}\hat{u}_\pm(t,\cdot) &= \mathcal{E}_\pm\hat{u}(t,\cdot), \quad u_\pm(t,x,y) = \hat{u}_\pm(t,T_{h(t)}(x,y)),\\ \pi_\pm(t,\cdot) &= \mathcal{E}_\pm\pi(t,\cdot), \quad q_\pm(t,x,y) = \pi_\pm(t,T_{h(t)}(x,y)).\end{aligned} \tag{45}$$

Then the mappings

$$(\hat{u},\pi,[\pi],h) \in \mathbb{E}(t_0) \mapsto u_\pm \in L_p\Big(J;H^1_p\big(\mathbb{R}^{n+1},\mathbb{R}^{n+1}\big)\Big), \tag{46}$$

$$(\hat{u},\pi,[\pi],h) \in \mathbb{E}(t_0) \mapsto q_\pm \in L_p\big(J;L_p\big(\mathbb{R}^{n+1}\big)\big) \tag{47}$$

are continuously differentiable with derivative

$$(\delta\hat{u}, \delta\pi, [\delta\pi], \delta h) \in \mathbb{E}(t_0) \mapsto \begin{pmatrix} \delta u_\pm \\ \delta q_\pm \end{pmatrix}(t,x,y)$$
$$= \begin{pmatrix} \delta\hat{u}_\pm \\ \delta\pi_\pm \end{pmatrix}(t,x,y-h(t,x)) - \partial_y \begin{pmatrix} \hat{u}_\pm \\ \pi_\pm \end{pmatrix}(t,x,y-h(t,x))\delta h(t,x).$$

Proof Define as in the previous proof the C^2-diffeomorphisms $T_{h(t)} : (x,y) \mapsto (x, y-h(t,x))$. Then $u(t,x,y) = \hat{u}(t, T_{h(t)}(x,y))$. Let $(\hat{u}, \pi, [\pi], h), (\delta\hat{u}, \delta\pi, [\delta\pi], \delta h) \in \mathbb{E}(t_0)$ be arbitrary. We recall the well known fact that for any $v \in C(\bar{J};L_{\tilde{p}}(\mathbb{R}^{n+1}))$, $p \le \tilde{p} < \infty$, it holds

$$\sup_{t\in J} \left\| v\big(t, T_{(h+\delta h)(t)}(\cdot)\big) - v\big(t, T_{h(t)}(\cdot)\big) \right\|_{L_{\tilde{p}}(\mathbb{R}^{n+1})} \to 0 \quad \text{as } \|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))} \to 0, \tag{48}$$

which can be shown by an approximation of v through a sequence of continuous functions with compact support. Similarly, for $v \in L_p(J;L_p(\mathbb{R}^{n+1}))$ one has

$$\int_J \left\| v\big(t, T_{(h+\delta h)(t)}(\cdot)\big) - v\big(t, T_{h(t)}(\cdot)\big) \right\|^p_{L_p(\mathbb{R}^{n+1})} dt \to 0 \quad \text{as } \|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))} \to 0. \tag{49}$$

Consider the remainder term

$$\begin{aligned} R_u(t,x,y) :=& (\hat{u} + \delta\hat{u})\big(t, T_{(h+\delta h)(t)}(x,y)\big) - \hat{u}\big(t, T_{h(t)}(x,y)\big) \\ &- \delta\hat{u}\big(t, T_{h(t)}(x,y)\big) + \partial_y\hat{u}\big(t, T_{h(t)}(x,y)\big)\delta h(t,x). \end{aligned} \tag{50}$$

Let $p \le \tilde{p} < \infty$ be arbitrary. We obtain

$$\begin{aligned} \|R_u\|_{C(\bar{J};L_{\tilde{p}}(\mathbb{R}^{n+1}))} &\le \sup_{t\in J} \left\| \int_0^1 (\partial_y \delta\hat{u}(t, T_{(h+\tau\delta h)(t)}(\cdot))\, d\tau\, \delta h(t,\cdot) \right\|_{L_{\tilde{p}}(\mathbb{R}^{n+1})} \\ &+ \sup_{t\in J} \left\| \int_0^1 (\partial_y \hat{u}(t, T_{(h+\tau\delta h)(t)}(\cdot)) - \partial_y\hat{u}(t, T_{h(t)}(\cdot)))\, d\tau\, \delta h(t,\cdot) \right\|_{L_{\tilde{p}}(\mathbb{R}^{n+1})} \\ &\le \|\delta\hat{u}\|_{C(\bar{J};H^1_{\tilde{p}}(\mathbb{R}^{n+1}))} \|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))} \\ &+ \sup_{t\in J, \tau\in[0,1]} \left\| \partial_y\hat{u}(t, T_{(h+\tau\delta h)(t)}(\cdot)) - \partial_y\hat{u}(t, T_{h(t)}(\cdot)) \right\|_{L_{\tilde{p}}(\mathbb{R}^{n+1})} \|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))} \\ &= \|\delta\hat{u}\|_{C(\bar{J};H^1_{\tilde{p}}(\mathbb{R}^{n+1}))} \|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))} + o\big(\|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))}\big) \\ &= o\big(\|\delta\hat{u}\|_{\mathbb{E}_1(t_0)} + \|\delta h\|_{\mathbb{E}_4(t_0)}\big). \end{aligned}$$

Here, we have used (47) and the imbeddings (40), (41). This shows that (43) is Fréchet differentiable. The continuity of the derivative follows from the fact that for $(\hat{u}_1, \pi_1, [\pi_1], h_1) \to (\hat{u}, \pi, [\pi], h)$ in $\mathbb{E}(t_0)$ we have

$$
\begin{aligned}
&\sup_{t\in J}\left\|\delta\hat{u}\big(t,T_{h_1(t)}(\cdot)\big)-\delta\hat{u}\big(t,T_{h(t)}(\cdot)\big)\right\|_{L_{\bar{p}}(\mathbb{R}^{n+1})}\\
&\quad=\sup_{t\in J}\left\|\int_0^1\partial_y\delta\hat{u}\big(t,T_{(h+\tau(h_1-h))(t)}(\cdot)\big)\,d\tau\,(h_1-h)(t,\cdot)\right\|_{L_{\bar{p}}(\mathbb{R}^{n+1})}\\
&\quad\le\|\delta\hat{u}\|_{C(\bar{J};H^1_{\bar{p}}(\mathbb{R}^{n+1}))}\|h_1-h\|_{C(\bar{J};BC(\mathbb{R}^n))}\\
&\quad\le C\|\delta\hat{u}\|_{\mathbb{E}_1(t_0)}\|h_1-h\|_{\mathbb{E}_4(t_0)}
\end{aligned}
$$

as well as $\|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))}\le C\|\delta h\|_{\mathbb{E}_4(t_0)}$ and

$$
\begin{aligned}
&\sup_{t\in J}\|\partial_y\hat{u}_1(t,T_{h_1(t)}(\cdot))-\partial_y\hat{u}(t,T_{h(t)}(\cdot))\|_{L_{\bar{p}}(\mathbb{R}^{n+1})}\\
&\quad\le\|\hat{u}_1-\hat{u}\|_{C(\bar{J};H^1_{\bar{p}}(\mathbb{R}^{n+1}))}+\sup_{t\in J}\|\partial_y\hat{u}_1(t,T_{h_1(t)}(\cdot))-\partial_y\hat{u}(t,T_{h(t)}(\cdot))\|_{L_{\bar{p}}(\mathbb{R}^{n+1})}\to 0,
\end{aligned}
$$

where we have used (47).

The continuous differentiability of (46) follows very similarly by using (48) instead of (47) and by applying (36), (37) and Theorem 10.

Finally, consider (45), (44). Then $\hat{u}_\pm,\delta\hat{u}_\pm\in L_p(J;H^2_p(\mathbb{R}^{n+1},\mathbb{R}^{n+1}))$. Define the remainder terms $R_{u_\pm}$ as in (49) with $\hat{u},\hat{\delta}u$ replaced by $\hat{u}_\pm,\delta\hat{u}_\pm$. After differentiation a calculation as above yields

$$
\begin{aligned}
&\|\nabla R_{u_\pm}\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
\le&\left\|\int_0^1(\partial_y\nabla\delta\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)^\top DT_{(h+\tau\delta h)(t)}\,d\tau\,\delta h(t,\cdot)\right\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
&+\left\|\int_0^1\partial_y\delta\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)\,d\tau\,\nabla\delta h(t,\cdot)^\top\right\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
&+\left\|\int_0^1\nabla\Big(\partial_y\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)-\partial_y\hat{u}_\pm\big(t,T_{h(t)}(\cdot)\big)\Big)^\top DT_{h(t)}\,d\tau\,\delta h(t,\cdot)\right\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
&+\left\|\int_0^1\partial_y\nabla\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)^\top\big(DT_{(h+\tau\delta h)(t)}-DT_{h(t)}\big)\,d\tau\,\delta h(t,\cdot)\right\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
&+\left\|\int_0^1\Big(\partial_y\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)-\partial_y\hat{u}_\pm\big(t,T_{h(t)}(\cdot)\big)\Big)\,d\tau\,\nabla\delta h(t,\cdot)^\top\right\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
\le&\|\delta\hat{u}_\pm\|_{L_p(J;H^2_p(\mathbb{R}^{n+1}))}\big(1+\|h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}+\|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}\big)\|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))}\\
&+\|\delta\hat{u}_\pm\|_{L_p(J;H^1_p(\mathbb{R}^{n+1}))}\|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}\\
&+\|\partial_y\nabla\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)-\partial_y\nabla\hat{u}_\pm\big(t,T_{h(t)}(\cdot)\big)\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\\
&\cdot\big(1+\|h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}\big)\|\delta h\|_{C(\bar{J};BC(\mathbb{R}^n))}+\|\hat{u}_\pm\|_{L_p(J;H^2_p(\mathbb{R}^{n+1}))}\|\delta h\|^2_{C(\bar{J};BC(\mathbb{R}^n))}\\
&+\|\partial_y\hat{u}_\pm\big(t,T_{(h+\tau\delta h)(t)}(\cdot)\big)-\partial_y\hat{u}_\pm\big(t,T_{h(t)}(\cdot)\big)\|_{L_p(J;L_p(\mathbb{R}^{n+1}))}\|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}\\
=&\,o\Big(\|\delta\hat{u}\|_{L_p(J;H^2_p(\mathbb{R}^{n+1}))}+\|\delta h\|_{C(\bar{J};BC^1(\mathbb{R}^n))}\Big)=o\big(\|\delta\hat{u}\|_{\mathbb{E}_1(t_0)}+\|\delta h\|_{\mathbb{E}_4(t_0)}\big).
\end{aligned}
$$

Here we have used (48) and the imbedding (41). The continuity of the derivative follows with very similar estimates. □

Similarly, we have

Lemma 12 *Let* $\mathbb{U}_c(t_0) = L_p(J; H_p^1(\mathbb{R}^{n+1}))$. *Then the mapping*

$$(c, h) \in \mathbb{U}_c(t_0) \times \mathbb{E}_4(t_0) \mapsto \hat{c}(c, h) \in \mathbb{U}_{\hat{c}}(t_0) \tag{51}$$

with $\hat{c}(c, h)(t, x, y) = c(t, x, y + h(t, x))$ *is continuously differentiable with derivative*

$$(\delta c, \delta h) \in \mathbb{U}_c(t_0) \times \mathbb{E}_4(t_0) \mapsto \delta c(t, x, y + h(t, x)) + \partial_y c(t, x, y + h(t, x))\delta h(t, x).$$

Proof The proof is the same as for (46). □

For the original data (u_0, h_0, c) we obtain the following existence and differentiability result.

Theorem 13 *Let* $p > n + 3$ *and* $\mathbb{U}_u(h_0), \mathbb{U}_c(t_0)$ *be defined by* (35). *Then for any* $t_0 > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(t_0) > 0$ *such that for all data*

$$(h_0, c) \in \mathbb{U}_h \times \mathbb{U}_c(t_0), \quad u_0 \in \mathbb{U}_u(h_0)$$

satisfying the compatibility condition (14) *as well as the smallness condition*

$$\|u_0\|_{\mathbb{U}_u(h_0)} + \|h_0\|_{\mathbb{U}_h} + \|c\|_{\mathbb{U}_c(t_0)} < \varepsilon_0 \tag{52}$$

there exists a unique solution of the transformed problem (10) *with*

$$(\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0).$$

Moreover, for any h_0 *with* $\|h_0\|_{\mathbb{U}_h} < \varepsilon_0$ *the mapping*

$$\{(u_0, c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) : (u_0, h_0, c) \text{ satisfy (14), (51) } \} \mapsto (\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$$

is continuously differentiable.

By the chain rule in Lemma 11, also the original state (u, q) *depends continuously differentiable on* (u_0, c) *with the spaces given in* (43), (45), (46).

Proof We adapt the fixed point argument in the proof of Theorem 8. Let

$$\hat{c}(c, h)(t, x, y) = c(t, x, y + h(t, x)). \tag{53}$$

The only difference compared to the situation in Theorem 8 results from the fact that $\hat{c}(c, h)$ depends now on h. Hence, the fixed point equation (32) changes to

$$L_0\tilde{z} = K\big(\tilde{z}; \hat{u}_0, h_0, \hat{c}\big(c, \tilde{z} + z^*(\hat{u}_0, h_0)\big)\big), \quad \tilde{z} \in {}_0\mathbb{E}(t_0). \tag{54}$$

Let $\hat{\varepsilon}_0 > 0$ be as in Theorem 8. We have

$$\|\hat{c}(c,h)\|_{\mathbb{U}_{\hat{u}}} = \|c\|_{\mathbb{U}_{\hat{u}}} \tag{55}$$

and the last estimate in (43) shows that for $\varepsilon_0 > 0$ small enough (51) implies (27).

Hence, for all (u_0, h_0, c) satisfying (51) we have $(\hat{u}_0, h_0, \hat{c}(c,h)) \in B_U(\hat{\varepsilon}_0)$ (note that (54) holds independently of h) and thus by (34)

$$\left\| L_0^{-1} K(\tilde{z};\hat{u}_0, h_0, \hat{c}) \right\|_{{}_0\mathbb{E}(t_0)} < \delta.$$

Finally, the Lipschitz constant of $K(\tilde{z};\hat{u}_0, h_0, \hat{c})$ with respect to $\hat{c}$ is 1 and the mapping (50), (52) is by Lemma 12 continuously differentiable and the Lipschitz constant with respect to h is bounded by $\|c\|_{\mathbb{U}_c(t_0)} < \varepsilon_0$. Hence, for $\varepsilon_0 > 0$ small enough, (53) is a contraction and the existence, uniqueness and continuous differentiability follow as in the proof of Theorem 8.

Lemma 11 and the chain rule yield now the continuous differentiability of the original state (u, q) with respect to (u_0, c) for the spaces given in (43), (45), (46). $\square$

3.3 Volume-of-fluid type formulation

Our aim is finally to derive a Volume-of-Fluid (VoF) type formulation with corresponding sensitivity equation that is satisfied by the solution (u, q) of the problem (1) and its sensitivities $(\delta u, \delta q)$. This provides an analytical foundation to derive and analyze appropriate numerical VoF schemes for sensitivity calculations.

Let $\alpha : \mathbb{R}^{n+1} \to [0,1]$ be a phase indicator satisfying the transport equation

$$\partial_t \alpha + u \cdot \nabla \alpha = 0 \quad \text{in } J \times \mathbb{R}^{n+1}, \quad \alpha(0) = 1_{\Omega_1(0)} \quad \text{on } \mathbb{R}^{n+1}. \tag{56}$$

We note that for $u \in L_1(J;W^1_\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$ with $\operatorname{div} u = 0$ a.e. any distributional solution $\alpha \in L_1(J;L_{1,loc}(\mathbb{R}^{n+1}))$ is also a distributional solution of

$$\partial_t \alpha + \operatorname{div}(u\alpha) = 0 \quad \text{in } J \times \mathbb{R}^{n+1}, \quad \alpha(0) = 1_{\Omega_1(0)} \quad \text{on } \mathbb{R}^{n+1}. \tag{57}$$

We define now

$$\rho(\alpha) = \alpha\rho_1 + (1-\alpha)\rho_2, \quad \mu(\alpha) = \alpha\mu_1 + (1-\alpha)\mu_2.$$

We will show that the unique solution (u, q) of (1) according to Theorem 13 satisfies the VoF-type formulation

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} & \big(\partial_t(\rho(\alpha)u) + \operatorname{div}(\rho(\alpha)u \otimes u)\big)(t,x,y)^\top \varphi(x,y) \\ & + S(u,q;\mu(\alpha))(t,x,y) : \nabla\varphi(x,y))\, d(x,y) \qquad \forall\, \varphi \in C_c^1\big(\mathbb{R}^{n+1};\mathbb{R}^{n+1}\big)\end{aligned} \tag{58}$$

$$
\begin{aligned}
&= -\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^{n+1}} \sigma \frac{\nu_\varepsilon(t,x,y)^\top}{|\nu_\varepsilon(t,x,y)|} (D\varphi - \operatorname{div}(\varphi) I)(x,y) \nabla \alpha(t,x,y)\, d(x,y), \\
&\int_{\mathbb{R}^{n+1}} \operatorname{div}(u)\, \psi\, dx = 0 \quad \forall \psi \in C^1_c\big(\mathbb{R}^{n+1}\big),
\end{aligned} \tag{59}
$$

$$
\alpha \text{ satisfies (56)}\,, \tag{60}
$$

where ν_ε is a suitable smoothed normal computed from $\nabla\alpha$, see (70) below.

In order to deal with the sensitivity equation, it will be beneficial to consider measure-valued solutions of the general equation

$$
\partial_t \delta\alpha + \operatorname{div}(u\, \delta\alpha) = b \quad \text{in } J \times \mathbb{R}^{n+1}, \quad \delta\alpha(0) = \delta\alpha_0 \quad \text{on } \mathbb{R}^{n+1}. \tag{61}
$$

For $u \in L_1(J; W^1_\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$ we can define uniquely the continuous mapping $(x,y) \mapsto X(t;x,y)$, where $X(t;\, x,\, y)$ satisfies the characteristic equation

$$
\partial_t X(t;x,y) = u(t, X(t;x,y)), \;\; t \in J, \quad X(0;x,y) = (x,y). \tag{62}
$$

In the following, we denote by $\mathcal{M}_{loc}(\mathbb{R}^{n+1})$ the space of locally bounded Radon measures.

Proposition 14 *Let $u \in L_1(J; W^1_\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$. Then for any $\delta\alpha_0 \in \mathcal{M}_{loc}(\mathbb{R}^{n+1})$ there exists a unique distributional solution of (60) in $C(\bar{J}; \mathcal{M}_{loc}(\mathbb{R}^{n+1}) -$ weak$^*)$, given by*

$$
\delta\alpha(t) = X(t)(\delta\alpha_0) + \int_0^t X(t-s)(b(s)). \tag{63}
$$

Here, X is the forward flow defined by (61) and $\delta\alpha_t = X(t)(\delta\alpha_0)$ is the measure satisfying

$$
\int_{\mathbb{R}^{n+1}} \phi(x,y)\, d\delta\alpha_t(x,y) = \int_{\mathbb{R}^{n+1}} \phi(X(t;x,y))\, d\delta\alpha_0(x,y) \quad \forall \phi \in C_c\big(\mathbb{R}^{n+1}\big).
$$

Proof For $u \in L_1(J; C^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$, see [21, Thm. 3.1 and 3.3]. Since the characteristics are unique and stable also for $u \in L_1(J; W^1_\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$, the proofs directly extend to this case, see also [3]. □

Proposition 15 *If $\hat{u} \in \mathbb{E}_1(t_0)$, $[\hat{u}] = 0$ and u is given by (42) then (55) as well as (56) have a unique solution given by*

$$
\alpha(t, X(t;x,y)) = 1_{\Omega_1(0)}(x,y) \tag{64}
$$

and thus $\alpha(t,\cdot) = 1_{\Omega_1(t)}$.

Moreover, for ε_0 from Theorem 13 and any h_0 with $\|h_0\|_{\mathbb{U}_h} < \varepsilon_0$ the mapping

$$\left\{(u_0, c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) : (u_0, h_0, c) \text{ satisfy (14), (51)} \right\} \mapsto \alpha \in C\big(\bar{J}; \mathcal{M}_{loc}\big(\mathbb{R}^{n+1}\big) - \text{ weak}^*\big)$$

is continuously differentiable. The derivative

$$(\delta u_0, \delta c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) \mapsto \delta\alpha \in C\big(\bar{J}; \mathcal{M}_{loc}\big(\mathbb{R}^{n+1}\big) - \text{ weak}^*\big)$$

is given by the unique measure-valued solution of

$$\partial_t \delta\alpha + \operatorname{div}(u\, \delta\alpha) = -\operatorname{div}(\delta u\, \alpha) \quad \text{in } J \times \mathbb{R}^{n+1}, \quad \delta\alpha(0) = 0 \quad \text{on } \mathbb{R}^{n+1}. \tag{65}$$

Finally, $\delta\alpha$ satisfies

$$\int_{\mathbb{R}^{n+1}} \phi(x, y)\, d\delta\alpha(t)(x, y) = \int_{\mathbb{R}^n} \phi(x, h(t, x))\delta h(t, x)\, dx. \tag{66}$$

Proof If $\hat{u} \in \mathbb{E}_1(t_0)$, $[\hat{u}] = 0$ and u is given by (42) then $u \in C(\bar{J}; W^1_\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}))$ by (38), (41). Now it is well known that (63) provides the unique weak solution of (55) in $L_{1,loc}(J \times \mathbb{R}^{n+1})$, see [3, Prop. 2.2] and [10, Cor. II.1]. Since $\operatorname{div}(u) = 0$ a.e., it is also a distributional solution of (56), which is unique by Proposition 14.

Let now $(u_0, h_0, c), (\delta u_0, 0, \delta c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_h \times \mathbb{U}_c(t_0)$ be such that (u_0, h_0, c) and $(u_0, h_0, c) + (\delta u_0, 0, \delta c)$ satisfy the conditions of Theorem 13. Denote by $(\hat{u}, \pi, [\pi], h)$ the unique solution of (10) for data (u_0, h_0, c) and by $(\hat{u}^s, \pi^s, [\pi^s], h^s)$ the one for data $(u_0, h_0, c) + s\,(\delta u_0, 0, \delta c)$. Let (u, q) and (u^s, q^s) be the corresponding states in physical coordinates according to (6) and let $\alpha = 1_{\Omega_1(t)}$, $\alpha^s = 1_{\Omega_1^s(t)}$ be the corresponding solutions of (56). Finally, let $(\delta u, \delta h, \delta q)$ be the directional derivatives (sensitivities) in direction $(\delta u_0, 0, \delta c)$ which exist by Theorem 13. We show that

$$\frac{\alpha^s - \alpha}{s} \to \delta\alpha \quad \text{in } C\big(\bar{J}; \mathcal{M}_{loc}\big(\mathbb{R}^{n+1}\big) - \text{ weak}^*\big) \text{ as } s \to 0, \tag{67}$$

where $\delta\alpha$ solves (64). Let $\phi \in C_c(\mathbb{R}^{n+1})$ be arbitrary. Then

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \frac{\alpha^s - \alpha}{s}(t, x, y)\phi(x, y)\, d(x, y) &= \int_{\mathbb{R}^n} \int_{h(t,x)}^{h^s(t,x)} \frac{1}{s}\phi(x, y)\, d(x, y) \\ &\to \int_{\mathbb{R}^n} \phi(x, h(t, x))\delta h(t, x)\, dx \end{aligned}$$

as $s \to 0$ uniformly in $t \in \bar{J}$, where we have used the differentiability result of Theorem 13. Moreover, it is obvious that the middle term is continuous with respect to t. Hence, (66) is proven and we have only to show that $\delta\alpha$ solves (64).

To this end, let $\varphi \in C^1_c(J \times \mathbb{R}^{n+1})$ be arbitrary. Since α, α^s are distributional solutions of (56), we have

$$\begin{aligned} 0 &= \int_J \int_{\mathbb{R}^{n+1}} -\Big(\big(\partial_t\varphi + (u \cdot \nabla)\varphi\big)\tfrac{\alpha^s - \alpha}{s} + \alpha^s\Big(\tfrac{u^s - u}{s} \cdot \nabla\Big)\varphi\Big)(t, x, y)\, d(x, y)\, dt \\ &\to \int_J \int_{\mathbb{R}^{n+1}} -\big(\big(\partial_t\varphi + (u \cdot \nabla)\varphi\big)\delta\alpha + \alpha(\delta u \cdot \nabla)\varphi\big)(t, x, y)\, d(x, y)\, dt \end{aligned}$$

as $s \to 0$. For the limit transition, we have used $u \in C(\bar{J};W^1_\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))$, (66) and that by Theorem 13$\alpha^s = 1_{\Omega^s(t)} \to \alpha = 1_{\Omega_1(t)}$ in $L_{2,loc}(J \times \mathbb{R}^{n+1})$ and $\frac{u^s-u}{s} \to \delta u$ in $C(\bar{J};L_p(\mathbb{R}^{n+1}))$. Hence, $\delta\alpha$ is a distributional solution of (64), which is unique by Proposition 14. □

The next step is to express the surface tension term by using the phase indicator α such that its sensitivities can be expressed by using the measure $\delta\alpha$.

We first rewrite the surface tension term in the weak formulation (2).

Lemma 16 *Let $\varphi \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Then with the curvature $\kappa(t)$ of $\Gamma(t)$ according to* (4) *one has the identity*

$$\begin{aligned}
&\int_{\Gamma(t)} (\sigma\kappa\nu)(t,x,y)^\top \varphi(x,y)\, dS(x,y)\\
&\quad= \int_{\mathbb{R}^n} \sigma\, \mathrm{div}_x\left(\frac{\nabla h(t,x)}{\sqrt{1+|\nabla h(t,x)|^2}}\right)\begin{pmatrix} -\nabla h(t,x)\\ 1\end{pmatrix}^\top \varphi(x,h(t,x))\, dx\\
&\quad= \int_{\mathbb{R}^n} \sigma \frac{(\nabla h(t,x)^\top, -1)}{\sqrt{1+|\nabla h(t,x)|^2}}(D\varphi(x,h(t,x)) -\ \mathrm{div}\,(\varphi)(x,h(t,x))I)\begin{pmatrix} \nabla h(t,x)\\ -1\end{pmatrix} dx.
\end{aligned} \tag{68}$$

Proof The first identity follows directly from (4). The second one follows from integration by parts and reflects the well known identity from differential geometry, see for example [7, Lem. 2.1]

$$\int_{\Gamma(t)} (\kappa\nu)(t,x,y)^\top \varphi(x,y)\, dS(x,y) = -\int_{\Gamma(t)} \nabla_T\, \mathrm{id}_{\Gamma(t)}(x,y) : \nabla_T\varphi(x,y)\, dS(x,y),$$

where $\nabla_T\varphi_i = \nabla\varphi_i - \nu^\top\nabla\varphi_i\nu$ is the tangential derivative. □

To compute the interface normal from $\nabla\alpha$, we use the following simple fact.

Lemma 17 *Let $\psi \in C^1_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Then*

$$\begin{aligned}
-\int_{\mathbb{R}^{n+1}} \psi(x,y)^\top \nabla\alpha(t,x,y)\, d(x,y) &= \int_{\Gamma(t)} \psi(x,y)^\top \nu(t,x,y)\, dS(x,y)\\
&= \int_{\mathbb{R}^n} \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla h(t,x)\\ 1\end{pmatrix} dx.
\end{aligned}$$

Proof By the definition of distributional derivatives one has

$$
\begin{aligned}
-\int_{\mathbb{R}^{n+1}} \psi(x,y)^\top \nabla\alpha(t,x,y)\,d(x,y) &= \int_{\mathbb{R}^{n+1}} \operatorname{div}(\psi)(x,y)\alpha(t,x,y)\,d(x,y) \\
&= \int_{\Omega_1(t)} \operatorname{div}(\psi)(x,y)\,d(x,y) = \int_{\Gamma(t)} \psi(x,y)^\top \nu(t,x,y)\,dS(x,y) \\
&= \int_{\mathbb{R}^n} \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} dx.
\end{aligned}
$$

□

Let now $\delta \in (0, 1/2)$ and

$$
\psi_\delta \in C_c^1((-1,1)), \quad \psi_\delta|_{[-1+\delta,1-\delta]} \equiv 1, \quad \psi_\delta(-s) = \psi_\delta(s) \quad \forall\, s \in \mathbb{R}, \int_{\mathbb{R}} \psi_\delta(s)\,ds = 1
$$

and set

$$
\phi_\varepsilon(x,y) = \frac{1}{\varepsilon^n}\psi_\delta(y/\varepsilon)\prod_{i=1}^n \psi_\delta(x_i/\varepsilon).
$$

To recover a mollified normal (not necessarily of unit length) we use

$$
\nu_\varepsilon(t,x,y) := -\int_{\mathbb{R}^{n+1}} \phi_\varepsilon((\tilde{x},\tilde{y}) - (x,y))\nabla\alpha(t,\tilde{x},\tilde{y})\,d(\tilde{x},\tilde{y}). \tag{69}
$$

Then by Lemma 17

$$
\begin{aligned}
\nu_\varepsilon(t,x,y) &= \int_{\Gamma(t)} \phi_\varepsilon((\tilde{x},\tilde{y}) - (x,y))\nu(t,\tilde{x},\tilde{y})\,dS(\tilde{x},\tilde{y}) \\
&= \int_{\mathbb{R}^n} \phi_\varepsilon((\tilde{x},h(t,\tilde{x})) - (x,y))\begin{pmatrix} -\nabla h(t,\tilde{x}) \\ 1 \end{pmatrix} d\tilde{x}.
\end{aligned}
$$

Now assume that

$$
|\nabla h| \le 1-\delta \qquad \text{on } x + [-\varepsilon,\varepsilon]^n. \tag{70}
$$

Then we have by the definition of ϕ_ε

$$
\nu_\varepsilon(t,x,h(t,x)) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \prod_{i=1}^n \psi_\delta((\tilde{x}_i - x_i)/\varepsilon)\begin{pmatrix} -\nabla h(t,\tilde{x}) \\ 1 \end{pmatrix} d\tilde{x}. \tag{71}
$$

Lemma 18 *Let (69) hold. If $h \in C(\bar{J}; BC^2(\mathbb{R}^n))$ then there is $C > 0$ such that*

$$
|\nu_\varepsilon(t,x,h(t,x)) - (-\nabla h(t,x),1)^\top| \le C\varepsilon \quad \forall\,(t,x) \in J \times \mathbb{R}^n.
$$

On compact subsets the error is $o(\varepsilon)$.

Proof Since ∇h has a uniform Lipschitz constant with respect to x the first assertion follows immediately from (70). Moreover, since $\nabla h(t,\tilde{x}) = \nabla h(t,x) + \nabla^2 h(t,x)(\tilde{x}-x) + o(\|\tilde{x}-x\|)$, the $o(\varepsilon)$ is obtained by the symmetry of ψ_δ. □

The variation of v_ε is

$$\begin{aligned}\delta v_\varepsilon(t,x,y) &:= -\int_{\mathbb{R}^{n+1}} \phi_\varepsilon((\tilde{x},\tilde{y})-(x,y))\nabla d\delta\alpha(t)(\tilde{x},\tilde{y}) \\ &= \int_{\mathbb{R}^{n+1}} \nabla\phi_\varepsilon((\tilde{x},\tilde{y})-(x,y))d\delta\alpha(t)(\tilde{x},\tilde{y})\end{aligned} \tag{72}$$

with the measure-valued solution of (64).

Lemma 19 *Let* (69) *hold. If* $\delta h \in C(\bar{J};BC^2(\mathbb{R}^n))$ *then there is* $C>0$ *such that*

$$|\delta v_\varepsilon(t,x,h(t,x)) - (-\nabla\delta h(t,x),0)^\top| \le C\varepsilon \quad \forall\,(t,x)\in J\times\mathbb{R}^n.$$

On compact subsets the error is $o(\varepsilon)$.

Proof Then by (65) and (71)

$$\delta v_\varepsilon(t,x,y) = \int_{\mathbb{R}^n} \nabla\phi_\varepsilon((\tilde{x},h(t,\tilde{x}))-(x,y))\delta h(t,\tilde{x})\,d\tilde{x}.$$

Setting $y=h(t,x)$ and using (69) we obtain

$$\begin{aligned}\delta v_\varepsilon(t,x,h(t,x)) &= \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\begin{pmatrix}\nabla_{\tilde{x}}\prod_{i=1}^n\psi_\delta((\tilde{x}_i-x_i)/\varepsilon)\\ 0\end{pmatrix}\delta h(t,\tilde{x})\,d\tilde{x} \\ &= \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\prod_{i=1}^n\psi_\delta((\tilde{x}_i-x_i)/\varepsilon)\begin{pmatrix}-\nabla\delta h(t,\tilde{x})\\ 0\end{pmatrix}d\tilde{x}.\end{aligned}$$

The remaining proof is identical to the one of Lemma 18. □

We are now in the position to show the following result.

Theorem 20 *If* (69) *holds for the solution* (u,q) *of* (1) *according to Theorem* 13 *(which is satisfied for* $\varepsilon_0>0$ *small enough) then it satisfies the VoF-type formulation* (57)–(59).

Let vice versa (u,q,α) *be a solution of the VoF-type formulation* (57)–(59), *where* $\alpha(t)$ *is the indicator function of a domain* $\Omega_1(t)=\{(x,y)\in\mathbb{R}^n\times\mathbb{R} : y=h(t,x)\}$. *If* (u,q,h) *has the regularity as in Theorem* 13, *then* (u,q,h) *coincides with the solution of* (1) *according to Theorem* 13.

Proof Let (u,q) be the solution of (1) according to Theorem 13. Then it solves clearly also the weak formulation (2)–(3). Since the solution of (56) is $\alpha = 1_{\Omega_1}(t)$

by Proposition 15, the formulations (57)–(59) and (2)–(3) are equivalent if the right hand side of (57) coincides with the surface tension force term (67). To show this we note that Lemma 17 yields for any $\varepsilon > 0$

$$
\begin{aligned}
&-\int_{\mathbb{R}^{n+1}} \sigma \frac{\nu_\varepsilon(t,x,y)^\top}{|\nu_\varepsilon(t,x,y)|}(D\varphi - \operatorname{div}(\varphi)I)(x,y)\nabla\alpha(t,x,y)\,d(x,y)\\
&= \int_{\mathbb{R}^n} \sigma \frac{\nu_\varepsilon(t,x,h(t,x))^\top}{|\nu_\varepsilon(t,x,h(t,x))|}(D\varphi - \operatorname{div}(\varphi)I)(x,h(t,x))\begin{pmatrix} -\nabla h(t,x)\\ 1\end{pmatrix} dx.
\end{aligned}
$$

Now the uniform convergence of $\nu_\varepsilon(t,x,h(t,x))$ to $\begin{pmatrix} -\nabla h(t,x)\\ 1\end{pmatrix}$ for $\varepsilon \searrow 0$ by Lemma 18 yields the convergence of the above term to (67).

Let vice versa (u, q, α) be a solution of the VoF-type formulation (57)–(59) such that $(u,\ q,\ h)$ satisfies the regularity assumptions of Theorem 13. Then again $\alpha = 1_{\Omega_1}(t)$, where the normal velocity of $\Gamma(t)$ is $u^\top \nu$. Moreover, $[u] = 0$ on $\Gamma(t)$ by the regularity of u and clearly the first two PDEs in (1) follow. Finally, the jump condition in the third line of (1) follow from (57)–(59) (or (2)–(3)) by choosing test functions of the form

$$
\varphi_\tau(t,x,y) = \phi(t,x)\psi_\tau(y - h(t,x)) \tag{73}
$$

with $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$, $\psi_\tau(s) = \psi(s/\tau)$, $\psi \in C_c^1((-1,1))$, $\psi \geq 0$, $\psi(0) = 1$, $\psi(-s) = \psi(s)$ and letting $\tau \searrow 0$. □

Finally, we can justify the following VoF-type formulation for computing the sensitivities $(\delta u, \delta q)$. Due to the limited spatial regularity of $\partial_t u$, we have to state time derivatives on the interface in weak form.

$$
\begin{aligned}
&\int_{J\times\mathbb{R}^{n+1}} \big(\partial_t(\rho(\alpha)\delta u) + \operatorname{div}(\rho(\alpha)(\delta u\otimes u + u\otimes \delta u)) + \delta c\big)^\top \varphi\, d(t,x,y)\\
&+ \int_{J\times\mathbb{R}^{n+1}} S(\delta u, \delta q; \mu(\alpha)) : \nabla\varphi\, d(t,x,y)\\
&+ \int_J \int_{\mathbb{R}^{n+1}} (\rho_2 - \rho_1)u^\top\big(\partial_t\varphi + u\cdot\nabla\varphi\big) d\delta\alpha(t)(x,y)\,dt\\
&- \int_J \int_{\mathbb{R}^{n+1}} \big[S(u,q;\mu(\alpha))\big] : \nabla\varphi\, d\delta\alpha(t)(x,y)dt =\\
&= \lim_{\varepsilon\searrow 0} - \int_{J\times\mathbb{R}^{n+1}} \sigma\left(\frac{\delta\nu_\varepsilon^\top}{|\nu_\varepsilon|} - \frac{\delta\nu_\varepsilon^\top \nu_\varepsilon \nu_\varepsilon^\top}{|\nu_\varepsilon|^3}\right)(D\varphi - \operatorname{div}(\varphi)I)\nabla\alpha\, d(t,x,y)\\
&\qquad - \int_{J\times\mathbb{R}^{n+1}} \sigma \frac{\nu_\varepsilon^\top}{|\nu_\varepsilon|}(D\varphi - \operatorname{div}(\varphi)I)\nabla d\delta\alpha(t)(x,y) \qquad \forall \varphi \in C_c^2(J\times\mathbb{R}^{n+1};\mathbb{R}^{n+1}),
\end{aligned}
\tag{74}
$$

$$
\int_{J\times\mathbb{R}^{n+1}} \operatorname{div}(\delta u)\,\psi\, d(t,x,y) = 0 \quad \forall \psi \in C_c^1\big(J\times\mathbb{R}^{n+1}\big), \tag{75}
$$

$$\delta\alpha \text{ satisfies (64) }, \tag{76}$$

$$\delta u(0) = \delta u_0, \tag{77}$$

where v_ε and δv_ε are given by (70) and (71).

We need the following Lemma

Lemma 21 *Let* $\psi \in C_c^1(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. *Then*

$$\begin{aligned}
&-\int_{\mathbb{R}^{n+1}} \psi(x,y)^\top \nabla d\delta\alpha(t)(x,y)\\
&= \int_{\mathbb{R}^n} \partial_y \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} \delta h(t,x) + \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla \delta h(t,x) \\ 0 \end{pmatrix} dx\\
&= \int_{\mathbb{R}^n} \operatorname{div}(\psi)(x,h(t,x))\delta h(t,x)\, dx.
\end{aligned}$$

Proof By the definition of distributional derivatives one has with (65)

$$\begin{aligned}
-\int_{\mathbb{R}^{n+1}} \psi(x,y)^\top \nabla d\delta\alpha(t)(x,y) &= \int_{\mathbb{R}^{n+1}} \operatorname{div}(\psi)(x,y) d\delta\alpha(t)(x,y)\\
&= \int_{\mathbb{R}^n} \operatorname{div}(\psi)(x,h(t,x))\delta h(t,x)\, dx.
\end{aligned}$$

On the other hand, integration by parts yields

$$\begin{aligned}
&\int_{\mathbb{R}^n} \delta h(t,x)\partial_y \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} + \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla \delta h(t,x) \\ 0 \end{pmatrix} dx\\
&= \int_{\mathbb{R}^n} \delta h(t,x)\partial_y \psi(x,h(t,x))^\top \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix}\\
&\quad + \left(\sum_{i=1}^n \partial_{x_i} \psi_i(x,h(t,x)) + \partial_y \psi(x,h(t,x))^\top \begin{pmatrix} \nabla h(t,x) \\ 0 \end{pmatrix} \right) \delta h(t,x)\, dx\\
&= \int_{\mathbb{R}^n} \operatorname{div}(\psi)(x,h(t,x))\delta h(t,x)\, dx.
\end{aligned}$$

$\square$

Theorem 22 *Let* (u, q) *be the solution of* (1) *according to Theorem* 13 *and let* (69) *hold (which is satisfied for* $\varepsilon_0 > 0$ *small enough). Moreover, let* $(\delta u, \delta q)$ *be the sensitivities of* (u, q) *in Theorem* 13 *corresponding to* $(\delta u_0, \delta c)$. *Then* $(\delta u, \delta q)$ *solve the linearized VoF-type system* (73)–(76).

Let vice versa (u, q, α) *be a solution of the VoF-type formulation* (57)–(59), *where* $\alpha(t)$ *is the indicator function of a domain* $\Omega_1(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = h(t,x)\}$. *If* (u, q, h) *has the regularity as in Theorem* 13 *and* $(\delta u, \delta q, \delta\alpha)$ *is a solution of*

(73)–(76) *such that* $(\delta u, \delta q)$ *has the regularity as in Theorem* 13, *then* $(\delta u, \delta q)$ *coincide with the sensitivities according to Theorem* 13.

Proof Let $(u_0, h_0, c), (\delta u_0, 0, \delta c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_h \times \mathbb{U}_c(t_0)$ be such that (u_0, h_0, c) and $(u_0, h_0, c) + (\delta u_0, 0, \delta c)$ satisfy the conditions of Theorem 13. Denote now by $(\hat{u}, \pi, [\pi], h)$ the unique solution of (10) for data (u_0, h_0, c) and by $(\hat{u}^s, \pi^s, [\pi^s], h^s)$ the one for data $(u_0, h_0, c) + s\,(\delta u_0, 0, \delta c)$. Let $(u,\ q)$ and (u^s, q^s) be the corresponding states in physical coordinates according to (6) and let $\alpha = 1_{\Omega_1(t)}, \alpha^s = 1_{\Omega_1^s(t)}$ be the corresponding solutions of (56). Finally, let $(\delta u, \delta h, \delta q)$ be the directional derivatives (sensitivities) in direction $(\delta u_0, 0, \delta c)$ which exist by Theorem 13. By the differentiability result of Theorem 13 we know that with the extensions $u_\pm, q_\pm$ in (44), see (43), (45), (46)

$$\frac{u^s - u}{s} \to \delta u \quad \text{in } C\big(\bar{J}; L_p\big(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}\big)\big), \tag{78}$$

$$\frac{u^s_\pm - u_\pm}{s} \to \delta u_\pm \quad \text{in } L_p\Big(J; H^1_p\big(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}\big)\Big), \tag{79}$$

$$\frac{q^s_\pm - q_\pm}{s} \to \delta q_\pm \quad \text{in } L_p\big(J; L_p\big(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}\big)\big). \tag{80}$$

We derive now the different terms in (73). Let

$$\Omega_s = \{(t,x,y) \,:\, \alpha^s = \alpha\}, \quad \Omega_s^c = \{(t,x,y) \,:\, \alpha^s(t) \neq \alpha\}.$$

We have for arbitrary $\varphi \in C^2_c(J \times \mathbb{R}^{n+1}; \mathbb{R}^{n+1})$

$$\begin{aligned}
&\int_{J\times\mathbb{R}^{n+1}} \frac{-1}{s}\big((\rho(\alpha^s)u^s - \rho(\alpha)u)^\top \partial_t\varphi + \rho(\alpha^s)(u^s)^\top(u^s\cdot\nabla\varphi) - \rho(\alpha)u^\top(u\cdot\nabla\varphi)\big)\, d(t,x,y)\\
&= \int_{\Omega_s} \frac{-1}{s}\big(\rho(\alpha)(u^s - u)^\top \partial_t\varphi + \rho(\alpha)\big((u^s)^\top(u^s\cdot\nabla\varphi) - u^\top(u\cdot\nabla\varphi)\big)\big)\, d(t,x,y)\\
&\quad + \int_{\Omega_s^c} \frac{-1}{s}\big((\rho(\alpha^s)u^s - \rho(\alpha)u)^\top \partial_t\varphi + \rho(\alpha^s)(u^s)^\top(u^s\cdot\nabla\varphi) - \rho(\alpha)u^\top(u\cdot\nabla\varphi)\big)\, d(t,x,y).
\end{aligned}$$

By (77), (78) one obtains

$$\begin{aligned}
&\int_{\Omega_s} \frac{-1}{s}\big(\rho(\alpha)(u^s - u)^\top \partial_t\varphi + \rho(\alpha)\big((u^s)^\top(u^s\cdot\nabla\varphi) - u^\top(u\cdot\nabla\varphi)\big)\big)\, d(t,x,y)\\
&\to \int_{J\times\mathbb{R}^{n+1}} -\big(\rho(\alpha)\delta u^\top \partial_t\varphi + \rho(\alpha)(\delta u^\top(u\cdot\nabla\varphi) + u^\top(\delta u\cdot\nabla\varphi))\big)\, d(t,x,y)\\
&= \int_{J\times\mathbb{R}^{n+1}} \big(\partial_t(\rho(\alpha)\delta u) + \operatorname{div}(\rho(\alpha)(\delta u \otimes u + u \otimes \delta u))\big)^\top \varphi\, d(t,x,y).
\end{aligned}$$

For the second summand we have by Theorem 13

$$
\begin{aligned}
&\int_{\Omega_s^c} \frac{-1}{s}\Big((\rho(\alpha^s)u^s-\rho(\alpha)u)^\top\partial_t\varphi+\rho(\alpha^s)(u^s)^\top(u^s\cdot\nabla\varphi)-\rho(\alpha)u^\top(u\cdot\nabla\varphi)\Big)\,d(t,x,y)\\
&=\int_{J\times\mathbb{R}^n}\frac{-1}{s}\int_{h(t,x)}^{\max(h(t,x),h^s(t,x))}\Big(\big(\rho_1u^s-\rho_2u\big)^\top\partial_t\varphi+\rho_1(u^s)^\top(u^s\cdot\nabla\varphi)-\rho_2u^\top(u\cdot\nabla\varphi)\Big)\,d(t,x,y)\\
&+\int_{J\times\mathbb{R}^n}\frac{-1}{s}\int_{h^s(t,x)}^{\max(h(t,x),h^s(t,x))}\Big(\big(\rho_2u^s-\rho_1u\big)^\top\partial_t\varphi+\rho_2(u^s)^\top(u^s\cdot\nabla\varphi)-\rho_1u^\top(u\cdot\nabla\varphi)\Big)\,d(t,x,y)\\
&\to\int_{J\times\mathbb{R}^n}(\rho_2-\rho_1)u^\top\big(\partial_t\varphi+u\cdot\nabla\varphi\big)(t,x,h(t,x))\delta h(t,x)\,d(t,x)\\
&=\int_J\int_{\mathbb{R}^{n+1}}(\rho_2-\rho_1)u^\top\big(\partial_t\varphi+u\cdot\nabla\varphi\big)d\delta\alpha(t)(x,y)\,dt,
\end{aligned}
$$

where we have used (65), (38) and (41) in the last step.

For the next term in (73) we note that

$$
\begin{aligned}
&\int_{J\times\mathbb{R}^{n+1}}\frac{1}{s}(S(u^s,q^s;\mu(\alpha^s))-S(u,q;\mu(\alpha)))\,:\,\nabla\varphi\,d(t,x,y)\\
&=\int_{\Omega_s}\frac{1}{s}(S(u^s-u,q^s-q;\mu(\alpha))\,:\,\nabla\varphi\,d(t,x,y)\\
&\quad+\int_{\Omega_s^c}\frac{1}{s}(S(u^s,q^s;\mu(\alpha^s))-S(u,q;\mu(\alpha))\,:\,\nabla\varphi\,d(t,x,y).
\end{aligned}
\tag{81}
$$

Now (78), (79) yield

$$
\int_{\Omega_s}\frac{1}{s}(S(u^s-u,q^s-q;\mu(\alpha))\,:\,\nabla\varphi\,d(t,x,y)\to\int_{J\times\mathbb{R}^{n+1}}S(\delta u,\delta q;\mu(\alpha))\,:\,\nabla\varphi\,d(t,x,y).
$$

Moreover, by using (38) and Theorem 8 we have

$$
\begin{aligned}
&\int_{\Omega_s^c}\frac{1}{s}(S(u^s,q^s;\mu(\alpha^s))-S(u,q;\mu(\alpha)))\,:\,\nabla\varphi\,d(t,x,y)\\
&=\int_{J\times\mathbb{R}^n}\frac{1}{s}\int_{h(t,x)}^{\max(h(t,x),h^s(t,x))}\big(S(u_-^s,q_-^s;\mu_1)-S(u_+,q_+;\mu_2)\big)\,:\,\nabla\varphi\,d(t,x,y)\\
&+\int_{J\times\mathbb{R}^n}\frac{1}{s}\int_{h^s(t,x)}^{\max(h(t,x),h^s(t,x))}\big(S(u_+^s,q_+^s;\mu_2)-S(u_-,q_-;\mu_1)\big)\,:\,\nabla\varphi\,d(t,x,y)\\
&\to-\int_{J\times\mathbb{R}^n}\big[S(u,q;\mu(\alpha))\big](t,x,h(t,x))\delta h(t,x)\,:\,\nabla\varphi(t,x,h(t,x))\,d(t,x)\\
&=-\int_J\int_{\mathbb{R}^{n+1}}\big[S(u,q;\mu(\alpha))\big]\,:\,\nabla\varphi\,d\delta\alpha(t)(x,y)dt.
\end{aligned}
$$

Here, we have used (65) and (38) in the last step.

Finally, the surface tension term (67) has with the abbreviations

$$\tilde{v}(t,x) = \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix}, \quad \delta\tilde{v}(t,x) = \begin{pmatrix} -\nabla \delta h(t,x) \\ 0 \end{pmatrix}$$

by Theorem 13 and (41) the directional derivative

$$\begin{aligned} &\int_{J\times\mathbb{R}^n} \sigma\left(\frac{\delta\tilde{v}^\top}{|\tilde{v}|} - \frac{\delta\tilde{v}^\top\tilde{v}\tilde{v}^\top}{|\tilde{v}|^3}\right)(t,x)(D\varphi - \operatorname{div}(\varphi)I)(t,x,h(t,x))\tilde{v}(t,x)\,d(t,x) \\ &+ \int_{J\times\mathbb{R}^n} \sigma\frac{\tilde{v}^\top}{|\tilde{v}|}\Big(\partial_y(D\varphi - \operatorname{div}(\varphi)I)(t,x,h(t,x))\delta h(t,x)\tilde{v}(t,x) \\ &\qquad + (D\varphi - \operatorname{div}(\varphi)I)(t,x,h(t,x))\delta\tilde{v}(t,x)\Big)\,d(t,x). \end{aligned} \tag{82}$$

Now the first integral on the right hand side of (73) converges to the first intergal in (81) by first applying Lemma 17 and then Lemmas 18 and 19 . By using first Lemma 21 (note that $v_\varepsilon(t,x,y)$ depends close to $\Gamma(t)$ only on x by (69), see (70)), and then Lemma 18 and the fact that $\nabla\delta h$ is continuous by (41), the second integral on the right hand side of (73) converges to the second intergal in (81).

(74), (76) are obvious and (75) follows by Proposition 15.

Let vice versa (u, q, α) be a solution of the VoF-type formulation (57)–(59) and $(\delta u, \delta q, \delta\alpha)$ a solution of (73)–(76) with the regularities as in Theorem 13. By Theorem 20 $(u,\ q,\ h)$ coincides with the solution of (1) in Theorem 13 and (64) implies by Proposition 15 that $\delta\alpha$ and δh correspond to each other via (65). Hence, (73)–(76) ensure that $(u,\ q)$ satisfy the linearization of (1) on $\Omega(t)$ and that $\delta\alpha$ provides for given δu the correct δh.

It remains to show that (73)–(76) implies the correct linearized jump condition. Denote the tested surface tension term from (67) for $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$ by

$$K(h;\phi) := \int_{J\times\mathbb{R}^n} \sigma\operatorname{div}_x\left(\tfrac{\nabla h(t,x)}{\sqrt{1+|\nabla h(t,x)|^2}}\right)\begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix}^\top \phi(t,x)\,d(t,x).$$

The jump condition in strong form is equivalent to

$$-\int_{J\times\mathbb{R}^n} \phi(t,x)^\top\left[S(u,q;\mu)(t,x,h(t,x))\right]\begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} d(t,x) = K(h;\phi) \tag{83}$$

for all $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$ with $S(u,q;\mu) = -qI + \mu(\nabla u + \nabla u^\top)$. In the transformed variables the jump condition reads

$$-\int_{J\times\mathbb{R}^n} \phi(t,x)^\top\left[\hat{S}(\hat{u},\pi,h;\hat{\mu})(t,x,0)\right]\begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} d(t,x) = K(h;\phi)$$

for all $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$, where with the notation of Remark 1$\hat{S}(\hat{u},\pi,h;\hat{\mu}) = -\pi I + \hat{\mu}\mathcal{D}(\hat{u},h)$. Thus, the sensitivities $\delta\hat{u}, \delta\pi, \delta h$ satisfy the linearized jump condition

$$
\begin{aligned}
&- \int_{J\times\mathbb{R}^n} \phi(t,x)^\top \Bigg(\big[\partial_{(\hat u,\pi,h)} \hat S(\hat u,\pi,h;\hat\mu)\cdot(\delta\hat u,\delta\hat\pi,\delta h)(t,x,0)\big] \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} \\
&\qquad + \big[\hat S(\hat u,\pi,h;\hat\mu)(t,x,0)\big] \begin{pmatrix} -\nabla \delta h(t,x) \\ 0 \end{pmatrix} \Bigg)\, d(t,x) = \partial_h K(h;\phi)\cdot\delta h
\end{aligned} \tag{84}
$$

for all $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$. We show now, that under the regularity ensured by Theorem 13, (83) is implied by the weak formulation (73) by using test functions of the form

$$
\varphi_\tau(t,x,y) = \phi(t,x)\psi_\tau(y-h(t,x)) \tag{85}
$$

with $\phi \in C_c^\infty(J\times\mathbb{R}^n;\mathbb{R}^{n+1})$, $\psi_\tau(s)=\psi(s/\tau)$, $\psi\in C_c^1((-1,1))$, $\psi\ge 0$, $\psi(0)=1$, $\psi(-s)=\psi(s)$ for $\tau \searrow 0$. Then

$$
\begin{aligned}
&\nabla\varphi_\tau(t,x,y) = \psi_\tau'(y-h(t,x)) \begin{pmatrix} -\nabla h(t,x) \\ 1 \end{pmatrix} \phi(t,x)^\top + \psi_\tau(y-h(t,x)) \begin{pmatrix} \nabla\phi(t,x) \\ 0 \end{pmatrix}, \\
&\partial_t\varphi_\tau(t,x,y) = -\phi(t,x)\psi_\tau'(y-h(t,x))\partial_t h(t,x) + \partial_t\phi(t,x)\psi_\tau(y-h(t,x)).
\end{aligned}
$$

We test the weak form (57) also in time and rewrite it in the transformed variables. This results in

$$
\begin{aligned}
&\int_{J\times\mathbb{R}^{n+1}} \Big(\big(\partial_t(\rho(\alpha)u) + \operatorname{div}(\rho(\alpha)u\otimes u)\big)(t,x,y)^\top \varphi + S(u,q;\mu) : \nabla\varphi \Big) d(t,x,y) = \\
&\int_{J\times\mathbb{R}^{n+1}} \Big(-\hat\rho(x,y)\hat u^\top \big(\partial_t\varphi(t,x,y+h(t,x)) + \hat u\cdot\nabla\varphi(t,x,y+h(t,x))\big) \\
&\quad + \hat S(\hat u,\pi,h;\hat\mu)(t,x,y) : \nabla\varphi(t,x,y+h(t,x)) \Big)\, d(t,x,y) = K(h;\varphi(\cdot,\cdot,h(\cdot,\cdot))).
\end{aligned} \tag{86}
$$

For the right hand side we have used that as in the proof of Theorem 20 the right hand side of (57) coincides with (67).

For the test function (84) we obtain

$$
\begin{aligned}
&\partial_t\varphi_\tau(t,x,y+h(t,x)) + \hat u\cdot\nabla\varphi_\tau(t,x,y+h(t,x)) \\
&\quad = -\phi(t,x)\psi_\tau'(y)(\partial_t h(t,x) + v^\top\nabla h(t,x) - w) + \psi_\tau(y)(\partial_t\phi(t,x) + v\cdot\nabla\phi(t,x)).
\end{aligned}
$$

Moreover, for any $(\bar x,\bar y)\in\Gamma(\bar t)$ and $(x(t),\ y(t))$ with $(x'(t),y'(t)) = u(x(t),y(t))$, $(x(\bar t),y(\bar t)) = (\bar x,\bar y)$ one has $y(t)-h(t,x(t))=0$ and thus

$$
\begin{aligned}
0 &= y'(t) - \partial_t h(t,x(t)) - \partial_x h(t,x(t))x'(t) \\
&= w(t,x(t),y(t)) - \partial_t h(t,x(t)) - v(t,x(t),y(t))^\top\nabla h(t,x(t)).
\end{aligned}
$$

Hence,

$$
\partial_t\varphi_\tau(t,x,y+h(t,x)) + \hat u\cdot\nabla\varphi_\tau(t,x,y+h(t,x)) = \psi_\tau(y)(\partial_t\phi(t,x) + v\cdot\nabla\phi(t,x)),
$$

and inserting φ_τ in (85) yields

$$\int_{J\times\mathbb{R}^{n+1}} \Big(-\hat\rho(x,y)\hat u^\top \psi_\tau(y)(\partial_t\phi(t,x)+v\cdot\nabla\phi(t,x))$$
$$+\hat S(\hat u,\pi,h;\hat\mu) : \Big(\phi(t,x)\psi_\tau'(y)\begin{pmatrix}-\nabla h(t,x)\\ 1\end{pmatrix} + \psi_\tau(y)\begin{pmatrix}\nabla\phi(t,x)\\ 0\end{pmatrix}\Big)\Big)\, d(t,x,y) = K(h;\phi).$$

We have already observed that the right hand side of (57) coincides with (67) and the right hand side of (73) with the derivative (81) of (67). Since $K(h;\phi)$ corresponds to using φ_τ in (67), $\partial_h K(h;\phi)\cdot\delta h$ can be expressed as the sum of the right hand side of (73) with $\varphi=\varphi_\tau$ and of the right hand side of (57) with $\varphi=-\partial_y\varphi_\tau\delta h$ (note again that φ_τ depends on h).

Similarly taking the derivative of the left hand side of (85) corresponds to the sum of the left hand side of (73) with $\varphi=\varphi_\tau$ and of the left hand side of (85) with $\varphi=-\partial_y\varphi_\tau\delta h$ (note that φ_τ depends on h) and is given by (we differentiate equivalently the middle term of (85) in transformed variables)

$$\int_{J\times\mathbb{R}^{n+1}} \Big(-\hat\rho(x,y)\psi_\tau(y)\big(\delta\hat u^\top(\partial_t\phi(t,x)+v\cdot\nabla\phi(t,x)) + \hat u^\top(\partial_t\phi(t,x)+\delta v\cdot\nabla\phi(t,x))\big)$$
$$+\partial_{(\hat u,\pi,h)}\hat S(\hat u,\pi,h;\hat\mu)(x,0)\cdot(\delta\hat u,\delta\hat\pi,\delta h) : \Big(\phi(t,x)\psi_\tau'(y)\begin{pmatrix}-\nabla h(t,x)\\ 1\end{pmatrix} + \psi_\tau(y)\begin{pmatrix}\nabla\phi(t,x)\\ 0\end{pmatrix}\Big)$$
$$+\hat S(\hat u,\pi,h;\hat\mu)(x,0) : \phi(t,x)\psi_\tau'(y)\begin{pmatrix}-\nabla\delta h(t,x)\\ 0\end{pmatrix}\Big)\, d(t,x,y) = \partial_h K(h;\phi)\cdot\delta h.$$

By the assumed regularity for $\tau\searrow 0$ all terms containing the factor $\psi_\tau(y)$ tend to zero and the remaining terms converge to

$$\int_{J\times\mathbb{R}^n} \Big(-\big[\partial_{(\hat u,\pi,h)}\hat S(\hat u,\pi,h;\hat\mu)(x,0)\cdot(\delta\hat u,\delta\hat\pi,\delta h)\big] : \begin{pmatrix}-\nabla h(t,x)\\ 1\end{pmatrix}$$
$$-\big[\hat S(\hat u,\pi,h;\hat\mu)(x,0)\big] : \begin{pmatrix}-\nabla\delta h(t,x)\\ 0\end{pmatrix}\Big)\phi(t,x)\, d(t,x) = \partial_h K(h;\phi)\cdot\delta h.$$

This is exactly the linearized jump condition (83). □

4 Analytical settings for the application of optimization methods

The results of this paper justify the application of derivative based optimization methods. We discuss now some possible settings. While some are canonical, the treatment of optimization problems involving the state across the interface, in particular the pressure or the position of the interface, requires care, since the pressure and phase indicator field are discontinuous across the interface.

Let $p>n+3$ and $\mathbb{U}_u(h_0), \mathbb{U}_c(t_0)$ be defined by (35). Let $h_0\in\mathbb{U}_h$ with $\|h_0\|_{\mathbb{U}_h}$ small enough

$$U_{ad} := \left\{(u_0, c) \in \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) : (u_0, h_0, c) \text{ satisfy (14), (51) }\right\}$$

and consider the control-to-state mapping

$$(u_0, c) \in U_{ad} \subset \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) \mapsto (\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$$

given by (10), which is differentiable at least for sufficiently small controls by Theorem 13, and the corresponding original state

$$(u, q)(t, x, y) = (\hat{u}, \pi)(1, x, y - h(t, x))$$

solving (1). Denote the solution operator by

$$(u, q) = (\mathcal{S}^u(u_0, c), \mathcal{S}^q(u_0, c)) = \mathcal{S}(u_0, c).$$

Now we are interested in optimization problems of the form

$$\min_{(u_0,c)} \mathcal{J}(\mathcal{S}(u_0, c), u_0, c) \text{ s.t. } (u_0, c) \in U_{ad},$$

where additional constraints would be possible. We discuss now analytical settings, for which the continuous differentiability of the reduced objective function $(u_0, c) \in U_{ad} \mapsto \mathcal{J}(\mathcal{S}(u_0, c), u_0, c)$ is ensured.

4.1 Objective functions involving the velocity field

We consider first the simpler case, where the state-dependence of the objective function involves only the velocity, i.e.,

$$(u_0, c) \in U_{ad} \mapsto \mathcal{J}(\mathcal{S}(u_0, c), u_0, c) = \mathcal{J}^u(\mathcal{S}^u(u_0, c), u_0, c). \tag{87}$$

Using the differentiability results of Theorem 13 with the space given in (43), the reduced objective functional (86) is continuously differentiable, if the mapping

$$(u, u_0, c) \in C\big(\bar{J}; L_{\tilde{p}}\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\big) \times \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0) \mapsto \mathcal{J}^u(u, u_0, c)$$

is continuously differentiable for some $\tilde{p} \in [p, \infty)$. This applies for example in the case of least squares functionals or many other types of tracking functionals. The derivative is easily obtained by the chain rule and by using the sensitivities δu, where δu can be obtained by the VoF-type formulation (73) or by using the linearization of the transformed problem (10) together with Lemma 11 and Lemma 12.

If the objective function (86) evaluates the velocity field u only in an open observation domain $J \times \Omega_o$ with positive distance from the interface $\bigcup_{t \in J}(\{t\} \times \Gamma(t))$, then it is by Theorem 13 sufficient if the mapping

$$(u, u_0, c) \in \Big(C\big(\bar{J}; L_{\tilde{p}}\big(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}\big)\big) \cap L_p\Big(J; H_p^1\big(\Omega_o, \mathbb{R}^{n+1}\big)\Big)\Big) \times \mathbb{U}_u(h_0) \times \mathbb{U}_c(t_0)$$
$$\mapsto \mathcal{J}^u(u, u_0, c)$$

is continuously differentiable for some $\tilde{p} \in [p, \infty)$.

4.2 Objective functions involving the pressure or phase indicator

Since the pressure field is discontinuous across the interface, the differentiability results of Theorem 13 apply only for extensions $q_{\pm}$ of the pressure across the interface. We show now that certain types of objective functionals are nevertheless differentiable. As before we work with the unique representative of the pressure satisfying (36).

We consider objective functions of the form

$$\begin{aligned}(u_0, c) \in U_{ad} \mapsto \mathcal{J}(\mathcal{S}(u_0, c), u_0, c) &= \mathcal{J}^q(\mathcal{S}^q(u_0, c), u_0, c)\\ &= \int_J \int_{\mathbb{R}^{n+1}} \ell(\mathcal{S}^q(u_0, c)(t, x, y), t, x, y)\, d(x, y)\, dt.\end{aligned} \tag{88}$$

Here, $\ell : \mathbb{R} \times J \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a continuous function, twice differentiable with respect to q and y such that there are $R > 0, C_\ell > 0$ with (other settings are possible)

$$\begin{aligned}&\operatorname{supp}(\ell(q, \cdot)) \subset \{(t, x, y) \in J \times \mathbb{R}^n \times \mathbb{R} : |(x, y)| \le R\} \quad \forall\, q \in \mathbb{R},\\ &|\partial^2_{(q,y)} \ell(q, t, x, y)| \le C_\ell\left(1 + |q|^{p-2}\right) \quad \forall\, (t, x, y) \in J \times \mathbb{R}^n \times \mathbb{R}.\end{aligned} \tag{89}$$

This implies the estimate

$$|\partial_q \ell(q, t, x, y)| \le |\partial_q \ell(0, t, x, y)| + \left| \int_0^q \partial^2_{qq} \ell(s, t, x, y)\, ds \right| \le C'_\ell (1 + |q|^{p-1})$$

with the constant $C'_\ell = \|\partial_q \ell(0, \cdot)\|_{L_\infty(\mathbb{R}^{n+1})} + 2C_\ell$ as well as

$$|\ell(q, x, y)| \le |\ell(0, x, y)| + \left| \int_0^q \partial_q \ell(s, x, y)\, ds \right| \le C''_\ell (1 + |q|^p) \tag{90}$$

with $C''_\ell = \|\ell(0, \cdot)\|_{L_\infty(\mathbb{R}^{n+1})} + 2C'_\ell$.

Now, using the transformation $(\hat{x}, \hat{y}) = T_{h(t)}(x, y) = (x, y - h(t, x))$ we can rewrite (87) as

$$\begin{aligned}\mathcal{J}^q(q, u_0, c) &= \int_J \int_{\mathbb{R}^{n+1}} \ell(q(t, x, y), t, x, y)\, d(x, y)\, dt\\ &= \int_J \int_{\mathbb{R}^{n+1}} \ell(\pi(t, x, y - h(t, x)), t, x, y)\, d(x, y)\, dt\\ &= \int_J \int_{\mathbb{R}^{n+1}} \ell(\pi(t, \hat{x}, \hat{y}), t, \hat{x}, \hat{y} + h(t, \hat{x}))\, d(\hat{x}, \hat{y})\, dt\\ &= \int_J \int_{\dot{R}^{n+1}} \ell(\pi(t, \hat{x}, \hat{y}), t, \hat{x}, \hat{y} + h(t, \hat{x}))\, d(\hat{x}, \hat{y})\, dt.\end{aligned} \tag{91}$$

We will now show the following result.

Theorem 23 *Let* (88) *hold. Then with the convention* (36) *the objective function* (87) *is continuously differentiable with derivative*

$$
\begin{aligned}
(\mathcal{J}^q)'\big(\mathcal{S}^q(u_0,c),u_0,c\big)\cdot\begin{pmatrix}\delta u_0\\ \delta c\end{pmatrix} &= \int_J\int_{\dot{\mathbb{R}}^{n+1}} \partial_q \ell(q(t,x,y),t,x,y)\delta q(t,x,y)\,d(x,y)\,dt\\
&\quad - \int_J\int_{\mathbb{R}^n} \big[\ell(q(t,x,h(t,x)),t,x,h(t,x))\big]\,\delta h(t,x)\,dx\,dt\\
&= \int_J\int_{\dot{\mathbb{R}}^{n+1}} \partial_q \ell(q(t,x,y),t,x,y)\delta q(t,x,y)\,d(x,y)\,dt\\
&\quad - \int_J\int_{\mathbb{R}^n} \big[\ell(q(t,x,h(t,x)),t,x,h(t,x))\big]\,d\delta\alpha(t)(x,y)\,dt,
\end{aligned}
$$

where $[\ell(q(t,x,h(t,x)),t,x,h(t,x))]$ *is the jump across* $\Gamma(t)$ *at* $(x,h(t,x))$ *and the sensitivities* δq, $\delta\alpha$ *can be obtained by the VoF-type formulation* (73) *or* $\delta\pi$, δh *can be computed by using the linearization of the transformed problem* (10) *together with Lemma* 12 *and* δq *by using Lemma* 11.

Remark 24 If we consider an objective function $\mathcal{J}^\alpha$ of the form (87) with the pressure q replaced by the phase indicator α then an analogue of Theorem 23 holds and the derivative simplifies to

$$
\begin{aligned}
&(\mathcal{J}^\alpha)'(\mathcal{S}^\alpha(u_0,c),u_0,c)\cdot\begin{pmatrix}\delta u_0\\ \delta c\end{pmatrix}\\
&\quad = -\int_J\int_{\mathbb{R}^n} [\ell(\alpha(t,x,h(t,x)),t,x,h(t,x))]\,d\delta\alpha(t)(x,y)\,dt,
\end{aligned}
$$

since $\delta\alpha$ has its support on the complement of $\dot{\mathbb{R}}^{n+1}$.

Remark 25 A practically relevant example satisfying (88) is a tracking type functional for the pressure q or the phase indicator α of the form

$$
\mathcal{J}^q\big(\mathcal{S}^q(u_0,c),u_0,c)\big) = \int_J\int_{\mathbb{R}^{n+1}} \psi(x,y)\big(\mathcal{S}^q(u_0,c)-q_d\big)^2(t,x,y)\,d(x,y)\,dt,
$$

with desired pressure field $q_d \in BC^2(J\times\mathbb{R}^{n+1})$ and a weighting function $\psi\in C_c^2(\mathbb{R}^{n+1})$ (or similarly with $\mathcal{S}^\alpha$ and desired phase indicator $\alpha_d\in BC^2(J\times\mathbb{R}^{n+1};[0,1])$). Note that discontinuous q_d can lead to nonsmoothness if the jump set of q and q_d coincide on a set of positiv surface measure.

In order to cover discontinuous q_d (or α_d) a variant with smoothed observation of the form

$$
\mathcal{J}^q\big(\mathcal{S}^q(u_0,c),u_0,c)\big) = \int_J\int_{\mathbb{R}^{n+1}} \psi(x,y)\big(\phi * \big(\mathcal{S}^q(u_0,c)-q_d\big)\big)^2(t,x,y)\,d(x,y)\,dt
$$

with a mollifier $\phi\in C_c^2(\mathbb{R}^{n+1})$ (the convolution is only in space) can be shown to be continuously differentiable similarly as for (87) under assumption (88).

We use the following auxiliary result.

Lemma 26 *Let* (88) *hold. Then the mapping*

$$\begin{aligned}&(\pi,h)\in L_p(J;L_\infty(\mathbb{R}^{n+1}))\times L_p(J;L_\infty(\mathbb{R}^n))\\&\mapsto \widehat{\mathcal{J}}^q(\pi,h):=\int_J\int_{\dot{R}^{n+1}}\ell(\pi(t,\hat{x},\hat{y}),t,\hat{x},\hat{y}+h(t,\hat{x}))\,d(\hat{x},\hat{y})\,dt\end{aligned}$$

is continuously differentiable with derivative

$$\begin{aligned}&(\widehat{\mathcal{J}}^q)'(\pi,h)\cdot\begin{pmatrix}\delta\pi\\\delta h\end{pmatrix}\\&=\int_J\int_{\dot{R}^{n+1}}\begin{pmatrix}\partial_q\ell\\\partial_y\ell\end{pmatrix}(\pi(t,\hat{x},\hat{y}),t,\hat{x},\hat{y}+h(t,\hat{x}))^\top\begin{pmatrix}\delta\pi(t,\hat{x},\hat{y})\\\delta h(t,\hat{x})\end{pmatrix}d(\hat{x},\hat{y})\,dt.\end{aligned}$$

Proof We have by using (88) and its consequence (89)

$$\begin{aligned}\widehat{\mathcal{J}}^q(\pi,h)&\le\int_J\int_{\dot{R}^{n+1}}|\ell(\pi(t,x,y-h(t,x)),t,x,y)|\,d(x,y)\,dt\\&\le\int_J\int_{|(x,y)|\le R}C''_\ell(1+|\pi(t,x,y-h(t,x))|^p)\,d(x,y)\,dt\\&\le C''_\ell(2R)^{n+1}\int_J(1+\|\pi(t,\cdot)\|^p_{L_\infty(\mathbb{R}^{n+1})})\le C''_\ell(2R)^{n+1}(t_0+\|\pi\|^p_{L_p(J;L_\infty(\mathbb{R}^{n+1}))}).\end{aligned}$$

To show the differentiability, we note that by Taylor expansion and (89)

$$\begin{aligned}&\Bigg|\ell((\pi+\delta\pi)(t,\hat{x},\hat{y}),t,\hat{x},\hat{y}+(h+\delta h)(t,\hat{x}))-\ell(\pi(t,\hat{x},\hat{y}),t,\hat{x},\hat{y}+h(t,\hat{x}))\\&\quad-\begin{pmatrix}\partial_q\ell\\\partial_y\ell\end{pmatrix}(\pi(t,\hat{x},\hat{y}),t,\hat{x},\hat{y}+h(t,\hat{x}))^\top\begin{pmatrix}\delta\pi(t,\hat{x},\hat{y})\\\delta h(t,\hat{x})\end{pmatrix}\Bigg|\\&\le C_\ell(1+(|\pi(t,\hat{x},\hat{y})|+|\delta\pi(t,\hat{x},\hat{y})|)^{p-2})(|\delta\pi(t,\hat{x},\hat{y})|^2+|\delta h(t,\hat{x})|^2).\end{aligned}$$

Hence, we obtain with the asserted derivative by using (88)

$$\begin{aligned}&\left|\widehat{\mathcal{J}}^q(\pi+\delta\pi,h+\delta h)-\widehat{\mathcal{J}}^q(\pi,h)-(\widehat{\mathcal{J}}^q)'(\pi,h)\cdot\begin{pmatrix}\delta\pi\\\delta h\end{pmatrix}\right|\\&\le\int_J\int_{\substack{\dot{\mathbb{R}}^{n+1}\cap\\ [-R,R]^{n+1}}}C_\ell\Big(1+(|\pi(t,\hat{x},\hat{y})|+|\delta\pi(t,\hat{x},\hat{y})|)^{p-2}\Big)\Big(|\delta\pi(t,\hat{x},\hat{y})|^2+|\delta h(t,\hat{x})|^2\Big)\,d(\hat{x},\hat{y})\,dt\\&\le(2R)^{n+1}C_\ell\int_J\Big(1+\||\pi(t,\cdot)|+|\delta\pi(t,\cdot)|\|^{p-2}_{L_\infty}\Big)\cdot\Big(\|\delta\pi(t,\cdot)\|^2_{L_\infty}+\|\delta h(t,\cdot)\|^2_{L_\infty}\Big)\,dt\\&\le(2R)^{n+1}C_\ell\Big(t_0^{(p-2)/p}+\||\pi|+|\delta\pi|\|^{p-2}_{L_p(J;L_\infty)}\Big)\Big(\|\delta\pi\|^2_{L_p(J;L_\infty)}+\|\delta h\|^2_{L_p(J;L_\infty)}\Big)\\&=O\Big(\|\delta\pi\|^2_{L_p(J;L_\infty)}+\|\delta h\|^2_{L_p(J;L_\infty)}\Big),\end{aligned}$$

where we have applied Hölder's inequality in the last step. The continuity of the derivative follows with (89) by very similar calculations. □

Proof of Theorem 23 From Theorem 8, Theorem 10 and Lemma 12 we know that

$$(u_0, c) \in U_{ad} \mapsto (\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0)$$

is continuously differentiable and with the convention (36) the mapping $(\hat{u}, \pi, [\pi], h) \in \mathbb{E}(t_0) \mapsto \pi \in L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1}))$ is linear and continuous by (37). Moreover, by $p > n+3$ we have clearly $L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1})) \hookrightarrow L_p(J;L_\infty(\mathbb{R}^{n+1}))$ and $\mathbb{E}_4(t_0) \hookrightarrow L_p(J;L_\infty(\mathbb{R}^n))$. Hence, the mapping

$$(u_0, c) \in U_{ad} \mapsto (\pi, h) \in L_p(J;L_\infty(\mathbb{R}^{n+1})) \times L_p(J;L_\infty(\mathbb{R}^n))$$

is continuously differentiable and thus (87) is continuously differentiable by Lemma 26 and the last representation of (87) in (90). The derivative is given by Lemma 26. Using that

$$\begin{aligned}\frac{d}{d\hat{y}}\ell(\pi(t,\hat{x},\hat{y}), t, \hat{x}, \hat{y} + h(t,\hat{x})) &= \partial_y \ell(\pi(t,\hat{x},\hat{y}), t, \hat{x}, \hat{y} + h(t,\hat{x})) \\ &\quad + \partial_q \ell(\pi(t,\hat{x},\hat{y}), t, \hat{x}, \hat{y} + h(t,\hat{x}))\partial_{\hat{y}}\pi(t,\hat{x},\hat{y}),\end{aligned}$$

integration by parts yields

$$\begin{aligned}(\hat{\mathcal{J}}^q)'(\pi, h) \cdot \begin{pmatrix} \delta\pi \\ \delta h \end{pmatrix} &= \int_J \int_{\dot{R}^{n+1}} \begin{pmatrix} \partial_q \ell \\ \partial_y \ell \end{pmatrix} (\pi(t,\hat{x},\hat{y}), t, \hat{x}, \hat{y} + h(t,\hat{x}))^\top \begin{pmatrix} \delta\pi(t,\hat{x},\hat{y}) \\ \delta h(t,\hat{x}) \end{pmatrix} d(\hat{x},\hat{y})\, dt \\ &= \int_J \int_{\dot{R}^{n+1}} \partial_q \ell(\pi(t,\hat{x},\hat{y}), t, \hat{x}, \hat{y} + h(t,\hat{x}))\big(-\partial_{\hat{y}}\pi(t,\hat{x},\hat{y})\delta h(t,\hat{x}) + \delta\pi(t,\hat{x},\hat{y})\big)\, d(\hat{x},\hat{y})\, dt \\ &\quad - \int_J \int_{\mathbb{R}^n} [\ell(\pi(t,\hat{x},0), t, \hat{x}, h(t,\hat{x}))]\delta h(t,\hat{x})\, d\hat{x}\, dt \\ &= \int_J \int_{\mathbb{R}^{n+1}} \partial_q \ell(q(t,x,y), t, x, y)\delta q(t,x,y)\, d(\hat{x},\hat{y})\, dt \\ &\quad - \int_J \int_{\mathbb{R}^n} \big[\ell(q(t,x,h(t,x)), t, x, h(t,x))\big]\delta h(t,x)\, dx\, dt.\end{aligned}$$

Here, we have used Lemma 11 and (65) in the last step. □

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Authors and Affiliations

Elisabeth Diehl[1] · Johannes Haubner[2] · Michael Ulbrich[2] · Stefan Ulbrich[1]

Elisabeth Diehl
diehl@mathematik.tu-darmstadt.de

Johannes Haubner
haubnerj@ma.tum.de

Michael Ulbrich
mulbrich@ma.tum.de

[1] Department of Mathematics, TU Darmstadt, Dolivostr. 15, 64293 Darmstadt, Germany

[2] Department of Mathematics, TU München, Boltzmannstr. 3, 85748 Garching b. München, Germany