Simulation-Based Analysis of Delay-Reducing Quantum Data Plane Repeater Protocols

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***Abstract*—Emerging communication systems are expected to integrate new types of communication networks including quantum networks. In contrast to classical networks, the communication in quantum networks is accomplished with qubits and their attribute entanglement. Entanglement is a type of correlation between multiple qubits with no counterpart in classical communication. With the help of entanglement new protocols are developed to perform classical and quantum network tasks more efficiently. The main task of quantum networks is to share entanglement to distant communication partners with the help of intermediate nodes and repeated entanglement swapping, the so called Repeater Protocol. Due to the decoherence of quantum states, establishing entanglement has to be accomplished with minimal delay to preserve the states of qubits. This paper proposes a new Repeater Protocol FRP, which reduces delay by up to 51% and on average by 13% in establishing end-to-end entanglement compared with the direct approach CRP. Additionally, the provided solution lowers hardware requirements for quantum routers. Both protocols, CRP and FRP are evaluated in terms of end-to-end delay and fidelity in extensive simulations with a parameter search on 3 different topologies.**

***Index Terms*—Quantum Networks, Quantum Routing, Quantum Data Plane**

I. Introduction

Quantum communication provides a new way of processing and distributing information. The key ingredients of quantum communication are qubits and their attribute entanglement. Entanglement is a special form of correlation between two or more qubits, which has no counterpart in classical communications [1]. With the help of entanglement, new forms of communication protocols have been developed, namely quantum key distribution (QKD) [2], distributed quantum computing [3], quantum clock synchronization [4] and distributed quantum sensing [5]. To enable these quantum application protocols (QAP), quantum networks are necessary for providing the required quantum and classical communication.

The goal of quantum networks is to provide high quality or fidelity entanglement between distant communication partners to allow the execution of QAPs. To achieve this goal, the nodes in a quantum network can be categorized into quantum clients and quantum routers [6]. In quantum networks, the clients are capable of executing QAPs, while quantum routers enable the necessary quantum and classical communication to provide high fidelity entanglement for different QAPs. However, the quantum and classical communication between nodes in networks need to follow structured and cohesive rules in order to ensure correct transmission of information. The collection of rules in a quantum network are named quantum network protocols (QNP) [7]. QNPs are different protocols employed for specific networking tasks, for example, quantum routing [8] or quantum error correction [9]. The quantum routers then employ multiple different QNPs to provide the communication for a specific QAP with certain fidelity requirements.

The main task in a quantum network is quantum routing. Quantum routing is capable of finding a path and establishing End-to-End (E2E) entanglement from a source to a destination over an arbitrary network topology. To achieve the establishment of entanglement between two clients, first entanglement between neighboring routers are established. Then, a path through the network over multiple quantum routers is calculated. To finally establish E2E entanglement, the repeated application of entanglement swapping [10] is employed, the so called Repeater Protocol. Entanglement swapping is the process of establishing entanglement between two nodes interconnected by an intermediate router. This procedure requires a Bell-State measurement (BSM) at the router, sending the measurement results to the next node using a classical communication channel and an application of a Pauli gate at the next router to recover the original Bell-State.

However, establishing E2E entanglement is challenging due to multiple reasons. First, entanglement is susceptible to decoherence [11], a degrading of fidelity over time, requiring the storage of qubits in quantum memories with low error rates or equivalently high coherence times. Second, due to the degradation of fidelity over time, the quantum and classical communication between clients needs to be achieved with minimal delay, requiring fast E2E entanglement establishment. Last, the hardware requirements for network components, such as quantum routers, needs to be as low as possible, in order to provide communication resources for future demands. Therefore, QNPs, in particular quantum routing, need to be fast and efficient in providing high fidelity entanglement, while being optimized for low hardware requirement network components. However, current solutions to quantum routing [8], [12]–[14] focus on the quantum control plane, i.e the communication of control information. The solutions are mainly centralized approaches, proposing new path metrics, focusing on maximizing E2E entanglement rate and/or fidelity. These approaches may be viable for small scale networks, with distances in the low kilometer ranges, but large scale quantum networks with distances in the $100\,\text{km}$ to $1000\,\text{km}$ require distributed approaches to satisfy delay and fidelity requirements of QAPs.

Therefore, in this paper we present a new distributed Repeater Protocol operating in the quantum data plane, aiming to reduce E2E delay in establishing entanglement and lowering hardware requirements for quantum routers. The main idea in the new Repeater Protocol is the skipping of Pauli gate applications at routers in the path. That way only the measurement results of BSMs at each router need to be collected and can be reduced to an application of one Pauli gate at the receiver. Therefore, quantum routers only need to apply BSMs in order to establish E2E entanglement between a source and destination. The two main messages of this paper are:

- The BSM and Pauli gate duration have a significant influence on the E2E delay.
- The coherence time of quantum memories saturates around $1\,\text{s}$.

In particular the contributions are as follows:

- We introduce a new repeater protocol FRP i.e. a QNP, in order to reduce E2E delay.
- We evaluate our approach and compare it to a baseline protocol in an extensive parameter search.
- We present formulas for calculating the resulting fidelity from the initial fidelities of one and multiple BSMs (Eq. 3 and Eq. 4).

The remainder of this paper is structured as follows. First, a short introduction in Sec. II to the mathematical representation of qubits and the entanglement swapping is given. Next, the current state-of-the-art of quantum routing is presented in Sec. III. Then, the new Repeater Protocol is proposed in Section IV, followed by a thorough evaluation in Sec. V. Finally, Sec. VI concludes this paper.

II. Background

This section gives a short description of fidelity [15] and entanglement swapping [10], [16].

A. Fidelity

The fidelity F between two qubit systems, ϕ, ψ computes the closeness between their respective states [15]. The fidelity is a measure between 0 and 1 and is calculated as follows:

$$F(\phi,\psi) = tr(\sqrt{\sqrt{\phi}\psi\sqrt{\phi}})^2, \tag{1}$$

where ϕ and ψ is the density matrix of the first and second quantum state, respectively. The fidelity of an arbitrary state can be measured with Eq. 1 if the state is compared to a noiseless state.

B. Entanglement Swapping

Entanglement swapping enables establishment of entanglement over larger distances by using intermediate repeaters/routers [10], [16]. Entanglement swapping requires at minimum 4 qubits q_1, q_2, q_3, q_4, where qubits q_1 and q_2 are entangled in $|\Phi^+\rangle$ and q_3 and q_4, both pairs in $|\Phi^+\rangle$.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2}$$

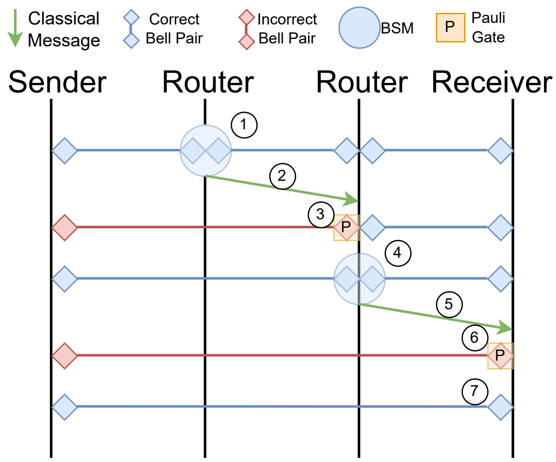
The goal is to entangle qubits q_1 and q_4 in the state $|\Phi^+\rangle$, without them interacting or sending qubit q_2 or q_3 to the receiver. First, q_1 is located at the sender and q_4 is located at the receiver. Qubits q_2 and q_3 are located at the intermediate router. The router then performs a BSM on q_2 and q_3, which results in two classical bits. The two classical bits indicate whether the amplitude or phase or both should be flipped. The measurement result is then sent to the receiver, where he applies a Pauli gate based on the measurement result to q_4, to recover $|\Phi^+\rangle$, resulting in entanglement between q_1 and q_4. Entanglement swapping can be performed on any of the four Bell-States, however the mapping of measurement result to Pauli gate changes depending on the used states. Furthermore, the deferred measurement principle [17] cannot be employed, as it would require CNOT and CZ gates between separated qubits. The application of CNOT/CZ gates can be either accomplished by sending one of the qubits to the receiver, defeating the purpose of entanglement swapping or remote CNOT/CZ gates, which require additional entanglement and the same protocol as entanglement swapping.

III. Related Work

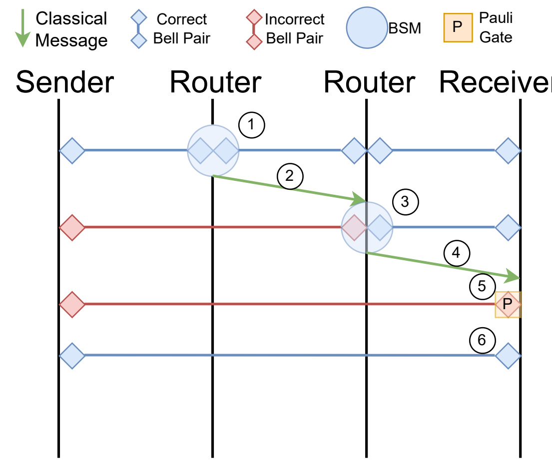
Current work focuses on QNPs operating in the control plane, i.e. the distribution of control information, for example routing information. For example, the authors of [12] design a routing algorithm (EFiRAP) considering throughput as well as fidelity, while incorporating purification strategies for each link. EFiRAP increases throughput by 50% compared to state-of-the-art protocols, while ensuring fidelity requirements. However, the number of SD-pairs is limited to 20 and the evaluated network scale and topology is not mentioned, while claiming throughput improvements. Furthermore, the duration of BSM or Pauli gates are not mentioned, as they can significantly influence E2E delay and fidelity.

The authors of [18] focus on the order in which BSMs are performed. They incorporate the entanglement generation probability in conjunction with the BSM success probability to proof that the order, in which BSMs are performed has an impact on the resulting expected number of entanglements at end nodes. However, to take order into account, a request from a host has to be potentially forwarded through a router without performing the Repeater Protocol, while only improving throughput marginally. This approach induces unnecessary overhead in establishing entanglement as routers might have to wait for requests from routers further down the path.

The authors of [19] present algorithms to reliably and fairly distribute entanglement in large-scale networks. In their algorithms, a requests is sent from the source to the destination and back to allocate entanglement along the path. Then the entanglement swapping in the path is performed, while the measurement results are sent back to the sender. They evaluate their approach based on fairness, throughput and fidelity, in



(a) Classical Repeater Protocol (CRP).



(b) Fast Repeater Protocol (FRP).

Fig. 1: Repeater protocols

a linear, asymmetric-dumbbell-shaped and large-scale topology. However, the forward-backward propagation of requests introduces significant E2E delay for quantum clients, as the requests have to be sent multiple times from the source to the destination. Furthermore, their largest network, the large-scale topology containing 50 nodes, is limited to 17 km in diameter, while serving 9 entanglement requests.

Our work differs from the presented work by focusing on the optimization of entanglement swapping in the quantum data plane in arbitrary networks to reduce delay and hardware requirements at routers. The core idea is to skip applying Pauli gates before a BSM and only applying a Pauli gate at the receiver, reducing E2E delay and hardware requirements. Furthermore, the simulation incorporates imperfect entanglement generation and distribution, imperfect BSMs and the resulting classical communication. The parameters influencing E2E delay and fidelity, namely the BSM and Pauli gate duration, BSM success probability and coherence times of quantum memories are varied in a large parameter search. The skipping of Pauli gates is also shortly described in [12] and [19], however, the presented algorithms are not compared to the direct approach. Furthermore, the protocols are not evaluated in large-scale networks with up to 100 nodes and 300 km network diameters, while important parameters, such as BSM and Pauli gate duration, are not mentioned.

IV. Repeater Protocols

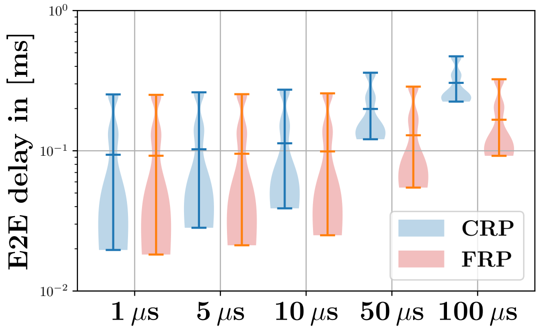
This section proposes a new Repeater Protocol, named Fast Repeater Protocol (FRP). *FRP* aims to reduce delay in establishing End-to-End (E2E) entanglement over the direct approach, while also reducing hardware requirements for routers. The Classical Repeater Protocol (CRP) [16] is the direct implementation of the Repeater Protocol in a network as shown in Fig. 1a. The Repeater Protocol variants based on the doubling architecture [20] or processing BSMs in parallel are not considered, as they require a central controller only suitable for small scale networks.

Both protocols assume pre-established entanglement between neighboring nodes on the path from the sender to the receiver. *CRP* begins by performing a BSM at the router closest to the sender (1). The classical result from the BSM is then forwarded to the next router in the path (2). Before the next router performs the BSM, it applies a corrective gate to the sender sided qubit based on the received measurement result from the previous router (3). This procedure continues until the last router has performed the BSM (4) and has forwarded the result to the receiver (5). The receiver applies a corrective gate to his qubit (6) and establishes E2E entanglement with the sender (7).

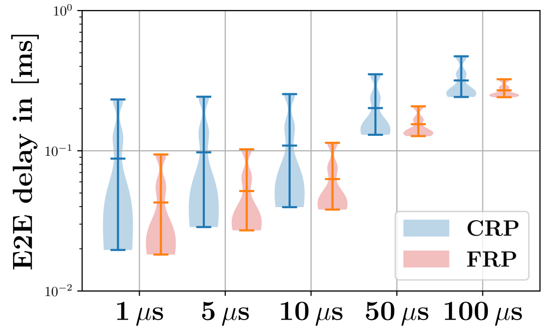
FRP (depicted in Fig. 1b) skips the application of gates at each router and directly applies a BSM (3) to the qubits. *CRP* has to apply as many Pauli gates as there are BSMs in the path, while *FRP* only has to apply one Pauli gate at the receiver to recover the original Bell-State. The validity of *FRP* can be easily verified via the commutativity of identity matrices in the construction of BSM operators. As the measurement results from a BSM consist of an amplitude and/or phase flip, the classical communication for *FRP* has to only keep track, whether the amplitude and/or the phase need to be flipped. Therefore, the hardware implementation of Pauli gates is not necessary on routers, as only BSMs need to be implemented. The implementation of Pauli gates is only necessary at clients.

V. Evaluation

Both Repeater Protocols are evaluated and compared in custom implemented Python simulations on three different network topologies, namely the Dense Random Graph, Sparse Random Graph and German Backbone topology [21]. The protocols are evaluated based on the E2E delay and secondary based on the E2E fidelity. The distribution of qubits are assumed as photons over optical channels, while the storage qubits are atoms. Each simulation consists of a *distribution phase* and a *request phase*. The *distribution phase* distributes entanglement between all adjacent nodes in a way that each entanglement request can be fulfilled. In the *request phase*,

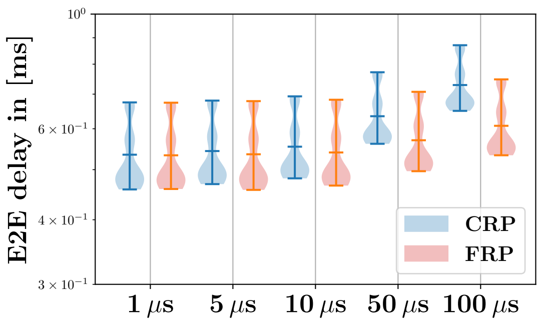


(a) E2E Delay for Pauli duration.

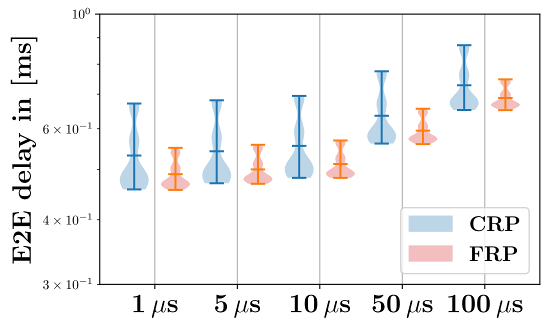


(b) E2E Delay for BSM duration.

Fig. 2: E2E Delay for Dense Random Graph.



(a) E2E Delay for Pauli duration.



(b) E2E Delay for BSM duration.

Fig. 3: E2E Delay for Sparse Random Graph.

each host requests to establish entanglement with another random host.

The paths for each request are pre-calculated and are based on the Dijkstra-algorithm [22] with the distances as the metric. The routers perform one of the Repeater Protocols and store the measurement results in the request and forward the request to the next router. The simulation finishes when all receivers have the requests from the senders. To make bidirectional communication possible, each router has a send and receive quantum memory for each connection. The qubits in the individual memories are added and removed in a FIFO manner. The qubits for the BSM are then easy to select, as the first qubit is taken from the receive memory of the connection the request came from, while the second qubit is taken from the send memory where the request should be forwarded.

To incorporate realistic network conditions, the simulation applies errors to stored qubits. The error model is based on [23], which is a combination of depolarization and dephasing errors with a common coherence time. The initial fidelities of a connection are uniformly distributed from 0.75 to 1.0 with a variance of 0.01. The duration of a Pauli gate, the duration of a BSM, the success probability of projecting onto a Bell-State and coherence time are varied in a parameter search. To unify the protocols for all network nodes, the Bell-State established between nodes and the E2E Bell-State are $|\Phi^+\rangle$. The parameter ranges are listed in Tab. I. For a simulation setting the topology and parameters are fixed and the parameters are the same for each network node.

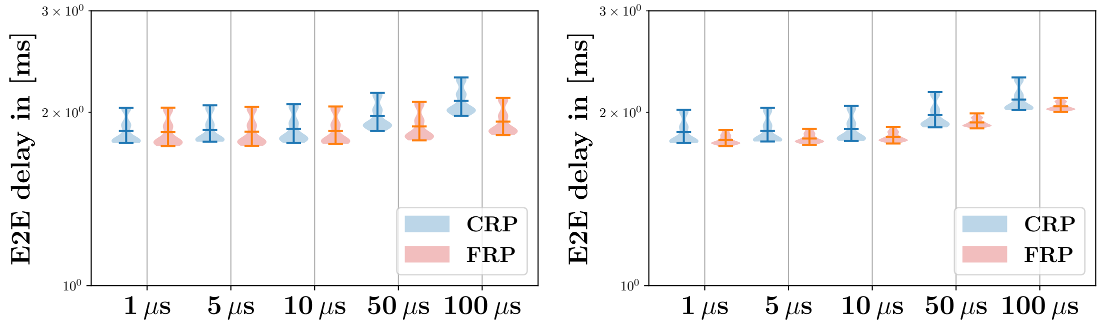
Parameter	Values
Pauli gate duration	$1\,\mu$s, $5\,\mu$s, $10\,\mu$s, $50\,\mu$s, $100\,\mu$s
BSM duration	$1\,\mu$s, $5\,\mu$s, $10\,\mu$s, $50\,\mu$s, $100\,\mu$s
BSM probability	0.5, 0.6, 0.7, 0.8, 0.9, 1.0
Coherence time	1 ms, 10 ms, 100 ms, 1 s, 10 s, 100 s, 1000 s

TABLE I: Simulation Parameters

This results in 1050 different parameter settings for a topology, while each of the parameter settings are evaluated 128 times.

Both random graph topologies consist of 20 routers and 80 hosts. The routers are in a dense core topology, while each host is connected to exactly one router. The difference of topologies stems from the distances in the core and periphery. The core topology is created with the Barabasi-Albert graph-model [24], with the number of edges connecting to existing nodes set to 3. Each host connects randomly to one router in the core, forming the periphery topology. The distances in the core topology of the Dense Random Graph are uniformly distributed from 0.2 km to 5 km, while the distances in the periphery are uniformly distributed from 0.1 km to 0.5 km. The distances in the core topology of the Sparse Random Graph are uniformly distributed from 10 km to 100 km, while the distances in the periphery range from 1 km to 10 km.

The German Backbone is based on the real german backbone network [21]. The German Backbone consists of 17 routers forming the core network with distances ranging form 36 km to 353 km. Then, each of the 80 hosts are randomly



(a) E2E Delay for Pauli duration.

(b) E2E Delay for BSM duration.

Fig. 4: E2E Delay for german backbone graph.

connected to one of the routers forming the periphery of the network, with distances ranging from 1 km to 5 km.

A. Delay in Dense Random Graph

In the Dense Random Graph, *FRP* has consistently lower delay than *CRP* as shown in Fig. 2a. If the duration of the Pauli gates is increased from 1 μs to 100 μs, the delay increases from on average 10μ s to 180 μs in the *FRP* case and 10 μs to 350 μs in the *CRP* case. The difference between *CRP* and *FRP* gets more pronounced, as the duration of the BSM increases as depicted in Fig. 2b. For low Pauli gate durations, (1 μs), the average delay for *FRP* is 45 μs and *CRP* is 90 μs, while for high Pauli gate durations, (100 μs) the average delay for *FRP* is 200 μs and for *CRP* is 300 μs.

In the Dense Random Graph, the delay mainly consists of the quantum operations, BSM duration and Pauli gate duration, while the transmission delay of classical packets only plays a minor role.

B. Delay in Sparse Random Graph

Similar to the Dense Random Graph topology, in the Sparse Random Graph topology *FRP* performs consistently better in terms of delay than *CRP* as depicted in Fig. 3a and Fig. 3b. For low BSM durations *FRP* has lower E2E delay at around 500 μs compared to *CRP* with 550 μs, while increasing the BSM duration only increases the averages of the individual protocols not the relative difference. For varying Pauli gate durations, delay of *FRP* and *CRP* starts of similar, but as the duration is increased *FRP* at 600 μs has lower delay than *CRP* at 700 μs.

C. Delay in German Backbone

In the german backbone topology, *CRP* and *FRP* perform more similar in terms of delay as shown in Fig. 4a and 4b. For increasing BSM durations *FRP* has lower delay compared to *CRP* and the average delay for both protocols increases, while the relative difference stays the same. For varying Pauli gate durations the delay is initially the same, but *FRP* has lower delay for increasing Pauli gate durations compared to *CRP*. The relative difference between the protocols in sparser networks is due to the transmission delays of classical packets, as they proportionally increase as the networks get sparser.

The simulations show *FRP* reduces E2E delay by up to 51% (compare 1 μs Fig. 2b) compared to *CRP* and on average by 13% averaged over all topologies and parameter settings.

D. Fidelity in Dense Random Graph

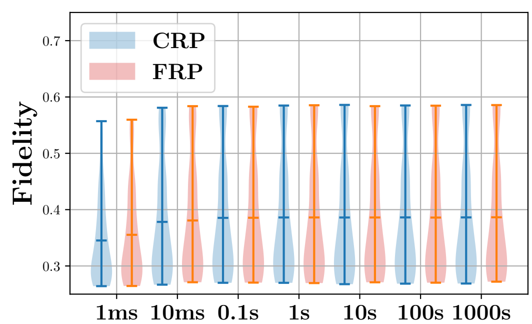


Fig. 5: Fidelity for coherence in Dense Random Graph.

The average fidelity in the Dense Random Graph topology, if the coherence time is varied, only increases up until a coherence of 1 s for both protocols from 0.35 to 0.38 as shown in Fig. 5. This means, increasing the coherence time of quantum memories above 1 s is not necessary, as there are only minor fidelity improvements for stored qubits. This indicates a relationship between coherence time and network density. Also, only a small portion of Bell-pairs are above the 0.5 fidelity threshold. The overall low fidelity in all parameter cases for both protocols is due to storage errors and BSMs. The resulting fidelity from a single BSM can be calculated as follows:

$$F_r = \frac{4F_sF_t - F_s - F_t + 1}{3}, \quad (3)$$

where F_s is the fidelity of the source pair and F_t is the fidelity of the target pair of the BSM. Based on Eq. 3, the E2E fidelity for a path can then be calculated as follows:

$$F_{E2E} = \frac{1}{3^{n-1}}\left(\frac{3^{n-2}+1}{2} + \frac{1}{4}\left(\prod_{i=1}^{n}(4F_i - 1) + (-1)^{n-1}\right)\right), \quad (4)$$

where n is the number of Bell-pairs on the path with the individual fidelities F_i. As QAPs require a fidelity above 0.5, most established Bell-pairs are not suitable for quantum communication.

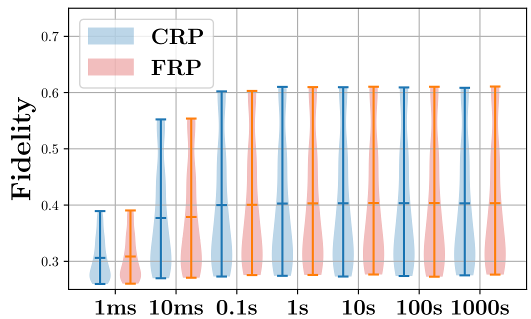


Fig. 6: Fidelity for coherence in Sparse Random Graph.

E. Fidelity in Sparse Random Graph

Again the fidelity for both protocols in the Sparse Random Graph is low at around 0.3 and only increases slightly as the coherence times are increased as depicted in Fig. 6. The low fidelity is again due to storage errors and BSMs.

F. Fidelity in German Backbone

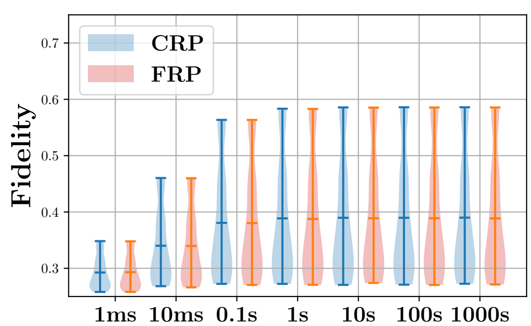


Fig. 7: Fidelity of coherence in german backbone graph.

The fidelity in both protocols for low coherence parameter settings is around 0.29 and increases to 0.39 as depicted in Fig. 7. The difference in fidelity compared to Dense and Sparse Random Graph is due to the increased delay in the German Backbone, as the delay has an influence on the resulting fidelity of qubits.

As the simulations show, quantum networks even with high coherence times, lead to low fidelity of E2E entanglement, as the fidelity of qubits reduces not only with storage errors, but also with the number of Bell-State measurements. Therefore, to improve the fidelity of qubits, error correction in the form of purification on a link and global level needs to be considered. Furthermore, as the resulting fidelity of a BSM depends on the input fidelities, the number of BSMs for an E2E entanglement establishment needs to be considered.

VI. Conclusion

The main task of quantum networks is the distribution of high fidelity entanglement over large distances. In order to establish long distance entanglement, quantum networks require low latency to reduce the effects of decoherence, while also keeping hardware requirements low for network components. It is therefore crucial to perform entanglement swapping at each router in a fast and efficient manner. This paper presents a new Repeater Protocol FRP operating in the quantum data plane, which reduces delay up to 51% in dense networks and on average by 13% in all networks to establish long distance entanglement, while not affecting the fidelity. Furthermore, hardware requirements are lowered for quantum network components, as an implementation of Pauli gates at routers is not required. As the simulations show, quantum networks without purification on a link and global level, lead to low E2E fidelity for most Bell-pairs, not usable for QAPs, as a fidelity of below 50% indicates pure classical correlations.

Future work will include simulations of quantum network protocols handling qubit loss and purification on a link and physical layer, while also restricting the number of qubits in quantum memories.

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