Modulated Estimate Correction for the White Gaussian Broadcast Channel with Feedback

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Abstract — **Modulated estimate correction strategies for the K-user white Gaussian broadcast channel with feedback are presented. The strategies are based on the discrete Fourier transform of length K. A recursion for finding good rates is derived and calculations show that the achievable sum rate increases with K. An outer bound shows that not much further improvement in sum rate is possible for small K.**

I. Introduction

The capacity region of the K-user white Gaussian broadcast channel with feedback (BC-FB) is unknown. A good strategy for two users was presented in [1] based on earlier results of [2]. However, it seemed that this strategy does not generalize to more than two users [3]. We show that there is a natural generalization by using an approach recently found to work well for the K-user white Gaussian multiple-access channel [4].

II. Modulated Estimate Correction

The best method for coding for single-user Gaussian channels with full feedback seems to involve *estimate correction*, i.e., one maps the message onto a point on the real line and transmits by correcting the receiver's linear minimum-mean-squared-error (LMMSE) estimate of this point at each channel use [5]. Similar techniques had been suggested for discrete memoryless channels (DMCs) as early as [6] and for the white Gaussian channel in [7]. In [5], where Gaussian channels with colored noise are considered, we first find the idea of *modulating* the estimate correction. This is done to "align" the transmitted symbols with the corresponding noise symbols, thereby increasing the mutual information between the transmitter and receiver symbol sequences. We will call this transmitting technique *modulated estimate correction* or simply MEC. One advantage of adopting MEC, apart from its excellent performance, is that the only design issue is the choice of the modulating sequence.

The MEC approach was adapted to the 2-user multiple-access channel with feedback (MAC-FB) in [2]. The idea is to let the first user correct the receiver's estimate of her message point *without* modulation, while the second user corrects the receiver's estimate of his message point *with* modulation by ± 1. The purpose of the modulation is, as for the single-user case, to align the two transmitted symbols to increase the mutual information between the input and output symbol sequences. It was found that, in the capacity achieving steady-state, the second user's modulating sequence is $+1, -1, +1, -1, \ldots$ [2]. In fact, this turns out to be the modulating sequence right from the very first transmission and not just in steady-state.

The above technique was generalized to the K-user MAC-FB by considering complex noise rather than real noise and by using complex modulating sequences [4]. We here describe a similar generalization for the K-user broadcast channel with feedback.

III. Channel Model

The K-user white Gaussian broadcast channel is a $K+1$ terminal channel with one input X and K outputs Y_1, Y_2, $\ldots, Y_K$ such that

$$Y_k = X + Z_k, \tag{1}$$

$k = 1, 2, \ldots, K$, where the Z_k are independent Gaussian noise random variables with zero mean (note that [1] considers a slightly more general model with an additional common noise random variable). We will consider both real noise with variance σ_k^2 and complex noise whose real and imaginary parts are independent and have variance σ_k^2. For the real case we will use the channel in blocks of length two to simulate a complex channel, and for the complex case we will allow X to be a complex random variable.

The broadcast channel is used N times and we denote the nth input and outputs by X_n and Y_{kn}. There is a block energy constraint on the inputs:

$$\sum_{n=1}^{N} \mathrm{E}[|X_n|^2]/N \leq P\ . \tag{2}$$

We assume *full* feedback is available from all K receivers so that X_n is a function of the messages at the input terminal and the K output sequences

$$Y_k^{n-1} := Y_{k1}, Y_{k2}, \ldots, Y_{k(n-1)}, \qquad k = 1, 2, \ldots, K. \tag{3}$$

We here consider only the scenario where one independent message is sent to each user, i.e., the transmitting terminal sends one B_k-bit message to receiving terminal k. The rate of the kth message is thus $R_k = B_k/N$ bits per use or $(B_k/N)\ln(2)$ nats per use. The *capacity region* $\mathcal{R}$ of the BC-FB is the closure of the set of rate-tuples $(R_1, R_2, \ldots, R_K)$ at which the receiving terminals can decode their messages with arbitrarily small positive error probability.

IV. Transmission and LMMSE Estimation

The transmitting terminal maps the K messages onto the complex plane as points θ_k, $k = 1, \ldots, K$. Without loss of essential generality, we assume that the θ_k are Gaussian distributed with zero mean and variance 1 for the real and imaginary parts. After the nth channel use, receiver k estimates θ_k by using the LMMSE estimate $\hat{\theta}_{kn}$ given Y_k^n. We write the estimation error as $\epsilon_{kn} = \hat{\theta}_{kn} - \theta_k$, where $\hat{\theta}_{k0} = 0$.

At time n the transmitting terminal sends

$$X_n = \sum_{k=1}^{K} X_{kn}, \tag{4}$$

where

$$X_{kn} = \sqrt{P_k/\sigma^2_{k(n-1)}} \cdot \epsilon_{k(n-1)} \cdot m^*_{kn}, \tag{5}$$

$\sigma^2_{kn} = \mathrm{E}[|\epsilon_{kn}|^2]$ is the variance of receiver k's estimation error, P_k is a real number, and m_{kn} is a complex *modulation coefficient*. Both the P_k and the m_{kn} are chosen before transmission and are known to all receivers. Of course, the choice of the P_k and m_{kn} must take into account the block energy constraint. The m_{kn} are complex conjugated to simplify the ensuing expressions somewhat.

The performance analysis of the "full" LMMSE strategies proves difficult because successive channel outputs are *not* independent. This is different than for the MAC-FB [4]. We thus introduce a class of simpler, but suboptimal, strategies. At time n, let receiver k estimate $\epsilon_{k(n-1)}$ by using the LMMSE estimate $\hat{\epsilon}_{k(n-1)}$ given $Y_k^{n-M..n} := Y_{k(n-M)}, \ldots, Y_{kn}$. The estimate of θ_k is then the recursive $\hat{\theta}_{kn} = \hat{\theta}_{k(n-1)} - \hat{\epsilon}_{k(n-1)}$. The transmitter will continue to use MEC and sends the sum of modulated versions of the ϵ_{kn}.

The receiver thus uses an *error-estimate* of *memory-M* and we call it an "E(M) estimate". Note that an E(N) estimate (or E(∞) estimate) is equivalent to a "full" LMMSE estimate. We will here consider only E(0) estimation, as was done in [1].

V. Performance Analysis for E(0) Estimates

Our analysis follows along the lines of [1]. User k's E(0) estimates are simply

$$\hat{\epsilon}_{k(n-1)} = \frac{\mathrm{E}[\epsilon_{k(n-1)} Y^*_{kn}]}{V_{Y_{kn}}} \cdot Y_{kn}, \tag{6}$$

where $V_{Y_{kn}} = \mathrm{E}[|Y^2_{kn}|]$ is the variance of Y_{kn}. In the appendix we show that

$$\sigma^2_{kn} = \sigma^2_{k(n-1)} \cdot \frac{V_{Y_{kn}|X_{kn}}}{V_{Y_{kn}}}, \tag{7}$$

where $V_{Y_{kn}|X_{kn}}$ is the variance of Y_{kn} given X_{kn}. The "instantaneous rate" is thus

$$\begin{aligned} R_{kn} &= \log\left(\sigma^2_{k(n-1)}/\sigma^2_{kn}\right) \\ &= \log\left(V_{Y_{kn}}/V_{Y_{kn}|X_{kn}}\right) \\ &= I(X_{kn}; Y_{kn}). \end{aligned} \tag{8}$$

One can derive similar results for E(M) estimates, but there are additional conditioning random variables $Y_k^{n-M..n-1}$.

As in [1], we are especially interested in the correlation coefficient

$$\rho_{k\ell n} = \frac{\mathrm{E}[\epsilon_{kn}\epsilon^*_{\ell n}]}{\sqrt{\sigma^2_{kn}\sigma^2_{\ell n}}}. \tag{9}$$

The recursion for $\rho_{k\ell n}$ is (see the appendix)

$$\begin{aligned} \rho_{k\ell n} = \; & \frac{1}{\sqrt{V_{Y_{kn}|X_{kn}} V_{Y_{\ell n}|X_{\ell n}}}} \Bigg[\rho_{k\ell(n-1)} \sqrt{V_{Y_{kn}} V_{Y_{\ell n}}} \\ & - c_{kn} c^*_{\ell n} \cdot \left(\sqrt{\frac{V_{Y_{kn}}}{V_{Y_{\ell n}}}} + \sqrt{\frac{V_{Y_{\ell n}}}{V_{Y_{kn}}}} - d_{k\ell n}\right)\Bigg], \end{aligned} \tag{10}$$

where

$$\begin{aligned} c_{kn} &= \mathrm{E}[\epsilon_{k(n-1)} X^*_n] \big/ \sqrt{\sigma^2_{k(n-1)}} && (11) \\ d_{k\ell n} &= \mathrm{E}[Y_{kn} Y^*_{\ell n}] \big/ \sqrt{V_{Y_{kn}} V_{Y_{\ell n}}}. && (12) \end{aligned}$$

VI. Equal Noise Powers

From here on we consider only the symmetric case with equal noise powers, i.e., $\sigma^2_k = \sigma^2$ for all k. Assuming that $\mathrm{E}[|X_n|^2] = P$, we have

$$\mathrm{E}[Y_{kn} Y^*_{\ell n}] = P + \sigma^2 \cdot \delta[k-\ell], \tag{13}$$

where $\delta[k]$ is the Kronecker-delta function taking on the value 1 if $k = 0$ and 0 otherwise. The recursion (10) simplifies to

$$\begin{aligned} \rho_{k\ell n} = \; & \frac{1}{\sqrt{V_{Y_{kn}|X_{kn}} V_{Y_{\ell n}|X_{\ell n}}}} \Big[\rho_{k\ell(n-1)} \left(P + \sigma^2\right) \\ & - c_{kn} c^*_{\ell n} \cdot \left(\frac{P + \sigma^2(2 - \delta[k-\ell])}{P + \sigma^2}\right)\Big]. \end{aligned} \tag{14}$$

We collect the $\rho_{k\ell n}$ to make a matrix recursion

$$\bar{Q}_{\underline{\epsilon}_n} = \odot \frac{\bar{Q}_{\underline{\epsilon}_{n-1}}(P + \sigma^2) - \underline{c}_n \underline{c}_n^H \odot A}{\sqrt{\underline{V}_{Y_n^K|X_n^K} \underline{V}^T_{Y_n^K|X_n^K}}}, \tag{15}$$

where

$$\bar{Q}_{\underline{\epsilon}_n} = \begin{bmatrix} 1 & \rho_{12n} & \rho_{13n} & \cdots & \rho_{1Kn} \\ \rho^*_{12n} & 1 & \rho_{23n} & \cdots & \rho_{2Kn} \\ \rho^*_{13n} & \rho^*_{23n} & 1 & \cdots & \rho_{3Kn} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^*_{1Kn} & \rho^*_{2Kn} & \rho^*_{3Kn} & \cdots & 1 \end{bmatrix}, \tag{16}$$

$$A = \begin{bmatrix} 1 & a & \ldots & a \\ a & 1 & \ldots & a \\ \vdots & & \ddots & \vdots \\ a & a & \ldots & 1 \end{bmatrix}, \quad a = \frac{P + 2\sigma^2}{P + \sigma^2}, \tag{17}$$

$\underline{c}_n$ and $\underline{V}_{Y_n^K|X_n^K}$ are the length K column vectors of the respective c_{kn} and $V_{Y_{kn}|X_{kn}}$, the "$\odot$" in front of the fraction denotes *term-by-term division* (a "Hadamard quotient" or a "Schur quotient" [8, Chapter 5]), the "$\odot$" in front of the A denotes *term-by-term multiplication* (a Hadamard product or a Schur product), and the square-root in the denominator denotes taking *term-by-term* square roots. The bar on top of the $\bar{Q}_{\underline{\epsilon}_n}$ emphasizes that this is a correlation *coefficient* matrix, i.e., that the absolute values of its entries are between -1 and $+1$.

We derive two more identities that will prove useful. First, by inserting (1), (4) and (5) into (11), we have

$$\underline{c}_n = \bar{Q}_{\underline{\epsilon}_{n-1}} \cdot \left(\sqrt{\underline{P}} \odot \underline{m}_n\right), \tag{18}$$

where $\sqrt{\underline{P}}$ and $\underline{m}_n$ are the length K column vectors of the $\sqrt{P_k}$ and m_{kn}, respectively, and "$\odot$" again denotes a Hadamard product. Next, we have

$$\begin{aligned} V_{Y_{kn}|X_{kn}} &= V_{Y_{kn}} - \left|\mathrm{E}[X_{kn} Y^*_{kn}]\right|^2 / (P_k \cdot |m_{kn}|^2) \\ &= (P + \sigma^2) - |c_{kn}|^2, \end{aligned} \tag{19}$$

so that

$$\underline{V}_{Y_n^K|X_n^K} = (P + \sigma^2) \cdot \underline{1} - |\underline{c}_n|^2, \tag{20}$$

where $\underline{1}$ is the length K all-ones column vector and $|\underline{c}_n|^2$ is the length K column vector with entries $|c_{kn}|^2$.

VII. Fourier Modulation

We now distribute the transmit power equally amongst the K messages, i.e., $P_k = P_1$ and $|m_{kn}| = 1$ for all k and n. We proceed by induction. Consider the recursion (15) and $n = 1$. The matrix $\bar{Q}_{\underline{\epsilon}_0}$ is simply the $K \times K$ identity matrix, which is of course a *circulant* matrix. But circulant matrices have as one possible set of eigenvectors the columns of the Fourier transform matrix F whose entry at row k and column ℓ is $e^{j2\pi(k-1)(\ell-1)/K}$. We denote the kth column of F by $\underline{f}_k$.

Suppose that at time n the matrix $\bar{Q}_{\underline{\epsilon}_{n-1}}$ has the eigenvectors $\underline{f}_k$ and the corresponding eigenvalues λ_{kn}, $k = 1, \ldots, K$. We choose some eigenvalue $\hat{\lambda}_n$ and make $\underline{m}_n$ the corresponding eigenvector $\hat{\underline{f}}_n$. We then have

$$\begin{aligned} \underline{c}_n &= \bar{Q}_{\underline{\epsilon}_{n-1}} \cdot \sqrt{P_1}\underline{m}_n = \sqrt{P_1}\hat{\lambda}_n \hat{\underline{f}}_n, && (21) \\ V_{Y_{kn}|X_{kn}} &= V_{Y_{1n}} - |c_{1n}|^2 \\ &= (P+\sigma^2) - P_1\hat{\lambda}_n^2, && (22) \\ \mathrm{E}[|X_n|^2] &= P_1 \cdot \underline{m}_n^H \bar{Q}_{\underline{\epsilon}_{n-1}} \underline{m}_n \\ &= P_1 \cdot K\hat{\lambda}_n. && (23) \end{aligned}$$

With these identities, and setting $\mathrm{E}[|X_n|^2] = P$, we can simplify (15) to

$$\bar{Q}_{\underline{\epsilon}_n} = \frac{\bar{Q}_{\underline{\epsilon}_{n-1}}(P+\sigma^2) - (P/K)\hat{\lambda}_n \hat{\underline{f}}_n \hat{\underline{f}}_n^H \odot A}{(P+\sigma^2) - (P/K)\hat{\lambda}_n}. \qquad (24)$$

Note that the Hadamard quotient has become a scalar division. Multiplying by $F^{-1} = F^H/K$ on the left and by F on the right, we find that $F^{-1}\bar{Q}_{\underline{\epsilon}_n}F$ is a *diagonal* matrix with entries

$$\lambda_{k(n+1)} = \begin{cases} \frac{\lambda_{kn}(P+\sigma^2) - (1+(K-1)a)(P/K)\hat{\lambda}_n}{(P+\sigma^2)-(P/K)\hat{\lambda}_n} & \text{if } \hat{\underline{f}}_n = \underline{f}_k, \\ \frac{\lambda_{kn}(P+\sigma^2) - (1-a)(P/K)\hat{\lambda}_n}{(P+\sigma^2)-(P/K)\hat{\lambda}_n} & \text{else.} \end{cases} \qquad (25)$$

This implies that $\bar{Q}_{\underline{\epsilon}_n}$ also has the columns of F as eigenvectors and that its eigenvalues are given by (25). Equation (25) is thus the recursion for the eigenvalues of $\bar{Q}_{\underline{\epsilon}_n}$.

Experiments show that (25) converges to a "periodic steady-state" by choosing the modulation coefficient vectors cyclically from the columns of the Fourier transform matrix F, i.e., $\underline{m}_n$ is column $[(n-1) \bmod K] + 1$ of F. However, it seems difficult to *prove* that convergence actually occurs. To bypass this problem, any "fixed point" of the recursion can be reached by appropriately initializing the transmission by prior agreement between the users. Alternatively, it should be possible to modify the transmit powers for the first $K-1$ channel uses to set the K eigenvalues before the Kth channel use. This would be a generalization of the approach described in [1].

VIII. An Outer Bound

An outer bound to the $K = 2$ capacity region was given in [1], where the channel was made *physically* degraded by giving user 1 both Y_1 and Y_2. This enlarges the capacity region because user 1 could simply ignore Y_2. Furthermore, the capacity region of this physically degraded broadcast channel is not increased by feedback (the proofs of [9, 10] can be extended to vector reception) and is the set of rate pairs (R_1, R_2) satisfying

$$\begin{aligned} 0 \leq\ & R_1 \leq \tfrac{1}{2}\log\left(1 + \tfrac{2P\alpha}{\sigma^2}\right) \\ 0 \leq\ & R_2 \leq \tfrac{1}{2}\log\left(1 + \tfrac{P(1-\alpha)}{\sigma^2+P\alpha}\right), \end{aligned} \qquad (26)$$

where α takes on any value between 0 and 1. The 1/2 in front of the log is needed for the real *and* complex cases if we normalize the rates by the number of real dimensions.

One can generalize this outer bound for $K > 2$. The idea is to give user k the outputs $Y_k, Y_{k+1}, \ldots, Y_K$ to get a K user physically degraded broadcast channel. The capacity region of this channel with and without feedback is the set of rate-tuples $(R_1, \ldots, R_K)$ satisfying

$$0 \leq R_k \leq \frac{1}{2}\log\left(1 + \frac{(K-k+1)P\,\alpha_k}{\sigma^2 + (K-k+1)P\sum_{\ell=1}^{k-1}\alpha_\ell}\right), \qquad (27)$$

for $k = 1, \ldots, K$, $\alpha_k \geq 0$ and $\sum_{k=1}^K \alpha_k = 1$. The proof of this result follows from generalizations of the results of [10, 11, 12].

We would like to find the best equal-rate point $R = R_1 = \ldots = R_K$ in (27). We do this by setting

$$\frac{KP\alpha_1}{\sigma^2} = \frac{(K-k+1)P\alpha_k}{\sigma^2 + (K-k+1)\sum_{\ell=1}^{k-1}\alpha_\ell} \qquad (28)$$

for $2 \leq k \leq K$. From this one can derive the recursion

$$\alpha_k = \alpha_{k-1}\left[1 + \frac{PK\alpha_1}{\sigma^2}\right] + \frac{K\alpha_1}{(K-k+1)(K-k+2)} \qquad (29)$$

for $2 \leq k \leq K$. Note that the α_k increase with k, and that they are all zero if $\alpha_1 = 0$. Thus, there is a unique α_1, $0 < \alpha_1 \leq 1/K$, so that $\sum_{k=1}^K \alpha_k = 1$. This value can easily be found by binary search and gives an outer bound on the equal-rates.

IX. An Example

We extend the example of [1] to K users. Let $P = 10$ and $\sigma^2 = 1$. The results of applying the eigenvalue recursion (25) are shown in Fig. 1, where the sum-of-equal-rates KR is plotted as a function of K. The recursion converges quite rapidly for small K but requires, e.g., about 800 channel uses to converge to within the 2nd decimal place of $KR = 1.7228$ for $K = 100$.

The outer bound was calculated with the method described in section VIII, and does not seem tight for large K. However, the bound does show that one cannot improve much over E(0) estimation for, say, $K \leq 4$.

X. Concluding Remarks

We have not discussed many important issues. First, one will get a double exponential decrease in error probability with the number of channel uses N (see [1]). Next, non-equal rate points and non-equal noise powers deserve to be considered in more detail. Third, in section VII one could have used any $K \times K$ orthogonal matrix whose entries have absolute value one, e.g., a Walsh-Hadamard transform matrix. This will give somewhat more efficient strategies for the real noise case. Fourth, we expect that the $E(M)$ estimates with $M > 0$ will improve the rates found here, and we guess that the full LMMSE estimates will achieve capacity in the steady state. It is also interesting to consider what rates are achievable as $K \to \infty$. Finally, we note that the coding techniques presented here will also work for the white Gaussian interference

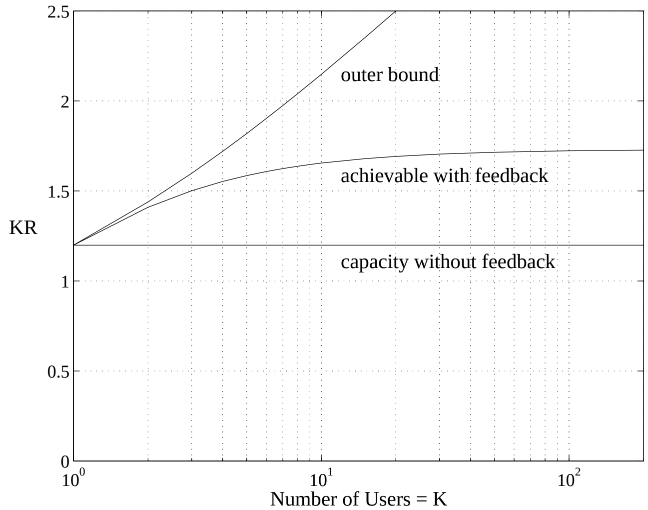

Fig. 1: Sum-of-equal-rates achievable with $P = 10$, $\sigma^2 = 1$ and E(0) estimation. The rate units are nats/use/dimension.

channel with full feedback from the receiving terminals to their transmitting terminals.

We end the paper with a remark that it is likely no accident that Fourier modulating coefficients work well. After all, if one uses MEC transmission and LMMSE reception the entire system is *linear*.

Appendix: E(0) Estimation

The variances of the errors are

$$\sigma_{kn}^2 = \mathrm{E}\left[\left|\epsilon_{k(n-1)} - \frac{\mathrm{E}[\epsilon_{k(n-1)}Y_{kn}^*]}{V_{Y_{kn}}} \cdot Y_{kn}\right|^2\right] = \sigma_{k(n-1)}^2 - \frac{|\mathrm{E}[\epsilon_{k(n-1)}Y_{kn}^*]|^2}{V_{Y_{kn}}}. \quad (30)$$

But from (5) we have

$$\mathrm{E}[\epsilon_{k(n-1)}Y_{kn}^*] = \mathrm{E}[X_{kn}Y_{kn}^*] \cdot \frac{1}{m_{kn}^*}\sqrt{\frac{\sigma_{k(n-1)}^2}{P_k}}. \quad (31)$$

Using $V_{X_{kn}} = P_k|m_{kn}^2|$ and inserting (31) into (30), we have

$$\sigma_{kn}^2 = \sigma_{k(n-1)}^2 \cdot \frac{V_{Y_{kn}} - \mathrm{E}[X_{kn}Y_{kn}^*]^* V_{X_{kn}}^{-1} \mathrm{E}[X_{kn}Y_{kn}^*]}{V_{Y_{kn}}} = \sigma_{k(n-1)}^2 \cdot \frac{V_{Y_{kn}|X_{kn}}}{V_{Y_{kn}}}, \quad (32)$$

which proves (7).

The cross-correlations can be expanded as

$$\begin{aligned}\rho_{k\ell n} &= \frac{\mathrm{E}[\epsilon_{kn}\epsilon_{\ell n}^*]}{\sqrt{\sigma_{kn}^2\sigma_{\ell n}^2}} \\ &= \frac{\mathrm{E}[[\epsilon_{k(n-1)} - \hat{\epsilon}_{k(n-1)}][\epsilon_{\ell(n-1)} - \hat{\epsilon}_{\ell(n-1)}]^*]}{\sqrt{\sigma_{k(n-1)}^2\sigma_{\ell(n-1)}^2}\sqrt{V_{Y_{kn}|X_{kn}}V_{Y_{\ell n}|X_{\ell n}}/V_{Y_{kn}}V_{Y_{\ell n}}}} \\ &= \sqrt{\frac{V_{Y_{kn}}V_{Y_{\ell n}}}{V_{Y_{kn}|X_{kn}}V_{Y_{\ell n}|X_{\ell n}}}} \cdot \big[\rho_{k\ell(n-1)} \\ &\quad - \tilde{c}_{\ell\ell n}^*\tilde{c}_{k\ell n} - \tilde{c}_{kkn}\tilde{c}_{\ell kn}^* + \tilde{c}_{kkn}\tilde{c}_{\ell\ell n}^* d_{k\ell n}\big], \end{aligned} \quad (33)$$

where

$$\tilde{c}_{k\ell n} = \mathrm{E}[\epsilon_{k(n-1)}Y_{\ell n}^*] \Big/ \sqrt{\sigma_{k(n-1)}^2 V_{Y_{\ell n}}} \quad (34)$$

$$d_{k\ell n} = \mathrm{E}[Y_{kn}Y_{\ell n}^*] \Big/ \sqrt{V_{Y_{kn}}V_{Y_{\ell n}}} \quad (35)$$

are the correlation coefficients of $\epsilon_{k(n-1)}$ and $Y_{\ell n}$, and Y_{kn} and $Y_{\ell n}$, respectively. To simplify (33) somewhat, let

$$\begin{aligned} c_{kn} &= \tilde{c}_{k\ell n} \cdot \sqrt{V_{Y_{\ell n}}} \\ &= \mathrm{E}[\epsilon_{k(n-1)}(X_n + Z_{\ell n})^*] \Big/ \sqrt{\sigma_{k(n-1)}^2} \\ &= \mathrm{E}[\epsilon_{k(n-1)}X_n^*] \Big/ \sqrt{\sigma_{k(n-1)}^2}. \end{aligned} \quad (36)$$

Note that c_{kn} does not depend on ℓ. We now have

$$\begin{aligned}\rho_{k\ell n} = \frac{1}{\sqrt{V_{Y_{kn}|X_{kn}}V_{Y_{\ell n}|X_{\ell n}}}} \Bigg[\rho_{k\ell(n-1)}\sqrt{V_{Y_{kn}}V_{Y_{\ell n}}} \\ - c_{kn}c_{\ell n}^* \cdot \left(\sqrt{\frac{V_{Y_{kn}}}{V_{Y_{\ell n}}}} + \sqrt{\frac{V_{Y_{\ell n}}}{V_{Y_{kn}}}} - d_{k\ell n}\right)\Bigg], \end{aligned} \quad (37)$$

which is the same as (10).

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