

Learning-based Prescribed-Time Safety for Control of Unknown Systems with Control Barrier Functions

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Abstract—In many control system applications, state constraint satisfaction needs to be guaranteed within a prescribed time. While this issue has been partially addressed for systems with known dynamics, it remains largely unaddressed for systems with unknown dynamics. In this paper, we propose a Gaussian process-based time-varying control method that leverages backstepping and control barrier functions to achieve safety requirements within prescribed time windows for control affine systems. It can be used to keep a system within a safe region or to make it return to a safe region within a limited time window. These properties are cemented by rigorous theoretical results. The effectiveness of the proposed controller is demonstrated in a simulation of a robotic manipulator.

Index Terms—Machine learning, data-based control, uncertain systems, safety-critical control, robotics

I. INTRODUCTION

CONTROL systems with active constraints during limited time windows are ubiquitous. For example, in a robot-human handover scenario [1], contact constraints are only relevant as long as the object is being handed over. This interaction temporarily modifies the safety region, reflecting the dynamic nature of the environment. The time-limited nature of the constraints in such settings allows for considerable flexibility, which can be leveraged to improve control performance. However, most existing algorithms aim to enforce safety at all times based on the initial safe condition, as opposed to relaxing these requirements when permissible, yielding potentially overly conservative behavior. Recent works have addressed a less stringent notion of safety than typically found in the literature, where safety constraints are only considered for a finite time window [2]. The goal is to guarantee that

the system returns and remains in the safe region within a pre-specified time interval. While this type of task has been addressed in settings with known dynamics, the considerably more challenging problem with unknown dynamics remains largely unaddressed.

Recently, several techniques addressing the issue of enforcing the system in the safe region have been proposed within the temporal constraints. Model predictive control [3] is widely adopted in dynamic systems to achieve optimal performance while satisfying multiple constraints. However, this approach demands substantial computational resources since it requires solving a sequence of constrained optimization problems within a finite time horizon at each discrete time step. Safe reinforcement learning [4] is another powerful tool to address safety issues even under an uncertain environment. Nevertheless, its practical deployment is challenging due to the large sim-to-real gaps and theoretical guarantee of safety. Control barrier functions (CBFs) [5] are increasingly utilized to ensure safety in systems, employing a quadratic program (QP) with linear constraints at every discrete time step. Benefiting by the QP framework, CBFs method could be applied as a real-time optimization-based controller, and the safety is guaranteed by rigorous proof. However, vanilla CBF lacks the consideration of temporal constraints. In settings where the system has to be within a safe region after a pre-specified time, the prescribed-time safety (PTSf) controller is devised in [6] based on a CBF framework. With a design of time-varying gain in the CBFs, the safety of the system is guaranteed in the specified time horizon starting from an initially safe condition. Compared to other CBF frameworks with time requirements such as finite-time [7] and fixed-time safety controller [8], the PTSf controller benefits from its simple design and independence of the initial system states. However, the design of the PTSf controller crucially relies on the availability of accurate system dynamics, restricting its practical usage in cases with unknown dynamics or environmental uncertainties. In addition, the capability of returning to the safe regions from the unsafe initial condition for PTSf controllers is not shown in previous work.

Supervised machine learning techniques are increasingly promising for identifying unknown dynamical systems from data. However, adequately accounting for model uncertainty remains an open problem for safe control. In [9], a neural network model is used to estimate model uncertainty, which

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is then leveraged together with robust CBFs to guarantee the safety of the closed-loop system. The work of [10] proposes a CBF-based imitation learning approach, where a deep neural network mimics the outcome of CBF-based controllers. In [11], a Gaussian process (GP) model is employed to formulate a robust CBF, which is then leveraged to derive a control law that renders the system safe. The works in [12] and [13] employ GP regression to model an elastic-joint robot, which is then rendered safe using a robust CBF. However, none of the techniques mentioned above address the initial unsafe condition that results from the temporary alteration of the safe region. Instead, these works always maintain the safety from a safe state at all times.

In this paper, we consider an unknown high-order control affine system with controllable canonical form, which requires to stay in a pre-defined safe set within a prescribed time. To this end, We propose a novel, robust Gaussian process-based framework for prescribed time control barrier functions incorporating the probabilistic uncertainty quantification. Our rigorous proof verifies that the system reliably remains in or returns to the safe set in a pre-defined time horizon with a high probability, regardless of whether the initial system state is safe or unsafe. The effectiveness of our method is demonstrated using a numerical simulation of a two-link robot manipulator.

The remainder of this paper is structured as follows. In Section II, the system setting and the safety requirements are introduced. In Section III, a time-varying Gaussian process-based control barrier function framework is proposed with mathematical proof. The effectiveness of the proposed method is shown via simulation for a two-link robot manipulator in Section IV, followed by the conclusion in Section V.

II. PROBLEM SETTING AND PRELIMINARIES

A. System Description

Consider a nonlinear continuous-time system with unknown dynamics in the controllable canonical form as

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{x}_{i+1}, & i &= 1, \dots, n-1, \\ \dot{\mathbf{x}}_n &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} + \mathbf{d}(\mathbf{x}), \end{aligned} \quad (1)$$

where $\mathbf{x} = [\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top]^\top \in \mathbb{X} \subset \mathbb{R}^{mn}$ with $\mathbf{x}_i = [x_{i,1}, \dots, x_{i,m}]^\top \in \mathbb{R}^m, \forall i = 1, \dots, n$ represents the system state in the compact domain \mathbb{X} and $\mathbf{u} \in \mathbb{U} \subseteq \mathbb{R}^m$ denotes the control input at time $t \in \mathbb{R}_+$. The functions $\mathbf{f} : \mathbb{X} \rightarrow \mathbb{R}^m$ and $\mathbf{g} : \mathbb{X} \rightarrow \mathbb{R}^{m \times m}$ are locally Lipschitz continuous functions that represent the known components of the system dynamics, the function $\mathbf{d} : \mathbb{X} \rightarrow \mathbb{R}^m$ in (1) encodes all state-dependent model uncertainties from, e.g., environmental effects and unmodeled parts of the system dynamics. This form is a common structure in many practical systems such as robot manipulators [14]. Moreover, we make the following assumption regarding \mathbf{g} .

Assumption 1: For all $\mathbf{x} \in \mathbb{X}$, $\mathbf{g}(\mathbf{x})$ is non-singular.

Assumption 1 is reasonable for various types of systems, e.g., manipulators, and is frequently satisfied by control-affine systems [15]. It implies that we can generate control inputs to compensate for nonlinearities in an arbitrary direction.

In order to design a control law that ensures safety, we require an adequate model of the uncertainty \mathbf{d} . The data-driven

model is employed to infer the uncertainty, where a noisy measurement data \mathbb{D} specified by the following assumption is leveraged.

Assumption 2: The data set \mathbb{D} consists of $N \in \mathbb{N}$ training pairs $\{\mathbf{x}^{(k)}, \mathbf{y}^{(k)}\}$ with $\mathbf{y}^{(k)} = \mathbf{d}(\mathbf{x}^{(k)}) + \boldsymbol{\epsilon}^{(k)}, \forall k = 1, \dots, N$, where $\boldsymbol{\epsilon}^{(k)}$ is i.i.d. zero-mean Gaussian noise with covariance $\boldsymbol{\Sigma}_o = \text{diag}(\sigma_{o,1}^2, \dots, \sigma_{o,m}^2), \sigma_{o,j} \in \mathbb{R}_+, \forall j = 1, \dots, m$.

Assumption 2 is a mild assumption often encountered in learning-based control settings [16]. It allows for Gaussian distributed measurement noise, which can be due to, e.g., numerical differentiation. In some settings, the requirements for the measurement noise distribution can be relaxed, e.g., by restricting it to be bounded [17], but this is out of the scope of this paper.

B. Prescribed-Time Safety (PTSf)

In this paper, a safe set $\mathcal{C} \subseteq \mathbb{X}$ is defined by a known, continuously differentiable *control barrier function* (CBF) $h(\mathbf{x}) : \mathbb{X} \rightarrow \mathbb{R}$ as $\mathcal{C} = \{\mathbf{x} \in \mathbb{X} : h(\mathbf{x}) \geq 0\}$. Specifically, the system is considered safe if $\mathbf{x} \in \mathcal{C}$, and unsafe otherwise.

If the system (1) is known perfectly, then the CBF h can be leveraged to compute certifiably safe control inputs [5], [18]. For many practical systems, it is possible to derive an appropriate CBF with only imperfect knowledge of the system at hand, e.g., adaptive cruise control system [19].

We now introduce the notion of *prescribed-time safety*, which is the main focus of this paper. Based on [6], we distinguish between PTSf for systems that are initially safe and unsafe. The latter case corresponds to rescuing safety within a prescribed time [6].

Definition 1 (PTSf for initially safe system): Consider the system (1). If the initial state is safe, i.e., $\mathbf{x}(t_0) \in \mathcal{C}$, then the system is said to be PTSf with a prescribed time $T_{\text{pre}} \in \mathbb{R}_+$ if $h(\mathbf{x}(t)) \geq 0, \forall t \in [t_0, t_0 + T_{\text{pre}}]$.

Definition 2 (PTSf for initially unsafe system): Consider the system (1). If the initial state is unsafe, i.e., $\mathbf{x}(t_0) \notin \mathcal{C}$, then the system is said to be PTSf with a prescribed time $T_{\text{pre}} \in \mathbb{R}_+$ if $h(\mathbf{x}(t_0 + T_{\text{pre}})) \geq 0$.

Our goal is then to design a control algorithm that guarantees PTSf for the system (1) whenever the initial state is either safe or unsafe.

III. LEARNING-BASED CONTROL WITH PRESCRIBED-TIME SAFETY

To address the PTSf problem, Gaussian process regression is adopted as a data-driven approach to approximate the uncertainty $\mathbf{d}(\mathbf{x})$ in (1). Based on GP regression, we propose a learning-based PTSf controller to guarantee the safety objectives defined in Definition 1 and Definition 2 with system uncertainty.

A. Gaussian Process Regression

Gaussian process regression, as a non-parametric method, is widely used to approximate unknown continuous functions due to its modeling flexibility. In order to learn the m -dimensional unknown function $\mathbf{d}(\cdot) = [d_1(\cdot), \dots, d_m(\cdot)]^\top$ from data set \mathbb{D}

satisfying Assumption 2, each component d_j is represented as a GP $d_j \sim \mathcal{GP}(m_j(\cdot), k_j(\cdot, \cdot)), \forall j = 1, \dots, m$, which is specified by the prior mean $m_j(\cdot) : \mathbb{X} \rightarrow \mathbb{R}$ and Lipschitz covariance function $k_j(\cdot, \cdot) : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_{0,+}$. The mean function $m_j(\cdot)$ encodes the prior knowledge of the system, which in our case is included in $f(\cdot)$ resulting in $m_j(\mathbf{x}) = 0$ for $\forall \mathbf{x} \in \mathbb{X}$. The covariance function, also called kernel function, resulting in $k_j(\mathbf{x}, \mathbf{x}')$ reflects the correlation between evaluations of $d_j(\cdot)$ at state \mathbf{x} and \mathbf{x}' with $\mathbf{x}, \mathbf{x}' \in \mathbb{X}$. The unknown function $\mathbf{d}(\mathbf{x})$ now is expressed as m GP models

$$\mathbf{d}(\mathbf{x}) = \begin{cases} d_1 \sim \mathcal{GP}(0, \kappa_1(\cdot, \cdot)) \\ \vdots \\ d_m \sim \mathcal{GP}(0, \kappa_m(\cdot, \cdot)) \end{cases} \quad (2)$$

Utilizing the data set \mathbb{D} satisfying Assumption 2 with $|\mathbb{D}| = N$ and Bayesian principle, the value of $\mathbf{d}(\mathbf{x})$ follows a Gaussian distribution characterized by the posterior mean $\boldsymbol{\mu}(\mathbf{x}) = [\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})]^\top$ and variance $\boldsymbol{\Sigma}(\mathbf{x}) = \text{diag}(\sigma_1^2(\mathbf{x}), \dots, \sigma_m^2(\mathbf{x}))$ with

$$\mu_j(\mathbf{x}) = \mathbf{k}_{\mathbf{X}_j}^\top(\mathbf{x})(\mathbf{K}_j + \sigma_n^2 \mathbf{I})^{-1} \mathbf{y}_j, \quad (3)$$

$$\sigma_j^2(\mathbf{x}) = \kappa_j(\mathbf{x}, \mathbf{x}) - \mathbf{k}_{\mathbf{X}_j}^\top(\mathbf{x})(\mathbf{K}_j + \sigma_n^2 \mathbf{I})^{-1} \mathbf{k}_{\mathbf{X}_j}(\mathbf{x}), \quad (4)$$

where the kernel vector and gram matrix are specified as $\mathbf{k}_{\mathbf{X}_j}(\mathbf{x}) = [\kappa_j(\mathbf{x}^{(1)}, \mathbf{x}), \dots, \kappa_j(\mathbf{x}^{(N)}, \mathbf{x})]^\top$ and $\mathbf{K}_j = [\kappa_j(\mathbf{x}^{(i)}, \mathbf{x}^{(k)})]_{i,k=1,\dots,N}$, respectively. The vector $\mathbf{y}_j = [y_j^{(1)}, \dots, y_j^{(N)}]^\top$ with $y_j^{(k)}$ representing the j^{th} dimension of $\mathbf{y}^{(k)}$ is the concatenation of the output values in data set \mathbb{D} . The posterior mean function $\boldsymbol{\mu}(\cdot)$ serves as a prediction model of the unknown function $\mathbf{d}(\cdot)$, whereas the variance $\boldsymbol{\Sigma}(\cdot)$ is employed as an indicator of epistemic uncertainty, which is shown as follows.

Lemma 1 ([20]): Consider an unknown function $d_j(\cdot)$ for $\forall j = 1, \dots, m$ and a data set satisfying Assumption 2. Choose $\tau \in \mathbb{R}_+$ and $\delta \in (0, 1) \subset \mathbb{R}$, then

$$\Pr \{ |d_j(\mathbf{x}) - \mu_j(\mathbf{x})| \leq \eta_j(\mathbf{x}), \forall \mathbf{x} \in \mathbb{X} \} \geq 1 - \delta, \quad (5)$$

$$\eta_j(\mathbf{x}) = \sqrt{\beta_\delta(\tau) \sigma_j(\mathbf{x})} + \gamma_\delta(\tau),$$

where $\gamma_\delta(\tau) = (L_{d,j} + \sqrt{\beta_\delta(\tau)} L_{\sigma,j} + L_{\mu,j})\tau$ and

$$\beta_\delta(\tau) = 2 \sum_{j=1}^{mn} \log \left(\frac{0.5\sqrt{mn}(\bar{x}_j - \underline{x}_j)}{\tau\delta} + \frac{1}{\delta} \right), \quad (6)$$

and $\bar{x}_j = \max_{\mathbf{x} \in \mathbb{X}} x_j, \underline{x}_j = \min_{\mathbf{x} \in \mathbb{X}} x_j$ with x_j referring to the j -th dimension of \mathbf{x} . The constants $L_{\mu,j}, L_{\sigma,j}, L_{d,j} \in \mathbb{R}_+$ are the Lipschitz constants of mean $\mu_j(\cdot)$, standard deviation $\sigma_j(\cdot)$ and function $d_j(\cdot)$, respectively.

Lemma 1 provides a probabilistic bound for the prediction error from GP regression, which is widely used in safety-critical applications. The detailed expressions of $L_{\mu,j}$ and $L_{\sigma,j}$ can be found in [20], and the Lipschitz constant $L_{d,j}$ for the unknown function $d_j(\cdot)$ can be approximated as in [20].

B. PTSf Design with Learning Uncertainty

Recalling the system as described in (1), the aim of this subsection is to develop a safety controller for the control input, ensuring that (1) satisfies Definition 1 and 2 with a n^{th}

order differentiable CBF $\mathbf{h}(\mathbf{x}_1) : \mathbb{R}^m \rightarrow \mathbb{R}^{d_h}$ with $d_h \in \mathbb{N}_+$. Without loss of generality, we consider the scalar CBF with $d_h = 1$, whose result can be extended to high dimensional $\mathbf{h}(\cdot)$.

The design of the prescribed-time safety controller incorporates a blow-up function, which is defined as follows:

$$\varphi(t) = \frac{T_{\text{pre}}^2 + \alpha((t - t_0)^2 - (t - t_0)T_{\text{pre}})^2}{(T_{\text{pre}} + t_0 - t)^2}, \quad t \geq t_0, \quad (7)$$

where $\alpha \in \mathbb{R}_{0,+}$ represents a scalar tunable parameter to the convergence speed. Note that $\varphi(\cdot)$ is an increasing positive function in the time horizon $[t_0, t_0 + T_{\text{pre}})$, which provides the flexibility to achieve PTSf in our following design. With the definition of the time-varying function $\varphi(\cdot)$ in (7), a series of barrier functions are designed as

$$h_1(\mathbf{x}_1) = h(\mathbf{x}_1), \quad (8)$$

$$h_{i+1}(t, \mathbf{x}_{1:i+1}) = \dot{h}_i(t, \mathbf{x}_{1:i}) + c_i \varphi(t) h_i(t, \mathbf{x}_{1:i}), \quad (9)$$

for $i = 1, \dots, n-1$, where $\mathbf{x}_{1:i} = [\mathbf{x}_1^\top, \dots, \mathbf{x}_i^\top]^\top$ and $c_i \in \mathbb{R}_+$ are positive constants to be determined later. The time derivatives \dot{h}_i for $i = 1, \dots, n$ are explicitly written according to (1) as

$$\dot{h}_i(t, \mathbf{x}_{1:i}) = \sum_{j=1}^i \frac{\partial h_i(t, \mathbf{x}_{1:i})}{\partial \mathbf{x}_j} \mathbf{x}_{j+1} + \frac{\partial h_i(t, \mathbf{x}_{1:i})}{\partial t} \quad (10)$$

for $i = 1, \dots, n-1$ and

$$\begin{aligned} h_{n+1}(t, \mathbf{x}, \mathbf{u}) &= \dot{h}_n(t, \mathbf{x}) + c_n \varphi(t) h_n(t, \mathbf{x}) \\ &= \sum_{j=1}^{n-1} \frac{\partial h_n(t, \mathbf{x})}{\partial \mathbf{x}_j} \mathbf{x}_{j+1} + \frac{\partial h_n}{\partial \mathbf{x}_n} (\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} + \mathbf{d}(\mathbf{x})) \\ &\quad + c_n \varphi(t) h_n(t, \mathbf{x}) \end{aligned} \quad (11)$$

Note that, the control input \mathbf{u} and uncertainty of the system $\mathbf{d}(\cdot)$ are included in (11), making $h_{n+1}(t, \mathbf{x}, \mathbf{u})$ impossible to evaluate. Instead, the posterior mean $\boldsymbol{\mu}(\cdot)$ and variance $\boldsymbol{\Sigma}(\cdot)$ obtained from Gaussian process in (3) and (4) are employed to approximate $h_{n+1}(\mathbf{x}_{n+1}, t)$. For notational simplicity, we denote $h_i(t) := h_i(t, \mathbf{x}_{1:i}(t))$ and $\dot{h}_i(t) := \dot{h}_i(t, \mathbf{x}_{1:i}(t))$ and the control performance is shown in the following theorem.

Theorem 1: Consider the system (1) and let Assumption 1 and 2 hold. Let $\boldsymbol{\mu}(\cdot)$ and $\eta_j(\cdot), j = 1, \dots, m$ be as in (3) and (5) respectively and choose $\delta \in (0, 1/m)$. Let \mathbf{u}_{nom} be the control input provided by other nominal controllers, e.g., PID controller, feedback linearization, and \mathbf{u}_{safe} be obtained by solving the quadratic programming (QP) as

$$\mathbf{u}_{\text{safe}} = \arg \min_{\mathbf{u} \in \mathbb{U}} \|\mathbf{u} - \mathbf{u}_{\text{nom}}\|^2 \quad (12a)$$

$$\text{s.t. } h_{n+1}^*(t, \mathbf{x}, \mathbf{u}) \geq 0, \quad (12b)$$

in which

$$\begin{aligned} h_{n+1}^*(t, \mathbf{x}, \mathbf{u}) &= \sum_{j=1}^{n-1} \frac{\partial h_n}{\partial \mathbf{x}_j} \mathbf{x}_{j+1} + \frac{\partial h_n}{\partial t} + c_n \varphi(t) h_n(t) \\ &\quad + \frac{\partial h_n}{\partial \mathbf{x}_n} (\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} + \boldsymbol{\mu}(\mathbf{x})) - \sum_{j=1}^m \left| \frac{\partial h_n}{\partial x_{n,j}} \right| |\eta_j(\mathbf{x})| \end{aligned} \quad (13)$$

with initial gains c_i for $i = 1, \dots, n$ satisfying $c_n > 0$ and

$$c_i > \max\{0, -h_i^{-1}(t_0) \dot{h}_i(t_0)\}, i = 1, \dots, n-1, \quad (14)$$

If the QP (12) is feasible for all $\mathbf{x} \in \mathbb{X}$ and all $t \geq t_0$, then the control input $\mathbf{u} = \mathbf{u}_{\text{safe}}$ in (1) guarantees PTSf according to Definition 1 and 2 with probability of at least $1 - m\delta$.

Proof: Our proof is structured into two parts. In the first part, we will show that $h_{n+1}(t) \geq 0, \forall t \geq t_0$ is satisfied with a high probability. Consider the subtraction of $h_{n+1}^*(t)$ in (13) from $h_{n+1}(t)$ in (11), which is written as

$$\begin{aligned} h_{n+1}(t) - h_{n+1}^*(t) & \quad (15) \\ &= \frac{\partial h_n}{\partial \mathbf{x}_n}(\mathbf{d}(\mathbf{x}) - \boldsymbol{\mu}(\mathbf{x})) + \sum_{j=1}^m \left| \frac{\partial h_n}{\partial x_{n,j}} \right| |\eta_j(\mathbf{x})|. \end{aligned}$$

Note that (15) is only related to \mathbf{x} , such that by applying the uniform probabilistic error bound in Lemma 1 to (15) and the extension to m dimension through Boole's inequality [21], [22], the positivity of $h_{n+1}(t) - h_{n+1}^*(t)$ is guaranteed within a probability bound as

$$\Pr \{h_{n+1}(t) - h_{n+1}^*(t) \geq 0, \forall \mathbf{x} \in \mathbb{X}\} \geq 1 - m\delta. \quad (16)$$

By picking the values of $\delta \leq 1/m \in \mathbb{R}^+$, a high probability of (16) and the following equation can be guaranteed.

(16) implies that $h_{n+1}(t) \geq h_{n+1}^*(t)$ with a probability of at least $1 - m\delta$ for $\forall t \geq t_0$ with $\mathbf{x}(t) \in \mathbb{X}$. Considering that in the compact domain \mathbb{X} , $h_{n+1}^*(t) \geq 0$ for $\forall t \geq t_0$ is guaranteed in (12), this consequently proves that

$$\begin{aligned} \Pr \{h_{n+1}(t) \geq 0, \forall t \geq t_0\} & \geq 1 - m\delta. \\ h_{n+1} & := h_{n+1}(t, \mathbf{x}_n(t)), \forall \mathbf{x}_n(t) \in \mathbb{X} \end{aligned} \quad (17)$$

A similar procedure is also employed in [11, Th4.3]. In the second part of our proof, we aim to guarantee the prescribed-time safety, regardless of whether the initial system condition is safe or unsafe. Based on (17), it implies from (11) that

$$\Pr \{\dot{h}_n(t) \geq -c_n \varphi(t) h_n(t), \forall t \geq t_0\} \geq 1 - m\delta. \quad (18)$$

By applying the variation of constants formula and the comparison lemma [23], for the time horizon of $[t_0, t_0 + T_{\text{pre}})$, the solution of (18) is derived as

$$\Pr \left\{ h_n(t) \geq h_n(t_0) e^{-c_n \int_{t_0}^t \varphi(s) ds}, \forall t \geq t_0 \right\} \geq 1 - m\delta. \quad (19)$$

Similarly, the analytical solutions of $h_i(t)$ for $i = 1, \dots, n-1$ in (9) are also reformulated as

$$h_i(t) = \int_{t_0}^t e^{-c_i \int_{\tau}^t \varphi(s) ds} h_{i+1}(\tau) d\tau + h_i(t_0) e^{-c_i \int_{t_0}^t \varphi(s) ds}. \quad (20)$$

Moreover, by substituting (19) into (20), it leads to

$$\begin{aligned} h_i(t) & \geq h_{i+1}(t_0) \int_{t_0}^t e^{-[c_i \int_{\tau}^t \varphi(s) ds + c_{i+1} \int_{t_0}^{\tau} \varphi(s) ds]} d\tau \\ & \quad + h_i(t_0) e^{-c_i \int_{t_0}^t \varphi(s) ds} \end{aligned} \quad (21)$$

with probability of at least $1 - m\delta$ for $i = n-1$.

We now start to demonstrate that the prescribed-time safety for an initially unsafe system is guaranteed. A series of auxiliary gains c_i^* for $\forall i = 1, \dots, n$ are defined as

$$c_i^* = \begin{cases} \bar{c}, & \text{if } h_i(t_0) > 0, \\ \underline{c}, & \text{otherwise} \end{cases}, \quad (22)$$

where $\bar{c} = \max\{c_1, \dots, c_n\}$, and $\underline{c} = \min\{c_1, \dots, c_n\}$. Due to the choice of c_i in Theorem 1, the auxiliary gains c_i^* are non-negative. Next, we rewrite (21) for $i = n-1$ as

$$h_i(t) \geq h_{i+1}(t_0) \int_{t_0}^t e^{-c_i^* \int_{\tau}^t \varphi(s) ds} d\tau + h_i(t_0) e^{-c_i^* \int_{t_0}^t \varphi(s) ds}, \quad (23)$$

which inherits the probability of at least $1 - m\delta$. By applying the induction step with i to (20) from $n-1$ to 1 recursively, the inequality of $h_i(t)$ for $\forall i = 1, \dots, n-1$ is expressed as

$$h_i(t) \geq \sum_{j=i}^n h_j(t_0) \frac{(t-t_0)^{j-i}}{(j-i)!} e^{c_j^* \int_{t_0}^t \varphi(\tau) d\tau}. \quad (24)$$

As a result, the value of $h_1(t)$ is bounded as

$$h_1(t) \geq \sum_{j=1}^n h_j(t_0) \frac{(t-t_0)^{j-1}}{(j-1)!} e^{-c_j^* \int_{t_0}^{T^*} \varphi(\tau) d\tau} e^{-c_j^* \int_{T^*}^t \varphi(\tau) d\tau}, \quad (25)$$

where $T^* = t_0 + T_{\text{pre}}$. Note that with the definition of $\varphi(\cdot)$ in (7) and the positivity of c_j^* , it has $e^{-c_j^* \int_{t_0}^{T^*} \varphi(\tau) d\tau} = 0$, such that $h_1(t) \geq 0, \forall t \geq t_0 + T_{\text{pre}}$ holds with a probability of at least $1 - m\delta$, which concludes the proof for an initial unsafe condition. We now prove that the prescribed-time safety for an initially safe system is also guaranteed. With $\varphi(t_0) = 1$ at the initial time, the equality in (9) for $i=2, \dots, n$ is carried out as

$$h_i(t_0) = \dot{h}_{i-1}(t_0) + c_{i-1} h_{i-1}(t_0). \quad (26)$$

Through the design of initial gains in Theorem 1, the initial value $h_i(t_0)$ follows that $h_i(t_0) > 0, i = 2, \dots, n$. Additionally, leveraging the positive nature of the exponential integral, the inequality (21) is further written as

$$h_i(t) \geq h_i(t_0) e^{-c_i \int_{t_0}^t \varphi(s) ds} > 0, i = n-1, \quad (27)$$

which inherits the probability of at least $1 - m\delta$. By substituting (27) to (20) for each steps backwards from $i = n-1$ to 1, the subsequent formula can be expressed as

$$h_1(t) \geq h_1(t_0) e^{-c_1 \int_{t_0}^t \varphi(s) ds} \quad (28)$$

Consider a system that initially remains within the safe area, i.e., $h_1(t_0) \geq 0$, and proceeding from (28), it is shown that

$$h_1(t) \geq h_1(t_0) e^{-c_1 \int_{t_0}^t \varphi(s) ds} \geq 0, \forall t \in [t_0, t_0 + T_{\text{pre}}) \quad (29)$$

with at least $1 - m\delta$ probability. Combining the proof for initially safe and unsafe cases, it is proven that the prescribed-time safety is guaranteed with a high probability. ■

Theorem 1 shows the prescribed time safety in Definition 1 and 2 is achieved with high probability by using the proposed GP-based safety controller in (12), relaxing the requirement of the known accurate model as in [6]. Despite probabilistic safety, the proposed controller only requires Lipschitz continuity of $\mathbf{d}(\cdot)$, which is common in nonlinear control [23] and less restrictive than other methods based on e.g., neural networks [10]. To guarantee the safety with higher probability, a more conservative approximation of the prediction error is non-negligible according to Lemma 1, inducing larger $\eta_j(\cdot), j = 1, \dots, m$ and causing potential infeasibility of the QP problem

in (12). To reduce the prediction errors and improve the feasibility, the incorporation of distributed GP [24] and online learning [25] is an efficient and promising way, which can be directly integrated into our framework.

Remark 1: The barrier functions h_i , $i = 1, \dots, n+1$ in (8)-(10), can be extended to a multi-dimensional function with $d_h > 1$, which represents multiple safety constraints. With respect to the extension of barrier functions, system safety is proven with a similar process from (15) to (29).

Remark 2: In this paper, the proposed method only guarantees the PTSf for the system (1) with state-dependent unknown dynamics, i.e., the uncertainty affected by control input is not included in the unknown dynamics. To broaden the applicability to a larger range of unknown systems e.g., consider an unknown $\mathbf{g}(\mathbf{x})\mathbf{u}$, the compound kernel trick [26] can be integrated to learn the unknown dynamics of $\mathbf{g}(\mathbf{x})\mathbf{u}$. However, how to sufficiently and safely excite the system for accurate predictions by choosing \mathbf{u} in the training dataset is still an open question, which is considered for future research.

Remark 3: In this paper, we propose a control method to pursue the PTSf, which is guaranteed if the QP form (12) is feasible for the system (1). In future extensions, this feasibility assumption could be relaxed by several techniques, for example, a back-up control law [12] or the online learning strategy [25], [27] can be designed to maintain feasibility.

IV. NUMERICAL EVALUATION

In this section, we consider a two-link robotic manipulator [28] with unit masses and unit length for each link. Based on the robot dynamics, the state space model as (1) is written as a second-order dynamics with $n = m = 2$ and

$$\mathbf{f}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x})(-\mathbf{C}(\mathbf{x}) - \mathbf{G}(\mathbf{x})), \quad \mathbf{g}(\mathbf{x}) = \mathbf{M}^{-1}(\mathbf{x}),$$

where $\mathbf{M}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$ are nominal inertia matrix, Coriolis and centrifugal term, and gravitational term from [28], respectively. The system states $\mathbf{x} = [\mathbf{x}_1^\top, \mathbf{x}_2^\top]^\top$ represents joint positions and joint velocities, which are expressed as $\mathbf{x}_1 = [q_1, q_2]^\top \in [-2\pi, 2\pi]^2$ and $\mathbf{x}_2 = [\dot{q}_1, \dot{q}_2]^\top \in [-10, 10]^2$. We consider unknown dynamics in (1) is $\mathbf{d}(\mathbf{x}) = [d_1(\mathbf{x}), d_2(\mathbf{x})]^\top = [5 \sin(q_1) + 3 \cos(q_2), 3 \cos(q_1) + 5 \sin(q_2) + 30]^\top$. To identify the system uncertainty, GP regression is used with the squared exponential kernel, i.e., $\kappa(\mathbf{x}, \mathbf{x}') = \sigma_f^2 \exp(-0.5l^{-2}\|\mathbf{x} - \mathbf{x}'\|^2)$, where $\sigma_f = 1$ and $l = 0.4$. The parameters of the error bound are chosen by $\delta = 0.01$ and $\tau = 10^{-10}$. For training the models, a data set \mathbb{D} with 900 data pairs are collected equally distributed on the domain $q_1, q_2 \in [-2\pi, 2\pi]$.

The safe region is defined such that the entire robot manipulator belongs to the first quadrant of Cartesian space in task space, which is equivalent to the green zone in Fig. 1 in joint space. Moreover, the safe region is also expressed through the functions $h_1^{(i)} : \mathbb{R}^2 \rightarrow \mathbb{R}$ for $\forall i = 1, \dots, 4$ as

$$\begin{aligned} h_1^{(1)}(\mathbf{x}_1) &= q_1, & h_1^{(3)}(\mathbf{x}_1) &= q_1 + q_2, \\ h_1^{(2)}(\mathbf{x}_1) &= -q_1 + \pi/2, & h_1^{(4)}(\mathbf{x}_1) &= -q_1 - q_2 + \pi/2. \end{aligned}$$

Each function $h_1^{(i)}(\cdot)$ introduces a constraint in (12), inducing a QP problem with 4 constraints.

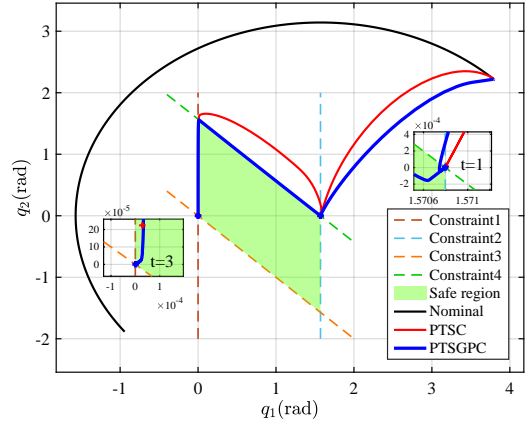


Fig. 1. Trajectory for robot manipulator under PTSC and PTSGPC.

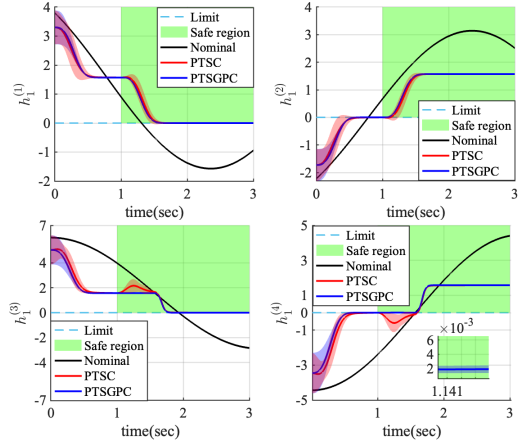


Fig. 2. Results of $h_1^{(i)}$ with $i = 1, \dots, 4$ for PTSC and PTSGPC in $t = [0, 3]$.

The nominal control task is to track a desired reference $\mathbf{q}_d(t) = [\pi \cos(t - \pi/4) + 0.5\pi, \pi \sin(t - \pi/4)]^\top$, such that the nominal control law \mathbf{u}_{nom} is designed in PD form as $\mathbf{u}_{nom} = \mathbf{K}_p(\mathbf{q}_d - \mathbf{x}_1) + \mathbf{K}_d(\dot{\mathbf{q}}_d - \dot{\mathbf{x}}_2)$ with $\mathbf{K}_p = \text{diag}(50, 50)$ and $\mathbf{K}_d = \text{diag}(25, 25)$. Set the simulation time as $t \in [0, 3]$, and notably the resultant path lies outside the safe region for $t \in [0, 3]$ as shown in Fig. 1. The initial state $\mathbf{x}(0)$ at $t_0 = 0$ is set as $\mathbf{x}(0) = [\mathbf{q}_d(0)^\top, \mathbf{0}_{1 \times 2}]^\top$ satisfying $\mathbf{x}(0) \notin \mathcal{C}$. The safety filter is designed as in (12) with $\alpha = 400$ and two time periods $t = [t_0^{(1)}, t_0^{(1)} + T_{pre}^{(1)}]$ as well as $t = [t_0^{(2)}, t_0^{(2)} + T_{pre}^{(2)}]$ with $t_0^{(1)} = 0, T_{pre}^{(1)} = t_0^{(2)} = 1, T_{pre}^{(2)} = 3$. In the first time period, starting from an unsafe initial condition, the safety objective is returning to the safe region. Then, the system states maintain within the safe region during the second time period, i.e., $h_1^{(i)} \in \mathbb{R}^+$, with $i = 1, \dots, 4$ for $\forall t \in [1, 3]$. In order to illustrate the validity of our proposed approach, the simulation is repeated 100 times to account for the randomness in unknown dynamics and initial states of the system, which are randomized uniformly in the range of $[\mathbf{d}(\mathbf{x}) - 15, \mathbf{d}(\mathbf{x}) + 15]$ and $[-1 + \mathbf{q}_d(0), \mathbf{q}_d(0)]$, respectively.

To demonstrate the superiority of the proposed prescribed-time safe Gaussian process control (PTSGPC), the prescribed-time safe control (PTSC) proposed in [6] is used for comparison. The desired reference trajectory and the state trajectory of PTSC and PTSGPC are shown in Fig. 1. The proposed PTSGPC properly addresses uncertainty, ensuring

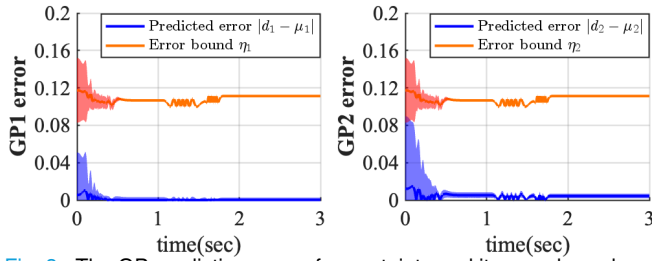


Fig. 3. The GP prediction error of uncertainty and its error bound.

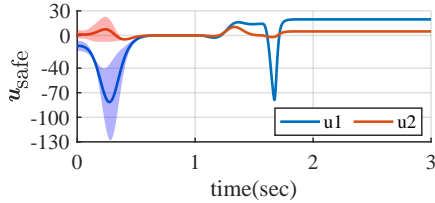


Fig. 4. The control signals of PTSCGP.

that the robot manipulator achieves the safety objective, which is also close to the nominal trajectory throughout the entire process. In contrast, although the PTSC enables the system to return to the safe region in a specified time, it fails to maintain the safety condition in the subsequent period due to the impact of uncertainty. Notably, the result of PTSC in Fig. 2 shows the negative value of $h_1^{(4)}$, which violates the 4th safety constraint, with high probability during the time period $t \in [1, 3]$. Conversely, in the case of PTSGPC, all values of $h_1^{(i)}$ turn positive with a 95% probability setting of $1 - m\delta$ after $t = 1$ attributed to the learning of uncertainty, which validates the Theorem 1 even if the system dynamics is partially known. The performance of unknown dynamics quantification is demonstrated in Fig. 3, which illustrates that the prediction error from GP regression is under the probabilistic error bound with 95% probability. The control input $\mathbf{u} = [u_1, u_2]^T$ from the proposed control law in Theorem 1 for the robot manipulator is shown in Fig. 4, where u_1 and u_2 are the control inputs in the first and second joints, respectively.

V. CONCLUSIONS

In this paper, we propose a safe learning control for control affine systems, that ensures the safety condition in a given prescribed time, independent of the initial state. By integrating the time-varying design and Gaussian process regression in the barrier function, the guarantee for system safety with a high probability is shown. The result shows that the system achieves the safety objective using our designed controller.

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