



Interacting Edge-Reinforced Random Walks

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Abstract. We consider the edge-reinforced random walk with multiple (but finitely many) walkers which influence the edge weights together. The walker which moves at a given time step is chosen uniformly at random, or according to a fixed order. First, we consider 2 walkers with linear reinforcement on a line graph comprising three nodes. We show that the edge weights evolve similarly to the setting with a single walker which corresponds to a Pólya urn. In particular, the left edge weight proportion is a martingale at certain stopping times, showing that a (random) limiting proportion exists. We then look at an arbitrary number of walkers on \mathbb{Z} with very general reinforcement. We show that in this case, the behaviour is also the same as for a single walker: either all walkers are recurrent or all walkers have finite range. In the particular case of reinforcements of “sequence type”, we give a criterion for recurrence.

1. Introduction

1.1. *Model.* In this paper, we study the edge-reinforced random walk with multiple walkers. The edge-reinforced random walk (ERRW) with a single walker is a stochastic process in discrete time defined on a graph. The edges in the graph are weighted, and the probability to leave a node via one of the incident edges is proportional to the respective edge weight compared to the weights of the other incident edges. Each time an edge is crossed, its weight is increased according to some reinforcement scheme. Thus, it becomes more likely to visit parts of the graph which have already been visited before and the process is not a Markov process. If the initial edge weights are the same everywhere and the edge weight increment is 1 (upon traversal), then the walk is called linearly edge-reinforced random walk (LERRW), the most commonly studied model of reinforced random walks.

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We extend the above model to include multiple walkers. There is a common set of edge weights for all walkers, and at every time step, one of the walkers moves (the walker to move may be selected at random, or according to some fixed order). The probability to take a specific edge is still proportional to the weight, and we keep the reinforcement: the weight of an edge is increased whenever any of the walkers crosses it. The main problem arising when studying multiple walkers is that the process is no longer partially exchangeable, in contrast to the single walker case, such that methods used previously are no longer applicable. More details are given in the following literature overview.

1.2. *Literature.* The ERRW (as well as its counterpart, the vertex-reinforced random walk) has been studied extensively, the first papers dating back to 1987, when the model was introduced by Coppersmith and Diaconis. Even before, [Diaconis and Freedman \(1980\)](#) showed that the LERRW (if certain assumptions are satisfied) has a representation as a mixture of Markov chains, i.e. a Markov chain with random transition probabilities, also known as random walk in random environment. In [Diaconis and Freedman \(1980\)](#), it was proved that the existence of such a representation as mixture of Markov chains is equivalent to the random walk being partially exchangeable, which means that two finite path fragments which the random walker can take starting from the same state have the same probability if the (directed) edges in the path are traversed the same number of times. Much later, [Merkl and Rolles \(2007\)](#) showed that this representation can be used for the LERRW on any graph, and [Merkl et al. \(2008\)](#) even gave a formula for the so-called mixing measure on finite graphs. The mixing measure simply is the distribution of the random transition probabilities in the mixture of Markov chains.

Relatively early, results for the LERRW on trees were obtained. [Pemantle \(1988\)](#); [Lyons and Pemantle \(1992, 2003\)](#) showed that there is a phase transition between recurrence and transience in the initial edge weights. The interesting case of \mathbb{Z}^d remained open much longer until [Angel et al. \(2014\)](#) showed that the LERRW on \mathbb{Z}^2 and on any graph of bounded degree is recurrent for sufficiently small initial weights, and until [Angel et al. \(2014\)](#); [Sabot and Tarrès \(2015\)](#); [Disertori et al. \(2015\)](#) showed that for $d \geq 3$, there is again a transition from recurrence to transience in the initial weights. [Poudevigne-Auboiron \(2024\)](#) proved that this transition is sharp, i.e. there is a certain critical initial edge weight such that for smaller initial weights, the random walk is recurrent, and transient for larger weights. [Poudevigne-Auboiron \(2024, Theorem 4\)](#) also demonstrated that the LERRW on \mathbb{Z}^2 is recurrent for any choice of initial edge weight (if every edge has the same initial weight), and thereby answered one of the most interesting remaining questions on the LERRW on the lattice \mathbb{Z}^d .

These results were obtained by exploiting a connection of the LERRW and the so-called vertex-reinforced jump process (VRJP). The VRJP is a continuous-time process $(Y_t)_{t \geq 0}$ on a graph with conductances c_e on every edge e . If the process is at vertex v at time t , then, conditional on $(Y_s, s \leq t)$, the process jumps to neighbor u of v at rate $c_{\{v,u\}} \cdot L_u(t)$ where $\{v,u\}$ is the edge between v and u , and where

$$L_u(t) = 1 + \int_0^t \mathbb{1}_{Y_s=u} \, ds.$$

When using independent Gamma-distributed conductances, the distribution of the VRJP at jump times corresponds to the distribution of the LERRW. The VRJP, in turn, and similarly to the LERRW, is distributed as a mixture of time-changed Markov jump processes which are more amenable to analysis on \mathbb{Z}^d than the mixing measure of the LERRW.

An overview of results on reinforced processes in general can be found in [Pemantle \(2007\)](#); [Kozma \(2012\)](#). These surveys also show that reinforced random walks are closely related to urn processes, which have a very similar reinforcement component to the linearly reinforced walks: in most urn models, when a ball of a certain color is drawn, a fixed number of balls of the same color is added

to the urn. This is also a type of linear reinforcement, and urns have been used on many occasions to analyze reinforced walks (see, for example, Pemantle (1988)).

Results for multiple interacting reinforced random walks are scarce. In Erhard and Reis (2021), the authors consider multiple ant random walks. However, the ant random walk is defined via a reinforcement rule on oriented edges instead of on non-oriented edges (as for the ERRW). Erhard and Reis (2021) also consider the case of superlinear reinforcement and compare the reinforced walk with Pólya urns.

Edge-Reinforced walks with multiple walkers on non-oriented graphs have (to the best of our knowledge) not been studied yet. For multiple walkers, there is no representation as a mixture of Markov chains, even in the case with linear reinforcement. This is due to the fact that the interacting random walks are no longer partially exchangeable. Therefore, most of the methods which have been used to study the LERRW cannot be applied to the case of multiple walkers. However, our paper is partially based on methods which have been used to study urn processes as there still is a similarity to urn models where multiple balls are drawn from the urn. In particular, Crimaldi (2016) studied urns where multiple balls are drawn and replaced, and new balls are added according to a random distribution depending on the balls which were drawn. The methods used to study the limiting proportions of balls in the urn are also useful to study two edge-reinforced walkers on a 3-node segment ($\bullet \text{---} \bullet \text{---} \bullet$) and the limiting proportions of the corresponding edge weights. A comprehensive overview of urn models and methods to analyse them may be found in Mahmoud (2009). In particular, Mahmoud (2009, Chapter 10) also considers urns where multiple balls are drawn. As will become apparent later, however, none of the previously studied urn models exactly match the model we study.

Apart from using methods from the theory on urn models, we can also fall back on Davis (1990) who studied the ERRW on \mathbb{Z} with very general reinforcement (and not just linear reinforcement) without using the representation as a mixture of Markov chains. Davis (1990) showed that a single walker is either recurrent or visits only a finite part of \mathbb{Z} . The methods used there can be adapted for multiple walkers, but the method is specific to \mathbb{Z} .

1.3. *Results.* In the following, the random walkers will be denoted by sequences $(X_n^{(i)})_{n \geq 0}$, where $X_n^{(i)}$ is equal to the position of walker i at time n , so $X_n^{(i)}$ is a node of the graph. We will show the following four main results:

- **Theorem 3.1:** consider two walkers on a 3-node segment with linear reinforcement where the walkers move alternately. Then, the proportion of the left edge weight converges to a random limit which has no atoms. Specifically, if we denote the weights of the left and right edges at time n by $w(n, -1)$ and $w(n, 0)$, respectively, then the fraction $\frac{w(n, -1)}{w(n, -1) + w(n, 0)}$ converges to a random limit M_∞ which satisfies $\mathbb{P}[M_\infty = a] = 0$ for any $a \in [0, 1]$.
- **Theorem 3.3:** consider two walkers on a 3-node segment with linear reinforcement where the walker to move is selected uniformly at random at every step. Then, the proportion of the left edge weight converges to a random limit. Furthermore, the left edge weight proportion is a martingale if looked at at certain stopping times. Specifically, the fraction $\frac{w(n, -1)}{w(n, -1) + w(n, 0)}$ converges to a random limit, which is identical to the limit of the martingale $\frac{w(\tau_n, -1)}{w(\tau_n, -1) + w(\tau_n, 0)}$, where $\tau_n = \inf \left\{ m > \tau_{n-1} : X_m^{(1)} \text{ and } X_m^{(2)} \text{ are both in the center node} \right\}$.
- **Theorem 4.1:** consider K walkers on \mathbb{Z} with very general reinforcement, but where all but finitely many initial weights are 1, and where the walker to move is selected uniformly at random. Then, either all walkers are recurrent or all walkers have finite range (visit only a

finite part of \mathbb{Z}) a.s. That is, under these assumptions,

$$\mathbb{P} \left[\forall i : X^{(i)} \text{ is recurrent} \right] + \mathbb{P} \left[\forall i : X^{(i)} \text{ has finite range} \right] = 1.$$

- **Theorem 4.2:** consider K walkers on \mathbb{Z} with sequence type reinforcement where the walker to move is selected uniformly at random. Then, we can characterize the two possible behaviors depending on the sequence of edge weight increments. “Sequence type” means that all initial edge weights are 1, and that the edge weights are then increased by a fixed sequence of increments $\mathbf{a} = (a_k)_{k \geq 1}$ which do not depend on the location of the edge. Specifically, if we set

$$\phi(\mathbf{a}) = \sum_{k=1}^{\infty} \left(1 + \sum_{l=1}^k a_l \right)^{-1},$$

then, if $\phi(\mathbf{a}) = \infty$, all walkers are recurrent a.s., and if $\phi(\mathbf{a}) < \infty$, all walkers have finite range a.s.

1.4. *Methods.* Section 3.1 treats the case of two walkers on the 3-node segment. When considering alternating walkers, the analysis of the left edge weight proportion $\frac{w(n,-1)}{w(n,-1)+w(n,0)}$ mostly relies on relatively straightforward calculations which show that the proportion is a martingale at times $4n$. Since this martingale is bounded, the existence of a limit can be derived. We will use the notation $M_n = \frac{w(4n,-1)}{w(4n,-1)+w(4n,0)}$, and use M_∞ to denote the (random) limit. Subsequently, methods taken from Crimaldi (2016) allow us to show the following central limit type result in Theorem 3.1 (ii):

$$\sqrt{n}(M_n - M_\infty) \rightarrow \mathcal{N} \left(0, \frac{1}{2} M_\infty (1 - M_\infty) \right)$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes a normal distribution. This result, and the comparison to the case with a single walker (which is in 1-to-1 correspondence to a Pólya urn where two balls of the drawn color are added after every draw) allow us to show that M_∞ has no atoms. The question of whether M_∞ has a density remains open.

For the 3-node segment with two walkers where the walker to move is selected randomly, the straightforward calculations from the previous case become much more complicated. Using a recursive formula for the conditional expectation, we can show that $\frac{w(\tau_n,-1)}{w(\tau_n,-1)+w(\tau_n,0)}$ is a martingale, where $\tau_n = \inf \left\{ m > \tau_{n-1} : X_m^{(1)} \text{ and } X_m^{(2)} \text{ are both in the center node} \right\}$. This again results in the existence of a limit random variable. The method we used for the alternating walkers to show that no atoms exist might be applicable in this case as well, but we could not carry through some of the necessary calculations.

Section 4 treats a finite number of walkers K on \mathbb{Z} . To show that all walkers are recurrent or all have finite range, we use a martingale technique which originates from Davis (1990). In particular, if you sum up the inverse of the edge weights between 0 and the position of one of the walkers, the resulting sum is a nonnegative supermartingale. By convergence of this supermartingale, we can deduce that every walker either reaches 0 at some point or only visits finitely many nodes which have not been visited before by any of the other walkers. In addition, you can exchange tails of the paths of walkers after they met in a certain node without affecting the law of the overall process. These two key lemmata allow us to show that all walkers will have the same behavior.

Finally, in order to characterize the two possible behaviors for sequence type reinforcements, we only have to adapt the method from Davis (1990) slightly in order to apply it to multiple walkers simultaneously. The proof idea here is to show that all walkers get stuck on a single edge with positive probability if the weight increments grow fast enough. On the other hand, one can show

that the neighbor of a node which is visited infinitely often is visited infinitely often as well if the increments grow slower.

2. Preliminaries

2.1. *Graphs.* Throughout this paper, random walks on \mathbb{Z} and on the 3-node segment will be considered. Recall that \mathbb{Z} has node set $V = \mathbb{Z}$ and edge set $E = \{\{u, v\} \mid u, v \in V \text{ and } |u - v| = 1\}$. The 3-node segment is the subgraph P_3 of \mathbb{Z} with vertex set $\{-1, 0, 1\}$ and edges $\{\{-1, 0\}, \{0, +1\}\}$.

2.2. *Edge-reinforced random walk.* We define the edge-reinforced random walk, which is the main object of this paper.

Definition 2.1 (Edge-Reinforced Random Walk). For this definition we consider only the graph \mathbb{Z} . We define the crossing number induced by the **edge-reinforced random walk** (ERRW) $(X_n)_{n \geq 0}$ on the edge $\{x, x+1\}$ (which we will identify with x for more concise notation) as follows: $c(0, x) = 0$, and

$$\text{for } n \geq 1 : c(n, x) = \sum_{i=1}^n \mathbb{1}_{\{\{X_{i-1}, X_i\} = \{x, x+1\}\}}, \quad w(n, x) := W_x(c(n, x)),$$

i.e. c counts the number of edge traversals and w describes the evolution of the edge weights. For $x \in \mathbb{Z}$, $W_x : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}_{>0}$ is a non-decreasing function, called the reinforcement function for the edge $\{x, x+1\}$. The choice of reinforcement $w(n, x) = 1 + c(n, x)$ is called **linearly edge-reinforced random walk** (LERRW).

Given the history $\mathcal{H}_n = \sigma(X_m : m \leq n)$, we have

$$\begin{aligned} \mathbb{P}[X_{n+1} = X_n + 1 \mid \mathcal{H}_n] &= \frac{w(n, X_n)}{w(n, X_n - 1) + w(n, X_n)} \\ \mathbb{P}[X_{n+1} = X_n - 1 \mid \mathcal{H}_n] &= \frac{w(n, X_n - 1)}{w(n, X_n - 1) + w(n, X_n)}. \end{aligned}$$

In words, the probability to take the transition from X_n to $X_n + 1$ at time n is proportional to the weight $w(n, X_n)$ associated to the edge $\{X_n, X_n + 1\}$, which depends on the number of traversals of the edge.

In the following, we will generalize the edge-reinforced random walk to include multiple walkers which move on the same graph and influence the edge weights together. We start by looking at a 3-node segment with two walkers in [Section 3](#), and then consider \mathbb{Z} with an arbitrary but finite number of walkers in [Section 4](#).

2.3. Almost sure conditional convergence.

Definition 2.2. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Consider a real random variable X on Ω and a sub- σ -field \mathcal{F} of \mathcal{G} . We define the **conditional distribution** $\mu_{X|\mathcal{F}}$ of X given \mathcal{F} as follows: $\mu_{X|\mathcal{F}}$ is a mapping from Ω to the set of probability measures on \mathbb{R} (with the Borel σ -algebra). Let $A \subseteq \mathbb{R}$ be a measurable set. Then

$$\mu_{X|\mathcal{F}}(A) = \mathbb{E}[\mathbb{1}_{X \in A} \mid \mathcal{F}]$$

Note that $\omega \mapsto \mu_{X|\mathcal{F}}(A)(\omega)$ is \mathcal{F} -measurable for every measurable set A .

Consider now a sequence X_n of random variables on Ω and a sequence of sub- σ -fields \mathcal{F}_n . Denote by μ_n the conditional distribution of X_n given \mathcal{F}_n . Let μ be a random probability measure on \mathbb{R} , where μ maps from Ω to the set of probability measures on \mathbb{R} , and $\omega \mapsto \mu(A)(\omega)$ is \mathcal{G} -measurable for every set $A \subseteq \mathbb{R}$ which is Borel-measurable. We say that X_n converges to μ in the sense of **almost sure conditional convergence** with respect to the conditioning system \mathcal{F}_n if μ_n converges weakly

to μ for almost every $\omega \in \Omega$. Notice that $X_n \rightarrow \mu$ in the sense of almost sure conditional convergence with respect to $(\mathcal{F}_n)_{n \geq 0}$ if, and only if, for all $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous and bounded it holds that

$$\mathbb{E}[f(X_n) | \mathcal{F}_n](\omega) \rightarrow \int f(\bar{\omega}) \mu_\omega(d\bar{\omega}), \quad (2.1)$$

where μ_ω represents a realization of the random measure μ .

The notion of almost sure conditional convergence is stronger than the notion of weak convergence: in the setting of [Definition 2.2](#), using the dominated convergence theorem, (2.1) implies that, for all continuous and bounded function f ,

$$\mathbb{E}[f(X_n)] \rightarrow \int \int f(\bar{\omega}) \mu_\omega(d\bar{\omega}) \mathbb{P}(d\omega),$$

that is, the sequence $(X_n)_{n \geq 0}$ converges weakly to the distribution defined through

$$\bar{\mu}(A) := \int \mu_\omega(A) \mathbb{P}(d\omega).$$

The following lemma will be useful.

Lemma 2.3. *Let μ be a random probability measure and*

$$\bar{\mu}(A) := \int \mu_\omega(A) \mathbb{P}(d\omega).$$

Assume that $(X_n)_{n \geq 0}$ converges to μ in the sense of almost sure conditional convergence. Let $(Y_n)_{n \geq 0}$ be a sequence of random variables such that

$$\lim_{n \rightarrow \infty} |X_n - Y_n| = 0, \text{ almost surely.}$$

Then the laws of Y_n converge weakly to $\bar{\mu}$ for $n \rightarrow \infty$.

Proof: Observe that

$$|\mathbb{E}[f(X_n)] - \mathbb{E}[f(Y_n)]| \leq \mathbb{E}[|f(X_n) - f(Y_n)|],$$

which goes to zero using the dominated convergence theorem. \square

In this work $\mathcal{N}(0, V)$ denotes the distribution of a normal random variable with mean zero and (random) variance V . We write $\bar{\mathcal{N}}(0, V)$ for the mixture of the laws $\mathcal{N}(0, V)$ when we average on the randomness of V :

$$\bar{\mathcal{N}}(0, V)(A) := \int \mathcal{N}(0, V)(A) d\mathcal{L}(V).$$

Theorem 2.4. *Let M_n be a bounded martingale with respect to a filtration \mathcal{F}_n and let V be a random variable. Let M_∞ be the random variable such that $M_n \rightarrow M_\infty$ a.s. and in L^1 . Assume that*

- (i) $\mathbb{E} \left[\sup_{n \geq 1} \sqrt{n} |M_{n-1} - M_n| \right] < \infty$
- (ii) $n \sum_{j \geq n} (M_{j-1} - M_j)^2 \rightarrow V$ a.s.

Then, the following convergence holds in the sense of almost sure conditional convergence with respect to \mathcal{F}_n :

$$\sqrt{n}(M_n - M_\infty) \rightarrow \mathcal{N}(0, V).$$

Proof: See [Crimaldi \(2016, Theorem A.1 \(A.3 in the arxiv version\)\)](#). The stated version is just a simplified version of it. \square

Lemma 2.5. *Let X_n be a sequence of random variables adapted to the filtration \mathcal{F}_n and let X be a random variable. Assume that*

- (i) $\sum_{n \geq 1} n^{-2} \mathbb{E} [X_n^2] < \infty$ and
- (ii) $\mathbb{E} [X_n | \mathcal{F}_{n-1}] \rightarrow X$ a.s.

Then

$$n \sum_{j \geq n} \frac{X_j}{j^2} \rightarrow X \text{ a.s.}$$

Proof: See [Crimaldi \(2016, Lemma A.2\)](#). □

3. Two walkers on a segment of 3 nodes

3.1. *The model.* We start with a very simple model, the linearly edge-reinforced random walk with two walkers on a segment of \mathbb{Z} with three nodes. We use the notation shown in [Figure 3.1](#).

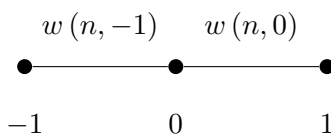


FIGURE 3.1. Edge weights of the line segment at time n

We can interpret this as a two-player urn: the left and right edge weights correspond to the number of balls in an urn, and the two walkers correspond to two players which draw balls from the urn and add more balls of the drawn color.

We adapt [Definition 2.1](#) for two walkers on the 3-node segment as follows:

- There are two walkers $X^{(1)}$ and $X^{(2)}$, which both start at the node in the center, i.e. $X_0^{(1)} = 0, X_0^{(2)} = 0$.
- Initially, both edge weights are 1. We denote the edge weight of the left edge at time n by $w(n, -1)$, the weight of the right edge by $w(n, 0)$.
- Whenever an edge is crossed by either of the walkers, its weight is increased by 1, which corresponds to the linear reinforcement function $W(n) = 1 + n$ in [Definition 2.1](#). More specifically, we define, for $\ell \in \{-1, 0\}$, $c(0, \ell) = 0$ and

$$\text{for } n \geq 1 : c(n, \ell) = \sum_{k=1}^2 \sum_{i=1}^n \mathbb{1}_{\{\{X_{i-1}^{(k)}, X_i^{(k)}\} = \{\ell, \ell+1\}\}}, \quad \text{and} \quad w(n, \ell) = 1 + c(n, \ell).$$

- When a walker at the node in the center is about to move, he chooses the edge to traverse with probability proportional to the respective edge weight. It remains to decide which walker jumps at each time.
- We will consider two variants of the order in which the walkers move:
 - **Alternating walkers:** the walkers move alternately, i.e. at odd time steps, walker 1 moves (in particular, walker 1 moves first at step 1) and at even time steps, walker 2 moves. Notice that the walkers will meet at the node in the center every four steps. As an example, given the history $\mathcal{H}_{4n} = \sigma(X_m^{(j)} : m \leq 4n, j = 1, 2)$, $X_{4n+1}^{(2)} = X_{4n}^{(2)}$ and $X_{4n+1}^{(1)} = X_{4n}^{(1)} + 1$ with probability

$$\frac{w(4n, 0)}{w(4n, -1) + w(4n, 0)}.$$

- **Random walker selection:** at every step, we choose uniformly at random (independently of all other steps) which of the two walkers moves. As an example, given the history $\mathcal{H}_n = \sigma(X_m^{(j)} : m \leq n, j = 1, 2)$, we select the walker 1 to jump with probability $\frac{1}{2}$, and assuming that $X_n^{(1)} = 0$, we have that $X_{n+1}^{(2)} = X_n^{(2)}$ and $X_{n+1}^{(1)} = X_n^{(1)} + 1$ with probability

$$\frac{w(n, 0)}{w(n, -1) + w(n, 0)}.$$

3.2. *Results.* We define

$$N_n := \frac{w(n, -1)}{w(n, -1) + w(n, 0)}$$

and let $\mathcal{G}_n := \sigma(N_0, \dots, N_n)$. Our goal is to prove limit theorems for the sequence $(N_n)_{n \geq 0}$.

To illustrate our methods, consider first the special case of just one walker on the 3-node graph. The walker decides to jump either to the right or to the left and afterwards comes back again to the center. As a consequence, the number of visits of each edge at times 0, 2, 4, ... has the same distribution as the number of balls in a Pólya urn where we add 2 more balls of the drawn color at every step. It is a classical result that the proportion of balls in the Pólya urn is a martingale.

The case of two alternating walkers can also be described in terms of an urn:

- (1) We start with one black and one white ball (the number of black balls corresponds to the left edge weight, and the number of white balls to the right edge weight).
- (2) We draw a ball from the urn, and replace it along with another ball of the same color (this corresponds to the first walker choosing an edge to traverse in outwards direction and incrementing the respective edge weight by 1). We then put another ball of the drawn color on hold, outside the urn.
- (3) We draw another ball from the urn, and replace it along with two more balls of the drawn color as well as with the ball on hold (this corresponds to the second walker choosing an edge to traverse in both directions and incrementing its weight by 2, and to the first walker returning to the central node in between). We then go back to (2) and repeat the procedure.

In contrast to previously studied urn models, we cannot describe our model as an urn where multiple balls are drawn and then immediately replaced along with additional balls whose colors depend on the sample drawn. Instead, we have to put some balls on hold before placing them in the urn. Again, we note that the sequence of drawn colors is no longer exchangeable as it is for the Pólya urn. However, we can still use some of the methods which have been applied to study standard urns – in particular the martingale property of the proportion of balls.

In the case of two walkers which move alternately, the walkers meet at node 0 after every 4 steps. As we will prove, the proportion of the left edge weight at times $4n$ is a martingale. This motivates the definition

$$M_n^{\text{alt}} := N_{4n}, \forall n \geq 0.$$

M_n^{alt} is the proportion of the weight of the left edge $\{-1, 0\}$ at times $(4n)_{n \geq 0}$.

Theorem 3.1. *Consider the model where the walkers jump alternately with the corresponding sequence $(N_n)_{n \geq 0}$ of random variables. Then*

- (i) $(M_n^{\text{alt}})_{n \geq 0}$ is a bounded martingale and we define

$$M_\infty^{\text{alt}} := \lim_{n \rightarrow \infty} M_n^{\text{alt}}.$$

We have that $\lim_{n \rightarrow \infty} |N_n - M_\infty^{\text{alt}}| = 0$ almost surely.

(ii) The following convergence holds in the sense of almost sure conditional convergence w.r.t. the filtration $(\mathcal{G}_n)_{n \geq 0}$:

$$\sqrt{n} \left(M_n^{\text{alt}} - M_\infty^{\text{alt}} \right) \rightarrow \mathcal{N} \left(0, \frac{1}{2} M_\infty^{\text{alt}} \left(1 - M_\infty^{\text{alt}} \right) \right). \tag{3.1}$$

As a consequence of Lemma 2.3, the following convergence holds in the weak sense:

$$\sqrt{n} \left(N_n - M_\infty^{\text{alt}} \right) \rightarrow \overline{\mathcal{N}} \left(0, \frac{1}{2} M_\infty^{\text{alt}} \left(1 - M_\infty^{\text{alt}} \right) \right).$$

(iii) $\mathbb{P} [M_\infty^{\text{alt}} = a] = 0$ for any $a \in [0, 1]$. As a consequence, M_∞^{alt} is not deterministic.

We furthermore believe that the following is true (partly based on simulations).

Conjecture 3.2. $M_\infty^{\text{alt}} \in [0, 1]$ has a density w.r.t. the Lebesgue measure on $[0, 1]$. M_∞^{alt} is not Beta-distributed.

For the case of two walkers and random walker selection, we define the following sequence of stopping times: $\tau_0 = 0$ and, recursively,

$$\tau_n = \inf \left\{ m > \tau_{n-1} : X_m^{(1)} = X_m^{(2)} = 0 \right\}.$$

Furthermore, we define the sequence $(M_n^{\text{rd}})_{n \geq 0}$ as follows:

$$M_n^{\text{rd}} = N_{\tau_n}, \forall n \geq 0.$$

Theorem 3.3. Consider the model where the walker to move is selected randomly with the corresponding sequence $(N_n)_{n \geq 0}$ of random variables. Then

(i) $(M_n^{\text{rd}})_{n \geq 0}$ is a bounded martingale and we define

$$M_\infty^{\text{rd}} := \lim_{n \rightarrow \infty} M_n^{\text{rd}}.$$

(ii) The sequence of stopping times $(\tau_n)_{n \geq 0}$ satisfies the following:

$$\begin{aligned} \mathbb{P} [\tau_{n+1} - \tau_n = 2l] &= 2^{-l} \text{ for all } l \geq 1, \\ \mathbb{E} [\tau_{n+1} - \tau_n] &= 4. \end{aligned}$$

(iii) $\lim_{n \rightarrow \infty} N_n = M_\infty^{\text{rd}}$ almost surely.

We prove Theorem 3.1 and Theorem 3.3 in Section 3.3 and Section 3.4.

3.3. Proofs: alternating walkers (Theorem 3.1).

Proof of Theorem 3.1 (i): In this section, we write M_n instead of M_n^{alt} . We first prove that $(M_n)_{n \geq 0}$ is a martingale. We assume that at time $4n$, the edge weights are given by $w(4n, -1) = a, w(4n, 0) = b$. Recall that the walkers move alternately and meet at the center node every four steps. Therefore, on the time interval $\{4n + 1, 4n + 2, 4n + 3, 4n + 4\}$, the following possibilities occur:

- $X^{(1)}$ and $X^{(2)}$ move alternately from 0 to -1 and back with probability

$$\frac{a}{a+b} \frac{a+1}{a+b+1}$$

and the new edge weights are $w(4n + 4, -1) = a + 4, w(4n + 4, 0) = b$.

- $X^{(1)}$ and $X^{(2)}$ move alternately from 0 to 1 and back with probability

$$\frac{b}{a+b} \frac{b+1}{a+b+1}$$

and the new edge weights are $w(4n + 4, -1) = a, w(4n + 4, 0) = b + 4$.

- $X^{(1)}$ moves from 0 to 1 and $X^{(2)}$ moves from 0 to -1 . $X^{(1)}$ and $X^{(2)}$ move back to 0 with probability 1. This event happens with probability

$$\frac{b}{a+b} \frac{a}{a+b+1}$$

and the new edge weights are $w(4n+4, -1) = a+2, w(4n+4, 0) = b+2$.

- $X^{(1)}$ moves from 0 to -1 and $X^{(2)}$ moves from 0 to 1. After that $X^{(1)}$ and $X^{(2)}$ move back to 0 with probability 1. This event happens with probability

$$\frac{a}{a+b} \frac{b}{a+b+1}$$

and the new edge weights are $w(4n+4, -1) = a+2, w(4n+4, 0) = b+2$.

Therefore,

$$\begin{aligned} & \mathbb{E} \left[\frac{w(4n+4, -1)}{w(4n+4, -1) + w(4n+4, 0)} \mid w(4n, -1) = a, w(4n, 0) = b \right] \\ &= \frac{a}{a+b} \frac{a+1}{a+b+1} \frac{a+4}{a+b+4} + \frac{a}{a+b} \frac{b}{a+b+1} \frac{a+2}{a+b+4} \\ & \quad + \frac{b}{a+b} \frac{a}{a+b+1} \frac{a+2}{a+b+4} + \frac{b}{a+b} \frac{b+1}{a+b+1} \frac{a}{a+b+4} \\ &= \frac{a}{a+b} \cdot \frac{(a+1)(a+4) + 2b(a+2) + b(b+1)}{(a+b+1)(a+b+4)} =: \frac{a}{a+b} E_{ab}. \end{aligned}$$

To see that $E_{ab} = 1$, it is enough to count how many terms of the form a^2 (respectively, b^2, b, a , and ab) appear in the numerator and in the denominator.

The martingale $(M_n)_{n \geq 0}$ is bounded and therefore $M_n = \frac{w(4n, -1)}{4n+2}$ converges a.s. to a random variable M_∞ . We claim that $N_n = \frac{w(n, -1)}{n+2}$ also converges to M_∞ . Indeed, for $i \in \{1, 2, 3\}$, we have

$$\lim_{n \rightarrow \infty} \left| \frac{w(4n+i, -1)}{4n+i+2} - \frac{w(4n, -1)}{4n+2} \right| = 0,$$

because $|w(4n+i, -1) - w(4n, -1)| \leq i$. □

Proof of Theorem 3.1 (ii): In this section, we write again M_n instead of M_n^{alt} . The proof of Theorem 3.1 (ii) uses Theorem 2.4. Indeed, we only have to verify the two assumptions of Theorem 2.4, with $V = \frac{1}{2}M_\infty(1 - M_\infty)$.

First step / condition (i): it holds that

$$\mathbb{E} \left[\sup_{n > 1} \sqrt{n} |M_{n-1} - M_n| \right] \leq 1 < \infty. \tag{3.2}$$

Since $|N_n - N_{n-1}|$ is bounded by $\frac{1}{(n-1)+2}$, we get that $|M_{n-1} - M_n| \leq \frac{4}{4(n-1)+2}$. Thus

$$\sqrt{n} |M_{n-1} - M_n| \leq \frac{4\sqrt{n}}{4(n-1)+2} < 1$$

which implies that (3.2) is true.

Second step / condition (ii): the following limit holds almost surely:

$$n \sum_{j \geq n} (M_{j-1} - M_j)^2 \rightarrow \frac{1}{2} M_\infty (1 - M_\infty). \tag{3.3}$$

To prove (3.3), we use Lemma 2.5 with $X_n := n^2 (M_{n-1} - M_n)^2$. Condition (i) of Lemma 2.5 is a direct consequence of $|M_{n-1} - M_n| \leq \frac{4}{4(n-1)+2}$.

Our goal (showing condition (ii) of Lemma 2.5): to show that the following holds almost surely, where $\mathcal{F}_n := \mathcal{G}_{4n} = \sigma(N_0, \dots, N_{4n})$:

$$\mathbb{E} \left[n^2 (M_{n-1} - M_n)^2 \mid \mathcal{F}_{n-1} \right] \rightarrow \frac{1}{2} M_\infty (1 - M_\infty) .$$

We claim that

$$\mathbb{E} [M_n^2 \mid \mathcal{F}_{n-1}] = \left(1 - \frac{1}{2n^2} \right) M_{n-1}^2 + \frac{1}{2n^2} M_{n-1} + \Delta_n \quad \text{with} \quad n^2 \Delta_n \xrightarrow{\text{a.s.}} 0 . \tag{3.4}$$

Assuming the claim it follows that

$$\begin{aligned} \mathbb{E} \left[n^2 (M_{n-1} - M_n)^2 \mid \mathcal{F}_{n-1} \right] &= n^2 \mathbb{E} [M_n^2 \mid \mathcal{F}_{n-1}] - n^2 M_{n-1}^2 \\ &= \left(n^2 - \frac{1}{2} \right) M_{n-1}^2 + \frac{1}{2} M_{n-1} + n^2 \Delta_n - n^2 M_{n-1}^2 \\ &= \frac{1}{2} M_{n-1} (1 - M_{n-1}) + n^2 \Delta_n \xrightarrow{\text{a.s.}} \frac{1}{2} M_\infty (1 - M_\infty) , \end{aligned}$$

as we wanted to prove.

We proceed to show (3.4). By definition,

$$n^2 \Delta_n = n^2 \left(\mathbb{E} [M_n^2 \mid \mathcal{F}_{n-1}] - \left(1 - \frac{1}{2n^2} \right) M_{n-1}^2 - \frac{1}{2n^2} M_{n-1} \right) .$$

We condition on $w(4(n-1), -1) = a, w(4(n-1), 0) = b$, similarly to the proof of Theorem 3.1 (i), and get

$$\begin{aligned} \mathbb{E} [M_n^2 \mid \mathcal{F}_{n-1}] &= \frac{a}{a+b} \cdot \frac{a+1}{a+b+1} \cdot \frac{(a+4)^2}{(a+b+4)^2} + 2 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b+1} \cdot \frac{(a+2)^2}{(a+b+4)^2} \\ &\quad + \frac{b}{a+b} \cdot \frac{b+1}{a+b+1} \cdot \frac{a^2}{(a+b+4)^2} \\ &= \frac{a}{a+b} \cdot \left(\frac{a+1}{a+b+1} \cdot \frac{(a+4)^2}{(a+b+4)^2} + \frac{2b}{a+b+1} \cdot \frac{(a+2)^2}{(a+b+4)^2} \right. \\ &\quad \left. + \frac{b(b+1)}{a+b+1} \cdot \frac{a}{(a+b+4)^2} \right) . \end{aligned}$$

Furthermore, under the same conditioning, and using that $n = \frac{a+b+2}{4}$,

$$\begin{aligned} \left(1 - \frac{1}{2n^2} \right) M_{n-1}^2 &= \left(1 - \frac{8}{(a+b+2)^2} \right) \frac{a^2}{(a+b)^2} \\ \frac{1}{2n^2} M_{n-1} &= \frac{8}{(a+b+2)^2} \cdot \frac{a}{a+b} . \end{aligned}$$

Putting this together (see <https://bit.ly/3I8rEWL>), we arrive at

$$\begin{aligned} \Delta_n &= - \underbrace{\frac{8}{(a+b+2)^2}}_{\frac{1}{2n^2}} \cdot \underbrace{\frac{a}{a+b} \cdot \frac{b}{a+b}}_{M_{n-1}(1-M_{n-1})} \cdot \frac{3a^2 + 6ab + 12a + 3b^2 + 12b + 8}{(a+b+1)(a+b+4)^2} , \\ n^2 \Delta_n &= - \frac{1}{2} M_{n-1} (1 - M_{n-1}) \cdot \frac{3a^2 + 6ab + 12a + 3b^2 + 12b + 8}{(a+b+1)(a+b+4)^2} . \end{aligned}$$

Using again $n = \frac{a+b+2}{4}$, we can rewrite the fraction on the right (see <http://bit.ly/3ICUJLz>) to get to the final conclusion

$$\begin{aligned} n^2 \Delta_n &= n^2 \left(\mathbb{E} [M_n^2 \mid \mathcal{F}_{n-1}] - \left(1 - \frac{1}{2n^2}\right) M_{n-1}^2 - \frac{1}{2n^2} M_{n-1} \right) \\ &= -\frac{1}{2} M_{n-1} (1 - M_{n-1}) \left(\frac{7}{9(n + \frac{1}{2})} - \frac{1}{6(n + \frac{1}{2})^2} - \frac{1}{36(n - \frac{1}{4})} \right) \xrightarrow{n \rightarrow \infty} 0. \quad \square \end{aligned}$$

Proof of Theorem 3.1 (iii): In this section, we write again M_n instead of M_n^{alt} . Assuming Theorem 3.1 (ii), we now prove Theorem 3.1 (iii).

We first prove that $\mathbb{P}(M_\infty = a) = 0$ if $a \in (0, 1)$. Afterwards we prove that $\mathbb{P}(M_\infty = 0) = 0$. The proof of $\mathbb{P}(M_\infty = 1) = 0$ follows by symmetry since we can swap the labels of the left and the right node to see that the law of the proportion of the right edge weight is equal to the law of the proportion of the left edge weight.

We follow the proof of Crimaldi (2016, Corollary 3). Set $A = \{M_\infty = a\}$ for $a \in (0, 1)$. First observe that, as a consequence of Theorem 3.1 (ii) (recall that $\mathcal{F}_n := \mathcal{G}_{4n} = \sigma(N_0, \dots, N_{4n})$),

$$\mathbb{E} [\exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] \xrightarrow{\text{a.s.}} \exp\left(-\frac{t^2 \cdot \frac{1}{2} M_\infty (1 - M_\infty)}{2}\right).$$

We also have that $I_n := \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}_n] \rightarrow \mathbb{1}_A$ a.s.. Therefore, for any $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} [I_n \exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] &= I_n \mathbb{E} [\exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] \\ &\xrightarrow{\text{a.s.}} \mathbb{1}_A \cdot \exp\left(-\frac{t^2 \cdot \frac{1}{2} M_\infty (1 - M_\infty)}{2}\right). \end{aligned} \tag{3.5}$$

Since $\mathbb{1}_A - I_n \rightarrow 0$ almost surely, it follows from the dominated convergence theorem for conditional expectation (see Crimaldi (2016, Lemma A.1)) that

$$\mathbb{E} [(\mathbb{1}_A - I_n) \exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] \xrightarrow{\text{a.s.}} 0. \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$\mathbb{E} [\mathbb{1}_A \exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] \xrightarrow{\text{a.s.}} \mathbb{1}_A \cdot \exp\left(-\frac{t^2 \cdot \frac{1}{2} M_\infty (1 - M_\infty)}{2}\right). \tag{3.7}$$

Therefore,

$$\begin{aligned} \mathbb{1}_A \cdot \exp\left(-\frac{t^2 \cdot \frac{1}{2} a (1 - a)}{2}\right) &= \mathbb{1}_A \cdot \exp\left(-\frac{t^2 \cdot \frac{1}{2} M_\infty (1 - M_\infty)}{2}\right) \\ &\text{(using (3.7))} = \lim_n \mathbb{E} [\mathbb{1}_A \exp(\mathbf{it}\sqrt{n}(M_n - M_\infty)) \mid \mathcal{F}_n] \\ &\text{(definition of } A) = \lim_n \mathbb{E} [\mathbb{1}_A \exp(\mathbf{it}\sqrt{n}(M_n - a)) \mid \mathcal{F}_n] \\ \text{(measurability of } M_n \text{ w.r.t. } \mathcal{F}_n) &= \lim_n I_n \exp(\mathbf{it}\sqrt{n}(M_n - a)) \\ \text{(using that } I_n \rightarrow \mathbb{1}_A) &= \lim_n \mathbb{1}_A \exp(\mathbf{it}\sqrt{n}(M_n - a)). \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathbb{1}_A &= \lim_n \mathbb{1}_A \left| \exp \left(\mathbf{it}\sqrt{n} (M_n - a) \right) \right| \\ &= \left| \lim_n \mathbb{1}_A \exp \left(\mathbf{it}\sqrt{n} (M_n - a) \right) \right| \\ &= \underbrace{\mathbb{1}_A \cdot \exp \left(-\frac{t^2 \cdot \frac{1}{2} a (1 - a)}{2} \right)}_{<1 \text{ since } a(1-a)>0}. \end{aligned}$$

Therefore, $\mathbb{P}[A] = 0$.

Now, it remains to prove that $\mathbb{P}(M_\infty = 0) = 0$. The idea of the proof is to fix $\epsilon > 0$ and to split the event $\{M_\infty = 0\}$ depending on whether $M_n \leq \epsilon\sqrt{n}$ or $M_n > \epsilon\sqrt{n}$. When $M_n > \epsilon\sqrt{n}$ we use the CLT convergence to prove that the probability goes to zero and when $M_n \leq \epsilon\sqrt{n}$ we use explicit bounds to show that the probability goes to zero.

More specifically, for any $\epsilon > 0$,

$$\mathbb{P}[M_\infty = 0] = \mathbb{P}\left[M_\infty = 0, M_n \leq \frac{\epsilon}{\sqrt{n}}\right] + \mathbb{P}\left[M_\infty = 0, M_n > \frac{\epsilon}{\sqrt{n}}\right] \tag{3.8}$$

By [Theorem 3.1 \(ii\)](#), i.e. [\(3.1\)](#), we know that $\sqrt{n}(M_n - M_\infty)$ converges in law to the Dirac measure in 0 if we condition on $M_\infty = 0$. As a consequence, $\mathbb{P}[M_\infty = 0, \sqrt{n}|M_n - M_\infty| > \epsilon] \rightarrow 0$ for $n \rightarrow \infty$. This implies in particular that the rightmost term in [\(3.8\)](#) converges to 0 for $n \rightarrow \infty$. On the other hand,

$$\mathbb{P}\left[M_\infty = 0, M_n \leq \frac{\epsilon}{\sqrt{n}}\right] \leq \mathbb{P}\left[M_n \leq \frac{\epsilon}{\sqrt{n}}\right] =: p_{n,\epsilon}.$$

The proof is then finished once we prove the following claim:

$$\limsup_{n \rightarrow \infty} p_{n,\epsilon} \leq 9\sqrt{\epsilon}. \tag{3.9}$$

[\(3.9\)](#) implies that the first term in [\(3.8\)](#) is upper bounded by a number arbitrarily close to 0 in the limit. We dedicate the remaining section to the proof of [\(3.9\)](#).

Notice that

$$p_{n,\epsilon} := \mathbb{P}\left[M_n \leq \frac{\epsilon}{\sqrt{n}}\right] = \mathbb{P}\left[\frac{w(4n, -1)}{w(4n, -1) + w(4n, 0)} \leq \frac{\epsilon}{\sqrt{n}}\right] = \mathbb{P}\left[w(4n, -1) \leq \frac{\epsilon(4n + 2)}{\sqrt{n}}\right].$$

Denote by l_n the number of crossings of the left edge in the outwards direction in the first $4n$ steps, and by r_n the number of crossings of the right edge in the outwards direction. It follows that $l_n + r_n = 2n$, and $w(4n, -1) = 1 + 2l_n$. Observe also that $\frac{\epsilon}{\sqrt{n}} \leq \frac{1}{2}$ for n large enough. As a consequence,

$$\begin{aligned} w(4n, -1) \leq \frac{\epsilon(4n + 2)}{\sqrt{n}} &\implies 2l_n \leq \frac{\epsilon(4n + 2)}{\sqrt{n}} - 1 \implies l_n \leq \frac{\epsilon(2n + 1)}{\sqrt{n}} - \frac{1}{2} \\ &\implies l_n \leq \lfloor 2\epsilon\sqrt{n} \rfloor. \end{aligned}$$

Therefore,

$$p_{n,\epsilon} \leq \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \mathbb{P}[l_n = j].$$

We claim that

$$\mathbb{P}[l_n = j] \leq \binom{2n}{j} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j - 1) \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2(2n - j) - 1)}{2 \cdot 3 \cdot 6 \cdot 7 \cdot 10 \cdot 11 \cdot \dots \cdot (4n - 2) \cdot (4n - 1)} \tag{3.10}$$

The reader is advised to write examples to convince herself/himself that [\(3.10\)](#) holds true. The binomial coefficient counts the number of possible walker movement sequences of length $4n$ which

end with $l_n = j$. The sequence in the denominator of the following fraction is just the sequence of total edge weights observed at the points in time at which one of the walkers moves from the node in the center to one of the outer nodes. For the sequence in the numerator, we just take an upper bound on the possible edge weights of the left and right edges before a walker crosses the respective edge in outwards direction.

Reordering the terms, and writing out the binomial coefficient, we get

$$\begin{aligned} p_{n,\epsilon} &\leq \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot 2n}{2 \cdot 3 \cdot 6 \cdot 7 \cdot \dots \cdot (4n - 1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2j - 1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot j} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2(2n - j) - 1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n - j)} \\ &= 2^{-2n} \cdot \prod_{i=1}^n \frac{4i}{4i - 1} \cdot \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \prod_{i=1}^j \frac{2i - 1}{i} \cdot \prod_{i=1}^{2n-j} \frac{2i - 1}{i} \\ &= \prod_{i=1}^n \frac{4i}{4i - 1} \cdot \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \prod_{i=1}^j \frac{2i - 1}{2i} \cdot \prod_{i=1}^{2n-j} \frac{2i - 1}{2i}. \end{aligned}$$

To bound the products, we do the following: let $m \in \{2, 4\}$ and note that $\log(1 - x) \geq -x - x^2$ for $x \in [0, \frac{1}{2}]$.

$$\begin{aligned} \prod_{i=1}^j \frac{m \cdot i}{m \cdot i - 1} &= \exp\left(-\sum_{i=1}^j \log\left(\frac{m \cdot i - 1}{m \cdot i}\right)\right) = \exp\left(-\sum_{i=1}^j \log\left(1 - \frac{1}{m \cdot i}\right)\right) \\ &\leq \exp\left(\sum_{i=1}^j \left(\frac{1}{m \cdot i} + \frac{1}{m^2 \cdot i^2}\right)\right) \leq \exp\left(\frac{1}{m}(\log(j) + 1) + \frac{\pi^2}{6m^2}\right) \leq 3j^{\frac{1}{m}}. \end{aligned}$$

Furthermore, using that $\log(1 - x) \leq -x$, we get that

$$\begin{aligned} \prod_{i=1}^j \frac{m \cdot i}{m \cdot i - 1} &\geq \exp\left(\sum_{i=1}^j \frac{1}{m \cdot i}\right) \geq \exp\left(\frac{1}{m} \log(j)\right) = j^{\frac{1}{m}} \\ \implies \prod_{i=1}^j \frac{m \cdot i - 1}{m \cdot i} &\leq j^{-\frac{1}{m}}. \end{aligned}$$

Hence (we let again n large enough such that $\frac{\epsilon}{\sqrt{n}} \leq \frac{1}{2}$)

$$\begin{aligned} p_{n,\epsilon} &\leq \prod_{i=1}^n \frac{4i}{4i - 1} \cdot \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \prod_{i=1}^j \frac{2i - 1}{2i} \cdot \prod_{i=1}^{2n-j} \frac{2i - 1}{2i} \leq 3n^{\frac{1}{4}} \cdot \sum_{j=0}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \frac{1}{\sqrt{j + \mathbb{1}_{j=0}}} \cdot \frac{1}{\sqrt{2n - j}} \\ &\leq 3n^{\frac{1}{4}} \cdot \frac{1}{\sqrt{2n - \lfloor 2\epsilon\sqrt{n} \rfloor}} \left(1 + \sum_{j=1}^{\lfloor 2\epsilon\sqrt{n} \rfloor} \frac{1}{\sqrt{j}}\right) \leq 3n^{\frac{1}{4}} \cdot \frac{1}{\sqrt{n}} \left(1 + \int_0^{2\epsilon\sqrt{n}} \frac{1}{\sqrt{j}} \, dj\right) \\ &= 3n^{\frac{1}{4}} \cdot \frac{1}{\sqrt{n}} \left(1 + 2\sqrt{2\epsilon\sqrt{n}}\right) \leq 3n^{-\frac{1}{4}} + 6\sqrt{2}\sqrt{\epsilon} \cdot n^{\frac{1}{4}} \cdot n^{-\frac{1}{2}} \cdot n^{\frac{1}{4}} \leq 3n^{-\frac{1}{4}} + 9\sqrt{\epsilon}. \end{aligned}$$

Therefore, $\limsup_{n \rightarrow \infty} p_{n,\epsilon} \leq 9\sqrt{\epsilon}$. □

3.4. *Proofs: random walker selection (Theorem 3.3).* Consider next the case with random walker selection. Lemma 3.5 shows that the expected time to meet again in the middle, if both walkers start in the center, is 4, just as in the case of alternating walkers. Of course, the difference now is that the next meeting time is random.

We want to prove [Theorem 3.3 \(iii\)](#), i.e. that the proportion of the left edge weight, N_n , converges a.s. for $n \rightarrow \infty$, in analogy to the case with alternating walkers, as well as in analogy with the case of a single walker which corresponds to a standard Pólya urn. The proof is based on the fact that the proportion of the left edge weight is a martingale at the times τ_n (recall that $\tau_n = \inf\{m > \tau_{n-1} : X_m^{(1)} = X_m^{(2)} = 0\}$), as we will see in [Lemma 3.4](#) (which also proves [Theorem 3.3 \(i\)](#)).

In order to prove [Theorem 3.3 \(iii\)](#), we need two lemmata. The main reason for convergence is the martingale property of the N_{τ_n} , and the remaining section will mostly be devoted to showing [Lemma 3.4](#). In order to make the step from the times τ_n to n , it is enough to see that the probability of $\{\tau_{n+1} - \tau_n = 2l\}$ decays exponentially in l in [Lemma 3.5](#) (which proves [Theorem 3.3 \(ii\)](#)), which allows us to conclude with a Borel-Cantelli argument.

In this section, we write M_n instead of M_n^{rd} , i.e.

$$M_n = M_n^{\text{rd}} = N_{\tau_n} = \frac{w(\tau_n, -1)}{w(\tau_n, -1) + w(\tau_n, 0)}.$$

Lemma 3.4. *The sequence of random variables $(M_n)_{n \geq 0}$ together with its natural filtration is a (bounded) martingale.*

Lemma 3.5. *For any $n \geq 0$ and any $l \geq 1$, it holds that $\mathbb{P}[\tau_{n+1} - \tau_n = 2l] = 2^{-l}$ and $\mathbb{E}[\tau_{n+1} - \tau_n] = 4$.*

Assuming [Lemma 3.4](#) and [Lemma 3.5](#), we can prove [Theorem 3.3 \(iii\)](#):

Proof of Theorem 3.3 (iii): Recall that

$$N_n := \frac{w(n, -1)}{w(n, -1) + w(n, 0)}, \quad M_n := N_{\tau_n} = \frac{w(\tau_n, -1)}{w(\tau_n, -1) + w(\tau_n, 0)}.$$

By [Lemma 3.4](#) there exists a random variable M_∞ such that

$$M_\infty := \lim_{n \rightarrow \infty} M_n. \tag{3.11}$$

Let $\varepsilon > 0$ and consider the events

$$A_n := \{|N_m - M_\infty| > \varepsilon \text{ for some } m \in [\tau_n, \tau_{n+1}]\}$$

It is sufficient to show that only finitely many of the events A_n can occur a.s. Further set

$$B_n := \left\{ |N_m - M_n| > \frac{\varepsilon}{2} \text{ for some } m \in [\tau_n, \tau_{n+1}] \right\}$$

Since M_n converges a.s. by [Lemma 3.4](#), there is some (random) $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that $|M_n - M_\infty| < \frac{\varepsilon}{2}$. For $n \geq N$, the occurrence of A_n implies that B_n occurs as well, so it is sufficient to show that only finitely many of the events B_n can occur.

Now, at time τ_n , the random walkers must have moved at least $2n$ times, so $w(\tau_n, -1) + w(\tau_n, 0) \geq 2n$. If $|N_m - M_n| = |N_m - N_{\tau_n}| > \frac{\varepsilon}{2}$ for some $m \in [\tau_n, \tau_{n+1}]$, it is therefore necessary that at least εn steps were made by the walkers between time τ_n and time m , since every step changes the value of N_m by at most $\frac{1}{2n}$. Thus

$$B_n \subseteq \{\tau_{n+1} - \tau_n \geq \varepsilon n\}$$

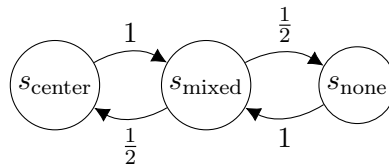
We have

$$\begin{aligned} \mathbb{P}[\tau_{n+1} - \tau_n \geq \varepsilon n] &= \sum_{l \geq \lceil \frac{\varepsilon n}{2} \rceil} \mathbb{P}[\tau_{n+1} - \tau_n = 2l] \stackrel{\text{Lemma 3.5}}{=} \sum_{l \geq \lceil \frac{\varepsilon n}{2} \rceil} 2^{-l} \\ &\leq 2^{-\frac{\varepsilon n}{2}} \sum_{l \geq 0} 2^{-l} = 2^{1-\frac{\varepsilon n}{2}} \\ \implies \sum_{n \geq 1} \mathbb{P}[\tau_{n+1} - \tau_n \geq \varepsilon n] &\leq \sum_{n \geq 1} 2^{1-\frac{\varepsilon n}{2}} = 2 \cdot \sum_{n \geq 1} \left(2^{-\frac{\varepsilon}{2}}\right)^n < \infty \end{aligned}$$

By the Borel-Cantelli lemma, it follows that only finitely many of the events $\{\tau_{n+1} - \tau_n \geq \varepsilon n\}$ can occur, and therefore also only finitely many of the events B_n . This concludes the proof. \square

We proceed by first showing the exponential decay of $\mathbb{P}[\tau_{n+1} - \tau_n = 2l]$ in l . This is easy to see since this probability can be calculated independently of any edge weights.

Proof of Lemma 3.5 and Theorem 3.3 (ii): We consider a Markov chain consisting of three states and coupled with the edge-reinforced random walk. The Markov chain is in state s_{center} if both walkers are in the center, in state s_{mixed} if one walker is in the center and the other in either of the two outer nodes, and in state s_{none} if none of the walkers is in the center. It is easy to verify that this is indeed a Markov chain with the following transition probabilities:



Let $n \geq 0$. At time τ_n , both walkers are in the center and the Markov chain is therefore in state s_{center} . The time $\tau_{n+1} - \tau_n$ corresponds to the time needed to return again to the state s_{center} . At uneven time steps, the Markov chain will always be in state s_{mixed} , so $\mathbb{P}[\tau_{n+1} - \tau_n = 2l]$ corresponds to the probability that at every uneven time step, starting with $\tau_n + 1$ and up to $\tau_{n+1} - 3$, the chain transitions to the state s_{none} and that the chain will finally go to s_{center} in step $\tau_{n+1} - 1$. Since the probabilities of these transitions are $\frac{1}{2}$, we get $\mathbb{P}[\tau_{n+1} - \tau_n = 2l] = 2^{-l}$. Consequently,

$$\mathbb{E}[\tau_{n+1} - \tau_n] = \sum_{l \geq 1} 2^{-l} \cdot 2l = \sum_{l \geq 1} l \cdot \left(\frac{1}{2}\right)^{l-1} = \frac{1}{\left(1 - \frac{1}{2}\right)^2} = 4 \quad \square$$

We next want to prove Lemma 3.4, i.e. that the proportion of the left edge weight, that is $\frac{w(\tau_n, -1)}{w(\tau_n, -1) + w(\tau_n, 0)}$, is a martingale, as it was in the previous case with alternating walkers and as in the Pólya urn with the difference that we look at the proportion not at every time step, but at certain stopping times. The proof is a bit more involved, but relies only on basic calculations. The main idea is to find a recursive formula for the following quantities:

- (1) We look at the expectation of the proportion of the left edge weight multiplied by the indicator of the event that the walkers need $2l$ steps to meet again:

$$\mathbb{E}_{a,b,l} := \mathbb{E} \left[\frac{w(\tau_{n+1}, -1)}{w(\tau_{n+1}, -1) + w(\tau_{n+1}, 0)} \cdot \mathbb{1}_{\{\tau_{n+1} - \tau_n = 2l\}} \mid w(\tau_n, -1) = a, w(\tau_n, 0) = b \right]$$

- (2) We also consider the probability that the last walker which returns to the center comes from the left node, again intersected with the event that the walkers need $2l$ steps to meet again:

$$\begin{aligned} L_n &:= \{\text{the walker returning to the center at time } \tau_{n+1} \text{ comes from the left node}\} \\ q_{a,b,l} &:= \mathbb{P}[L_n \text{ occurs and } \tau_{n+1} - \tau_n = 2l \mid w(\tau_n, -1) = a, w(\tau_n, 0) = b] \end{aligned}$$

Lemma 3.6. *We have the following recursive equations:*

$$\begin{aligned} \mathbb{E}_{a,b,l+1} &= \frac{1}{2}\mathbb{E}_{a,b,l} + \frac{1}{(a+b+2l-1)(a+b+2l+2)}(\mathbb{E}_{a,b,l} - q_{a,b,l}) \\ q_{a,b,l+1} &= \frac{1}{2}q_{a,b,l} + \frac{1}{4} \cdot \frac{a+b+2l}{a+b+2l-1}(\mathbb{E}_{a,b,l} - q_{a,b,l}) \end{aligned}$$

We will prove [Lemma 3.6](#) later. First, we can complete the proof of [Lemma 3.4](#) as follows:

Corollary 3.7. *The expectation of the proportion of the left edge weight and the probability that the last walker returning to the center comes from the left coincide, and:*

$$\mathbb{E}_{a,b,l} = q_{a,b,l} = \frac{1}{2^l} \cdot \frac{a}{a+b}$$

Proof: We use [Lemma 3.6](#). Let us first calculate $\mathbb{E}_{a,b,1}$ and $q_{a,b,1}$. There are four possible paths of length 2 which end again with both walkers in the center: first, we choose which of the two walker moves, and this walker can then either move left or right and then back to the center. Since it is irrelevant which walker we choose in the beginning, we can disregard which walker moves. We thus get:

- With probability $\frac{a}{a+b} \cdot \frac{1}{2}$ the walker which moves in the first step moves left and is then chosen again to move back to the center in the next step. In this case, L_n occurs and the resulting edge weight ratio is $\frac{a+2}{a+b+2}$.
- With probability $\frac{b}{a+b} \cdot \frac{1}{2}$ the walker which moves in the first step moves right and is then chosen again to move back to the center in the next step. In this case, L_n does not occur and the resulting edge weight ratio is $\frac{a}{a+b+2}$.

We see directly that $q_{a,b,1} = \frac{1}{2} \cdot \frac{a}{a+b}$ and that

$$\mathbb{E}_{a,b,1} = \frac{1}{2} \cdot \left(\frac{a}{a+b} \cdot \frac{a+2}{a+b+2} + \frac{b}{a+b} \cdot \frac{a}{a+b+2} \right) = \frac{1}{2} \cdot \frac{a}{a+b}$$

The remaining proof is now a simple induction using [Lemma 3.6](#), where it should be noted that $\mathbb{E}_{a,b,l} - q_{a,b,l} = 0$ under the induction assumption. □

Proof of [Lemma 3.4](#) and [Theorem 3.3 \(i\)](#): We have by [Corollary 3.7](#):

$$\begin{aligned} &\mathbb{E} \left[\frac{w(\tau_{n+1}, -1)}{w(\tau_{n+1}, -1) + w(\tau_{n+1}, 0)} \cdot \mathbb{1}_{\{\tau_{n+1} - \tau_n = 2l\}} \mid w(\tau_n, -1) = a, w(\tau_n, 0) = b \right] = 2^{-l} \cdot \frac{a}{a+b} \\ \implies &\mathbb{E} \left[\frac{w(\tau_{n+1}, -1)}{w(\tau_{n+1}, -1) + w(\tau_{n+1}, 0)} \mid w(\tau_n, -1) = a, w(\tau_n, 0) = b \right] = \frac{a}{a+b} \cdot \sum_{l \geq 1} 2^{-l} = \frac{a}{a+b} \end{aligned}$$

as we wanted to prove. □

This concludes the proof of [Theorem 3.3](#), up to the proof of [Lemma 3.6](#), which follows now. In order to prove [Lemma 3.6](#), we will use a recursive path construction technique. We define a path of the two walkers as a sequence of the symbols $1_l, 1_r, 2_l, 2_r$ which correspond to the first (respectively second) walker moving left and right, where we assume that both walkers start in the center. The set Path_{2l} contains all the paths of length $2l$ (a sequence of $2l$ symbols) such that the first time at which both walkers are in the center at the same time again is at the end of the path. Note that any such path must be of even length since each walker can only be in the center after having made an even number of movements.

For $\rho \in \text{Path}_{2l}$, we set $d(\rho)$ to be the number of traversals of the left edge when the path ρ is taken and we can now write

$$\mathbb{E}_{a,b,l} = \sum_{\rho \in \text{Path}_{2l}} \underbrace{\mathbb{P}[\text{the walkers move according to } \rho \mid w(\tau_n, -1) = a, w(\tau_n, 0) = b]}_{=:\mathbb{P}_{\rho,a,b}} \cdot \frac{a + d(\rho)}{a + b + 2l}$$

Consider $\rho \in \text{Path}_{2l}$. If the walkers move according to ρ , then, at any uneven time step, there will be one walker which is in the **center** and one which is in one of the **outer** nodes (this corresponds to s_{mixed} in the proof of [Lemma 3.5](#)). In addition, at any even time step, except for the beginning and end, both walkers have to be in the outer nodes, not necessarily the same one (state s_{none}). The possible walker locations and edge weights after $2l - 1$ steps of the path ρ are depicted in [Figure 3.2](#), where we assume that we start with edge weights a and b . They will be relevant for the recursive path construction which we present now.

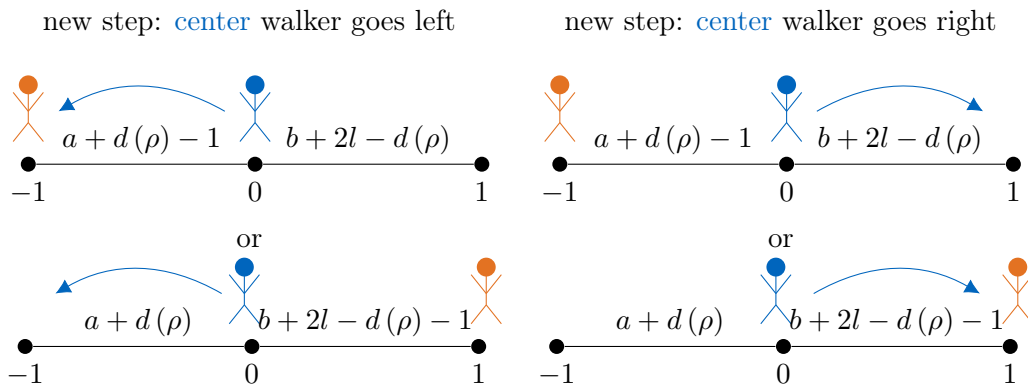


FIGURE 3.2. Possible walker locations after $2l - 1$ steps of a path ρ of length $2l$

For $\rho \in \text{Path}_{2l}$, we define $p_k(\rho) \in \text{Path}_{2(l+1)}$, $1 \leq k \leq 4$ as follows:

- delete the last symbol from ρ ,
- add three new symbols to ρ , using the following rules. After the $2l - 1$ steps of ρ which we did not delete, the walkers are in two possible configurations: there is one walker which is in the **center**, and the other walker can be in either of the **outer** nodes. We now add a symbol to the path such that the **center** walker moves either left or right in the next step, and then we add two more symbols such that both walkers return to the center in either order. This leaves us a total of $2 \times 2 = 4$ choices for our new path (2 choices for which way the **center** walker moves, and 2 choices for the order in which the walkers return). We can therefore construct 4 new paths, which we will label as follows:
 - $p_1(\rho)$ is the new path where the **center** walker moves left, and then returns immediately to the center before the other walker
 - $p_2(\rho)$ is the new path where the **center** walker moves left, and then returns to the center only after the other walker
 - $p_3(\rho)$ is the new path where the **center** walker moves right, and then returns immediately to the center before the other walker
 - $p_4(\rho)$ is the new path where the **center** walker moves right, and then returns to the center only after the other walker

This construction is summarized in [Figure 3.2](#).

Lemma 3.8. *We can write the set $Path_{2(l+1)}$ as a disjoint union over the modifications of the paths ρ in $Path_{2l}$:*

$$Path_{2(l+1)} = \bigcup_{\rho \in Path_{2l}} \{p_1(\rho), p_2(\rho), p_3(\rho), p_4(\rho)\}$$

Proof: To see that we get indeed all paths of length $2(l + 1)$ if we take the given union, note that the paths in $Path_{2l}$ must fulfil the condition that at any even time step (except for the beginning and end), both walkers have to be in the outer nodes, not necessarily the same one, and that the walkers are in the center at the beginning and at the end of the path. This is the only condition, so we will get all possible prefixes of paths of length $2(l + 1)$ by taking all $\rho \in Path_{2l}$ and deleting the last symbol. We then only have to complete the paths by all possible suffixes, but the possible suffixes are just the 4 given in the construction above by the constraints imposed on the paths. The union is disjoint because we do not construct any path twice: any two different paths in $Path_{2l}$ must already differ somewhere in the first $2l - 1$ steps. \square

The edge weights at time $2l - 1$ depend on where the **outer** walker is at step $2l - 1$. Therefore, we will also use the notation $left(\rho) = 1$ if the **outer** walker is on the left after step $2l - 1$, and $left(\rho) = 0$ otherwise. Finally, in order to state [Lemma 3.9](#), we also need to define $f_{\rho,a,b}$ which is the left edge weight divided by the sum of the edge weights, where we plug in the edge weights which we get if we start with weights a and b and then execute the path ρ .

Lemma 3.9. *The path probabilities and edge weight ratios of the paths $p_k(\rho)$ change as follows compared to ρ :*

- if $left(\rho) = 1$, then:

$$\begin{aligned} \mathbb{P}_{p_1(\rho),a,b} &= \mathbb{P}_{p_2(\rho),a,b} = \mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{a + d(\rho) - 1}{a + b + 2l - 1} \\ f_{p_1(\rho),a,b} &= f_{p_2(\rho),a,b} = \frac{a + d(\rho) + 2}{a + b + 2l + 2} \\ \mathbb{P}_{p_3(\rho),a,b} &= \mathbb{P}_{p_4(\rho),a,b} = \mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{b + 2l - d(\rho)}{a + b + 2l - 1} \\ f_{p_3(\rho),a,b} &= f_{p_4(\rho),a,b} = \frac{a + d(\rho)}{a + b + 2l + 2} \end{aligned}$$

- and if $left(\rho) = 0$, then:

$$\begin{aligned} \mathbb{P}_{p_1(\rho),a,b} &= \mathbb{P}_{p_2(\rho),a,b} = \mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{a + d(\rho)}{a + b + 2l - 1} \\ f_{p_1(\rho),a,b} &= f_{p_2(\rho),a,b} = \frac{a + d(\rho) + 2}{a + b + 2l + 2} \\ \mathbb{P}_{p_3(\rho),a,b} &= \mathbb{P}_{p_4(\rho),a,b} = \mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{b + 2l - d(\rho) - 1}{a + b + 2l - 1} \\ f_{p_3(\rho),a,b} &= f_{p_4(\rho),a,b} = \frac{a + d(\rho)}{a + b + 2l + 2} \end{aligned}$$

Proof: We look at how following the path $p_k(\rho)$ changes the outcome (i.e. the final edge weights) and the probability of the path, compared to the original path $\rho \in Path_{2l}$.

- If the **outer** walker is on the left after $2l - 1$ steps (this is a condition on ρ : $left(\rho) = 1$):
 - If the **center** walker should move left in step $2l$ (this is the choice for creating the new paths $p_1(\rho), p_2(\rho)$): the probability $\mathbb{P}_{\rho,a,b}$ of ρ is a product over the probability that the walker indicated in ρ is chosen at the respective step (which is always $\frac{1}{2}$) and the probability that the walker moves in the direction indicated by ρ (this can either be 1,

if the walker moves back to the center, or a fraction depending on the edge weights). In our new modified paths $p_1(\rho), p_2(\rho)$ of length $2(l+1)$, the first difference is that we choose a different walker to move in step $2l$. This event has probability $\frac{1}{2}$, but this is the same as choosing the original walker, hence this part is already included in the product giving the probability $\mathbb{P}_{\rho,a,b}$ of ρ .

Next, the probability for the center walker to go left, if he is chosen to move at step $2l$, is given by $\frac{a+d(\rho)-1}{a+b+2l-1}$ (compare with Figure 3.2). This factor is new and has to be added to the product. In the next step, one of the two walkers is chosen and will move back to the center. We get a new factor of $\frac{1}{2}$, since the probability that the walker indicated by $p_1(\rho)$ (or $p_2(\rho)$ respectively) is chosen is $\frac{1}{2}$. Finally, in step $2(l+1)$, the walker which is still in an outer node has to be chosen to move back into the center. This happens with probability $\frac{1}{2}$, and this is the final factor to be added to the product. Our new paths therefore have probability

$$\mathbb{P}_{p_1(\rho),a,b} = \mathbb{P}_{p_2(\rho),a,b} = \mathbb{P}_{\rho,a,b} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{a+d(\rho)-1}{a+b+2l-1}$$

The new paths will end with the following ratio of the left edge weight divided by the total edge weights:

$$f_{p_1(\rho),a,b} = f_{p_2(\rho),a,b} = \frac{a+d(\rho)+2}{a+b+2l+2}$$

since the left edge will be traversed twice more by the center walker.

- If the center walker should move right in step $2l$ (paths $p_3(\rho), p_4(\rho)$):

new probability: $\mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{b+2l-d(\rho)}{a+b+2l-1}$, new outcome: $\frac{a+d(\rho)}{a+b+2l+2}$

- If the outer walker is on the right after $2l-1$ steps ($\text{left}(\rho) = 0$):

- If the center walker should move left in step $2l$ (paths $p_1(\rho), p_2(\rho)$):

new probability: $\mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{a+d(\rho)}{a+b+2l-1}$, new outcome: $\frac{a+d(\rho)+2}{a+b+2l+2}$

- If the center walker should move right in step $2l$ (paths $p_3(\rho), p_4(\rho)$):

new probability: $\mathbb{P}_{\rho,a,b} \cdot \frac{1}{4} \cdot \frac{b+2l-d(\rho)-1}{a+b+2l-1}$, new outcome: $\frac{a+d(\rho)}{a+b+2l+2}$ □

Proof of Lemma 3.6: By Lemma 3.8 and Lemma 3.9, we have

$$\begin{aligned} \mathbb{E}_{a,b,l+1} &= \sum_{\rho \in \text{Path}_{2(l+1)}} \mathbb{P}_{\rho,a,b} \cdot \frac{a+d(\rho)}{a+b+2l+2} = \sum_{\rho \in \text{Path}_{2l}} \sum_{k=1}^4 \mathbb{P}_{p_k(\rho),a,b} \cdot f_{p_k(\rho),a,b} = \\ & \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \frac{1}{2} \cdot \left(\text{left}(\rho) \cdot \left(\frac{a+d(\rho)-1}{a+b+2l-1} \cdot \frac{a+d(\rho)+2}{a+b+2l+2} + \frac{b+2l-d(\rho)}{a+b+2l-1} \cdot \frac{a+d(\rho)}{a+b+2l+2} \right) \right) \\ & + (1 - \text{left}(\rho)) \cdot \left(\frac{a+d(\rho)}{a+b+2l-1} \cdot \frac{a+d(\rho)+2}{a+b+2l+2} + \frac{b+2l-d(\rho)-1}{a+b+2l-1} \cdot \frac{a+d(\rho)}{a+b+2l+2} \right) \right) \end{aligned}$$

Using this expression for $\mathbb{E}_{a,b,l+1}$, we can calculate (a simple, but longer calculation which we skip here, see <https://bit.ly/3BIMorx> and <http://bit.ly/3PgLLPq> – the calculation is done by expanding

the involved fractions to the same denominator):

$$\begin{aligned}
 \mathbb{E}_{a,b,l+1} - \frac{1}{2}\mathbb{E}_{a,b,l} &= \mathbb{E}_{a,b,l+1} - \sum_{\rho \in \text{Path}_{2l}} \frac{1}{2} \cdot \mathbb{P}_{\rho,a,b} \cdot \frac{a+d(\rho)}{a+b+2l} \\
 &= \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \frac{1}{(a+b+2l-1)(a+b+2l+2)} \cdot \\
 &\quad \left(\text{left}(\rho) \cdot \frac{d(\rho) - b - 2l}{a+b+2l} + (1 - \text{left}(\rho)) \cdot \frac{a+d(\rho)}{a+b+2l} \right) \\
 &= \frac{1}{(a+b+2l-1)(a+b+2l+2)} \cdot \\
 &\quad \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \left(\text{left}(\rho) \cdot \frac{-a-b-2l}{a+b+2l} + \frac{a+d(\rho)}{a+b+2l} \right) \\
 &= \frac{1}{(a+b+2l-1)(a+b+2l+2)} \cdot \left(\mathbb{E}_{a,b,l} - \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \text{left}(\rho) \right) \\
 &= \frac{1}{(a+b+2l-1)(a+b+2l+2)} \cdot (\mathbb{E}_{a,b,l} - q_{a,b,l})
 \end{aligned}$$

This proves the first equation in [Lemma 3.6](#). For the second equation, we use the same strategy. For our newly constructed paths, we already know if L_n occurs or not (whether the last walker to return to the center comes from the left node):

- if $\text{left}(\rho) = 1$, then: L_n occurs for $p_1(\rho), p_2(\rho), p_4(\rho)$, and does not occur for $p_3(\rho)$
- and if $\text{left}(\rho) = 0$, then: L_n occurs for $p_2(\rho)$, and does not occur for $p_1(\rho), p_3(\rho), p_4(\rho)$.

Therefore, again by [Lemma 3.8](#) and [Lemma 3.9](#),

$$\begin{aligned}
 q_{a,b,l+1} - \frac{1}{2}q_{a,b,l} &= \sum_{\rho \in \text{Path}_{2(l+1)}} \mathbb{P}_{\rho,a,b} \cdot \text{left}(\rho) - \frac{1}{2} \cdot \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \text{left}(\rho) \\
 &= \sum_{\rho \in \text{Path}_{2l}} (\mathbb{P}_{p_1(\rho),a,b} + \mathbb{P}_{p_2(\rho),a,b} + \mathbb{P}_{p_4(\rho),a,b}) \cdot \text{left}(\rho) \\
 &\quad + \mathbb{P}_{p_2(\rho),a,b} \cdot (1 - \text{left}(\rho)) - \frac{1}{2} \cdot \mathbb{P}_{\rho,a,b} \cdot \text{left}(\rho) \\
 &= \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \frac{1}{2} \cdot \\
 &\quad \left(\frac{a+d(\rho)-1}{a+b+2l-1} \cdot \text{left}(\rho) + \frac{1}{2} \cdot \frac{b+2l-d(\rho)}{a+b+2l-1} \cdot \text{left}(\rho) \right. \\
 &\quad \left. + \frac{1}{2} \cdot \frac{a+d(\rho)}{a+b+2l-1} \cdot (1 - \text{left}(\rho)) - \text{left}(\rho) \right) \\
 &= \frac{1}{2} \cdot \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \\
 &\quad \left(-\frac{1}{2} \cdot \frac{a+b+2l}{a+b+2l-1} \cdot \text{left}(\rho) + \frac{1}{2} \cdot \frac{a+d(\rho)}{a+b+2l-1} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \cdot \frac{a+b+2l}{a+b+2l-1} \cdot \sum_{\rho \in \text{Path}_{2l}} \mathbb{P}_{\rho,a,b} \cdot \left(\frac{a+d(\rho)}{a+b+2l} - \text{left}(\rho) \right) \\
&= \frac{1}{4} \cdot \frac{a+b+2l}{a+b+2l-1} \cdot (\mathbb{E}_{a,b,l} - q_{a,b,l}) \quad \square
\end{aligned}$$

This proof of [Lemma 3.6](#) completes the proof of [Theorem 3.3](#). As a next step, one would analyze the (random) limit of the fraction $\frac{w(n,-1)}{w(n,-1)+w(n,0)}$ and compare it to the single walker case where the corresponding limit is Beta-distributed. We believe that the following holds.

Conjecture 3.10. *Theorem 3.1 (ii) and (iii) (the CLT and the no atoms property) are also true for the case where we randomly select the walker which moves.*

Conjecture 3.11. $M_\infty^{\text{rd}} \in [0, 1]$ (the limit of the left edge weight proportion, with random walker selection) has a density w.r.t. the Lebesgue measure on $[0, 1]$. M_∞^{rd} is not Beta-distributed, and its distribution is different from the distribution of the limit in [Conjecture 3.2](#) (where we considered alternating walkers).

As in [Conjecture 3.2](#), simulations showed that the distributions of M_∞^{rd} and M_∞^{alt} are probably not identical.

4. A Finite Number of Walkers on \mathbb{Z}

So far, we have looked at the linearly edge-reinforced random walk with multiple walkers only on a very simple graph and only with 2 walkers. We now consider a finite number K of edge-reinforced random walkers on \mathbb{Z} . As before, the transition probabilities depend on the edge weights $w(n, z) > 0$ for $n \geq 0, z \in \mathbb{Z}$ where we will abuse notation and use z to denote the edge from z to $z + 1$.

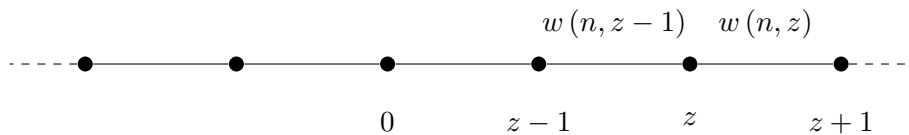


FIGURE 4.3. Edge weights on \mathbb{Z} at time n

4.1. *Model and Results.* We have K sequences (for the K walkers) $(X_n^{(i)})_{n \geq 0}$ (where $1 \leq i \leq K$) of interacting nearest neighbour processes with the following dynamics, adapted from [Definition 2.1](#):

If \mathcal{G}_n denotes $\sigma(\{X_m^{(i)} : 0 \leq m \leq n, 1 \leq i \leq K\} \cup \{w(m, z) : 0 \leq m \leq n, z \in \mathbb{Z}\})$ i.e. the history of the random walkers and edge weights up to and including time n , then we define, conditional on \mathcal{G}_n , the following transition probabilities:

- At every time step, the random walker i ($1 \leq i \leq K$) which is going to jump is chosen uniformly at random (independently of \mathcal{G}_n) amongst the K walkers.
- If the chosen random walker is at position z (i.e. $X_n^{(i)} = z$), then he jumps
 - to the right (i.e. $X_{n+1}^{(i)} = z + 1$) with probability $\frac{w(n,z)}{w(n,z-1)+w(n,z)}$,
 - to the left (i.e. $X_{n+1}^{(i)} = z - 1$) with probability $\frac{w(n,z-1)}{w(n,z-1)+w(n,z)}$,
 i.e. the jump probabilities are proportional to the corresponding edge weights.
- If z^* is the traversed edge ($z^* = z$ if the walker jumps to the right, $z^* = z - 1$ if he jumps to the left), then for $y \neq z^*$, $w(n+1, y) = w(n, y)$ and $w(n+1, z^*) \geq w(n, z^*)$, i.e. the weight of the traversed edge may be increased according to some reinforcement scheme.

We consider schemes where the increment $w(n + 1, z^*) - w(n, z^*)$ only depends on z^* , and the number of times the edge was crossed up to time n . In other words, $w(n, z)$ can still be written in terms of the weight function $W_z(k)$ as in [Definition 2.1](#).

- The initial edge weights can be chosen arbitrarily, but all of them must be strictly positive. The results given below only apply to the case where all but finitely many initial weights are 1.
- The initial positions of the K walkers can be chosen arbitrarily.

We call a walker recurrent if he visits all integers infinitely often, and we say that he has finite range if he only visits finitely many integers. We have the following main result:

Theorem 4.1. *Assume the edge-reinforced random walk with K walkers starts with an initial configuration of the weights $w(0, z)$ such that all but finitely many of them are 1. Then, we have the following:*

$$\mathbb{P} \left[\forall i : X^{(i)} \text{ is recurrent} \right] + \mathbb{P} \left[\forall i : X^{(i)} \text{ has finite range} \right] = 1.$$

We can further characterize these two possible cases for a particular class of reinforcement schemes, called “sequence type”, which means that all initial edge weights are 1, and that the edge weights are then increased by a fixed sequence of increments $\mathbf{a} = (a_k)_{k \geq 1}$ which do not depend on the location of the edge.

Theorem 4.2. *Consider the edge-reinforced random walk with K walkers and a reinforcement scheme of sequence type. Set*

$$\phi(\mathbf{a}) = \sum_{k=1}^{\infty} \left(1 + \sum_{l=1}^k a_l \right)^{-1}.$$

If $\phi(\mathbf{a}) = \infty$, then all walkers are recurrent a.s., and if $\phi(\mathbf{a}) < \infty$, then all walkers have finite range a.s.

4.2. Recurrence or Finite Range on \mathbb{Z} .

Definition 4.3. For each $i \in \{1, \dots, K\}$, we say that walker i

- is **transient**, if he visits every integer only finitely often, that is, every integer appears only finitely often in the sequence $(X_n^{(i)})_{n \geq 0}$
- is **recurrent**, if he visits every integer infinitely often, that is, every integer appears infinitely often in the sequence $(X_n^{(i)})_{n \geq 0}$
- has **finite range**, if he only visits finitely many integers, that is, the number of distinct integers appearing in the sequence $(X_n^{(i)})_{n \geq 0}$ is finite

Our main result shows that either all walkers are recurrent or all walkers have finite range ([Theorem 4.1](#)). This was already known for a single walker. The proof for the single walker case has to be adapted and additional steps are needed to show that all walkers show the same behavior: we will see that it cannot be the case that one walker has finite range, while another is recurrent. Of course, this is very plausible.

The proof of [Theorem 4.1](#) follows the same strategy as [Davis \(1990\)](#), with some changes in the details. More precisely, [Lemma 4.4](#) uses the methods from [Davis \(1990\)](#) and modifies them to fit the case with multiple walkers. As a result, we see that every walker either reaches 0 or visits only finitely many nodes which have not been visited before by any other walker. [Lemma 4.5](#) is new for the case with multiple walkers and shows that any two walkers which meet infinitely often are either both recurrent or do both have finite range. Combining the lemmas in the proof of [Theorem 4.1](#)

allows us to conclude that the behavior of all walkers is the same: either all of the walkers are recurrent, or all have finite range.

Lemma 4.4. *Assume the edge-reinforced random walk with K walkers starts with an initial configuration of the weights $w(0, z)$ such that all but finitely many of them are 1. Then, for every random walker i ($1 \leq i \leq K$) we have the following:*

$$\begin{aligned} & \mathbb{P} \left[X_n^{(i)} = 0 \text{ for some } n \geq 0 \right] \\ & + \mathbb{P} \left[X_n^{(i)} \neq 0 \text{ for all } n \geq 0 \text{ and } X^{(i)} \text{ only visits finitely many nodes} \right. \\ & \quad \left. \text{which have not been visited before by any other walker} \right] = 1. \end{aligned}$$

Lemma 4.5. *Assume the edge-reinforced random walk with $K \geq 2$ walkers starts with an arbitrary initial configuration of the weights $w(0, z)$. Assume further that $X^{(1)}$ and $X^{(2)}$ meet infinitely often. Then, almost surely,*

- (i) *if one of the walkers $X^{(1)}$ and $X^{(2)}$ does not have finite range, then both $X^{(1)}$ and $X^{(2)}$ do not have finite range.*
- (ii) *if some integer z is visited infinitely often by one of the walkers $X^{(1)}$ or $X^{(2)}$, then both $X^{(1)}$ and $X^{(2)}$ visit z infinitely often.*
- (iii) *if every integer is visited infinitely often by one of the walkers $X^{(1)}$ or $X^{(2)}$, then both $X^{(1)}$ and $X^{(2)}$ are recurrent.*

Proof of Theorem 4.1: We have to show the following: if at least one of the walkers does not have finite range, then, almost surely, all of them are recurrent. We look at the (random) set

$$A_{\text{rec}} := \left\{ i : 1 \leq i \leq K \text{ and } X^{(i)} \text{ is recurrent} \right\}.$$

The complement can be written as

$$\begin{aligned} A_{\text{rec}}^c &= \{1, \dots, K\} \setminus A_{\text{rec}} \\ &= \bigcup_{z \in \mathbb{Z}} \left\{ i : 1 \leq i \leq K \text{ and } X^{(i)} \text{ visits } z \text{ only finitely often} \right\} \\ &= \bigcup_{z \in \mathbb{Z}} \bigcup_{n \geq 0} \left\{ i : 1 \leq i \leq K \text{ and } X^{(i)} \text{ does not visit } z \text{ at any time } m \geq n \right\}. \end{aligned}$$

Since at time n , all but finitely many edge weights are still 1, we can apply Lemma 4.4 to the random walk from time n onwards. By relabeling, Lemma 4.4 can also be applied to any node z and not just 0, and with arbitrary initial walker position. Hence, a.s.

$$\begin{aligned} & \left\{ i : 1 \leq i \leq K \text{ and } X^{(i)} \text{ does not visit } z \text{ at any time } m \geq n \right\} \\ &= \left\{ i : 1 \leq i \leq K, X^{(i)} \text{ does not visit } z \text{ at any time } m \geq n \text{ and } X^{(i)} \text{ only visits} \right. \\ & \quad \left. \text{finitely many nodes which have not been visited before by any other walker} \right\}. \end{aligned}$$

Since A_{rec}^c is a union of sets of this type, we conclude that, a.s.

$$\begin{aligned} A_{\text{rec}}^c &\subseteq \left\{ i : 1 \leq i \leq K \text{ and } X^{(i)} \text{ only visits finitely many nodes} \right. \\ & \quad \left. \text{which have not been visited before by any other walker} \right\}. \end{aligned}$$

Now, $A_{\text{rec}}^c = \{1, \dots, K\}$ implies that only finitely many nodes are visited overall, so the event $\{\forall i : X^{(i)} \text{ has finite range}\}$ would occur. Hence, it suffices to show that $\{\forall i : X^{(i)} \text{ is recurrent}\}$ occurs whenever $A_{\text{rec}}^c \neq \{1, \dots, K\}$ a.s.

$A_{\text{rec}}^c \neq \{1, \dots, K\}$ implies that there is at least one recurrent walker, say $X^{(i_1)}$. Now, take, for a contradiction, any walker $X^{(i_2)}$ with $i_2 \in A_{\text{rec}}^c$. If i_2 has finite range, then i_2 will meet the recurrent walker i_1 infinitely often (recall that only one walker moves in each time step!), but this is a contradiction to Lemma 4.5 (i). So i_2 must have infinite range. Since i_2 is not recurrent, there must be some integer z which is never visited by i_2 , w.l.o.g. we assume that i_2 only visits nodes to the right of z . The walkers in A_{rec}^c , including i_2 , only visit finitely many nodes not visited before by any other walker, so at least one walker i_3 in A_{rec} must be to the right of i_2 infinitely often in order to “free the path” for i_2 (recall that we consider the case where i_2 has infinite range and must thus go infinitely far to the right). As the walkers in A_{rec} are all recurrent, and as i_2 only visits nodes to the right of z , this implies that i_2 meets this walker i_3 infinitely often, which is a contradiction to Lemma 4.5 (iii). Hence, there is a contradiction if we assume that a walker $i_2 \in A_{\text{rec}}^c$ exists.

We conclude that $A_{\text{rec}}^c = \emptyset$ (which implies that all walkers are recurrent) a.s. if $A_{\text{rec}}^c \neq \{1, \dots, K\}$. □

Proof of Lemma 4.4: We follow the proof of Davis (1990, Lemma 3.0). Consider a fixed random walker i and assume w.l.o.g. $X_0^{(i)} \geq 0$. We now define:

$$\begin{aligned}
 F(n, z) &:= \begin{cases} \sum_{y=0}^{z-1} \frac{1}{w(n, y)} & \text{if } z > 0 \\ 0 & \text{if } z \leq 0, \end{cases} & \tau^{(i)} &:= \inf \left\{ n \geq 0 : X_n^{(i)} \leq 0 \right\}, \\
 M_n^{(i)} &:= F\left(n \wedge \tau^{(i)}, X_{n \wedge \tau^{(i)}}^{(i)}\right) \\
 H_n^{(i)} &:= M_n^{(i)} + \sum_{m=1}^n \left(\frac{1}{w(m-1, X_{m-1}^{(i)})} - \frac{1}{w(m, X_{m-1}^{(i)})} \right) \cdot \mathbb{1}_{X_m^{(i)} > X_{m-1}^{(i)}, m \leq \tau^{(i)}}, \\
 &+ \underbrace{\sum_{m=1}^n \sum_{j=1, j \neq i}^K \sum_{z=0}^{\infty} \left(\frac{1}{w(m-1, z)} - \frac{1}{w(m, z)} \right) \cdot \mathbb{1}_{\{X_{m-1}^{(j)}, X_m^{(j)}\} = \{z, z+1\}, m \leq \tau^{(i)}, z < X_m^{(i)}}}_{=1 \text{ for at most one pair of } j, z}.
 \end{aligned}$$

$M_n^{(i)}$ is nonnegative by definition of F , and $H_n^{(i)} \geq M_n^{(i)} \geq 0$ since edge weights can only increase and therefore, all terms in the sums in the definition of $H_n^{(i)}$ are nonnegative. $H_n^{(i)}$ is a martingale: setting

$$\begin{aligned}
 d_n^{(i)} &:= H_n^{(i)} - H_{n-1}^{(i)} = \underbrace{M_n^{(i)} - M_{n-1}^{(i)}}_{:=e_n^{(i)}} + \underbrace{\left(\frac{1}{w(n-1, X_{n-1}^{(i)})} - \frac{1}{w(n, X_{n-1}^{(i)})} \right) \cdot \mathbb{1}_{X_n^{(i)} > X_{n-1}^{(i)}, n \leq \tau^{(i)}}}_{:=f_n^{(i)}} \\
 &+ \underbrace{\sum_{j=1, j \neq i}^K \sum_{z=0}^{\infty} \left(\frac{1}{w(n-1, z)} - \frac{1}{w(n, z)} \right) \cdot \mathbb{1}_{\{X_{n-1}^{(j)}, X_n^{(j)}\} = \{z, z+1\}, n \leq \tau^{(i)}, z < X_n^{(i)}}}_{:=g_n^{(i)}}
 \end{aligned}$$

we have to show that $\mathbb{E} \left[d_n^{(i)} \mid \mathcal{G}_{n-1} \right] = 0$. We have:

- if $n - 1 \geq \tau^{(i)}$, then $d_n^{(i)} = 0$. Hence, it suffices to consider the case $X_{n-1}^{(i)} = z > 0$ and $\tau^{(i)} \geq n$.
- with probability $\frac{1}{K}$, the walker i jumps at time $n - 1$. In this case, $g_n^{(i)} = 0$ since no other walker can jump and the indicator variable in $g_n^{(i)}$ is therefore 0. If he jumps to

the right (with probability $\frac{1}{K} \cdot \frac{w(n-1,z)}{w(n-1,z-1)+w(n-1,z)}$), then $e_n^{(i)} = w(n,z)^{-1}$ and $f_n^{(i)} = w(n-1,z)^{-1} - w(n,z)^{-1}$, hence $d_n^{(i)} = w(n-1,z)^{-1}$. If he jumps left (with probability $\frac{1}{K} \cdot \frac{w(n-1,z-1)}{w(n-1,z-1)+w(n-1,z)}$), then $e_n^{(i)} = -w(n-1,z-1)^{-1}$ and $f_n^{(i)} = 0$, hence $d_n^{(i)} = -w(n-1,z-1)^{-1}$.

- with probability $\frac{K-1}{K}$, the walker i does not jump. In this case, $f_n^{(i)} = 0$ since the indicator variable in $f_n^{(i)}$ is therefore 0. The value of $M_{n-1}^{(i)}$ now changes (that is, $e_n^{(i)} \neq 0$) if one of the other $K-1$ walkers crosses one of the edges between the nodes 0 and z . At the same time, $g_n^{(i)} \neq 0$ only in this exact case. Now assume the walker $j \neq i$ crosses the edge y with $0 \leq y < z$. Then $e_n^{(i)} = \frac{1}{w(n,y)} - \frac{1}{w(n-1,y)}$ and $g_n^{(i)} = \frac{1}{w(n-1,y)} - \frac{1}{w(n,y)}$, hence $d_n^{(i)} = 0$.
- conditioned on $X_{n-1}^{(i)} = z > 0$ and $\tau^{(i)} \geq n$ (both events measurable w.r.t. \mathcal{G}_{n-1}), we can therefore conclude

$$\begin{aligned} \mathbb{E} \left[d_n^{(i)} \mid \mathcal{G}_{n-1} \right] &= \frac{1}{K} \cdot \frac{1}{w(n-1,z-1) + w(n-1,z)} \cdot \left(\frac{w(n-1,z)}{w(n-1,z)} - \frac{w(n-1,z-1)}{w(n-1,z-1)} \right) \\ &= \frac{1}{K} \cdot \frac{1}{w(n-1,z-1) + w(n-1,z)} \cdot (1 - 1) = 0. \end{aligned}$$

By the same arguments, but only considering $e_n^{(i)}$, we can show that $M_n^{(i)}$ is a supermartingale.

As a nonnegative martingale, $H_n^{(i)}$ converges almost surely.

We just showed this for all walkers i ($1 \leq i \leq K$) with $X_0^{(i)} \geq 0$. Further observe that for such a walker i , we have, on the event $B_n^{(i)} = \left\{ X_n^{(i)} > X_{n-1}^{(i)}, n \leq \tau^{(i)}, w(n-1, X_{n-1}^{(i)}) = 1 \right\}$, that $e_n^{(i)} = w(n, X_{n-1}^{(i)})^{-1}$, $f_n^{(i)} = w(n-1, X_{n-1}^{(i)})^{-1} - w(n, X_{n-1}^{(i)})^{-1}$, $g_n^{(i)} = 0$ and hence $d_n^{(i)} = 1$.

Thus, by convergence, only a finite number of the events $B_n^{(i)}$ can occur for every such walker i .

Now define Γ to be the set of edges between two nonnegative integers to the right of the integer $\max \left\{ X_0^{(i)} : 1 \leq i \leq K \right\}$ for which the initial weight was 1 (all but finitely many edges meet the latter criterion), and further define the event

$$D_n = \left\{ \exists j : X_0^{(j)} \geq 0, \text{ an edge in } \Gamma \text{ is crossed between time } n-1 \text{ and } n \text{ for the first time by any walker, the crossing walker is } j \text{ and } n \leq \tau^{(j)} \right\}.$$

Clearly, $D_n \subseteq B_n^{(i)}$ for some random walker i with $X_0^{(i)} \geq 0$, hence only a finite number of the events D_n can occur.

Now the proof cannot be continued along Davis (1990, Lemma 3.0) since the walkers starting to the left of 0 and the walkers which reach 0 can later cross edges to the right of 0 without triggering D_n and the other walkers can then follow them without triggering D_n . So, we only proved that walkers which never go to 0 and start to the right of 0 cannot visit infinitely many edges which have not been visited before by any other walker. \square

Proof of Lemma 4.5: The proof idea is the following: whenever $X^{(1)}$ and $X^{(2)}$ meet, we can randomly exchange their labels, i.e. we can randomly decide whether we want to rename $X^{(1)}$ to $X^{(2)}$ and vice versa, and the law of the edge-reinforced random walk with the two walkers is invariant under such relabelings because the only distinguishing feature of a random walker is his position. But now, to construct counterexamples to the three statements in Lemma 4.5, we would have to choose a fixed labeling for infinitely many times at which the walkers meet. But if we randomize the labeling with a sequence of independent Bernoulli random variables, then the probability of

choosing a certain fixed labeling at infinitely many points in the sequence is 0, and since the law was invariant under random relabeling, it follows that the probability of any such counterexample is 0. We continue with the formal proof.

Set $\tau_1 := \inf \{n \geq 0 : X_n^{(1)} = X_n^{(2)}\}$ and $\tau_{m+1} := \inf \{n > \tau_m : X_n^{(1)} = X_n^{(2)}\}$. If $X^{(1)}, X^{(2)}$ meet infinitely often, then $\forall n : \tau_n < \infty$, but the construction also works if this is not the case. Let $(\omega_m)_{m \geq 1}$ be a sequence of iid random variables with $\mathbb{P}[\omega_m = 1] = \frac{1}{2} = \mathbb{P}[\omega_m = 0]$ (the ω_m are also independent of \mathcal{G}_n for all n , i.e. independent of the edge-reinforced random walk). Define $\tilde{X}_n^{(1)}$ and $\tilde{X}_n^{(2)}$ as follows (with $\omega_0 = 0$ and $\tau_0 = -1$):

$$\begin{aligned} \tilde{X}_n^{(1)} &:= X_n^{(1,\omega)} = \sum_{m \geq 0} \left((1 - \omega_m) X_n^{(1)} + \omega_m X_n^{(2)} \right) \cdot \mathbb{1}_{\tau_m < n \leq \tau_{m+1}}, \\ \tilde{X}_n^{(2)} &:= X_n^{(2,\omega)} = \sum_{m \geq 0} \left((1 - \omega_m) X_n^{(2)} + \omega_m X_n^{(1)} \right) \cdot \mathbb{1}_{\tau_m < n \leq \tau_{m+1}}. \end{aligned}$$

Note that the sums consist of a single term. $\omega_m = 1$ means that we switch the labels of $X^{(1)}$ and $X^{(2)}$ during the time interval $(\tau_m, \tau_{m+1}]$.

If we consider $(X^{(1)}, X^{(2)})$ and $(\tilde{X}^{(1)}, \tilde{X}^{(2)})$ as sequences of pairs of integers, then we have

$$\left(\tilde{X}_n^{(i)} \right)_{1 \leq i \leq 2, n \geq 0} \stackrel{d}{=} \left(X_n^{(i)} \right)_{1 \leq i \leq 2, n \geq 0}. \tag{4.1}$$

The equality in distribution follows from the above-mentioned invariance of the law of the random walk under relabelings at meeting times which is quite intuitive, and could be proved formally by looking at cylinder events, for example.

We now show that any counterexamples to statements (i), (ii) or (iii) have probability 0:

- (i) Let A be the event that one of the walkers $X^{(1)}, X^{(2)}$ has finite range while the other one has infinite range, and that they meet infinitely often. It suffices to show that $\mathbb{P}[A] = 0$. Denote by \mathbf{P} the probability measure induced by the edge reinforced random walk alone and by \mathbf{Q} the probability measure induced by the sequence $(\omega_m)_{m \geq 1}$ alone. Then, by (4.1), we have

$$\begin{aligned} \mathbb{P}[A] &= \int \int \mathbb{1}_B \, d\mathbf{Q} \, d\mathbf{P} \\ \text{where } B &:= \left\{ \text{one of } \tilde{X}^{(1)}, \tilde{X}^{(2)} \text{ has finite range while} \right. \\ &\quad \left. \text{the other has infinite range, they meet infinitely often} \right\}. \end{aligned}$$

We have to show that the inner integral is 0 almost surely with respect to \mathbf{P} . Consider fixed walker sequences $X^{(1)}$ and $X^{(2)}$. If one of $\tilde{X}^{(1)}, \tilde{X}^{(2)}$ should have finite range while the other has infinite range, then, by definition of $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$, at least one of $X^{(1)}, X^{(2)}$ must have infinite range. Of course, by definition, we also have that $\tilde{X}^{(1)}, \tilde{X}^{(2)}$ meet infinitely often if, and only if, $X^{(1)}, X^{(2)}$ meet infinitely often. Hence, the indicator variable in the integral above can only be 1 in the case where one of the walkers $X^{(1)}, X^{(2)}$ has infinite range and the two walkers meet infinitely often, so we only need to show that in this particular case, the inner integral is still 0 almost surely.

Assume $X^{(1)}$ does not have finite range (w.l.o.g.). Then, for every n , one can find m such that between times τ_m and τ_{m+1} (all τ_m are finite if the two walkers meet infinitely often), $X^{(1)}$ visits a node at distance at least n from the integer 0. Call these times τ_{m_n} with m_n strictly increasing in n (w.l.o.g.).

Now consider the walkers $\tilde{X}^{(1)}, \tilde{X}^{(2)}$. One of them can have finite range only if the following holds. The same argument works for both walkers, we do it here for $\tilde{X}^{(1)}$ w.l.o.g.

$\tilde{X}^{(1)}$ can only have finite range if there exists N such that for all $n \geq N$ we have $\omega_{m_n} = 1$. Assume to the contrary that no such N exists. Then we can find arbitrarily large n such that $\omega_{m_n} = 0$ which means that the labels of $X^{(1)}$ and $X^{(2)}$ are *not* exchanged in the interval $(\tau_{m_n}, \tau_{m_n+1}]$. Since $X^{(1)}$ visits a node at distance at least n from 0 in this time interval, the same holds then for $\tilde{X}^{(1)}$, so $\tilde{X}^{(1)}$ would not have finite range.

But the probability that the sequence ω_{m_n} is 1 for all $n \geq N$ is 0 for any N (since the choice of m_n only depends on the edge-reinforced random walk, i.e. is independent of the ω_m , and since the probability of ω being constantly 1 on any fixed infinite subset of the integers is 0 by the choice of ω). Hence, the probability that such N exists is 0, and therefore the probability that $\tilde{X}^{(1)}$ has finite range is 0 as well, and the same arguments give that the probability for $\tilde{X}^{(2)}$ having finite range is 0 as well (both with respect to the measure \mathbf{Q}).

So the indicator variable in the integral above is 0 almost surely w.r.t. \mathbf{Q} , and hence the inner integral is always 0, which implies that the outer integral is also 0 and hence $\mathbb{P}[A] = 0$.

- (ii) Similar to (i): Let A now be the event that the integer z is visited infinitely often by at least one of the walkers $X^{(1)}, X^{(2)}$, that they meet infinitely often, and that one of them does *not* visit z infinitely often. Then, we have again:

$$\mathbb{P}[A] = \int \int \mathbb{1}_B \, d\mathbf{Q} \, d\mathbf{P}$$

where $B := \left\{ z \text{ visited } \infty \text{ often by at least one of } \tilde{X}^{(1)}, \tilde{X}^{(2)}, \right. \\ \left. \text{they meet } \infty \text{ often, one of them visits } z \text{ only finitely often} \right\}$.

We see that the indicator variable can be 1 only if at least one of $X^{(1)}, X^{(2)}$ visits z infinitely often, and w.l.o.g. assume that this holds for $X^{(1)}$. As before, we can construct a strictly increasing sequence m_n such that in the time interval $(\tau_{m_n}, \tau_{m_n+1}]$, $X^{(1)}$ visits z . Again as before, one of $\tilde{X}^{(1)}, \tilde{X}^{(2)}$, take $\tilde{X}^{(1)}$ w.l.o.g., can visit z only finitely often only if $\omega_{m_n} = 1$ for all $n \geq N$ for some N , an event which has again probability 0 w.r.t. \mathbf{Q} .

- (iii) Apply (ii) to every integer z . □

4.3. Sequence-Type Reinforcement. We now consider the following class of reinforcement schemes, called “sequence type” in Davis (1990). Let $\mathbf{a} = (a_k)_{k \geq 1}$ be a sequence of nonnegative numbers. The reinforcement is of sequence type if $W_z(k) = 1 + \sum_{l=1}^k a_l$ (see Definition 2.1) for all edges z . In words, we require that all initial edge weights are 1 (this is an additional property called “initially fair” in Davis (1990)), and that the edge weights are then increased by a fixed sequence of increments which do not depend on the location of the edge.

We want to show Theorem 4.2, i.e. if

$$\phi(\mathbf{a}) = \sum_{k=1}^{\infty} \left(1 + \sum_{l=1}^k a_l \right)^{-1}$$

then all walkers are recurrent a.s. if $\phi(\mathbf{a}) = \infty$, and all walkers have finite range a.s. otherwise. This result is not surprising as the case $K = 1$ is known due to Davis (1990).

Proof of Theorem 4.2: • Case $\phi(\mathbf{a}) < \infty$: let $z_0 = \max \{ X_0^{(i)} : 1 \leq i \leq K \} + 1$. Consider $z \geq z_0$. Define a stopping time by $T_z := \inf \{ n \geq 0 : X_n^{(i)} = z \text{ for some } i \}$. We want to bound $\mathbb{P}[T_{z+2} < \infty \mid T_z < \infty]$ uniformly away from 1. Once this is proven, we can conclude that $\mathbb{P}[\forall z \geq z_0 : T_z < \infty] = 0$ and therefore, by Theorem 4.1, all walkers must have finite range.

Call E_z the event that any walker reaching z at some point in time only traverses the edge $\{z, z + 1\}$ forever afterwards. For the uniform upper bound, first note that

$$\mathbb{P}[T_{z+2} < \infty \mid T_z < \infty] \leq 1 - \mathbb{P}[E_z].$$

Hence we want to a uniform lower bound for $\mathbb{P}[E_z]$. Consider the following situation. A given number of walkers are currently located at the two neighboring nodes z and $z + 1$ which are incident to the edge $e = \{z, z + 1\}$, which has been traversed k times so far. Set $\alpha_k := 1 + \sum_{l=1}^k a_l$ (this is the weight of edge e). $\phi(\mathbf{a}) < \infty$ implies $\sum_{k \geq 0} \alpha_k^{-1} < \infty$. We do not assume anything about the weights of the adjacent edges at this point and call them w_0 and w_1 .

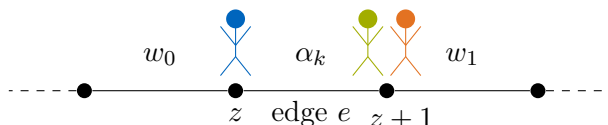


FIGURE 4.4. Some walkers are located at neighboring nodes

We now give a lower bound for the probability that the edge e will be traversed in the next step, if one of the walkers at nodes $z, z + 1$ is chosen to move:

$$\mathbb{P}[e \text{ traversed in next step}] \geq \min \left\{ \frac{\alpha_k}{w_0 + \alpha_k}, \frac{\alpha_k}{w_1 + \alpha_k} \right\} = \frac{\alpha_k}{w_m + \alpha_k} = \frac{1}{w_m \alpha_k^{-1} + 1}$$

where $w_m = \max \{w_0, w_1\}$.

Using that $\frac{1}{x} \geq e^{1-x}$, we further get

$$\mathbb{P}[e \text{ traversed in next step}] \geq \exp(-w_m \alpha_k^{-1}) = (e_m)^{\alpha_k^{-1}}$$

where $e_m = \exp(-w_m) > 0$.

We give a lower bound for $\mathbb{P}[E_z]$ as follows. Define $E_z^{(n)}$ as the event that up to time n , the behavior of the walkers is consistent with the event E_z , i.e. up to time n , any walker which reached z only traversed the edge $e = \{z, z + 1\}$ afterwards. Then $E_z = \bigcap_{n \geq 1} E_z^{(n)}$, and the situation looks as follows if $E_z^{(n)}$ occurs:

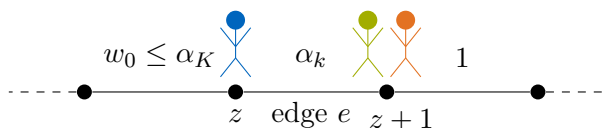


FIGURE 4.5. The walkers' behavior is still consistent with the event E_z

Note:

- z is to the right of the initial walker positions, so the weight of $\{z + 1, z + 2\}$ must be 1 as long as the walkers' behavior is consistent with E_z , since they arrive at z from the left and are trapped in $e = \{z, z + 1\}$ if E_z occurs.
- The weight w_0 of the edge $\{z - 1, z\}$ is at most α_K if the behavior is consistent: the edge can have been traversed at most K times, since there are only K walkers, and since a walker can never go back to $z - 1$ if he reaches z and E_z occurs.
- The weight of e depends on the number of traversals of e , which can be arbitrary. We assume k traversals, so the edge weight is α_k .

We have $w_m = \max\{w_0, w_1\} \leq \max\{\alpha_K, 1\} = \alpha_K$, so $e_m \geq \exp(-\alpha_K)$ in any such situation.

$$\mathbb{P}[E_z] = \prod_{n \geq 1} \mathbb{P}\left[E_z^{(n)} \mid E_z^{(n-1)}\right] \text{ since } E_z^{(n)} \subseteq E_z^{(n-1)} \text{ and where } \mathbb{P}\left[E_z^{(0)}\right] = 1.$$

But $\mathbb{P}\left[E_z^{(n)} \mid E_z^{(n-1)}\right]$ is the probability that e is traversed if one of the walkers incident to e is selected to move. Thus

$$\prod_{n \geq 1} \mathbb{P}\left[E_z^{(n)} \mid E_z^{(n-1)}\right] \geq \prod_{k \geq 0} \exp(-\alpha_K)^{\alpha_k^{-1}} = \exp(-\alpha_K)^{\sum_{k \geq 0} \alpha_k^{-1}} > 0,$$

since $\sum_{k \geq 0} \alpha_k^{-1} < \infty$. Note that the bound does not depend on z , so this is indeed the desired uniform lower bound.

- Case $\phi(\mathbf{a}) = \infty$: we set again $\alpha_k := 1 + \sum_{l=1}^k a_l$, so we have $\sum_{k \geq 0} \alpha_k^{-1} = \infty$ in this case. We want to prove recurrence. By [Theorem 4.1](#), it suffices to show that there is at least one node which is visited infinitely often by at least one of the walkers. Assume for a contradiction that not every node is visited infinitely often and let z be the largest node which is visited infinitely often. Consider the last time n at which a node to the right of z is occupied (visited) by any of the walkers. We will now look at the random walk from time $n + 1$ onwards, and we will show that the probability that $z + 1$ is never visited again is 0, which is a contradiction.

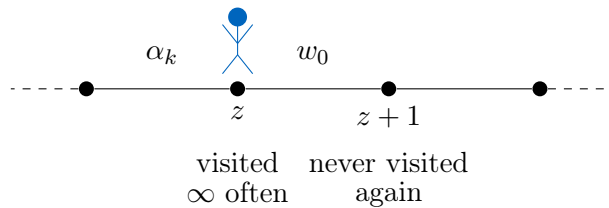


FIGURE 4.6. z is visited infinitely often, but not its right neighbor

We assume that $\{z - 1, z\}$ has been traversed k times and therefore has weight α_k , and we call w_0 the arbitrary weight of $\{z, z + 1\}$ at time $n + 1$. Let $\tau_1, \tau_2, \tau_3, \dots$ be the times at which a walker located at z is selected to move after time n . There will be infinitely many such times since there are infinitely many visits to z . We call t_m the number of traversals of $\{z - 1, z\}$ at time τ_m . Note that $t_m \leq k + K - 1 + 2(m - 1)$. The reason for this is as follows: at time n , there was still one walker to the right of z by definition of the time n , and he must have moved to z at time $n + 1$. Hence, there is at least one walker at z at time $n + 1$. Before a walker at z is selected to move, the weight of $\{z - 1, z\}$ could still be increased by walkers moving to z from the left, but since there are only K walkers, the number of traversals can increase by at most $K - 1$. At subsequent times τ_m , the only other possibility for the weight of $\{z - 1, z\}$ to increase is that a walker leaves z and then (possibly) returns, adding an additional two traversals. This leads to the given upper bound. Then, at each

time τ_m , we have (recall that at time τ_m , a walker at z was selected to move)

$$\begin{aligned} \mathbb{P}[\text{the selected walker moves to } z-1] &= \frac{\alpha_{t_m}}{\alpha_{t_m} + w_0} \\ &\leq \frac{\alpha_{k+K+2m-3}}{\alpha_{k+K+2m-3} + w_0} \\ &\implies \end{aligned}$$

$$\begin{aligned} \mathbb{P}[\text{at all times } \tau_m, \text{ the selected walker moves to } z-1] &\leq \prod_{m \geq 1} \frac{\alpha_{(k+K-3)+2m}}{\alpha_{(k+K-3)+2m} + w_0} \\ &\stackrel{\circledast}{\leq}_{l=k+K-3} \exp\left(-\sum_{m \geq 1} \frac{w_0}{\alpha_{l+2m} + w_0}\right) \end{aligned}$$

where \circledast holds since $\forall x \in \mathbb{R} : x \leq \exp(x-1)$

Since $\alpha_k \geq 1$ for all k , we have that $\alpha_{l+2m} + w_0 \leq (1+w_0)\alpha_{l+2m}$, and we can analyze the sum as follows:

$$\begin{aligned} \sum_{m \geq 1} \frac{w_0}{\alpha_{l+2m} + w_0} &\geq \frac{w_0}{1+w_0} \cdot \sum_{m \geq 1} \frac{1}{\alpha_{l+2m}} \stackrel{\alpha_m \text{ increasing in } m}{\geq} \frac{w_0}{1+w_0} \cdot \frac{1}{2} \cdot \sum_{m \geq 2} \frac{1}{\alpha_{l+m}} \\ &= \frac{w_0}{2(1+w_0)} \cdot \sum_{m \geq l+2} \alpha_m^{-1} = \infty. \end{aligned}$$

Therefore,

$$\mathbb{P}[\text{at all times } \tau_m, \text{ the selected walker moves to } z-1] \leq \exp\left(-\sum_{m \geq 1} \frac{w_0}{\alpha_{l+2m} + w_0}\right) = 0.$$

Hence, the probability that $z+1$ is never visited again after time n is 0, which concludes the proof. \square

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