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Studying the impact of the Asymmetry in Convex Geometry

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Für meine Eltern.

Zusammenfassung

Das Ziel dieser Dissertation ist, einige grundliegende Probleme der konvexen Geometrie mithilfe eines Asymmetriemaß konvexer Körper anzugehen. Ein großer Teil der Konvexen Geometrie beschäftigt sich mit geometrischen Ungleichungen auf Funktionalen der konvexen Körper, wie beispielsweise dem In- und Umradius, dem Volumen, dem Durchmesser oder der Dicke. Viele bekannte geometrische Ungleichungen wurden zuerst für den Fall einer symmetrischen konvexen Menge gezeigt und anschließend für nichtsymmetrische konvexe Körper verallgemeinert. Ein Asymmetriekoeffizient hilft dabei, eine Brücke zwischen diesen Ergebnissen zu schlagen. Darüber hinaus, stellen viele klassische geometrische Ungleichungen Grenzen für Funktionale dar, die von der Dimension des Raums abhängen. Wir ersetzen diese Grenzen durch Koeffizienten des Asymmetriemaßes. Da schließlich der Wert des Asymmetriemaßes die Dimension nicht überschreiten kann, schärfen die erhaltenen Ergebnisse die ursprünglichen.

Abstract

The purpose of this thesis is to tackle several fundamental problems in Convex Geometry using an Asymmetry measure of convex bodies. A big part of Convex Geometry is dedicated to geometric inequalities on functionals of convex bodies, such as, for instance, the in- and circumradius, the volume, the diameter or the width. Many well-known geometric inequalities have been first shown for the case of symmetric convex set, and afterwards generalized for non-symmetric convex bodies. The asymmetry coefficient helps to build a bridge between those results. Moreover, many classical geometric inequalities present bounds for functionals depending on the dimension of the space. We replace those bounds with coefficients on the asymmetry measure. As an advantage of the Minkowski asymmetry is that it is computable in polynomial time for (reasonably given) polytopes. Finally, since the value of the asymmetry measure cannot exceed the dimension, the obtained results sharpen the original ones.

"Tomorrow – more".

——— Bernardo González Merino

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Index of included publications

This is a publication-based doctoral thesis that joins the independent scientific articles that were accepted/published in international peer-reviewed mathematical journals. In addition, it contains a recent work, that has been uploaded to arXiv as a preprint. The presented work was carried out at the Research Unit M9 of Discrete Mathematics and Applied Geometry of the Technical University of Munich (TUM) under the supervision of PD Dr. René Brandenberg and appeared as a result of international collaboration with my second advisor Assoc. Prof. Bernardo González Merino.

The following is included in this cumulative dissertation as ‘Core-Publication’:

(i) The publication [**BDG**], “Relating Symmetrizations of Convex Bodies: Once More the Golden Ratio” by R. Brandenberg, K. von Dichter, B. González Merino, that appeared in Volume 129, No 4 of American Mathematical Monthly in 2022, is included as a ‘Core-Publication’. The original publication is available at www.tandfonline.com. DOI:110.1080/00029890.2022.2043113.

(ii) The publication [**BDG1**], “Tightening and reversing the arithmetic-harmonic mean inequality for symmetrizations of convex sets” by R. Brandenberg, K. von Dichter, B. González Merino, that is accepted for publication in Communications in Contemporary Mathematics and appeared in the ‘online first’ format in 2022 is included as a ‘Core-Publication’. The original publication is available at www.worldscientific.com. DOI:10.1142/S0219199722500456.

This manuscript is included as an additional publication:

(iii) The publication [**BDG2**] “From inequalities relating symmetrizations of convex bodies to the Diameter-width ratio for complete and pseudo-complete convex sets” by R. Brandenberg, K. von Dichter, B. González Merino is included as an additional publication, preprint arXiv:2306.11460.

Papers [**BDG**] and [**BDG1**] are reprinted with permissions from the publishers.

Einverständniserklärung zur publikationsbasierten Promotion[1]

Anlage 6 (für § 6 Abs. 2)

Hiermit erkläre ich mein Einverständnis, dass die Dissertation als publikationsbasierte Dissertation eingereicht wird. Sie erfüllt die nachfolgenden Kriterien:

1. Einleitungs- und Methodenteil (20 Seiten). Ein themenübergreifender Diskussionsteil mit Reflexion zur bestehenden Literatur.
2. Kumulative Einbindung von mindestens zwei akzeptierten Erstautorenveröffentlichungen (full paper in einem englischsprachigen, international verbreiteten Publikationsorgan, peer reviewed)
3. Die eingebundenen Veröffentlichungen müssen federführend vom Doktoranden abgefasst sein.
4. Eingebunden muss sein: je eine einseitige Zusammenfassung der jeweiligen Veröffentlichungen unter Hervorhebung der individuellen Leistungsbeiträge des Kandidaten.
5. Einbindung von ausgewählten Originalveröffentlichungen nur mit einem separaten schriftlichen „Erlaubnisschreiben des jeweiligen Verlags". Alle anderen Originalveröffentlichungen werden unter Nennung der bibliografischen Angaben aufgelistet. In den Exemplaren für die Mitglieder der Prüfungskommission sind alle Originalveröffentlichungen separat dazu abzugeben.

20.6.23

Datum

1 Zur Vorlage bei der Einreichung der Dissertation.

Contents

CHAPTER 1

Introduction

1.1. **Motivation.** Convex geometry focuses on studying properties of convex bodies: compact, convex subsets of Euclidean spaces. A given set in a real vector space is called convex, if for any two points of the set the segment joining them is also contained in the set. Convexity is a fundamental concept in mathematics. It naturally appears in many different areas of mathematics, such as Linear Programming, Combinatorics, Probability Theory, Functional Analysis, Partial Differential Equations, Information Theory and the Geometry of Numbers. For instance, density functions of some of the most important probability measures, like gaussians, exponential, or uniform densities over convex domains, are logarithmically (or at least quasi) concave functions. In particular, this implies that all their level sets are convex (c.f. [**Sch**]). The beauty of convexity lies in its simple formulation and the surprisingly rich structure, that the convex bodies possess. Not only the results, but also the methods of convex geometry are particularly relevant, for instance, in optimization theory and in stochastic geometry (see e.g. [**GrK**]).

The main subject of this thesis are geometric functionals, geometric inequalities between them and extreme relations between convex sets in general, which play a central role in Convex geometry and have various applications in Asymptotic Geometric Analysis, Banach Space Theory and Computational Geometry (see e.g. [**AGM**], [**GrK**], [**Sch**]). The goal is to get a better understanding of the geometric functionals and to sharpen geometric inequalities using an asymmetry measure.

An aim is to take into account an asymmetry measure, i.e., a functional that measures how far a convex body K is away from being symmetric. The most common asymmetry measure, which also is best suited to our purposes, is the so called Minkowski (measure of) asymmetry. It is defined by

$$s(K) := \inf\{\rho > 0 : K - c \subset \rho(c - K),\, c \in \mathbb{R}^n\}.$$

We would like to present a motivation, which comes from dimension reduction. Many research areas in fields, such as biology, physics, astronomy, and meteorology have to frequently deal with computations involving high-dimensional data. Thus, in order to store m vectors in $\mathbb{R}^n$, one would need nm bits for the exact storage. To find a more efficient way to store information one can find an upper bound for the Banach-Mazur distance in conjunction with the Johnson-Lindenstrauss Flattening Lemma (see [**Ve**]). The latter is a helpful tool used to project m points in a (high) dimensional space n to an $\mathrm{O}(\log m)$-dimensional space, such that the distances between the points are (almost) perserved. The

Banach-Mazur distance between two convex compact sets K and C, tells us how similar these sets are, i.e., how one of the two sets can be inscribed into and circumscribed around the other, allowing a linear transformation of one of the sets.

Finding bounds on the Banach-Mazur distance can help with dimension reduction. We will show how the Banach-Mazur distance is connected with the Minkowski Asymmetry and present results on the bounds on the Banach-Mazur distance involving the asymmetry measure.

1.2. **Outline of the Results.** We focus on the generalizations of means of numbers for convex bodies. Namely, we consider the arithmetic and harmonic means of convex bodies, together with the minimum and maximum. Also, similar to the case of means of numbers, these four means of convex sets K and C form a chain of inclusions [**Fir**]. We restrict ourselves to the means of suitably centered convex sets K and $-K$, which then define symmetrizations of the convex set K, and relate all the considered means with each other in terms of (optimal, when possible) inclusions with the help of the Minkowski asymmetry measure. We use the Minkowski center, which is a suitable translation from the definition of the Minkowski asymmetry. For a Minkowski centered convex body K we define the factors $\alpha(K)$ and $\beta(K)$, to be the smallest possible factor to cover $K \cap (-K)$ by $\mathrm{conv}(K \cup (-K))$ and the harmonic mean by the arithmetic mean, respectively.

In the first paper we show that the inclusion of the harmonic in the arithmetic means of the convex sets K and $-K$ is tight if and only if the inclusion of the minimum in the maximum is. Thus, we have $\alpha(K) = 1$ if and only if $\beta(K) = 1$. The main contribution from the first paper is that for the planar convex body K we have: if the asymmetry value of K is greater than the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$, then $\alpha(K) < 1$. Moreover, we show that when $s(K) = \varphi$, then $\alpha(K) = 1$ holds if and only if K is (up to a linear transformation) a special pentagon, which we call the golden house.

In the second paper we continue our analysis on symmetrizations of means of convex sets extending our research on higher dimensions. We show all reverse inclusions for any two considered symmetrizations, involving the Minkowski asymmetry. We also derive bounds for the factors $\alpha(K)$ and $\beta(K)$ in the planar case, as well as in the higher dimensions. Moreover, we show a stability result on the α-value for "very asymmetric" convex sets, i.e., when K is almost a simplex.

In the last paper we give a complete description of the possible α-values of K in the planar case in dependence of its Minkowski asymmetry. In particular, we show that for the planar convex body K we have: if the asymmetry value of K is greater than the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$, then $\alpha(K) \leq \frac{s(K)}{s(K)^2-1}$. Moreover, we derive the family of convex bodies that fulfill with equality this upper bound of $\alpha(K)$.

The breadth of a convex body K within a normed space $\mathbb{R}^n$ in direction of a given hyperplane H is the distance between the two supporting hyperplanes of K parallel to H, measured in the underlying norm. The minimal and the maximal value over all non-zero directions are called minimal width and diameter of K. The body K is complete if every body $K \subsetneq K^*$ has a larger diameter. We present an application of the diagram of the α-values of K for the diameter-width ratio for complete sets, improving the ones given in a recent result of Richter [**Ri**].

CHAPTER 2

Background

2.1. **Notation and the Minkowski asymmetry.** First of all we introduce notations, which are mostly standard concepts in convex geometry and can be found in [**Sch**].

The Minkowski sum of two sets $A, B \subset \mathbb{R}^n$, which is given by

$$A + B = \{x + y : x \in A,\, y \in B\}\,,$$

is used in a wide range of fields in pure and applied mathematics. For instance, in motion planning one may use the Minkowski sum of an obstacle and the robot to define the region, where the robot may run into the obstacle.

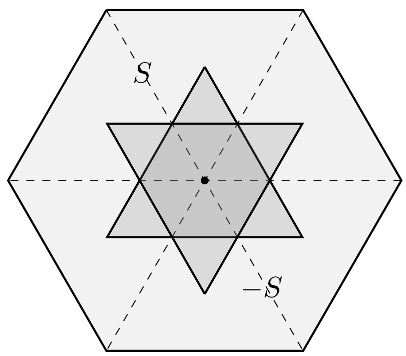

FIGURE 1. Minkowski sum (light gray) of the regular 0-centered triangles S and $-S$ (gray) in $\mathbb{R}^2$.

For any $A, B \subset \mathbb{R}^n$ let $A \subset_t B$ denote that there exists a translate of A being a subset of B, $\rho K := \{\rho x : x \in K\}$ and $-K := (-1)K$. By $\mathbb{B}_p$ we denote the unit ball of an n-dimensional ℓ_p-space. Let $\mathcal{K}^n := \{K \subset \mathbb{R}^n : K \text{ full-dimensional convex and compact}\}$ be the family of convex bodies. For any set $K \in \mathcal{K}^n$, we say that K is symmetric if $K =_t -K$, and 0-symmetric if $K = -K$. We denote the family of 0-symmetric bodies by $\mathcal{K}^n_0$. Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$, which is defined by $h_K(a) = \sup_{x \in K} a^T x$. Let $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$ be the gauge function of a set $K \in \mathcal{K}^n$ with $0 \in \operatorname{int}(K)$, which is defined by $\|x\|_K = \inf\{\rho \geq 0 : x \in \rho K\}$. Let $K^\circ = \{x \in \mathbb{R}^n : x^T y \leq 1\, \forall y \in K\}$ be the polar of $K \in \mathcal{K}^n$.

The main object of this thesis is the Minkowski asymmetry. Although the idea of an asymmetry measure was mentioned at the end of the 19th century by Minkowski [**Mi**], the formal definition appeared only in 1963 by Grünbaum [**Gr**]. He proposed the concept of measures of asymmetry for convex bodies, as a real-valued function, which "for any convex compact sets gives its value, reflecting how far the set is from being centrally symmetric". The research on measures of asymmetry is still going on. Thus, some of the studies focus on describing new measures of asymmetry, which naturally appeared in applications, while others are trying to get a deeper understanding of the nature of the well-known ones (see [**GuK**] for an overview).

We focus on the Minkowski asymmetry of $K \in \mathcal{K}^n$, which is the most common asymmetry measure and is defined as

$$s(K) = \min \{\rho > 0 : K \subset_t -\rho K\}.$$

Note that since K is a convex compact set, we can write min / max instead of inf / sup in the definitions of the considered geometric functionals.

If $c-K \subset s(K)(K-c)$, we call c to be a Minkowski center of K, and if $c = 0$ is possible, we say that K is Minkowski centered. It is well-known (see e.g. [**Gr**]) that for all $K \in \mathcal{K}^n$ we have $s(K) \in [1,n]$ with $s(K) = 1$ if and only if K is symmetric, and $s(K) = n$ if and only if K is an n-simplex, i.e., the convex hull of $n+1$ affinely independent points. Moreover, the Minkowski asymmetry $s : \mathcal{K}^n \to [1,n]$ is continuous with respect to the Hausdorff metric (see [**Gr**]) and invariant under non-singular affine transformations.

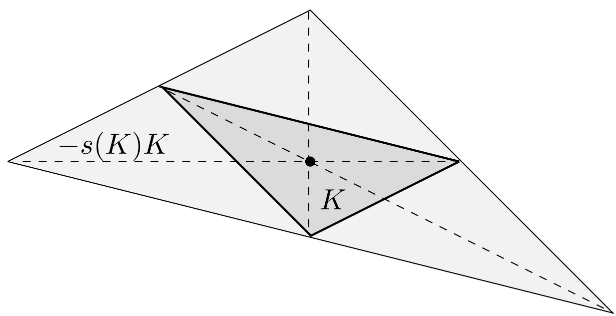

FIGURE 2. $K-c \subset s(K)(c-K)$ for the Minkowski center $c \in \mathbb{R}^n$: K (grey), $-s(K)K$ (light grey).

2.2. **Optimal containment under homothety.** We introduce some standard geometric functionals.

Let $K, C \in \mathcal{K}^n$. The circumradius $R(K,C)$ (of K w.r.t. C) is defined as

$$R(K,C) = \min \{\rho \geq 0 : K \subset_t \rho C\},$$

with $R(K,C) = \infty$ if and only if $\operatorname{aff}(K) \not\subset \operatorname{aff}(C)$ and $R(K,C) = 0$ if and only if K is a singleton. Moreover, if $K \subset c + R(K,C)C$, we call c a circumcenter of K w.r.t. C.

The inradius $r(K,C)$ is defined as $r(K,C) := \max \{\rho \geq 0 : \rho C \subset_t K\}$, and if $c + r(K,C)C \subset K$, c is an incenter of K w.r.t. C. Note that if $r(K,C)$ is not 0 or infinity, $r(K,C) := R(C,K)^{-1}$.

2.3. **Means of Convex bodies.** Naturally, the Minkowski sum of two convex sets defines a mean of those bodies. In the 1960s Firey, in his sequence of works [**Fir, Fir2, Fir3**], has introduced and studied different means of convex sets, the so-called p-means. This investigation continues even nowadays (see [**MiRo, MiRo2, MiMiRo**]).

The harmonic, geometric and arithmetic means of real numbers a and b are known as the Pythagorean means. Together with minimum and maximum they form the extended arithmetic-geometric-harmonic mean inequality

$$\min\{a,b\} \leq \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \max\{a,b\} \tag{1}$$

for any $a, b > 0$, with equality in any/all of the inequalities if and only if $a = b$ (see [**Sch**]).

Convexity is very closely related to the notion of means. For $K, C \in \mathcal{K}^n$ the arithmetic is defined as $\frac{K+C}{2}$. Moreover, for $K, C \in \mathcal{K}^n$, such that $0 \in \mathrm{int}(K) \cap \mathrm{int}(C)$ harmonic means is defined as $\left(\frac{K^\circ + C^\circ}{2}\right)^\circ$, respectively. The inversion operation $x \to 1/x$ can be replaced in higher-dimensional spaces by polarity (see [**MiRo**]). The minimum and maximum of $K, C \in \mathcal{K}^n$ is defined as $K \cap C$ and $\mathrm{conv}(K \cup C)$, respectively. In order to keep the convexity of the considered means, we need to involve the convex hull in the definition of the maximum of K and C.

Note that we can interpret the means of real numbers a and b as means of convex bodies, too. To do so, we associate them with the segments $[-a, a]$ and $[-b, b]$. Now, any mean of a and b corresponds to a particular segment. Thus, for instance, the arithmetic mean of a and b is associated with the segment $[-\frac{1}{2}(a+b), \frac{1}{2}(a+b)] =: \frac{1}{2}([-a,a]+[-b,b])$.

Notice that the considered symmetrizations of a convex body K, i.e., $K \cap (-K)$, $\frac{K-K}{2}$, $\mathrm{conv}(K \cup (-K))$, are frequently used in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [**Bo**], which bounds from above the ratio of the circumradius ($\min_{x \in \mathbb{R}^n} \max_{y \in K} \|x-y\|$) and the diameter ($\max_{x,y \in K} \|x-y\|$) of convex bodies in arbitrary normed spaces endowed with a norm $\|\cdot\|$ by $n/(n+1)$, and for which equality is reached in spaces with $S \cap (-S)$ or $\frac{1}{2}(S-S)$ as the unit ball [**BrKo**] where S is a 0-centered regular simplex. These means also appear in characterizations of spaces, for which K is complete or reduced [**BGJM**, Prop. 3.5 – 3.10].

Firey has shown that similarly to the Pythagorean means, the means of convex sets can be ordered in terms of inclusions [**Fir**].

Proposition 2.3.1. *For all $K, C \in \mathcal{K}^n$ with 0 in their interior we have*

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K+C}{2} \subset \mathrm{conv}(K \cup C). \tag{2}$$

We would like also to mention that

$$\|x\|_{\left(\frac{K^\circ + C^\circ}{2}\right)^\circ} = \frac{1}{2}\left(\|x\|_K + \|x\|_C\right).$$

Let $x \in \mathbb{R}^n$. Then the well-known l_p-norm inequality in particular states

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \quad \text{and} \quad \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty. \tag{3}$$

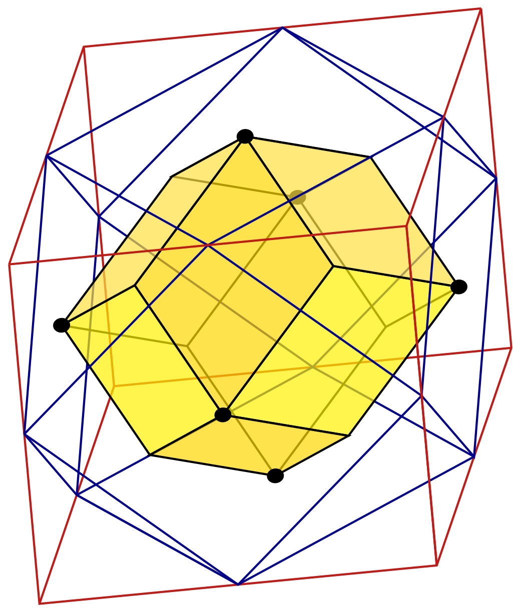

FIGURE 3. Symmetrizations of a regular simplex $S \subset \mathbb{R}^3$: $S \cap (-S)$ (convex hull of black points), $\left(\frac{S^\circ - S^\circ}{2}\right)^\circ$ (yellow), $\left(\frac{S-S}{2}\right)$ (blue), and $\operatorname{conv}(S \cup (-S))$ (red).

Notice that, for instance, in $\mathbb{R}^3$ the symmetrizations of a regular Minkowski centered simplex $S \in \mathcal{K}^3$ are exactly $S \cap (-S) = \mathbb{B}_1$, $\operatorname{conv}(S \cup (-S)) = \mathbb{B}_\infty$. Thus, in 3-space the inequalities (3) can be read as

$$\|x\|_{\operatorname{conv}(S\cup(-S))} \leq \|x\|_2 \leq \|x\|_{S\cap(-S)} \quad \text{and} \quad \|x\|_{S\cap(-S)} \leq \sqrt{n}\|x\|_2 \leq n\|x\|_{\operatorname{conv}(S\cup(-S))}.$$

In general, the ℓ_p-norm chain of inequalities can be generalized using (2) to

$$\|x\|_{\operatorname{conv}(K\cup C)} \leq \|x\|_{\frac{K+C}{2}} \leq \|x\|_{\left(\frac{K^\circ+C^\circ}{2}\right)^\circ} \leq \|x\|_{K\cap C}. \tag{4}$$

In order to reverse the ℓ_p-norm inequalities, we focus on reversing (2) using scaling factors that depend on the Minkowski asymmetry. We show those results in the next section.

Observe also that even for a non-symmetric unit ball K, one may approximate the gauge function by the norms induced from symmetrizations of K

$$\|x\|_{\operatorname{conv}(K\cup(-K))} \leq \|x\|_K \leq \|x\|_{K\cap(-K)}. \tag{5}$$

As mentioned in Section 2, for a Minkowski centered convex compact set K we define the factor $\alpha(K)$ to be the smallest possible factor to cover $K \cap (-K)$ by $\operatorname{conv}(K \cup (-K))$, i.e.,

$$\alpha(K) := \inf\{\rho > 0 : K \cap (-K) \subset \rho \operatorname{conv}(K \cup (-K))\}.$$

Notice that there always exists some $x \in \mathbb{R}^n$ such that $\alpha(K)\|x\|_{K\cap(-K)} = \|x\|_{\mathrm{conv}(K\cup(-K))}$, which means that we have equality in the complete chain in (5) for that x if $\alpha(K) = 1$.

Observe that $\alpha(K) = R\,(K \cap (-K), \mathrm{conv}(K \cup (-K)))$, and we also define

$$\beta(K) := R\left(\left(\frac{K^\circ - K^\circ}{2}\right)^\circ, \frac{K-K}{2}\right),$$

$$\tau(K) := R\left(K \cap (-K), \frac{K-K}{2}\right).$$

In the following section we will show bounds on the parameters $\alpha(K)$, $\beta(K)$ and $\tau(K)$.

2.4. **Diameters.** Note that the diameter has several different definitions, which are equivalent in normed spaces, i.e., when the gauge is symmetric. Leichweiss in [**Le**] defined the diameter of K w.r.t. C in a natural way, as the maximal distance between two points in the set (w.r.t. the gauge)

$$D_{\max}(K,C) := \max_{x,y \in K} \|x-y\|_C.$$

Another way to define the diameter of K w.r.t. C is

$$D(K,C) := 2 \max_{x,y \in K} R(\{x,y\},C).$$

(see [**DGK**] for details). The latter definition allows to see the diameter as a best 2-point approximation of the circumradius of the whole set K. Another advantage is that it is translation invariant in both arguments, which is not the case for $D_{\max}$. Choosing C with 0 close to its boundary, the ratio of $\frac{R(K,C)}{D_{\max}(K,C)}$ may even get arbitrarily small. Note that

$$D_{\max}(K,C) = D(K, C \cap (-C)) = 2R\left(\frac{K-K}{2}, C \cap (-C)\right), \quad \text{while}$$
$$D(K,C) = D\left(K, \frac{C-C}{2}\right) = 2R\left(\frac{K-K}{2}, \frac{C-C}{2}\right).$$

We see that different means of C and $-C$ naturally appear in the definition of the diameters.

Moreover, if C is Minkowski centered, the results above show us, that we can bound those diameters in terms of the other and therefore also the circumradius-diameter ratio for the maximal diameter.

The width of K w.r.t. C is defined by

$$w(K,C) = 2 \min_{a \in \mathbb{R}^n \setminus \{0\}} \frac{h_K(a) + h_K(-a)}{h_C(a) + h_C(-a)}.$$

Observe that

$$\frac{w(K,C)}{2} = \left(\frac{D(C,K)}{2}\right)^{-1}.$$

Let $K, C \in \mathcal{K}^n$ and $s \in \mathbb{R}^n \setminus \{0\}$. The s-breadth of K w.r.t. C is defined as

$$b_s(K,C) := 2 \frac{\max_{x,y \in K} s^T(x-y)}{\max_{x,y \in C} s^T(x-y)}.$$

For symmetric gauges C the s-breadth $b_s(K,C)$ of a body K is defined as the maximal distance of the two supporting hyperplanes of K with outer and inner normal s, where the gauge body is involved by choosing s to be a unit vector of the polar space [**GrK**]. Thus,

$$b_s(K,C) = h_{K-K}(s), \quad s \in C^\circ,$$

which is equivalent to

$$b_s(K,C) = \frac{h_{K-K}(s)}{h_C(s)}, \quad s \in \mathbb{R}^n \setminus \{0\}.$$

Using the definition of the s-breadth,

$$w(K,C) = \min_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K,C) \quad \text{and} \quad D(K,C) = \max_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K,C).$$

One can show that (since C is symmetric),

$$D(K,C) = 2\max_{x,y\in K} R(\{x,y\},C) = \max_{x,y\in K} \|x-y\|_C = D_{\max}(K,C).$$

Observe that the choice of the diameter depends on the needed properties. Thus, if based on the applications, one needs in the general case to have the possibility to measure the directional length from x to y to be different than the one from y to x, one would choose to calculate the distance as $\|x-y\|_C$ and consider $D_{\max}(K,C)$.

Instead of averaging the directional breadthes, one may also take their minimal value. This way we arrive in the minimal diameter

$$D_{\min}(K,C) = \max_s \min\left\{\frac{h_{K-K}(s)}{h_C(s)}, \frac{h_{K-K}(s)}{h_{-C}(s)}\right\}.$$

It turns out that any of the four diameters defined this way corresponds to a different symmetrization of C :

$$\begin{aligned}
D_{\min}(K,C) &= D(K, \operatorname{conv}(C\cup(-C))),\\
D_{\max}(K,C) &= D(K, C\cap(-C)),\\
D(K,C) &= D\left(K, \frac{C-C}{2}\right), \quad \text{and}\\
D_{\text{mean}}(K,C) &= D\left(K, \left(\frac{C^\circ - C^\circ}{2}\right)^\circ\right).
\end{aligned}$$

In general, from Proposition 2.3.1 we have for $C \in K^n$ that

$$C\cap(-C) \subset \left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset \frac{C-C}{2} \subset \operatorname{conv}(C\cup(-C)),$$

which easily implies an order on the diameters

$$D_{\min}(K,C) \le D_{\text{mean}}(K,C) \le D(K,C) \le D_{\max}(K,C).$$

For general $C \in \mathcal{K}^n$ a backward relation of considered diameters is not possible, since for a fixed size of $D(K,C)$, the two first may be arbitrarily small, while the last may be arbitrarily big. This is due to a possibly badly chosen position of C as $D(K,C)$ is the only of the four diameters being translation-invariant w.r.t. C. However, if we restrict to Minkowski centered C we can easily bound each of the symmetrizations of C by any of the smaller ones using the Minkowski asymmetry (see [**BDG1**]).

Enumerating them along the ascending set-containment chain

$$\begin{aligned}
C_1 &= C\cap(-C),\\
C_2 &= \left(\frac{C^\circ - C^\circ}{2}\right)^\circ,\\
C_3 &= \frac{C-C}{2}, \quad \text{and}\\
C_4 &= \operatorname{conv}(C\cup(-C)),
\end{aligned}$$

we obtain, (see [**BDG1**, Theorem 1.3])

$$C_{i+1} \subset^{opt} \frac{2s(C)}{s(C)+1} C_i, \quad i = 1, 3, \text{ and}$$
$$C_{i+2} \subset^{opt} \frac{s(C)+1}{2} C_i, \quad i = 1, 2.$$

Again, this immediately implies geometric inequalities between the according diameters.

2.5. **The Brunn-Minkowski and Rogers-Shephard inequalities.** The Brunn-Minkowski theory concerns the issue of understanding the relation between two basic notions: the Minkowski addition of $K, C \in \mathcal{K}^n$ and the (fulldimensional) volume denoted by $\mathrm{vol}(\cdot)$.

The well-known Brunn-Minkowski inequality [**Bru, Mi**] states that

$$\mathrm{vol}(K+C)^{\frac{1}{n}} \geq \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(C)^{\frac{1}{n}}, \tag{6}$$

with equality if and only if K and C are homotheties of each other.

Observe that even if $\mathrm{vol}(K) = \mathrm{vol}(C) = 0$, $\mathrm{vol}(K+C)$ can be arbitrarily big: consider, for instance, K to be a segment and C a line, which is orthogonal to the segment.

In particular, choosing $K = -C$ one obtains a particular variant of the inequality

$$\mathrm{vol}\left(\frac{K-K}{2}\right) \geq \mathrm{vol}(K), \tag{7}$$

with equality if and only if K is symmetric [**Gr**].

The reverse to the Brunn-Minkowski inequality (7) is the socalled Rogers and Shephard inequality [**RoSh**], which in case $K = -C$ states

$$\mathrm{vol}\left(\frac{K-K}{2}\right) \leq \frac{1}{2^n}\binom{2n}{n}\mathrm{vol}(K), \tag{8}$$

with equality if and only if K is an n-dimensional simplex.

This is equivalent to

$$\mathrm{vol}\left(\frac{K-K}{2}\right) \leq \frac{(2n-1)!!}{n!}\mathrm{vol}(K), \tag{9}$$

Here the double factorial $(2n-1)!! := 1 \cdot 3 \cdot \dots \cdot (2n-1)$.

Combining (7) and (9), we obtain

$$1 \leq \frac{\mathrm{vol}\left(\frac{K-K}{2}\right)}{\mathrm{vol}(K)} \leq \frac{(2n-1)!!}{n!}. \tag{10}$$

Moreover, equality in the left inequality holds if and only if K is symmetric and in the right one if and only if K is an n-dimensional simplex.

This is the reason why the quotient

$$\frac{\mathrm{vol}\left(\frac{K-K}{2}\right)}{\mathrm{vol}(K)}$$

is itself a measure of asymmetry which is also known as the difference measure of symmetry (c.f. [**Gr, Ta**]).

An important consequence of the Brunn-Minkowski inequality is the isoperimentric inequality, which in the planar case states that among all subsets of $\mathbb{R}^2$ with some fixed perimeter, the disc has the greatest area. This result was known even in the ancient Greece and in $\mathbb{R}^3$ explains why soap bubbles are round.

Let $K \subset \mathbb{R}^n$ and C be an Euclidean ball with an infinitely small inradius. Then (6) leads to the isoperimetric inequality in dimension n, which states the following.

Proposition 2.5.1. *Let $K \in \mathcal{K}^n$ and $P(K)$ be the surface area of K. Then*

$$P(K) \geq n(\mathrm{vol}(\mathbb{B}_2))^{\frac{1}{n}}(\mathrm{vol}(K))^{\frac{n-1}{n}}.$$

Recently, there have been several new conjectures regarding inequalities, which involve volumes of convex bodies, based on the idea of replacing the arithmetic mean $\frac{K+C}{2}$ by other means of convex bodies K and $-K$, which provides an extra motivation to study symmetrizations of convex bodies, such as, e.g., $K \cap (-K)$ or $\left(\frac{K^\circ - K^\circ}{2}\right)^\circ$.

We define the constants $c(s(K)), C(s(K))$ with

$$2n \leq c(s(K)) \leq C(s(K)) \leq \binom{2n}{n},$$

such that for every convex body K such that $1 \leq s(K) \leq n$ holds

$$c(s(K)) \leq \frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \leq C(s(K)).$$

If $s(K)$ grows, then the difference measure of symmetry should be *reasonably* big, and vice-versa, and the same should hold for small values of the asymmetries. This intuition is partly verified in [**Di**], where it was shown slightly sharpened versions of (7) and (8) using the Minkowski asymmetry.

Indeed, if $K \in \mathcal{K}^n$, then

$$c(s(K)) \geq 2^n \left(1 + \frac{1}{n\,4^{n-1}}\left(\frac{(s(K)-1)^n \mathrm{vol}_{n-1}(\mathbb{B}_2^{n-1})}{2^{n-1}n^{2n}\mathrm{vol}(\mathbb{B}_2)}\right)^2\right)^n \tag{11}$$

and

$$C(s(K)) \leq (1+s(K))^n. \tag{12}$$

Observe that in (11) we use the notation of $\mathbb{B}_2^{n-1}$, since it is important to stress the dimension of the considered Euclidean ball.

Moreover, if $n - \frac{1}{4n} < s(K) < n$, then

$$c(s(K)) \geq \binom{2n}{n}(1-4n^2(n-s(K))) \quad \text{and} \quad C(s(K)) \leq \binom{2n}{n}\left(1 - \frac{n-s(K)}{n^{1+50n^2}}\right). \tag{13}$$

It is worth mentioning that (11) and (12) are especially good when $s(K)$ is close to 1, whereas (13) is better when $s(K)$ is close to n.

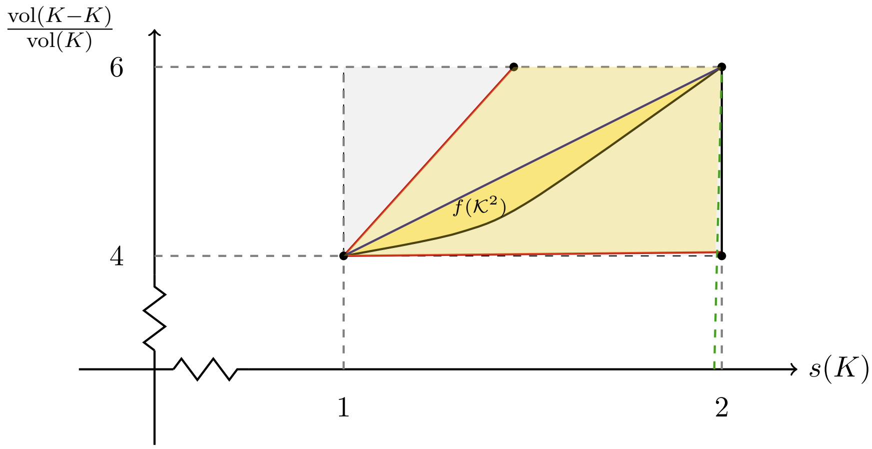

FIGURE 4. The diagram show the impact of the asymmetry $s(K)$ (x-axis) onto the quotient $\mathrm{vol}(K-K)/\mathrm{vol}(K)$ in the planar case. While the slightly shaded area depicts the still possible combinations between the proven boundaries, the darker grey area shows those combinations for which explicit examples can be given.

2.6. **The Banach-Mazur distance.** For any $K, C \in \mathcal{K}^n$ the Banach-Mazur distance between K and C is defined as

$$d_{BM}(K,C) := \min\{\rho \geq 1 : A(K) \subset_t C \subset_t \rho A(K) \text{ with } A \text{ a linear map}\}.$$

Reformulating slightly, one can easily see the fundamental (yet elementary) link between the Banach-Mazur distance and the radii measures introduced before:

$$d_{BM}(K,C) = \min\left\{\frac{R(A(K),C)}{r(A(K),C)} : A \text{ a linear map}\right\}. \tag{14}$$

Calculating the Banach-Mazur distance between convex bodies is important in many field, e.g. in data reduction as mentioned in the introduction, or in functional analysis, where it is used as one of the most important metrics between normed spaces (see [**Con**]).

A major open task is to determine the maximal possible Banach-Mazur distance between any two n-dimensional convex sets, which is the diameter of the Banach-Mazur Compactum [**TJ**], i.e., finding

$$\max\{d_{BM}(K,C) : K, C \in \mathcal{K}^n_0\}.$$

Observe that the Minkowski asymmetry $s(K)$ can be understood as the minimal distance of K to the family of symmetric bodies (as already recognized in [**Gr**])

$$s(K) = \min\{d_{BM}(K,C) : C =_t -C\}. \tag{15}$$

Very recently [**KoVa**] has shown that for every $s \in [\frac{7}{4}, 2]$ there exists $K \in \mathcal{K}^2$ with $s(K) = s$ such that

$$d_{BM}(K,C) = s.$$

Moreover, since the Minkowski asymmetry is affine-invariant, we obtain from (23) that

$$d_{BM}(K,C) \geq \max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\}. \tag{16}$$

It directly follows from a result in [**BrG3**] that

$$d_{BM}(K,C) \leq s(K)s(C),$$

whenever K is complete w.r.t. C, which together with (16) implies

$$s(K) = d_{BM}(K,C),$$

whenever K is complete w.r.t. $C =_t -C$. Undoubtedly, John's theorem [**Jo**] is the result with highest impact until today towards the Banach-Mazur distance.

Thus, for $K \in \mathcal{K}^n$ holds

$$d_{BM}(K, \mathbb{B}_2) \leq n$$

and for $K \in \mathcal{K}^n_0$

$$d_{BM}(K, \mathbb{B}_2) \leq \sqrt{n}.$$

Moreover, if both bodies K and C are symmetric, we have

$$d_{BM}(K,C) \leq n. \tag{17}$$

Also in the case of two non-necessarily symmetric convex bodies, the upper bound n^2 obtained from John's theorem is not best possible. Indeed, so far no sequences of convex bodies K_n and C_n could be proven to exist, fulfilling $d_{BM}(K_n, C_n) > cn^{1+\epsilon}$, for any absolute constants $c, \epsilon > 0$ for arbitrarily large n. Thus, it may indicate that there may exists an absolute constant $c > 0$ such that $d_{BM}(K,C) \leq cn$ for any two n-dimensional convex bodies K, C.

The so far best general upper bound known is given by Rudelson [**Ru**]. It states

$$d_{BM}(K,C) < cn^{4/3}\log(n)^{\alpha}, \tag{18}$$

where $c, \alpha > 0$ are absolute constants.

If one allows to use K and $-K$ for the coverage of $A(C)$ we arrive in a variant (and lower bound) of the Banach-Mazur distance called Grünbaum distance [**Gr**]:

$$d_{Gr}(K,C) := \min\left\{|\rho| : K \subset_t A(C) \subset_t \rho K \text{ with } A \text{ a linear map}\right\}.$$

In [**GLMP**] it is shown through the generalized John position between K and C that

$$d_{Gr}(K,C) \leq n.$$

Using the fact that $d_{Gr}(K,C) = d_{BM}(K,C)$ whenever $C =_t -C$ combined with (15) now directly implies

$$d_{BM}(K,C) \leq n \min\left\{s(K), s(C)\right\}. \tag{19}$$

This may in general be weaker than the one of Rudelson. However, whenever the asymmetry of one of the sets is small (e.g. not depending on n), it gives the better bound.

Pukhov [**Pu**] shows that if K is an n-simplex then

$$d_{BM}(K,C) \leq n+2$$

(c.f. Perelman [**Pe**] and Lassak [**La**]).

In [**Sch1**] a stability result for the simplex is derived involving the Minkowski asymmetry. It shows that for every $K, C \in \mathcal{K}^n$, and every $\varepsilon \in [0, \frac{1}{n}]$, such that $s(C) = n$ and $s(K) > n - \varepsilon$ it holds

$$d_{BM}(K,C) < 1 + \frac{(n+1)\varepsilon}{1 - n\varepsilon}. \tag{20}$$

There are several results in the planar case. In [**Str**] it is shown that in case when K and C are both symmetric,

$$d_{BM}(K,C) \leq \frac{3}{2}$$

with equality for the square and the regular hexagon.

Dropping the symmetry condition, [**La**] shows that for $K, C \in \mathcal{K}^n$ holds

$$d_{BM}(K, C) \leq 3$$

and if C is a triangle,

$$d_{BM}(K, C) \leq \frac{5}{2}.$$

[**La**] also conjectures that both bounds get tight for the value $1 + \frac{\sqrt{5}}{2} \approx 2.118$, with equality for the triangle and the regular pentagon.

Recently, the general bound in the planar case was improved to $\frac{19-\sqrt{75}}{4} \approx 2.585$ [**BPP**]).

Based on the results from [**BDG1**], we observe the following. Let $C \in \mathcal{K}_0^n$ and $K \in \mathcal{K}^n$. Then, since $L(C)$ is symmetric, we get from the definition of the Banach-Mazur distance that

$$d_{BM}(K, C) \leq s(K)$$

with equality if and only if there exists a transformation $L \in GL(n)$ and $t^1, t^2 \in \mathbb{R}^n$ such that

$$\operatorname{conv}((K + t^1) \cup (-K - t^1)) \subset L(C) \subset s(K)((K + t^2) \cap (-K - t^2)).$$

For a Minkowski concentric K we now immediately obtain that all four choices

$$C \in \left\{ K \cap (-K), \left(\frac{K^\circ - K^\circ}{2} \right)^\circ, \frac{K - K}{2}, \operatorname{conv}(K \cup (-K)) \right\}$$

of symmetrizations of K considered in this paper fulfill

$$d_{BM}(K, C) = s(K)$$

and are therefore minimizers for the Banach-Mazur distance between K and $\mathcal{K}_0^n$.

Observe, that based on the results from [**BDG1**, Theorem 1.3], we also obtain bounds on the Banach-Mazur distance between different symmetrizations, such as, for instance,

$$d_{BM}(K \cap (-K), \operatorname{conv}(K \cup (-K)) \leq s(K) \quad \text{or}$$
$$d_{BM}\left(K \cap (-K), \left(\frac{K^\circ - K^\circ}{2} \right)^\circ \right) \leq \frac{2s(K)}{s(K) + 1}.$$

2.7. **Constant width, Completeness and Pseudo-completeness.** For $K, C \in \mathcal{K}^n$ we define K to be of constant width (w.r.t. C) if $D(K,C) = w(K,C)$ and (diametrically) complete (w.r.t. C) if $D(K',C) > D(K,C)$ for all $K' \supset K$.

In some cases, for instance, when C is the Euclidean ball or C is a planar set, those two concepts are equal, but even in dimension three it is possible to find a set $C \in \mathcal{K}_0^3$ such that there exists a convex set K, which is complete, but not of constant width w.r.t. C (see [**Egg**]). Due to the invariance under symmetrization, completeness of K w.r.t. C implies completeness of K w.r.t. $C-C$. However, as long as K is not of constant width, K complete will not imply $K-K$ complete (c.f. [**BGJM**]).

One should recognize that constant width always implies completeness, while the opposite is not true. In fact, constant width is equivalent to

$$\frac{K-K}{2} = \frac{D(K,C)}{2} \cdot \frac{C-C}{2}.$$

Constant width sets and complete bodies have been studied in numerous works, including [**Gro**], [**HMa**] (in Euclidean spaces); [**Egg**], [**MoSch**] (in general Minkowski spaces). Minkowski spaces in which all complete sets are of constant width are called perfect. Characterizing those spaces is still a major open question in convex geometry (see [**Egg**] and [**MoSch**]).

The Jung constant j_C of the normed space induced by the gauge C measures the maximal ratio between the circumradius and the diameter of arbitrary bodies in that space.

In [**Bo**] and [**Le**] it is shown that for symmetric $C \in \mathcal{K}^n$ we have

$$j_C := \max_{K \in \mathcal{K}^n} \frac{R(K,C)}{D(K,C)} \leq \frac{n}{n+1} \quad \text{and} \quad \max_{K \in \mathcal{K}^n} \frac{w(K,C)}{r(K,C)} \leq n+1. \tag{21}$$

As mentioned in Section 2, using the facts that for $K, C \in \mathcal{K}^n$ it holds that

$$r(K,C) = R(C,K)^{-1} \quad \text{and} \quad D(K,C) = 4w(C,K)^{-1},$$

these two results (21) have been unified and tightened in [**BrK2**] with help of the Minkowski asymmetries of K and C:

$$\frac{2R(K,C)}{D(K,C)} = \frac{w(C,K)}{2r(C,K)} \leq \frac{s(K)(s(C)+1)}{s(K)+1}. \tag{22}$$

While in general $R(K,C)/r(K,C)$ can only be bounded from below by 1 (with equality iff K and C are homothets of each other) it is long known that for an n-simplex S it holds

$$\frac{R(S,\mathbb{B}_2)}{r(S,\mathbb{B}_2)} \geq n.$$

Again using the Minkowski asymmetry this inequality was generalized in [**BrK2**] to

$$\frac{R(K,C)}{r(K,C)} \geq \max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\}. \tag{23}$$

The well-known concentricity inequalities states for symmetric C that

$$w(K,C) \le r(K,C) + R(K,C) \le D(K,C). \tag{24}$$

By (22) and (23), equation (24) can be extended to a whole inequality chain using $s(K)$:

$$w(K,C) \le (1+s(K))r(K,C) \le r(K,C) + R(K,C) \le \frac{1+s(K)}{s(K)}R(K,C) \le D(K,C). \tag{25}$$

Note that the following are equivalent:

$$(1+s(K))r(K,C) = \frac{1+s(K)}{s(K)}R(K,C),$$
$$(1+s(K))r(K,C) = r(K,C) + R(K,C),$$
$$r(K,C) + R(K,C) = \frac{1+s(K)}{s(K)}R(K,C). \tag{26}$$

The notion of pseudo-complete sets was introduced in [**BrG2**] as the family of sets $K \in \mathcal{K}^n$ and $C \in \mathcal{K}^n_0$ that fulfill the equation

$$r(K,C) + R(K,C) = D(K,C).$$

Notice that any complete set is pseudo-complete and that the opposite does not always hold [**MoSch**]. In [**BrG2**] it also shown that a set is pseudo-complete if and only if there exists a circumcenter $c \in \mathbb{R}^n$ of K w.r.t. $C \in \mathcal{K}^n_0$, such that

$$c + (D(K,C) - R(K,C))C \subset K.$$

It is shown in [**BrG2**] that K complete only implies the following equivalent facts:

(i) $r(K,C-C) + R(K,C-C) = D(K,C-C)$ and
(ii) for every incenter c of K w.r.t. $C-C$ it holds

$$K - K \subset 1/2D(K,C)(C-C) \subset (s(K)+1)(K-c) \cap (-(K-c)). \tag{27}$$

One of the most important characterizations of completeness, the spherical intersection property [**Egg**], holds true only for symmetric C, too:

Let $K \in \mathcal{K}^n$, $C \in \mathcal{K}^n_0$ be such that K complete w.r.t. C. Then

$$K = \bigcap_{x \in K} (x + D(K,C)C).$$

It is shown in [**GuK**] for $C = \mathbb{B}_2$ and in [**BrG2**] for $C \in \mathcal{K}^n_0$, that the Jung constant can be expressed as follows

$$j_C = \max_{K \text{ complete w.r.t. } C} \frac{s(K)}{s(K)+1}. \tag{28}$$

For constant width sets the diameter-width-ratio is equal to one. [**Ri**] recently has shown how big this ratio can get for complete sets.

Proposition 2.7.1. *Let $K \in \mathcal{K}^n$, $C \in \mathcal{K}^n_0$ and K be complete w.r.t. C. Then*

$$\frac{D(K,C)}{w(K,C)} \leq \frac{n+1}{2}.$$

Moreover, this bound is sharp, if n is odd, and asymptotically sharp, if n is even.

We will show in the next chapter how the bound from Proposition 2.7.1 can be sharpened involving the asymmetry measure of Minkowski.

CHAPTER 3

Results

3.1. **Summary of the first paper.** First of all, we characterize optimal containment of the means of K and $-K$.

Theorem 3.1.1. *Let $K \in \mathcal{K}^n$ be Minkowski centered. Then the following are equivalent:*

(i) $\alpha(K) = 1$,
(ii) $\beta(K) = 1$,
(iii) there exist $p, -p \in \mathrm{bd}(K)$ and two parallel halfspaces $H^{\leq}_{a,\rho}$ and $H^{\leq}_{-a,\rho}$, supporting K at p and $-p$, respectively.

We show that in 2-space the greatest value of the Minkowski asymmetry such that the harmonic mean can be optimally contained in the arithmetic mean is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$.

Theorem 3.1.2. *Let $K \in \mathcal{K}^2$ be Minkowski centered such that $\alpha(K) = 1$.*

Then $s(K) \leq \varphi$. Moreover, if $s(K) = \varphi$, there exists a non-singular linear transformation L, such that

$$L(K) = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\\varphi\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}\right)$$

is the golden house $\mathbb{GH}$.

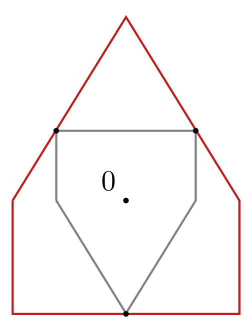

FIGURE 1. The Golden house $\mathbb{GH}$ (red) and $-\frac{1}{s(\mathbb{GH})}\mathbb{GH}$ (gray).

Observe that we also present a family of sets $K_s \in \mathcal{K}^2$ with $s(K_s) = s \in [1, \varphi]$, such that $\alpha(K) = 1$, thus, showing that for any $s \in [1, 2]$ there exists a Minkowski centered $K \in \mathcal{K}^2$ with $s(K) = s$ and $\alpha(K) = 1$.

I was significantly involved in the essential phases of brainstorming and in the elaboration of all parts of the work on this paper.

3.2. **Summary of the second paper.** We describe the inclusions between any two symmetrizations in the reverse direction.

Theorem 3.2.1. *Let $K \in \mathcal{K}^n$ be Minkowski centered. Then*

(i) $\mathrm{conv}(K \cup (-K)) \subset^{opt} s(K)(K \cap (-K))$,
(ii) $\mathrm{conv}(K \cup (-K)) \subset^{opt} \frac{2s(K)}{s(K)+1} \frac{K-K}{2}$,
(iii) $\left(\frac{K^\circ - K^\circ}{2}\right)^\circ \subset^{opt} \frac{2s(K)}{s(K)+1}(K \cap (-K))$,
(iv) $\frac{K-K}{2} \subset^{opt} \frac{s(K)+1}{2}(K \cap (-K))$, *and*
(v) $\mathrm{conv}(K \cup (-K)) \subset^{opt} \frac{s(K)+1}{2} \left(\frac{K^\circ - K^\circ}{2}\right)^\circ$.

Moreover, for the following containment chain always applies:

(vi) $\frac{K-K}{2} \subset \frac{s(K)+1}{2} \left(\frac{K^\circ - K^\circ}{2}\right)^\circ$, *and for all $s \in [1,n]$ there exists a Minkowski centered $K \in \mathcal{K}^n$ with $s(K) = s$, such that this containment is optimal.*

As a consequence we show that for a Minkowski concentric K holds that

$$C \in \left\{ K \cap (-K), \left(\frac{K^\circ - K^\circ}{2}\right)^\circ, \frac{K-K}{2}, \mathrm{conv}(K \cup (-K)) \right\}$$

fulfills $d_{BM}(K, C) = s(K)$, and is therefore a minimizer for the Banach-Mazur distance between K and $\mathcal{K}^n_0$.

We also show an example, for which the inclusion from Theorem 3.2.1, (vi) is not tight.

We show bounds for the parameters $\alpha(K) = R\left(K \cap (-K), \mathrm{conv}(K \cup (-K))\right)$ and $\beta(K) = R\left(\left(\frac{K^\circ - K^\circ}{2}\right)^\circ, \frac{K-K}{2}\right)$.

Theorem 3.2.2. *Let $K \in \mathcal{K}^n$ be Minkowski centered with $s(K) = s$. Then*

$$\frac{2}{s+1} \leq \alpha(K) \leq 1 \quad \textit{and} \quad \frac{4s}{(s+1)^2} \leq \beta(K) \leq 1.$$

Moreover, for all $s \in [1,2]$ there exist Minkowski centered $K_1, K_2 \in \mathcal{K}^n$ with $s(K_1) = s(K_2) = s$, such that $\alpha(K_1) = \frac{2}{s+1}$ and $\beta(K_2) = \frac{4s}{(s+1)^2}$.

Finally, we show upper bounds for both factors $\alpha(K)$ and $\beta(K)$ for Minkowski centered $K \in \mathcal{K}^n$ with $s(K) \approx n$, i.e., in case, when K is almost a simplex.

We define $\psi := \frac{(n-s+1)(s+1)}{1-(n-s)(n+s(n+1))} - n$, and $\zeta := (n+1)\left(\left(1 + \frac{sn}{s+1}\right) \frac{1+n-s}{1-n(n-s)} - n\right)$.

Theorem 3.2.3. *Let n be even and $C \in \mathcal{K}^n$ Minkowski centered with $s(C) = s$. Then*

(i) $K \cap (-K) \subset \psi \dfrac{n}{n+1} \mathrm{conv}(K \cup (-K))$, *and*
(ii) $\left(\dfrac{K^\circ - K^\circ}{2}\right)^\circ \subset \zeta \dfrac{n(n+2)}{(n+1)^2} \dfrac{K-K}{2}$.

Observe that we also present the parameters $\gamma_2 = \gamma_2(s,n)$ and $\gamma_3 = \gamma_3(s,n)$ such that $\psi\frac{n}{n+1} < 1$ for all $s > \gamma_2$, while $\zeta\frac{n(n+2)}{(n+1)^2} < 1$ for all $s > \gamma_3$. Note that for every even n holds $n - \frac{1}{n} < \gamma_2 < \gamma_3 < n$ and $\psi = \zeta = 1$ in case $s = n$.

I was significantly involved in the essential phases of brainstorming and in the elaboration of all parts of the work on this paper.

3.3. **Summary of the third paper.** We extend the results on the bounds for $\alpha(K)$ from Theorem 3.2.2, and now completely describe the region of all possible values for this parameter for Minkowski centered $K \in \mathcal{K}^2$ in dependence of the asymmetry of K.

Theorem 3.3.1. *Let $K \in \mathcal{K}^2$ be Minkowski centered. Then*

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2-1}\right\}.$$

Moreover, for every pair (α, s), such that $\frac{2}{s+1} \leq \alpha \leq \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered $K \in \mathcal{K}^2$, such that $s(K) = s$ and $\alpha(K) = \alpha$.

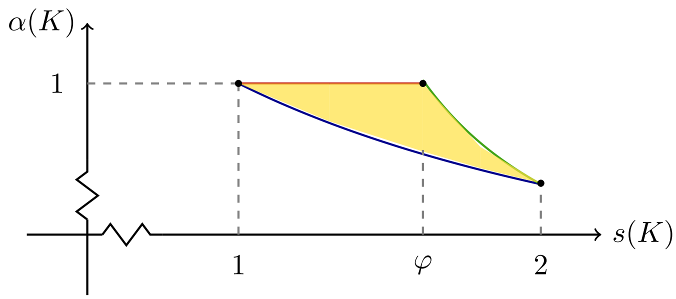

FIGURE 2. Region of possible values for the parameter $\alpha(K)$ for Minkowski centered $K \in \mathcal{K}^2$ (yellow): $\alpha(K) \geq \frac{2}{s+1}$ (blue); $\alpha(K) \leq 1$ for $s \leq \varphi$ (red), $\alpha(K) \leq \frac{s}{s^2-1}$ for $s \geq \varphi$ (green).

Surprisingly, we were able to describe the number of intersection points of the boundaries of a convex set K and its negative $-K$, when its asymmetry is greater than the golden ratio.

Theorem 3.3.2. *Let $K \in \mathcal{K}^2$ be Minkowski centered with $s(K) \geq \varphi$. Then the set* $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *consists of exactly 6 points.*

However, when the asymmetry is less than the golden ratio, $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ can consist of countable or uncountable number of points, as well as of a small one.

As mentioned in Section 2, [**Ri**] has shown that the $\frac{D}{w}$-ratio for complete sets is bounded from above by $\frac{n+1}{2}$. We present an improved quantitative result on the $\frac{D}{w}$-ratio for pseudo-complete (and therefore, for complete) sets.

Theorem 3.3.3. *Let $K \in \mathcal{K}^n$, $C \in \mathcal{K}^n_0$, K be pseudo-complete w.r.t. C. Then*

$$\frac{D(K,C)}{w(K,C)} \leq \frac{s(K)+1}{2}.$$

Moreover, for $n > 2$ odd and any $s \in [1, n]$ or for $n > 2$ even and any $s \in [1, n-1]$ there exist $K \in \mathcal{K}^2$, $C \in \mathcal{K}^2_0$, K being complete w.r.t. C with $s(K) = s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$.

We sharpen the bound for pseudo-complete sets from Proposition 2.7.1 using the Minkowski asymmetry measure in the planar case.

Theorem 3.3.4. *Let $K \in \mathcal{K}^2$, $C \in \mathcal{K}^2_0$, K be pseudo-complete w.r.t. C and $s(K) = s$. Then*

$$\frac{D(K,C)}{w(K,C)} \leq \min\left\{\frac{s+1}{2}, \frac{s^2}{s^2-1}\right\} \leq 1.42.$$

Since we always have $s \leq n$ and equality holds if and only if K is a n-simplex, Theorem 3.3.4 sharpens the bound from Proposition 2.7.1.

We show that for every pair (ρ, s), with $s \in [1,2]$ and $1 \leq \rho \leq \min\left\{\frac{s+1}{2}, \frac{s}{2(s-1)}\right\}$, there exists some Minkowski centered K, s.t. $s(K) = s$ and a set C, s.t. K is pseudo-complete w.r.t. C and $\frac{D(K,C)}{w(K,C)} = \rho$ (c.f. Figure 3). Observe also that $\min\left\{\frac{s(K)+1}{2}, \frac{s(K)}{2(s(K)-1)}\right\} \leq \frac{D(\mathbb{GH},\mathbb{GH}\cap(-\mathbb{GH}))}{w(\mathbb{GH},\mathbb{GH}\cap(-\mathbb{GH}))} = \frac{\varphi+1}{2} \approx 1.31$.

Finally, we show a result for pseudo-complete set in the Euclidean planar case.

Theorem 3.3.5. *Let $K \in \mathcal{K}^2$ be pseudo-complete w.r.t. $\mathbb{B}_2$ and $s(K) = s$. Then*

$$\frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \leq \frac{D(\mathbb{H},\mathbb{B}_2)}{w(\mathbb{H},\mathbb{B}_2)} = \frac{s(\mathbb{H})+1}{2} \approx 1.135,$$

where $\mathbb{H}$ denotes a special construction called the hood with $s(\mathbb{H}) \approx 1.27$.

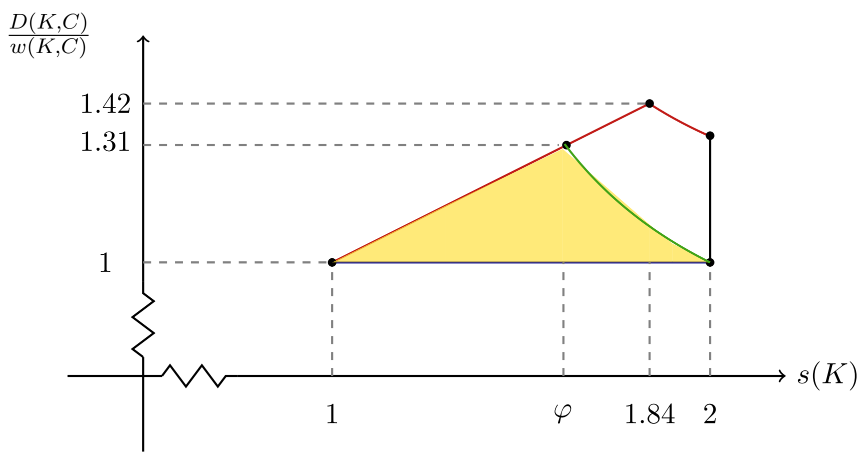

Figure 3. Region of all possible values for the diameter-width ratio for pseudo complete sets $K \in \mathcal{K}^2$ in dependence of $s(K)$: $\frac{D(K,C)}{w(K,C)} \geq 1$ (blue); $\frac{D(K,C)}{w(K,C)} \leq \min\left\{\frac{s(K)+1}{2}, \frac{s(K)^2}{s(K)^2-1}\right\}$ (red). $\left\{\frac{D(K,C_\lambda)}{w(K,C_\lambda)}, 0 \leq \lambda \leq 1\right\} = \left[1, \min\left\{\frac{s(K)+1}{2}, \frac{s(K)}{2(s(K)-1)}\right\}\right]$ (yellow, with $\frac{s(K)}{2(s(K)-1)}$ in green).

I was significantly involved in the essential phases of brainstorming and in the elaboration of all parts of the work on this paper.

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Relating Symmetrizations of Convex Bodies: Once More the Golden Ratio

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Abstract. Similar to the arithmetic-harmonic mean inequality for numbers, the harmonic mean of two convex sets K and C is always contained in their arithmetic mean. The harmonic and arithmetic means of C and $-C$ define two different symmetrizations of C, each keeping some useful properties of the original set. We investigate the relations of such symmetrizations, involving a suitable measure of asymmetry—the Minkowski asymmetry, which, besides other advantages, is polynomial time computable for (reasonably given) polytopes. The Minkowski asymmetry measures the minimal dilatation factor needed to cover a set C by a translate of its negative. Its values range between 1 and the dimension $\dim(C)$ of C, attaining 1 if and only if C is symmetric and $\dim(C)$ if and only if C is a simplex. Restricting to planar compact, convex sets, positioned so that the translation in the definition of the Minkowski asymmetry is 0, we show that if the asymmetry of C is greater than the golden ratio $(1+\sqrt{5})/2 \approx 1.618$, then the harmonic mean of C and $-C$ is a subset of a dilatate of their arithmetic mean with a dilatation factor strictly less than 1; and for any asymmetry less than the golden ratio, there exists a set C with the given asymmetry value, such that the considered dilatation factor cannot be less than 1.

The *golden ratio* $\varphi = (1+\sqrt{5})/2 \approx 1.618$ has a history of 2400 years and wide roots in mathematics, music, architecture, biology, and philosophy (see, e.g., [**16**]). It was first studied by the ancient Greeks because of its frequent appearance in geometry. For example, if one considers a regular pentagon of edge-length 1, its diagonals have length φ. No wonder that the regular pentagram was the Pythagorean symbol [**16**]. The first known definition is given in Euclid's Elements, II.11: "If a straight line is cut in extreme and mean ratio, then as the whole line is to the greater segment, the greater is to the lesser segment." Expressed algebraically, this transfers to the (probably) best-known definition of the golden ratio:

$$\text{if } a > b > 0 \text{ such that } \frac{a+b}{a} = \frac{a}{b}, \quad \text{then} \quad \frac{a}{b} = \varphi. \tag{1}$$

Among the fundamental inequalities in mathematics, a special place is reserved for the arithmetic-geometric-harmonic mean inequality, which in the two-argument case, together with the minimum and maximum, states that

$$\min\{a,b\} \le \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \le \sqrt{ab} \le \frac{a+b}{2} \le \max\{a,b\} \tag{2}$$

for any real numbers $a, b > 0$ (see [**13**, **21**]). We may identify means of numbers with means of segments by associating $a, b > 0$ with $[-a,a]$ and $[-b,b]$. By doing so we identify, e.g., the arithmetic mean of a and b with the segment $[-\frac{1}{2}(a+b), \frac{1}{2}(a+b)] =: \frac{1}{2}([-a,a]+[-b,b])$. In this way means of convex bodies can be introduced.

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Let $\mathcal{K}^n$ denote the set of *convex bodies*, i.e., full-dimensional compact convex sets in $\mathbb{R}^n$. For $X \subset \mathbb{R}^n$ let $\mathrm{conv}(X)$ (respectively, $\mathrm{pos}(X)$ or $\mathrm{aff}(X)$) be the *convex hull* (respectively, *positive hull* or *affine hull*) of X, i.e., the smallest convex set in $\mathbb{R}^n$ (respectively, convex cone or affine subspace) containing X. A *line segment* is the convex hull of a two-point set $\{x, y\} \subset \mathbb{R}^n$, which we denote by $[x, y]$. For any $K, C \subset \mathbb{R}^n$, $\rho \in \mathbb{R}$, let $K + C = \{a + b : a \in K, b \in C\}$ be the *Minkowski sum* of K, C and $\rho C = \{\rho x : x \in C\}$ the *ρ-dilatation* of C. We abbreviate $(-1)C$ by $-C$.

Now, the *arithmetic mean* of compact convex bodies K and C is defined by $\frac{1}{2}(K + C)$, the *minimum* by $K \cap C$, and the *maximum* by $\mathrm{conv}(K \cup C)$. For any $K \in \mathcal{K}^n$ let $K^\circ = \{a \in \mathbb{R}^n : a^T x \leq 1,\ x \in K\}$ be the *polar* of K. Since the polarity can be regarded as the higher-dimensional replacement of the inversion operation $x \to 1/x$ (see [**17**]), the *harmonic mean* of K and C is defined by $\left(\frac{1}{2}(K^\circ + C^\circ)\right)^\circ$. The geometric mean has been extended in several ways (see [**4**] or [**17**]); thus it would need a separate, more involved treatment, which is the reason why we focus on the four other means here. The study of means of convex bodies started in the 1960s [**8**–**10**], but there also exist several recent papers [**17**, **18**, **19**].

Probably the most essential result of Firey is the extension of the harmonic-arithmetic mean inequality from positive numbers to convex bodies containing 0 in their interior in [**8**]. Moreover, one can easily show that Firey's inequality again may be extended involving the minimum and maximum:

Proposition 1. *For all $K, C \in \mathcal{K}^n$ with 0 in their interior we have*

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K + C}{2} \subset \mathrm{conv}(K \cup C). \tag{3}$$

Let us mention an application given in [**11**]. For two positive definite symmetric matrices $A, B \in \mathbb{R}^{n\times n}$ we denote by $A \succcurlyeq B$ if $A - B$ is also positive definite. Moreover, $A \succcurlyeq B$ is strict if $A - B$ is not a zero matrix. Since means of ellipsoids correspond to combinations of the corresponding matrices, (3) also results in a (generalized) harmonic-arithmetic mean inequality:

$$(1 - \lambda)A + \lambda B \succcurlyeq ((1 - \lambda)A^{-1} + \lambda B^{-1})^{-1}$$

for any $\lambda \in [0, 1]$. This inequality is strict, except of the trivial cases $A = B$ or $\lambda \in \{0, 1\}$.

Moreover, the well-known Brunn–Minkowski determinantal inequality [**14**]

$$((1 - \lambda)\det(A) + \lambda \det(B))^{\frac{1}{n}} \geq \det((1 - \lambda)A)^{\frac{1}{n}} + \det(\lambda B)^{\frac{1}{n}},$$

can be further developed using the means of convex bodies as follows [**11**]: Let $k \in \{1, \ldots, n\}$ and $|A|_k$ denote the product of the k greatest eigenvalues of A; then

$$|(1 - \lambda)A^{-1} + \lambda B^{-1}|_k^{-\frac{1}{k}} \leq ((1 - \lambda)|A|_k^{-\frac{1}{k}} + \lambda|B|_k^{-\frac{1}{k}})^{-1}.$$

For any $K, C \in \mathcal{K}^n$ we say that K is *optimally contained* in C, and denote it by $K \subset^{opt} C$, if $K \subset C$ and $K \not\subset t + \rho C$ for any $0 \leq \rho < 1$ and $t \in \mathbb{R}^n$. If $C = t - C$ for some $t \in \mathbb{R}^n$, we say C is *symmetric*, and if $C = -C$, we say C is 0-*symmetric*. The family of 0-symmetric convex bodies is denoted $\mathcal{K}^n_0$. By $T \in \mathcal{K}^n$ we denote a regular simplex with (bary-)center 0.

The goal of this article is to consider optimal containments of means of C and $-C$ of a convex body C, i.e., symmetrizations of C. These kinds of symmetrizations are used frequently in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [3], which bounds from above the ratio of the circumradius ($\min_{x\in\mathbb{R}^n}\max_{y\in K}|x-y|$) and the diameter ($\max_{x,y\in K}|x-y|$) of convex bodies for general norms $|\cdot|$ by $n/(n+1)$, and for which equality is reached in spaces with $T\cap(-T)$ or $\frac{1}{2}(T-T)$ as the unit ball [7].

Or consider the characterization of normed spaces in which C is complete or reduced, if the unit ball is sandwiched between suitable rescalings of two different means of C and $-C$ [6, Propositions 3.5–3.10].

Also, well-known geometric inequalities have been re-investigated, replacing one mean by another. Consider, e.g., the Rogers–Shephard-type inequalities, which bound the ratio of the products of the volumes of the maximum and harmonic (respectively, arithmetic) means of K and C with the product of their volumes [1, 2, 20].

Notice that for any $C\in\mathcal{K}^n$ we have

$$C\cap(-C)\subset^{opt}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ \quad\text{and}\quad \frac{C-C}{2}\subset^{opt}\operatorname{conv}(C\cup(-C)).$$

Moreover,

$$\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\subset^{opt}\frac{C-C}{2}$$

is also possible, i.e., all containments in (3) may be optimal at the same time even for nonsymmetric C. In particular, if $T\in\mathcal{K}^3$ is a regular simplex with center 0, then we have the nice situation that the four means are a cross polytope (minimum), a rhombic dodecahedron (harmonic mean), a cuboctahedron (arithmetic mean), and a cube (maximum), such that even the cross polytope is optimally contained in the cube.

However, in the planar case, optimal containment of the harmonic mean of T and $-T$ in their arithmetic mean for an equilateral triangle T implies that the center of the triangle cannot be 0. In contrast, for the equilateral triangle $T\subset\mathbb{R}^2$ with center 0, we have

$$\left(\frac{T^\circ-T^\circ}{2}\right)^\circ\subset^{opt}\frac{8}{9}\cdot\frac{T-T}{2}\quad\text{and}\quad T\cap(-T)\subset^{opt}\frac{2}{3}\cdot\operatorname{conv}(T\cup(-T)).$$

Clearly, symmetrizations of a symmetric C should coincide with C, which is always true for the arithmetic mean of C and $-C$, but for the other means, which we consider, this holds only if 0 is the center of symmetry of C. This indicates the need to fix a meaningful center for every convex body first and then concentrate on translates with that center at 0.

Since we want to investigate the optimality of the inequality chain (3) in dependence of asymmetry, we will introduce one of the most common asymmetry measures, which is best suited to our purposes, and choose the center definition matching it. The *Minkowski asymmetry* of C is defined by $s(C):=\inf\{\rho>0 : C-c\subset\rho(c-C), c\in\mathbb{R}^n\}$ [12] and a *Minkowski center* of C is any $c\in\mathbb{R}^n$ such that $C-c\subset s(C)(c-C)$ [5]. Moreover, if $c=0$ is a Minkowski center, we say C is *Minkowski centered*. Note that $s(C)\in[1,n]$ for $C\in\mathcal{K}^n$, where $s(C)=1$ if and only if C is centrally symmetric, while $s(C)=n$ if and only if C is an n-dimensional simplex [12]. Moreover, the Minkowski asymmetry $s:\mathcal{K}^n\to[1,n]$ is continuous with respect to the Hausdorff metric (see [12, 21] for some basic properties) and invariant under nonsingular affine transformations.

The main contribution of this article is that the golden ratio is the largest asymmetry such that (3) can be optimal in the planar case.

Theorem 2. *Let $C \in \mathcal{K}^2$ be Minkowski centered such that*

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{C - C}{2};$$

then $s(C) \leq \varphi$. Moreover, if $s(C) = \varphi$, then there exists a nonsingular linear transformation L such that $L(C) = \mathbb{GH} := \mathrm{conv}(\{p^1, \ldots, p^5\})$, where $p^1 = (-1, -1)^T$, $p^2 = (-1, 0)^T$, $p^3 = (0, \varphi)^T$, $p^4 = (1, 0)^T$, $p^5 = (1, -1)^T$ form the golden house.

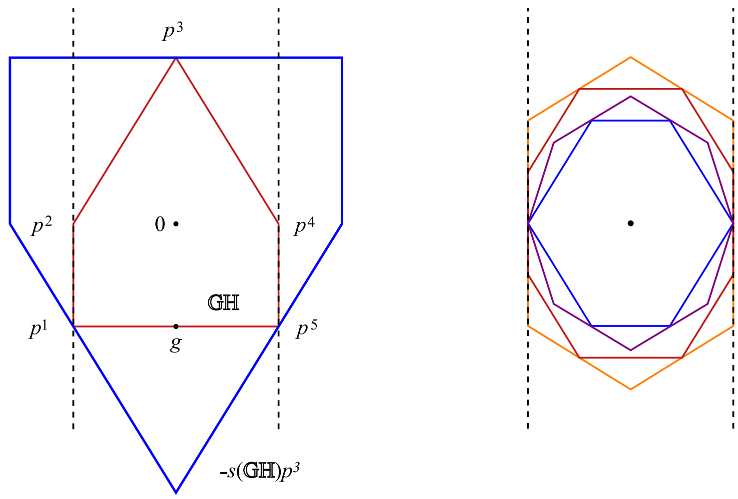

Figure 1. Left: $\mathbb{GH}$ (red), $-s(\mathbb{GH})\mathbb{GH}$ (blue), and parallel supporting halfspaces in p^2 and $p^4 = -p^2$ (dashed). **Right:** $\mathrm{conv}(\mathbb{GH} \cup (-\mathbb{GH}))$ (orange), $\frac{1}{2}(\mathbb{GH} - \mathbb{GH})$ (red), $\left(\frac{1}{2}(\mathbb{GH}^\circ + (-\mathbb{GH})^\circ)\right)^\circ$ (violet), and $\mathbb{GH} \cap (-\mathbb{GH})$ (blue). The golden house and its symmetrizations.

The important facts about the construction of the golden house are the following:

1. $p^2 = -p^4$;
2. $\|p^2 - p^3\| = \|p^4 - p^3\|$;
3. $\mathrm{conv}(\{p^1, -s(\mathbb{GH})p^3, p^5\})$ and $\mathrm{conv}(\{p^2, p^3, p^4\})$ are similar up to reflection.

Let $g := [p^1, p^5] \cap [p^3, -s(\mathbb{GH})p^3]$, $\alpha := \|p^3 - g\|$, and $\beta := \|p^3\|$. Then we have on the one hand

$$s(\mathbb{GH}) = \frac{\| - s(\mathbb{GH})p^3\|}{\|p^3\|} = \frac{\|p^3 - g\|}{\|p^3\|} = \frac{\alpha}{\beta}, \tag{4}$$

and on the other hand

$$s(\mathbb{GH}) = \frac{\| - s(\mathbb{GH})p^3 - p^3\|}{\|p^3 - g\|} = \frac{\alpha + \beta}{\alpha}. \tag{5}$$

Combining (4) and (5) we see that $s(\mathbb{GH}) = \varphi$ (see Left in Figure 1).

To the best of our knowledge, this is the first explicit mention of a set with the properties of the golden house. Theorem 3 demonstrates that items 1 and 2 above suffice to show that, in the case of the golden house (and its negative), optimal containment is reached in (3) throughout the full chain (see Right in Figure 1). Even more: from Theorem 3 it directly follows that the minimum is optimally contained in the maximum.

For any $C \in \mathcal{K}^n$ let $\mathrm{bd}(C)$ be the *boundary* of C and for any $a \in \mathbb{R}^n \setminus \{0\}$ and $\rho \in \mathbb{R}$, let $H^{\leq}_{a,\rho} = \{x \in \mathbb{R}^n : a^T x \leq \rho\}$ denote a *halfspace*. We say that the halfspace $H^{\leq}_{a,\rho}$ *supports* $C \in \mathcal{K}^n$ at $q \in C$ if $C \subset H^{\leq}_{a,\rho}$ and $q \in \mathrm{bd}(H^{\leq}_{a,\rho})$.

Theorem 3. *Let $C \in \mathcal{K}^n$ be Minkowski centered. Then the following are equivalent:*
1. $C \cap (-C) \subset^{opt} \mathrm{conv}(C \cup (-C))$;
2. $\left(\frac{1}{2}(C^\circ - C^\circ)\right)^\circ \subset^{opt} \frac{1}{2}(C - C)$;
3. there exist $p, -p \in \mathrm{bd}(C)$, parallel halfspaces $H^{\leq}_{a,\rho}$ and $H^{\leq}_{-a,\rho}$ supporting C at p and $-p$, respectively.

Let us mention that for any regular Minkowski centered $(2n+1)$-gon P, the vertices of $-\frac{1}{s(P)}P$ are the midpoints of the edges of P. Hence, they obviously do not satisfy part (iii) of Theorem 3. By letting n grow, we see that there exist Minkowski centered $C \in \mathcal{K}^2$ with $s(C)$ arbitrary close to 1 such that not all containments in the inequality chain (3) are optimal for C. Furthermore, one may observe that a Minkowski centered regular pentagon has asymmetry $2/\varphi \approx 1.236 < \varphi$.

1. CHARACTERIZATIONS OF OPTIMAL CONTAINMENT.

Let us first collect some simple set identities under affine transformations.

Lemma 4. *Let $K, C \in \mathcal{K}^n$ and A be a nonsingular affine transformation. Then*

$$A(K) \cap A(C) = A(K \cap C),$$
$$(((A(K))^\circ - (A(C))^\circ)/2)^\circ = A\left((K^\circ - C^\circ)/2\right)^\circ,$$
$$(A(K) + A(C))/2 = A\left((K + C)/2\right),$$
$$\mathrm{conv}\left(A(K) \cup (A(C)\right) = A\left(\mathrm{conv}(K \cup C)\right).$$

The following proposition characterizes the optimal containment $K \subset^{opt} C$ between two convex sets $K, C \in \mathcal{K}^n$ in terms of common boundary points and corresponding supporting halfspaces (see [**7**, Theorem 2.3]).

Proposition 5. *Let $K, C \in \mathcal{K}^n$ and $K \subset C$. Then the following are equivalent:*
1. $K \subset^{opt} C$;
2. There exist $k \in \{2, \dots, n+1\}$, $p^j \in K \cap \mathrm{bd}(C)$, a^j outer normals of supporting halfspaces of K and C at p^j, $j = 1, \dots, k$, such that $0 \in \mathrm{conv}(\{a^1, \dots, a^k\})$.
Moreover, in case that $K, C \in \mathcal{K}^n_0$, items 1, 2 are equivalent to $K \cap \mathrm{bd}(C) \neq \emptyset$.

Lemma 4 together with Proposition 5 obviously yield the following corollary.

Corollary 6. *Let $C \in \mathcal{K}^n$ and let L be a nonsingular linear transformation. Then*
1. C is Minkowski centered if and only if $L(C)$ is Minkowski centered.
2. $C \cap (-C) \subset^{opt} \mathrm{conv}(C \cup (-C))$ if and only if $L(C) \cap L(-C) \subset^{opt} \mathrm{conv}(L(C) \cup L(-C))$.

Let us now add a proposition that is a result of Klee [**15**] reduced to the two-dimensional case.

Proposition 7. *Let $P, C \in \mathcal{K}^2$, where P is a polygon and C is 0-symmetric, such that $P \subset^{opt} C$. Then $0 \in P$.*

Taking the two preceding propositions together we obtain the corollary below.

Corollary 8. *Let $C \in \mathcal{K}^2$ be Minkowski centered, but not 0-symmetric. Then there exist* $p^1, p^2, p^3 \in \mathrm{bd}(C) \cap (-s(C)\mathrm{bd}(C))$ *such that* $0 \in \mathrm{conv}(\{p^1, p^2, p^3\})$.

Proof. Let us first mention that the existence of two or three such touching points of $\mathrm{bd}(C) \cap (-s(C)\mathrm{bd}(C))$ is a direct consequence of Proposition 5, and if it were only two it would follow that $s(C) = 1$.

Now let S be the intersection of the three common supporting halfspaces of C and $-s(C)C$ at the points p^i, $i = 1, 2, 3$. In addition, C (together with $-1/s(C)C$) is also supported in $1/s(C)p^i$ by halfspaces with outer normals being the negatives of the outer normals of the starting three. Hence, we obtain that $\mathrm{conv}(\{p^1, p^2, p^3\})$ is optimally contained in the minimum $S \cap (-S)$ of S and $-S$ and therefore, by Proposition 7, that $0 \in \mathrm{conv}(\{p^1, p^2, p^3\})$. ■

Proof of Theorem 3. (1) $\Rightarrow$ (2): This part of the proof follows directly from Proposition 1.

(2) $\Rightarrow$ (3): Assuming that $(\frac{C^\circ - C^\circ}{2})^\circ \subset^{opt} \frac{C-C}{2}$, we obtain from Proposition 5 that there exists a common boundary point p of the two sets. Let $\rho_1, \rho_2 > 0$ be the smallest factors such that $\frac{1}{\rho_1} p \in \mathrm{bd}(C)$ and $\frac{1}{\rho_2} p \in \mathrm{bd}(-C)$, respectively. On the one hand, this implies

$$\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) p \in \frac{C - C}{2}$$

and since $p \in \mathrm{bd}\left(\frac{C-C}{2}\right)$, we have that

$$1 \leq \left(\frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right)^{-1}. \tag{6}$$

On the other hand, from $\frac{1}{\rho_1} p \in \mathrm{bd}(C)$ it follows that $C^\circ \subset \{a \in \mathbb{R}^n : a^T p \leq \rho_1\}$ and that there exists some $a^1 \in \mathrm{bd}(C^\circ)$ such that $(a^1)^T p = \rho_1$. Similarly, we obtain $-C^\circ \subset \{a \in \mathbb{R}^n : a^T p \leq \rho_2\}$ and the existence of $a^2 \in \mathrm{bd}(-C^\circ)$ such that $(a^2)^T p = \rho_2$. Hence, $\frac{1}{2}(C^\circ - C^\circ) \subset \{a \in \mathbb{R}^n : a^T p \leq \frac{1}{2}(\rho_1 + \rho_2)\}$ and $\frac{1}{2}(a^1 + a^2) \in \mathrm{bd}\left(\frac{1}{2}(C^\circ - C^\circ)\right)$ with $\frac{1}{2}(a^1 + a^2)^T p = \frac{1}{2}(\rho_1 + \rho_2)$. This means

$$\frac{2}{\rho_1 + \rho_2} p \in \mathrm{bd}\left(\frac{C^\circ - C^\circ}{2}\right)^\circ,$$

which by the fact that $p \in \mathrm{bd}\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ implies

$$\frac{2}{\rho_1 + \rho_2} = 1. \tag{7}$$

Combining (6) and (7), we obtain that the arithmetic mean is not greater than the harmonic mean of ρ_1 and ρ_2, thus $\rho_1 = \rho_2 = 1$. This proves $p \in C \cap (-C)$.

Finally, let $H^{\leq}_{a,\rho}$ be a supporting half space of $(C - C)/2$ at p and assume that $H^{\leq}_{a,\rho}$ does not support C. Hence there would exist some $q \in C$ with $a^T q > \rho$. Now, since $p \in -C$, we obtain $(p + q)/2 \in (C - C)/2$, which, because of $a^T\left(\frac{p+q}{2}\right) > \rho$, contradicts the fact that $(C - C)/2 \subset H^{\leq}_{a,\rho}$. This proves that $C \subset H^{\leq}_{a,\rho}$ and analogously one obtains $-C \subset H^{\leq}_{a,\rho}$. However, the convexity of halfspaces now implies $\mathrm{conv}(C \cup (-C)) \subset H^{\leq}_{a,\rho}$, which shows that condition 3 is satisfied.

(3) $\Rightarrow$ (1): Assuming that C is supported by $H^{\leq}_{a,\rho}$, $H^{\leq}_{-a,\rho}$ at $p, -p$, respectively, the same holds for $-C$. Hence, we have $p, -p \in C \cap (-C)$ and $\mathrm{conv}(C \cup (-C))$

is supported by $H^{\leq}_{a,\rho}$, $H^{\leq}_{-a,\rho}$ at $p, -p$, respectively. By Proposition 5 this means that $C \cap (-C) \subset^{opt} \operatorname{conv}(C \cup (-C))$. ■

2. THE MAIN RESULT.

Proof of Theorem 2. Let $C \in \mathcal{K}^2$ be Minkowski centered, with $s := s(C) > 1$, such that

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{C - C}{2}.$$

By Theorem 3 this optimality condition is equivalent to $C \cap (-C) \subset^{opt} \operatorname{conv}(C \cup (-C))$ and to the existence of $-p$, $p \in \operatorname{bd}(-sC)$, as well as parallel halfplanes $-H$, H supporting $-sC$ at $-p$ and p, respectively. Since C is Minkowski centered, we have $C \subset^{opt} -sC$ and therefore by Proposition 5 we obtain the existence of $k \in \{2, 3\}$, $q^1, \dots, q^k \in \operatorname{bd}(C) \cap \operatorname{bd}(-sC)$ and outer normals of supporting halfplanes $a^1, \dots, a^k$ with $0 \in \operatorname{conv}(\{a^1, \dots, a^k\})$. Moreover, from $s > 1$ it easily follows that $k = 3$.

It cannot be that $\pm p \notin \{q^1, q^2, q^3\}$. Otherwise, let, e.g., $q^2 = p$. Then we have $q^2 \in C \cap \operatorname{bd}(-sC)$ and thus $-sp = -sq^2 \in \operatorname{bd}(-sC)$, which would imply $s(C) = 1$.

By Corollary 8 we have $0 \in \operatorname{conv}(\{q^1, q^2, q^3\})$. Hence, we can assume without loss of generality that q^1 is located is on one side, while q^2, q^3 are on the other side of the line $\operatorname{aff}(\{-p, p\})$, and moreover even that $-p \in \operatorname{pos}\{q^1, q^3\}$ and $p \in \operatorname{pos}\{q^1, q^2\}$. Observe that the lines $\operatorname{aff}(\{-p, q^3\})$ and $\operatorname{aff}(\{p, q^2\})$ intersect in some point d^1. Otherwise, we would have $q^3 \in \operatorname{bd}(-H)$ and $q^2 \in \operatorname{bd}(H)$ and therefore $[q^3, -p], [q^2, p] \subset \operatorname{bd}(-sC)$. This would imply that the segment $[-\frac{1}{s}q^2, -\frac{1}{s}p]$, which is parallel to $[q^3, -p]$, belongs to $\operatorname{bd}(C) \cap \operatorname{int}(-sC)$. Together with $q^1 \in \operatorname{bd}(C)$ and $s > 1$, this would contradict the convexity of C.

We choose $d^2 \in \operatorname{bd}(-H)$, $d^3 \in \operatorname{bd}(H)$ such that $q^1 \in [d^2, d^3]$ and $[d^2, d^3]$ is parallel to $[q^2, q^3]$.

Let us first prove that

$$-sq^1 \in \operatorname{conv}(\{q^2, q^3, d^1\}). \tag{8}$$

Since $0 \in \operatorname{conv}(\{q^1, q^2, q^3\})$, we have $-sq^1 \in \operatorname{pos}(\{q^2, q^3\})$. Thus using the fact that $q^2, q^3, -sq^1 \in \operatorname{bd}(-sC)$, the convexity of $-sC$ implies that $-sq^1 \in \operatorname{conv}(\{q^2, q^3, d^1\})$.

The next fact we want to see is

$$-sq^3 \in \operatorname{conv}(\{p, q^1, d^3\}). \tag{9}$$

To see this, remember that $-H$ supports $-sC$ at $-p$. Moreover, directly from $-p \in \operatorname{pos}(\{q^1, q^3\})$ we obtain $-sq^3 \in \operatorname{pos}(\{p, q^1\})$. Now, since $p, q^1, -sq^3 \in \operatorname{bd}(-sC)$, the convexity of $-sC$ implies $-sq^3 \notin \operatorname{int}(\operatorname{conv}(\{0, p, q^1\}))$. Collecting the facts that $q^1, -sq^2, -sq^3 \in \operatorname{bd}(-sC)$, $q^1 \in \operatorname{pos}(\{-sq^2, -sq^3\})$, and the parallelism of $[-sq^2, -sq^3]$ and $[d^2, d^3]$, we obtain $-sq^3 \in \operatorname{conv}(\{p, q^1, d^3\})$.

Similarly to (9), one may prove

$$-sq^2 \in \operatorname{conv}(\{-p, q^1, d^2\}). \tag{10}$$

See Figure 2 for an illustration of the construction and the validness of (8)–(10).

Our goal in the following is to determine the greatest possible s such that $C \cap (-C) \subset^{opt} \operatorname{conv}(C \cup (-C))$ is still satisfied. We say that the points q^1, q^2, q^3 present a *valid situation* if they satisfy conditions (8), (9), and (10). We make the

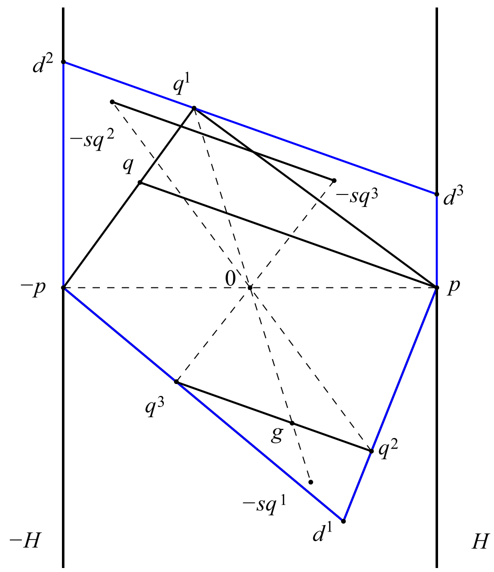

Figure 2. Construction used in the proof of Theorem 2.

following changes to q^1, q^2, q^3, so that after each step (see Figure 3), we still have a valid situation for the given asymmetry s:

1. Replace q^2 (respectively, q^3) by the point in $[q^2, p]$ (respectively, $[q^3, -p]$) such that $-sq^2 \in -H$ (respectively, $-sq^3 \in H$). Since $s > 1$, q^2 belongs in the strip between H and $-H$, and $-sp$ belongs outside the same strip and is closer to $-H$ than to H, we have that $-s[q^2, p] = [-sq^2, -sp]$ intersects $-H$ at a point $-s\widetilde{q}^2$. Let us replace q^2 by $\widetilde{q}^2$.
2. Replace q^1 by μq^1, for some $\mu < 1$, such that $\mu q^1 \in [-sq^2, -sq^3]$.
3. Substitute q^1 by $-\gamma d^1 \in [-sq^2, -sq^3]$, for some $\gamma > 0$.

Recognize that $s\gamma d^1 = -sq^1 \in \operatorname{conv}(\{d^1, q^2, q^3\})$ implies $s\gamma \leq 1$.

Now we can study the maximal possible value for s, which means we want to characterize the situation in which s becomes maximal such that $s\gamma \leq 1$. Thus we need to know the explicit value of γ (depending on s).

To do so, after a suitable linear transformation, suppose that $p = (1, 0)$, and H and $-H$ are vertical lines (perpendicular to $[-p, p]$). Because of step 1 above we may furthermore assume that $q^2 = (1/s, -a)^T$ and $q^3 = (-1/s, -1)^T$ for some $a \in (0, 1]$. Now we need the coordinates of d^1, which is the intersection of the lines $\operatorname{aff}\{p, q^2\}$ and $\operatorname{aff}\{-p, q^3\}$. We obtain

$$d_2^1 = -\frac{1}{1 - \frac{1}{s}}(d_1^1 + 1) \quad \text{and} \quad d_2^1 = \frac{a}{1 - \frac{1}{s}}(d_1^1 - 1),$$

resulting in

$$d^1 = \left(\frac{a - 1}{a + 1}, \frac{-2a}{(1 - \frac{1}{s})(a + 1)} \right)^T.$$

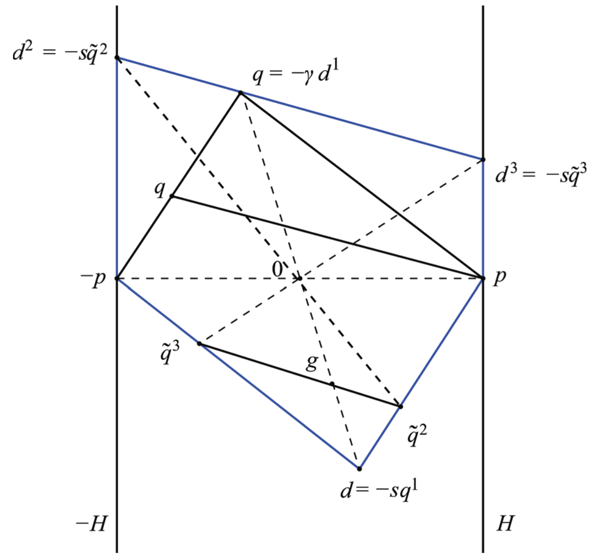

Figure 3. Construction used in the proof of Theorem 2 after applying the transformation described in steps 1–3 obtaining the new valid situation.

Now we compute γ such that condition (3) is satisfied, i.e.,

$$-\gamma d^1 \in [-sq^2, -sq^3] = [(-1, sa)^T, (1, s)^T].$$

Hence, for some $\lambda \in [0, 1]$, we have

$$\begin{aligned} -\gamma \left(\frac{a-1}{a+1}, \frac{-2a}{(1-1/s)(a+1)} \right) &= (1-\lambda)(-1, sa)^T + \lambda(1, s)^T \\ &= (-1+2\lambda, s((1-\lambda)a+\lambda))^T \end{aligned}$$

and it is easy to check that this implies

$$\gamma = \frac{(s-1)(a+1)^2}{4a-(s-1)(a-1)^2}.$$

Thus the problem of finding the maximal s under the condition $s\gamma \leq 1$ may be rewritten as

$$\max s, \ \text{ such that } \frac{s(s-1)(a+1)^2}{4a-(s-1)(a-1)^2} \leq 1.$$

The above condition is easily rewritten as

$$(s^2-1)(a+1)^2 - 4as \leq 0.$$

We are interested in the maximum s, i.e., in the larger of the two roots of the equation $(s^2-1)(a+1)^2 - 4as = 0$, which is

$$s = \frac{2a}{(a+1)^2} + \sqrt{1 + \frac{4a^2}{(a+1)^4}} =: h(a),$$

$a \in (0, 1]$. Hence, the maximum of s coincides with the maximum of $h(a)$ with $a \in (0, 1]$. It is straightforward to verify that $h(a)$ is increasing in $(0, 1]$, and thus we can conclude that

$$\max s = \max_{a \in (0,1]} h(a) = h(1) = \frac{1 + \sqrt{5}}{2} = \varphi.$$

Now, note that equality holds if and only if $a = 1$, $\gamma = \varphi - 1$, and $d^1 = (0, -\varphi/(\varphi - 1))^T$. Moreover, in the extreme case we have $\varphi\gamma = 1$, which is true if and only if $-\varphi q^1 = d^1$, $q^2 = (1/\varphi, -1)^T$, and $q^3 = (-1/\varphi, -1)^T$. Since $q^2 \in \mathrm{bd}(-\varphi C) \cap [d^1, p]$, we have $[d^1, p] \subset \mathrm{bd}(-\varphi C)$. The same reasoning with q^3 replacing q^2 shows that $[d^1, -p] \subset \mathrm{bd}(-\varphi C)$. Moreover, $q^1 = (0, 1/(\varphi - 1))^T = -\gamma d^1 \in [-\varphi q^2, -\varphi q^3]$. Thus $q^1 \in \mathrm{bd}(-\varphi C)$ implies $[-\varphi q^2, -\varphi q^3] \subset \mathrm{bd}(-\varphi C)$. Since it is also clear that $[p, -\varphi q^3], [-p, -\varphi q^2] \subset \mathrm{bd}(-\varphi C)$, we obtain a complete description of the boundary of $-\varphi C$, thus proving

$$-\varphi C = \mathrm{conv}(\{d^1, \pm p, -\varphi q^2, -\varphi q^3\}).$$

Finally, since $\varphi = 1/(\varphi - 1)$, we obtain

$$C = \mathrm{conv}\left(\left\{\left(0, \frac{1}{\varphi - 1}\right)^T, \left(\pm\frac{1}{\varphi}, 0\right)^T, \left(\pm\frac{1}{\varphi}, -1\right)^T\right\}\right) = \begin{pmatrix} \frac{1}{\varphi} & 0 \\ 0 & 1 \end{pmatrix} \mathbb{GH},$$

which concludes the proof of our theorem. ■

Remark. For every $s \in [1, \varphi]$ there exists $C \in \mathcal{K}^2$, Minkowski centered with $s(C) = s$, such that

$$C \cap (-C) \subset^{opt} \mathrm{conv}(C \cup (-C)).$$

To see this, we perform a symmetrization process: making a hexagon from the pentagon $\mathbb{GH}$ by adding the point $(0, -\tau)^T$ for $\tau \in [1, \varphi^2]$ and translating the whole set in the direction of $(1, 0)^T$ such that it is Minkowski centered again. In this way we obtain a continuously monotonely shrinking Minkowski asymmetry with growing τ, ending in a 0-symmetric hexagon when $\tau = \varphi^2$, while keeping property 3 of Theorem 3 true.

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TIGHTENING AND REVERSING THE ARITHMETIC-HARMONIC MEAN INEQUALITY FOR SYMMETRIZATIONS OF CONVEX SETS

RENÉ BRANDENBERG, KATHERINA VON DICHTER, AND BERNARDO GONZÁLEZ MERINO

Abstract. This paper deals with four symmetrizations of a convex set C: the intersection, the harmonic and the arithmetic mean, and the convex hull of C and $-C$. A well-known result of Firey shows that those means build up a subset-chain in the given order. On the one hand, we determine the dilatation factors, depending on the asymmetry of C, to reverse the containments between any of those symmetrizations. On the other hand, we tighten the relations proven by Firey and show a stability result concerning those factors near the simplex.

1. Introduction and Notation

The arithmetic-geometric-harmonic mean inequality together with minimum and maximum (which can be seen as the extreme means) states in the two-argument case

$$\min\{a,b\} \leq \left(\frac{a^{-1}+b^{-1}}{2}\right)^{-1} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \max\{a,b\} \tag{1}$$

for any $a, b > 0$, with equality in any/all of the inequalities if and only if $a = b$ (see [HLP, Sch]).

For any $X \subset \mathbb{R}^n$ let $\operatorname{conv}(X)$ denote the *convex hull*, i.e., the smallest convex set containing X. A *segment* is the convex hull of $\{x, y\} \subset \mathbb{R}^n$, which we abbreviate by $[x, y]$. For any $X, Y \subset \mathbb{R}^n$, $\rho \in \mathbb{R}$ let $X + Y = \{x + y : x \in X, y \in Y\}$ be the *Minkowski sum* of X, Y and $\rho X = \{\rho x : x \in X\}$ the *ρ-dilatation* of X. We abbreviate $(-1)X$ by $-X$. The family of all *(convex) bodies* (compact convex sets) is denoted by $\mathcal{K}^n$ and for any $C \in \mathcal{K}^n$ we write $C^\circ = \{a \in \mathbb{R}^n : a^T x \leq 1,\, x \in C\}$ for the *polar* of C.

All the means above can be generalized for convex sets. One may identify means of numbers by means of segments via associating $a, b > 0$ with $[-a, a]$ and $[-b, b]$. Thus, e.g., the arithmetic mean of a and b is identified with $[-\frac{1}{2}(a+b), \frac{1}{2}(a+b)] = \frac{1}{2}([-a,a]+[-b,b])$. In general, the *arithmetic mean* of $K, C \in \mathcal{K}^n$ is defined by $\frac{1}{2}(K+C)$, the *minimum* by $K \cap C$, and the *maximum* by $\operatorname{conv}(K \cup C)$. Since polarity can be regarded as the higher-dimensional counterpart of the inversion operation $x \mapsto 1/x$ (cf. [MR]), the *harmonic mean*

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of K and C is defined by $\left(\frac{1}{2}(K^\circ + C^\circ)\right)^\circ$. The geometric mean has been extended in several ways (cf. [BLYZ] or [MR]). It would need a separate, more involved treatment. Here we focus only on the four other means. The study of means of convex bodies has been started by Firey in the 1960's [Fi, Fi2, Fi3], but there also exist several recent papers (see, e.g., [MR, MR2, MMR]).

Perphaps the most essential result of Firey is the extension of the harmonic-arithmetic mean inequality from positive numbers to convex bodies with 0 in their interior in [Fi] (see [MR] for a nice and short proof). Moreover, Firey's inequality may again be extended involving minimum and maximum.

Proposition 1.1. *Let $C, K \in \mathcal{K}^n$ with 0 in their interior. Then*

$$K \cap C \subset \left(\frac{K^\circ + C^\circ}{2}\right)^\circ \subset \frac{K+C}{2} \subset \mathrm{conv}(K \cup C), \tag{2}$$

with equality between any of the means if and only if $K = C$.

In the following we analyze sharpness of the set-containment inequalities with respect to optimal containment (instead of equality of sets): For any $C, K \in \mathcal{K}^n$ we say K is *optimally contained* in C, and denote it by $K \subset^{opt} C$, if $K \subset C$ and $K \not\subset \rho C + t$ for any $\rho \in [0,1)$ and $t \in \mathbb{R}^n$. For $C_1, \dots, C_k \in \mathcal{K}^n$ we say $C_1 \subset \dots \subset C_k$ is *left-to-right optimal*, if $C_1 \subset^{opt} C_k$.

The starting point of our investigation is the following generalization of [BDG, Theorem 1.3] for arbitrary convex sets with 0 in their interior.

Theorem 1.2. *Let $C, K \in \mathcal{K}^n$ with 0 in their interior. Then*

$$K \cap C \subset^{opt} \mathrm{conv}(K \cup C) \iff \left(\frac{1}{2}(K^\circ + C^\circ)\right)^\circ \subset^{opt} \frac{1}{2}(K + C).$$

Note that Theorem 1.2 implies that left-to-right optimality in (2) depends solely on the optimal containment of the harmonic in the arithmetic mean.

If $C = -C + t$ for some $t \in \mathbb{R}^n$, we say C is *symmetric*, and if $C = -C$, we say C is 0-*symmetric*. The family of 0-symmetric convex bodies is denoted by $\mathcal{K}^n_0$.

A special focus in our study lies on optimal containments of means of C and $-C$ of a convex body C, which are all symmetrizations of C. Symmetrizations are frequently used in convex geometry, e.g., as extreme cases of a variety of geometric inequalities. Consider, e.g., the Bohnenblust inequality [Bo], which bounds the ratio of the circumradius and the diameter of convex bodies in arbitrary normed spaces. The equality case in this inequality is reached in normed spaces with $S \cap (-S)$ or $\frac{1}{2}(S - S)$ as their unit balls [BK], where S denotes an n-simplex with barycenter 0. These means also appear in characterizations of spaces, for which C is complete or reduced [BGJM, Prop. 3.5 – 3.10]. We provide more motivation for considering optimal containments between symmetrizations of C in the Appendix.

A major part of this paper is devoted to a better understanding of the optimal containments between those symmetrizations depending on the asymmetry of the initial body. We naturally require all symmetrizations of an already symmetric C to coincide with C up to translations. This is always true for the arithmetic mean $\frac{1}{2}(C-C)$, but 0 has to be the center of symmetry in case of the other three considered means. This indicates the need of fixing a meaningful center for every convex body. The most common choice of an asymmetry measure and a corresponding center are the *Minkowski asymmetry* of $C \in \mathcal{K}^n$, which is defined by

$$s(C) := \inf\{\rho > 0 : C - c \subset \rho(c - C), c \in \mathbb{R}^n\},$$

and the (not necessarily unique) *Minkowski center* of C, which is any $c \in \mathbb{R}^n$ fulfilling $C - c \subset s(C)(c - C)$ [Gr, BG2]. If 0 is a Minkowski center, we say C is *Minkowski centered*.

Note that, if C is not a singleton, $s(C) \in [1, n]$, with $s(C) = 1$ if and only if C is symmetric, and $s(C) = n$ if and only if C is an n-dimensional simplex [Gr]. Moreover, the Minkowski asymmetry $s : \mathcal{K}^n \to [1, n]$ is continuous w.r.t. the Hausdorff metric (see [Gr], [Sch] for some basic properties) and invariant under non-singular affine transformations. We believe that the Minkowski asymmetry is most suitable for studying optimal containments and consequently focus on Minkowski centered convex sets.

The classical norm relations $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ with $x \in \mathbb{R}^n$ can be naturally reversed by the inequalities $\|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$, which both transfer to left-to-right optimal containments between the corresponding unit ball of these ℓ_p-spaces. Defining the *gauge function* of a general $C \in \mathcal{K}^n$ in x by $\|x\|_C = \inf\{\rho > 0 : x \in \rho C\}$ one may consider the gauge functions induced by the means of K and C. Doing so, (2) can be read as

$$\|x\|_{\mathrm{conv}(K\cup C)} \leq \|x\|_{\frac{K+C}{2}} \leq \|x\|_{\left(\frac{K^\circ+C^\circ}{2}\right)^\circ} \leq \|x\|_{K\cap C}. \tag{3}$$

In order to reverse this chain of inequalities, we need to provide a chain of (optimal) inclusions, which is reverse to (2). This is not possible in general, since the needed scaling factors of the reverse inclusions cannot always be bounded. However, assuming Minkowski centeredness of the considered body, this problem can be fixed.

Theorem 1.3. *Let $C \in \mathcal{K}^n$ be Minkowski centered. Then*

(i) $\mathrm{conv}(C \cup (-C)) \subset^{opt} s(C)(C \cap (-C))$,
(ii) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{2s(C)}{s(C)+1}\frac{C-C}{2}$,
(iii) $\left(\frac{C^\circ-C^\circ}{2}\right)^\circ \subset^{opt} \frac{2s(C)}{s(C)+1}(C \cap (-C))$,
(iv) $\frac{C-C}{2} \subset^{opt} \frac{s(C)+1}{2}(C \cap (-C))$, *and*
(v) $\mathrm{conv}(C \cup (-C)) \subset^{opt} \frac{s(C)+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ$.

Moreover,

(vi) $\frac{C-C}{2} \subset \frac{s(C)+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ$, *and*

for all $s \in [1, n]$ there exists a Minkowski centered $C \in \mathcal{K}^n$ with $s(C) = s$, such that this containment is optimal.

After the proof of Theorem 1.3 we also provide an example that shows that the containment in Part (vi) above may not be optimal and derive a lower bound for the minimal dilatation factor needed for this covering.

The following containment chains are a direct consequence of Theorem 1.3.

Corollary 1.4. *Let $C \in \mathcal{K}^n$ be Minkowski centered.Then the following containment chains are both left-to-right optimal:*

(i) $\mathrm{conv}(C \cup (-C)) \subset \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset s(C)(C \cap (-C))$, *and*
(ii) $\mathrm{conv}(C \cup (-C)) \subset \frac{2s(C)}{s(C)+1} \frac{C-C}{2} \subset s(C)(C \cap (-C))$.

Moreover, the following two containment chains always apply:

(iii) $\frac{C-C}{2} \subset \mathrm{conv}(C \cup (-C)) \subset \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$,
(iv) $\frac{C-C}{2} \subset \frac{s(C)+1}{2} C \cap (-C) \subset \frac{s(C)+1}{2} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$, *and*

for all $s \in [1, n]$ there exist $C \in \mathcal{K}^n$ with $s(C) = s$, for which they are left-to-right optimal.

Based on this corollary, one obtains, e.g., that the following reverse inequality chain of (3) is sharp w.r.t. $s(C)$:

$$\|x\|_{C \cap (-C)} \leq \frac{s(C)+1}{2} \|x\|_{\frac{C-C}{2}} \leq s(C) \|x\|_{\mathrm{conv}(C \cup (-C))}.$$

In Lemma 3.1 we show that two of the containments of symmetrizations in forward direction are always optimal:

$$\frac{C-C}{2} \subset^{opt} \mathrm{conv}(C \cup (-C)) \quad \text{and} \quad C \cap (-C) \subset^{opt} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ.$$

Using Proposition 1.2, we see that (2) may be left-to-right optimal even for non-symmetric C. In particular, consider a regular Minkowski centered simplex $S \in \mathcal{K}^3$, whose four means are an octahedron (minimum), a rhombic dodecahedron (harmonic mean), a cube octahedron (arithmetic mean), and a cube (maximum). They build a left-to-right optimal chain of containments (see Figure 1). This property remains true for the four symmetrizations of a regular Minkowski centered simplex in any odd dimension.

In contrast, for a regular Minkowski centered simplex S in even dimensions we show in Lemma 3.2 that

$$S \cap (-S) \subset^{opt} \frac{n}{n+1} \mathrm{conv}(S \cup (-S)) \quad \text{and} \quad \left(\frac{S^\circ - S^\circ}{2}\right)^\circ \subset^{opt} \frac{n(n+2)}{(n+1)^2} \frac{S-S}{2}.$$

Concerning the above, we proceed with a stability result. First we introduce several parameters depending on $n \in \mathbb{N}$, $n \geq 2$ and $s \in [1, n]$, which we need throughout the upcoming results.

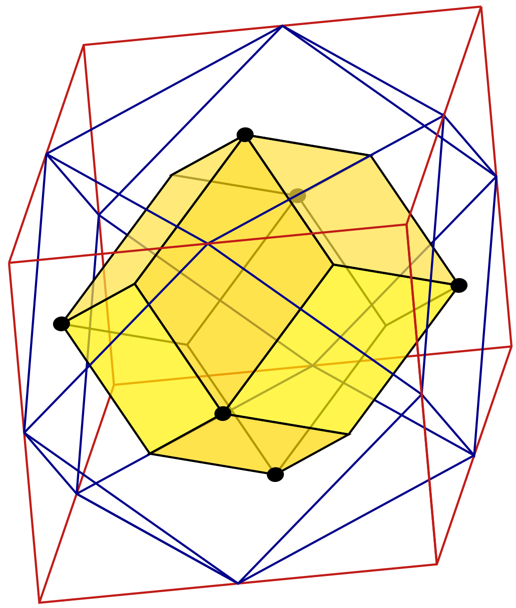


FIGURE 1. Symmetrizations of a regular simplex $S \subset \mathbb{R}^3$: $S \cap (-S)$ an octahedron (convex hull of black points), $\left(\frac{S^\circ - S^\circ}{2}\right)^\circ$ a rhombic dodecahedron (yellow), $\left(\frac{S-S}{2}\right)$ a cube-octahedron (blue), and $\mathrm{conv}(S \cup (-S))$ a cube (red).

$$
\begin{aligned}
\psi &:= \psi(n,s) := \frac{(n-s+1)(s+1)}{1-(n-s)(n+s(n+1))} - n,\\
\zeta &:= \zeta(n,s) = (n+1)\left(\left(1+\frac{sn}{s+1}\right)\frac{1+n-s}{1-n(n-s)} - n\right),\\
\gamma_1 &:= \gamma_1(n) := \frac{1}{2}\left(n-1+\sqrt{(n-2)n+5}\right),\\
\gamma_2 &:= \gamma_2(n) := \frac{n^4+n^3+2n^2+\sqrt{\delta_2}}{2(n^3+2n^2+3n+1)},\\
\gamma_3 &:= \gamma_3(n) := \frac{n^5+2n^4+2n^3+2n^2-2n-1+\sqrt{\delta_3}}{2n(n^3+3n^2+4n+3)},\\
\delta_2 &:= \delta_2(n) := n^8+6n^7+17n^6+28n^5+28n^4+12n^3-4n^2-12n-4,\\
\delta_3 &:= \delta_3(n) := n^{10}+8n^9+28n^8+56n^7+72n^6+66n^5+44n^4+16n^3-4n^2-8n+1.
\end{aligned}
$$

One can check that for every even n holds $n - \frac{1}{n} < \gamma_2 < \gamma_3 < n$ and $\psi = \zeta = 1$ in case $s = n$. Moreover, we will see that $\psi \frac{n}{n+1} < 1$ for all $s > \gamma_2$, while $\zeta \frac{n(n+2)}{(n+1)^2} < 1$ for all $s > \gamma_3$.

Theorem 1.5. *Let n be even and $C \in \mathcal{K}^n$ Minkowski centered with $s(C) = s$. Then*

(i) $C \cap (-C) \subset \psi \dfrac{n}{n+1} \mathrm{conv}(C \cup (-C))$, *if* $s \geq \gamma_2(n)$, *and*

(ii) $\left(\dfrac{C^\circ - C^\circ}{2}\right)^\circ \subset \zeta \dfrac{n(n+2)}{(n+1)^2} \dfrac{C-C}{2}$, *if* $s \geq \gamma_3(n)$.

Whenever (2) is left-to-right optimal for some Minkowski centered convex body C there also exists a series of Minkowski centered convex bodies with any smaller asymmetry providing a left-to-right optimality for the full chain (see Lemma 3.4). Thus, we aim to determine the smallest number $\gamma(n) \in [n-1,n]$ such that $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ is not optimally contained in $\frac{C-C}{2}$ for every Minkowski centered $C \in \mathcal{K}^n$ with $s(C) > \gamma(n)$, and call this value the *asymmetry threshold of means*. Notice that [BDG, Theorem 1.2] already shows that $\gamma(2) = \frac{1+\sqrt{5}}{2} =: \varphi$ is the golden ratio. Moreover, from Part (i) of Lemma 3.2 below we directly get $\gamma(n) = n$, whenever n is odd.

We present a result on the asymmetry threshold for arbitrary even dimension.

Theorem 1.6. *Let n be even. Then*

$$n - 1 < \gamma_1(n) \leq \gamma(n) \leq \gamma_2(n) < n.$$

One may recognize the following: it is well-known that the golden ratio can be obtained from solving the equation $\frac{a+b}{a} = \frac{a}{b}$ with $a > b > 0$, and it is also $\gamma(2)$. However, one can similarly obtain the values of γ_1 from solving the equation $\frac{(n-1)a+b}{a} = \frac{a}{b}$, and therefore may consider the values of γ_1 as a generalized golden ratio.

The asymmetry threshold provides us with a lower bound for the values of s such that (2) cannot be left-to-right optimal. In the following we go one step further and determine the possible values for the contraction factors $\alpha(C)$ and $\beta(C)$ for which the minimum is optimally contained in the according contraction of the maximum and for which the harmonic mean is optimally contained in the contraction of the arithmetic mean, respectively.

Let $C \in \mathcal{K}^n$ and $\alpha(C) > 0$ be such that $C \cap (-C) \subset^{opt} \alpha(C)\,\mathrm{conv}(C \cup (-C))$. For $s \in [1,n]$ we define

$$\alpha_1(s) := \inf\{\alpha(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\} \quad \text{and}$$
$$\alpha_2(s) := \sup\{\alpha(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\}.$$

Similarly, let $\beta(C) > 0$ be such that $\left(\frac{1}{2}(C^\circ - C^\circ))\right)^\circ \subset^{opt} \beta(C)\,\frac{1}{2}(C - C)$. Then

$$\beta_1(s) := \inf\{\beta(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\} \quad \text{and}$$
$$\beta_2(s) := \sup\{\beta(C) : C \in \mathcal{K}^n \text{ Minkowski centered}, s(C) = s\}.$$

Theorem 1.7. *Let $C \in \mathcal{K}^n$ be Minkowski centered with $s(C) = s$. Then*

a) *(i)* $\alpha_1(s) \geq \frac{2}{s+1}$, *with equality at least for* $s \leq 2$.
(ii) $\alpha_2(s) = 1$ *for* $s \leq \gamma_1$, $\alpha_2(s) \leq \psi\frac{n}{n+1}$, *for* $s > \gamma_2$, *and* $\alpha_2(s) \geq \frac{s}{s^2-1}$ *for* $n = 2$ *and* $s \geq \varphi$.

b) *(i)* $\beta_1(s) \geq \frac{4s}{(s+1)^2}$, *with equality at least for* $s \leq 2$.
(ii) $\beta_2(s) = 1$ *for* $s \leq \gamma_1$, $\beta_2(s) \leq \zeta\frac{n(n+2)}{(n+1)^2}$ *for* $s > \gamma_3$, $\beta_2(s) < 1$ *for* $s > \gamma_2$, *and* $\beta_2(s) \geq \max\left\{\frac{s}{s^2-1}, \frac{4s}{(s+1)^2}\right\}$ *for* $n = 2$ *and* $s \geq \varphi$.

Let us denote the *canonical basis* of $\mathbb{R}^n$ by $e^1, \dots, e^n \in \mathbb{R}^n$, the *Euclidean norm* of $x \in \mathbb{R}^n$ by $\|x\|$, and the *Euclidean unit ball* by $\mathbb{B}_2 = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. For any $C, K \in \mathcal{K}^n$ the

Euclidean distance is denoted by $d(C,K)$ and in case $C=\{p\}$ is a singleton, we abbreviate $d(\{p\},K)$ by $d(p,K)$. For any $C,K\in\mathcal{K}^n$ the *Banach-Mazur distance* between C and K is defined by $d_{BM}(C,K)=\inf\{\rho\ge 1 : t^1+K\subset L(C)\subset t^2+\rho K,\ L\in \mathrm{GL}(n),\ t^1,t^2\in\mathbb{R}^n\}$. For every $X\subset\mathbb{R}^n$ let $\mathrm{bd}(X)$ and $\mathrm{int}(X)$ denote the *boundary* and *interior* of X, respectively. For $C\in\mathcal{K}^n$ and $a\in\mathbb{R}^n$ let $h_C(a)=\sup\{a^Tx : x\in C\}$ be the *support function* of C in a. Notice that $\|\cdot\|_C$ is a norm in the classic sense if and only if $C\in\mathcal{K}^n_0$ and remember that $\|x\|_C=h_{C^\circ}(x)$ for every $C\in\mathcal{K}^n$ and $x\in\mathbb{R}^n$ (see [MR]). For any $a\in\mathbb{R}^n\setminus\{0\}$ and $\rho\in\mathbb{R}$, $H^{\le}_{a,\rho}=\{x\in\mathbb{R}^n : a^Tx\le\rho\}$ denotes the *halfspace* with outer normal a and right-hand side ρ. We say that the halfspace $H^{\le}_{a,\rho}$ *supports* $C\in\mathcal{K}^n$ at $q\in C$, if $C\subset H^{\le}_{a,\rho}$ and $q\in\mathrm{bd}(H^{\le}_{a,\rho})$. For any $C\in\mathcal{K}^n$ and $p\in\mathrm{bd}(C)$, the *outer normal cone* of K at p is defined as $N(C,p)=\{a\in\mathbb{R}^n : a^Tp\ge a^Tx \text{ for all } x\in C\}$. For every $X\subset\mathbb{R}^n$ let us denote by $\mathrm{pos}(X)$ and $\mathrm{aff}(X)$ the *positive* and *affine hull* of X, respectively, while the *relative interior* of X is denoted by $\mathrm{relint}(X)$. In case $u^1,\dots,u^{n+1}\in\mathbb{R}^n$ are affinely independent, we say that $\mathrm{conv}(\{u^1,\dots,u^{n+1}\})$ is an *n-simplex*. Throughout the paper we abbreviate $[n]:=\{1,2,\dots,n\}$ for any $n\in\mathbb{N}$.

2. Preliminary results and lemmas

We recall the characterization of the optimal containment under homothety in terms of the touching conditions (see [BK, Theorem 2.3]).

Proposition 2.1. *Let $K,C\in\mathcal{K}^n$ and $K\subset C$. The following are equivalent:*

(i) $K\subset^{opt} C$.

(ii) There exist $k\in\{2,\dots,n+1\}$, $p^j\in K\cap\mathrm{bd}(C)$, $u^j\in N(C,p^j)$, $j=1,\dots,k$, such that $0\in\mathrm{conv}(\{u^1,\dots,u^k\})$.

Moreover, if $K,C\in\mathcal{K}^n_0$, then (i) and (ii) are also equivalent to $K\cap\mathrm{bd}(C)\neq\emptyset$.

One may choose u^j, $j=1,\dots,k$, to be extreme points of C° above, which in case of a polytopal C means that they are facet normals.

The next lemma shows that all the considered means are affine invariant.

Lemma 2.2. *Let $K,C\in\mathcal{K}^n$ and A be a non-singular affine transformation. Then*

$$A(K)\cap A(C)=A(K\cap C),\qquad (((A(K))^\circ-(A(C))^\circ)/2)^\circ=A\left((K^\circ-C^\circ)/2\right)^\circ,$$
$$(A(K)+A(C))/2=A\left((K+C)/2\right),\qquad \mathrm{conv}\left(A(K)\cup(A(C)\right)=A\left(\mathrm{conv}(K\cup C)\right).$$

Proof. From the fact that $A(C^\circ)=((A^{-1})^T(C))^\circ$, we obtain

$$\left(\frac{(A(K))^\circ-(A(C))^\circ}{2}\right)^\circ=\left(\frac{(A^{-1})^T(K^\circ)-(A^{-1})^T(C^\circ)}{2}\right)^\circ$$
$$=\left(\frac{(A^{-1})^T(K^\circ-C^\circ)}{2}\right)^\circ=\left((A^{-1})^T\left(\frac{K^\circ-C^\circ}{2}\right)\right)^\circ=A\left(\left(\frac{K^\circ-C^\circ}{2}\right)^\circ\right).$$

The other identities are trivially true. $\square$

The next result is a straightforward corollary of Lemma 2.2.

Corollary 2.3. *Let $C \in \mathcal{K}^n$ be Minkowski centered, $A \in \mathbb{R}^{n\times n}$ a regular linear transformation and $\alpha \in \mathbb{R}$. Then*

$$C \cap (-C) \subset^{opt} \alpha \cdot conv(C \cup (-C))$$

if and only if

$$A(C) \cap A(-C) \subset^{opt} \alpha \cdot conv(A(C) \cup A(-C)).$$

The following proposition is a corollary of Proposition 2.1, which is a (variant of a) known result, given in [GrK, (1.1)] in a more general version, and will be used in the proof of Lemma 3.1.

Proposition 2.4. *Let $C, K \in \mathcal{K}^n_0$. Then $C \subset^{opt} K$ if and only if $K^\circ \subset^{opt} C^\circ$. Moreover, the touching points of C to the boundary of K become the outer normals of supporting halfspaces of the touching points of K° to the boundary of C° and vice versa.*

Let us mention that while the containment in Proposition 2.4 holds for any C, K with 0 in their interior, the optimality of this containment may in general be lost even in case of Minkowski centered C and K (see Figure 2).

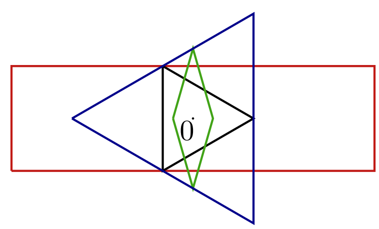


FIGURE 2. Minkowski centered C (black) and K (red), s.t. $C \subset^{opt} K$ but K° (green) is not optimal contained in C° (blue).

As mentioned in the introduction, (2) is not left-to-right optimal for regular Minkowski centered simplices in even dimensions. The following lemma prepares us to prove this fact in Lemma 3.2.

Lemma 2.5. *Let $P \in \mathcal{K}^n_0$ be a polytope and $v \in \mathrm{bd}(P)$ such that v is also an outer normal of a facet of P, closest to the origin 0. Then*

$$P^\circ \subset^{opt} \frac{1}{\|v\|^2} P.$$

Proof. Let $0 < t_1 \leq \cdots \leq t_m$ and $u^1, \ldots, u^m \in \mathbb{S}^{n-1}$ be such that $P = \{x \in \mathbb{R}^n : |(u^i)^T x| \leq t_i,\ i \in [m]\}$. Then $P^\circ = \mathrm{conv}(\{\pm u^1/t_1, \ldots, \pm u^m/t_m\})$ and $t_1 u^1 \in t_1\mathbb{B}_2 \cap \mathrm{bd}(P)$, which implies $t_1\mathbb{B}_2 \subset^{opt} P$ by Proposition 2.1. Since $\frac{1}{t_1}u^1 \in P^\circ \cap \mathrm{bd}(\frac{1}{t_1}\mathbb{B}_2)$, we have $P^\circ \subset^{opt} \frac{1}{t_1}\mathbb{B}_2 \subset^{opt} \frac{1}{t_1^2}P$ and $\frac{1}{t_1}u^1$ is a touching point of P° to the boundary of $\frac{1}{t_1^2}P$. Thus, by Part (iii) of Proposition 2.1, we have $P^\circ \subset^{opt} \frac{1}{t_1^2}P$. Choosing $v = t_1 u^1$ finishes the proof. $\square$

We recall a stability result for the Banach-Mazur distance in the near-simplex case, given in [Sch2, Theorem 2.1].

Proposition 2.6. *Let $S \in \mathcal{K}^n$ be an n-simplex and $C \in \mathcal{K}^n$ such that $s(C) = n - \varepsilon$, with $\varepsilon \in (0, \frac{1}{n})$. Then*

$$d_{BM}(C, S) \leq 1 + \frac{(n+1)\varepsilon}{1 - n\varepsilon}. \tag{4}$$

3. Optimality in Firey's inequality chain

As mentioned in the introduction, two of the containments in Proposition 1.1 are always optimal for symmetrizations.

Lemma 3.1. *Let $C \in \mathcal{K}^n$. Then*

(i) $\frac{C-C}{2} \subset^{opt} \mathrm{conv}(C \cup (-C))$ *and*
(ii) $C \cap (-C) \subset^{opt} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$*, if also* $0 \in C$.

Proof. We start proving (i). By (2), we have $\frac{C-C}{2} \subset \mathrm{conv}(C \cup (-C))$. Now, there exists an extreme point x of C and an extreme point y of $-C$, s.t. $[x, y] \subset \mathrm{bd}(\mathrm{conv}(C \cup (-C)))$. This implies $\frac{x+y}{2} \in \frac{C-C}{2} \cap \mathrm{bd}(\mathrm{conv}(C \cup (-C))$ and, since $\frac{C-C}{2}$ and $\mathrm{conv}(C \cup (-C))$ are 0-symmetric, we can conclude the proof of (i) by Proposition 2.1.

If $0 \in C$, we have $C \cap (-C) \neq \emptyset$. Thus, applying (i) and Proposition 2.4 implies (ii). □

We are now ready to prove Lemma 3.2.

Lemma 3.2. *Let S be a Minkowski centered regular n-simplex. Then*

(i) $S \cap (-S) \subset^{opt} \mathrm{conv}(S \cup (-S))$*, if n is odd,*
(ii) $S \cap (-S) \subset^{opt} \frac{n}{n+1}\mathrm{conv}(S \cup (-S))$*, if n is even, and*
(iii) $\left(\frac{S^\circ - S^\circ}{2}\right)^\circ \subset^{opt} \frac{n(n+2)}{(n+1)^2} \cdot \frac{S-S}{2}$*, if n is even.*

Proof. In order to simplify the calculations, we assume w.l.o.g. that $S = \mathrm{conv}(\{p^1, \ldots, p^{n+1}\})$ with $p^j \in \mathbb{R}^n$, such that $\|p^j\| = 1$, $j \in [n+1]$.

(i) Let $n \geq 1$ be odd and $p = \frac{2}{n+1}\left(p^1 + \cdots + p^{\frac{n+1}{2}}\right)$. Since $\sum_{i=1}^{n+1} p^i = 0$, we have $-p = \frac{2}{n+1}\left(p^{\frac{n+3}{2}} + \cdots + p^{n+1}\right) \in S$, and thus $p \in S \cap (-S)$. Define

$$H_1^{\leq} := \left\{x \in \mathbb{R}^n : (p^1 + \cdots + p^{\frac{n+1}{2}})^T x \leq \frac{n+1}{2n}\right\},$$
$$H_2^{\leq} := \left\{x \in \mathbb{R}^n : (-p^1 - \cdots - p^{\frac{n+1}{2}})^T x \leq \frac{n+1}{2n}\right\}.$$

Then we obtain for $j = 1, \ldots, (n+1)/2$

$$(p^1 + \cdots + p^{\frac{n+1}{2}})^T p^j = 1 - \left(\frac{n+1}{2} - 1\right)\frac{1}{n} = \frac{n+1}{2n},$$

and for $j = (n+3)/2, \dots, n+1$

$$(-p^1 - \cdots - p^{\frac{n+1}{2}})^T p^j = \frac{n+1}{2n},$$

and therefore $S \subset H_1^{\leq} \cap H_2^{\leq}$. Moreover,

$$|(p^1 + \cdots + p^{\frac{n+1}{2}})^T p| = \frac{n+1}{2n},$$

which shows that $H_1^{\leq}$ and $H_2^{\leq}$ support S at p and $-p$, respectively. Hence, $S \cap (-S) \subset^{opt} \mathrm{conv}(S \cup (-S))$.

(ii) We start by observing that $S = \{x \in \mathbb{R}^n : (-p^j)^T x \leq \frac{1}{n}, j \in [n+1]\}$, and that therefore

$$S \cap (-S) = \left\{ x \in \mathbb{R}^n : |(p^j)^T x| \leq \frac{1}{n}, j \in [n+1] \right\}.$$

Now, let $p = \frac{2}{n+1}p^1 + \cdots + \frac{2}{n+1}p^{\frac{n}{2}} + \frac{1}{n+1}p^{n+1} \in S \cap (-S)$. Then, for $j = 1, \dots, \frac{n}{2}$ we have

$$(p^j)^T p = \frac{2}{n+1} - \left(\left(\frac{n}{2} - 1 \right) \frac{2}{n+1} + \frac{1}{n+1} \right) = \frac{2}{n+1} - \frac{1}{n} \cdot \frac{n-1}{n+1} = \frac{1}{n},$$

and for $j = \frac{n}{2} + 1, \dots, n$

$$(p^j)^T p = -\frac{1}{n} \left(\frac{n}{2} \cdot \frac{2}{n+1} + \frac{1}{n+1} \right) = -\frac{1}{n}.$$

Thus, n of the constraints $|(p^j)^T x| \leq \frac{1}{n}, j \in [n+1]$ are active in p and p is a vertex of $S \cap (-S)$. Moreover,

$$\|p\|^2 = \frac{1}{(n+1)^2} \left(\left(\frac{n}{2} \cdot 4 + 1 \right) + \left(\frac{n}{2} \cdot \left(\frac{n}{2} - 1 \right) \cdot 4 + \frac{n}{2} \cdot 2 \right) \left(-\frac{1}{n} \right) \right) = \frac{1}{n+1},$$

which shows $(n+1)p \in \mathrm{bd}((S \cap (-S))^\circ)$ using the Cauchy-Schwarz inequality. Finally, since $S^\circ = -nS$, we have

$$(S \cap (-S))^\circ = \mathrm{conv}(S^\circ \cup (-S)^\circ) = n \cdot \mathrm{conv}(S \cup (-S)),$$

implying $\frac{n+1}{n} p \in \mathrm{bd}(\mathrm{conv}(S \cup (-S)))$. Hence,

$$S \cap (-S) \subset^{opt} \frac{n}{n+1} \mathrm{conv}(S \cup (-S)).$$

(iii) Notice that taking differences of any of the first $\frac{n}{2}$ vertices and any of the last $\frac{n}{2}+1$ induces a facet of $\frac{S-S}{2}$, which is closest to the origin. Moreover,

$$v := \frac{1}{2} \left(\sum_{i=1}^{\frac{n}{2}} \frac{p^i}{\frac{n}{2}} - \sum_{i=\frac{n}{2}+1}^{n+1} \frac{p^i}{\frac{n}{2}+1} \right) \in \mathrm{bd} \left(\frac{S-S}{2} \right)$$

belongs to that facet and is also an outer normal of it.

We compute

$$\begin{aligned}\|v\|^2 &= \frac{1}{4}\left(\frac{n}{2}\frac{1}{(\frac{n}{2})^2}+\left(\frac{n}{2}+1\right)\frac{1}{\left(\frac{n}{2}+1\right)^2}+\frac{n}{2}\left(\frac{n}{2}-1\right)\frac{1}{(\frac{n}{2})^2}\left(-\frac{1}{n}\right)\right.\\ &\quad\left.+\left(\frac{n}{2}+1\right)\frac{n}{2}\frac{1}{(\frac{n}{2}+1)^2}\left(-\frac{1}{n}\right)-2\frac{n}{2}\left(\frac{n}{2}+1\right)\frac{1}{\frac{n}{2}}\frac{1}{\frac{n}{2}+1}\left(-\frac{1}{n}\right)\right)\\ &= \frac{1}{4}\left(\frac{2}{n}+\frac{2}{n+2}-\frac{n-2}{n^2}-\frac{1}{n+2}+\frac{2}{n}\right)=\frac{(n+1)^2}{n^2(n+2)}.\end{aligned}$$

Using Lemma 2.5 and the identity $S^\circ = -nS$ again, we obtain

$$\left(\frac{S^\circ - S^\circ}{2}\right)^\circ = \frac{1}{n}\left(\frac{S-S}{2}\right)^\circ \subset^{opt} \frac{1}{n\|v\|^2}\frac{S-S}{2} = \frac{n(n+2)}{(n+1)^2}\frac{S-S}{2}.$$

□

As mentioned in the introduction, Theorem 1.2 states that left-to-right optimality in (2) depends only on the optimal containment of the harmonic in the arithmetic mean.

Proof of Theorem 1.2. The forward direction directly follows from Proposition 1.1. Thus, we only have to show the backward direction.

Let $\left(\frac{K^\circ+C^\circ}{2}\right)^\circ \subset^{opt} \frac{K+C}{2}$. By Proposition 2.1 there exist $k \in \{2,\dots,n+1\}$, $p^j \in \left(\frac{K^\circ+C^\circ}{2}\right)^\circ \cap \mathrm{bd}\left(\frac{K+C}{2}\right)$, $u^j \in N(\frac{K+C}{2}, p^j)$, $j \in [k]$, such that $0 \in \mathrm{conv}(\{u^1,\dots,u^k\})$. Choose any $p = p^j$, $u = u^j$, $j \in [k]$, and $\beta \in \mathbb{R}$, such that $H_{u,\beta}$ is a hyperplane supporting $\frac{K+C}{2}$ in p. Since $p \in \left(\frac{K^\circ+C^\circ}{2}\right)^\circ \cap \mathrm{bd}\left(\frac{K+C}{2}\right)$, we have $\|p\|_{\left(\frac{K^\circ+C^\circ}{2}\right)^\circ} = \|p\|_{\frac{K+C}{2}} = 1$.

On the one hand, using $\frac{p}{\|p\|_K} \in K$ and $\frac{p}{\|p\|_C} \in C$, we see

$$\frac{1}{2}\left(\frac{1}{\|p\|_K}+\frac{1}{\|p\|_C}\right)p \in \frac{K+C}{2},$$

and therefore,

$$\|p\|_{\frac{K+C}{2}} \le \left(\frac{1}{2}\left(\frac{1}{\|p\|_K}+\frac{1}{\|p\|_C}\right)\right)^{-1}.$$

On the other hand, since $h_{C^\circ} = \|\cdot\|_C$, we have

$$\frac{1}{2}\left(\|p\|_K+\|p\|_C\right) = \frac{1}{2}\left(h_{K^\circ}(p)+h_{C^\circ}(p)\right) = h_{\frac{K^\circ+C^\circ}{2}} = \|p\|_{\left(\frac{K^\circ+C^\circ}{2}\right)^\circ}.$$

Applying the arithmetic-harmonic mean inequality (for numbers - restating the main argument for Proposition 1.1), we obtain

$$\|p\|_{\frac{K+C}{2}} \le \left(\frac{1}{2}\left(\frac{1}{\|p\|_K}+\frac{1}{\|p\|_C}\right)\right)^{-1} \le \frac{1}{2}\left(\|p\|_K+\|p\|_C\right) = \|p\|_{\left(\frac{K^\circ+C^\circ}{2}\right)^\circ} = \|p\|_{\frac{K+C}{2}}.$$

This means that we have equality between the harmonic and arithmetic mean of $\|p\|_K$ and $\|p\|_C$, which implies

$$\|p\|_K = \|p\|_C = \|p\|_{\frac{K+C}{2}} = 1,$$

and as a direct implication

$$\|p\|_{K\cap C} = \max\{\|p\|_K, \|p\|_C\} = 1.$$

Now, it suffices to show that $H_{u,\beta}$ also supports $\operatorname{conv}(K \cup C))$ in p. Assume that the latter is wrong. This would imply, that there exists $q \in K \setminus C$ or $q \in C \setminus K$ such that $u^T q > \beta$. Say, w.l.o.g., $q \in K \setminus C$. However, this would imply $u^T\left(\frac{p+q}{2}\right) > \beta$, contradicting the fact that $H_{u,\beta}$ supports $\frac{K+C}{2}$. Hence, $H_{u,\beta}$ supports also $\operatorname{conv}(K \cup C)$ at p.

Altogether, we see that $p^j \in (K \cap C) \cap \operatorname{bd}(\operatorname{conv}(K \cup C)))$ with $u^j \in N(\operatorname{conv}(K \cup C), p^j)$, $j \in [k]$ and $0 \in \operatorname{conv}(\{u^1, \dots, u^k\})$. Using Proposition 2.1, we obtain the optimal containment of $K \cap C$ in $\operatorname{conv}(K \cup C)$. $\square$

The following proposition (see [BDG, Theorem 1.3]) is a direct consequence of Theorem 1.2, applied to C and $-C$.

Proposition 3.3. *Let $C \in \mathcal{K}^n$ be Minkowski centered. Then the following are equivalent:*

(i) $C \cap (-C) \subset^{opt} \operatorname{conv}(C \cup (-C))$,
(ii) $\left(\frac{1}{2}(C^\circ - C^\circ))\right)^\circ \subset^{opt} \frac{1}{2}(C - C)$,
(iii) there exist $p, -p \in \operatorname{bd}(C)$ and parallel halfspaces $H^{\leq}_{a,1}$ and $H^{\leq}_{-a,1}$ supporting C at p and $-p$, respectively.

In case, when the containment in (2) is left-to-right optimal for some Minkowski centered C and $K = -C$, there also exist Minkowski centered bodies with an arbitrary smaller asymmetry, providing left-to-right optimality in the full chain.

Lemma 3.4. *Let $C \in \mathcal{K}^n$ be Minkowski centered. If*

$$C \cap (-C) \subset^{opt} \operatorname{conv}(C \cup (-C)),$$

for every $s \in [1, s(C)]$ there exists a Minkowski centered $C_s \in \mathcal{K}^n$ with $s(C_s) = s$, such that

$$C_s \cap (-C_s) \subset^{opt} \operatorname{conv}(C_s \cup (-C_s)).$$

Proof. Since C is Minkowski centered, we obtain from Proposition 2.1 that there exist $p^1, \dots, p^k \in -\frac{1}{s(C)} C \cap \operatorname{bd}(C)$ with $k \in \{2, \dots, n+1\}$ and $u^j \in N(C, p^j)$, $j \in [k]$, such that $0 \in \operatorname{conv}(\{u^1, \dots, u^k\})$. By Part (iii) of Proposition 3.3 there also exist $p, -p \in (C \cap (-C)) \cap \operatorname{bd}(\operatorname{conv}(C \cup (-C)))$. For $\lambda \in [0,1]$ let us define

$$K_\lambda := \operatorname{conv}\left(\{p^1, \dots, p^k, \alpha_\lambda p^1, \dots, \alpha_\lambda p^k, \pm p\}\right) \quad \text{with} \quad \alpha_\lambda := -((1-\lambda)s(C) + \lambda).$$

One may recognize, that since $\alpha_0 p^j = -s(C)p^j \in C$, $j \in [k]$, we have $K_\lambda \subset C$. By the fact that $\pm p \in K_\lambda$, we have $\pm p \in (K_\lambda \cap (-K_\lambda)) \cap \operatorname{bd}(\operatorname{conv}(K_\lambda \cup (-K_\lambda))$ for all $\lambda \in [0,1]$. Hence, K_λ fulfills Part (iii) of Proposition 3.3 and therefore the optimal containment. Moreover, $K_\lambda \subset \alpha_\lambda K_\lambda$ with $\alpha_\lambda p^j \in K_\lambda \cap \operatorname{bd}(\alpha_\lambda K_\lambda)$ and $-u^j \in N(\alpha_\lambda K_\lambda, \alpha_\lambda p^j)$, $j \in [k]$. Thus, by Proposition 2.1 K_λ is optimally contained in $\alpha_\lambda K_\lambda$, which shows that $s(K_\lambda) = -\alpha_\lambda = (1-\lambda)s(C) + \lambda \in [1, s(C)]$. Choosing $C_s := K_{\frac{s(C)-s}{s(C)-1}}$ concludes the proof. $\square$

Using Proposition 2.6 we now prove Theorem 1.5.

Proof of Theorem 1.5. Let us mention that we split the proof in (i) and (ii) only after a while. Let $S = \mathrm{conv}(\{p^1, \dots, p^{n+1}\})$ be a regular Minkowski centered n-simplex with $\|p^i\| = n$, $i \in [n+1]$ and $\rho = d_{BM}(C, S)$. Since $\gamma_3 > \gamma_2 > n - \frac{1}{n}$, C is under the conditions of Proposition 2.6. Hence, we may use (4) with $\varepsilon := n - s$ to obtain

$$\rho \leq \rho_* = 1 + \frac{(n+1)(n-s)}{1-n(n-s)} = \frac{n+1-s}{1-n(n-s)}. \tag{5}$$

Let $F_i = \mathrm{conv}(\{p^j : j \neq i\})$ be the facet of S, opposing p^i and $L_i := \mathrm{aff}(F_i)$ with $i \in [n+1]$. Since $\rho = d_{BM}(C, S)$, there exists a regular linear transformation L, such that $c^1 + S \subset L(C) \subset c^2 + \rho S$ for some $c^1, c^2 \in \mathbb{R}^n$. Since by Corollary 2.3, $L(C) \cap (-L(C)) \subset \gamma \cdot \mathrm{conv}(L(C) \cup (-L(C)))$ for some $\gamma > 0$ is equivalent to $C \cap (-C) \subset \gamma \cdot \mathrm{conv}(C \cup (-C))$, we can (w.l.o.g.) replace C by $L(C)$, and thus obtain

$$c^1 + S \subset C \subset c^2 + \rho S. \tag{6}$$

We now show by contradiction that $0 \in c^1 + S$. Note that since S is Minkowski centered, we have

$$S = \bigcap_{j \in [n+1]} \left\{ x : -\frac{1}{n}(p^j)^T x \leq 1 \right\}.$$

Assume $0 \notin c^1 + S$. Then $-c^1 \notin S$, which means $(-p^k/n)^T(-c^1) = (p^k/n)^T c^1 > 1$ for some $k \in [n+1]$. Since $\|p^k\| = n$, this implies that the distance from $c^1 + p^k$ to $\{x : (p^k)^T x = 0\}$ equals $(p^k/n)^T(p^k + c^1) = n + 1 + t$ for some $t > 0$.

However, since C is Minkowski centered and $c^1 + p^k \in C$, we also have $-(c^1 + p^k)/s \in C$. This implies that the breadth of C in direction of p^k is at least $n + 1 + t + (n + 1 + t)/s$. Moreover, since $C \subset c^2 + \rho S$, we see that $n + 1 + t + (n + 1 + t)/s$ is not larger than the breath of ρS in direction of p^k. Using (5) we obtain

$$(n+1+t)\left(1 + \frac{1}{s}\right) \leq \rho(n+1) \leq (n+1)\left(1 + \frac{(n+1)(n-s)}{1-n(n-s)}\right).$$

Since $t > 0$, this implies

$$1 + \frac{1}{s} < 1 + \frac{(n+1)(n-s)}{1-n(n-s)},$$

which is equivalent to $(n+1)s^2 - n^2 s + 1 - n^2 < 0$. Hence,

$$\frac{n^2 - \sqrt{n^4 - 4(n+1)(1-n^2)}}{2(n+1)} < s < \frac{n^2 + \sqrt{n^4 - 4(n+1)(1-n^2)}}{2(n+1)}.$$

However, for both parts, (i) and (ii), of the theorem we assume that $s \geq \gamma_2$ and it is not hard to check that $\gamma_2 > \frac{n^2 + \sqrt{n^4 - 4(n+1)(1-n^2)}}{2(n+1)}$, which contradicts the above.

Knowing that $0 \in c^1 + S$, we assume w.l.o.g. that the minimal distance $\bar{\mu}$ from 0 to the facets of $c^1 + S$ is attained at $c^1 + L_1$.

Since C is Minkowski centered and $c^1 + S \subset C$, we have $z := c^1 + p^1 \in C$ and, using the fact that $0 \in (\frac{p^1}{n} + L_1) \cap (c^1 + S)$, it follows that

$$\begin{aligned} d(z, \frac{p^1}{n} + L_1) + \bar{\mu} &= d(z, \frac{p^1}{n} + L_1) + d(\frac{p^1}{n} + L_1, c^1 + L_1) \\ &= d(z, c^1 + L_1) = n + 1. \end{aligned}$$

Taking into account that $\frac{-z}{s} \in C$, we obtain

$$\xi := d(\frac{-z}{s}, \frac{p^1}{n} + L_1) = \frac{d(z, \frac{p^1}{n} + L_1)}{s} = \frac{n + 1 - \bar{\mu}}{s}. \tag{7}$$

From $c^1 + S \subset c^2 + \rho S$ and considering the breadths of each simplex orthogonal to L_1 one can deduce

$$d(c^2 + \rho L_1, c^1 + L_1) \leq (n + 1)\rho - (n + 1) = (n + 1)(\rho - 1).$$

Since we also have $\frac{-z}{s} \in C \subset c^2 + \rho S$ it follows

$$\begin{aligned} \xi &\leq d(c^2 + \rho L_1, \frac{p^1}{n} + L_1) \\ &= d(c^2 + \rho L_1, c^1 + L_1) + d(c^1 + L_1, \frac{p^1}{n} + L_1) \\ &\leq (n + 1)(\rho - 1) + \bar{\mu}. \end{aligned} \tag{8}$$

Combining (7) and (8), we obtain

$$\frac{n + 1 - \bar{\mu}}{s} \leq (n + 1)(\rho - 1) + \bar{\mu},$$

which is equivalent to

$$\bar{\mu} \geq \mu := \mu(\rho) := \frac{n + 1}{s + 1} \left(1 - s(\rho - 1)\right). \tag{9}$$

Thus, $d(0, c^2 + \rho L_j) \geq d(0, c^1 + L_j) \geq \mu$ for all $j \in [n + 1]$ and therefore

$$\mu S \subset c^1 + S. \tag{10}$$

as well as $0 \in c^2 + (\rho - \mu)S$.

Hence,

$$d(0, c^2 + \rho L_j) \leq d(c^2 + (\rho - \mu)p^j, c^2 + \rho L_j) = \rho + n(\rho - \mu)$$

for all $j \in [n + 1]$ and therefore

$$c^2 + \rho S \subset (\rho + n(\rho - \mu))S. \tag{11}$$

Now, we go on proving the two separate parts of the theorem.

(i) Combining (11) and (10) with Part (ii) of Lemma 3.2 directly implies

$$\begin{aligned}C\cap(-C) &\subset (c^2+\rho S)\cap(-c^2-\rho S)\\ &\subset (\rho+n(\rho-\mu))(S\cap(-S))\\ &\subset \frac{n}{n+1}(\rho+n(\rho-\mu))\mathrm{conv}(S\cup(-S))\\ &\subset \frac{n}{n+1}\frac{(\rho+n(\rho-\mu))}{\mu}\mathrm{conv}((c^1+S)\cup(-c^1-S))\\ &\subset \frac{n}{n+1}\frac{(\rho+n(\rho-\mu))}{\mu}\mathrm{conv}(C\cup(-C)).\end{aligned}$$

Since $\frac{\rho+n(\rho-\mu(\rho))}{\mu(\rho)}$ is increasing in ρ, it follows

$$C\cap(-C)\subset\frac{n}{n+1}\frac{(\rho_*+n(\rho_*-\mu(\rho_*)))}{\mu(\rho_*)}\mathrm{conv}(C\cup(-C)).$$

Finally, from (5) and (9), we obtain

$$C\cap(-C)\subset\psi\frac{n}{n+1}\mathrm{conv}(C\cup(-C)),$$

where

$$\begin{aligned}\psi &= \frac{(n+1)\rho_*}{\frac{n+1}{s+1}(1-s(\rho_*-1))}-n=\frac{(n+1)\frac{n+1-s}{1-n(n-s)}}{\frac{n+1}{s+1}\left(1-s\frac{(n+1)(n-s)}{1-n(n-s)}\right)}-n\\ &=\frac{(n-s+1)(s+1)}{1-(n-s)(n+s(n+1))}-n.\end{aligned}$$

Solving $\psi\frac{n}{n+1}=1$ becomes a quadratic equation in s and the unique positive root is the given constant γ_2 and therefore $\psi\frac{n}{n+1}<1$, whenever $s(C)>\gamma_2$.

(ii) From (6), (11), and (10) we obtain

$$\mu S\subset c^1+S\subset C\subset c^2+\rho S\subset(\rho+n(\rho-\mu))S.$$

Thus, using (iii) of Lemma 3.2 and the fact $\frac{1}{2}(S-S)=\frac{1}{2}((c^1+S)-(c^1+S))\subset\frac{1}{2}(C-C)$, it follows

$$\begin{aligned}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ &\subset\left(\frac{(c^2+\rho S)^\circ-(c^2+\rho S)^\circ}{2}\right)^\circ\subset(\rho+n(\rho-\mu))\left(\frac{S^\circ-S^\circ}{2}\right)^\circ\\ &\subset(\rho+n(\rho-\mu))\frac{n(n+2)}{(n+1)^2}\frac{S-S}{2}\subset(\rho+n(\rho-\mu))\frac{n(n+2)}{(n+1)^2}\frac{C-C}{2}.\end{aligned}$$

Again, using that $\rho+n(\rho-\mu(\rho))$ is increasing in ρ, one obtains

$$\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\subset(\rho_*+n(\rho_*-\mu(\rho_*)))\frac{n(n+2)}{(n+1)^2}\frac{C-C}{2}.$$

Finally, combining (5) and (9), results in

$$\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\subset\zeta\frac{n(n+2)}{(n+1)^2}\frac{C-C}{2},$$

where

$$\begin{aligned}\zeta &= \frac{(n+1)(n+1-s-\frac{n}{s+1}(1-(n-s)(n+s(n+1))))}{1-n(n-s)}\\ &= (n+1)\left(\left(1+\frac{sn}{s+1}\right)\frac{1+n-s}{1-n(n-s)}-n\right).\end{aligned}$$

Similar as in (i), we conclude that

$$\frac{n(n+2)}{(n+1)^2}\zeta = \frac{n(n+2)}{(n+1)}\frac{n+1-s-\frac{n}{s+1}(1-(n-s)(n+s(n+1)))}{1-n(n-s)} < 1,$$

whenever s is bigger than the given constant γ_3.

□

Proof of Theorem 1.6. From Theorem 1.5 we directly obtain $\gamma(n) \leq \gamma_2(n)$ for even n.

In order to obtain a lower bound on $\gamma(n)$ in even dimensions, we provide a suitable family of sets with left-to-right optimal containment in (2) and asymmetry equal to $\gamma_1(n) > n-1$ by extending the construction of the Golden House from [BDG]. Note that this construction also holds for odd $n \geq 3$.

Let $S = \operatorname{conv}(\{p^1,\dots,p^{n+1}\})$ with $p^j \in \mathbb{R}^n$ be a Minkowski centered regular n-simplex such that $\|p^j\| = 1$ and $(p^i)^T p^j = -1/n$ for $1 \leq i < j \leq n+1$. Now, define $C = S \cap H^{\pm}$ with $H^{\pm} := \{x \in \mathbb{R}^n : \pm(p^1-p^2)^T x \leq \eta\}$, where $\eta = (p^1-p^2)^T p \in (0, 1+\frac{1}{n})$ and $p = (1-\lambda)p^1 + \lambda p^2 \in \operatorname{bd}(H^+)$ for some $\lambda \in [0,\frac{1}{2}]$. Then $\eta = 1-\lambda+\frac{1-\lambda}{n}-\frac{\lambda}{n}-\lambda = (1+\frac{1}{n})(1-2\lambda)$ and therefore

$$\lambda = \frac{1+\frac{1}{n}-\eta}{2(1+\frac{1}{n})}. \tag{12}$$

We will show that there exist $\nu \in \mathbb{R}$ and $s \in [1,n]$ such that

$$\nu(p^1+p^2) - \frac{1}{s}C \subset^{opt} C.$$

This can be rewritten as

$$-\frac{1}{s}(C-c) \subset^{opt} C-c \quad \text{with} \quad c = \frac{s}{s+1}\nu(p^1+p^2), \tag{13}$$

which means that c is the Minkowski center of C.

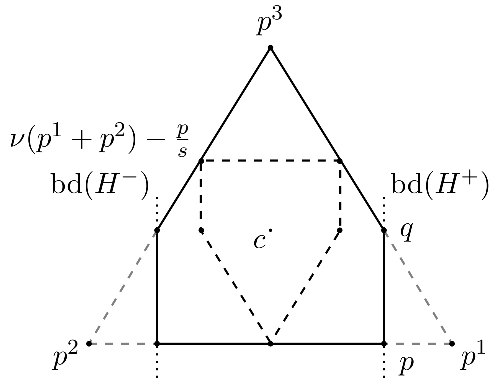


FIGURE 3. Construction from the proof of Theorem 1.6 for $n = 2$: C (black), $\nu(p^1+p^2) - \frac{1}{s}C$ (dashed) and $\mathrm{bd}(H^+)$, $\mathrm{bd}(H^-)$ (dotted).

Our starting point is to compute the vertices of $\nu(p^1 + p^2) - \frac{1}{s}C$ that belong to the boundary of C. Particularly, our goal is that $\nu(p^1+p^2) - \frac{1}{s}((1-\lambda)p^i + \lambda p^j)$, $i, j \in \{1, 2\}$, $i \neq j$, and $\nu(p^1 + p^2) - \frac{1}{s}p^k$, $k = 3, \dots, n+1$, belong to the facets of C with outer normals $-p^1, -p^2$ and $-p^3, \dots, -p^{n+1}$, respectively. Realize that it is sufficient to make sure that $\nu(p^1 + p^2) - \frac{p}{s}$ and $\nu(p^1 + p^2) - \frac{p^3}{s}$ belong to the facets of C, that are contained in facets of S with outer normals $-p^1$ and $-p^3$, respectively. This means we need

$$\left(\nu(p^1+p^2) - \frac{p}{s}\right)^T (-p^1) = \frac{1}{n} \quad \text{and} \quad \left(\nu(p^1+p^2) - \frac{p^3}{s}\right)^T (-p^3) = \frac{1}{n}.$$

The latter two conditions translate into

$$-\left(\nu - \frac{1-\lambda}{s}\right) + \left(\nu - \frac{\lambda}{s}\right)\frac{1}{n} = \frac{1}{n} \quad \text{and} \quad \frac{2\nu}{n} + \frac{1}{s} = \frac{1}{n},$$

which can be simplified to

$$s = n - 2\lambda \quad \text{and} \quad \nu = \frac{\lambda}{2\lambda - n}. \tag{14}$$

Inserting (12), results in

$$s = \frac{n - \frac{1}{n} + \eta}{1 + \frac{1}{n}} \quad \text{and} \quad \nu = \frac{1 + \frac{1}{n} - \eta}{2\left(\frac{1}{n} - n - \eta\right)}. \tag{15}$$

Next, let $q = c + \xi(p^1 - p^2)$ with $\xi > 0$, such that q fulfills both conditions: $q \in \mathrm{bd}(H^+)$ and q belongs to the facet of S with the outer normal $-p^2$ (c.f. Figure 3).

Combining this with the expression of c in (13), we obtain

$$\eta = q^T(p^1 - p^2) = (c + \xi(p^1 - p^2))^T(p^1 - p^2) = \left(\frac{s}{s+1}\nu(p^1+p^2) + \xi(p^1 - p^2)\right)^T (p^1 - p^2)$$

and

$$\frac{1}{n} = q^T(-p^2) = (c + \xi(p^1 - p^2))^T(-p^2) = \left(\frac{s}{s+1}\nu(p^1+p^2) + \xi(p^1 - p^2)\right)^T (-p^2).$$

Using $(p^1+p^2)^T(p^1-p^2)=0$ and $(p^1-p^2)^T(p^1-p^2)=2\left(1+\frac{1}{n}\right)$, gives

$$\left(\frac{s}{s+1}\nu+\xi\right)\frac{1}{n}-\left(\frac{s}{s+1}\nu-\xi\right)=\frac{1}{n} \quad \text{and} \quad \eta=2\xi\left(1+\frac{1}{n}\right). \tag{16}$$

Inserting η from (16) into (15) leads to

$$s=n-1+2\xi \quad \text{and} \quad \nu=\frac{1+\frac{1}{n}-2\xi\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}-n-2\xi\left(1+\frac{1}{n}\right)\right)}=\frac{1-2\xi}{2(1-2\xi-n)}$$

and the obtained ν in (16) gives us

$$\left(\frac{s}{s+1}\frac{1-2\xi}{1-2\xi-n}+\xi\right)\frac{1}{n}-\left(\frac{s}{s+1}\frac{1-2\xi}{1-2\xi-n}-\xi\right)=\frac{1}{n}.$$

Finally, solving for ξ and s results into

$$\xi=\frac{1-n+\sqrt{(n-2)n+5}}{4} \quad \text{and} \quad s=n+2\xi-1=\frac{n-1+\sqrt{(n-2)n+5}}{2}=\gamma_1.$$

One can also verify that every vertex of $\nu(p^1+p^2)-\frac{1}{s}C$ not considered above is contained in $\operatorname{int}(C)$. Together with Proposition 2.1 this guarantees the correctness of (13). Since condition (iii) of Proposition 3.3 is fulfilled for the Minkowski centered $C-c$ at the points $\pm\xi(p^1-p^2)$, (2) gets optimal using $C-c$ and $c-C$ as the arguments, as desired.

Finally, by Lemma 3.4 we see that for $s\leq\gamma_1$ there exists a Minkowski centered $C\in\mathcal{K}^n$ such that $C\cap(-C)\subset^{opt}\operatorname{conv}(C\cup(-C))$ with $s(C)=s$, proving $\gamma(n)\geq\gamma_1$. □

4. Reverse containment

In this section we prove Theorem 1.3. While the proof of Parts (i)-(v) is straightforward, understanding (vi) needs some additional effort: on the one hand, we show that $C=S\cap(-sS)$, where S is a Minkowski centered regular simplex, provides optimality in (vi) for each $s\in[1,n]$, while on the other hand, we find a more intriguing family of sets not fulfilling that optimality (see Example 4.3).

Remark 4.1. *Let $C=S\cap(-sS)$, where $S=\operatorname{conv}(\{p^1,\dots,p^{n+1}\})$ with $\|p^i\|=1$, $i\in[n+1]$, is a regular Minkowski centered simplex and $s\in[1,n]$. Notice that $C=S\cap(-sS)\subset s^2S\cap(-sS)=-sC$. Let $F_i=\{x\in S:(p^i)^Tx=-\frac{1}{n}\}$ and $G_i=S\cap(-sF_i)$ with $i\in[n+1]$. Then $F_i\subset\operatorname{bd}(C)$ and therefore $G_i\subset\operatorname{bd}(-sC)$. Moreover, the points $\frac{s}{n}p^i$ belong to G_i with p^i being a normal vector of $-sC$ in $\frac{s}{n}p^i$, $i\in[n+1]$. Thus, by Proposition 2.1, we conclude that $C\subset^{opt}-sC$ and therefore that C is Minkowski centered with $s(C)=s$.*

Proof of Theorem 1.3. Let $s:=s(C)$. First, one should recognize that Part (i) directly follows from the left-to-right optimality in

$$-C\subset\operatorname{conv}(C\cup(-C))\subset s\cdot(C\cap(-C))\subset sC.$$

For the remaining parts of the proof, we begin by showing the correctness of the containments in Parts (ii) - (v).

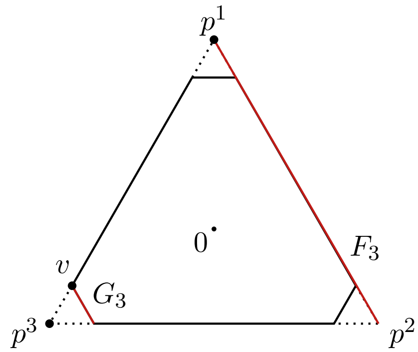


FIGURE 4. Construction from the proof of Part (vi) of Theorem 1.3 for $s = s(C) = 1.5$, $n = 2$: $C = S \cap (-sS)$ (black), S (dotted), F_3 and G_3 (red)

For Part (ii) notice that $-C \subset sC$ directly implies $(s+1)(-C) \subset s(C-C)$, and therefore $-C \subset \frac{2s}{s+1}\frac{C-C}{2}$. Using the 0-symmetry of $\frac{C-C}{2}$, we obtain $\mathrm{conv}(C \cup (-C)) \subset \frac{2s}{s+1}\frac{C-C}{2}$.

Since $-C \subset sC$, we have $\frac{C-C}{2} \subset \frac{s+1}{2}C$ and, again using its symmetry, also $\frac{C-C}{2} \subset \frac{s+1}{2}(C \cap (-C))$, which yields Part (iv).

Parts (iii) and (v) then follow from (ii) and (iv) using Proposition 2.4.

Since we have optimality in Part (i) for every C and since either joining Parts (ii),(iv) or (iii),(v) recovers (i), each of the Parts (ii)-(v) must be optimal for every C, too.

The proof of Part (vi) is a bit more subtle. Let $\frac{C-C}{2} \subset^{opt} \omega \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ for some $\omega \geq 1$. Since

$$\frac{C-C}{2} \subset \frac{s+1}{2}(C \cap (-C)) \subset \frac{s+1}{2}\left(\frac{C^\circ - C^\circ}{2}\right)^\circ,$$

we immediately see that $\omega \leq \frac{s+1}{2}$.

Now, let $C = S \cap (-sS)$ for $s \in [1, n]$ as given in Remark 4.1, q^n, q^{n+1} be the centers of the $(n-2)$-dimensional facets of G_n and G_{n+1}, respectively, which do not contain a vertex belonging to the line segment $[p^n, p^{n+1}]$, and v be a vertex of G_{n+1}. Then v belongs to an edge connecting p^{n+1} with F_{n+1}. Thus, $v = (1-\lambda)p^i + \lambda p^{n+1}$ for some $\lambda \in [0,1]$ and $i \neq n+1$. Since $v \in G_{n+1}$, we know by Remark 4.1 that

$$\frac{s}{n} = ((1-\lambda)p^i + \lambda p^{n+1})^T p^{n+1} = (1-\lambda)\left(\frac{-1}{n}\right) + \lambda.$$

This implies $\lambda = \frac{s+1}{n+1}$ and that the vertices of F_{n+1} are $\frac{n-s}{n+1}p^i + \frac{s+1}{n+1}p^{n+1}$, $i \in [n]$. Hence, using the fact $\sum_{i=1}^{n-1} p^i = -p^n - p^{n+1}$, we obtain

$$\begin{aligned} q^{n+1} &= \frac{1}{n-1}\sum_{i=1}^{n-1}\left(\frac{n-s}{n+1}p^i + \frac{s+1}{n+1}p^{n+1}\right) \\ &= \frac{1}{n-1}\left(\frac{(n-1)(s+1)}{n+1}p^{n+1} + \frac{n-s}{n+1}(-p^n - p^{n+1})\right) \\ &= \frac{1}{n^2-1}((ns-1)p^{n+1} - (n-s)p^n). \end{aligned}$$

For the same reasons

$$-q^n = \frac{1}{n^2-1}((n-s)p^{n+1} - (ns-1)p^n).$$

Now, let $z := \frac{q^{n+1}-q^n}{2} \in \frac{C-C}{2}$. Then

$$z = \frac{1}{2(n^2-1)}\left((ns+n-s-1)p^{n+1} - (ns+n-s-1)p^n\right) = \frac{s+1}{2(n+1)}(p^{n+1}-p^n).$$

For the final argument, let us stress the dependence on s by denoting the points q^n and q^{n+1} by $q^n(s)$ and $q^{n+1}(s)$ as well as C by $C(s)$. On the one hand, it is immediate to check from the above that $q^{n+1}(1) = -q^n(1) = \frac{1}{n+1}(p^{n+1}-p^n)$. On the other hand, since $C(s)$ is increasing in s w.r.t set containment, we have $q^n(1) \in F_{n+1} \cap C(s) \subset \mathrm{bd}(C(s))$, independently of $s \in [1,n]$. Moreover, for the same reasons $-q^n(1) \in \mathrm{bd}(-C(s))$ for every $s \in [1,n]$. It follows

$$\begin{aligned}\|z\|_{\frac{s+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ} &= \frac{2}{s+1}\left(\frac{\|z\|_C + \|z\|_{-C}}{2}\right) = \frac{1}{2}\left(\|\frac{p^{n+1}-p^n}{n+1}\|_C + \|\frac{p^{n+1}-p^n}{n+1}\|_{-C}\right)\\ &= \frac{1}{2}\left(\|q^{n+1}(1)\|_C + \|-q^n(1)\|_{-C}\right) = 1,\end{aligned}$$

where we omitted the s in $C(s)$ again. However, this yields $z \in \mathrm{bd}\left(\frac{s+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right)$, as desired.

□

Only in Part (vi) of Theorem 1.3 the containment may not always be optimal. Below we give two examples: the first one shows that in 2-space all regular k-gons achieve optimality, while the second one provides a construction of sets in arbitrary dimensions where the containment is not optimal.

Example 4.2. *Let $C \subset \mathbb{R}^2$ be a Minkowski centered regular k-gon with odd k. Then*

$$\frac{C-C}{2} \subset^{opt} \frac{s(C)+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ.$$

Proof. Let $r(C)$ and $R(C)$ be the euclidean in- and circumradius of C, respectively. Assume w.l.o.g. that $R(C)=1$. Since the Minkowski center of C coincides with the in- and circumcenter, we can easily conclude that $s(C) = \frac{R(C)}{r(C)}$. Now, since for any k-gon $r(C) = R(C)\cos\left(\frac{\pi}{k}\right)$, it follows

$$s := s(C) = \frac{R(C)}{R(C)\cos\left(\frac{\pi}{k}\right)} = \frac{1}{\cos\left(\frac{\pi}{k}\right)}.$$

We choose a vertex $p = \frac{u-v}{2}$ of $\frac{C-C}{2}$, where u and v are vertices of C. Then, since C is Minkowski centered and $R(C)=1$, we have $\|u\| = \|v\| = 1$, $C^\circ = \rho(-C)$ for some $\rho > 0$, $r(C^\circ) = 1$, and $\rho = s$. Since C° is again a regular k-gon, any edge of $\left(\frac{C^\circ-C^\circ}{2}\right)^\circ$ is of the same distance from the origin.

Using $C^\circ = s(-C)$, we have that $w = sp$ is a vertex of $\frac{C^\circ - C^\circ}{2}$ and therefore $\{x \in \mathbb{R}^n : w^T x = \|w\|^2\}$ determines an edge of $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ with outer-normal vector w. This implies $\frac{C^\circ - C^\circ}{2} \subset^{opt} \|w\|^2 \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$. Since $\cos\left(\frac{\pi}{k}\right) = \frac{1}{s}$ and $R(C) = 1$, we obtain

$$\begin{aligned}\|w\|^2 &= \frac{s^2}{4}\|u-v\|^2 = \frac{s^2}{4}\left(\|u\|^2 + \|v\|^2 - 2u^T v\right) = \frac{s^2}{4}\left(2 + 2\cos\left(\frac{\pi}{k}\right)\right) \\ &= \frac{s^2}{2}\left(1 + \frac{1}{s}\right).\end{aligned}$$

Altogether,

$$\frac{C-C}{2} = \frac{1}{s}\frac{C^\circ - C^\circ}{2} \subset^{opt} \frac{1}{s}\frac{s^2}{2}\left(1+\frac{1}{s}\right)\left(\frac{C^\circ - C^\circ}{2}\right)^\circ = \frac{s+1}{2}\left(\frac{C^\circ - C^\circ}{2}\right)^\circ.$$

□

Example 4.3. *In the following we provide a planar construction, which can simply be embedded in higher dimensions, keeping the Minkowski center to be 0. This keeps its asymmetry value and the same factor for the containment of the arithmetic mean within the harmonic mean. Thus, the construction essentially provides a family of sets in arbitrary dimensions with asymmetry $s \in (1,2)$, such that the arithmetic mean is contained in the interior of the harmonic mean scaled by $\frac{s+1}{2}$.*

Let $K = S \cap (-sS)$, where S is a Minkowski centered regular triangle and $s \in (1,2)$. By $p^1, \dots, p^6$ we denote the vertices of K, counted in clockwise order, such that $[p^i, p^{i+1}]$ with $i = 1,3,5$ are the shorter edges of K. Let

$$C = \operatorname{conv}(\{p^2, p^4, p^6, \frac{p^1+p^2}{2}, \frac{p^3+p^4}{2}, \frac{p^5+p^6}{2}, \frac{p^1-p^4}{s+1}, \frac{p^3-p^6}{s+1}, \frac{p^5-p^2}{s+1}\})$$

(c.f. Figure 5).

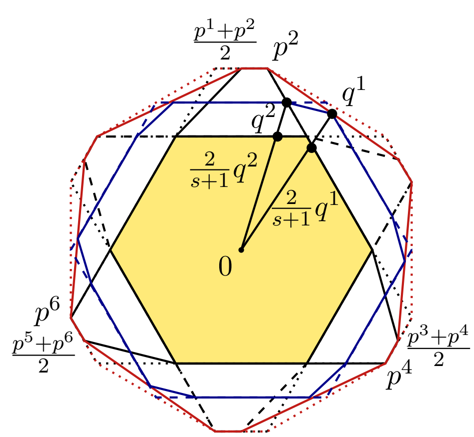


FIGURE 5. Construction from Example 4.3, $s = 1.5$: C (black), $-C$ (black dashed), K, $-K$ (black dotted), $\operatorname{conv}(C \cup (-C))$ (red), $\operatorname{conv}(K \cup (-K))$ (red dotted), $\frac{C-C}{2}$ (blue), $\frac{K-K}{2}$ (blue dashed), $C \cap (-C) = K \cap (-K)$ (yellow).

Note that

$$\frac{K-K}{2} = \operatorname{conv}\left(\left\{\pm\frac{p^1-p^4}{2}, \pm\frac{p^2-p^5}{2}, \pm\frac{p^3-p^6}{2}\right\}\right) \quad \text{and}$$

$$K\cap(-K) = \operatorname{conv}\left(\left\{\pm\frac{p^1-p^4}{s+1}, \pm\frac{p^2-p^5}{s+1}, \pm\frac{p^3-p^6}{s+1}\right\}\right).$$

Moreover, K and C are Minkowski centered with $s(K) = s(C) = s$ and

$$\frac{C-C}{2} = \operatorname{conv}\left(\left\{\pm\frac{2p^2-p^5-p^6}{4}, \pm\frac{2p^4-p^1-p^2}{4}, \pm\frac{2p^6-p^3-p^4}{4}\right.\right.,$$
$$\left.\left.\pm\frac{(s+2)p^2-p^5}{2(s+1)}, \pm\frac{(s+2)p^6-p^3}{2(s+1)}, \pm\frac{(s+2)p^4-p^1}{2(s+1)}\right\}\right).$$

Now, assume

$$\frac{C-C}{2} \subset^{opt} \frac{s+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ.$$

Then there would exist a vertex of $\frac{C-C}{2}$, which also belongs to $\operatorname{bd}\left(\frac{s+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right)$. Since the vertices of $\frac{C-C}{2}$ have two different types, we have to consider both of them. Denote $q^1 := \frac{2p^2-p^5-p^6}{4}$ and $q^2 := \frac{(s+2)p^2-p^5}{2(s+1)}$. Then, for both $i=1,2$,

$$\frac{2}{s+1}q^i \in \frac{2}{s+1}\frac{K-K}{2} = K\cap(-K) = C\cap(-C) = \operatorname{conv}\left(\left\{\pm\frac{p^2-p^5}{s+1}, \pm\frac{p^3-p^6}{s+1}, \pm\frac{p^4-p^1}{s+1}\right\}\right).$$

Remember, that by Part (ii) of Lemma 3.1, we have $C\cap(-C) \subset^{opt} \left(\frac{C^\circ-C^\circ}{2}\right)^\circ$. Thus, we would obtain

$$\pm\frac{p^2-p^5}{s+1}, \pm\frac{p^3-p^6}{s+1}, \pm\frac{p^4-p^1}{s+1} \in (C\cap(-C))\cap\operatorname{bd}\left(\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right).$$

If $\frac{2}{s+1}q^1 \in \operatorname{bd}\left(\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right)$ would be true, then

$$\frac{p^2-p^5}{s+1}, \frac{p^3-p^6}{s+1}, \frac{2}{s+1}q^1 \in (C\cap(-C))\cap\operatorname{bd}\left(\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right),$$

which is only possible if the full edge $[\frac{p^2-p^5}{s+1}, \frac{p^3-p^6}{s+1}] \subset (C\cap(-C))\cap\operatorname{bd}\left(\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right)$. By the symmetries of C, we would conclude that $C\cap(-C) = \left(\frac{C^\circ-C^\circ}{2}\right)^\circ$, obtaining a contradiction to Part (iii) of Theorem 1.3. In case $\frac{2}{s+1}q^2 \in \operatorname{bd}\left(\left(\frac{C^\circ-C^\circ}{2}\right)^\circ\right)$, we obtain a similar conclusion (c.f. Figure 5). Thus, C fulfills $\frac{C-C}{2} \subset \operatorname{int}(\frac{s+1}{2}\left(\frac{C^\circ-C^\circ}{2}\right)^\circ)$.

Remark 4.4. *Let $C\in\mathcal{K}^n$ with $s(C)=s$ and $\omega(C)>0$ be such that $\frac{C-C}{2} \subset^{opt} \omega(C)\left(\frac{C^\circ-C^\circ}{2}\right)^\circ$. Then there exists $x\in\mathbb{R}^n$ such that*

$$\omega(C)\|x\|_{\frac{C-C}{2}} = \|x\|_{\left(\frac{C^\circ-C^\circ}{2}\right)^\circ}.$$

On the one hand, $\frac{x}{\|x\|_C} \in \operatorname{bd}(C)$ and $\frac{x}{\|x\|_{-C}} \in \operatorname{bd}(-C)$, implying $\frac{1}{2}\left(\frac{1}{\|x\|_C}+\frac{1}{\|x\|_{-C}}\right)x \in \frac{C-C}{2}$. It follows $\|x\|_{\frac{C-C}{2}} \le \left(\frac{\frac{1}{\|x\|_C}+\frac{1}{\|x\|_{-C}}}{2}\right)^{-1}$.

On the other hand, $\|x\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ} = \frac{\|x\|_C + \|x\|_{-C}}{2}$. *Therefore,*

$$\omega(C) = \frac{\|x\|_{\left(\frac{C^\circ - C^\circ}{2}\right)^\circ}}{\|x\|_{\frac{C-C}{2}}} \geq \frac{\|x\|_C + \|x\|_{-C}}{2} \frac{\frac{1}{\|x\|_C} + \frac{1}{\|x\|_{-C}}}{2} = \frac{(\|x\|_C + \|x\|_{-C})^2}{4\|x\|_C\|x\|_{-C}}.$$

Let w.l.o.g. $\|x\|_C \geq \|x\|_{-C}$ *and* $\rho := \frac{\|x\|_C}{\|x\|_{-C}}$. *Then* $\omega(C) \geq \frac{(\rho+1)^2}{4\rho}$ *and since* C *is Minkowski centered, we have* $1 \leq \rho \leq s$, *which implies that* $\frac{(\rho+1)^2}{4\rho}$ *attains its maximum for* $\rho = s$. *We actually achieve this upper bound, whenever* $x \in C \cap \mathrm{bd}(-sC)$.

Remark 4.5. *Let* $P \subset \mathbb{R}^n$ *be a Minkowski centered polytope. By Proposition 2.1 (and the remark after it) there exist* $a^i \in \mathbb{R}^n$ *and vertices* x^i, *such that* $H^{\leq}_{a^i,1}$ *define facets of* P, $i \in [k+1]$ *and* $s(P) \leq k \leq n$, *such that* $0 \in \mathrm{conv}(\{a^1, \dots, a^{k+1}\})$ *and* $-x^i \in H_{a^i,s(P)}$, $i \in [k+1]$. *Now, we do not only have* $\frac{P-P}{2} \subset^{opt} \frac{(s(P)+1)}{2}(P \cap (-P))$ *from Theorem 1.3, but also the fact that for any vertex* y *of the facet* $P \cap H_{a^i,1}$ *of* P, *we have* $\frac{y-x^i}{2} \in H_{a^i,\frac{s(P)+1}{2}} \cap \frac{P-P}{2} \cap \frac{s(P)+1}{2}(P \cap (-P))$, *i.e.* $\frac{P-P}{2}$ *touches* $\frac{s(P)+1}{2}(P \cap (-P))$ *in all the facets* $\frac{s(P)+1}{2}(P \cap (-P)) \cap H_{a^i,\frac{s(P)+1}{2}}$ *with a full facet (c.f.* [BG2, Lemma 2.8], *where this fact is shown for simplices).*

Remark 4.6. *It is well known, that* $s(K) = \inf_{C \in \mathcal{K}^n_0} d_{BM}(K,C)$ *for every* $K \in \mathcal{K}^n$ *(see, e.g.,* [Gr]*). Furthermore, in* [BG, Prop. 3.1] *it is shown that this infimum is always attained by* $C = \frac{K-K}{2}$. *In general, if* $C \in \mathcal{K}^n_0$ *and* $K \in \mathcal{K}^n$, *we see from the definition of the Banach-Mazur distance that*

$$d_{BM}(K,C) = s(K) \Longleftrightarrow \exists\, L \in GL(n), t^1, t^2 \in \mathbb{R}^n \text{ s.t. } -K - t^1 \subset L(C) \subset s(K)K + t^2.$$

Since $L(C)$ *is symmetric, we may symmetrize and replace the right-hand side above by*

$$\exists\, L \in GL(n), t^1, t^2 \in \mathbb{R}^n \text{ s.t. } \mathrm{conv}((K+t^1) \cup (-K-t^1)) \subset L(C) \subset s(K)((K+t^2) \cap (-K-t^2)).$$

For a Minkowski concentric K *we now immediately obtain that all four choices*

$$C \in \left\{ K \cap (-K), \left(\frac{K^\circ - K^\circ}{2}\right)^\circ, \frac{K-K}{2}, \mathrm{conv}(K \cup (-K)) \right\}$$

of symmetrizations of K *considered in this paper fulfill* $d_{BM}(K,C) = s(K)$ *and are therefore minimizers for the Banach-Mazur distance between* K *and* $\mathcal{K}^n_0$.

Moreover, with the help of the reverse containments from Theorem 1.3 we obtain some upper bounds on the Banach-Mazur distances of pairs of these symmetrizations, e.g.

$$d_{BM}(K \cap (-K), \mathrm{conv}(K \cup (-K)) \leq s(K) \quad \text{or}$$
$$d_{BM}(K \cap (-K), \left(\frac{K^\circ - K^\circ}{2}\right)^\circ) \leq \frac{2s(K)}{s(K)+1}$$

However, this bounds do not even have to be tight when the containments between those sets are. E.g. is the Banach-Mazur distance of any two of the symmetrizations of a regular triangle in the plane exactly 1, as they are all regular hexagons.

5. Improving the containment factors in the forward direction

Proof of Theorem 1.7. We start showing Part a).

(i) First, we show that $\alpha_1(s) \geq \frac{2}{s+1}$ independently of n. By the definition of $\alpha(C)$ and Part (v) of Theorem 1.3 we have $C \cap (-C) \subset^{opt} \alpha(C) \cdot \operatorname{conv}(C \cup (-C)) \subset^{opt} \frac{\alpha(C)(s+1)}{2} \cdot \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$, while Part (ii) of Lemma 3.1 gives $C \cap (-C) \subset^{opt} \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$. Hence, $\alpha(C)$ must always be at least $\frac{2}{s+1}$.

Next, we show $\alpha_1(s) = \frac{2}{s+1}$ in any dimension if $s \leq 2$. First, let $n = 2$ and $C = S \cap (-sS)$ with $s \in [1,2]$, where $S = \operatorname{conv}\left(\{p^1, p^2, p^3\}\right)$ is a Minkowski centered regular triangle with $\|p^i\| = 1, i = 1,2,3$. Now, let v be the vertex of $C \cap (-C)$ with $v \in \operatorname{pos}\left(\{p^2, -p^3\}\right)$ and $\mu \geq 1$, such that $\mu v \in \operatorname{bd}(\operatorname{conv}(C \cup (-C)))$. Finally, let q be a vertex of $\operatorname{conv}(C \cup (-C))$, such that $q \in [p^2, v]$.

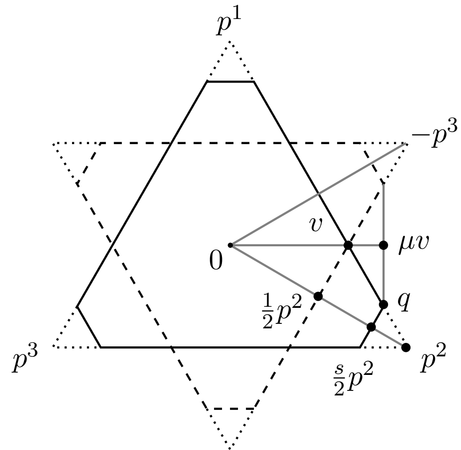


Figure 6. Construction from the proof of Part (i) of Theorem 1.7 for $s = 1.5$: $C = S \cap (-sS)$ (black), $-C$ (dashed), S and $-S$ (dotted).

Since $\|p^2\| = 1$, we have

$$\|v - q\| = \frac{\left\|p^2 - \left(\frac{1}{2}p^2\right)\right\| - \left\|p^2 - \left(\frac{s}{2}p^2\right)\right\|}{\cos(\pi/6)} = \frac{2}{\sqrt{3}}\left(\frac{1}{2} - \left(1 - \frac{s}{2}\right)\right) = \frac{s-1}{\sqrt{3}}$$

Since $\|v\| = R((S \cap (-S)) = \frac{1}{\sqrt{3}}$, we obtain

$$\frac{\|\mu v\|}{\|v\|} = 1 + \frac{\|\mu v - v\|}{\|v\|} = 1 + \frac{\|v - q\| \sin(\pi/6)}{\|v\|} = 1 + \frac{\frac{1}{2}\|v - q\|}{\frac{1}{\sqrt{3}}} = 1 + \frac{s}{2} - \frac{1}{2},$$

which implies

$$\alpha(C) = \frac{1}{1 + \frac{s}{2} - \frac{1}{2}} = \frac{2}{s+1}.$$

For $n \geq 3$ we can simply embed the above C keeping the Minkowski center to be still 0 into n-space. This keeps its asymmetry value and also the correct factor for the containment between $C \cap (-C)$ and $\operatorname{conv}(C \cup (-C))$.

(ii) By the definition of $\gamma(n)$, we have $\alpha_2(s) = 1$ for $s \le \gamma_1(n)$, while $\alpha_2(s) \le \psi\frac{n}{n+1}$, if $s > \gamma_2(n)$, follows from Part (i) of Theorem 1.5 (see Figure 8).

We now show that if $n = 2$ and $s \ge \varphi$ then there exists a Minkowski centered C with $s(C) = s$ for which $\alpha(C) = \frac{s}{s^2-1}$. To do so, consider the pentagon

$$C = \operatorname{conv}\left(\left\{\begin{pmatrix} \pm 1 \\ (\varphi+1)(2-s-\frac{1}{s+1}) \end{pmatrix}, \begin{pmatrix} \pm 1 \\ -\frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix}\right\}\right),$$

with $s \ge \varphi$ (see Figure 7).

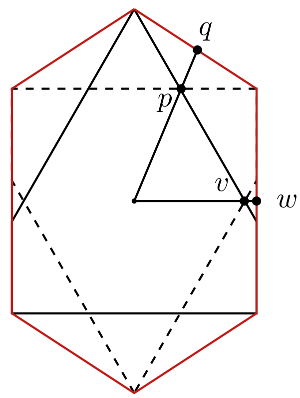


FIGURE 7. Construction from the proof of Part a) (ii) of Theorem 1.7 for $s = 1.7$: C (black), $-C$ (black dashed), and $\operatorname{conv}(C \cup (-C))$ (red).

Since $-\frac{1}{s}C \subset C$ with

$$-\frac{1}{s}\begin{pmatrix} \pm 1 \\ -\frac{\varphi+1}{s+1} \end{pmatrix} \in \left[\begin{pmatrix} \mp 1 \\ (\varphi+1)(2-s-\frac{1}{s+1}) \end{pmatrix}, \begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix}\right]$$

and

$$-\frac{1}{s}\begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix} \in \left[\begin{pmatrix} 1 \\ -\frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} -1 \\ -\frac{\varphi+1}{s+1} \end{pmatrix}\right],$$

we obtain from Proposition 2.1 that C is Minkowski centered and $s(C) = s$.

Let p and v be vertices of $C \cap (-C)$, such that

$$\{p\} = \left[\begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} 1 \\ (\varphi+1)(2-s-\frac{1}{s+1}) \end{pmatrix}\right] \cap \left[\begin{pmatrix} 1 \\ \frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{\varphi+1}{s+1} \end{pmatrix}\right] \quad \text{and}$$

$$\{v\} = \left[\begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} 1 \\ (\varphi+1)(2-s-\frac{1}{s+1}) \end{pmatrix}\right] \cap \left[\begin{pmatrix} 0 \\ -s\frac{\varphi+1}{s+1} \end{pmatrix}, \begin{pmatrix} 1 \\ -(\varphi+1)(2-s-\frac{1}{s+1}) \end{pmatrix}\right],$$

while $q = \nu_1 p, w = \nu_2 v \in \operatorname{bd}(\operatorname{conv}(C \cup (-C)))$ for some $\nu_1, \nu_2 > 1$.

On the one hand, for some $x \in \mathbb{R}$ and some $\lambda \in [0,1]$, we have

$$p = \begin{pmatrix} x \\ \frac{\varphi+1}{s+1} \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \frac{s(\varphi+1)}{s+1} \end{pmatrix} + (1-\lambda)\begin{pmatrix} 1 \\ \frac{(\varphi+1)(-s^2+s+1)}{s+1} \end{pmatrix}.$$

Solving for λ and x results in $\lambda = \frac{s}{s+1}$, $x = \frac{1}{s+1}$, and

$$p = \frac{1}{s+1}\begin{pmatrix} 1 \\ \varphi+1 \end{pmatrix}.$$

On the other hand, there exists $\lambda \in [0,1]$, such that

$$q = \lambda \begin{pmatrix} 0 \\ \frac{s(\varphi+1)}{s+1} \end{pmatrix} + (1-\lambda)\begin{pmatrix} 1 \\ \frac{\varphi+1}{s+1} \end{pmatrix} = \nu_1 \begin{pmatrix} \frac{1}{s+1} \\ \frac{\varphi+1}{s+1} \end{pmatrix},$$

which results in $\lambda = \frac{1}{2}$, $\nu_1 = \frac{s+1}{2}$, and

$$q = \frac{1}{2}\begin{pmatrix} 1 \\ \varphi+1 \end{pmatrix}.$$

In order to compute v, notice that

$$v = \begin{pmatrix} x \\ 0 \end{pmatrix} = (1-\lambda)\begin{pmatrix} 0 \\ s\frac{\varphi+1}{s+1} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ (\varphi+1)\left(2 - s - \frac{1}{s+1}\right) \end{pmatrix},$$

for some $x \in \mathbb{R}$ and $\lambda \in [0,1]$. Solving for x and λ again gives $\lambda = x = \frac{s}{s^2-1}$, and $v = \begin{pmatrix} \frac{s}{s^2-1} \\ 0 \end{pmatrix}$. Hence,

$$w = \frac{1}{2}\left(\begin{pmatrix} 1 \\ -\frac{\varphi+1}{s+1} \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{\varphi+1}{s+1} \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \nu_2 \begin{pmatrix} \frac{s}{s^2-1} \\ 0 \end{pmatrix},$$

which means that $\nu_2 = \frac{s^2-1}{s}$.

Due to the symmetries of C, we conclude

$$\alpha(C) = \max\left\{\frac{1}{\nu_1}, \frac{1}{\nu_2}\right\} = \max\left\{\frac{2}{s+1}, \frac{s}{s^2-1}\right\} = \frac{s}{s^2-1}.$$

We now proceed with Part b).

(i) On the one hand, by Part (iv) of Theorem 1.3, we have

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \beta(C)\frac{C-C}{2} \subset^{opt} \beta(C)\frac{s+1}{2}C \cap (-C).$$

On the other hand, from Part (iii) of Theorem 1.3, we know

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{2s}{s+1}C \cap (-C).$$

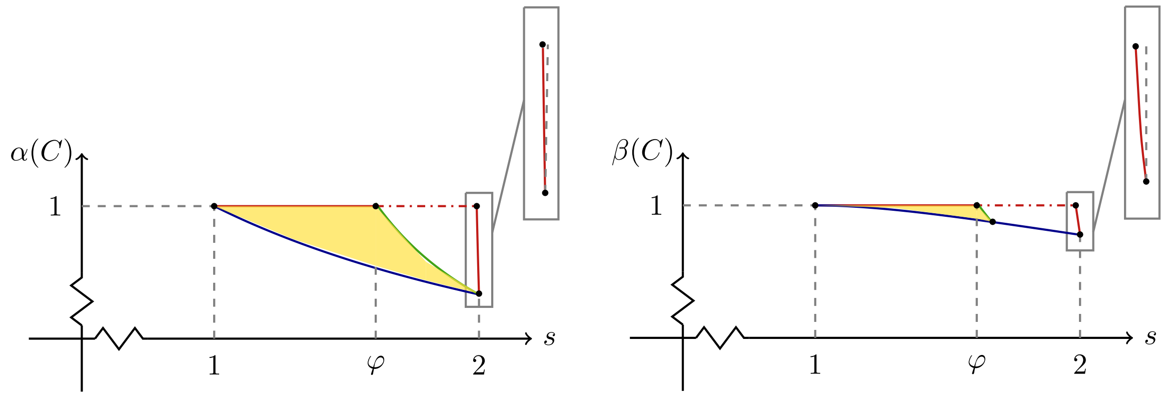
Hence, $\frac{2s}{s+1} \le \beta(C)\frac{s+1}{2}$, which implies $\beta_1(s) \ge \frac{4s}{(s+1)^2}$.

Now, consider the hexagon

$$C = \mathrm{conv}\left(\left\{\begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(1-\frac{s}{2}\right) \\ \frac{s}{2} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(\frac{s+1}{2}\right) \\ \frac{1-s}{2} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix}\right\}\right), \quad s \in [1,2]$$

(see left part of Figure 9). Since $-\frac{1}{s}C \subset C$ with

$$-\frac{1}{s}\begin{pmatrix} \frac{\sqrt{3}}{3}\left(1-\frac{s}{2}\right) \\ \frac{s}{2} \end{pmatrix} \in \left[\begin{pmatrix} -\frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix}\right]$$



$\alpha_1(s) = \frac{2}{s+1}$ (blue); $\alpha_2(s) = 1$ for $s \le \varphi$ (red), $\alpha_2(s) \ge \frac{s}{s^2-1}$ for $s \ge \varphi$ (green), while $\alpha_2(s) < 1$ for $s > \varphi$ and $\alpha_2 \le \frac{2}{3}\psi$ for $s \ge \gamma_2$ (red).

$\beta_1(s) = \frac{4s}{(s+1)^2}$ (blue); $\beta_2(s) = 1$ for $s \le \varphi$ (red), $\beta_2(s) \ge \max\{\frac{s}{s^2-1}, \frac{4s}{(s+1)^2}\}$ for $s \ge \varphi$ (green/blue), $\beta_2(s) < 1$ for $s > \varphi$ and $\beta_2(s) \le \frac{8}{9}\zeta$ for $s \ge \gamma_3$ (red).

FIGURE 8. Regions of possible values for the parameters $\alpha(C)$ and $\beta(C)$ for Minkowski centered $C \in \mathcal{K}^2$ with $s(C) = s$ from Theorem 1.7.

and

$$-\frac{1}{s}\begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix} \in \left[\begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(1-\frac{s}{2}\right) \\ \frac{s}{2} \end{pmatrix}, \begin{pmatrix} \mp\frac{\sqrt{3}}{3}\left(\frac{s+1}{2}\right) \\ \frac{1-s}{2} \end{pmatrix}\right],$$

we obtain from Proposition 2.1 that C is Minkowski centered and $s(C) = s$. Since C is a hexagon with three pairs of parallel edges, it turns out that its arithmetic mean stays to be a hexagon:

$$\frac{C-C}{2} = \mathrm{conv}\left(\left\{\begin{pmatrix} \pm\frac{\sqrt{3}}{12}(s+1) \\ \frac{s+1}{4} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{12}(s+1) \\ -\frac{s+1}{4} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(\frac{s+1}{2}\right) \\ 0 \end{pmatrix}\right\}\right).$$

The next step to do is to calculate $C^\circ =: \mathrm{conv}(\{q^1,\dots,q^6\})$. Since the vertices of C are the outer normals of the edges of C° and $0 \in \mathrm{int}(C)$, we obtain the vertices of C° as the solution of pairs of inequalities of the form $(v^i)^T x = 1$, built from consecutive vertices v^i of C. Moreover, we make use of the fact that C, and therefore also C°, is symmetric w.r.t. the y-axis. Hence, it suffices to calculate four of the vertices of C°. Let q^1 fulfill the equations

$$(q^1)^T\begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(1-\frac{s}{2}\right) \\ \frac{s}{2} \end{pmatrix} = 1,$$

which obviously needs $q^1 = \begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix}$. For q^2 we demand

$$(q^2)^T\begin{pmatrix} \frac{\sqrt{3}}{3}\left(1-\frac{s}{2}\right) \\ \frac{s}{2} \end{pmatrix} = (q^2)^T\begin{pmatrix} \frac{\sqrt{3}}{3}\left(\frac{s+1}{2}\right) \\ \frac{1-s}{2} \end{pmatrix} = 1,$$

and obtain $q^2 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$. The third vertex q^3 should fulfill

$$(q^3)^T \begin{pmatrix} \frac{\sqrt{3}}{3}(\frac{s+1}{2}) \\ \frac{1-s}{2} \end{pmatrix} = (q^3)^T \begin{pmatrix} \frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix} = 1,$$

resulting in $q^3 = \begin{pmatrix} \frac{\sqrt{3}}{s} \\ -\frac{1}{s} \end{pmatrix}$. Finally, for q^4 we have to solve

$$(q^4)^T \begin{pmatrix} \pm\frac{\sqrt{3}}{3}\left(s-\frac{1}{2}\right) \\ -\frac{1}{2} \end{pmatrix} = 1,$$

which gives $q^4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

Altogether, we obtain

$$C^\circ = \operatorname{conv}\left(\left\{\begin{pmatrix} 0 \\ \frac{2}{s} \end{pmatrix}, \begin{pmatrix} \pm\sqrt{3} \\ 1 \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{s} \\ -\frac{1}{s} \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right\}\right).$$

Since C° has no parallel edges, $\frac{C^\circ - C^\circ}{2}$ is a 12-gon, which computes to

$$\frac{C^\circ - C^\circ}{2} = \operatorname{conv}\left(\left\{\begin{pmatrix} 0 \\ \pm\frac{s+1}{s} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{2} \\ \pm\frac{3}{2} \end{pmatrix}, \begin{pmatrix} \pm\sqrt{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\sqrt{3}\frac{s+1}{2s} \\ \pm\frac{s+1}{2s} \end{pmatrix}\right\}\right).$$

Note that six vertices of $\frac{C^\circ - C^\circ}{2}$ are rescales of the outer-normals of $\frac{C-C}{2}$, thus the 12-gon $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ has six edges parallel to the corresponding edges of $\frac{C-C}{2}$. Since $\begin{pmatrix} 0 \\ \pm\frac{s+1}{s} \end{pmatrix}$ is a vertex of $\frac{C^\circ - C^\circ}{2}$, there exists an edge of $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ with the outer-normal $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{s}{s+1} \end{pmatrix} \in \operatorname{bd}\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$. Moreover, since $\begin{pmatrix} \pm\frac{\sqrt{3}}{12}(s+1) \\ \frac{s+1}{4} \end{pmatrix}$ are vertices of $\frac{C-C}{2}$, there exists an edge of $\frac{C-C}{2}$ with the outer-normal $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \frac{s+1}{4} \end{pmatrix} \in \operatorname{bd}\left(\frac{C-C}{2}\right)$. Thus, $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ \subset^{opt} \frac{4s}{(s+1)^2}\frac{C-C}{2}$, proving that $\beta_1(s) = \frac{4s}{(s+1)^2}$ for all $s \in [1,2]$. For higher dimensions we can embed the above construction keeping the Minkowski center 0.

(ii) By the definition of $\gamma(n)$, we have $\beta_2(s) = 1$ for $s \le \gamma_1(n)$, while $\beta_2(s) \le \zeta\frac{n(n+2)}{(n+1)^2}$ for $s > \gamma_3(n)$ follows from Part (ii) of Theorem 1.5.

Recognize that the factor $\zeta\frac{n(n+2)}{(n+1)^2}$ in Part (ii) of Theorem 1.5 becomes greater than 1 for $s < \gamma_3$. However, for $s \in (\gamma_2, \gamma_3]$ the factor $\psi\frac{n}{n+1}$ from Part (i) of Theorem 1.5 is strictly less than 1. In such a case $C \cap (-C)$ is not optimally contained in $\operatorname{conv}(C \cup (-C))$, which by Theorem 1.2 implies that $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ is not optimally contained in $\frac{C-C}{2}$ either. Hence $\beta_2(s) < 1$ for $s > \gamma_2$.

Finally, we provide a construction of sets in $\mathbb{R}^2$, showing that $\beta_2(s) \ge \max\{\frac{s}{s^2-1}, \frac{4s}{(s+1)^2}\}$.

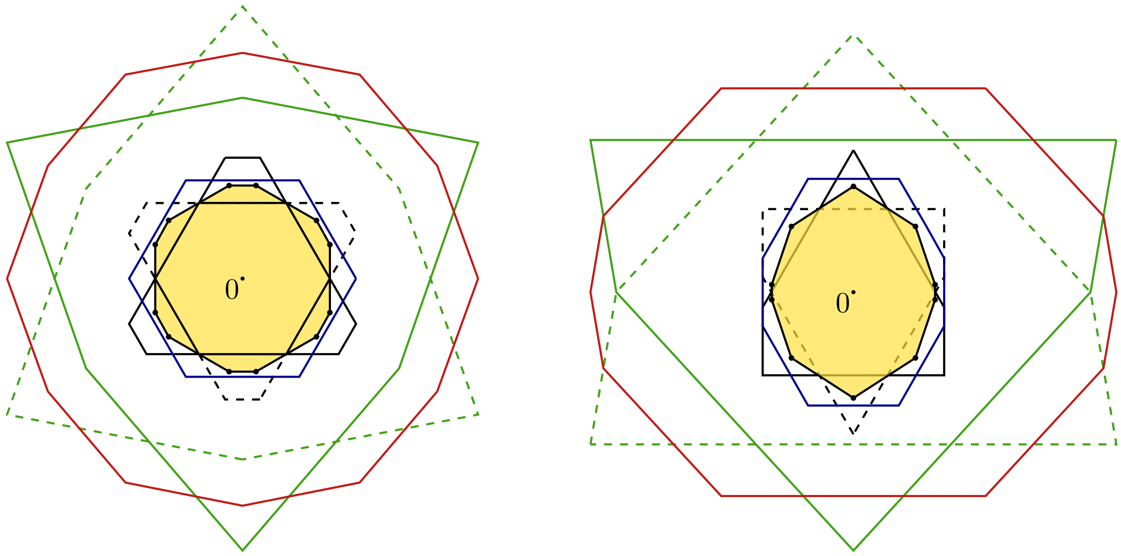


FIGURE 9. The two constructions from the proof of Part b) of Theorem 1.7 for $s = 1.7$: C (black) and $-C$ (black dashed), $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ (yellow), $\frac{C-C}{2}$ (blue), C° (green), $-C^\circ$ (green dashed) and $\frac{C^\circ - C^\circ}{2}$ (red).

Let $C = \mathrm{conv}\left(\left\{\begin{pmatrix} \pm\frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{2}\left(-\frac{1}{s+1}+2-s\right)\end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)}\end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3s}{2(s+1)}\end{pmatrix}\right\}\right)$, $s \in [\varphi, 2]$ (see right part of Figure 9).

Since $-\frac{1}{s}C \subset C$ with

$$-\frac{1}{s}\begin{pmatrix} 0 \\ \frac{3s}{2(s+1)}\end{pmatrix} \in \left[\begin{pmatrix} -\frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)}\end{pmatrix}, \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)}\end{pmatrix}\right] \quad \text{and}$$

$$-\frac{1}{s}\begin{pmatrix} \pm\frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)}\end{pmatrix} \in \left[\begin{pmatrix} \mp\frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{2}\left(-\frac{1}{s+1}+2-s\right)\end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3s}{2(s+1)}\end{pmatrix}\right],$$

Proposition 2.1 implies that C is Minkowski centered and $s(C) = s$.

Note that C has five edges, two of which are parallel. Hence, $\frac{C-C}{2}$ has four different pairs of parallel edges and eight vertices:

$$\begin{pmatrix} \pm\frac{\sqrt{3}}{2}(s-1) \\ \pm\frac{3}{4}(2-s)\end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{2}(s-1) \\ \mp\frac{3}{4}(2-s)\end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{4}(s-1) \\ \pm\frac{3}{4}\end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}}{4}(s-1) \\ \mp\frac{3}{4}\end{pmatrix}.$$

Next, we determine $C^\circ =: \mathrm{conv}(\{q^1, q^2, q^3, q^4, q^5\})$. Since the vertices of C are the outer normals of the edges of C° and $0 \in \mathrm{int}(C)$, we obtain the vertices of C° as the solution of pairs of inequalities of the form $(v^i)^T x = 1$, such that the v^i are consecutive vertices of C. Since C is symmetric w.r.t. the y-axis, so is C°. Hence, it suffices to calculate q^1, q^2, q^3.

Let q^1 be such that

$$(q^1)^T \begin{pmatrix} 0 \\ \frac{3s}{2(s+1)} \end{pmatrix} = (q^1)^T \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{2}\left(-\frac{1}{s+1}+2-s\right) \end{pmatrix} = 1.$$

This gives $q^1 = \begin{pmatrix} \frac{2}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \frac{2}{3}\left(\frac{s+1}{s}\right) \end{pmatrix}$. Now, assume q^2 to be such that

$$(q^2)^T \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)} \end{pmatrix} = (q^2)^T \begin{pmatrix} \frac{-\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)} \end{pmatrix} = 1.$$

One obtains $q^2 = \begin{pmatrix} 0 \\ -\frac{2}{3}(s+1) \end{pmatrix}$. For q^3 we assume

$$(q^3)^T \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{2}\left(-\frac{1}{s+1}+2-s\right) \end{pmatrix} = (q^3)^T \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{2(s+1)} \end{pmatrix} = 1.$$

Then $q^3 = \begin{pmatrix} \frac{2}{\sqrt{3}(s-1)} \\ 0 \end{pmatrix}$ and altogether

$$C^\circ = \operatorname{conv}\left(\left\{\begin{pmatrix} \pm\frac{2}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \frac{2}{3}\left(\frac{s+1}{s}\right) \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{2}{3}(s+1) \end{pmatrix}, \begin{pmatrix} \pm\frac{2}{\sqrt{3}(s-1)} \\ 0 \end{pmatrix}\right\}\right).$$

Note that C° has five edges, none of which are parallel, thus $\frac{C^\circ - C^\circ}{2}$ must have five different pairs of parallel edges and ten vertices:

$$\begin{pmatrix} \pm\frac{1}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \pm\frac{1}{3}\frac{(s+1)^2}{s} \end{pmatrix}, \begin{pmatrix} \pm\frac{1}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \mp\frac{1}{3}\frac{(s+1)^2}{s} \end{pmatrix}, \begin{pmatrix} \pm\frac{2}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\frac{1}{\sqrt{3}}\frac{s^2+s-1}{s(s-1)} \\ \pm\frac{1}{3}\frac{s+1}{s} \end{pmatrix}, \begin{pmatrix} \pm\frac{1}{\sqrt{3}}\frac{s^2+s-1}{s(s-1)} \\ \mp\frac{1}{3}\frac{s+1}{s} \end{pmatrix}.$$

The last set to be calculated is $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$.

Let v^1 be such that

$$(v^1)^T \begin{pmatrix} \frac{1}{\sqrt{3}}\frac{s+1}{s} \\ \frac{1}{3}\frac{(s+1)^2}{s} \end{pmatrix} = (v^1)^T \begin{pmatrix} \frac{1}{\sqrt{3}}\frac{s^2+s-1}{s(s-1)} \\ \frac{1}{3}\frac{s+1}{s} \end{pmatrix} = 1.$$

Then $v^1 = \begin{pmatrix} \frac{\sqrt{3}(s-1)}{s+1} \\ \frac{3}{(s+1)^2} \end{pmatrix}$. Assuming

$$(v^2)^T \begin{pmatrix} \frac{2}{\sqrt{3}}\frac{s+1}{s} \\ 0 \end{pmatrix} = (v^2)^T \begin{pmatrix} \frac{1}{\sqrt{3}}\frac{s^2+s-1}{s(s-1)} \\ \frac{1}{3}\frac{s+1}{s} \end{pmatrix} = 1$$

results in $v^2 = \begin{pmatrix} \frac{\sqrt{3}s}{2(s+1)} \\ \frac{3s(s^2-s-1)}{2(s+1)(s^2-1)} \end{pmatrix}$ and

$$(v^3)^T \begin{pmatrix} \frac{1}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \frac{1}{3}\frac{(s+1)^2}{s} \end{pmatrix} = (v^3)^T \begin{pmatrix} -\frac{1}{\sqrt{3}}\left(\frac{s+1}{s}\right) \\ \frac{1}{3}\frac{(s+1)^2}{s} \end{pmatrix} = 1$$

in $v^3 = \begin{pmatrix} 0 \\ \frac{3s}{(s+1)^2} \end{pmatrix}$. From the symmetries of $\frac{C^\circ - C^\circ}{2}$ we conclude that

$$\left(\frac{C^\circ - C^\circ}{2}\right)^\circ = \operatorname{conv}\left(\left\{\begin{pmatrix} \pm\frac{\sqrt{3}(s-1)}{s+1} \\ \frac{3}{(s+1)^2} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}(s-1)}{s+1} \\ -\frac{3}{(s+1)^2} \end{pmatrix}, \begin{pmatrix} 0 \\ \pm\frac{3s}{(s+1)^2} \end{pmatrix},\right.\right.$$
$$\left.\left.\begin{pmatrix} \pm\frac{\sqrt{3}s}{2(s+1)} \\ \frac{3s(s^2-s-1)}{2(s+1)(s^2-1)} \end{pmatrix}, \begin{pmatrix} \pm\frac{\sqrt{3}s}{2(s+1)} \\ -\frac{3s(s^2-s-1)}{2(s+1)(s^2-1)} \end{pmatrix}\right\}\right).$$

The next thing to do is to compute scaling factors μ_1, μ_2, μ_3 with respect to the three different types of vertices of $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ mapping them to the boundary of $\frac{C-C}{2}$. For μ_1 we have to solve

$$\mu_1 v^1 = \mu_1 \begin{pmatrix} \frac{\sqrt{3}(s-1)}{s+1} \\ \frac{3}{(s+1)^2} \end{pmatrix} = \lambda \begin{pmatrix} \frac{\sqrt{3}}{4}(s-1) \\ \frac{3}{4} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{4}(2-s) \end{pmatrix}$$

for some $\lambda \in [0,1]$. We obtain $\lambda = \frac{s-1}{s}$ and $\mu_1 = \frac{(s+1)^2}{4s}$. For μ_2 exactly one of the following equations has to be fulfilled for some $\lambda \in [0,1]$:

$$\mu_2 v^2 = \mu_2 \begin{pmatrix} \frac{\sqrt{3}}{2}\frac{s}{s+1} \\ \frac{3s(s^2-s-1)}{2(s+1)(s^2-1)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{4}(2-s) \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{\sqrt{3}}{4}(s-1) \\ \frac{3}{4} \end{pmatrix}$$

or

$$\mu_2 v^2 = \mu_2 \begin{pmatrix} \frac{\sqrt{3}}{2}\frac{s}{s+1} \\ \frac{3s(s^2-s-1)}{2(s+1)(s^2-1)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ \frac{3}{4}(2-s) \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{\sqrt{3}}{2}(s-1) \\ -\frac{3}{4}(2-s) \end{pmatrix}$$

It turns out that for $\frac{1}{6}(3+\sqrt{57}) \le s \le 2$ holds the first equation, from which we obtain $\lambda = \frac{-s^2+2s+2}{2s^2-s-2}$ and $\mu_2 = \frac{(s^2-1)(s+1)}{2(2s^2-s-2)}$. For $1 \le s \le \frac{1}{6}(3+\sqrt{57})$ the second system is the right one, which gives us $\lambda = \frac{s(s-1)}{2(2-s)(s+1)}$ and $\mu_2 = \frac{s^2-1}{s}$. For the last factor, μ_3, we have to solve

$$\mu_3 v^3 = \mu_3 \begin{pmatrix} 0 \\ \frac{3s}{(s+1)^2} \end{pmatrix} = \lambda \begin{pmatrix} -\frac{\sqrt{3}}{4}(s-1) \\ \frac{3}{4} \end{pmatrix} + (1-\lambda) \begin{pmatrix} \frac{\sqrt{3}}{4}(s-1) \\ \frac{3}{4} \end{pmatrix}$$

for some $\lambda \in [0,1]$, which leads to $\lambda = \frac{1}{2}$ and $\mu_3 = \frac{(s+1)^2}{4s}$.

Finally, since $\mu_1 = \mu_3$ we conclude

$$\beta(C) = \max\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}, \frac{1}{\mu_3}\right\} = \max\left\{\frac{1}{\mu_1}, \frac{1}{\mu_2}\right\}.$$

Notice that for $1 \le s \le \frac{1}{6}(3+\sqrt{57})$ holds $\frac{1}{\mu_2} = \frac{s}{s^2-1}$, and in this case $\beta(C) = \frac{s}{s^2-1}$, if $1 \le s \le \frac{5}{3}$ and $\beta(C) = \frac{4s}{(s+1)^2}$, if $\frac{5}{3} \le s \le \frac{1}{6}(3+\sqrt{57})$.

In case of $\frac{1}{6}(3+\sqrt{57}) \le s \le 2$ we have $\frac{1}{\mu_2} = \frac{2(2s^2-s-2)}{(s^2-1)(s+1)}$, and thus $\beta(C) = \frac{4s}{(s+1)^2}$. All in all, $\beta(C) = \frac{s}{s^2-1}$ for $s \in [1, \frac{5}{3}]$ and $\beta(C) = \frac{4s}{(s+1)^2}$ for $s \in [\frac{5}{3}, 2]$.

□

Let us remark that for any $s \in [1,2]$ and $\alpha \in [\alpha_1(s), \alpha_2(s)]$ or $\beta \in [\beta_1(s), \beta_2(s)]$ from Theorem 1.7, respectively, there exists a Minkowski centered set C with asymmetry $s(C) = s$ such that $C \cap (-C) \subset^{opt} \alpha \operatorname{conv}(C \cup (-C))$ or $(\frac{C^\circ - C^\circ}{2})^\circ \subset^{opt} \beta \, \frac{C-C}{2}$, respectively.

Finally, we would also like to mention that similar to the upper bounds $\psi \frac{n}{n+1}$ and $\zeta \frac{n(n+2)}{(n+1)^2}$ on $\alpha(C)$ and $\beta(C)$, respectively, for $s(C)$ close to n, one may use the ideas from the proof of Theorem 1.5 to derive also lower bounds on $\alpha(C)$ and $\beta(C)$ for $s(C)$ close to n, i.e., $\alpha_1(s) \geq f_1(s)$ and $\beta_1(s) \geq f_2(s)$ for some continuous functions f_1, f_2 fulfilling $f_1(n) = \frac{n}{n+1}$ and $f_2(n) = \frac{n(n+1)}{(n+1)^2}$.

Appendix

The *circumradius* of K w.r.t. the gauge body C is defined as

$$R(K,C) = \min\{\rho \geq 0 : K \subset \rho C + t, \; t \in \mathbb{R}^n\}.$$

Surprisingly, the definition of a diameter with respect to a (possibly) non-symmetric gauge body $C \in \mathcal{K}^n$ (with $0 \in \operatorname{int}(C)$) is not unified. While in [Le] it is defined as

$$D_{\max}(K,C) = \max_{x,y \in K} \|x - y\|_C,$$

which we call the *maximal diameter*, and which at first view is the most natural generalization of a diameter for non-symmetric gauges; others (see c.f. [DGK]) preferred and partly argued to choose the following definition of the *diameter* of K w.r.t. C:

$$D(K,C) = 2 \max_{x,y \in K} R(\{x,y\}, C).$$

The latter definition allows to see the diameter as a best 2-point approximation of the circumradius of the whole set K. Another advantage of it is that it is translation invariant in both arguments. In contrast, for the maximal diameter choosing C with 0 close to the boundary of C, the circumradius-diameter ratio may get arbitrarily small.

However, the choice of a definition should always fit its desired properties. For instance, if choosing an asymmetric gauge body is motivated by the desire to measure the distance from x to y different than that from y to x, the latter should possibly be reflected in the length measurements (instead of measuring the length of the segment $[x,y]$ independently of its direction). Thus there may be applications where we would prefer to measure the distance from x to y by $\|x - y\|_C$, which then leads to the maximal diameter.

And, this is part of our motivation for the investigation above, one can see that

$$D_{\max}(K,C) = D(K, C \cap (-C)), \quad \text{while} \quad D(K,C) = D\left(K, \frac{C-C}{2}\right).$$

Moreover, if C is Minkowski centered, the results above show us, that we can bound those diameters in terms of the other and therefore also the circumradius-diameter ratio for the maximal diameter.

Finally, it is easy to see that there are also well motivated definitions of lengths of segments or directional breadths w.r.t. a given gauge C that lead to diameters that depend on the harmonic mean $\left(\frac{C^\circ - C^\circ}{2}\right)^\circ$ or the maximum $\mathrm{conv}(C \cup (-C))$.

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FROM INEQUALITIES RELATING SYMMETRIZATIONS OF CONVEX BODIES TO THE DIAMETER-WIDTH RATIO FOR COMPLETE AND PSEUDO-COMPLETE CONVEX SETS

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ABSTRACT. For a Minkowski centered convex compact set K we define $\alpha(K)$ to be the smallest possible factor to cover $K \cap (-K)$ by a rescalation of $\operatorname{conv}(K \cup (-K))$ and give a complete description of the possible values of $\alpha(K)$ in the planar case in dependence of the Minkowski asymmetry of K. As a side product, we show that, if the asymmetry of K is greater than the golden ratio, the boundary of K intersects the boundary of its negative $-K$ always in exactly 6 points. As an application, we derive bounds for the diameter-width-ratio for pseudo-complete and complete sets, again in dependence of the Minkowski asymmetry of the convex bodies, tightening those depending solely on the dimension given in a recent result of Richter [10].

1. Introduction and Notation

Any set $A \subset \mathbb{R}^n$ fulfilling $A = t - A$ for some $t \in \mathbb{R}^n$ is called *symmetric* and *0-symmetric* if $t = 0$. We denote the family of all *(convex) bodies* (full-dimensional compact convex sets) by $\mathcal{K}^n$ and the family of 0-symmetric bodies by $\mathcal{K}^n_0$. For any $K \in \mathcal{K}^n$ the *gauge function* $\|\cdot\|_K : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\|x\|_K = \inf\{\rho > 0 : x \in \rho K\}.$$

In case $K \in \mathcal{K}^n_0$, we see that $\|\cdot\|_K$ defines a norm. However, even for a non-symmetric unit ball K, one may approximate the gauge function by the norms induced from symmetrizations of K

$$\|x\|_{\operatorname{conv}(K\cup(-K))} \le \|x\|_K \le \|x\|_{K\cap(-K)}. \tag{1}$$

It is natural to request that $K \cap (-K) = K = \operatorname{conv}(K \cup (-K))$ if K is symmetric, which is true if and only if 0 is the center of symmetry of K. This motivates the definition of a meaningful center for general bodies K. We introduce one of the most common asymmetry measures, which is best suited to our purposes, and choose the center matching it.

The *Minkowski asymmetry* of K is defined as

$$s(K) := \inf\{\rho > 0 : K - c \subset \rho(c - K),\ c \in \mathbb{R}^n\},$$

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and a *Minkowski center* of K is any $c \in \mathbb{R}^n$ such that $K - c \subset s(K)(c - K)$ [4]. Moreover, if 0 is a Minkowski center, we say K is *Minkowski centered*. It is well-known that $s(K) \in [1, n]$ for all $K \in \mathcal{K}^n$, with $s(K) = 1$ if and only if K is symmetric and $s(K) = n$ if and only if K is a fulldimensional simplex [7].

For $K \in \mathcal{K}^n$ we define $\alpha(K) > 0$ such that $K \cap (-K) \subset^{opt} \alpha(K) \operatorname{conv}(K \cup (-K))$. Notice that there always exists some $x \in \mathbb{R}^n$ such that $\alpha(K)\|x\|_{K\cap(-K)} = \|x\|_{\operatorname{conv}(K\cup(-K))}$, which means that we have equality for that x in the complete chain in (1) if $\alpha(K) = 1$.

In [2] we started an investigation of the region of all possible values for the parameter $\alpha(K)$ for Minkowski centered $K \in \mathcal{K}^n$ in dependence of the asymmetry of K. It has been shown in [2, Lemma 3.2] that for a Minkowski centered fulldimensional simplex S we have

$$\alpha(S) = \begin{cases} 1, & \text{if } n \text{ is odd, and} \\ \frac{n}{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

Moreover, it is shown in [2, Theorem 1.7] that $\alpha(K) \geq \frac{2}{s(K)+1}$ for all Minkowski centered $K \in \mathcal{K}^n$, and that in the planar case $\alpha(K) = 1$ implies $s(K) \leq \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61$ denotes the *golden ratio*.

The main result of the present work is a complete description of the possible α values of K in dependence of its asymmetry (c.f. Figure 1).

Theorem 1.1. *Let $K \in \mathcal{K}^2$ be Minkowski centered. Then*

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2 - 1}\right\}.$$

Moreover, for every pair (α, s), such that $1 \leq s \leq 2$ and $\frac{2}{s+1} \leq \alpha \leq \min\left\{1, \frac{s}{s^2-1}\right\}$, there exists a Minkowski centered $K \in \mathcal{K}^2$, such that $s(K) = s$ and $\alpha(K) = \alpha$.

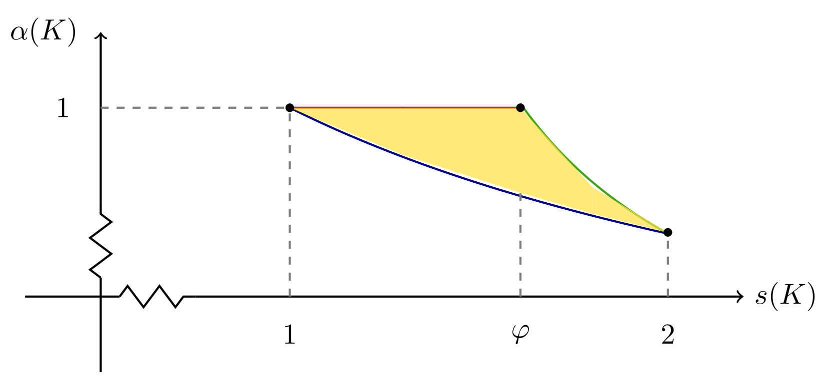


FIGURE 1. Region of possible values for the parameter $\alpha(K)$ for Minkowski centered $K \in \mathcal{K}^2$ (yellow): $\alpha(K) \geq \frac{2}{s+1}$ (blue); $\alpha(K) \leq 1$ for $s \leq \varphi$ (red), $\alpha(K) \leq \frac{s}{s^2-1}$ for $s \geq \varphi$ (green). Vertices are given by 0-symmetric $K \in \mathcal{K}^2$ ($s = 1$), the Golden House $\mathbb{GH}$ (see definition in Proposition 2.4) ($s = \varphi$) and triangles ($s = 2$).

Observe that from Theorem 1.1 directly follows for Minkowski centered $K \in \mathcal{K}^2$ that

$$\frac{1}{s(K)} \leq \frac{2}{s(K)+1} \leq \alpha(K), \tag{2}$$

with equality if and only if $s(K) = 1$.

While developing the proof of Theorem 1.1, we also made another interesting observation, for which we believe a separate theorem is justified.

Theorem 1.2. *Let $K \in \mathcal{K}^2$ be Minkowski centered with $s(K) \geq \varphi$. Then the set* $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *consists of exactly 6 points.*

Consider $K \in \mathcal{K}^n$ and $C \in \mathcal{K}^n_0$. For $s \in \mathbb{R}^n \setminus \{0\}$ the *s-breadth* of K w.r.t. C is the distance between the two parallel supporting hyperplanes of K with normal vector s, i.e.,

$$b_s(K,C) := \frac{\max_{x,y \in K} s^T(x-y)}{\max_{x \in C} s^T x}.$$

The minimal s-breadth

$$w(K,C) := \min_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K,C)$$

and the maximal s-breadth

$$D(K,C) := \max_{s \in \mathbb{R}^n \setminus \{0\}} b_s(K,C)$$

are called *width* and *diameter* of K w.r.t. C, respectively. K is said to be of *constant width* with respect to C, if $b_s(K,C)$ is constant in dependence of $s \in \mathbb{R}^n \setminus \{0\}$, i.e., $w(K,C) = D(K,C)$ and $K - K = D(K,C)C$ [6]. Finally, K is called *(diametrically) complete* w.r.t. C, if any proper superset of K has a greater diameter than K.

The most famous example of a set of constant width w.r.t. the euclidean ball is the Reuleaux triangle (see, e.g. [3]). One should recognize that constant width always implies completeness, but not the other way around [6]. Minkowski spaces, in which all complete sets are of constant width are called perfect. Characterizing such spaces is still a major open question in convex geometry [6, 9].

The *circumradius* and the *inradius* of $K \in \mathcal{K}^n$ w.r.t. $C \in \mathcal{K}^n$ are defined as

$$R(K,C) := \inf\{\rho > 0 : K \subset_t \rho C\} \quad \text{and} \quad r(K,C) := \sup\{\rho > 0 : \rho C \subset_t K\},$$

where we write $K \subset_t C$, if there exists $t \in \mathbb{R}^n$, such that $K \subset C + t$. Whenever C is symmetric, we have $D(K,C) = 2\max_{x,y \in K} R(\{x,y\},C) = \max_{x,y \in K} \|x-y\|_C$ and

$$\frac{w(K,C)}{2} \leq \frac{s(K)+1}{2} r(K,C) \leq \frac{r(K,C)+R(K,C)}{2} \leq \frac{s(K)+1}{2s(K)} R(K,C) \leq \frac{D(K,C)}{2}$$

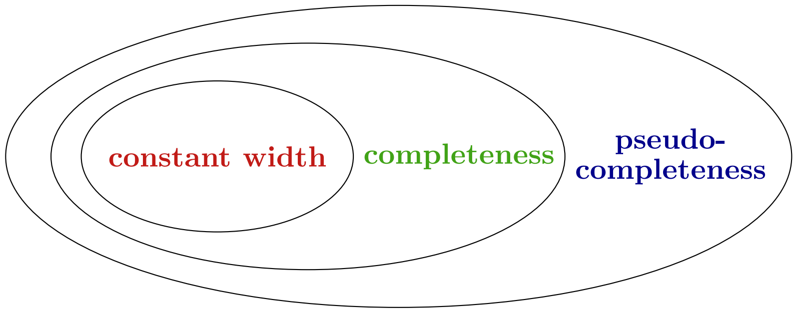
(see [5]).

While $w(K,C) = D(K,C)$ characterizes constant width of K w.r.t. C, we know from [4] that all sets K, which are complete w.r.t. C fulfill the following chain of equalities

$$\frac{s(K)+1}{2} r(K,C) = \frac{r(K,C)+R(K,C)}{2} = \frac{s(K)+1}{2s(K)} R(K,C) = \frac{D(K,C)}{2}.$$

However, this property does not characterize completeness, not even in the planar case. For instance, the sliced Reuleaux triangle and the hood, as defined in [3] are pseudo-complete w.r.t. the euclidean ball, but not of constant width.

We say that K is *pseudo-complete* w.r.t. C if $r(K,C)+R(K,C)=D(K,C)$, and denote by $\mathcal{K}^n_{ps,C}$ or $\mathcal{K}^n_{comp,C}$ the families of all $K \in \mathcal{K}^n$, which are pseudo-complete or complete w.r.t. C, respectively.



Recently, it has been shown in [10] that the diameter-width-ratio for complete sets $K \in \mathcal{K}^n$ is bounded from above by $\frac{n+1}{2}$. We sharpen this result for pseudo-complete (and therefore, for complete) sets taking the asymmetry $s(K)$ of K into account.

Theorem 1.3. *Let $K \in \mathcal{K}^n_{ps,C}$. Then*

$$\frac{D(K,C)}{w(K,C)} \leq \frac{s(K)+1}{2}.$$

Moreover, for $n>2$ odd and any $s \in [1,n]$ or for $n>2$ even and any $s \in [1,n-1]$ there exists $K \in \mathcal{K}^n_{comp,C}$ with $s(K)=s$, such that $\frac{D(K,C)}{w(K,C)} = \frac{s+1}{2}$.

Euclidean spaces of any dimension as well as general planar Minkowski spaces are perfect. Thus, obviously, in all those spaces the diameter-width-ratio of complete sets is equal to one. However, as an application of Theorem 1.1, we are able to state a better bound than the one given in Theorem 1.3 for pseudo-complete sets in the planar case.

Theorem 1.4. *Let $K \in \mathcal{K}^2_{ps,C}$. Then*

$$\frac{D(K,C)}{w(K,C)} \leq \min\left\{\frac{s(K)+1}{2}, \frac{s(K)^2}{s(K)^2-1}\right\}.$$

We will also show that Theorem 1.4 improves the absolute upper bound for the diameter-width-ratio of pseudo-complete planar sets from $\frac{3}{2}$ (derived in Theorem 1.3) down to $\frac{1}{6}\left(4+\sqrt[3]{19-3\sqrt{33}}+\sqrt[3]{19+3\sqrt{33}}\right) \approx 1.42$.

Finally, we use the results on the 3-dimensional Blaschke-Santaló diagram w.r.t. the circumradius, inradius, diameter, and width for convex bodies in the euclidean plane [3], to derive the optimal absolut upper bound for the diameter-width-ratio of pseudo-complete sets in that case.

Theorem 1.5. *Let $K \in \mathcal{K}^2_{ps,\mathbb{B}_2}$. Then*

$$\frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \frac{1}{2}\left(1+\frac{1}{r}\right) \approx 1.135,$$

where

$$r = \frac{\sqrt{t}}{2} - 1 + \sqrt{\frac{16}{\sqrt{t}} - t}, \quad \text{and} \quad t = \sqrt[3]{\frac{32}{9}}\left(\sqrt[3]{9+\sqrt{69}} + \sqrt[3]{9-\sqrt{69}}\right).$$

Moreover, equality holds if K is a hood (see Section 6 for details).

2. Definitions and Propositions

For any $X, Y \subset \mathbb{R}^n$, $\rho \in \mathbb{R}$ let $X + Y = \{x + y : x \in X, y \in Y\}$ be the *Minkowski sum* of X, Y and $\rho X = \{\rho x : x \in X\}$ the *ρ-dilatation* of X, and abbreviate $(-1)X$ by $-X$. For any $X \subset \mathbb{R}^n$ let $\mathrm{conv}(X), \mathrm{pos}(X)$, $\mathrm{lin}(X)$, and $\mathrm{aff}(X)$ denote the *convex hull*, the *positive hull*, the *linear hull*, and the *affine hull* of X, respectively. A *segment* is the convex hull of $\{x, y\} \subset \mathbb{R}^n$, which we abbreviate by $[x, y]$. With $u^1, \dots, u^n$ we denote the *standard unit vectors* of $\mathbb{R}^n$. For every $X \subset \mathbb{R}^n$ let $\mathrm{bd}(X)$ and $\mathrm{int}(X)$ denote the *boundary* and *interior* of X, respectively. Let us denote the *Euclidean norm* of $x \in \mathbb{R}^n$ by $\|x\|$, the *Euclidean unit ball* by $\mathbb{B}_2 = \{x \in \mathbb{R}^n : \|x\| \le 1\}$. In case $u^1, \dots, u^{n+1} \in \mathbb{R}^n$ are affinely independent, we say that $\mathrm{conv}(\{u^1, \dots, u^{n+1}\})$ is an *n-simplex*.

For any $K, C \in \mathcal{K}^n$ we say K is *optimally contained* in C, and denote it by $K \subset^{opt} C$, if $K \subset C$ and $K \not\subset_t \rho C$ for any $\rho \in [0, 1)$.

We recall the characterization of the optimal containment under homothety in terms of the touching conditions [5, Theorem 2.3].

Proposition 2.1. *Let $K, C \in \mathcal{K}^n$ and $K \subset C$. Then the following are equivalent:*

(i) $K \subset^{opt} C$.
(ii) There exist $k \in \{2, \dots, n+1\}$, $p^j \in K \cap \mathrm{bd}(C)$, $j = 1, \dots, k$, and a^j outer normals of supporting halfspaces of K and C at p^j, such that $0 \in \mathrm{conv}(\{a^1, \dots, a^k\})$.

In the planar case Proposition 2.1 implies the following corollary (c.f. [1, Prop. 2.5]).

Corollary 2.2. *Let $K \in \mathcal{K}^2$ with $s(K) > 1$. Then K is Minkowski centered if and only if there exist $p^1, p^2, p^3 \in \mathrm{bd}(K) \cap (-\frac{1}{s(K)}K)$ and a^i, $i = 1, 2, 3$, outer normals of supporting halfspaces of K in p^i, such that $0 \in \mathrm{int}(\mathrm{conv}(\{p^1, p^2, p^3\}))$ and $0 \in \mathrm{conv}(\{a^1, a^2, a^3\})$.*

For $K \in \mathcal{K}^n$ Minkowski centered we call any $p \in \mathrm{bd}(K) \cap \mathrm{bd}\left(-\frac{1}{s(K)}K\right)$ an *asymmetry point* of K, and any triple of asymmetry points, with the properties as stated in Corollary 2.2, to be *well-spread*.

For any $a \in \mathbb{R}^n \setminus \{0\}$ and $\rho \in \mathbb{R}$, let $H^{\le}_{a,\rho} = \{x \in \mathbb{R}^n : a^T x \le \rho\}$ denote a *halfspace* with its boundary being the *hyperplane* $H^{=}_{a,\rho} = \{x \in \mathbb{R}^n : a^T x = \rho\}$. Analogously, we define

$H^{\geq}_{a,\rho}, H^{<}_{a,\rho}, H^{>}_{a,\rho}$. We say that the halfspace $H^{\leq}_{a,\rho}$ *supports* $K \in \mathcal{K}^n$ in $q \in K$, if $K \subset H^{\leq}_{a,\rho}$ and $q \in H^{=}_{a,\rho}$. We denote the set of all *extreme points* of K by $\mathrm{ext}(K)$.

Note that $\alpha(K) = R\,(K \cap (-K), \mathrm{conv}(K \cup (-K)))$, and we also define

$$\tau(K) := R\left(K \cap (-K), \frac{K-K}{2}\right).$$

The following proposition combines [2, Corollary 2.3] and [1, Theorem 1.3].

Proposition 2.3. *Let $K \in \mathcal{K}^n$ be Minkowski centered with $s(K) = s$ and let L be a regular linear transformation. Then the following are true:*

(i) $\alpha(K) = \alpha(L(K))$ *and* $\tau(K) = \tau(L(K))$.
(ii) $\alpha(K) = \tau(K) = 1$ *if and only if there exist* $p, -p \in \mathrm{bd}(K)$ *and parallel halfspaces* $H^{\leq}_{a,\rho}$ *and* $H^{\leq}_{-a,\rho}$ *supporting* K *in* p *and* $-p$*, respectively.*

We recall another result from [1] based on Proposition 2.3 about the equality cases for the upper bound in the inequality $\alpha(K) \leq 1$ in the planar case.

Proposition 2.4. *Let $K \in \mathcal{K}^2$ be Minkowski centered such that $\alpha(K) = \tau(K) = 1$. Then $s(K) \leq \varphi$. Moreover, if $s(K) = \varphi$, there exists a linear transformation L such that $L(K) = \mathbb{GH}$, where*

$$\mathbb{GH} := \mathrm{conv}\left(\left\{\begin{pmatrix}\pm 1\\ 0\end{pmatrix}, \begin{pmatrix}\pm 1\\ -1\end{pmatrix}, \begin{pmatrix}0\\ \varphi\end{pmatrix}\right\}\right)$$

is the golden house.

3. Geometry of the boundaries of sets

In this section we give the proof of Theorem 1.2, describing the number of the intersection points of $\mathrm{bd}(K)$ and $\mathrm{bd}(-K)$ for a Minkowski centered $K \in \mathcal{K}^2$ with $s(K) > \varphi$. In order to do so, we provide the necessary definitions and show a lemma, which describes the locations of the asymmetry points from Corollary 2.2. After this, we focus on the geometry of the touching points of $K \cap (-K)$ and $\alpha(K)\,\mathrm{conv}(K \cup (-K))$, which is needed for the proof of the main theorem.

For $K \in \mathcal{K}^2$ we call $z^1, z^2 \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *consecutive*, if there exists no point $z \in \mathrm{bd}(K) \cap \mathrm{bd}(-K) \cap \mathrm{int}(\mathrm{pos}\{z^1, z^2\})$.

Even so we assume $s(K) > \varphi$ in this section, the arguments keep valid as long as there is just a finite amount of points in $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$. In case of an *infinite* sequence of points $\{z^i\}_{i \in \mathbb{N}} \subset \mathrm{bd}(K) \cap \mathrm{bd}(-K)$, there exists at least a subsequence of it, converging to a common boundary point z^0, such that K and $-K$ are commonly supportable in z^0. However, the latter would anyway imply $\alpha(K) = 1$ by Proposition 2.3, and thus by Proposition 2.4 is not possible for $s(K) > \varphi$.

Lemma 3.1. *Let $K \in \mathcal{K}^2$ be Minkowski centered with $s(K) > \varphi$, let $z^1, z^2 \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$ be consecutive, and $H^{\leq}_{a^1,\rho_1}$, $H^{\leq}_{a^2,\rho_2}$ be two halfspaces supporting K in z^1 and z^2, respectively. Moreover, let p^1, p^2, p^3 be well-spread asymmetry points of K. Then the following are true:*

(i) If $p^i \in \mathrm{pos}\{z^1, z^2\}$, then $s(K)p^i \in H^{>}_{a^1,\rho_1} \cap H^{>}_{a^2,\rho_2}$.
(ii) There exists either an asymmetry point of K or of $-K$ in $\mathrm{pos}\{z^1, z^2\}$.

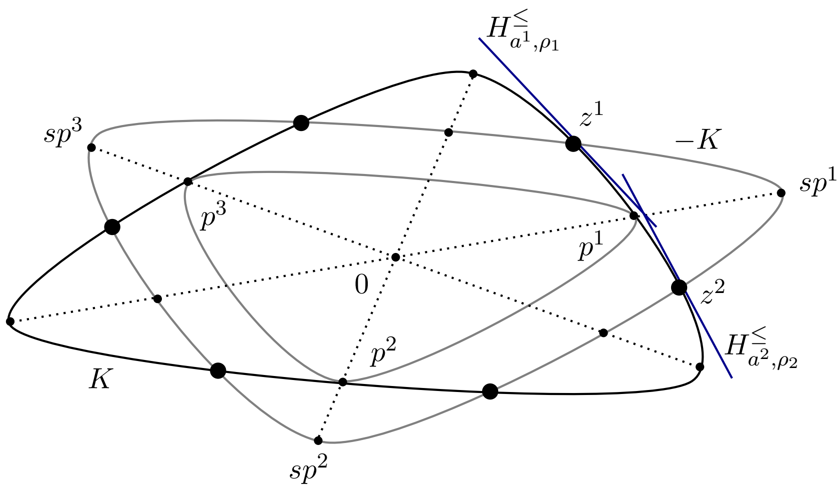


FIGURE 2. Construction from Lemma 3.1 and Theorem 1.2: K (black), $-K$ and $-\frac{1}{s(K)}K$ (gray), $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ (big black dots), halfspaces $H^{\leq}_{a^1,\rho_1}$, $H^{\leq}_{a^2,\rho_2}$ supporting K at consecutive points z^1 and z^2, respectively, and p^1, p^2, p^3 well-spread asymmetry points of K.

Proof. We start proving (i) and define $s := s(K)$. Notice that $(-K) \cap H^{>}_{a^1,\rho_1} \subset \mathrm{pos}(\{z^1, z^2\})$ and that, since $0 \in \mathrm{conv}(\{p^1, p^2, p^3\})$, between any two of those three asymmetry points, we find points from $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$. To check the latter, consider e.g. p^1 and p^2. Since $sp^1, -p^3 \in \mathrm{bd}(-K) \cap \mathrm{pos}(\{p^1, p^2\})$ and $p^1, -sp^3 \in \mathrm{bd}(K) \cap \mathrm{pos}(\{p^1, p^2\})$, we find a point from $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ in $\mathrm{int}(\mathrm{pos}(\{p^1, -p^3\})) \subset \mathrm{int}(\mathrm{pos}\{p^1, p^2\})$.

For the aim of a contradiction to (i), assume w.l.o.g. that $p^1 \in \mathrm{pos}\{z^1, z^2\}$ and $sp^1 \in H^{\leq}_{a^1,\rho_1}$. Because of the observation before it follows that $p^2, p^3 \notin \pm\mathrm{pos}(\{z^1, z^2\})$. Defining $K' := K \cap (-H^{\leq}_{a^1,\rho_1})$, we obtain $-sp^j \in K'$, $j = 1, 2, 3$, and therefore $-\frac{1}{s}K' \subset K'$, with p^1, p^2, p^3 being well-spread asymmetry point of K', too. It follows from Proposition 2.1 that K' is still Minkowski centered and $s(K') = s$. Moreover, the halfspaces $H^{\leq}_{\pm a^1,\rho_1}$ support K' in the points $\pm z^1$, respectively, implying $\alpha(K') = 1$ by Proposition 2.3. However, this means K' is Minkowski centered with $s(K') > \varphi$ and $\alpha(K') = 1$, contradicting Proposition 2.4.

The proof of (ii) is completely analogous to the one above, starting here with the assumption that there is no asymmetry point between z^1, z^2. $\square$

Proof of Theorem 1.2. Since $s := s(K) > 1$, there exists a triple of well-spread asymmetry points p^1, p^2, p^3 of K. Then $-p^i \in \mathrm{pos}(\{p^j, p^k\})$, whenever $\{i,j,k\} = \{1,2,3\}$. In particular, this means that we find at least one point from $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ within each cone $\mathrm{int}(\mathrm{pos}(\{p^i, -p^j\}))$, $i \neq j$, which shows that $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ contains at least 6 points.

Now, let us assume that $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ contains more than 6 points. Using the pigeonhole principle, we see that there must exist two consecutive points $z^1, z^2 \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$, such that $\pm p^i \notin \mathrm{pos}(\{z^1, z^2\})$, $i = 1,2,3$. Let us assume w.l.o.g. that there exists some $z \in \mathrm{bd}(K) \cap \mathrm{int}(-K) \cap \mathrm{int}(\mathrm{pos}(\{z^1, z^2\}))$. Let $H^{\leq}_{a,\rho}$ be a supporting halfspace of K in z. Then, $\mathrm{int}(-K) \cap H^{>}_{a,\rho} \subset \mathrm{pos}(\{z^1, z^2\})$, and therefore $K \setminus \{-\mathrm{pos}(\{z^1, z^2\})\} \subset K \cap (-H^{\leq}_{a,\rho}) =: K'$. It follows that $p^i, -sp^i \in K'$, $i = 1,2,3$, and therefore that $s(K') = s \geq \varphi$ by the assumption of the theorem. Moreover, since $\pm H^{\leq}_{a,\rho}$ are parallel supporting halfspaces of K' in $\pm z$, we have $\alpha(K') = 1$ by Proposition 2.3.

For $s > \varphi$ this would directly contradict Proposition 2.4. In case of $s = \varphi$ we have that K' is Minkowski centered with $\alpha(K') = 1$ and $s(K') = \varphi$. Thus, by the equality case of Proposition 2.4, we obtain that K' equals the golden house $\mathbb{GH}$ up to a linear transformation. However, this constradicts our assumption of more than 6 points in $\mathrm{bd}(K') \cap \mathrm{bd}(-K')$. $\square$

In case of $1 < s < \varphi$, the set $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ may consist of an uncountably infinite amount of points, an arbitrarily large finite amount, or a small number of points as well.

Example 3.2. *(i) For any $s \in [1, \varphi)$ there exists a Minkowski centered $K \in \mathcal{K}^2$ with $s(K) = s$, such that the set* $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *is uncountable.*

Let $K_t := \mathrm{conv}\left(\left\{\binom{\pm 1}{0}, \binom{\pm 1}{-1}, \binom{0}{t}\right\}\right) \in \mathcal{K}^2$ with $t \in [0, \varphi)$. It is not hard to verify that $K_t - x_t \subset^{opt} s(-K_t + x_t)$, where

$$s = \frac{t + \sqrt{9t^2 + 12t + 4}}{2(t+1)} \quad \text{and} \quad x_t = \begin{pmatrix} 0 \\ \frac{t-s}{s+1} \end{pmatrix}.$$

Thus, we obtain $\{s(K_t) : t \in [0, \varphi)\} = [1, \varphi)$, and since $\pm u^1 \in \mathrm{bd}(K_t)$, $t \in [0, \varphi)$, with $H^{=}_{u^1, \pm 1}$ supporting K_t in $\pm u^1$, we have $\alpha(K_t) = 1$ for every $t \in [0, \varphi)$.

Finally, $\left[\begin{pmatrix} 1 \\ \frac{t-s}{s+1} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{s-t}{s+1} \end{pmatrix}\right] \subset \mathrm{bd}(K_t) \cap \mathrm{bd}(-K_t)$, thus the boundaries of $\pm K_t$ possess infinitely *many common points.*

(ii) Let $K \in \mathcal{K}^2$ be a Minkowski centered regular k-gon with $k \geq 5$ odd. By [2, Example 4.2]*, $s(K) = \frac{1}{\cos(\frac{\pi}{k})} < \varphi$ and* $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *consists of $2k$ points.*

(iii) Let S be a (regular) simplex and $K = S \cap (-sS)$ with $s \in [1, \varphi)$. It is easy to see that $s(K) = s$, that $K \cap (-K) = S \cap (-S)$, and that $\mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *contains exactly 6 points.*

We now discuss the locations of the touching points of $K \cap (-K)$ and $\alpha(K)\,\mathrm{conv}(K \cup (-K))$.

Lemma 3.3. *Let $K \in \mathcal{K}^2$ be Minkowski centered with $s(K) > 1$ and $p \in \mathrm{bd}(K \cap (-K)) \cap \mathrm{bd}(\alpha(K)\,\mathrm{conv}(K \cup (-K)))$. Then one of the following must be true:*

(i) $p \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$ *or*
(ii) $\frac{1}{\alpha(K)}p \notin \mathrm{conv}(K \cup (-K)) \setminus (K \cup (-K))$.

Moreover, if (i) is fulfilled, there exist halfspaces $H^{\leq}_{a,\rho}$ *and* $H^{\leq}_{-a,\rho}$*, such that each of them supports both,* K *and* $-K$*, while* $H^{\leq}_{a,\alpha(K)\rho}$ *and* $H^{\leq}_{-a,\alpha(K)\rho}$ *support* $K \cap (-K)$ *in* p *and* $-p$*, respectively.*

Note that Case (ii) of Lemma 3.3 is equivalent to one of the following two statements getting true:

$$p \in \mathrm{bd}(K) \text{ with } \frac{1}{\alpha(K)}p \in \mathrm{bd}(-K) \quad \text{or} \quad p \in \mathrm{bd}(-K) \text{ with } \frac{1}{\alpha(K)}p \in \mathrm{bd}(K).$$

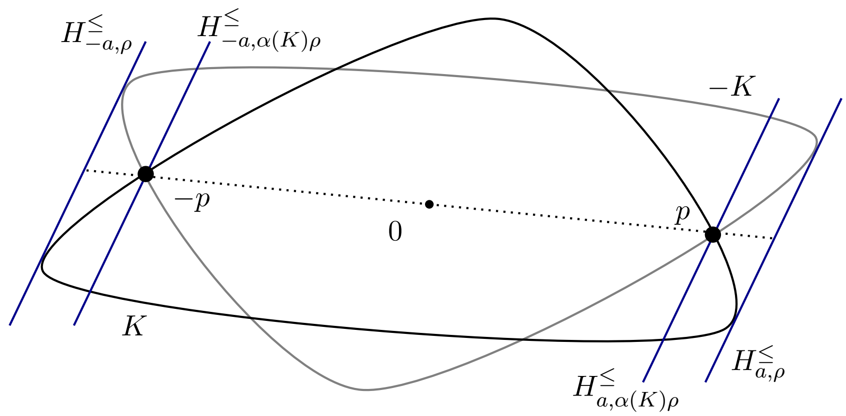


FIGURE 3. Construction for Case (i) of Lemma 3.3: K (black), $-K$ (gray), $H^{\leq}_{a,\rho}$, $H^{\leq}_{-a,\rho}$ and $H^{\leq}_{a,\alpha(K)\rho}$, $H^{\leq}_{-a,\alpha(K)\rho}$ (blue) .

Proof of Lemma 3.3. Let us start observing that if $\alpha := \alpha(K) = 1$, we have $p \in \mathrm{bd}(\mathrm{conv}(K \cup (-K))$. Hence, $p \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$, i.e. (i) applies. Hence, we may assume $\alpha < 1$ for the rest of the proof.

We first show the "moreover"-part of the statement. Thus, assume $p \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$ and let $s := s(K) > 1$. Since $\alpha < 1$, we have $\pm\frac{1}{\alpha}p \notin K$, but there exist $x, -y \in \mathrm{ext}(K)$ such that $\frac{1}{\alpha}p \in [x, y] \subset \mathrm{bd}(\mathrm{conv}(K \cup (-K)))$ and a halfspace $H^{\leq}_{a,\rho}$ supporting $\mathrm{conv}(K \cup (-K))$ in $\frac{1}{\alpha}p$ and thus in $[x, y]$.

Since $K \cap (-K)$ and $\mathrm{conv}(K \cup (-K))$ are both symmetric, the halfspace $H^{\leq}_{-a,\rho}$ supports $\mathrm{conv}(K \cup (-K))$ in $-\frac{1}{\alpha}p$. Using $\frac{1}{\alpha}(K \cap (-K)) \subset \mathrm{conv}(K \cup (-K)) \subset H^{\leq}_{a,\rho}$, we see that $H^{\leq}_{\pm a,\alpha\rho}$ support $K \cap (-K)$ in $\pm p$.

Now, for the sake of a contradiction, let us assume (i) and (ii) are wrong. Doing so, we may assume w.l.o.g. that $p \in \mathrm{bd}(-K) \cap \mathrm{int}(K)$ and $\frac{1}{\alpha}p \in \mathrm{conv}(K \cup (-K)) \setminus K$. Then there exist $x, -y \in \mathrm{bd}(-K) \setminus K$, such that $\frac{1}{\alpha}p \in [x, y] \subset \mathrm{conv}(K \cup (-K))$. Let $z \in \mathrm{bd}(K) \cap \mathrm{bd}(-K) \cap \mathrm{pos}(\{x, y\})$ and $\xi \in \mathbb{R}$ be such that $\xi z \in [x, y]$. Since $\frac{1}{\alpha}p \in [x, y] \setminus K$, we have $\xi > 1$. Obviously, $z \in \mathrm{bd}(K) \cap \mathrm{bd}(-K)$ and $x \in \mathrm{bd}(-K) \setminus K$ together imply $(\mathrm{bd}(-K) \cap \mathrm{int}(\mathrm{pos}(\{x, z\}))) \cap K = \emptyset$.

Since $p \in \mathrm{bd}(-K) \cap \mathrm{int}(K)$, we obtain $p, x, z \in \mathrm{bd}(-K)$ with $z \in \mathrm{pos}(\{p, x\})$, and, due to the convexity of $-K$, $z \in x + \mathrm{pos}(\{p - x, \frac{1}{\alpha}p - x\})$. It follows that $\frac{1}{\alpha\xi}p \notin -K$ and, since $p \in \mathrm{bd}(-K)$, we have $\frac{1}{\alpha} > \xi$. Together with $\frac{1}{\alpha}(K \cap (-K)) \subset \mathrm{conv}(K \cup (-K))$, $z \in K \cap (-K)$, and $\xi z \in \mathrm{bd}(\mathrm{conv}(K \cup (-K))$, this implies the desired contradiction. $\square$

We present a family of sets, where the situation described in Lemma 3.3 (ii) happens.

Example 3.4. *Let $S = \mathrm{conv}(\{p^1, p^2, p^3\})$ be a regular Minkowski centered triangle with $R(S, \mathbb{B}_2) = 1$ and $D := \left\{ \binom{\rho_1}{\rho_2} : 1 \leq \rho_1 \leq 2, \frac{\rho_1^2 - \rho_1 + 1}{\rho_1 + 1} \leq \rho_2 \leq \frac{\rho_1}{2} \right\}$. We define*

$$K := K_{\rho_1, \rho_2} := \mathrm{conv}(((-S) \cap \rho_1 S) \cup \rho_2 S) \text{ with } \binom{\rho_1}{\rho_2} \in D.$$

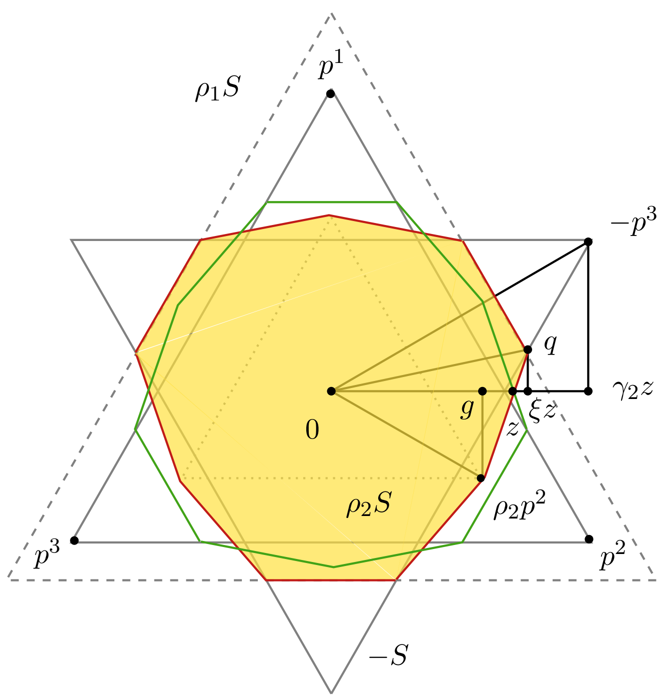


FIGURE 4. Constructions used in Example 3.4: K (yellow), $-K$ (green), $S/-S$ (gray), $\rho_2 S$ (gray dotted), $\rho_1 S$ (gray dashed).

Let further $z \in \mathrm{bd}(K) \cap \mathrm{bd}(-K) \cap \mathrm{pos}(\{p^2, -p^3\})$ with $\xi z \in \mathrm{bd}(\mathrm{conv}(K \cup (-K)))$ for some $\xi > 1$. Let $\gamma_1 \leq 1$ and $\gamma_2 \geq 1$ be such that $\gamma_1 z \in \mathrm{bd}(S) \cap \mathrm{bd}(-S)$ and $\gamma_2 z \in \mathrm{bd}(\mathrm{conv}(S \cup (-S)))$, respectively. Then, since $R(S, \mathbb{B}_2) = 1$, we have $\|\gamma_1 z\| = \frac{\sqrt{3}}{3}$ and $\|\gamma_2 z\| = \frac{\sqrt{3}}{2}$. Choose q to be the vertex of K, such that $q \in \mathrm{int}(\mathrm{pos}(\{p^2, -p^3\}))$.

Now, since $\frac{\|-p^3 - q\|}{\|-p^3 - \gamma_1 z\|} = \frac{\|-p^3 - (-\frac{\rho_1}{2}p^3)\|}{\|-p^3 - (-\frac{1}{2}p^3)\|}$, we have

$$\frac{\| - p^3 - q\|}{\| - p^3 - \gamma_1 z\|} = \frac{1 - \frac{\rho_1}{2}}{\frac{1}{2}} = 2 - \rho_1.$$

Therefore,

$$\frac{\|\gamma_2 z - \gamma_1 z\|}{\|\xi z - \gamma_1 z\|} = 1 + \frac{\|-p^3 - q\|}{\|q - \gamma_1 z\|} = 1 + \frac{2-\rho_1}{\rho_1 - 1} = \frac{1}{\rho_1 - 1}.$$

We have

$$\|\xi z\| = \|\gamma_1 z\| + \|\xi z - \gamma_1 z\| = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{6}(\rho_1 - 1) = \frac{\sqrt{3}}{6}(\rho_1 + 1).$$

Let g be the orthogonal projection of $\rho_2 p^2$ on $\operatorname{lin}(\{z\})$.

Since $\|-\frac{\rho_1}{2}p^3 - q\| = \tan(\frac{\pi}{6}) \cdot \|-p^3 - \left(-\frac{\rho_1}{2}p^3\right)\| = \frac{\sqrt{3}}{3}\left(1 - \frac{\rho_1}{2}\right)$, we obtain from the Pythagorean theorem

$$\|q\| = \sqrt{\left(\frac{\rho_1}{2}\right)^2 + \left(\frac{\sqrt{3}}{3}\left(1 - \frac{\rho_1}{2}\right)\right)^2} = \sqrt{\frac{\rho_1^2 - \rho_1 + 1}{3}}.$$

Now, we calculate

$$\|q - \xi z\| = \sqrt{\|q\|^2 - \|\xi z\|^2} = \sqrt{\frac{\rho_1^2 - \rho_1 + 1}{3} - \left(\frac{\sqrt{3}}{6}(\rho_1 + 1)\right)^2} = \frac{\rho_1 - 1}{2}.$$

Since $\|g\| = \frac{\sqrt{3}}{2}\rho_2$, we have

$$\|g - \rho_2 p^2\| = \sqrt{\|\rho_2 p^2\|^2 - \|g\|^2} = \sqrt{\rho_2^2 - \frac{3}{4}\rho_2^2} = \frac{\rho_2}{2}.$$

Since $q, z, \rho_2 p^2$ are collinear, we have $\frac{\|g-z\|}{\|z-\xi z\|} = \frac{\|g-\rho_2 p^2\|}{\|q-\xi z\|}$, and thus

$$\frac{\|g - z\|}{\|z - \xi z\|} = \frac{\frac{\rho_2}{2}}{\frac{\rho_1 - 1}{2}} = \frac{\rho_2}{\rho_1 - 1}.$$

Using this fact, we obtain

$$\|\xi z\| = \|g\| + \|g - z\| + \|\xi z - z\| = \frac{\sqrt{3}}{2}\rho_2 + \left(\frac{\rho_2}{\rho_1 - 1} + 1\right)\|\xi z - z\|.$$

Therefore, remembering that $\|g\| = \frac{\sqrt{3}}{2}\rho_2$,

$$\|\xi z - z\| = \frac{\|\xi z\| - \|g\|}{\frac{\rho_2}{\rho_1 - 1} + 1} = \frac{\|\xi z\| - \frac{\sqrt{3}}{2}\rho_2}{\frac{\rho_2}{\rho_1 - 1} + 1}$$

and

$$\begin{aligned}\frac{\|z\|}{\|\xi z\|} &= \frac{\|\xi z\| - \|\xi z - z\|}{\|\xi z\|} = 1 - \frac{1}{\frac{\rho_2}{\rho_1 - 1} + 1} + \frac{\frac{\sqrt{3}}{2}\rho_2}{\left(\frac{\rho_2}{\rho_1 - 1} + 1\right)\frac{\sqrt{3}}{6}(\rho_1 + 1)} \\ &= \frac{\rho_2(\rho_1 - 1)}{\rho_1 + \rho_2 - 1}\left(\frac{1}{\rho_1 - 1} + \frac{3}{\rho_1 + 1}\right) = \frac{2\rho_2(2\rho_1 - 1)}{(\rho_1 + \rho_2 - 1)(\rho_1 + 1)}.\end{aligned}$$

Since $\rho_2 p^2 \in \operatorname{bd}(K)$ is a vertex of $\rho_2 S$ and $\frac{\rho_1}{2}p^2 \in \operatorname{bd}(\rho_1 S) \cap \operatorname{bd}(-K)$, we have $\frac{1}{\alpha(K)} \leq \frac{\rho_1}{2\rho_2}$.

Thus, notice that

$$\frac{1}{\alpha(K)} = \min\left\{\xi, \frac{\rho_1}{2\rho_2}\right\} = \min\left\{\frac{(\rho_1 + \rho_2 - 1)(\rho_1 + 1)}{2\rho_2(2\rho_1 - 1)}, \frac{\rho_1}{2\rho_2}\right\}.$$

Since

$$\frac{\rho_1}{2\rho_2} \leq \frac{(\rho_1+\rho_2-1)(\rho_1+1)}{2\rho_2(2\rho_1-1)},$$

is equivalent to

$$\frac{\rho_1^2-\rho_1+1}{\rho_1+1} \leq \rho_2, \tag{3}$$

we obtain $\alpha(K) = \frac{\rho_1}{2\rho_2}$, *whenever* (3) *is fulfilled. Since* $\frac{\rho_1^2-\rho_1+1}{\rho_1+1} \leq \frac{\rho_1}{2}$, *for any fixed* $\rho_1 \in [1,2]$ *we can select* ρ_2, *such that* $\frac{\rho_1^2-\rho_1+1}{\rho_1+1} \leq \rho_2 \leq \frac{\rho_1}{2}$, *and thus selecting any* $\binom{\rho_1}{\rho_2} \in D$ *would give us examples of bodies fulfilling the condition above.*

4. The Proof of Theorem 1.1

Lemma 4.1. *Let* $K \in \mathcal{K}^2$ *be Minkowski centered with* $s(K) > \varphi$ *and* p^1, p^2, p^3 *a triple of well-spread asymmetry points of* K.

a) Then there exists a Minkowski centered $K' \in \mathcal{K}^2$ *with*
  (i) $\operatorname{conv}(K' \cup (-K'))) = \operatorname{conv}(\{\pm sp^1, \pm sp^2, \pm sp^3\})$,
  (ii) $\alpha(K') \geq \alpha(K)$,
  (iii) $s(K') = s(K)$, *and*
  (iv) $p \in \operatorname{bd}(K' \cap (-K')) \cap \alpha(K')(\operatorname{bd}(\operatorname{conv}(K' \cup (-K'))))$ *implies* $p \in \operatorname{bd}(K') \cap \operatorname{bd}(-K')$.

b) Taking p *as above and defining* d^1 *to be the intersection point of* $\operatorname{aff}(\{-p, p^3\})$ *and* $\operatorname{aff}(\{p, p^2\})$, *there exists* $\gamma \leq 1$, *such that the set*

$$\bar{K} := \operatorname{conv}(\{\pm p, p^2, p^3, \gamma s(K')d^1, -s(K')p^2, -s(K')p^3\})$$

fulfills
  (i) $\alpha(\bar{K}) \geq \alpha(K')$,
  (ii) $s(\bar{K}) = s(K')$, *and*
  (iii) $p \in \operatorname{bd}(\bar{K} \cap (-\bar{K})) \cap \alpha(K')(\operatorname{bd}(\operatorname{conv}(\bar{K} \cup (-\bar{K}))))$.

Proof. a) Let $s := s(K)$. Since $s > \varphi$, there exists a triple of well-spread asymmetry points p^1, p^2, p^3 of K by Corollary 2.2 and by Theorem 1.2, we know that $\operatorname{bd}(K) \cap \operatorname{bd}(-K)$ consists of exactly 6 points. Moreover, by Lemma 3.1 there exist consecutive points $z^{i,1}, z^{i,2} \in \operatorname{bd}(K) \cap \operatorname{bd}(-K)$ with $-p^i \in \operatorname{pos}\{z^{i,1}, z^{i,2}\}$, $i = 1,2,3$. Let the halfspaces $H^{\leq}_{a^{i,1},\rho_{i,1}}$, $H^{\leq}_{a^{i,2},\rho_{i,2}}$ be defined such that $z^{i,1}, -sp^i \in H^{=}_{a^{i,1},\rho_{i,1}}$ and $z^{i,2}, -sp^i \in H^{=}_{a^{i,2},\rho_{i,2}}$, $i = 1, \dots, 3$. We define

$$K' := K \cap \bigcap_{i=1,\dots,3} \left(H^{\leq}_{a^{i,1},\rho_{i,1}} \cap H^{\leq}_{a^{i,2},\rho_{i,2}} \right). \tag{4}$$

We now show the properties (i)-(iv) for the set K'.

Obviously, $\pm sp^i \in \operatorname{bd}(\operatorname{conv}(K' \cup (-K')))$, $i = 1,2,3$. Assume $z \in (\operatorname{bd}(K') \cup \operatorname{bd}(-K')) \setminus (\{\pm sp^1, \pm sp^2, \pm sp^3\} \cup \operatorname{bd}(K' \cap (-K')))$, w.l.o.g. $z \in \operatorname{bd}(K') \cap \operatorname{int}(\operatorname{pos}(\{-sp^1, z^{1,1}\}))$. The way we constructed K', this implies $z \in [-sp^1, z^{1,1}]$. Since $-sp^1 \in \operatorname{bd}(\operatorname{conv}(K' \cup (-K')))$

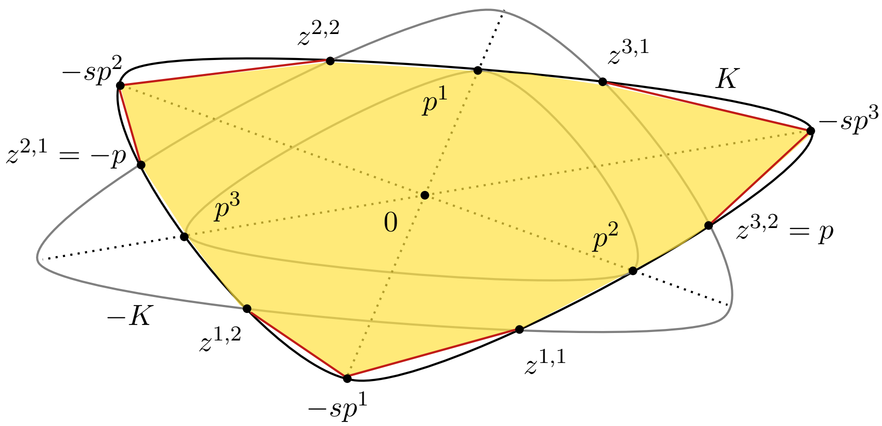


FIGURE 5. Construction from Lemma 4.2: K (black), $-K$ and $-\frac{1}{s(K)}K$ (gray), K' (yellow), segments $[-sp^i, z^{i,j}]$, $i = 1, 2, 3$ and $j = 1, 2$ (red).

and $z^{1,1} \in \mathrm{int}(\mathrm{conv}(K' \cup (-K')))$, we obtain $z \in \mathrm{int}(\mathrm{conv}(K' \cup (-K')))$, a contradiction. Thus, $\mathrm{conv}(K' \cup (-K'))) = \mathrm{conv}(\{\pm sp^1, \pm sp^2, \pm sp^3\})$.

Moreover, we also have $K' \cap (-K') = K \cap (-K)$, which together with $K' \subset K$ implies $\alpha(K') \geq \alpha(K)$.

We now need to show that $s(K') = s(K)$. Since $p^i, -sp^i \in K'$, $i = 1, 2, 3$, we only need to check the (non-obvious) fact that $-\frac{1}{s}K' \subset K'$.

Notice that for any $i = 1, 2, 3$ we have

$$\begin{aligned}-(K' \cap \mathrm{pos}(\{-z^{i,1}, -z^{i,2}\})) = -K' \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) &= -K \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) \\ &= -(K \cap \mathrm{pos}(\{-z^{i,1}, -z^{i,2}\})).\end{aligned}$$

On the one hand, since $K' \subset K$, we have $-\frac{1}{s}K' \subset -\frac{1}{s}K \subset K$, and therefore

$$\left(-\frac{1}{s}K'\right) \cap \mathrm{pos}(\{-z^{i,1}, -z^{i,2}\}) \subset K \cap \mathrm{pos}(\{-z^{i,1}, -z^{i,2}\}) = K' \cap \mathrm{pos}(\{-z^{i,1}, -z^{i,2}\}),$$

for every $i = 1, 2, 3$.

On the other hand, we need to show

$$(-K') \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) \subset K' \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}), \quad i = 1, 2, 3.$$

Clearly,

$$(-K') \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) = (-K) \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) \subset K \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}), \quad i = 1, 2, 3.$$

To show the needed inclusion, we additionally prove

$$(-K') \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) \subset \bigcap_{j=1,2} H^{\leq}_{a^{i,j}, \rho^i_j}, \quad i = 1, 2, 3.$$

Let $H^{\leq}_{b^{ij}, \mu_{i,j}}$ be a supporting halfspace of $-K$ in $z^{i,j}$, $i = 1, 2, 3$ and $j = 1, 2$. By Part (i) of Lemma 3.1, we know that $-sp^i \in H^{>}_{b^{ij}, \mu_{i,j}}$. Thus,

$$H^{>}_{a^{i,j}, \rho_{i,j}} \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}) \subset H^{>}_{b^{ij}, \mu_{i,j}} \cap \mathrm{pos}(\{z^{i,1}, z^{i,2}\}).$$

We conclude

$$-K'\cap \operatorname{pos}(\{z^{i,1},z^{i,2}\}) = -K\cap \operatorname{pos}(\{z^{i,1},z^{i,2}\}) \subset \bigcap_{j=1,2} H^{\leq}_{b^{i,j},\mu^i_j}\cap \operatorname{pos}(\{z^{i,1},z^{i,2}\})$$
$$\subset \bigcap_{j=1,2} H^{\leq}_{a^{i,j},\rho^i_j}\cap \operatorname{pos}(\{z^{i,1},z^{i,2}\}) = K'\cap \operatorname{pos}(\{z^{i,1},z^{i,2}\}).$$

Hence $-\frac{1}{s}K'\cap \operatorname{pos}(\{z^i_1,z^i_2\})\subset K'\cap \operatorname{pos}(\{z^i_1,z^i_2\})$, $i=1,2,3$ and therefore $-\frac{1}{s}K'\subset K'$.

Since the points p^1,p^2,p^3 also build a triple of well-spread asymmetry points of K', we obtain $s(K')=s(K)$.

Now, let $p\in \operatorname{bd}(K'\cap(-K'))\cap\alpha(K')\operatorname{bd}(\operatorname{conv}(K'\cup(-K')))$. Within the next paragraph we show by contradiction that $p\in \operatorname{bd}(K')\cap\operatorname{bd}(-K')$. To do so, assume $p\notin \operatorname{bd}(K')\cap \operatorname{bd}(-K')$. Then, by Lemma 3.3, we obtain $\frac{1}{\alpha(K')}p\notin \operatorname{conv}(K'\cup(-K'))\setminus(K'\cup(-K'))$, and therefore that either $\frac{1}{\alpha(K')}p$ belongs to $\operatorname{bd}(K')$ or $\operatorname{bd}(-K')$. Thus, by our cutting-offs of K and $-K$ above,

$$\frac{1}{\alpha(K')}p\in \operatorname{bd}(\operatorname{conv}(K'\cup(-K')))\cap(\operatorname{bd}(K')\cup\operatorname{bd}(-K'))=\{\pm sp^1,\pm sp^2,\pm sp^3\}.$$

Since by (2) $\frac{1}{\alpha(K')}<s$, this implies that one of the points $\pm p^1,\pm p^2,\pm p^3$ is not in $K'\cap(-K')\cap\alpha(K')(\operatorname{bd}(\operatorname{conv}(K'\cup(-K'))))$, a contradiction. Thus, $p\in\operatorname{bd}(K')\cap\operatorname{bd}(-K')$, and hence we may assume w.l.o.g. in the following that $p:=z^{3,2}$ and $-p:=z^{2,1}$.

b) Notice that, because of $0\in\operatorname{conv}(\{p^1,p^2,p^3\})$ and the way we have chosen the points $-p,p$, we know that p^1 is located on the one side, while p^2,p^3 are on the other side of $\operatorname{aff}(\{-p,p\})$. Moreover, $p\in\operatorname{pos}\{p^1,p^2\}$ and $-p\in\operatorname{pos}\{p^1,p^3\}$ and again, by Lemma 3.3, there exists a pair of halfspaces $H^{\leq}_{\pm u,\frac{1}{\alpha(K')}\rho}$ supporting $\operatorname{conv}(K'\cup(-K'))$ in $\pm\frac{1}{\alpha(K')}p$.

Hence, $-sp^2,-sp^3\in H^{\leq}_{u,\frac{1}{\alpha(K')}\rho}\cap H^{\leq}_{-u,\frac{1}{\alpha(K')}\rho}$ and, because $\frac{1}{\alpha(K')}<s$, this implies $p^2,p^3\in H^{<}_{-u,\rho}\cap H^{<}_{u,\rho}$. Together with $p\in H^{=}_{u,\rho}$, we obtain the existence of some $d^1\in\operatorname{aff}(\{-p,p^3\})\cap\operatorname{aff}(\{p,p^2\})$. Moreover, since $p^1,-sp^2,-sp^3\in\operatorname{bd}(K')$ with $p^1\in\operatorname{pos}(\{-sp^2,-sp^3\})$, we have $p^1\notin\operatorname{int}(\operatorname{conv}(\{0,-sp^2,-sp^3\}))$.

Now, we show $-sp^1\in\operatorname{conv}(\{p^2,p^3,d^1\})\setminus[p^2,p^3]$.

Since $0\in\operatorname{conv}(\{p^1,p^2,p^3\})$, we have $-sp^1\in\operatorname{pos}(\{p^2,p^3\})$, and since all the points $-p,p,p^2,p^3$ belong to $\operatorname{bd}(K')$, with $p^3\in\operatorname{pos}(\{-p,-sp^1\})$, and $p^2\in\operatorname{pos}(\{p,-sp^1\})$, we obtain $-sp^1\in\operatorname{conv}(\{-p,p,d^1\})$. Finally, $-sp^1\in\operatorname{bd}(K')$ and $s>1$ imply $-sp^1\notin\operatorname{conv}(\{-p,p,p^2,p^3\})$.

Thus, combining the results from above, we obtain

$$-sp^1\in\operatorname{conv}(\{p^2,p^3,d^1\})\setminus[p^2,p^3]\quad\text{and}\quad p^1\notin\operatorname{int}(\operatorname{conv}(\{0,-sp^2,-sp^3\})).\tag{5}$$

In the following, we say that a well-spread triple of asymmetry points p^1,p^2,p^3 presents a *valid situation*, if they satisfy (5).

Notice that since K' is of the form (4), we also have $[-sp^2,-p],[p,-sp^3]\subset\operatorname{bd}(K')$.

The next part of the proof describes a three-step transformation of K' into $\bar K$, such that $s(\bar K)=s$, $p\in\operatorname{bd}(\bar K\cap(-\bar K))\cap\alpha(K')\left(\operatorname{bd}(\operatorname{conv}(\bar K\cup(-\bar K)))\right)$, and $\bar p^1,p^2,p^3\in\operatorname{bd}(\bar K)\cap\operatorname{bd}\left(-\frac{1}{s}\bar K\right)$ is a well spread triple of asymmetry points that presents a valid situation.

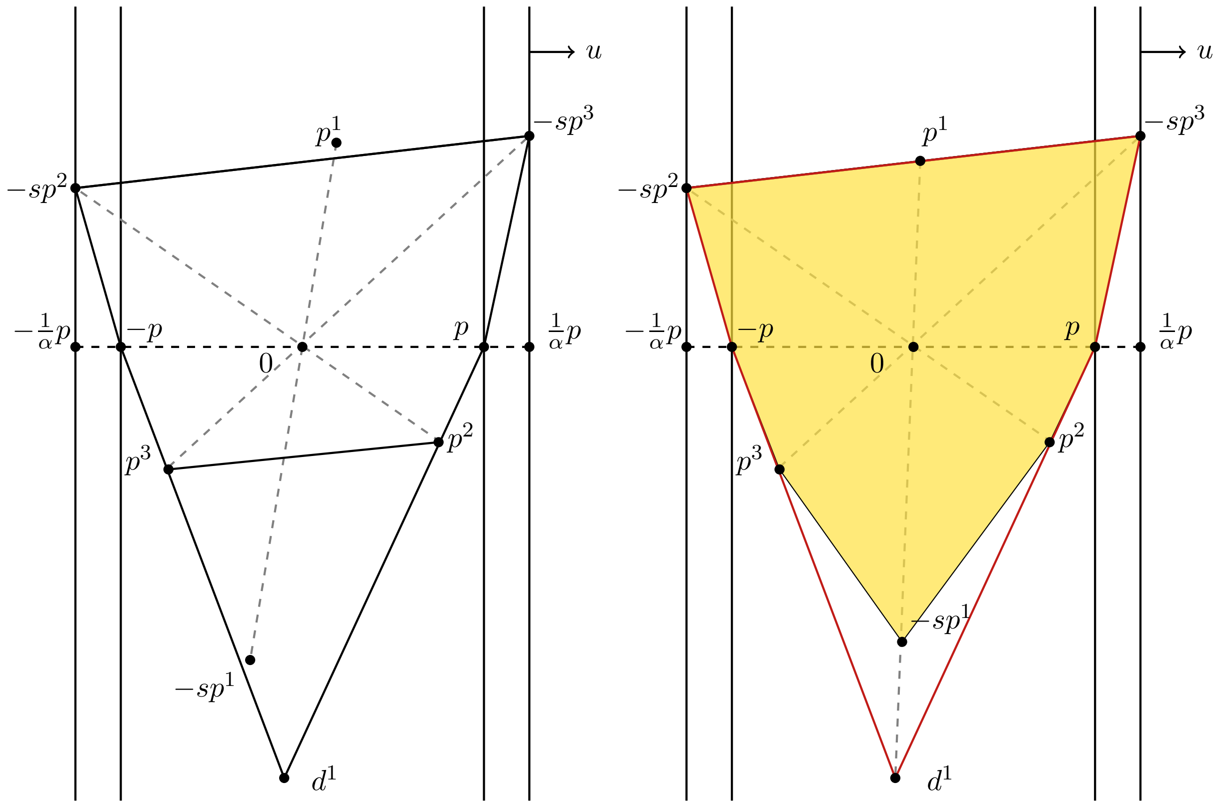


FIGURE 6. Construction used in the proof of Lemma 4.1b). On the left: the construction before applying the three-step transformation and on the right the set after the transformation (yellow). In red: $K_{\min,s}$ (see definition before Lemma 4.3).

Step 1 Replace p^1 by $\widetilde{p}^1 := \mu p^1$, for some $\mu \leq 1$, such that $\widetilde{p}^1 \in [-sp^2, -sp^3]$.
Obviously, $\widetilde{p}^1 \notin \mathrm{int}(\mathrm{conv}(\{0, -sp^2, -sp^3\}))$, and since $\mu < 1$ and $s > 1$, we obtain directly from $-sp^1 \in \mathrm{conv}(\{p^2, p^3, d^1\})$ that also $-s\widetilde{p}^1 \in \mathrm{conv}(\{p^2, p^3, d^1\}) \setminus [p^2, p^3]$. Hence, the points $\widetilde{p}^1, p^2, p^3$ stay to present a valid situation.

Step 2 Replace $\widetilde{p}^1$ by $\bar{p}^1 := -\gamma d^1 \in [-sp^2, -sp^3]$, for some $\gamma > 0$.
Obviously again, $\bar{p}^1 \notin \mathrm{int}(\mathrm{conv}(\{0, -sp^2, -sp^3\}))$. Moreover, since replacing $\widetilde{p}^1$ by $\bar{p}^1$ means moving $-s\widetilde{p}^1$ onto $-s\bar{p}^1$ parallel to $[p^2, p^3]$ such that it belongs to $\mathrm{lin}\{d^1\}$. Thus, $s\gamma d^1 = -s\bar{p}^1 \in \mathrm{conv}(\{d^1, p^2, p^3\}) \setminus [p^2, p^3]$, again. In particular, we obtain $s\gamma \leq 1$, which shows that the points $\bar{p}^1, p^2, p^3$ stay to present a valid situation.

Step 3 Replace K' by $\bar{K} := \mathrm{conv}(\{\pm p, p^2, p^3, -s\bar{p}^1, -sp^2, -sp^3\})$.
The points $\bar{p}^1, p^2, p^3$ still form a well-spread triple of asymmetry points of $\bar{K}$, which implies also $s(\bar{K}) = s$. Moreover, we still have $p \in \mathrm{bd}(\bar{K} \cap (-\bar{K})) \cap \alpha(K') \left(\mathrm{bd}(\mathrm{conv}(\bar{K} \cup (-\bar{K})))\right)$, which implies that $\alpha(\bar{K}) \geq \alpha(K')$.

□

Let us explicitely mention, that we do not show (since not needed and possibly false in some cases) that $\alpha(K) = \alpha(\bar{K})$ is always true.

Lemma 4.2. *Let $K \in \mathcal{K}^2$ be Minkowski centered such that $s(K) > \varphi$. Then*

$$\alpha(K) \leq \frac{s(K)}{s(K)^2 - 1}. \tag{6}$$

Proof. In order to show (6), we determine for any fixed $\alpha \in \left[\frac{2}{3}, 1\right]$ below the maximal $s = s(\alpha)$, such that there exists a Minkowski centered $K \in \mathcal{K}^2$ with $\alpha(K) = \alpha$ and $s(K) = s$. This way we show that $s \leq \frac{1}{2\alpha} + \sqrt{1 + \frac{1}{4\alpha^2}}$, which is equivalent to $\alpha \leq \frac{s}{s^2-1}$.

Considering K' to be defined as in Lemma 4.1, we have $\alpha(K) \leq \alpha(K')$ and $s(K) = s(K')$. Hence, if K' fulfills (6), then

$$\alpha(K) \leq \alpha(K') \leq \frac{s(K')}{s(K')^2 - 1} = \frac{s(K)}{s(K)^2 - 1},$$

i.e., K also fulfills (6). Thus, it suffices to show (6) for K'.

In the following, instead of directly showing (6) for K', we simplify the task by considering $\bar{K}$ as given in Lemma 4.1. By Lemma 4.1 b) we have $s(K') = s(\bar{K})$ and $p \in \mathrm{bd}(\bar{K} \cap (-\bar{K})) \cap \alpha(K')(\mathrm{bd}(\mathrm{conv}(\bar{K} \cup (-\bar{K}))))$. Thus, we can study the maximal possible value for $s := s(K) = s(\bar{K})$, while keeping $p \in \mathrm{bd}(\bar{K} \cap (-\bar{K})) \cap \alpha(K') \left(\mathrm{bd}(\mathrm{conv}(\bar{K} \cup (-\bar{K})))\right)$.

Observe that from the convexity of $\bar{K}$ we directly obtain that $s\gamma \leq 1$. Thus, we want to characterize the situation, in which s becomes maximal, under the condition $s\gamma \leq 1$. To do this, we compute the explicit value of γ in dependence of s.

By Lemma 3.3, there exists a pair of halfspaces $H^{\leq}_{\pm u, \frac{1}{\alpha}\rho}$ supporting $\mathrm{conv}(\bar{K} \cup (-\bar{K}))$ in $\pm\frac{1}{\alpha}p$. In the following, we assume w.l.o.g. that $p = \left(\begin{smallmatrix}\alpha \\ 0\end{smallmatrix}\right)$ and, since we may choose the above hyperplanes to be orthogonal to $[-p, p]$, $u = \left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right)$ and $\rho = \alpha$.

Remembering that we have $\mathrm{conv}(\bar{K} \cup (-\bar{K})) = \mathrm{conv}(\{\pm sp^1, \pm sp^2, \pm sp^3\})$ and that, following the notations from Lemma 4.1, we assumed $p = z^{3,2}$, $-p = z^{2,1}$, we see that $\frac{1}{\alpha}p \in [sp^2, -sp^3]$ and $-\frac{1}{\alpha}p \in [-sp^2, sp^3]$. Hence, $sp^2 \in H^{\leq}_{\left(\begin{smallmatrix}1 \\ 0\end{smallmatrix}\right),1}$ and $sp^3 \in H^{\leq}_{\left(\begin{smallmatrix}-1 \\ 0\end{smallmatrix}\right),1}$ and therefore the first coordinates of p^2 and p^3 are $\frac{1}{s}$ and $-\frac{1}{s}$, respectively. Knowing this fact and remembering that p^2, p^3 are located on the same side of $\mathrm{aff}\{-p, p\}$, we may further assume that $p^2 = \begin{pmatrix}1/s \\ -a\end{pmatrix}$ and $p^3 = \begin{pmatrix}-1/s \\ -1\end{pmatrix}$ for some $a \in (0, 1]$.

Let d^2 denote the intersection point of $H^{=}_{\left(\begin{smallmatrix}-1 \\ 0\end{smallmatrix}\right),1}$ and $\mathrm{aff}(\{-p, p^3\})$. Hence, $d^2 = -p + \mu(-p + p^3) = \left(\begin{smallmatrix}-\alpha \\ 0\end{smallmatrix}\right) + \mu\begin{pmatrix}-\alpha + \frac{1}{s} \\ 1\end{pmatrix}$ for some $\mu > 0$ and we directly see that $d^2_2 = \mu$. Now, since we have $d^2_1 = -1$ we obtain $d^2_2 = \frac{s(1-\alpha)}{s\alpha - 1}$. Altogether,

$$d^2 = \begin{pmatrix}-1 \\ \frac{s(1-\alpha)}{s\alpha-1}\end{pmatrix}.$$

By definition, d^2 should be in $[-\frac{1}{\alpha}p, -sp^2]$, which implies, because of $-sp^2 = \left(\begin{smallmatrix}-1 \\ sa\end{smallmatrix}\right)$, that $\frac{s(1-\alpha)}{s\alpha-1} \leq sa$ or, equivalently, $a \geq \frac{1-\alpha}{s\alpha-1}$.

Now, we calculate the coordinates of d^1 as the intersection point of the lines $\operatorname{aff}\{p,p^2\}$ and $\operatorname{aff}\{-p,p^3\}$. We obtain that those coordinates satisfy the following system of equations:

$$d_2^1 = \frac{sa}{s\alpha-1}d_1^1 - \frac{sa\alpha}{s\alpha-1},$$
$$d_2^1 = \frac{-s}{s\alpha-1}d_1^1 - \frac{s\alpha}{s\alpha-1}.$$

Solving, gives us

$$d^1 = \begin{pmatrix} \frac{\alpha(a-1)}{a+1} \\ \frac{-2sa\alpha}{(s\alpha-1)(a+1)} \end{pmatrix}.$$

Now, we compute γ such that

$$-\gamma d^1 \in [-sp^2, -sp^3] = \left[\begin{pmatrix} -1 \\ sa \end{pmatrix}, \begin{pmatrix} 1 \\ s \end{pmatrix}\right].$$

This means we are looking for some $\lambda \in [0,1]$, such that

$$-\gamma \begin{pmatrix} \frac{\alpha(a-1)}{a+1} \\ \frac{-2sa\alpha}{(s\alpha-1)(a+1)} \end{pmatrix} = (1-\lambda)\begin{pmatrix} -1 \\ sa \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ s \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ s((1-\lambda)a+\lambda) \end{pmatrix},$$

and it is easy to verify that this implies

$$\gamma^{-1} = \frac{2\alpha}{(a+1)^2}\left(\frac{2a}{s\alpha-1} - \frac{(a-1)^2}{2}\right).$$

One may check that, since $a \geq \frac{1-\alpha}{s\alpha-1}$, we have $\gamma^{-1} \geq 0$.

Thus, finding the maximal s with $s\gamma \leq 1$ may now be rewritten as

$$\max s, \text{ such that } s \leq \frac{-2\alpha}{(a+1)^2}\left(\frac{2a}{1-s\alpha} + \frac{(a-1)^2}{2}\right),$$

which can be transformed into

$$s^2 + \left(\alpha\left(\frac{a-1}{a+1}\right)^2 - \frac{1}{\alpha}\right)s - 1 \leq 0.$$

We are interested in the maximal s, i.e., in the larger of the two roots of the quadratic on the left, which is (independently of α and a)

$$s = \frac{1}{2}\left(\frac{1}{\alpha} - \alpha\left(\frac{a-1}{a+1}\right)^2 + \sqrt{4 + \left(\alpha\left(\frac{a-1}{a+1}\right)^2 - \frac{1}{\alpha}\right)^2}\right) =: s(a,\alpha).$$

Hence, we obtain the maximal s (in dependence of α) from maximizing $s(a,\alpha)$ over a, where $\frac{1-\alpha}{s\alpha-1} \leq a \leq 1$. It is straightforward to verify that $s(\cdot,\alpha)$ is increasing in $(0,1]$. We conclude

$$\max s = s(1,\alpha) = \frac{1}{2\alpha} + \sqrt{1 + \frac{1}{4\alpha^2}}.$$

□

For any fixed $s \in (\varphi, 2]$ we define

$$K_{min,s} := \operatorname{conv}\left(\left\{\begin{pmatrix} \pm\frac{s}{s^2-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -s^2 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ s \end{pmatrix}\right\}\right) \quad \text{and}$$

$$K_{max,s} := \operatorname{conv}\left(\left\{\begin{pmatrix} \pm 1 \\ s(s^2-s-1) \end{pmatrix}, \begin{pmatrix} 0 \\ -s^2 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ s \end{pmatrix}\right\}\right).$$

The following lemma deals with the equality case of Lemma 4.2.

Lemma 4.3. *Let $K \in \mathcal{K}^2$ be Minkowski centered such that $s(K) > \varphi$ and $\alpha(K) = \frac{s(K)}{s(K)^2-1}$. Then there exists a non-singular linear transformation L such that*

$$K_{min,s(K)} \subset L(K) \subset K_{max,s(K)}.$$

Proof. We know from the proof of the previous lemma that $s := s(K) > \varphi$ and $\alpha := \alpha(K) = \frac{s}{s^2-1}$ imply $\alpha \in [\frac{2}{3}, 1]$ and $s = \frac{1+\sqrt{1+4\alpha^2}}{2\alpha}$. Moreover, equality holds in the maximization process of Lemma 4.2 if and only if $a = 1$, $\gamma = \frac{1}{s}$. If we use these values in the respective formulas above, we obtain for the set $\bar{K}$ from Lemma 4.1 that

$$\bar{K} := \operatorname{conv}\left(\left\{\begin{pmatrix} \pm\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} \pm\frac{1}{s} \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -s^2 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ s \end{pmatrix}\right\}\right).$$

However, since $\begin{pmatrix} \pm 1/s \\ -1 \end{pmatrix} = \frac{1}{s^2}\begin{pmatrix} 0 \\ -s^2 \end{pmatrix} + (1-\frac{1}{s^2})\begin{pmatrix} \pm s/(s^2-1) \\ 0 \end{pmatrix}$, we essentially have $\bar{K} = K_{min,s}$.

Thus, we have shown that the only Minkowski centered convex body with $s(K) > \varphi$ and $\alpha(K) = \frac{s(K)}{s(K)^2-1}$ of the form given in Lemma 4.1b) is $K_{min,s}$.

Next, we show that the only Minkowski centered convex body K' of the form (4) with $s(K') > \varphi$ and $\alpha(K') = \frac{s(K')}{s(K')^2-1}$ is still $K_{min,s}$. Notice that before Step 3 of Part (b) in Lemma 4.1, both segments, $[-sp^2, -p]$ and $[-sp^3, p]$ already belong to $\operatorname{bd}(K')$ and that from the previous paragraph the lines $\operatorname{aff}(\{-sp^1, p\})$, $\operatorname{aff}(\{-sp^1, -p\})$, $\operatorname{aff}(\{-sp^2, -sp^3\})$ support K'. Thus, K' must equal $K_{min,s}$ also before Step 3. In Step 2, nothing can change (as otherwise $-sp^1$ would lie outside K') and the same holds true in Step 1 (as only for $\mu = 1$ we have $-sp^1 \in K'$).

Finally, we investigate the freedom in the design of K before one applies the transformation (4). Let $d^2 = \begin{pmatrix} -1 \\ s(s^2-s-1) \end{pmatrix}$ denote the intersection point of $H^{=}_{\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right),1}$ and $\operatorname{aff}(\{-p, p^3\})$, as in Lemma 4.2. Moreover, let d^3 be defined, s.t. $\{d^3\} = H^{=}_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),1} \cap \operatorname{aff}(\{p, p^2\})$. The only possible freedom we have in choosing the original set K is to replace the linear boundaries $[p, -sp^3]$ and $[-p - sp^2]$, s.t. $\operatorname{bd}(K) \cap \operatorname{pos}(\{-sp^3, p\}) \subset \operatorname{conv}(\{-sp^3, p, d^3\})$ and $\operatorname{bd}(K) \cap \operatorname{pos}(\{-sp^2, -p\}) \subset \operatorname{conv}(\{-sp^2, -p, d^2\})$, respectively (c.f. Figure 7). However, since we need to ensure that $s(K) = s$, we also have to fulfill

$$\operatorname{bd}(K) \cap \operatorname{pos}(\{-sp^3, p\}) \not\subset \operatorname{int}(\operatorname{conv}(\{0, -sp^3, p\})) \quad \text{and}$$

$$\operatorname{bd}(K) \cap \operatorname{pos}(\{-sp^2, -p\}) \not\subset \operatorname{int}(\operatorname{conv}(\{0, -sp^2, -p\})).$$

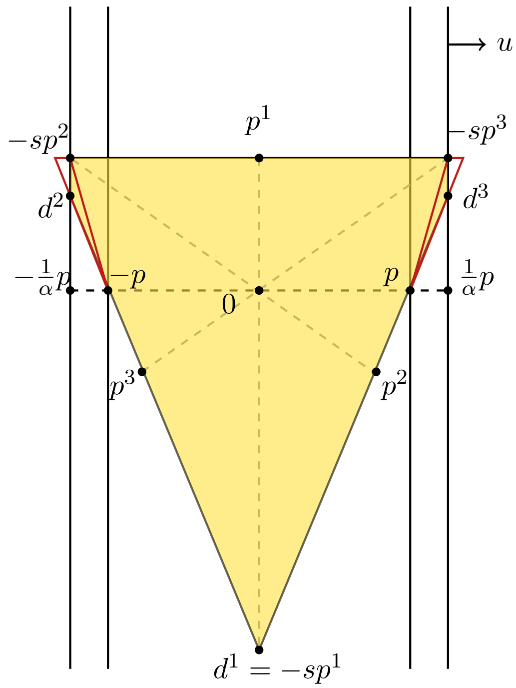


FIGURE 7. Construction used in the proof of Lemma 4.3, $K_{max,s}$ (yellow).

Assuming now, there exists some $x \in \mathrm{bd}(K) \cap \mathrm{pos}(\{-sp^3, p\}) \setminus \mathrm{conv}(\{-sp^3, p, d^3\})$, we would have that $[sp^2, x] \subset \mathrm{conv}(K \cup (-K))$. This would imply $x, \frac{1}{\alpha}p, sp^2 \in \mathrm{bd}(\mathrm{conv}(K \cup (-K))$ with $\frac{1}{\alpha}p \in \mathrm{pos}(\{sp^2, x\})$ and therefore $p \in \mathrm{bd}(K)$ as well as $\frac{1}{\alpha}p \in \mathrm{int}(\mathrm{conv}(\{p, x, sp^2\})) \subset \mathrm{int}(\mathrm{conv}(K \cup (-K)))$ and, since $\pm\frac{1}{\alpha}p$ were the only points in $\frac{1}{\alpha}K \cap (-K) \cap \mathrm{conv}(K \cup (-K))$, this would imply $\alpha(K) < \alpha(K') = \alpha$ in contradiction to our assumption that $\alpha(K) = \alpha$. Thus, $\mathrm{bd}(K) \cap \mathrm{pos}(\{-sp^3, p\}) \subset \mathrm{conv}(\{-sp^3, p, d^3\})$. Using a similar argument, it follows that $\mathrm{bd}(K) \cap \mathrm{pos}(\{-sp^2, -p\}) \subset \mathrm{conv}(\{-sp^2, -p, d^2\})$.

On the other hand, observe that we may choose $K = \mathrm{conv}\left(\{-sp^1, d^2, -sp^2, -sp^3, d^3\}\right)$, which equals $K_{max,s}$ up to a linear transformation (such that $K' = K_{min,s}$). All in all, we have shown that there always exists some linear transformation L such that $K_{min,s} \subset L(K) \subset K_{max,s}$.

Finally, let us realize that for any K with $s = s(K)$, $s \in (\varphi, 2]$, such that $K_{min,s} \subset K \subset K_{max,s}$ we have $\alpha(K) = \frac{s}{s^2-1}$. Notice that in the proof of [2, Theorem 1.7 a),(ii)] it is shown that $\alpha(K_{max,s}) = \frac{s}{s^2-1}$. Since $K_{min,s} \cap (-K_{min,s}) = K_{max,s} \cap (-K_{max,s})$ and $\mathrm{conv}(K_{min,s} \cup (-K_{min,s}))) = \mathrm{conv}(K_{max,s} \cup (-K_{max,s})))$, we have $\alpha(K_{min,s}) = \frac{s}{s^2-1}$, too. Thus, for any $K_{min,s} \subset K \subset K_{max,s}$ holds $\alpha(K) = \frac{s}{s^2-1}$. □

Now, we are ready to prove the main Theorem.

Proof of Theorem 1.1. In [2, Theorem 1.7] it is shown that $\frac{2}{s(K)+1} \leq \alpha(K) \leq 1$.

Combining this with Lemma 4.2, we obtain

$$\frac{2}{s(K)+1} \leq \alpha(K) \leq \min\left\{1, \frac{s(K)}{s(K)^2-1}\right\}.$$

Now, for any given $s \in [1,2]$ consider the set $K_{max,s}$ and notice that $u^1 \in \mathrm{bd}(K_{max,s}) \cap \mathrm{bd}(-K_{max,s}) \cap \mathrm{bd}(\mathrm{conv}(K_{max,s} \cup (-K_{max,s}))$ if $s \leq \varphi$, which shows $\alpha(K_{max,s}) = 1$. On the other hand, for $s > \varphi$, Lemma 4.2 shows that $\alpha(K_{max,s}) = \frac{s(K)}{s(K)^2-1}$.

We now show that for every $s \in [1,2]$ and $\alpha \in \left[\frac{2}{s+1}, \min\{1, \frac{s}{s^2-1}\}\right]$ there exists $K_{s,\alpha} \in \mathcal{K}^2$, such that $s(K_{s,\alpha}) = s$ and $\alpha(K_{s,\alpha}) = \alpha$. To do so, let $S = \mathrm{conv}(\{p^1, p^2, p^3\})$ be a regular Minkowski centered triangle and $K_s = S \cap (-sS)$, $s \in [1,2]$. By [2, Remark 4.1] we have that K_s is Minkowski centered with $s(K_s) = s$ and $\alpha(K_s) = \frac{2}{s+1}$. Moreover, defining q^i, $i = 2,3$, to be the vertices of K_s, which are the intersection point of the edges with the normal vectors $\frac{s}{2}p^i$ and $-\frac{1}{2}p^1$, respectively, we see that $-\frac{1}{2}p^1, -\frac{1}{s}q^2, -\frac{1}{s}q^3$ is a well-spread triple of asymmetry points of K_s.

The idea is to define a continuous transformation $f : \{K_s : s \in [1,2]\} \times [0,1] \to \mathcal{K}^2$ with $s(f(K_s,t)) = s$ for all $t \in [0,1]$, while $f(K_s, 0) = K_s$ and $\alpha(f(K_s,1)) = \min\left\{1, \frac{s(K)}{s(K)^2-1}\right\}$.

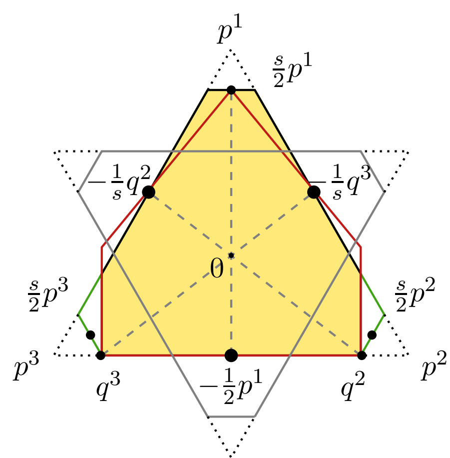


FIGURE 8. Transformation within the proof of Theorem 1.1: $-S \cap (sS)$ (gray), $K_s = S \cap (-sS)$ before the transformations (green), the transformed set after Step 1 (filled yellow), the transformed set after Step 2 (red), the asymmetry points $-\frac{1}{2}p^1, -\frac{1}{s}q^2, -\frac{1}{s}q^3$ (big black dots) of $f(K_s,t)$, $t \in [0,1]$, and S as well as $-S$ (dotted).

This is done in two steps:

Step 1 For $t \in [0, \frac{1}{2}]$, continuously rotate the lines containing the edges of K_s supporting the point $\frac{s}{2}p^2$ around q^2 and $\frac{s}{2}p^3$ around q^3, respectively, such that at the end of the transformation, the new edges are both orthogonal to the edge containing $-\frac{1}{2}p^1$.

Step 2 For $t \in [\frac{1}{2}, 1]$, continuously rotate the lines containing the edges of K_s, which contain $-\frac{1}{s}q^2$ and $-\frac{1}{s}q^3$, respectively, around those points, s.t. at the end of the transformation, the new edges intersect in $\frac{s}{2}p^1$.

For every $u \in \mathrm{bd}(\mathbb{B}_2)$ let $\rho(u) > 0$ be defined such that $\rho(u)u \in \mathrm{bd}(K_s)$. It is very simple to verify that $\rho(u)/\rho(-u) \in [1/s, s]$ for all $u \in \mathrm{bd}(\mathbb{B}_2)$ after each step. Thus, $-\frac{1}{s}f(K_s,t) \subset f(K_s,t)$, $t \in [0,1]$. Moreover, these transformations are done in a way that the asymmetry points $-\frac{1}{2}p^1, -\frac{1}{s}q^2, -\frac{1}{s}q^3 \in \mathrm{bd}(f(K_s,t))$, $t \in [0,1]$ are kept to be asymmetry points. Hence, by Proposition 2.1, $s(f(K_s,t)) = s$ for every $s \in [\varphi, 2]$. Recognize that $f(K_s,t)$ equals (up to a linear transformation) the corresponding $K_{max,s}$. Since the transformation f is continuous, $\alpha(f(K_s,0)) = \alpha(K_s) = \frac{2}{s+1}$, and $\alpha(f(K_s,1)) = \min\{1, \frac{s}{s^2+1}\}$, we conclude that $\{\alpha(f(K_s,t)) : t \in [0,1]\} = \left[\frac{2}{s+1}, \min\left\{1, \frac{s}{s^2-1}\right\}\right]$, for every $s \in [1,2]$, as desired. □

5. Diameter-width-ratio for (pseudo-)complete sets

For $C_1,\dots,C_k \in \mathcal{K}^n$ we say $C_1 \subset \dots \subset C_k$ is left-to-right optimal, if $C_1 \subset^{opt} C_k$.

We recall the characterization of pseudo-completeness from [4].

Proposition 5.1. *Let $K, C \in \mathcal{K}^n$ with $s(C) = 1$. Then the following are equivalent:*

(i) K is pseudo-complete w.r.t. C,
(ii) $(s(K)+1)r(K,C) = r(K,C) + R(K,C) = \frac{s(K)+1}{s(K)}R(K,C) = D(K,C)$, and
(iii) for every incenter c of K we have

$$\frac{s(K)+1}{2s(K)}(-(K-c)) \subset \frac{K-K}{2} \subset \frac{1}{2}D(K,C)C \subset \frac{s(K)+1}{2}(K-c)$$

is left-to-right optimal, which implies that c is also a circumcenter and a Minkowski center of K.

For $K \in \mathcal{K}^n$ a regular supporting slab of K is a pair of opposing supporting hyperplanes of K, such that at least one of the two hyperplanes supports K in a smooth boundary point. In case when K is a polytope, the latter means that at least one of the hyperplanes supports K in a whole facet.

In[9] a characterization of complete sets using the concept of regular supporting slabs is presented.

Proposition 5.2. *Let $K, C \in \mathcal{K}^n$. Then the following are equivalent:*

(i) K is complete w.r.t. C,
(ii) $b_s(K,C) = D(K,C)$ for all s such that s is the normal vector of a regular supporting slab of K.

Now, we are ready to prove the general dimension bound on the diameter-width ratio for (pseudo-)complete sets.

Recall that $\tau(K) = R\left(K \cap (-K), \frac{K-K}{2}\right) = r\left(\frac{K-K}{2}, K \cap (-K)\right)^{-1}$.

Proof of Theorem 1.3. We assume w.l.o.g. that K is Minkowski centered and $r(K,C)=1$. Abbreviating $s:=s(K)$ again, we obtain $D(K,C)=(s+1)r(K,C)=s+1$ and

$$\frac{K-K}{2}\subset\frac{D(K,C)}{2}C=\frac{s+1}{2}C\subset\frac{s+1}{2}K\cap(-K)$$

from Proposition 5.1. Thus, $C\subset K\cap(-K)$, which implies $w(K,C)\geq w(K,K\cap(-K))$ and since $w(K,K\cap(-K))=w\left(\frac{K-K}{2},K\cap(-K)\right)=2r\left(\frac{K-K}{2},K\cap(-K)\right)$ (see [11] for basic properties of the width)

$$\frac{D(K,C)}{w(K,C)}=\frac{s+1}{w(K,C)}\leq\frac{s+1}{2r\left(\frac{K-K}{2},K\cap(-K)\right)}=\frac{s+1}{2}\tau(K)\leq\frac{s+1}{2}. \tag{7}$$

Moreover, remember that $\alpha(K)=1$ if and only if $\tau(K)=1$ by Proposition 2.3 (ii).

Now, consider first the case n odd. Let $S=\operatorname{conv}(\{p^1,\dots,p^{n+1}\})$ be a regular Minkowski centered simplex, and for any $s\in[1,n]$ we define the sets $K=S\cap(-sS)$ and $C=S\cap(-S)$. Then $s(K)=s$, $K\cap(-K)=S\cap(-S)=C$ (see [2, Remark 4.1]) and since $\frac{K-K}{2}\subset^{opt}\frac{s+1}{2}(K\cap(-K))$ (see [2, Theorem 1.3]), we have

$$D(K,C)=2R\left(\frac{K-K}{2},C\right)=2R\left(\frac{K-K}{2},K\cap(-K)\right)=s+1.$$

Since all facets of K are facets of S or $-sS$, the normal vectors of the facets of K and C are exactly $\pm p^i$, $i\in\{1,\dots,n+1\}$ and therefore all the regular supporting slabs of K have those normal vectors. Now, since $\frac{s}{2}p^i,-\frac{1}{2}p^i\in\operatorname{bd}(K)$, while $\pm\frac{1}{2}p^i\in\operatorname{bd}(C)$ it follows that $b_{p^i}(K,C)=s+1=D(K,C)$ for all $i\in\{1,\dots,n\}$, and therefore the completeness of K w.r.t. C by Proposition 5.2.

Moreover, from [2], we know $S\cap(-S)\subset\frac{S-S}{2}\subset\operatorname{conv}(S\cup(-S))$ is left-to-right optimal for odd n. Using $K\cap(-K)=S\cap(-S)$ we obtain

$$K\cap(-K)\subset^{opt}\frac{S-S}{2}.$$

and since $K\subset S$ implies $\frac{K-K}{2}\subset\frac{S-S}{2}$ we conclude that

$$K\cap(-K)\subset^{opt}\frac{K-K}{2},$$

i.e., $\tau(K)=1$.

Hence, we see that, with this choice of K and $C=K\cap(-K)$, we have equality all through the inequality chain (7) for all $s\in[1,n]$.

Finally, for even n, let $K':=K\times[-1,1]$ and $C':=C\times\left[-\frac{2}{s+1},\frac{2}{s+1}\right]$ with $K,C\in\mathcal{K}^{n-1}$ as above. Then, $K',C'\in\mathcal{K}^n$, and we easily see that $s(K')\leq n-1$.

By [2, Theorem 1.3],

$$\frac{K-K}{2}\subset^{opt}\frac{s+1}{2}K\cap(-K)=\frac{s+1}{2}C.$$

Moreover, $\pm u^n \in \mathrm{bd}\left(\frac{K'-K'}{2}\right) \cap \mathrm{bd}\left(\frac{s+1}{2}C'\right)$. Hence, $\frac{K'-K'}{2} \subset^{opt} \frac{s+1}{2}C'$. Thus,

$$\frac{D(K',C')}{2} = R\left(\frac{K'-K'}{2}, C'\right) = \frac{s+1}{2}.$$

Notice, that the set of all regular supporting slabs of K' consists of those of K and the new additional one in the direction u^n. For all normal vectors u of such regular supporting slabs of K' we have $b_u(K',C') = s+1 = D(K',C')$. Hence, K' is complete w.r.t. C' by Proposition 5.2. □

Next, we state the proof for the even tighter diameter-width ratio bound for pseudo-complete sets in the planar case.

Proof of Theorem 1.4. Again, we may assume w.l.o.g. that K is Minkowski centered and use the abbreviation $s := s(K)$. It is easy to see that $\mathrm{conv}(K \cup (-K)) \subset \frac{2s}{s+1}\frac{K-K}{2}$ (c.f. [2, Theorem 1]). Thus,

$$K \cap (-K) \subset \alpha(K)\mathrm{conv}(K \cup (-K)) \subset \alpha(K)\frac{2s}{s+1}\frac{K-K}{2},$$

which implies $\tau(K) \leq \frac{2s}{s+1}\alpha(K)$. Using the inequality chain (7), we obtain the tighter bound

$$\frac{D(K,C)}{w(K,C)} \leq \frac{s+1}{2}\tau(K) \leq \frac{s+1}{2}\min\left\{1, \frac{2s}{s+1}\frac{s}{s^2-1}\right\} = \min\left\{\frac{s+1}{2}, \frac{s^2}{s^2-1}\right\}.$$

Finally, let $\tilde{s} := \max_{s\in[1,2]}\min\left\{\frac{s+1}{2}, \frac{s^2}{s^2-1}\right\}$. Since $\frac{s+1}{2}$ is increasing and $\frac{s^2}{s^2-1}$ decreasing in $s \in [1,2]$, we see that $\tilde{s}$ is the solution of the equation $\frac{s+1}{2} = \frac{s^2}{s^2-1}$ and has the value

$$\tilde{s} = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \approx 1.8393.$$

Thus, we obtain for all pseudo-complete K

$$\frac{D(K,C)}{w(K,C)} \leq \frac{\tilde{s}+1}{2} \approx 1.42,$$

independently of the asymmetry of K. □

Remark 5.3. *Of course, there exist Minkowski centered $K \in \mathcal{K}^2$ with $\alpha(K) \neq \tau(K)$* [2, Example 4.3]. *Observe, however, that for any $s \in [\varphi, 2]$ and K such that $K_{min,s} \subset K \subset K_{max,s}$ with $s(K) = s$, we have $\alpha(K) = \tau(K)$.*

In the proof of Theorem 1.1 we have shown that $\alpha(K) = \frac{s}{s^2-1}$. Note that $K \cap (-K)$ is a hexagon, which we can denote by $\mathrm{conv}\{-p, p, q^1, q^2, q^3, q^4\}$, where p is defined as in the proof of Theorem 1.1. Hence, the touching points of $K \cap (-K)$ and $\mathrm{bd}\left(\tau(K)\frac{K-K}{2}\right)$ must be vertices of $K \cap (-K)$. From the proof of [2, Theorem 1.7 a),(ii)] *we also have $p \in \frac{s}{s^2-1}\mathrm{bd}\left(\frac{K-K}{2}\right)$ and $q^i \in \frac{s+1}{2}\mathrm{bd}\left(\frac{K-K}{2}\right)$, $1 \leq i \leq 4$. Since $s \geq \varphi$, this implies $\tau(K) = \min\left\{\frac{s}{s^2-1}, \frac{2}{s+1}\right\} = \frac{s}{s^2-1}$.*

Observe that if one could show

$$\tau(K) \le \min\left\{1, \frac{s(K)}{s(K)^2-1}\right\},$$

the bound in Theorem 1.4 could be improved to

$$\frac{D(K,C)}{w(K,C)} \le \min\left\{\frac{s(K)+1}{2}, \frac{s(K)}{2(s(K)-1)}\right\} \le \frac{D(\mathbb{GH}, \mathbb{GH}\cap(-\mathbb{GH}))}{w(\mathbb{GH}, \mathbb{GH}\cap(-\mathbb{GH}))} = \frac{\varphi+1}{2} \approx 1.31.$$

Remark 5.4. *We show that for every pair (ρ, s), with $s \in [1,2]$ and $1 \le \rho \le \min\left\{\frac{s+1}{2}, \frac{s}{2(s-1)}\right\}$, there exists some Minkowski centered K, s.t. $s(K)$=s and a set C, s.t. K is pseudo-complete w.r.t. C and $\frac{D(K,C)}{w(K,C)} = \rho$ (c.f. Figure 9).*

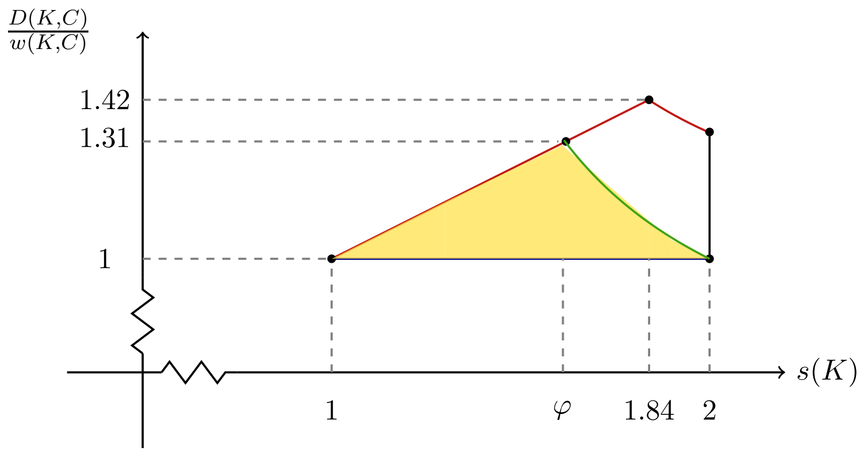


FIGURE 9. Region of all possible values for the diameter-width ratio for pseudo complete sets K in dependence of their Minkowski asymmetry $s(K)$: $\frac{D(K,C)}{w(K,C)} \ge 1$ (blue); $\frac{D(K,C)}{w(K,C)} \le \min\left\{\frac{s(K)+1}{2}, \frac{s(K)^2}{s(K)^2-1}\right\}$ (red). Construction from Remark 5.4: $\left\{\frac{D(K,C_\lambda)}{w(K,C_\lambda)}, 0 \le \lambda \le 1\right\} = \left[1, \min\left\{\frac{s(K)+1}{2}, \frac{s(K)}{2(s(K)-1)}\right\}\right]$ (yellow, with $\frac{s(K)}{2(s(K)-1)}$ in green).

To do so, let $K \in \mathcal{K}^2$ be Minkowski centered with $K := K_{max,s}$ (where $K_{max,s}$ is defined as in Section 5) and $s := s(K) \in [1,2]$. Then define $C_\lambda = (1-\lambda)\left(\frac{K-K}{2}\right) + \lambda\frac{s+1}{2}(K\cap(-K))$ with $\lambda \in [0,1]$. This way C_λ is a convex combination of $\frac{K-K}{2}$ and $\frac{s+1}{2}(K\cap(-K))$, and therefore $K \in \mathcal{K}^2_{ps,C_\lambda}$ with $D(K, C_\lambda) = 2$ by Proposition 5.1.

By Remark 5.3, we have $\tau(K) = \min\left\{1, \frac{s}{s^2-1}\right\}$ and

$$w(K,C_1) = w\left(\frac{K-K}{2}, \frac{s+1}{2}K\cap(-K)\right) = \frac{2}{s+1}w\left(\frac{K-K}{2}, \frac{s+1}{2}K\cap(-K)\right)$$
$$= \frac{4}{s+1}r\left(\frac{K-K}{2}, \frac{s+1}{2}K\cap(-K)\right) = \frac{4}{s+1}\frac{1}{\tau(K)}.$$

Thus,

$$\frac{D(K,C_1)}{w(K,C_1)} = \frac{s+1}{2}\tau(K).$$

Moreover, $\frac{D(K,C_0)}{w(K,C_0)} = 1$. *Hence,*

$$\left\{\frac{D(K,C_\lambda)}{w(K,C_\lambda)}, 0 \le \lambda \le 1\right\} = \left[1, \min\left\{\frac{s+1}{2}, \frac{s}{2(s-1)}\right\}\right].$$

Note that $\frac{s+1}{2}$ *is increasing, while* $\frac{s}{2(s-1)}$ *is decreasing on* $s \in [1,2]$. *Thus,* $\min\left\{\frac{s+1}{2}, \frac{s}{2(s-1)}\right\}$ *attains its maximum of* $\frac{\varphi+1}{2} \approx 1.31$, *when* $s = \varphi$.

We conclude the paper with a consideration of the diameter-width ratio of pseudo-complete sets in the euclidean plane. We do this by first recalling the definition of the hood from [3] ($\mathbb{H}_{min}$ there).

The *hood* may be defined by

$$\mathbb{H} := \operatorname{conv}\left(\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}r\\\sqrt{1-r^2}\end{pmatrix}, \begin{pmatrix}-r\\\sqrt{1-r^2}\end{pmatrix}\right\} \cup (r \cdot \mathbb{B}_2)\right),$$

where

$$r = \frac{\sqrt{t}}{2} - 1 + \sqrt{\frac{16}{\sqrt{t}} - t}, \quad \text{and} \quad t = 2\left(\frac{2}{3}\right)^{\frac{2}{3}}\left((9+\sqrt{69})^{\frac{1}{3}} + (9-\sqrt{69})^{\frac{1}{3}}\right).$$

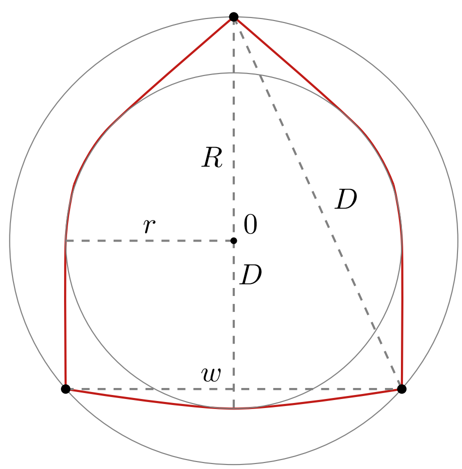


FIGURE 10. The hood $\mathbb{H}$ (red); $r(\mathbb{H},\mathbb{B}_2)\mathbb{B}_2$ and $R(\mathbb{H},\mathbb{B}_2)\mathbb{B}_2$ (gray).

As shown in [3], the hood has the following properties. First of all,

$$r(\mathbb{H},\mathbb{B}_2)\mathbb{B}_2 \subset \mathbb{H} \subset R(\mathbb{H},\mathbb{B}_2)\mathbb{B}_2,$$

with $R(\mathbb{H},\mathbb{B}_2) = 1$ and $r(\mathbb{H},\mathbb{B}_2) = r \approx 0.7935$.

The triangle $\operatorname{conv}\left(\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}r\\\sqrt{1-r^2}\end{pmatrix}, \begin{pmatrix}-r\\\sqrt{1-r^2}\end{pmatrix}\right\}\right)$ is isosceles, with the long edges of length $D(\mathbb{H},\mathbb{B}_2) = r+1$ and the short of length $w(\mathbb{H},\mathbb{B}_2) = 2r$. Moreover, we have

$$r(\mathbb{H},\mathbb{B}_2) + R(\mathbb{H},\mathbb{B}_2) = D(\mathbb{H},\mathbb{B}_2),$$

thus $\mathbb{H} \in \mathcal{K}^2_{ps,\mathbb{B}_2}$ by Proposition 5.1 and $s(\mathbb{H}) = \frac{R(\mathbb{H},\mathbb{B}_2)}{r(\mathbb{H},\mathbb{B}_2)} = \frac{1}{r(\mathbb{H},\mathbb{B}_2)} \approx 1.27$. Thus,

$$\frac{D(\mathbb{H},\mathbb{B}_2)}{w(\mathbb{H},\mathbb{B}_2)} = \frac{r+1}{2r} = \frac{\frac{1}{s(\mathbb{H})}+1}{2\frac{1}{s(\mathbb{H})}} = \frac{s(\mathbb{H})+1}{2} \approx 1.135.$$

Proof of Theorem 1.5. Let $K \in \mathcal{K}^2_{ps,\mathbb{B}_2}$, i.e. $r(K,\mathbb{B}_2) = D(K,\mathbb{B}_2) - R(K,\mathbb{B}_2)$. W.l.o.g., we assume $R(K,\mathbb{B}_2) = 1$.

Now, on the one hand, since $2r(K,\mathbb{B}_2) \le w(K,\mathbb{B}_2)$, it follows

$$\frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \frac{D(K,\mathbb{B}_2)}{2r(K,\mathbb{B}_2)} = \frac{D(K,\mathbb{B}_2)}{2(D(K,\mathbb{B}_2) - R(K,\mathbb{B}_2))} = \left(2\left(1 - \frac{1}{D(K,\mathbb{B}_2)}\right)\right)^{-1}$$

However, $2\left(1-\frac{1}{x}\right)$ is an increasing function for any positive values of x, and therefore

$$\max_{D(K,\mathbb{B}_2)\in[D(\mathbb{H},\mathbb{B}_2),2]} \frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \left(2\left(1 - \frac{1}{D(\mathbb{H},\mathbb{B}_2)}\right)\right)^{-1} = \frac{D(\mathbb{H},\mathbb{B}_2)}{w(\mathbb{H},\mathbb{B}_2)}.$$

On the other hand, since $r(K,\mathbb{B}_2) = D(K,\mathbb{B}_2) - 1$, we obtain from [3, Theorem 3.2]

$$\begin{aligned}
\frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \Bigg(& 2\sqrt{1-\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right)^2}\cos\Bigg[\arccos\left(\frac{D(K,\mathbb{B}_2)}{2(D(K,\mathbb{B}_2)-r(K,\mathbb{B}_2))}\right) \\
&+\arccos\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right) - \arcsin\left(\frac{r(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)-r(K,\mathbb{B}_2)}\right)\Bigg]\Bigg)^{-1} \\
&= \left(2\sqrt{1-\left(\frac{D(K,\mathbb{B}_2)}{2}\right)^2}\cos\left(2\arccos\left(\frac{D(K,\mathbb{B}_2)}{2}\right) - \arcsin\left(r(K,\mathbb{B}_2)\right)\right)\right)^{-1} \\
&= \left(\sqrt{4-D(K,\mathbb{B}_2)^2}\cos\left(2\arccos\left(\frac{D(K,\mathbb{B}_2)}{2}\right) - \arcsin\left(D(K,\mathbb{B}_2)-1\right)\right)\right)^{-1}.
\end{aligned}$$

It is easy to verify that $\sqrt{4-x^2}\cos(2\arccos(\frac{x}{2}) - \arcsin(x-1))$ is decreasing for $x \ge \sqrt{3}$. Thus,

$$\max_{D(K,\mathbb{B}_2)\in[\sqrt{3},D(\mathbb{H},\mathbb{B}_2)]} \frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \frac{D(\mathbb{H},\mathbb{B}_2)}{w(\mathbb{H},\mathbb{B}_2)}.$$

Since $[\sqrt{3},2]$ covers the full range of possible diameters for pseudo-complete sets with circumradius 1 [3] and because $\mathbb{H}$ attains equality in each of the two inequalities derived above, we conclude

$$\frac{D(K,\mathbb{B}_2)}{w(K,\mathbb{B}_2)} \le \frac{D(\mathbb{H},\mathbb{B}_2)}{w(\mathbb{H},\mathbb{B}_2)} = \frac{s(\mathbb{H})+1}{2} \approx 1.135$$

for $K \in \mathcal{K}^2_{ps,\mathbb{B}_2}$. □

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