Recurrence for the frog model with drift on $\mathbb{Z}$

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Abstract

The frog model is a system of interacting random walks. We give a criterion for recurrence/transience for the frog model in a particular example. In our case, the underlying random walk is an asymmetric random walk on $\mathbb{Z}$, and the starting configuration is i.i.d. We also show that in this example, the probability to visit the origin infinitely often satisfies a $0-1-$law.

KEYWORDS: Frog model, recurrence and transience, interacting random walks, $0-1-$law.

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1 Introduction

The frog model can be described as follows. Let G be a graph and take one vertex to be the origin. Initially there is a number of sleeping particles ("frogs") at each site of the graph G except at the origin. The origin contains one active frog. The active frog then starts a discrete-time simple random walk on the vertices of G. Each time an active frog visits a site with sleeping frogs the latter become active and start moving according to the same random walk as the active frogs, independently from everything else. An interpretation of the model is the distribution of information: Active frogs hold some information and share them with sleeping frogs as soon as they meet. The sleeping frogs become active and start helping in the process of spreading the information (cf. [4]). The frog model can also be interpreted as a "once-branching" random walk, i.e. a branching random walk where branching takes only place in a site which is visited for the first time.

In this note we consider the example "frogs with drift" given as follows: $G = \mathbb{Z}$ and the underlying random walk is an asymmetric random walk. Fix $p \in \left(\frac{1}{2}, 1\right)$: in each step an active frog moves to the right with probability p and to the left with probability $1-p$. For $x \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ we denote by η_x the number of sleeping frogs initially in x. We are interested in recurrence and transience, i.e. whether the probability of having infinitely many visits to the origin is 1 or strictly less than 1. We give a necessary and sufficient condition for the starting configuration to ensure recurrence. In the case of an i.i.d. starting configuration, we show that a $0-1-$law holds: The probability of having infinitely many visits to the origin is either 0 or 1. For a symmetric random walk on $\mathbb{Z}^d$, the frog model (starting with one frog at each site) is known to be recurrent, cf. [2] and [3]. There are variants of the model where the frogs have random lifetimes, and one is interested in survival/extinction of the process (and its dependence on the parameters), see [4]. Another question which has been investigated for the model is the existence of shape theorems. We give some open problems in Section 3. For background on the model and further open problems, we refer to [4].

2 Results

For a fixed starting configuration $\eta = (\eta_x)_{x\in\mathbb{Z}^*}$, we denote the probability measure for the evolution of the frog configuration by P_η.

Definition 2.1. The frog model with starting configuration η is recurrent if the P_η-probability that the origin is occupied infinitely often (with an active frog) is 1. Otherwise the model is transient.

First, we present a necessary and sufficient condition for recurrence for an arbitrary starting configuration η. Let

$$\rho := \frac{1-p}{p}\,.$$

Theorem 2.1. *Consider frogs with drift with starting configuration η. The model is recurrent (in the sense of Definition 2.1) if and only if*

$$\sum_{j=1}^{\infty} \eta_j \rho^j = \infty\,.$$

Remark 2.1. The weight ρ^j is the probability of a frog starting in $j \in \mathbb{N}$ to ever visit the origin. Notice that the frog starting at the origin activates all the frogs on the right of the origin as it performs a random walk with drift to the right.

Proof. (i) We show that the condition $\sum_{j=1}^{\infty} \eta_j \rho^j = \infty$ is sufficient for recurrence. Consider the probability that the site $-x$, $x \in \mathbb{N}$, is visited by a frog starting on the right of the origin. It suffices to show that for each $x \in \mathbb{N}$ this probability is 1. Fix $x \in \mathbb{N}$. Then

$$P_\eta\left[\text{one of the frogs starting on the right of the origin visits } -x\right]$$

$$= 1 - \prod_{j=1}^{\infty} P_\eta \left[\text{none of the frogs starting in } j \text{ visits } -x\right]$$
$$= 1 - \prod_{j=1}^{\infty} \left(1 - \rho^{x+j}\right)^{\eta_j}$$

and the infinite product equals 0, since $\sum_{j=1}^{\infty} \eta_j \rho^j = \infty$.
(ii) Assume $\sum_{j=1}^{\infty} \eta_j \rho^j < \infty$. We show that P_η [the origin is visited infinitely often] $<$ 1. It suffices to show that the probability that the vertex -1 is visited eventually is strictly less than 1. But

$$P_\eta \left[-1 \text{ is never visited}\right] = (1-\rho) \prod_{j=1}^{\infty} \left(1 - \rho^{j+1}\right)^{\eta_j} > 0 .$$

□

We now choose the starting configuration at random. Let $(\eta_x)_{x \in \mathbb{Z}^*}$ be a collection of i.i.d. nonnegative integer-valued random variables. Let μ and $\mathbb{E}$ denote the probability and expectation with respect to the initial configuration. We assume w.l.o.g. that $\mu[\eta_1 > 0] > 0$.

Theorem 2.2. *Consider frogs with drift. We have*

$$P_\eta \left[\textit{the origin is visited infinitely often}\right] = \begin{cases} 0 \ \mu\text{-a.s.} & \textit{if } \ \mathbb{E}\left[\log^+ \eta_1\right] < \infty \\ 1 \ \mu\text{-a.s.} & \textit{if } \ \mathbb{E}\left[\log^+ \eta_1\right] = \infty. \end{cases}$$

In particular, once the law of η_1 is given, the value of p does not change the probability to visit the origin infinitely often.

In order to prove Theorem 2.2, we will show the following.

Theorem 2.3. *Consider frogs with drift. Let*

$$A = \left\{ \eta : \sum_{j=1}^{\infty} \eta_j \rho^j < \infty \right\} .$$

Then we have

$$P_\eta \left[\textit{the origin is visited infinitely often}\right] = \begin{cases} 0 \ \mu\text{-a.s.} & \textit{on } A \\ 1 \ \mu\text{-a.s.} & \textit{on } A^c. \end{cases}$$

Proof of Theorem 2.2. We first explain how Theorem 2.2 follows from Theorem 2.3. Exercise 22.10 in [1] implies that $\mu(A)$ equals 1 or 0 depending on whether $\mathbb{E}\left[\log^+ \eta_1\right]$ is finite or not. More precisely, for i.i.d. random variables $Y_1, Y_2, \ldots$ with $P[Y_1 > 0] >$

0, the radius of convergence of the (random) power series $\sum\limits_{n=1}^{\infty} Y_n z^n$ is either $+\infty$ with probability one, if $E\left[\log^+|Y_1|\right] < \infty$, or 0 with probability one, if $E\left[\log^+|Y_1|\right] = \infty$. Sketch of proof: It follows from Kolmogorov's $0-1$-law that the radius of convergence equals, a.s., a constant r (possibly infinite). Use the Cauchy-Hadamard criterion $r = \left(\limsup |Y_n|^{1/n}\right)^{-1}$ and note that $\limsup \frac{1}{n}\log|Y_n| = 0$, a.s. if $E\left[\log^+|Y_1|\right] < \infty$ and $\limsup \frac{1}{n}\log|Y_n| = \infty$, a.s. if $E\left[\log^+|Y_1|\right] = \infty$. $\square$

We now proceed to the proof of Theorem 2.3.

Proof of Theorem 2.3. Taking into account Theorem 2.1, it remains to show that if $\sum_{j=1}^{\infty}\eta_j\rho^j < \infty$, μ-a.s., then for μ-a.a. η, the origin is visited P_η-a.s. only finitely many times. We will show that P_η-a.s. there exists an integer $M \in \mathbb{N}$ such that $-M$ is never visited. Then, using the Borel-Cantelli lemma, there is only a finite number of visits to the origin. For $x \in \mathbb{N}$ let B_x be the event that $-x$ is visited eventually.
(i) Let B_x^+ be the event that $-x$ is visited by a frog starting on the right of the origin. Then

$$P_\eta\left[B_x^+\right] \leq \sum_{i=1}^{\infty}\eta_i\rho^{x+i} = \rho^x\sum_{i=1}^{\infty}\eta_i\rho^i$$

Hence

$$\sum_{x=1}^{\infty}P_\eta\left[B_x^+\right] \leq \sum_{i=1}^{\infty}\eta_i\rho^i\sum_{x=1}^{\infty}\rho^x < \infty \qquad \mu\text{-a.s.}$$

The Borel-Cantelli lemma implies that a.s. only finitely many of the events B_x^+ occur. So a.s. there exists an integer M_1 such that $-M_1$ will not be visited by a frog starting on the right of the origin.
(ii) Fix N and consider the events B_{Nm} for $m \in \mathbb{N}$. We divide the interval $(-Nm, 0]$ into N subintervals, $I_k = (-km, -(k-1)m]$, $k = 1, \ldots, N$, of length m. Let $A_{k,m}$ be the event that a frog from the k-th interval reaches the interval I_{k+2}, i.e. goes to the left through the whole interval I_{k+1}, $k = 1, \ldots, N-2$. Then we have

$$\begin{aligned} P_\eta\left[A_{k,m}\right] &\leq \sum_{i=-km+1}^{-(k-1)m}\eta_i\rho^{i+(k+1)m} \\ &= \rho^m\sum_{i=-km+1}^{-(k-1)m}\eta_i\rho^{i+km} \end{aligned}$$

Hence for $A_m^{(N)} = \bigcup_{k=1}^{N-2}A_{k,m}$ we get

$$\begin{aligned} P_\eta\left[A_m^{(N)}\right] &\leq \sum_{k=1}^{N}\rho^m\sum_{i=-km+1}^{-(k-1)m}\eta_i\rho^{i+km} \\ &= \sum_{k=1}^{N}\rho^m Z_{k,m} \end{aligned}$$

where

$$Z_{k,m} = \sum_{i=-km+1}^{-(k-1)m} \eta_i \rho^{i+km}$$

The random variables $Z_{k,m}$, $k = 1, \dots, N$, are i.i.d. We want to prove the existence of a subsequence $(m_\ell)_{\ell \in \mathbb{N}}$ of $\mathbb{N}$ such that

$$\sum_{\ell=1}^{\infty} P_\eta \left[A_{m_\ell}^{(N)} \right] \leq \sum_{\ell=1}^{\infty} \sum_{k=1}^{N} \rho^{m_\ell} Z_{k,m_\ell} < \infty \qquad \mu\text{-a.s.} \tag{1}$$

If (1) holds, again by the Borel-Cantelli lemma, there is P_η-a.s. an index $L \in \mathbb{N}$, such that for $\ell \geq L$ the events $\left(A_{m_\ell}^{(N)} \right)^c$ occur. To prove the existence of a subsequence m_ℓ satisfying (1), it suffices (by Borel-Cantelli) to show that for some $\alpha > 1$ there is a subsequence m_ℓ such that

$$\sum_{\ell=1}^{\infty} \mu \left[\sum_{k=1}^{N} Z_{k,m_\ell} \geq \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] < \infty \, . \tag{2}$$

Since the Z_{k,m_ℓ}, $k = 1, \dots, N$, are i.i.d., we have

$$\begin{aligned} \mu \left[\sum_{k=1}^{N} Z_{k,m_\ell} \geq \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] &\leq N \cdot \mu \left[Z_{1,m_\ell} \geq \frac{1}{N} \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] \\ &= N \cdot \mu \left[\sum_{i=1}^{m_\ell} \eta_i \rho^i \geq \frac{1}{N} \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] \\ &\leq N \cdot \mu \left[\sum_{i=1}^{\infty} \eta_i \rho^i \geq \frac{1}{N} \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] \end{aligned}$$

According to our assumption, the random variable $\sum_{i=1}^{\infty} \eta_i \rho^i$ is finite, μ-a.s. This implies that we can choose the subsequence $(m_\ell)_{\ell \in \mathbb{N}} = (m_\ell(N))_{\ell \in \mathbb{N}}$ in such a way that

$$\mu \left[\sum_{i=0}^{\infty} \eta_i \rho^i \geq \frac{1}{N} \rho^{-m_\ell} \frac{1}{m_\ell^\alpha} \right] \leq \frac{1}{\ell^2}$$

for all $\ell \in \mathbb{N}$, and the right hand side is summable over ℓ, hence (2) holds for this choice of the subsequence.

(iii) If there is no interval I_k such that a frog from I_k makes it to the interval I_{k+2}, the frogs in $-Nm$ can only be activated by frogs from the intervals I_{N-1} and I_N (if we make N and m sufficiently large we can ignore the frogs starting on the right side of the origin because of (i)). Let $C_{N,m}$ be the event that no frog from I_{N-1} and I_N reaches $-Nm$. Now we choose a further subsequence $(m_{\ell_r})_{r \in \mathbb{N}}$ of $(m_\ell)_{\ell \in \mathbb{N}}$ such that the intervals $(-Nm_{\ell_r}, -(N-2)m_{\ell_r}]$ are disjoint for different r. Then we get the following estimate for the conditional probabilities of the $C_{N,m_{\ell_r}}$ (for $r, s \in \mathbb{N}$, $r > s$):

$$P_\eta \left[C_{N,m_{\ell_r}} \left| \bigcap_{j=s}^{r-1} C^c_{N,m_{\ell_j}} \right. \right] \geq \prod_{j=1}^{2m_{\ell_r}} \left(1-\rho^j\right)^{\eta(-Nm_{\ell_r}+j)} \tag{3}$$

since in the worst case all frogs in the interval $I_N \cup I_{N-1} = (-Nm_{\ell_r}, -(N-2)m_{\ell_r}]$ get activated. We will show that

$$\sum_{r=s}^{\infty} P_\eta \left[C_{N,m_{\ell_r}} \left| \bigcap_{j=s}^{r-1} C^c_{N,m_{\ell_j}} \right. \right] = \infty \qquad \forall\, s \in \mathbb{N} \qquad \mu\text{-a.s.} \tag{4}$$

Taking logarithms, we have

$$\log P_\eta \left[C_{N,m_{\ell_r}} \left| \bigcap_{j=s}^{r-1} C^c_{N,m_{\ell_j}} \right. \right] \geq \sum_{j=1}^{2m_{\ell_r}} \eta_{(-Nm_{\ell_r}+j)} \log\left(1-\rho^j\right) .$$

There is a constant $c(\rho) > 0$ such that $\log(1-x) \geq -c(\rho)x$ for $0 < x < \rho$, hence

$$\sum_{j=1}^{2m_{\ell_r}} \eta_{(-Nm_{\ell_r}+j)} \log\left(1-\rho^j\right) \geq -c(\rho) \sum_{j=1}^{2m_{\ell_r}} \eta_{(-Nm_{\ell_r}+j)} \rho^j ,$$

yielding

$$P_\eta \left[C_{N,m_{\ell_r}} \left| \bigcap_{j=s}^{r-1} C^c_{N,m_{\ell_j}} \right. \right] \geq \exp\left(-c(\rho) Y_r(\eta)\right) \tag{5}$$

where

$$Y_r(\eta) = \sum_{j=1}^{2m_{\ell_r}} \eta_{(-Nm_{\ell_r}+j)} \rho^j .$$

Since the intervals $(-Nm_{\ell_r}, -(N-2)m_{\ell_r}]$ are disjoint for different r, $(Y_r(\eta))_{r=1,2,\dots}$ are independent random variables, whose distribution is dominated by the law of $Y := \sum_{j=1}^{\infty} \eta_j \rho^j$. We conclude that for some constant C_Y, $\liminf_{r\to\infty} Y_r(\eta) \leq C_Y$ for μ-a.a. η. Hence, for infinitely many r, the r.h.s. of (5) is bounded away from zero, which yields (4).

The Borel-Cantelli Lemma implies that for μ-almost every starting configuration η, P_η-a.s. infinitely many of the events $C_{N,m_{\ell_r}}$ occur. Hence, we find μ-a.s. an R (depending on η) such that

$$\left(B^+_{Nm_{\ell_R}}\right)^c \cap \left(A^{(N)}_{m_{\ell_R}}\right)^c \cap C_{N,m_{\ell_R}}$$

occurs. But

$$\left(B^+_{Nm_{\ell_R}}\right)^c \cap \left(A^{(N)}_{m_{\ell_R}}\right)^c \cap C_{N,m_{\ell_R}} \subseteq \left(B_{Nm_{\ell_R}}\right)^c$$

Hence, for μ-a.a. initial configurations η, there is P_η-a.s. an index $M \in \mathbb{N}$ (take $M = N m_{\ell_R}$) such that the frogs in $-M$ never get activated. With the Borel-Cantelli lemma we conclude that

$$P_\eta\,[\text{the origin is visited infinitely often}] = 0 \qquad \mu\text{-a.s.}$$

$\square$

3 Open Problems

1. Consider the frog model with simple random walk. Not even for transitive graphs, starting with one frog everywhere, recurrence and transience are settled: an example of a graph where this question is open is the binary tree.
2. A natural conjecture is the following: Assume that the graph G is transitive, the underlying random walk is homogeneous and the initial configuration η is i.i.d. Then we have either

$$P_\eta\,[\text{the origin is visited infinitely often}] = 0 \qquad \mu\text{-a.s.}$$

or

$$P_\eta\,[\text{the origin is visited infinitely often}] = 1 \qquad \mu\text{-a.s.}$$

3. We conjecture that for random walk with drift on $\mathbb{Z}^d$ (i.e. with $p(e_j)-p(-e_j) \neq 0$ for some $1 \leq j \leq d$, where e_j, $j = 1, \ldots, d$ are the unit vectors in $\mathbb{Z}^d$ and $p(e_j)$ is the probability of the random walk to move from $x \in \mathbb{Z}^d$ to $x + e_j$), Theorem 2.2 still holds true.

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