

Hysteresis Operators ^{*}

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1 Models for Scalar Hysteresis

We discuss some basic examples of hysteresis operators. Hysteresis operators map time-dependent functions (here: u) to time-dependent functions (here: w). In this chapter, those functions are scalar-valued; the input functions u are assumed to belong to $C_{pm}[a, b]$, the space of continuous and piecewise monotone functions on $[a, b]$.

A partition $a = t_0 < \dots < t_N = b$ is called a **monotonicity partition** for $u \in C_{pm}[a, b]$ if u is monotone (that is, either nondecreasing or nonincreasing) on every partition interval $[t_{i-1}, t_i]$. The smallest such partition, defined by

$$t_i = \max\{t \in [a, b] : u \text{ is monotone on } [t_{i-1}, t]\},$$

is called the **standard monotonicity partition** for u .

The exposition in this chapter is based on Section 2.1 of [4].

Relay with hysteresis. Such a relay is characterized by two thresholds $\alpha < \beta$ and two output values, here ± 1 . Assume that $u \in C_{pm}[a, b]$ with standard monotonicity partition $\{t_i\}_{0 \leq i \leq N}$. Let $w_a \in \{-1, 1\}$ be given. We set

$$w(a) = \begin{cases} 1, & u(a) \geq \beta, \\ -1, & u(a) \leq \alpha, \\ w_a, & \alpha < u(a) < \beta, \end{cases} \quad (1.1)$$

and define w successively on $(t_{i-1}, t_i]$ by

$$w(t) = \begin{cases} 1, & u(t) \geq \beta, \\ -1, & u(t) \leq \alpha, \\ w(t_{i-1}), & \alpha < u(t) < \beta. \end{cases} \quad (1.2)$$

Then w is piecewise constant and switches at most N times on $[a, b]$. The operator $\mathcal{R}_{\alpha, \beta} : C_{pm}[a, b] \times \{-1, 1\} \rightarrow L^\infty(a, b)$ defined by

$$w = \mathcal{R}_{\alpha, \beta}[u; w_0] \quad (1.3)$$

is called the **relay operator** with thresholds α and β .

The relay operator behaves in a discontinuous manner. Consider $u : [0, 2] \rightarrow \mathbb{R}$, $u(t) = \beta - (t - 1)^2$. The functions $u_\varepsilon^\pm = u \pm \varepsilon$ converge uniformly to u for $\varepsilon \rightarrow 0$, but $w_\varepsilon^+ = \mathcal{R}_{\alpha, \beta}[u_\varepsilon^+; -1] = 1$ on $[1, 2]$, while $w_\varepsilon^- = \mathcal{R}_{\alpha, \beta}[u_\varepsilon^-; -1] = -1$ on $[1, 2]$. This leads to difficulties when analyzing differential equations containing a relay operator. One way to address this problem is to close the relay operator by passing to a set-valued extension, see [26].

The scalar play operator. It arises when the diagonal $w = u$ in the u - w -plane (which represents the identity operator $w = Iu$ on functions) is split into two parallel straight lines $w = u - r$ and $w = u + r$, where $r > 0$ is given. On the right line $w = u - r$ one can only ascend, on the left line one can only descend; in the region in between, w has to remain constant. For nondecreasing continuous input functions $u : [a, b] \rightarrow \mathbb{R}$ and an initial value $w(a)$ with $|u(a) - w(a)| \leq r$, this behaviour is described by

$$w(t) = \max\{w(a), u(t) - r\}, \quad t \geq a. \quad (1.4)$$

If instead u is nonincreasing, (1.4) is replaced by

$$w(t) = \min\{w(a), u(t) + r\}, \quad t \geq a. \quad (1.5)$$

Setting

$$f_r(x, y) = \max\{x - r, \min\{x + r, y\}\} \quad (1.6)$$

both (1.4) and (1.5) can be combined into the single formula

$$w(t) = f_r(u(t), w(a)), \quad t \geq a \quad (1.7)$$

which is valid for monotone continuous functions u .

If $u \in C_{pm}[a, b]$ with standard monotonicity partition $\{t_i\}_{0 \leq i \leq N}$, we define

$$w(t) = f_r(u(t), w(t_{i-1})), \quad t_{i-1} < t \leq t_i, \quad 0 < i \leq N. \quad (1.8)$$

It turns out to be convenient to specify the initial condition for $w(a)$ in the form

$$w(a) = u(a) - z_a, \quad z_a \in [-r, r] \text{ given.} \quad (1.9)$$

In this manner we obtain the scalar play operator

$$w = \mathcal{P}_r[u; z_a], \quad \mathcal{P}_r : C_{pm}[a, b] \times [-r, r] \rightarrow C_{pm}[a, b]. \quad (1.10)$$

The scalar stop operator. Let $r \geq 0$. For a given input $u \in C_{pm}[a, b]$ and initial value $z_a \in [-r, r]$, the scalar stop is defined by

$$\mathcal{S}_r[u; z_a] = u - \mathcal{P}_r[u; z_a], \quad \mathcal{S}_r : C_{pm}[a, b] \times [-r, r] \rightarrow C_{pm}[a, b]. \quad (1.11)$$

Thus, the output functions

$$z = \mathcal{S}_r[u; z_a]$$

and $w = \mathcal{P}_r[u; z_a]$ are related by

$$u(t) = w(t) + z(t), \quad t \in [a, b]. \quad (1.12)$$

In particular, $z(a) = z_a$ and $w(a) = u(a) - z_a$.

The Preisach model. Preisach [24] had the idea that the two-parameter family $\mathcal{R}_{\alpha, \beta}$ of relays, defined above, can serve as a basis of a mathematical model for a scalar version of the ferromagnetic constitutive law.

We replace the thresholds α and β by the mean value $s = (\beta + \alpha)/2$ and the half-width $r = (\beta - \alpha)/2$ as parameters and define

$$w(t) = \int_0^\infty \int_{-\infty}^\infty \rho(r, s) \mathcal{R}_{s-r, s+r}[u; w^a(r, s)](t) ds dr. \quad (1.13)$$

The function ρ plays the role of a density function. For the moment we assume that ρ has compact support, so the integral is well-defined.

Preisach [24] showed that the evolution in time of the family of relays can be characterized by the evolution of the curve in the half-plane $\mathbb{R}_+ \times \mathbb{R}$ which separates the regions

$$A_\pm(t) = \{(r, s) \in \mathbb{R}_+ \times \mathbb{R} : \mathcal{R}_{s-r, s+r}[u; w^a(r, s)](t) = \pm 1\}. \quad (1.14)$$

If we set the initial values of the relays to

$$w^a(r, s) = -\text{sign}(s), \quad (1.15)$$

then $A_+(a)$, $A_-(a)$ and the separating curve coincide with the right lower and upper quadrants and the r -axis, respectively.

We describe the evolution of the separating curve. As an example, let $u : [0, 3] \rightarrow \mathbb{R}$ be given which increases on $[0, 2]$ from $u(0) = 0$ to $u(2) = 2$ and then decreases on $[2, 3]$ to $u(3) = 1$. At $t \in [0, 2]$, the relays satisfying $s + r = u(t)$ switch from -1 to $+1$. No switches in the other direction occur. Let $\psi : [0, 3] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ denote the time-dependent separating curve, that is, the graph of the function

$$\psi(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$$

represents the curve separating $A_+(t)$ and $A_-(t)$. Then

$$\psi(t, r) = \max\{u(t) - r, \psi(0, r)\} = \max\{u(t) - r, 0\}, \quad t \in [0, 2].$$

In the time interval $[2, 3]$ where u decreases, the relays satisfying $s - r = u(t)$ switch from $+1$ to -1 . Therefore,

$$\psi(t, r) = \min\{u(t) + r, \psi(2, r)\}, \quad t \in [2, 3].$$

In general, let $u \in C_{pm}[a, b]$ with standard monotonicity partition $\{t_i\}$. Then the graph of the separating curve (with the choice (1.15) for the initial values of the relays) is given by

$$\begin{aligned} \psi(a, r) &= \max\{u(a) - r, \min\{u(a) + r, 0\}\}, & r \geq 0, \\ \psi(t, r) &= \max\{u(t) - r, \min\{u(t) + r, \psi(t_{i-1}, r)\}\}, & r \geq 0, \quad t \in (t_{i-1}, t_i]. \end{aligned} \quad (1.16)$$

Let now a density ρ be given. For any $u \in C_{pm}[a, b]$, (1.13) yields the time evolution of the Preisach model as a curve $t \mapsto (u(t), w(t))$ in the u - w -plane. Inserting the values ± 1 for the relays, the integral (1.13) becomes

$$w(t) = \int_0^\infty \int_{-\infty}^{\psi(t, r)} \rho(r, s) ds dr - \int_0^\infty \int_{\psi(t, r)}^\infty \rho(r, s) ds dr. \quad (1.17)$$

The Preisach model is capable to describe not only single, but also nested hysteresis loops. For example, let $\rho = 1/2$ in $[0, 2] \times [-2, 2]$ and consider $u : [0, 6] \rightarrow \mathbb{R}$ which linearly interpolates the values $(0, 2, -2, 1, -1, 1, 2)$ at successive times $t_i = i$, $0 \leq i \leq 6$. Then $u(t) = 2t$ on $[0, 1]$ and $u(t) = 6 - 4t$ on $[1, 2]$. Using (1.17) one computes that

$$\begin{aligned} w(t) - w(0) &= \frac{1}{2} u(t)^2 \cdot 2\rho = 2t^2, & t \in [0, 1], \\ w(t) - w(1) &= \frac{1}{4} (2 - u(t))^2 \cdot (-2\rho) = -4(t - 1)^2, & t \in [1, 2]. \end{aligned}$$

Continuing in this manner one sees that the curve $t \mapsto (u(t), w(t))$ consists of parabolic arcs connecting the points $P_i = (u(t_i), w(t_i))$. Its part $P_3 \rightarrow P_4 \rightarrow P_5 = P_3$ constitutes an inner hysteresis loop. Moreover, $\psi(5, \cdot) = \psi(3, \cdot)$. Consequently, the evolution for $t \geq 5$ is

identical to the one which arises if on $[3, 5]$ the function u is replaced by the constant 1. Thus, at $t = 5$ the model “forgets” that the inner loop was present. This feature of the Preisach model is called the **wiping out property**. The part $P_1 \rightarrow P_2 \rightarrow P_3 = P_5 \rightarrow P_6$ constitutes the outer hysteresis loop.

Play and Preisach model. Comparing (1.6) and (1.16) we see that for all $r \geq 0$

$$\begin{aligned}\psi(a, r) &= f_r(u(a), 0), \\ \psi(t, r) &= f_r(u(t), \psi(t_{i-1}, r)), \quad t \in (t_{i-1}, t_i].\end{aligned}\tag{1.18}$$

Let $\pi_r : \mathbb{R} \rightarrow [-r, r]$ denote the projection. Then for all $x \in \mathbb{R}$

$$\begin{aligned}f_r(x, 0) &= \max\{x - r, \min\{x + r, 0\}\} = x + \max\{-r, \min\{r, -x\}\} \\ &= x + \pi_r(-x) = x - \pi_r(x).\end{aligned}\tag{1.19}$$

Consequently, it follows from (1.8) and (1.9) that

$$\psi(t, r) = \mathcal{P}_r[u; \pi_r(u(a))](t), \quad t \in [a, b], \quad r \geq 0.\tag{1.20}$$

In particular, $\pi_r(u(a)) = 0$ for all $r \geq 0$ if $u(a) = 0$.

In view of (1.17) the Preisach model can be written in terms of the one parameter family of play operators as

$$w(t) = \int_0^\infty g(r, \mathcal{P}_r[u; \pi_r(u(a))](t)) dr + w_{00}.\tag{1.21}$$

Here, the function $g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$g(r, s) = 2 \int_0^s \rho(r, \sigma) d\sigma,\tag{1.22}$$

and the number w_{00} by

$$w_{00} = \int_0^\infty \int_{-\infty}^0 \rho(r, s) ds dr - \int_0^\infty \int_0^\infty \rho(r, s) ds dr.\tag{1.23}$$

For the mathematical analysis of equations including a Preisach nonlinearity, (1.21) is often more convenient than (1.13), because the discontinuities inherent in the relays are no longer present.

The model of Prandtl and Ishlinskiĭ. In 1928, Prandtl [23] proposed the following model for the scalar version of the elastoplastic constitutive law; it was later rediscovered by Ishlinskiĭ [8]. It has the form

$$w(t) = \int_0^\infty p(r) \mathcal{S}_r[u; \pi_r(u(a))](t) dr,\tag{1.24}$$

where \mathcal{S}_r is the scalar stop operator and p is a density function. Using (1.11) we may replace \mathcal{S}_r by \mathcal{P}_r ; (1.24) then becomes

$$w(t) = \int_0^\infty p(r) dr \cdot u(t) - \int_0^\infty p(r) \mathcal{P}_r[u; \pi_r(u(a))](t) dr.\tag{1.25}$$

In this manner, setting $g(r, s) = -p(r)s$ the model becomes a special case of the Preisach model, once we slightly generalize the latter to include terms like the first integral on the right side of (1.25).

Rate independence. All the models discussed above are rate independent in the following sense.

Let \mathcal{W} be an operator which maps functions u defined on some time interval $[a, b]$ with values in some set X to functions $w = \mathcal{W}[u]$ defined on the same time interval with values in some set Y . Such an operator \mathcal{W} is called **rate independent** if it commutes with all time transformations $\varphi : [a, b] \rightarrow [a, b]$,

$$\mathcal{W}[u \circ \varphi] = (\mathcal{W}[u]) \circ \varphi. \quad (1.26)$$

Here, $\varphi : [a, b] \rightarrow [a, b]$ is called a **time transformation** if it is nondecreasing and surjective; thus in particular, $\varphi(a) = a$, $\varphi(b) = b$, and φ is continuous. We do not require that φ is injective. The functions u are taken from some given set of functions ($C_{pm}[a, b]$ in the examples above).

The operator \mathcal{W} is said to be **causal** if for every t the value $w(t) = (\mathcal{W}[u])(t)$ does not depend upon the future values $u(s)$, $s > t$, of the function u . (It may depend on the past and present values $u(s)$, $s \leq t$, in an arbitrary manner.) A causal rate independent operator is called a **hysteresis operator**.

In the examples above, also initial values are involved, that is, $w = \mathcal{W}[u; q]$ with $q \in Q$ for some set Q . In that case, \mathcal{W} is called a hysteresis operator if $u \mapsto \mathcal{W}[u; q]$ is a hysteresis operator for every $q \in Q$.

During the period 1965 to 1985, a basic mathematical theory of hysteresis operators was developed by Krasnosel'skiĭ and his group, see the monograph [9]. Other monographs in this tradition are [16, 26, 4, 10]. There is also the collection [3]. Since around 2000, the so-called energetic approach has been developed by Mielke and others. There, the dynamics of a rate independent evolution is governed by the interplay between an energy and a dissipation functional. See the monograph [18].

2 The Scalar Play and Stop

In order to analyze mathematically equations or systems which include hysteresis operators, it is necessary to define the latter on standard function spaces and investigate their properties on those spaces.

We recall the definition of the scalar play and stop. For $u \in C_{pm}[a, b]$ with standard monotonicity partition $a = t_0 < \dots < t_N = b$, the functions $w, z \in C_{pm}[a, b]$ are defined by

$$\begin{aligned} w(t) &= f_r(u(t), w(t_{i-1})), \quad t_{i-1} < t \leq t_i, \quad 0 < i \leq N, \\ z(t) &= u(t) - w(t), \quad t \in [a, b], \end{aligned} \quad (2.1)$$

where

$$f_r(x, y) = \max\{x - r, \min\{x + r, y\}\}. \quad (2.2)$$

The initial value of w at $t_0 = a$ is given by

$$w(a) = u(a) - z_a, \quad z_a \in [-r, r]. \quad (2.3)$$

In order to cover arbitrary initial data $z_a \in \mathbb{R}$ it is convenient to replace (2.3) by

$$w(a) = u(a) - \pi_r(z_a). \quad (2.4)$$

where $\pi_r : \mathbb{R} \rightarrow [-r, r]$ denotes the projection; thus $z(a) = \pi_r(z_a)$. In this manner, (2.1) and (2.4) yield the scalar play and stop operators

$$\begin{aligned} w &= \mathcal{P}_r[u; z_a], \quad \mathcal{P}_r : C_{pm}[a, b] \times \mathbb{R} \rightarrow C_{pm}[a, b], \\ z &= \mathcal{S}_r[u; z_a], \quad \mathcal{S}_r : C_{pm}[a, b] \times \mathbb{R} \rightarrow C_{pm}[a, b]. \end{aligned} \quad (2.5)$$

The function u and the functions w, z are often called the **input** resp. **output** functions of \mathcal{P}_r and \mathcal{S}_r .

Lemma 2.1 *Let $u \in C_{pm}[a, b]$, $z_a \in \mathbb{R}$, $z = \mathcal{S}_r[u; z_a]$. Then $|z(t)| \leq r$ for all $t \in [a, b]$.*

Proof. This follows from (2.4) and (2.1) by induction over the monotonicity intervals of u , since $z(a) = \pi_r(z_a)$ and

$$f_r(x, y) - x = \max\{-r, \min\{r, y - x\}\} = \pi_r(y - x), \quad x - f_r(x, y) = \pi_r(x - y),$$

so $|z(t)| = |u(t) - f_r(u(t), w(t_{i-1}))| \leq r$ on $[t_{i-1}, t_i]$ for all i . \square

One may check that if u is monotone on $[a, b]$, (2.1) and (2.3) define the same functions w and z no matter which partition $\{t_i\}$ of $[a, b]$ is used. Therefore, for piecewise monotone u the definition of the play and stop does not depend on the choice of the monotonicity partition for u in (2.1).

Maximum norm estimate. The basic maximum norm estimate for the scalar play arises from a corresponding estimate for the function f_r .

Lemma 2.2 *Let $a, b, c, d \in \mathbb{R}$. Then*

$$|\max\{a, b\} - \max\{c, d\}| \leq \max\{|a - c|, |b - d|\}, \quad (2.6)$$

The estimate also holds if we replace “max” with “min” on the left side.

Proof. We may assume that $\max\{a, b\} \geq \max\{c, d\}$. Then

$$|\max\{a, b\} - \max\{c, d\}| = \max\{a - d, b - d\} - \max\{c - d, 0\}.$$

In order to prove (2.6), it therefore suffices to prove that

$$\max\{a, b\} - \max\{c, 0\} \leq \max\{|a - c|, |b|\} \quad (2.7)$$

for all $a, b, c \in \mathbb{R}$. Setting $x^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$, we obtain (2.7) from the estimate

$$\max\{a, b\} - c^+ = \max\{a - c^+, b - c^+\} \leq \max\{a^+ - c^+, b\} \leq \max\{|a - c|, |b|\},$$

where we have used that $|a^+ - c^+| \leq |a - c|$. As $\min\{x, y\} = -\max\{-x, -y\}$ for $x, y \in \mathbb{R}$, the second assertion follows. \square

Lemma 2.3 *We have*

$$|f_{\tilde{r}}(\tilde{x}, \tilde{y}) - f_r(x, y)| \leq \max\{|\tilde{x} - x| + |\tilde{r} - r|, |\tilde{y} - y|\} \quad (2.8)$$

for all $r, \tilde{r} \geq 0$ and all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$.

Proof. Using Lemma 2.2 twice, we obtain

$$|f_{\tilde{r}}(\tilde{x}, \tilde{y}) - f_r(x, y)| \leq \max\{|(\tilde{x} - \tilde{r}) - (x - r)|, |(\tilde{x} + \tilde{r}) - (x + r)|, |\tilde{y} - y|\}.$$

This implies the assertion. \square

Proposition 2.4 *The operators \mathcal{P}_r and \mathcal{S}_r can be extended uniquely to Lipschitz continuous operators*

$$\mathcal{P}_r, \mathcal{S}_r : C[a, b] \times \mathbb{R} \rightarrow C[a, b],$$

such that $|\mathcal{S}_r[u; z_a](t)| \leq r$ for all $t \in [a, b]$, $u \in C[a, b]$, $z_a \in \mathbb{R}$. Moreover, there holds

$$\|\mathcal{P}_{\tilde{r}}[\tilde{u}; \tilde{z}_a] - \mathcal{P}_r[u; z_a]\|_{\infty} \leq \|\tilde{u} - u\|_{\infty} + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.9)$$

$$\|\mathcal{S}_{\tilde{r}}[\tilde{u}; \tilde{z}_a] - \mathcal{S}_r[u; z_a]\|_{\infty} \leq 2\|\tilde{u} - u\|_{\infty} + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.10)$$

for all $\tilde{u}, u \in C[a, b]$, all $\tilde{z}_a, z_a \in \mathbb{R}$ and all $\tilde{r}, r \geq 0$.

Proof. Let $w = \mathcal{P}_r[u; z_a]$, $\tilde{w} = \mathcal{P}_{\tilde{r}}[\tilde{u}; \tilde{z}_a]$ and $z = \mathcal{S}_r[u; z_a]$, $\tilde{z} = \mathcal{S}_{\tilde{r}}[\tilde{u}; \tilde{z}_a]$. Since $w + z = u$ and $\tilde{w} + \tilde{z} = \tilde{u}$ by (2.1), (2.10) immediately follows from (2.9). Concerning (2.9), by a basic result of functional analysis it suffices to show that it holds for $\tilde{u}, u \in C_{pm}[a, b]$, because $C_{pm}[a, b]$ is dense in $C[a, b]$. By (2.4)

$$|\tilde{w}(a) - w(a)| \leq |\tilde{u}(a) - u(a)| + |\tilde{z}_a - z_a|.$$

Let $\{t_i\}$ be a partition of $[a, b]$ such that both \tilde{u} and u are monotone on all subintervals $[t_{i-1}, t_i]$. By Lemma 2.3, on each subinterval

$$|\tilde{w}(t) - w(t)| \leq \max\{|\tilde{u}(t) - u(t)| + |\tilde{r} - r|, |\tilde{w}(t_{i-1}) - w(t_{i-1})|\}$$

for all $t \in [t_{i-1}, t_i]$, $t > a$. Therefore,

$$|\tilde{w}(t_i) - w(t_i)| \leq \max\{\|\tilde{u} - u\|_\infty + |\tilde{r} - r|, |\tilde{w}(t_{i-1}) - w(t_{i-1})|\}, \quad 0 < i \leq N.$$

By induction over i , for all $t \in [a, b]$ we get

$$|\tilde{w}(t) - w(t)| \leq \max\{\|\tilde{u} - u\|_\infty + |\tilde{r} - r|, |\tilde{u}(a) - u(a)| + |\tilde{z}_a - z_a|\}.$$

This implies (2.9). Finally, that $z = \mathcal{S}_r[u; z_a]$ takes values only in $[-r, r]$ follows from Lemma 2.1 and a limit passage. \square

The constant 2 in (2.10) cannot be improved. For example, setting $[a, b] = [0, 2]$ and $\tilde{r} = r$, let \tilde{u} be the linear interpolate for $\tilde{u}(0) = 0$, $\tilde{u}(1) = r$ and $\tilde{u}(2) = -r$, set $u = 0$ and $\tilde{z}_a = z_a = r$. Then $\tilde{z} = r$ on $[0, 1]$ and $z(2) = -r$, while $z = r$ on $[0, 2]$. Thus $\|\tilde{z} - z\|_\infty = 2r = 2\|\tilde{u} - u\|_\infty$.

Variation norm estimate. By $C_{pl}[a, b]$ we denote the space of piecewise affine linear (or simply “piecewise linear”) functions on $[a, b]$.

Let $u \in C_{pl}[a, b]$, let

$$w = \mathcal{P}_r[u; z_a], \quad z = \mathcal{S}_r[u; z_a], \quad z_a \in \mathbb{R}.$$

One immediately checks from (2.1) that w and z are piecewise linear. By subdividing the standard monotonicity partition for u we can decompose $[a, b]$ into finitely many disjoint open intervals such that the union of their closures equals $[a, b]$ and that, on each such interval, the time derivatives \dot{u} , \dot{w} and \dot{z} are constant and satisfy

$$\begin{aligned} \text{either} \quad & |z| < r, \quad \dot{z} = \dot{u} \text{ and } \dot{w} = 0, \\ \text{or} \quad & z = -r, \quad \dot{w} = \dot{u} \leq 0 \text{ and } \dot{z} = 0, \\ \text{or} \quad & z = r, \quad \dot{w} = \dot{u} \geq 0 \text{ and } \dot{z} = 0. \end{aligned} \tag{2.11}$$

Lemma 2.5 *Let $u \in C_{pl}[a, b]$, $z_a \in \mathbb{R}$. Then*

$$\begin{aligned} \dot{w}(t)(z(t) - \zeta) &\geq 0, \quad \text{for all } \zeta \in [-r, r] \\ z(t) &\in [-r, r], \quad z(a) = \pi_r(z_a), \end{aligned} \tag{2.12}$$

for all except finitely many (for all, resp.) $t \in (a, b)$.

Proof. This is a direct consequence of (2.11), (2.4) and Lemma 2.1. \square

A system like (2.12) is called a **variational inequality**. Variational inequalities typically take care of case distinctions like those in (2.11).

Lemma 2.6 *Let $\tilde{u}, u \in C_{pl}[a, b]$, $\tilde{z}_a, z_a \in \mathbb{R}$. Let $w = \mathcal{P}_r[u; z_a]$, $\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a]$ and $z = \mathcal{S}_r[u; z_a]$, $\tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a]$. Then we have*

$$(\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{z}(t) - z(t)) \geq 0 \tag{2.13}$$

and

$$|\dot{\tilde{w}}(t) - \dot{w}(t)| + \frac{d}{dt}|\tilde{z}(t) - z(t)| \leq |\dot{\tilde{u}}(t) - \dot{u}(t)| \tag{2.14}$$

for all except finitely many $t \in (a, b)$.

Proof. By Lemma 2.5,

$$\dot{w}(t)(\tilde{z}(t) - z(t)) \geq 0, \quad \dot{w}(t)(z(t) - \tilde{z}(t)) \geq 0,$$

Adding those inequalities yields (2.13).

In order to prove (2.14), we decompose $[a, b]$ into intervals J such that in the interior of each J either $\tilde{z}(t) = z(t)$ for all t or $\tilde{z}(t) \neq z(t)$ for all t holds. On intervals where $\tilde{z} = z$ we have

$$\tilde{w} - w = \tilde{u} - \tilde{z} + z - u = \tilde{u} - u,$$

thus (2.14) holds with equality. On intervals where $\tilde{z} \neq z$ we obtain from (2.13) that

$$|\dot{w}(t) - \dot{w}(t)| = (\dot{w}(t) - \dot{w}(t))\text{sign}(\tilde{z}(t) - z(t)).$$

Moreover

$$\frac{d}{dt}|\tilde{z}(t) - z(t)| = (\dot{\tilde{z}}(t) - \dot{z}(t))\text{sign}(\tilde{z}(t) - z(t)).$$

Adding the previous two equations yields the assertion. \square

By $W^{1,1}(a, b)$ we denote the space of absolutely continuous functions on $[a, b]$. Equivalently, $W^{1,1}(a, b)$ is the space of functions $u \in L^1(a, b)$ whose distributional derivative belongs to $L^1(a, b)$. For such functions,

$$u(t) = u(a) + \int_a^t \dot{u}(s) ds, \quad \text{for all } t \in [a, b]. \quad (2.15)$$

For $u \in W^{1,1}(a, b)$ we define

$$\|u\|_{BV} = |u(a)| + \text{var}(u), \quad \text{var}(u) = \int_a^b |\dot{u}(t)| dt. \quad (2.16)$$

We will use the fact that $C_{pl}[a, b]$ is dense in both $(C[a, b], \|\cdot\|_\infty)$ and $(W^{1,1}(a, b), \|\cdot\|_{BV})$. The latter follows e.g. from the fact that the piecewise constant functions are dense in $L^1(a, b)$.

Proposition 2.7 *Let $\tilde{u}, u \in W^{1,1}(a, b)$, $\tilde{z}_a, z_a \in \mathbb{R}$. Then $w = \mathcal{P}_r[u; z_a]$ and $\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a]$ satisfy*

$$\text{var}(\tilde{w} - w) \leq \text{var}(\tilde{u} - u) + |\tilde{z}_a - z_a|. \quad (2.17)$$

Consequently,

$$\mathcal{P}_r : W^{1,1}(a, b) \times \mathbb{R} \rightarrow W^{1,1}(a, b)$$

is Lipschitz continuous, and the same is true for \mathcal{S}_r .

Proof. Assume first that \tilde{u}, u are piecewise linear. Setting $z = \mathcal{S}_r[u; z_a]$ and $\tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a]$ we obtain from (2.14)

$$\begin{aligned} \int_a^b |\dot{w}(t) - \dot{w}(t)| dt &\leq \int_a^b |\dot{\tilde{u}}(t) - \dot{u}(t)| dt - |\tilde{z} - z| \Big|_a^b \\ &\leq \int_a^b |\dot{\tilde{u}}(t) - \dot{u}(t)| dt + |\tilde{z}(a) - z(a)| \end{aligned}$$

which proves (2.17) as $|\tilde{z}(a) - z(a)| \leq |\tilde{z}_a - z_a|$. This and the estimate

$$|\tilde{w}(a) - w(a)| \leq |\tilde{u}(a) - u(a)| + |\tilde{z}_a - z_a|$$

yield (2.17) as well as the Lipschitz continuity of \mathcal{P}_r for piecewise linear functions. Since those functions are dense in $W^{1,1}(a, b)$, \mathcal{P}_r can be uniquely extended to a Lipschitz continuous operator on $W^{1,1}(a, b) \times \mathbb{R}$ satisfying (2.17). This extension coincides with that provided by Proposition 2.4 on the dense subspace $C_{pl}[a, b]$, so both extensions coincide on $W^{1,1}(a, b)$. As $z = u - w$ and $\tilde{z} = \tilde{u} - \tilde{w}$, the stop too is Lipschitz continuous on $W^{1,1}(a, b) \times \mathbb{R}$. \square

Together with Propositions 2.4 and 2.7, the density of $C_{pl}[a, b]$ in $C[a, b]$ and $W^{1,1}(a, b)$ often makes it possible to extend formulas like (2.14),

$$|\dot{\tilde{w}}(t) - \dot{w}(t)| + \frac{d}{dt}|\tilde{z}(t) - z(t)| \leq |\dot{\tilde{u}}(t) - \dot{u}(t)|,$$

directly from piecewise linear input functions to input functions in $W^{1,1}$ (resp. C if no time derivatives are present).

It is no coincidence that the maximum norm and the total variation norm enter the basic estimates in Propositions 2.4 and 2.7. Both norms are (in contrast to other usual norms) invariant w.r.t. time transformations $\varphi : [a, b] \rightarrow [a, b]$, that is,

$$\|u \circ \varphi\|_\infty = \|u\|_\infty, \quad \|u \circ \varphi\|_{BV} = \|u\|_{BV}.$$

The scalar stop as a variational inequality. Given $u : [a, b] \rightarrow \mathbb{R}$ and $z_a \in \mathbb{R}$, we look for $z : [a, b] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} (\dot{z}(t) - \dot{u}(t))(\zeta - z(t)) &\geq 0 \quad \forall \zeta \in [-r, r] \quad \text{holds a.e. in } (a, b), \\ z(t) &\in [-r, r] \quad \forall t \in [a, b], \quad z(a) = \pi_r(z_a). \end{aligned} \tag{2.18}$$

Proposition 2.8 *Let $u \in W^{1,1}(a, b)$, $z_a \in \mathbb{R}$. Then $z = \mathcal{S}_r[u; z_a]$ is the unique solution of (2.18) in $W^{1,1}(a, b)$.*

Proof. That $z = \mathcal{S}_r[u; z_a]$ belongs to $W^{1,1}(a, b)$ and satisfies $z(t) \in [-r, r]$ for all t as well as $z(a) = \pi_r(z_a)$ has already been proved in Propositions 2.4 and 2.7. For piecewise linear u , the inequalities in (2.18) have been obtained in Lemma 2.5. For arbitrary $u \in W^{1,1}(a, b)$ we choose a sequence u^n of piecewise linear functions on $[a, b]$ with $u^n \rightarrow u$ in $W^{1,1}(a, b)$. By Proposition 2.7, $z^n \rightarrow z$ in $W^{1,1}(a, b)$. Then $z^n \rightarrow z$ uniformly and, after passing to a subsequence, $\dot{u}^n \rightarrow \dot{u}$ and $\dot{z}^n \rightarrow \dot{z}$ pointwise a.e. Therefore, for any fixed $\zeta \in [-r, r]$

$$(\dot{z}(t) - \dot{u}(t))(\zeta - z(t)) \geq 0 \tag{2.19}$$

holds a.e. in (a, b) . Let $\{\zeta_j\}$ be a countable dense subset of $[-r, r]$. Then (2.19) holds for all ζ_j and all $t \in (a, b) \setminus N$ for some set N of zero measure. Since $\{\zeta_j\}$ is dense in $[-r, r]$, the inequalities in (2.18) hold for all $t \in (a, b) \setminus N$ and all $\zeta \in [-r, r]$.

Concerning uniqueness, let $\tilde{z}, z \in W^{1,1}(a, b)$ be solutions of (2.18). Inserting $\tilde{z}(t)$ for ζ in the inequality for z and vice versa and adding the resulting two inequalities gives

$$(\dot{z}(t) - \dot{\tilde{z}}(t))(\tilde{z}(t) - z(t)) \geq 0, \quad \text{for a.e. } t \in (a, b).$$

It follows that

$$\frac{1}{2} \frac{d}{dt} |\tilde{z}(t) - z(t)|^2 = (\dot{\tilde{z}}(t) - \dot{z}(t))(\tilde{z}(t) - z(t)) \leq 0, \quad \text{for a.e. } t \in (a, b).$$

Thus $t \mapsto |z(t) - \tilde{z}(t)|$ is nonincreasing. As $z(a) = \pi_r(z_a) = \tilde{z}(a)$ it follows that $z = \tilde{z}$. \square

Monotonicity properties of the play. The play operator is **order monotone**, that is, for $u, \tilde{u} \in C[a, b]$ and $z_a, \tilde{z}_a \in \mathbb{R}$

$$\mathcal{P}_r[u; z_a] \leq \mathcal{P}_r[\tilde{u}; \tilde{z}_a] \quad (2.20)$$

if $u \leq \tilde{u}$ pointwise and $z_a \geq \tilde{z}_a$. (The sign change w.r.t. the initial value occurs because we write the initial condition as $w(a) = u(a) - z_a$.) This follows since $(x, y) \mapsto f_r(x, y)$ is nondecreasing in both variables, and because (2.20) persists when passing from piecewise monotone to arbitrary continuous functions. However, the stop operator is not order monotone. For example, let $a = 0$, $z_a = r$, $\tilde{u} = 0$ and $u(t) = t$ on $[0, r]$, $u(t) = 2r - t$ on $[r, 2r]$. Then $\tilde{u} \leq u$ on $[0, 2r]$, $\tilde{z} = \mathcal{S}_r[\tilde{u}; z_a] = r$ and $z = \mathcal{S}_r[u; z_a]$ satisfies $z = r$ on $[0, r]$ and $z(t) = 2r - t < \tilde{z}(t)$ on $(r, 2r]$.

Moreover, the play and the stop operator are **piecewise monotone**, that is, if $u \in C[a, b]$ is monotone (nondecreasing or nonincreasing) on some subinterval $[s, t]$ of $[a, b]$, then the same is true for $\mathcal{P}_r[u; z_a]$ and $\mathcal{S}_r[u; z_a]$. For piecewise linear u this follows directly from (2.11), and a limit passage yields the result for arbitrary $u \in C[a, b]$.

On the other hand, the play operator is **not L^2 -monotone**, that is, it may happen that for $w = \mathcal{P}_r[u; z_a]$ and $\tilde{w} = \mathcal{P}_r[\tilde{u}; z_a]$

$$\int_a^b (w - \tilde{w})(u - \tilde{u}) dt < 0. \quad (2.21)$$

An example is given by $a = 0$, $z_a = r$, $\tilde{u} = r$ and

$$u(t) = \begin{cases} r + t, & t \in [0, r], \\ 3r - t, & t \in [r, 3r], \\ 0 & t \geq 3r. \end{cases}$$

Then $\tilde{w} = 0$, $w(t) = r$ for $t \geq r$ and, if b is large enough,

$$\int_0^b (w - \tilde{w})(u - \tilde{u}) dt = \int_0^{3r} (w - \tilde{w})(u - \tilde{u}) dt + \int_{3r}^b r \cdot (-r) dt < 0.$$

Regularization property of the play. For $u \in W^{1,1}(a, b)$ and $w = \mathcal{P}_r[u; z_a]$, we have

$$\text{var}(w) \leq \text{var}(u). \quad (2.22)$$

This follows immediately from (2.17) if we set $\tilde{z}_a = z_a$ and \tilde{u} to some constant (then \tilde{w} is constant, too). We will see that (in the nontrivial case $r > 0$) w is moreover piecewise monotone, even if u is only continuous; in particular, $\text{var}(w)$ remains finite even if $\text{var}(u)$ is infinite. Also, w turns out to be a minimizer of the variation on the set of functions $v : [a, b] \rightarrow \mathbb{R}$ which satisfy $\|u - v\|_\infty \leq r$ and $v(a) = w(a)$.

In order to obtain these and related results, a more detailed analysis of the behaviour of the play and stop is required. In the remainder of this section we generally assume that $r > 0$.

Let $(u, z_a) \in C[a, b] \times \mathbb{R}$ be given, let $w = \mathcal{P}_r[u; z_a]$ and $z = \mathcal{S}_r[u; z_a]$. The trajectories $\{(u(t), w(t)) : t \in [a, b]\}$ lie within the subset $D = \{(x, y) : |x - y| \leq r\}$ of the plane. They consist of parts which belong to the interior, the right or the left boundary of D . Accordingly, we decompose the time interval $[a, b]$ into the three disjoint sets

$$\begin{aligned} I_0 &= \{t \in [a, b] : |u(t) - w(t)| = |z(t)| < r\}, \\ I_+ &= \{t \in [a, b] : u(t) - w(t) = z(t) = r\}, \\ I_- &= \{t \in [a, b] : u(t) - w(t) = z(t) = -r\}. \end{aligned} \quad (2.23)$$

We also define

$$I_\partial = I_+ \cup I_- . \quad (2.24)$$

The set I_0 is a relatively open subset of $[a, b]$, the sets I_\pm and I_∂ are compact.

Lemma 2.9 *The function $w = \mathcal{P}_r[u; z_a]$ is locally constant on I_0 .*

Proof. Let J be a closed subinterval of I_0 . Then $r - \max_J |z| > 0$ on J . Let $u_n \in C_{pl}[a, b]$ with $u_n \rightarrow u$ uniformly and $r - \max_J |z_n| > 0$ on J for all n , where $z_n = \mathcal{S}_r[u_n; z_a]$. (Such a sequence exists due to Proposition 2.4.) It follows from (2.11) that $w_n = \mathcal{P}_r[u_n; z_a]$ is constant on J for all n ; thus so is w . \square

The regularization properties of the play are related to the fact that the minimum time δ_c to cross from one boundary to the other is strictly positive,

$$\delta_c = \min\{|t - s| : t \in I_+, s \in I_-\} > 0, \quad (2.25)$$

because z is continuous (and hence uniformly continuous) by Proposition 2.4.

A closed interval $J \subset [a, b]$ is called a **plus interval** for (u, z_a) if $J \subset I_+ \cup I_0$ and $J \cap I_+ \neq \emptyset$; it is called a **minus interval** for (u, z_a) if $J \subset I_- \cup I_0$ and $J \cap I_- \neq \emptyset$. Thus, on a plus interval the trajectory $\{(u(t), w(t))\}$ hits the right but not the left boundary of the domain D , and vice versa on a minus interval.

Due to the symmetry of the constraint $[-r, r]$,

$$\mathcal{P}_r[-u; -z_a] = -\mathcal{P}_r[u; z_a], \quad \mathcal{S}_r[-u; -z_a] = -\mathcal{S}_r[u; z_a]. \quad (2.26)$$

For $u \in C_{pl}[a, b]$ this directly follows from the variational inequality (2.18) or from the basic definition (2.1) – (2.4), for $u \in C[a, b]$ then by a limit passage according to Proposition 2.4. As a consequence, for a closed interval $J \subset [a, b]$

$$J \text{ is a minus interval for } (u, z_a) \iff J \text{ is a plus interval for } (-u, -z_a). \quad (2.27)$$

A partition $\Delta : a = \tau_0 < \dots < \tau_M = b$ is called **regular for (u, z_a)** if $z(\tau_i) = 0$ for all $0 < i < M$ and if, setting $J_i = [\tau_{i-1}, \tau_i]$, the intervals J_1, \dots, J_M form an alternating sequence of plus and minus intervals for (u, z_a) . This construction entails that $\tau_i \in I_0$ if $0 < i < M$; the points $\tau_0 = a$ and $\tau_M = b$ may or may not belong to I_0 .

If $I_0 = [a, b]$, then w is constant on $[a, b]$ by Lemma 2.9, so

$$\text{var}(w) = 0$$

and the results below (Propositions 2.12 and 2.13) become trivial.

Lemma 2.10 *Let $(u, z_a) \in C[a, b] \times \mathbb{R}$, let $I_0 \neq [a, b]$. Then there exists a regular partition for (u, z_a) .*

Proof. Starting from $\{a, b\}$ we successively add points $\tau \in (a, b)$ with $z(\tau) = 0$ as follows until we arrive at a partition Δ_{\pm} consisting only of plus and minus intervals for (u, z_a) . Let $\delta > 0$ be such that $|z(t) - z(s)| < r$ for all $s, t \in [a, b]$ with $|s - t| \leq \delta$. Let $\Delta = \{\tau_i\}$ be a partition of $[a, b]$ such that every interval $[\tau_{i-1}, \tau_i]$ includes at least one point from I_0 and that $z(\tau_i) = 0$ at all partition points $\tau_i \in (a, b)$. (The partition $\Delta = \{a, b\}$ has this property since we assumed that $I_0 \neq [a, b]$.) If not all intervals of Δ are plus or minus intervals, there exists an interval $J = [\tau_{k-1}, \tau_k]$ which includes points s, t with $z(s) = -r$ and $z(t) = r$. We choose τ' between s and t with $z(\tau') = 0$. All partition intervals of $\Delta' = \Delta \cup \{\tau'\}$ then include at least one point from I_0 . Since the distance of the new point τ' from s and t , and thus from all partition points of Δ , is at least δ , this process comes to an end at a desired partition Δ_{\pm} after a finite number of steps. If two adjacent intervals J_i and J_{i+1} of Δ_{\pm} are both plus or both minus intervals, we merge them into a single interval. Again this process terminates after a finite number of steps. The resulting partition is regular. \square

Lemma 2.11 *Let Δ be a regular partition for $(u, z_a) \in C[a, b] \times \mathbb{R}$, let $J = [\tau_{i-1}, \tau_i]$ be a plus interval of Δ . Then there exists an open ball B_i around (u, z_a) such that for all $(\tilde{u}, \tilde{z}_a) \in B_i$ we have*

$$\tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a] > -r \quad \text{on } J, \quad (2.28)$$

$$\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a] \quad \text{is nondecreasing on } J. \quad (2.29)$$

Moreover, setting $t_i = \max(J \cap I_+)$,

$$w = \max_{s \in J} w(s) \quad \text{on } [t_i, \tau_i]. \quad (2.30)$$

$$\max\{u(\tau_{i-1}), u(\tau_i)\} \leq u(t_i) = w(t_i) + r \quad (2.31)$$

Proof. Since $\min_J z > -r$, because of Proposition 2.4 we have $\min_J \tilde{z} > -r$ if the radius of B_i is small enough, so (2.28) holds. Then (2.29) follows from (2.11) and a limit passage. Next, $(t_i, \tau_i) \subset I_0$ by the definition of t_i if $t_i < \tau_i$ (which is the case if $t_i < b$), so w is constant on $[t_i, \tau_i]$ due to Lemma 2.9. In view of (2.29) this implies (2.30). Finally, using $t_i \in I_+$ as well as (2.29) and (2.30) we obtain (2.31) from the inequalities

$$\begin{aligned} u(\tau_{i-1}) &\leq w(\tau_{i-1}) + r \leq w(t_i) + r = u(t_i), \\ u(\tau_i) &\leq w(\tau_i) + r = w(t_i) + r = u(t_i). \end{aligned}$$

This concludes the proof. \square

If in the lemma above, $J = [\tau_{i-1}, \tau_i]$ is assumed to be a minus interval of Δ , due to the symmetry expressed in (2.26) the lemma remains true if we replace (2.28) to (2.31) with

$$\tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a] < r \quad \text{on } J, \quad (2.32)$$

$$\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a] \quad \text{is nonincreasing on } J, \quad (2.33)$$

and, setting $t_i = \max(J \cap I_-)$,

$$w = \min_{s \in J} w(s) \quad \text{on } [t_i, \tau_i]. \quad (2.34)$$

$$\min\{u(\tau_{i-1}), u(\tau_i)\} \geq u(t_i) = w(t_i) - r. \quad (2.35)$$

Proposition 2.12 *Let $(u, z_a) \in C[a, b] \times \mathbb{R}$, $w = \mathcal{P}_r[u; z_a]$. Then*

(i) *$w \in C_{pm}[a, b]$, in particular $\text{var}(w) < \infty$.*

(ii) *The mapping $(u, z_a) \mapsto \text{var}(w)$ is locally Lipschitz continuous on $C[a, b] \times \mathbb{R}$.*

Proof. Let $\Delta = \{\tau_i\}_{0 \leq i \leq M}$ be a regular partition for (u, z_a) , let B be the intersection of the balls B_i from Lemma 2.11. Applying (2.29) and (2.33) to the plus resp. minus intervals of Δ , we see that for all $(\tilde{u}, \tilde{z}_a) \in B$, the functions $\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a]$ have Δ as a common monotonicity partition; in particular, (i) holds.

In order to prove (ii), let $(\tilde{u}, \tilde{z}_a), (u, z_a) \in B$. Then Δ is a monotonicity partition for $\tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a]$ and $w = \mathcal{P}_r[u; z_a]$. Using the triangle as well as the inverse triangle inequality we obtain

$$\begin{aligned} |\text{var}(w) - \text{var}(\tilde{w})| &= \left| \sum_{i=1}^M |w(\tau_i) - w(\tau_{i-1})| - \sum_{i=1}^M |\tilde{w}(\tau_i) - \tilde{w}(\tau_{i-1})| \right| \\ &\leq \sum_{i=1}^M \left| |(w(\tau_i) - \tilde{w}(\tau_i))| - |(w(\tau_{i-1}) - \tilde{w}(\tau_{i-1}))| \right| \\ &\leq \sum_{i=1}^M |(w(\tau_i) - \tilde{w}(\tau_i)) - (w(\tau_{i-1}) - \tilde{w}(\tau_{i-1}))| \\ &\leq 2M \|w - \tilde{w}\|_\infty \leq 2M(\|u - \tilde{u}\|_\infty + |z_a - \tilde{z}_a|). \end{aligned}$$

by Proposition 2.4. Thus, the mapping $(u, z_a) \mapsto \text{var}(w)$ is Lipschitz continuous on B with Lipschitz constant $2M$. \square

The following result is due to Tronel and Vladimirov, see [25].

Proposition 2.13 *Let $(u, z_a) \in C[a, b] \times \mathbb{R}$, $w = \mathcal{P}_r[u; z_a]$. Then*

$$\text{var}(w) = \min_{v \in V} \text{var}(v), \quad (2.36)$$

where V is the set of all functions $v : [a, b] \rightarrow \mathbb{R}$ with $\|v - u\|_\infty \leq r$ and $v(a) = w(a) = u(a) - \pi_r(z_a)$. In particular, $\text{var}(w) \leq \text{var}(u)$.

Proof. Let $\Delta = \{\tau_i\}_{0 \leq i \leq M}$ be a regular partition for (u, z_a) , let $\{t_i\}_{1 \leq i \leq M}$ be the numbers $t_i = \max([\tau_{i-1}, \tau_i] \cap I_+)$ and $t_i = \max([\tau_{i-1}, \tau_i] \cap I_-)$ for plus and minus intervals of Δ , respectively.

We set $t_0 = a$, $t_{M+1} = b$ and claim that $\{t_i\}$, too, is a monotonicity partition for w . Indeed, it follows from (2.29), (2.30), (2.33) and (2.34) that w is monotone on $[a, t_1]$ and

constant on $[t_M, b]$ and that, for $1 \leq i < M$, w is nonincreasing (nondecreasing, resp.) on $[t_i, t_{i+1}]$ if $[\tau_{i-1}, \tau_i]$ is a plus (minus, resp.) interval. Therefore,

$$\text{var}(w) = \sum_{i=1}^M |w(t_i) - w(t_{i-1})|. \quad (2.37)$$

Now let $v \in V$ be arbitrary. Again, we use (2.29), (2.30), (2.33) and (2.34). If $J = [a, \tau_1]$ is a plus interval, then $t_1 \in I_+$ and w is nondecreasing on J , so

$$0 \leq w(t_1) - w(a) = u(t_1) - r - v(a) \leq v(t_1) - v(a). \quad (2.38)$$

Analogously,

$$0 \geq w(t_1) - w(a) = u(t_1) + r - v(a) \geq v(t_1) - v(a) \quad (2.39)$$

if J is a minus interval. If $M \geq 2$, $i \geq 1$ and $[\tau_{i-1}, \tau_i]$ is a plus interval, then $[\tau_i, \tau_{i+1}]$ is a minus interval and

$$0 \leq w(t_i) - w(t_{i+1}) = (u(t_i) - r) - (u(t_{i+1}) + r) \leq v(t_i) - v(t_{i+1}). \quad (2.40)$$

Analogously we get

$$0 \geq w(t_i) - w(t_{i+1}) = (u(t_i) + r) - (u(t_{i+1}) - r) \geq v(t_i) - v(t_{i+1}) \quad (2.41)$$

if the roles of the two intervals are interchanged. Putting together (2.37) – (2.41) we arrive at

$$\text{var}(w) = \sum_{i=1}^M |w(t_i) - w(t_{i-1})| \leq \sum_{i=1}^M |v(t_i) - v(t_{i-1})| \leq \text{var}(v)$$

which proves (2.36) since $v \in V$ was arbitrary and $w \in V$. Finally, setting $v = u - \pi_r(z_a)$, (2.36) yields that $\text{var}(w) \leq \text{var}(u)$. \square

Discrete and discontinuous input functions. Let S be the set of all finite sequences (u_0, \dots, u_M) of real numbers, let $z_a \in \mathbb{R}$. For $u^d = (u_0, \dots, u_M) \in S$ we define sequences $w^d, z^d \in S$ of the same length by

$$\begin{aligned} z_0 &= \pi_r(z_a), \quad w_0 = u_0 - z_0, \\ w_i &= f_r(u_i, w_{i-1}), \quad z_i = u_i - w_i, \quad 0 < i \leq M. \end{aligned} \quad (2.42)$$

In this manner we obtain the **discrete scalar play and stop**

$$\begin{aligned} w^d &= \mathcal{P}_r^d[u^d; z_a], \quad \mathcal{P}_r^d : S \times \mathbb{R} \rightarrow S, \\ z^d &= \mathcal{S}_r^d[u^d; z_a], \quad \mathcal{S}_r^d : S \times \mathbb{R} \rightarrow S. \end{aligned} \quad (2.43)$$

The set S is not a normed space; nevertheless for $u^d = (u_0, \dots, u_M)$ we write $\|u^d\|_\infty = \max_{1 \leq i \leq M} |u_i|$. As in Lemma 2.1 we see that $\|z^d\|_\infty \leq r$.

Let $u \in C_{pm}[a, b]$ with monotonicity partition $\{t_i\}_{0 \leq i \leq N}$. Comparing (2.1) and (2.4) with (2.42) we immediately obtain that

$$\begin{aligned} \mathcal{P}_r[u; z_a](a) &= \mathcal{P}_r^d[u(a); z_a], \\ \mathcal{P}_r[u; z_a](t) &= \mathcal{P}_r^d[(u(t_0), \dots, u(t_{i-1}), u(t)); z_a], \quad t \in (t_{i-1}, t_i], \quad 0 < i \leq N. \end{aligned} \quad (2.44)$$

Now consider $u : [0, 2] \rightarrow \mathbb{R}$,

$$u(t) = \begin{cases} u_0, & t < 1, \\ \tilde{u}, & t \geq 1, \end{cases} \quad \text{where } u_0, \tilde{u} \in \mathbb{R}.$$

The natural way to define $w = \mathcal{P}_r[u; z_a]$ on $[0, 2]$ is to set $w(t) = w_0 := u_0 - z_a$ for $t < 1$ and $w(t) = \tilde{w} := f_r(\tilde{u}, w_0)$ for $t \geq 1$.

We use this approach to define the play for **step functions** $u : [a, b] \rightarrow \mathbb{R}$, that is, functions which have finitely many different values and finitely many (or no) discontinuities. These functions have the form

$$u = \sum_{k=1}^N u_{k-1} \chi_{(t_{k-1}, t_k)} + \sum_{k=0}^N \hat{u}_k \chi_{\{t_k\}}. \quad (2.45)$$

Here, $\Delta = \{t_k\}_{0 \leq k \leq N}$ is a partition of $[a, b]$, χ_A for $A \subset [a, b]$ denotes the characteristic function which is 1 on A and 0 elsewhere, and u_k, \hat{u}_k are real numbers. We set

$$\begin{aligned} u^d &= (\hat{u}_0, u_0, \hat{u}_1, \dots, u_{N-1}, \hat{u}_N) \\ w^d &= \mathcal{P}_r^d[u^d; z_a] = (\hat{w}_0, w_0, \hat{w}_1, \dots, w_{N-1}, \hat{w}_N) \\ z^d &= \mathcal{S}_r^d[u^d; z_a] = (\hat{z}_0, z_0, \hat{z}_1, \dots, z_{N-1}, \hat{z}_N) \end{aligned}$$

and define the play $w = \mathcal{P}_r[u; z_a]$ and the stop $z = \mathcal{S}_r[u; z_a]$ by

$$w = \sum_{k=1}^N w_{k-1} \chi_{(t_{k-1}, t_k)} + \sum_{k=0}^N \hat{w}_k \chi_{\{t_k\}}, \quad z = \sum_{k=1}^N z_{k-1} \chi_{(t_{k-1}, t_k)} + \sum_{k=0}^N \hat{z}_k \chi_{\{t_k\}}. \quad (2.46)$$

For a given step function u , the choice of the partition Δ in the representation (2.45) is not unique. But one may check that w and z as defined in (2.46) do not depend on this choice. Thus, \mathcal{P}_r and \mathcal{S}_r are well defined by (2.46). By construction,

$$u = \mathcal{P}_r[u; z_a] + \mathcal{S}_r[u; z_a], \quad \|\mathcal{S}_r[u; z_a]\|_\infty \leq r. \quad (2.47)$$

The extension of the play to a larger class of discontinuous functions is based on the maximum norm estimate for the discrete play that corresponds to the one given in Proposition 2.4 for the play in continuous time.

Lemma 2.14 *Let $\tilde{u}^d, u^d \in \mathbb{R}$, let $\tilde{z}_a, z_a \in \mathbb{R}$ and $\tilde{r}, r \geq 0$. Then*

$$\|\mathcal{P}_{\tilde{r}}^d[\tilde{u}^d; \tilde{z}_a] - \mathcal{P}_r^d[u^d; z_a]\|_\infty \leq \|\tilde{u}^d - u^d\|_\infty + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.48)$$

$$\|\mathcal{S}_{\tilde{r}}^d[\tilde{u}^d; \tilde{z}_a] - \mathcal{S}_r^d[u^d; z_a]\|_\infty \leq 2\|\tilde{u}^d - u^d\|_\infty + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.49)$$

Proof. Based on the estimate (2.8) for f_r in Lemma 2.3, one checks that for the components of \tilde{u}^d, u^d and of \tilde{w}^d, w^d

$$\begin{aligned} |\tilde{w}_0 - w_0| &\leq |\tilde{u}_0 - u_0| + |\tilde{z}_a - z_a|, \\ |\tilde{w}_k - w_k| &\leq \max\{\|\tilde{u}^d - u^d\|_\infty + |\tilde{r} - r|, |\tilde{w}_{k-1} - w_{k-1}|\}, \quad k > 0, \end{aligned}$$

analogously as in the proof of Proposition 2.4, and one uses induction to arrive at (2.48).

□

The Lipschitz estimate (2.48) makes it possible to continuously extend the play to uniform limits of step functions. These form the space of **regulated functions** which on $[a, b]$ we denote by $G[a, b]$. It is a fact of real analysis that $G[a, b]$ is a Banach space when equipped with the norm

$$\|u\|_\infty = \sup_{t \in [a, b]} |u(t)|$$

and that $u : [a, b] \rightarrow \mathbb{R}$ belongs to $G[a, b]$ if and only if u possesses right and left limits at every point of $[a, b]$. On $G[a, b]$ the result corresponding to Proposition 2.4 holds.

Proposition 2.15 *The operators \mathcal{P}_r and \mathcal{S}_r can be extended uniquely to Lipschitz continuous operators*

$$\mathcal{P}_r, \mathcal{S}_r : G[a, b] \times \mathbb{R} \rightarrow G[a, b],$$

such that

$$u = \mathcal{P}_r[u; z_a] + \mathcal{S}_r[u; z_a], \quad \|\mathcal{S}_r[u; z_a]\|_\infty \leq r \quad (2.50)$$

hold for all $u \in G[a, b]$, $z_a \in \mathbb{R}$. Moreover, there holds

$$\|\mathcal{P}_{\tilde{r}}[\tilde{u}; \tilde{z}_a] - \mathcal{P}_r[u; z_a]\|_\infty \leq \|\tilde{u} - u\|_\infty + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.51)$$

$$\|\mathcal{S}_{\tilde{r}}[\tilde{u}; \tilde{z}_a] - \mathcal{S}_r[u; z_a]\|_\infty \leq 2\|\tilde{u} - u\|_\infty + \max\{|\tilde{r} - r|, |\tilde{z}_a - z_a|\} \quad (2.52)$$

for all $\tilde{u}, u \in G[a, b]$, all $\tilde{z}_a, z_a \in \mathbb{R}$ and all $\tilde{r}, r \geq 0$.

Proof. We first consider the case where \tilde{u} and u are step functions. Then (2.50) has already been obtained (2.47). The functions \tilde{u} and u have the form (2.45) with partitions $\Delta_1 = \{t_k^1\}$ and $\Delta_2 = \{t_k^2\}$, respectively. Let $\Delta = \Delta_1 \cup \Delta_2 = \{t_k\}$ be their common refinement. According to (2.45) and (2.46),

$$\|\tilde{u} - u\|_\infty = \|\tilde{u}^d - u^d\|_\infty \quad \text{and} \quad \|\tilde{w} - w\|_\infty = \|\tilde{w}^d - w^d\|_\infty.$$

Now (2.51) follows directly from Lemma 2.14. As the step functions are dense in $G[a, b]$, \mathcal{P}_r (and hence \mathcal{S}_r) can be extended to $G[a, b]$ and (2.50) – (2.52) continue to hold for this extension, called $\tilde{\mathcal{P}}_r$ for the moment. It remains to show that $\tilde{\mathcal{P}}_r$ coincides on $C[a, b] \times \mathbb{R}$ with \mathcal{P}_r as obtained in Proposition 2.4. By density, it suffices to check this on $C_{pm}[a, b] \times \mathbb{R}$. Let $u \in C_{pm}[a, b]$, $z_a \in \mathbb{R}$, $w = \mathcal{P}_r[u; z_a]$. Let $\Delta_n = \{t_i^n\}$ be a monotonicity partition for u such that $|\Delta_n| = \sup_i(t_i^n - t_{i-1}^n) \rightarrow 0$ as $n \rightarrow \infty$. Let $u^n : [a, b] \rightarrow \mathbb{R}$ be the piecewise constant function with $u^n(t) = u(t_i^n)$ for $t \in [t_i^n, t_{i+1}^n)$. Then $u^n \rightarrow u$ uniformly on $[a, b]$, so $\tilde{w}^n = \tilde{\mathcal{P}}_r[u^n; z_a] \rightarrow \tilde{w} = \tilde{\mathcal{P}}_r[u; z_a]$ uniformly on $[a, b]$. We claim that $\tilde{w}^n(t_i^n) = w(t_i^n)$ for all i, n . Indeed, for fixed n , this follows by induction over i , the induction step being

$$\tilde{w}^n(t_i^n) = f_r(u^n(t_i^n), \tilde{w}^n(t_{i-1}^n)) = f_r(u(t_i^n), w(t_{i-1}^n)) = w(t_i^n). \quad (2.53)$$

Let $t \in [a, b]$ be arbitrary, choose i such that $t \in [t_i^n, t_{i+1}^n)$. Then $\tilde{w}^n(t) = \tilde{w}^n(t_i^n)$ since u^n is constant on $[t_i^n, t]$ and therefore

$$\begin{aligned} w(t) - \tilde{w}(t) &= (w(t) - w(t_i^n)) + (w(t_i^n) - \tilde{w}^n(t_i^n)) + (\tilde{w}^n(t_i^n) - \tilde{w}^n(t)) + (\tilde{w}^n(t) - \tilde{w}(t)) \\ &= (w(t) - w(t_i^n)) + (\tilde{w}^n(t) - \tilde{w}(t)) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus $\tilde{\mathcal{P}}_r = \mathcal{P}_r$ on $C_{pm}[a, b] \times \mathbb{R}$. \square

Concatenation of inputs. Let $u^d = (u_0, \dots, u_N) \in S$ be given. Rewriting (2.42) we obtain $z^d = \mathcal{S}_r[u^d; z_a]$ from

$$\begin{aligned} z_0 &= \pi_r(z_a), \\ z_i &= u_i - f_r(u_i, u_{i-1} - z_{i-1}), \quad i > 0. \end{aligned} \quad (2.54)$$

We partition u^d into (u_0, \dots, u_L) and (u_L, \dots, u_N) where $0 < L < N$. We have $z_L = \pi_r(z_L)$ since $|z_L| \leq r$, and therefore

$$\mathcal{S}_r^d[(u_0, \dots, u_N); z_a]_i = z_i = \mathcal{S}_r^d[(u_L, \dots, u_N); z_L]_{i-L}, \quad L \leq i \leq N. \quad (2.55)$$

If $u : [a, b] \rightarrow \mathbb{R}$ is a step function, $z = \mathcal{S}_r[u; z_a]$ and $\tau \in (a, b)$, (2.55) becomes

$$\mathcal{S}_r[u; z_a](s) = \mathcal{S}_r[u|[\tau, b]; \mathcal{S}_r[u; z_a](\tau)](s), \quad \tau \leq s \leq b. \quad (2.56)$$

Introducing the time shift $t = s - \tau$ we arrive at

$$\mathcal{S}_r[u; z_a](\tau + t) = \mathcal{S}_r[u(\cdot + \tau); \mathcal{S}_r[u; z_a](\tau)](t), \quad 0 \leq t \leq b - \tau. \quad (2.57)$$

This is the **semigroup property** for the scalar stop operator. As the step functions are dense in $G[a, b]$ and \mathcal{S}_r is Lipschitz continuous by Proposition 2.15, the semigroup property is valid for arbitrary $u \in G[a, b]$. For the play operator, (2.57) becomes

$$\mathcal{P}_r[u; z_a](\tau + t) = \mathcal{P}_r[u(\cdot + \tau); u(\tau) - \mathcal{P}_r[u; z_a](\tau)](t), \quad 0 \leq t \leq b - \tau. \quad (2.58)$$

3 Models with Preisach Memory

The Preisach memory. We have explained in Chapter 1 how the Preisach model is based on the time evolution $t \mapsto \psi(t, \cdot)$ of the function $\psi(t, \cdot)$ whose graph separates the half-space $\mathbb{R}_+ \times \mathbb{R}$ into the two regions where the relays underlying the Preisach models have the values $+1$ and -1 respectively.

We define the **memory space** Ψ as the set of all functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ that are Lipschitz continuous with Lipschitz constant 1,

$$|\varphi(\tilde{r}) - \varphi(r)| \leq |\tilde{r} - r| \quad \text{for all } \tilde{r}, r \geq 0, \quad (3.1)$$

and vanish as $r \rightarrow \infty$. So

$$\Psi = \{\varphi | \varphi : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ satisfies (3.1), } \lim_{r \rightarrow \infty} \varphi(r) = 0\}. \quad (3.2)$$

The set Ψ is a closed convex subset of the Banach space of all bounded continuous functions on \mathbb{R}_+ , equipped with the maximum norm. Consequently, Ψ is a complete metric space w.r.t. the distance induced by the maximum norm. Moreover, the pointwise ordering “ $\tilde{\varphi} \leq \varphi$ if $\tilde{\varphi}(r) \leq \varphi(r)$ for all $r \geq 0$ ” defines a partial order on Ψ .

By a result of real analysis

$$\Psi = \{\varphi \mid \varphi \in W^{1,\infty}(0, \infty), |\varphi'(r)| \leq 1 \text{ a.e. on } (0, \infty), \lim_{r \rightarrow \infty} \varphi(r) = 0\}. \quad (3.3)$$

Given an input function $u \in C_{pm}[a, b]$ with monotonicity partition $\{t_i\}$ and an initial state $\psi_a \in \Psi$, we define the memory evolution by

$$\begin{aligned} \psi(a, r) &= f_r(u(a), \psi_a(r)), & r \geq 0, \\ \psi(t, r) &= f_r(u(t), \psi(t_{i-1}, r)), & r \geq 0, \ t \in (t_{i-1}, t_i] \end{aligned} \quad (3.4)$$

with

$$f_r(x, y) = \max\{x - r, \min\{x + r, y\}\}$$

as before. We write

$$\psi = \mathcal{P}[u; \psi_a], \quad \psi : [a, b] \times \mathbb{R}_+ \rightarrow \mathbb{R}. \quad (3.5)$$

Lemma 3.1 *Let $\psi = \mathcal{P}[u; \psi_a]$ and $\tilde{\psi} = \mathcal{P}[\tilde{u}; \tilde{\psi}_a]$ with $u, \tilde{u} \in C_{pm}[a, b]$ and $\psi_a, \tilde{\psi}_a \in \Psi$. Then*

$$|\tilde{\psi}(t, \tilde{r}) - \psi(t, r)| \leq \max\left\{\max_{a \leq \tau \leq t} |\tilde{u}(\tau) - u(\tau)| + |\tilde{r} - r|, |\tilde{\psi}_a(\tilde{r}) - \psi_a(r)|\right\} \quad (3.6)$$

for all $t \in [a, b]$ and all $\tilde{r}, r \geq 0$. Moreover,

$$|\psi(t, r) - \psi(s, r)| \leq \max_{s \leq \tau \leq t} |u(\tau) - u(s)| \quad (3.7)$$

holds for all $s, t \in [a, b]$ with $s \leq t$ and all $r \geq 0$.

Proof. Let $\{t_i\}$ be a monotonicity partition for both \tilde{u} and u . By Lemma 2.3, for all $\tilde{r}, r \geq 0$

$$|\tilde{\psi}(a, \tilde{r}) - \psi(a, r)| \leq \max\{|\tilde{u}(a) - u(a)| + |\tilde{r} - r|, |\tilde{\psi}_a(\tilde{r}) - \psi_a(r)|\}, \quad (3.8)$$

and

$$|\tilde{\psi}(t, \tilde{r}) - \psi(t, r)| \leq \max\{|\tilde{u}(t) - u(t)| + |\tilde{r} - r|, |\tilde{\psi}(t_{i-1}, \tilde{r}) - \psi(t_{i-1}, r)|\}, \quad t \in (t_{i-1}, t_i].$$

By induction over i we obtain (3.6). Setting $\tilde{r} = r$, $\tilde{\psi}_a = \psi_a$, $\tilde{u} = u$ on $[a, s]$ and $\tilde{u} = u(s)$ on $[s, t]$, (3.7) follows from (3.6). \square

We may interpret the mapping $(t, r) \mapsto \psi(t, r)$ as a mapping

$$t \mapsto \psi(t, \cdot), \quad \text{where } \psi(t, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}.$$

This viewpoint is used in the following proposition.

Proposition 3.2 *The mapping $(u, \psi_a) \mapsto \psi$ as defined above on $C_{pm}[a, b] \times \Psi$ can be uniquely extended to a Lipschitz continuous operator*

$$\mathcal{P} : C[a, b] \times \Psi \rightarrow C([a, b]; \Psi) \quad (3.9)$$

which satisfies

$$\|\mathcal{P}[u; \psi_a](t) - \mathcal{P}[u; \psi_a](s)\|_\infty \leq \max_{s \leq \tau \leq t} |u(\tau) - u(s)|, \quad a \leq s \leq t \leq b, \quad (3.10)$$

as well as

$$\|\mathcal{P}[\tilde{u}; \tilde{\psi}_a](t) - \mathcal{P}[u; \psi_a](t)\|_\infty \leq \max\{\max_{a \leq \tau \leq t} |\tilde{u}(\tau) - u(\tau)|, \|\tilde{\psi}_a - \psi_a\|_\infty\}, \quad (3.11)$$

$$\|\mathcal{P}[u; \psi_a](t)\|_\infty \leq \max\{\sup_{a \leq \tau \leq t} |u(\tau)|, \|\psi_a\|_\infty\} \quad (3.12)$$

for all $\tilde{u}, u \in C[a, b]$, all $\tilde{\psi}_a, \psi_a \in \Psi$ and all $t \in [a, b]$.

Proof. For $\tilde{u}, u \in C_{pm}[a, b]$ this is a direct consequence of Lemma 3.1 applied with $\tilde{r} = r$; in particular, $\mathcal{P}[u; \psi_a] : [a, b] \rightarrow \Psi$ is continuous. Since \mathcal{P} is thus Lipschitz continuous on $C_{pm}[a, b] \times \Psi$, it can be uniquely extended to an operator on $C[a, b] \times \Psi$ satisfying the same Lipschitz estimate. Then, again by density, (3.10) extends to $C[a, b] \times \Psi$ as well. \square

We call \mathcal{P} the **Preisach memory operator** or simply the **memory operator**.

Relation to the play operator. As we have seen in Chapter 1, the Preisach memory operator is closely related to the one-parameter family $\{\mathcal{P}_r\}_{r \geq 0}$ of play operators. Here we describe this correspondence for an arbitrary initial memory $\psi_a \in \Psi$. Since

$$f_r(x, y) = x - \pi_r(x - y),$$

for $\psi = \mathcal{P}[u; \psi_a]$ we have by the definition (2.4) of the initial value of \mathcal{P}_r

$$\psi(a, r) = f_r(u(a), \psi_a(r)) = u(a) - \pi_r(u(a) - \psi_a(r)) = \mathcal{P}_r[u; u(a) - \psi_a(r)](a).$$

It follows that

$$\psi(\cdot, r) = \mathcal{P}_r[u; u(a) - \psi_a(r)], \quad \text{for all } r \geq 0, \quad (3.13)$$

or expressed differently

$$(\mathcal{P}[u; \psi_a](t))(r) = \mathcal{P}_r[u; u(a) - \psi_a(r)](t), \quad \text{for all } r \geq 0, t \in [a, b], \quad (3.14)$$

since the evolutions for $t > r$ for both sides of (3.14) coincide if the initial values at $t = a$ coincide.

Since the play operator \mathcal{P}_r is causal and rate independent for all r , by (3.14) the same is true for the memory operator \mathcal{P} .

For the investigation of the Preisach memory, the definition of $\psi(a, \cdot)$ in (3.4) is natural; on the other hand, for the study of the single play and stop, the definition of $w(0)$ in (2.4) is natural when working with the variational inequality formulation.

The discrete memory evolution. As in the case of the play operator, for a given finite sequence $u^d = (u_0, \dots, u_M) \in S$ of input values and of an initial memory state $\psi_a \in \Psi$ we define the corresponding sequence $\psi^d = (\psi_0, \dots, \psi_M)$ of memory states $\psi_i \in \Psi$ by

$$\begin{aligned} \psi_0(r) &= f_r(u_0, \psi_a(r)), & r \geq 0, \\ \psi_i(r) &= f_r(u_i, \psi_{i-1}(r)), & r \geq 0, \quad 0 < i \leq M. \end{aligned} \quad (3.15)$$

We write

$$\psi^d = \mathcal{P}^d[u^d; \psi_a] = \mathcal{P}^d[u_0, \dots, u_M; \psi_a] \quad (3.16)$$

and call \mathcal{P}^d the **discrete memory operator**. We also define the **final state mapping** $\psi^f : S \times \Psi \rightarrow \Psi$ by

$$\psi^f(u^d; \psi_a) = \psi_M. \quad (3.17)$$

It immediately follows from the definitions that

$$\begin{aligned} \mathcal{P}[u; \psi_a](a) &= \mathcal{P}^d[u(a); \psi_a] \\ \mathcal{P}[u; \psi_a](t) &= \mathcal{P}^d[u(t_0), \dots, u(t_{i-1}), u(t); \psi_a], \quad t \in (t_{i-1}, t_i], \quad 0 < i \leq N \end{aligned} \quad (3.18)$$

holds for $u \in C_{pm}[a, b]$ with monotonicity partition $\{t_i\}_{0 \leq i \leq N}$. Moreover, the Lipschitz estimate (3.11) becomes

$$\|\mathcal{P}^d[\tilde{u}^d; \tilde{\psi}_a]_k - \mathcal{P}^d[u^d; \psi_a]_k\|_\infty \leq \max\{\max_{0 \leq i \leq k} |\tilde{u}_i - u_i|, \|\tilde{\psi}_a - \psi_a\|_\infty\}, \quad 0 \leq k \leq N. \quad (3.19)$$

The semigroup property. Let a sequence $u^d = (u_0, \dots, u_M)$ of input values and an initial memory state $\psi_a \in \Psi$ be given. It directly follows from the definition of \mathcal{P}^d that for $(\psi_0, \dots, \psi_M) = \mathcal{P}^d[u^d; \psi_a]$ the **semigroup property**

$$(\psi_L, \dots, \psi_M) = \mathcal{P}^d[u_L, \dots, u_M; \psi_L] \quad (3.20)$$

holds for all $0 \leq L \leq M$. For continuous inputs $u : [a, b] \rightarrow \mathbb{R}$ and $\tau \in [a, b]$, the semigroup property for \mathcal{P} becomes

$$\mathcal{P}[u; \psi_a](\tau + t) = \mathcal{P}\left[u(\cdot + \tau); \mathcal{P}[u; \psi_a](\tau)\right](t), \quad 0 \leq t \leq b - \tau. \quad (3.21)$$

This follows from (3.20) and (3.18) if u is piecewise monotone, and in the general case by a limit passage using Proposition 3.2.

Monotonicity and symmetry. These properties carry over from the play operator to the memory operator due to (3.14) and (3.18). More precisely, \mathcal{P} and \mathcal{P}^d are **order monotone**, that is, w.r.t. the pointwise ordering in Ψ we have pointwise (in $[a, b]$ resp. componentwise)

$$\mathcal{P}[u; \psi_a] \leq \mathcal{P}[\tilde{u}; \tilde{\psi}_a], \quad \mathcal{P}^d[u^d; \psi_a] \leq \mathcal{P}^d[\tilde{u}^d; \tilde{\psi}_a], \quad (3.22)$$

if $\psi_a \leq \tilde{\psi}_a$ and $u \leq \tilde{u}$ resp. $u^d \leq \tilde{u}^d$. Moreover, \mathcal{P} and \mathcal{P}^d are **piecewise monotone**, that is,

$$\mathcal{P}[u; \psi_a](s) \leq (\geq) \mathcal{P}[u; \psi_a](t) \quad (3.23)$$

if $s \leq t$ and u is nondecreasing (resp. nonincreasing) on $[s, t]$; for $\psi^d = \mathcal{P}^d[u^d; \psi_a]$ we have

$$\psi_{k-1} \leq \psi_k \quad \text{if } u_{k-1} \leq u_k, \quad \psi_{k-1} \geq \psi_k \quad \text{if } u_{k-1} \geq u_k. \quad (3.24)$$

In addition, the symmetry relations

$$\mathcal{P}[-u; -\psi_a] = -\mathcal{P}[u; \psi_a], \quad \mathcal{P}^d[-u; -\psi_a] = -\mathcal{P}^d[u; \psi_a] \quad (3.25)$$

hold for all $u \in C[a, b]$, $u^d \in S$ and $\psi_a \in \Psi$.

Memory erasure and periodicity. Here the discrete case provides the natural setting. In the following, for $x, y \in \mathbb{R}$ we use the notation $[x, y]$ for the interval between x and y , no matter whether $x \leq y$ or $y \leq x$.

Proposition 3.3 Let $u^d = (u_0, \dots, u_M) \in S$ with $u_k \in [u_0, u_M]$ for all $0 \leq k \leq M$, let $\psi_a \in \Psi$. Then

$$\psi^f(u^d; \psi_a) = \psi^f(u_0, u_M; \psi_a). \quad (3.26)$$

Proof. In view of the symmetry relation (3.25) we may assume that $u_0 \leq u_M$. Then by order monotonicity

$$\begin{aligned} \psi^f(u_0, u_M; \psi_a) &= \psi^f(u_0, \dots, u_0, u_M; \psi_a) \leq \psi^f(u^d; \psi_a) \\ &\leq \psi^f(u_0, u_M, \dots, u_M; \psi_a) = \psi^f(u_0, u_M; \psi_a). \end{aligned}$$

□

Thus, the last input value u_M completely erases the influence of the inputs u_k , $0 < k < M$, and of their corresponding memory states ψ_k on the final memory state $\psi_M = \psi^f(u^d; \psi_a)$.

Corollary 3.4 Let $u^d = (u_0, u_1, u_0, u_1, u_0, \dots)$ with $u_0, u_1 \in \mathbb{R}$, let $\psi_a \in \Psi$. Then $\psi^d = \mathcal{P}^d[u^d; \psi_a]$ has the form

$$\psi^d = (\psi_0, \psi_1, \psi_2, \psi_1, \psi_2, \dots). \quad (3.27)$$

In other words: if $u_{k+2} = u_k$ for all $k \geq 0$, then $\psi_{k+2} = \psi_k$ for all $k \geq 1$.

Proof. We have $\psi^f(u_0, u_1, u_0, u_1; \psi_a) = \psi^f(u_0, u_1; \psi_a)$ as well as $\psi^f(u_1, u_2, u_1, u_2; \psi_0) = \psi^f(u_1, u_2; \psi_0)$. Using the semigroup property we obtain (3.27). □

The continuous version of Proposition 3.3 reads as follows.

Proposition 3.5 Let $u \in C[a, b]$, $\psi_a \in \Psi$, $\psi = \mathcal{P}[u; \psi_a]$, $a \leq s \leq t \leq b$. If

$$u(s) = \min_{[s,t]} u, \quad u(t) = \max_{[s,t]} u \quad \text{or} \quad u(s) = \max_{[s,t]} u, \quad u(t) = \min_{[s,t]} u, \quad (3.28)$$

then

$$\psi(t) = \psi^f(u(s), u(t); \psi(s)) = \psi^f(u(t); \psi(s)). \quad (3.29)$$

Proof. If u is piecewise monotone on $[s, t]$, the assertion follows from Proposition 3.3, the semigroup property and (3.18). In the general case, we approximate u on $[s, t]$ by piecewise monotone functions u_n satisfying $u_n \rightarrow u$ uniformly, $u_n(s) = u(s)$, $u_n(t) = u(t)$ as well as (3.28) with u_n in place of u , and pass to the limit using Proposition 3.2. □

Proposition 3.6 Let $u \in C[a, \infty)$ be T -periodic, let $\psi_a \in \Psi$, $\psi = \mathcal{P}[u; \psi_a]$. Then ψ is T -periodic on $[a + T, \infty)$.

Proof. Let $t_*, t^* \in [a, a + T]$ with

$$u(t_*) = \min_{[a, a+T]} u, \quad u(t^*) = \max_{[a, a+T]} u.$$

First we assume that $t_* \leq t^*$. Applying Proposition 3.5 successively on $[t_*, t^*]$, $[t^*, t_* + T]$ and $[t_* + T, t^* + T]$ we obtain, using Proposition 3.3 as well as the semigroup property,

$$\begin{aligned} \psi(t^* + T) &= \psi^f(u(t_*), u(t^*), u(t_* + T), u(t^* + T); \psi(t_*)) \\ &= \psi^f(u(t_*), u(t^*), u(t_*), u(t^*); \psi(t_*)) = \psi^f(u(t_*), u(t^*); \psi(t_*)) \\ &= \psi(t^*). \end{aligned}$$

Since $u(t+T) = u(t)$ for all t , it follows that $\psi(t+T) = \psi(t)$ for all $t \geq t^*$.

In view of the symmetry (3.25), the case $t_* > t^*$ is reduced to the case above by replacing (u, ψ_a) by $(-u, -\psi_a)$. \square

The single memory update. When an input value $x \in \mathbb{R}$ acts on a given memory state $\varphi \in \Psi$, the new memory state $\varphi_* \in \Psi$ is given by

$$\begin{aligned}\varphi_*(r) &= f_r(x, \varphi(r)) = \max\{x - r, \min\{x + r, \varphi(r)\}\}, \quad \text{that is,} \\ \varphi_* &= \mathcal{P}^d[x; \varphi] = \psi^f(x; \varphi).\end{aligned}\tag{3.30}$$

In particular, $\varphi_*(0) = x$ and $\varphi_* = \varphi$ if $x = \varphi(0)$, that is,

$$\psi^f(\varphi(0); \varphi) = \varphi.\tag{3.31}$$

We set

$$r_*(x, \varphi) = \begin{cases} \sup\{r : r \geq 0, x - r > \varphi(r)\}, & x > \varphi(0), \\ \sup\{r : r \geq 0, x + r < \varphi(r)\}, & x < \varphi(0), \\ 0, & x = \varphi(0). \end{cases}\tag{3.32}$$

Thus $r_*(x, \varphi)$ is the smallest value of r for which the graphs of φ and of the straight line $r \mapsto x - r$ (resp. $r \mapsto x + r$) intersect. (An intersection point exists since $\varphi(r) \rightarrow 0$ for $r \rightarrow \infty$.) We have

$$r_*(-x, -\varphi) = r_*(x, \varphi) \quad \text{for all } x \in \mathbb{R}, \varphi \in \Psi,\tag{3.33}$$

since $x > \varphi(0)$ and $x - r > \varphi(r)$ if and only if $-x < -\varphi(0)$ and $-x + r < -\varphi(r)$. (The other case is analogous.)

We note that the function $x \mapsto r_*(x, \varphi)$ need not be continuous unless $|\varphi'| < 1$.

Lemma 3.7 *Let $\varphi \in \Psi$.*

(i) *The function $x \mapsto r_*(x, \varphi)$ is nonincreasing on $(-\infty, \varphi(0)]$ and nondecreasing on $[\varphi(0), +\infty)$.*

(ii) *The new memory state φ_* in (3.30) satisfies*

$$\varphi_*(r) = \begin{cases} x - r, & x \geq \varphi(0) \text{ and } r \leq r_*(x, \varphi), \\ x + r, & x \leq \varphi(0) \text{ and } r \leq r_*(x, \varphi), \\ \varphi(r), & r \geq r_*(x, \varphi). \end{cases}\tag{3.34}$$

(iii) *(Uniqueness property of φ_*) If $\hat{\varphi} \in \Psi$ with $\hat{\varphi}(0) = x$ and $\hat{\varphi} = \varphi$ on $[r_*(x, \varphi), \infty)$ then $\hat{\varphi} = \varphi_*$.*

Proof. (i) Let $x \leq \tilde{x} \leq \varphi(0)$. For $r \geq r_*(x, \varphi)$ we have $\tilde{x} + r \geq x + r \geq \varphi(r)$, so $r_*(\tilde{x}, \varphi) \leq r_*(x, \varphi)$. The other case is analogous.

(ii) Since $|\varphi'| \leq 1$ a.e., the functions $r \mapsto x \pm r - \varphi(r)$ are nondecreasing resp. nonincreasing. For $x \neq \varphi(0)$, the assertion then follows from (3.30). For $x = \varphi(0)$ we have $r_*(x, \varphi) = 0$ and $f_0(x, \varphi(0)) = \varphi(0)$.

(iii) By (3.34), $|\hat{\varphi}(r) - \hat{\varphi}(0)| = r$ for $r = r_*(x, \varphi)$. Since $|\hat{\varphi}'| \leq 1$ a.e. we must have $\hat{\varphi} = \varphi_*$ on $[0, r]$. \square

Lemma 3.8 Let $\varphi \in \Psi$, $u_0, u_1 \in \mathbb{R}$ and $(\psi_0, \psi_1) = \mathcal{P}^d[u_0, u_1; \varphi]$. Then $r_*(u_0, \psi_1) \leq r_*(u_1, \psi_0)$.

Proof. In view of (3.33) it suffices to consider the case $u_1 < u_0$, as $\psi_1 = \psi_0$ if $u_1 = u_0$. Let $r = r_*(u_1, \psi_0)$, then $\psi_0(r) = \psi_1(r)$ by (3.34). If $r < r_*(u_0, \psi_1)$ then $\psi_1(r) < u_0 - r = \psi_0(0) - r$ as $u_0 > u_1$. It follows that $\psi_0(0) - \psi_0(r) > r$ which is impossible since $|\psi'_0| \leq 1$. \square

Lemma 3.9 Let $\varphi \in \Psi$, let $u_0, u_1, u_2 \in \mathbb{R}$ and set $(\psi_0, \psi_1, \psi_2) = \mathcal{P}^d[u_0, u_1, u_2; \varphi]$. If $u_1 \leq u_2 \leq u_0$ then

$$\begin{aligned} 2r_*(u_2, \psi_1) &= u_2 - u_1, \\ \psi_2(r) &= u_2 - r, \quad \psi_1(r) = u_1 + r, & \text{for } 2r \leq u_2 - u_1, \\ \psi_2(r) &= \psi_1(r), & \text{for } 2r \geq u_2 - u_1. \end{aligned} \quad (3.35)$$

If $u_0 \leq u_2 \leq u_1$ then (3.35) holds with indices 1 and 2 interchanged.

Proof. In view of the symmetry (3.25) it suffices to consider the case $u_1 < u_0$. By Lemma 3.7(i) and Lemma 3.8,

$$0 = r_*(u_1, \psi_1) \leq \underbrace{r_*(u_2, \psi_1)}_{=: r_2} \leq r_*(u_0, \psi_1) \leq r_*(u_1, \psi_0).$$

By Lemma 3.7(ii), $\psi_1(r) = u_1 + r$ and $\psi_2(r) = u_2 - r$ for $r \leq r_2$ as well as $\psi_2(r) = \psi_1(r)$ for $r \geq r_2$. Thus $2r_2 = u_2 - u_1$ which proves (3.35). \square

An explicit formula for the memory state. Given $u \in C[a, b]$ and $\psi_a \in \Psi$, we want to derive a formula for $\psi = \mathcal{P}[u; \psi_a]$. Since \mathcal{P} is causal, it suffices to do this for the final state $\psi(b)$. Let

$$u_{max} = \max_{t \in [a, b]} u(t), \quad u_{min} = \min_{t \in [a, b]} u(t). \quad (3.36)$$

Proposition 3.10 Let $u \in C[a, b]$, $\psi_a \in \Psi$, $\psi = \mathcal{P}[u; \psi_a]$. Then

$$\max_{t \in [a, b]} r_*(u(t), \psi_a) = \max\{r_*(u_{max}, \psi_a), r_*(u_{min}, \psi_a)\} =: r_{max}, \quad (3.37)$$

and $\psi(t) = \psi_a$ on $[r_{max}, \infty)$ for all $t \in [a, b]$.

If $t \in [a, b]$ with $r_*(u(t), \psi_a) = r_{max}$, then $\psi(t) = \psi^f(u(t); \psi_a)$.

Proof. We obtain (3.37) directly from Lemma 3.7(i). Thus

$$\psi^f(u_{min}; \psi_a) = \psi_a = \psi^f(u_{max}; \psi_a)$$

on $[r_{max}, \infty)$ by Lemma 3.7(ii). Since \mathcal{P} is order monotone, for all $t \in [a, b]$

$$\psi^f(u_{min}; \psi_a) = \mathcal{P}[u_{min}; \psi_a](t) \leq \mathcal{P}[u; \psi_a](t) = \psi(t) \leq \mathcal{P}[u_{max}; \psi_a](t) = \psi^f(u_{max}; \psi_a).$$

It follows from (3.34) that $\mathcal{P}[u; \psi_a](t) = \psi_a$ on $[r_{max}, \infty)$. Now assume that $r_*(u(t), \psi_a) = r_{max}$. Since $\hat{\varphi} = \mathcal{P}[u; \psi_a](t)$ satisfies $\hat{\varphi}(0) = u(t)$ and $\hat{\varphi} = \psi_a$ on $[r_{max}, \infty)$, by Lemma 3.7(iii) we obtain that $\hat{\varphi} = \psi^f(u(t); \psi_a)$. \square

Corollary 3.11 *Let $u \in C[a, b]$, $\psi_a \in \Psi$, $\psi = \mathcal{P}[u; \psi_a]$. Then we have*

$$\text{supp}(\psi(t)) \subset \text{supp}(\psi_a) \cup [0, r_{\max}], \quad \text{for all } t \in [a, b]. \quad (3.38)$$

In particular, the support of ψ is compact if the support of ψ_a is compact.

Corollary 3.12 *Let $\varphi \in \Psi$ and $u_0, u_1 \in \mathbb{R}$ with $r_*(u_1, \varphi) \leq r_*(u_0, \varphi)$. Then we have $\psi^f(u_0, u_1, u_0; \varphi) = \psi^f(u_0; \varphi)$.*

Proof. We apply Proposition 3.10 to the piecewise linear interpolate of the values u_0, u_1, u_0 on $[a, b]$ with $t = b$, setting $\psi_a = \varphi$. \square

We now construct the final state $\psi(b) = \mathcal{P}[u; \psi_a](b)$. We begin by defining

$$\begin{aligned} r_0 &= r_{\max}, \quad t_0 = \max\{t : r_*(u(t), \psi_a) = r_0\}, \\ u_0 &= u(t_0), \quad \psi_0 = \psi^f(u_0; \psi_a), \end{aligned} \quad (3.39)$$

We consider the case $u_0 = u_{\max}$. By Proposition 3.10 and (3.34),

$$\psi(t_0) = \psi^f(u_0; \psi_a) = \psi_0, \quad \psi_0(r) = \begin{cases} u_0 - r, & r \leq r_0, \\ \psi_a(r), & r \geq r_0. \end{cases} \quad (3.40)$$

If $t_0 = b$ we are done. If not, for $k \geq 1$ with $t_{k-1} < b$ we define t_k to be the largest value for which u takes its minimum (maximum, resp.) on $[t_{k-1}, b]$,

$$\begin{aligned} t_k &= \max\{t : t \leq b, u(t) = \min_{t_{k-1} \leq s \leq b} u(s)\}, \quad \text{if } k \text{ is odd,} \\ t_k &= \max\{t : t \leq b, u(t) = \max_{t_{k-1} \leq s \leq b} u(s)\}, \quad \text{if } k \text{ is even.} \end{aligned} \quad (3.41)$$

Then $t_k > t_{k-1}$ and either $t_N = b$ for some N or $t_k \uparrow b$ as $k \rightarrow \infty$. We set

$$u_k = u(t_k), \quad \psi_k = \psi^f(u_k; \psi_{k-1}), \quad r_k = r_*(u_k, \psi_{k-1}). \quad (3.42)$$

Now let $k = 1$. Since $r_1 < r_0$ by definition of t_0 , we have $\psi_0(r) = u_0 - r$ on $[0, r_1]$ and $\mathcal{P}^d[u_0, u_1, u_0; \psi_a] = (\psi_0, \psi_1, \psi_0)$ by Corollary 3.12. Applying Lemma 3.9 with $u_2 = u_0$ and $\varphi = \psi_a$ we then obtain from (3.35) that

$$\begin{aligned} \psi_1(r) &= \begin{cases} u_1 + r, & r \leq r_1, \\ \psi_0(r), & r \geq r_1, \end{cases} \\ 2r_1 &= u_0 - u_1, \\ \psi(t_1) &= \psi^f(u_1; \psi_0) = \psi_1. \end{aligned} \quad (3.43)$$

Finally, for $k \geq 2$ we can apply Lemma 3.9 with u_{k-2}, u_{k-1}, u_k and ψ_{k-2} in place of u_0, u_1, u_2 and φ , as by construction (3.41) we have $u_k \in [u_{k-1}, u_{k-2}]$. From (3.35) we obtain

$$\begin{aligned} 2r_k &= (-1)^k(u_k - u_{k-1}) < 2r_{k-1}, \\ \psi_k(r) &= \begin{cases} u_k - (-1)^k r, & r \leq r_k, \\ \psi_{k-1}(r), & r \geq r_k, \end{cases} \\ \psi(t_k) &= \psi^f(u_k; \psi_{k-1}) = \psi_k. \end{aligned} \quad (3.44)$$

We now set

$$\sigma = \begin{cases} 1, & u(t_0) = u_{\max}, \\ -1, & u(t_0) = u_{\min}. \end{cases} \quad (3.45)$$

Lemma 3.13 *The memory state ψ_k from (3.42) satisfies $\psi_k = \psi_a$ on $[r_0, \infty)$ and*

$$\psi_k(r) = \psi_a(r_0) + \sigma \left[(-1)^j (r_j - r) + \sum_{i=0}^{j-1} (-1)^i (r_i - r_{i+1}) \right] \quad (3.46)$$

for $r \in [r_{j+1}, r_j]$, $0 \leq j \leq k$, with r_j given in (3.39) and (3.42).

Proof. For the case $\sigma = 1$ we use (3.40), (3.43) and (3.44) and proceed by induction. For $k = 0$ we have $\psi_0(r) = \psi_a(r_0) + \sigma(r_0 - r)$ on $[r_1, r_0]$ by (3.40). Let (3.46) hold for $k - 1$ in place of k . Since $\psi_k = \psi_{k-1}$ on $[r_k, \infty)$ by (3.44) resp. (3.43), (3.46) is satisfied for $r \in [r_{j+1}, r_j]$, $0 \leq j < k$. In particular, setting $j = k - 1$,

$$\begin{aligned} \psi_{k-1}(r_k) &= \psi_a(r_0) + \left[(-1)^{k-1} (r_{k-1} - r_k) + \sum_{i=0}^{k-2} (-1)^i (r_i - r_{i+1}) \right] \\ &= \psi_a(r_0) + \sum_{i=0}^{k-1} (-1)^i (r_i - r_{i+1}). \end{aligned}$$

Since $\psi'_k(r) = -(-1)^k$ on (r_{k+1}, r_k) by (3.44), (3.46) also holds on $[r_{k+1}, r_k]$. Replacing (u, ψ_a) by $(-u, -\psi_a)$ we reduce the case $\sigma = -1$ to the case $\sigma = 1$ in view of the symmetries (3.25) and (3.33). \square

Proposition 3.14 *Let $u \in C[a, b]$, $\psi_a \in \Psi$, $\psi = \mathcal{P}[u; \psi_a]$. Then $\psi(b, 0) = u(b)$, $\psi(b, r) = \psi_a(r)$ for $r \geq r_0$ and*

$$\psi(b, r) = \psi_a(r_0) + \sigma \left[(-1)^j (r_j - r) + \sum_{i=0}^{j-1} (-1)^i (r_i - r_{i+1}) \right] \quad (3.47)$$

if $r \in [r_{j+1}, r_j]$, $j \geq 0$, with r_j given in (3.39) and (3.42).

Proof. In the case $t_N = b$ we have $r_N = 0$ and $\psi(b, \cdot) = \psi_N$. Then (3.47) coincides with (3.46). If $t_k \uparrow b$ we have $u(t_k) \rightarrow u(b)$, $\psi_k = \psi(t_k, \cdot) \rightarrow \psi(b, \cdot)$ and $r_k \downarrow 0$ by (3.44) and Proposition 3.2. Since for every $r > 0$ the values $\psi_k(r)$ do not depend on k for k large enough, (3.47) again follows from (3.46). \square

From (3.47) we see that the state $\psi(t) = \mathcal{P}[u; \psi_a](t) \in \Psi$ at $t = b$ (and hence, also for all $t < b$) on $[0, r_0]$ is a piecewise linear function whose slope takes only the values 1 and -1, whereas on $[r_0, \infty)$ it coincides with the initial state ψ_a . Thus, unless $\psi_a = 0$, the state $\varphi = 0$ (called the “demagnetized state” in applications to ferromagnetism) cannot be reached exactly no matter how u is chosen. But on a bounded interval $[0, R]$ one may approximate it uniformly to arbitrary accuracy by choosing an oscillating input u whose amplitude is decreasing sufficiently slowly.

Operators of Preisach type. Let $Q : \Psi \rightarrow \mathbb{R}$ be a mapping. An operator \mathcal{W} of the form

$$\mathcal{W}[u; \psi_a](t) = Q(\psi(t)), \quad \psi = \mathcal{P}[u; \psi_a], \quad t \in [a, b], \quad (3.48)$$

is called an **operator of Preisach type**. For $u \in C[a, b]$ and $\psi_a \in \Psi$ it yields a function $\mathcal{W}[u; \psi_a] : [a, b] \rightarrow \mathbb{R}$.

If $Q : \Psi \rightarrow \mathbb{R}$ is continuous then

$$\mathcal{W} : C[a, b] \times \Psi \rightarrow C[a, b] \quad (3.49)$$

since $t \mapsto \psi(t)$ is continuous by Proposition 3.2. If Q is order monotone, that is, $Q(\varphi) \leq Q(\tilde{\varphi})$ if $\varphi \leq \tilde{\varphi}$, then \mathcal{W} is piecewise monotone, that is,

$$\mathcal{W}[u; \psi_a](s) \leq (\geq) \mathcal{W}[u; \psi_a](t) \quad (3.50)$$

if $s \leq t$ and u is nondecreasing (resp. nonincreasing) on $[s, t]$; this follows since the memory operator \mathcal{P} is piecewise monotone.

The discrete version

$$\mathcal{W}^d : S \times \Psi \rightarrow S \quad (3.51)$$

is defined by

$$\mathcal{W}^d[u^d; \psi_a] = w^d = (w_0, \dots, w_M), \quad w_k = Q(\psi_k), \quad 0 \leq k \leq M, \quad (3.52)$$

where $u^d = (u_0, \dots, u_M)$ and $\psi^d = \mathcal{P}^d[u^d; \psi_a]$.

Due to the definition (3.48), the periodicity result from Proposition 3.6 immediately carries over to operators of Preisach type.

Proposition 3.15 *Let \mathcal{W} be an operator of Preisach type, let $u \in C[a, \infty)$ be T -periodic and let $\psi_a \in \Psi$. Then $w = \mathcal{W}[u; \psi_a]$ is T -periodic on $[a + T, \infty)$. \square*

Examples. Setting $Q(\varphi) = \varphi(r)$ for a given $r \geq 0$ yields the play operator (see (3.13))

$$\mathcal{W}[u; \psi_a] = \mathcal{P}_r[u; z_a], \quad z_a = u(a) - \psi_a(r). \quad (3.53)$$

In Section 1 we have introduced the Preisach model for piecewise monotone input functions u and presented the formula (1.17)

$$w(t) = \int_0^\infty \int_{-\infty}^{\psi(t,r)} \rho(r, s) ds dr - \int_0^\infty \int_{\psi(t,r)}^\infty \rho(r, s) ds dr. \quad (3.54)$$

for the output function w . The given function $\rho : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is called the **Preisach density**. (If ρ belongs to $L^1(\mathbb{R}_+ \times \mathbb{R})$, the integrals are well-defined.) Setting

$$Q(\varphi) = \int_0^\infty \int_{-\infty}^{\varphi(r)} \rho(r, s) ds dr - \int_0^\infty \int_{\varphi(r)}^\infty \rho(r, s) ds dr, \quad (3.55)$$

the Preisach model thus becomes a special case of (3.48). If the Preisach density ρ is nonnegative, Q is order monotone and the Preisach operator \mathcal{W} given by (3.48) is piecewise monotone.

For the purpose of analysis, we also use a different representation of Q , namely

$$Q(\varphi) = \int_0^\infty g(r, \varphi(r)) dr. \quad (3.56)$$

Given $\rho \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ and $g_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$, the function

$$g(r, s) = 2 \int_0^s \rho(r, \sigma) d\sigma + g_0(r) \quad (3.57)$$

is well-defined on $\mathbb{R}_+ \times \mathbb{R}$. If

$$g_0(r) = - \int_0^\infty \rho(r, \sigma) - \rho(r, -\sigma) d\sigma, \quad (3.58)$$

the definitions of Q in (3.55) and (3.56) formally coincide. In particular, $g_0 = 0$ if $\rho(r, -s) = -\rho(r, s)$ for all $r \geq 0$ and $s \in \mathbb{R}$. In the following we will mainly work with the representation (3.56), (3.57) of the Preisach model, assuming that $g_0 \in L^1(\mathbb{R}_+)$ is a given function.

The model of Prandtl and Ishlinskii from (1.25) and (1.20),

$$w(t) = \int_0^\infty p(r) dr \cdot u(t) - \int_0^\infty p(r) \psi(t, r) dr, \quad (3.59)$$

can also be written in the form of (3.48) with

$$Q(\varphi) = q_0 \varphi(0) + \int_0^\infty q(r) \varphi(r) dr, \quad (3.60)$$

where

$$q_0 = \int_0^\infty p(r) dr, \quad q = -p. \quad (3.61)$$

The integral in (3.60) is a special case of (3.56) with $g(r, s) = q(r)s$.

Primary curve, reversal curves. Let \mathcal{W} be an operator of Preisach type. The graph of the function $\ell_0 : \mathbb{R} \rightarrow \mathbb{R}$,

$$\ell_0(x) = Q(\psi^f(x; \psi_a)), \quad (3.62)$$

is called the **primary curve** or **initial loading curve** of \mathcal{W} with respect to ψ_a . (A canonical choice is $\psi_a = 0$.) It describes the behaviour of the system for monotone inputs $u \in C[a, b]$, as then

$$w(t) = \mathcal{W}[u; \psi_a](t) = \ell_0(u(t)), \quad t \in [a, b].$$

For $x \rightarrow \pm\infty$, the values for positive and negative **saturation**

$$w_{+\infty} = \lim_{x \rightarrow +\infty} \ell_0(x), \quad w_{-\infty} = \lim_{x \rightarrow -\infty} \ell_0(x) \quad (3.63)$$

may or may not be finite; in many situations (e.g. for the Preisach model under natural assumptions) they do not depend on the initial state ψ_a .

Let $u_0 = \psi_a(0)$, so $\psi_0 = \psi^f(u_0; \psi_a) = \psi_a$. Let $\psi_1 = \psi^f(u_1; \psi_a)$ be the memory state attained by applying the input value u_1 to ψ_a , let $(u_1, Q(\psi_1))$ be the corresponding point on the primary curve. Setting

$$\ell_1(x) = Q(\psi^f(x; \psi_1)), \quad \psi_1 = \psi^f(u_1; \psi_a), \quad (3.64)$$

we obtain a family of functions $\ell_1 : (-\infty, u_1] \rightarrow \mathbb{R}$ for $u_1 > u_0$ and $\ell_1 : [u_1, +\infty) \rightarrow \mathbb{R}$ for $u_1 < u_0$. Their graphs are called **first-order reversal curves** of \mathcal{W} . For large enough $|x|$ they coincide with the primary curve; indeed, if $r_*(x, \psi_1) > r_*(u_1, \psi_a)$ then $\psi^f(x; \psi_1) = \psi^f(x; \psi_a)$ since $\psi_1 = \psi_a$ on $[r_*(u_1, \psi_a), \infty)$, and therefore $\ell_1(x) = \ell_0(x)$ for such x .

Higher order reversal curves are defined analogously, starting from a point on a reversal curve and proceeding in the opposite direction.

If Q is order monotone, the primary curve as well as all reversal curves are nondecreasing.

Hysteresis loops. We consider the following situation. Let \mathcal{W} be an operator of Preisach type. A given input $u \in C[a, b]$ with value $u(t_0) = u_0$ at some $t_0 < b$ decreases to $u(t_1) = u_1 < u_0$ on some interval $[t_0, t_1]$, increases to $u(t_2) = u_2$ with $u_2 \leq u_0$ on some $[t_1, t_2]$ and decreases back to $u(t_3) = u_1$ on some $[t_2, t_3]$. Let $\psi_i = \psi(t_i) = \mathcal{P}[u; \psi_a](t_i)$ be the corresponding memory states. (The choice of ψ_a is immaterial here.) Then $\psi(t_3) = \psi(t_1)$ since $\psi^f(u_0, u_1, u_2, u_1; \psi(t_0)) = \psi^f(u_0, u_1; \psi(t_0))$ by Proposition 3.3. The output function $w(t) = Q(\psi(t))$ satisfies

$$\begin{aligned} w(t) - w(t_1) &= Q(\psi(t)) - Q(\psi(t_1)), & t \in [t_1, t_2], \\ w(t) - w(t_2) &= Q(\psi(t)) - Q(\psi(t_2)), & t \in [t_2, t_3], \\ w(t_3) &= w(t_1). \end{aligned} \quad (3.65)$$

On $[t_1, t_3]$ we thus obtain a closed curve $t \mapsto (u(t), w(t))$ with $t \mapsto u(t)$ moving from u_1 to u_2 and back. Such a curve (as well as the one obtained in the corresponding situation where $u_0 \leq u_2 < u_1$) we call a **hysteresis loop**.

When traversing the loop given above, the memory state evolves according to the structure specified in Lemma 3.9. For $t \in [t_1, t_2]$,

$$\begin{aligned} \psi(t, r) &= u(t) - r, & \psi(t_1, r) &= u_1 + r, & \text{if } 2r \leq u(t) - u_1, \\ \psi(t, r) &= \psi(t_1, r), & & & \text{if } 2r \geq u(t) - u_1. \end{aligned} \quad (3.66)$$

For $t \in [t_2, t_3]$ we obtain from the same lemma that

$$\begin{aligned} \psi(t, r) &= u(t) + r, & \psi(t_2, r) &= u_2 - r, & \text{if } 2r \leq u_2 - u(t), \\ \psi(t, r) &= \psi(t_2, r), & & & \text{if } 2r \geq u_2 - u(t). \end{aligned} \quad (3.67)$$

Hysteresis loops in the Preisach model. Let \mathcal{W} be a Preisach operator written in the form (3.56), that is,

$$Q(\varphi) = \int_0^\infty g(r, \varphi(r)) dr.$$

We assume that g is sufficiently regular so that the following computations are valid.

Using (3.66) we get for $t \in [t_1, t_2]$

$$\begin{aligned} Q(\psi(t)) - Q(\psi(t_1)) &= \int_0^\infty g(r, \psi(t, r)) - g(r, \psi(t_1, r)) dr \\ &= \int_0^{(u(t)-u_1)/2} g(r, u(t) - r) - g(r, u_1 + r) dr. \end{aligned} \quad (3.68)$$

The first part of the loop is thus described by

$$w(t) = w(t_1) + h(u(t), u(t_1)), \quad t \in [t_1, t_2]. \quad (3.69)$$

with

$$h(x, y) = \int_0^{(x-y)/2} g(r, x - r) - g(r, y + r) dr. \quad (3.70)$$

An analogous computation for the second part of the loop, $t \in [t_2, t_3]$, yields

$$\begin{aligned} w(t) - w(t_2) &= Q(\psi(t)) - Q(\psi(t_2)) = \int_0^{(u_2-u(t))/2} g(r, u(t) + r) - g(r, u_2 - r) dr \\ &= -h(u(t_2), u(t)). \end{aligned} \quad (3.71)$$

Since for $t \in [t_2, t_3]$

$$w(t) - w(t_1) = w(t) - w(t_2) + w(t_2) - w(t_1) = -h(u(t_2), u(t)) + h(u(t_2), u(t_1))$$

and $w(t) - w(t_1) = h(u(t), u(t_1))$ on $[t_1, t_2]$, for fixed values of u_1 and u_2 all loops generated by oscillations between u_1 and u_2 are identical except for an additive constant given by the value of w at the beginning of the oscillation. This property has been called the **vertical congruency property** of the Preisach model.

In view of (3.69) and (3.71), the slope of the loop in the (u, w) -plane at $(u(t), w(t))$ is given by (recall that $\partial_s g = 2\rho$, the Preisach density)

$$\begin{aligned} \partial_x h(x, y) &= \int_0^{(x-y)/2} \partial_s g(r, x - r) dr = 2 \int_0^{(x-y)/2} \rho(r, x - r) dr, \\ -\partial_y h(x, y) &= \int_0^{(x-y)/2} \partial_s g(r, y + r) dr = 2 \int_0^{(x-y)/2} \rho(r, y + r) dr, \end{aligned} \quad (3.72)$$

at $(x, y) = (u(t), u(t_1))$ and $(x, y) = (u(t_2), u(t))$, respectively. It is therefore nonnegative if and only if the integrals on the right side of (3.72) are nonnegative. (This is a somewhat weaker condition than nonnegativity of ρ which yields order monotonicity of Q .) In particular, the loops start horizontally at their end points, since $\partial_x h(x, x) = \partial_y h(x, x) = 0$.

The second partial derivatives $\partial_{xx} h$ and $\partial_{yy} h$ become

$$\begin{aligned} \partial_{xx} h(x, y) &= \rho\left(\frac{x-y}{2}, \frac{x+y}{2}\right) + 2 \int_0^{(x-y)/2} \partial_s \rho(r, x - r) dr, \\ \partial_{yy} h(x, y) &= \rho\left(\frac{x-y}{2}, \frac{x+y}{2}\right) - 2 \int_0^{(x-y)/2} \partial_s \rho(r, y + r) dr. \end{aligned} \quad (3.73)$$

If $\rho(0, s) > 0$ for s lying in some interval $I \subset \mathbb{R}$ and if ρ is continuous, then $\partial_{xx} h(x, y)$ and $\partial_{yy} h(x, y)$ are positive if $x - y$ is small enough and $x, y \in I$. As a consequence,

loops generated by oscillations between input values u_1 and u_2 with a sufficiently small amplitude $u_2 - u_1$ become convex, that is, the two parts of the loop going from u_1 to u_2 and back enclose a convex set in the (u, w) -plane. Moreover, the loop is traversed counterclockwise as time increases.

The same is true if $\rho(0, s) < 0$ in I , except that the loop is now traversed clockwise.

For large amplitudes, however, in many cases the loops are not convex; this always happens e.g. if the saturation values (3.63) are finite.

Hysteresis loops in the Prandtl-Ishlinskiĭ model. Let \mathcal{W} be a Prandtl-Ishlinskiĭ operator written in the form (3.60), that is,

$$Q(\varphi) = q_0\varphi(0) + \int_0^\infty q(r)\varphi(r) dr ,$$

with $q_0 \in \mathbb{R}$ and $q \in L^1(\mathbb{R}_+)$. Setting $g(r, s) = q(r)s$ in (3.68) and (3.71) we see that $w(t) = Q(\psi(t))$ satisfies

$$\begin{aligned} w(t) &= w(t_1) + h(u(t), u(t_1)) , & t \in [t_1, t_2] , \\ w(t) &= w(t_2) - h(u(t_2), u(t)) , & t \in [t_2, t_3] , \end{aligned} \quad (3.74)$$

as above, with

$$h(x, y) = q_0(x - y) + \int_0^{(x-y)/2} (x - y - 2r)q(r) dr . \quad (3.75)$$

Since the value of h depends only on the difference $x - y$, all loops generated by oscillations between u_1 and u_2 with fixed amplitude $u_2 - u_1$ are identical except for an additive constant given by the value of w at the beginning of the oscillation. Thus, in addition to the vertical congruency property, the model of Prandtl and Ishlinskiĭ has the so-called **horizontal congruency property**.

As the model is specified (besides by the scalar q_0) by q which is a function of one variable only, it is not surprising that the shape of the hysteresis loops is linked to the shape of the primary curve. We assume for simplicity that $\psi_a = 0$. Then for $x \geq 0$, $\varphi = \psi^f(x; 0)$ satisfies $\varphi(r) = x - r$ if $r \leq x$ and $\varphi = 0$ on $[x, \infty)$. Therefore, the primary curve ℓ_0 becomes

$$\ell_0(x) = Q(\psi^f(x; 0)) = q_0x + \int_0^x (x - r)q(r) dr , \quad x \geq 0 . \quad (3.76)$$

Comparing this with (3.75) we see that for $x \geq y$

$$h(x, y) = 2\ell_0\left(\frac{x - y}{2}\right) . \quad (3.77)$$

Using (3.75) we obtain the first partial derivatives of h ,

$$\partial_x h(x, y) = q_0 + \int_0^{(x-y)/2} q(r) dr = -\partial_y h(x, y) . \quad (3.78)$$

As $\partial_x h(x, x) = -\partial_y h(x, x) = q_0$, the presence of the scalar q_0 allows for a nonzero slope at the beginning of a loop. Moreover, the slopes along the loops are nonnegative if the middle expression in (3.78) is nonnegative for all $x \geq y$. This is e.g. the case if $q_0 \geq \|q\|_1$, which holds for the original version (3.59) of the model if $p \geq 0$.

The second partial derivatives $\partial_{xx}h$ and $\partial_{yy}h$ become

$$\partial_{xx}h(x, y) = \frac{1}{2}q\left(\frac{x-y}{2}\right) = \partial_{yy}h(x, y) \quad (3.79)$$

If $q \geq 0$ then all loops are convex and are traversed counterclockwise; if $q \leq 0$ they are convex, too, but are traversed clockwise. The latter occurs in the original version (3.59) of the model if $p \geq 0$ since then $q = -p \leq 0$.

Regularity properties of operators of Preisach type. We consider

$$\mathcal{W}[u; \psi_a](t) = Q(\psi(t)), \quad \psi = \mathcal{P}[u; \psi_a], \quad t \in [a, b]. \quad (3.80)$$

We recall that the memory operator \mathcal{P} is Lipschitz continuous and that

$$|\psi(t, r) - \psi(s, r)| \leq \max_{s \leq \tau \leq t} |u(\tau) - u(s)| \quad (3.81)$$

holds for all $s, t \in [a, b]$ with $s \leq t$ and all $r \geq 0$, see Proposition 3.2.

Proposition 3.16 *Let $Q : \Psi \rightarrow \mathbb{R}$ be Lipschitz continuous with Lipschitz constant L_Q .*

(i) *The operator $\mathcal{W} : C[a, b] \times \Psi \rightarrow C[a, b]$ is Lipschitz continuous. The functions $w = \mathcal{W}[u; \psi_a]$ and $\tilde{w} = \mathcal{W}[\tilde{u}; \tilde{\psi}_a]$ satisfy*

$$|\tilde{w}(t) - w(t)| \leq L_Q \max\left\{\max_{a \leq \tau \leq t} |\tilde{u}(\tau) - u(\tau)|, \|\tilde{\psi}_a - \psi_a\|_\infty\right\} \quad (3.82)$$

$$|w(t) - w(s)| \leq L_Q \max_{s \leq \tau \leq t} |u(\tau) - u(s)|, \quad (3.83)$$

for any $\tilde{u}, u \in C[a, b]$, any $\tilde{\psi}_a, \psi_a \in \Psi$ and any $s, t \in [a, b]$ with $s \leq t$.

(ii) *Let $u \in W^{1,1}(a, b)$, $\psi_a \in \Psi$. Then $w = \mathcal{W}[u; \psi_a]$ and, for all $r \geq 0$, the functions $t \mapsto \psi(t, r)$ are absolutely continuous and satisfy*

$$\begin{aligned} |\partial_t \psi(t, r)| &\leq |\dot{u}(t)| \\ |\dot{w}(t)| &\leq L_Q |\dot{u}(t)| \end{aligned} \quad \text{for a.a. } t \in (a, b). \quad (3.84)$$

Proof. (i) For $\tilde{u}, u \in C[a, b]$, $\tilde{\psi}_a, \psi_a \in \Psi$ and $t \in [a, b]$ we have

$$\begin{aligned} |\mathcal{W}[\tilde{u}; \tilde{\psi}_a](t) - \mathcal{W}[u; \psi_a](t)| &\leq L_Q \|\mathcal{P}[\tilde{u}; \tilde{\psi}_a](t) - \mathcal{P}[u; \psi_a](t)\|_\infty \\ &\leq L_Q \max\left\{\sup_{a \leq \tau \leq t} |\tilde{u}(\tau) - u(\tau)|, \|\tilde{\psi}_a - \psi_a\|_\infty\right\} \end{aligned}$$

by Proposition 3.2. In the same way, (3.83) follows from (3.10).

(ii) The space $W^{1,1}(a, b)$ coincides with the space of absolutely continuous functions on $[a, b]$. Let u be absolutely continuous, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\sum_{i \in I} |t_i - s_i| < \delta \quad \Rightarrow \quad \sum_{i \in I} |u(t_i) - u(s_i)| < \frac{\varepsilon}{L_Q} \quad (3.85)$$

for all finite collections $\{[s_i, t_i]\}_{i \in I}$ of disjoint subintervals of $[a, b]$. By (3.81), for any such collection and any $i \in I$

$$\sup_{r \geq 0} |\psi(t_i, r) - \psi(s_i, r)| \leq |u(t_i) - u(s_i)|$$

holds for some $\tau_i \in [s_i, t_i]$. Using (3.85) it follows that

$$\sum_{i \in I} |w(t_i) - w(s_i)| \leq \sum_{i \in I} L_Q |u(\tau_i) - u(s_i)| \leq \varepsilon.$$

Thus w and $t \mapsto \psi(t, r)$ are absolutely continuous. Dividing both sides of (3.83) and (3.81) by $|t - s|$ and passing to the limit $s \rightarrow t$ we obtain (3.84). \square

Corollary 3.17 *Let $q \in L^1(\mathbb{R}_+)$, $q_0 \in \mathbb{R}$. Then for the Prandtl-Ishlinskiĭ operator $\mathcal{W} : C[a, b] \times \Psi \rightarrow C[a, b]$ given by*

$$Q(\varphi) = q_0 \varphi(0) + \int_0^\infty q(r) \varphi(r) dr \quad (3.86)$$

the assertions of Proposition 3.16 hold with $L_Q = |q_0| + \|q\|_1$. \square

We consider the Preisach operator as an operator of Preisach type with

$$Q(\varphi) = \int_0^\infty g(r, \varphi(r)) dr \quad (3.87)$$

where

$$g(r, s) = 2 \int_0^s \rho(r, \sigma) d\sigma + g_0(r), \quad (3.88)$$

with the Preisach density ρ .

Proposition 3.18 *Let $g_0 \in L^1(\mathbb{R}_+)$, let $\rho : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable, let*

$$L_Q = 2 \int_0^\infty \sup_{s \in \mathbb{R}} |\rho(r, s)| dr < \infty. \quad (3.89)$$

(i) The Preisach operator $\mathcal{W} : C[a, b] \times \Psi \rightarrow C[a, b]$ given by (3.87) and (3.88) is well-defined, and the assertions of Proposition 3.16 hold.

(ii) Let moreover ρ be bounded. Then

$$\dot{w}(t) = 2 \int_0^\infty \rho(r, \psi(t, r)) \partial_t \psi(t, r) dr. \quad (3.90)$$

Assumption (3.89) holds e.g. if ρ is bounded and has support in $[0, R] \times \mathbb{R}$ for some $R > 0$.

Proof. (i) The mapping $Q : \Psi \rightarrow \mathbb{R}$ is well-defined and Lipschitz continuous with Lipschitz constant L_Q because we have for $\varphi, \tilde{\varphi} \in \Psi$

$$|Q(\varphi)| \leq \int_0^\infty |g(r, \varphi(r))| dr \leq \int_0^\infty 2|\varphi(r)| \sup_{s \in \mathbb{R}} |\rho(r, s)| + |g_0(r)| dr < \infty$$

as well as

$$\begin{aligned} |Q(\tilde{\varphi}) - Q(\varphi)| &\leq \int_0^\infty |g(r, \tilde{\varphi}(r)) - g(r, \varphi(r))| dr \leq \int_0^\infty 2|\tilde{\varphi}(r) - \varphi(r)| \sup_{s \in \mathbb{R}} |\rho(r, s)| dr \\ &\leq L_Q \|\tilde{\varphi} - \varphi\|_\infty. \end{aligned}$$

(ii) We define $y : [a, b] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$y(t, r) = g(r, \psi(t, r)). \quad (3.91)$$

For fixed $r \geq 0$, the mapping $t \mapsto \psi(t, r)$ is absolutely continuous by Proposition 3.16(ii), and the mapping $s \mapsto g(r, s)$ is Lipschitz continuous since $\partial_s g = 2\rho$ is bounded. Therefore, the mapping $t \mapsto y(t, r)$ is absolutely continuous, and we can apply the chain rule to obtain

$$\partial_t y(t, r) = 2\rho(r, \psi(t, r))\partial_t \psi(t, r). \quad (3.92)$$

By (3.84),

$$|\partial_t y(t, r)| \leq 2|\dot{u}(t)| \sup_{s \in \mathbb{R}} |\rho(r, s)| \quad \text{a.e. in } (a, b).$$

Using (3.89) it follows that $\partial_t y \in L^1((a, b) \times \mathbb{R}_+)$ and that for all $t \in [a, b]$

$$\begin{aligned} w(t) - w(a) &= \int_0^\infty g(r, \psi(t, r)) - g(r, \psi(a, r)) dr = \int_0^\infty \int_a^t \partial_t y(\tau, r) d\tau dr \\ &= \int_a^t \int_0^\infty \partial_t y(\tau, r) dr d\tau. \end{aligned} \quad (3.93)$$

□

In (3.90), \dot{w} is expressed in terms of the partial time derivatives $\partial_t \psi(t, r)$ which coincide with the time derivatives of the outputs of the play operators \mathcal{P}_r according to (3.13).

Corollary 3.19 *Let \mathcal{W} be the Prandtl-Ishlinskii operator from Corollary 3.17, let $u \in W^{1,1}(a, b)$. Then $w = \mathcal{W}[u; \psi_a]$ satisfies*

$$\dot{w}(t) = q_0 \dot{u}(t) + \int_0^\infty q(r) \partial_t \psi(t, r) dr. \quad (3.94)$$

Proof. In the proof of part (ii) of the proposition above we replace the definition (3.91) of y by $y(t, r) = q(r)\psi(t, r)$. Then $\partial_t y(t, r) = q(r)\partial_t \psi(t, r)$ and $|\partial_t y(t, r)| \leq |q(r)||\dot{u}(t)|$ a.e., so again $\partial_t y \in L^1((a, b) \times \mathbb{R}_+)$ and the computation (3.93) applies. □

An initial value problem for an ODE with hysteresis. We look for a solution $y : [a, b] \rightarrow \mathbb{R}^n$ of the problem

$$\begin{aligned} \dot{y} &= f(t, y, w), \quad y(a) = y_a, \\ w &= \mathcal{W}[u; \psi_a], \quad u = h^T y. \end{aligned} \quad (3.95)$$

Here, $y_a, h \in \mathbb{R}^n$, $\psi_a \in \Psi$, $f : [a, b] \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and \mathcal{W} is a hysteresis operator of Preisach type with a Lipschitz continuous mapping Q .

An example is given by the second order equation including a forcing function α

$$\ddot{x} + f(x, w) = \alpha(t), \quad w = \mathcal{W}[x; \psi_a], \quad x(a) = x_a.$$

To treat the problem (3.95) one replaces it as usual by the integral equation

$$y(t) = y_a + \int_a^t f(s, y(s), \mathcal{W}[h^T y; \psi_a](s)) ds. \quad (3.96)$$

In order to obtain unique solvability of (3.96) we only use that \mathcal{W} is causal and Lipschitz continuous as expressed by (3.82) in Proposition 3.16. (The specific structure of the memory and the fact that \mathcal{W} is rate independent does not matter for this result.)

As we want to concentrate on the part played by \mathcal{W} , we use assumptions on f which keep the exposition simple. Specifically, let f be measurable in t and Lipschitz continuous in (y, w) , that is, there exists $L_f > 0$ such that

$$|f(t, \tilde{y}, \tilde{w}) - f(t, y, w)| \leq L_f(|\tilde{y} - y| + |\tilde{w} - w|), \quad \text{for all } \tilde{y}, y \in \mathbb{R}^n, \tilde{w}, w \in \mathbb{R}, \quad (3.97)$$

and let

$$|f(t, 0, 0)| \leq c_0(t), \quad \text{for a.a. } t \in (a, b) \quad (3.98)$$

with $c_0 \in L^1(a, b)$. It follows from (3.82) in Proposition 3.16 that for $u \in C[a, b]$ and $t \in [a, b]$

$$|\mathcal{W}[u; \psi_a](t)| \leq c_1 + L_Q \left(\sup_{a \leq \tau \leq t} |u(\tau)| + \|\psi_a\|_\infty \right), \quad c_1 = |\mathcal{W}[0; 0]|, \quad (3.99)$$

Together with (3.97) and (3.98) we obtain that for $y, \tilde{y} \in C[a, b]$

$$|f(t, y(t), \mathcal{W}[h^T y; \psi_a](t))| \leq c_2(t) + c_3 \sup_{a \leq \tau \leq t} |y(\tau)|, \quad \text{for all } t \in [a, b], \quad (3.100)$$

with $c_2 \in L^1(a, b)$ and $c_3 > 0$ independent from y , as well as, again using (3.82),

$$\begin{aligned} & |f(t, \tilde{y}(t), \mathcal{W}[h^T \tilde{y}; \psi_a](t)) - f(t, y(t), \mathcal{W}[h^T y; \psi_a](t))| \\ & \leq L_f |\tilde{y}(t) - y(t)| + L_f L_Q |h| \sup_{a \leq \tau \leq t} |\tilde{y}(\tau) - y(\tau)|. \end{aligned} \quad (3.101)$$

As a consequence, the operator T defined by the right side of the integral equation (3.96) is a contraction on $C[a, a + \varepsilon]$ if $\varepsilon > 0$ is sufficiently small, so (3.96) has a unique local solution. Integrating (3.101) shows that if a solution y exists on an interval $[a, t]$, it satisfies

$$\sup_{a \leq \tau \leq t} |\tilde{y}(\tau) - y(\tau)| \leq |y_a| + \|c_2\|_1 + c_3 \int_a^t \sup_{a \leq \tau \leq s} |\tilde{y}(\tau) - y(\tau)| ds.$$

Gronwall's lemma implies that

$$\sup_{a \leq \tau \leq t} |\tilde{y}(\tau) - y(\tau)| \leq (|y_a| + \|c_2\|_1) e^{c_2 t}. \quad (3.102)$$

The standard argument now yields that the solution exists on the whole interval $[a, b]$ and is unique there.

Some inequalities with monotone structure. Let us return for the moment to the play and stop operator. Given $u, \tilde{u} \in W^{1,1}(a, b)$ and $z_a, \tilde{z}_a \in \mathbb{R}$, the functions

$$w = \mathcal{P}_r[u; z_a], \quad \tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a], \quad z = \mathcal{S}_r[u; z_a], \quad \tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a] \quad (3.103)$$

satisfy

$$(\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{z}(t) - z(t)) \geq 0, \quad \text{a.e. in } (a, b), \quad (3.104)$$

see Lemma 2.6 resp. its extension to $W^{1,1}(a, b)$. Since $u = w + z$ and $\tilde{u} = \tilde{w} + \tilde{z}$ we immediately obtain the inequality

$$(\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{u}(t) - u(t)) \geq (\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{w}(t) - w(t)) = \frac{1}{2} \frac{d}{dt} (\tilde{w}(t) - w(t))^2 \quad (3.105)$$

a.e. in (a, b) . Integrating over $[a, b]$ yields

$$\int_a^b (\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{u}(t) - u(t)) dt \geq \frac{1}{2} (\tilde{w}(b) - w(b))^2 - \frac{1}{2} (\tilde{w}(a) - w(a))^2. \quad (3.106)$$

We note that the right side of (3.106) is nonnegative if $\tilde{w}(a) = w(a)$.

Inequality (3.105) can be extended to the Prandtl-Ishlinskii operator defined by

$$Q(\varphi) = q_0 \varphi(0) + \int_0^\infty q(r) \varphi(r) dr. \quad (3.107)$$

We first discuss the case where $q \geq 0$, that is, when the hysteresis loops are traversed counterclockwise.

Proposition 3.20 *Let $q \in L^1(\mathbb{R}_+)$ and $q_0 \in \mathbb{R}$ with $q \geq 0$ and $q_0 \geq 0$. Let $w = \mathcal{W}[u; \psi_a]$ and $\tilde{w} = \mathcal{W}[\tilde{u}; \tilde{\psi}_a]$, with $u, \tilde{u} \in W^{1,1}(a, b)$ and $\psi_a, \tilde{\psi}_a \in \Psi$, where \mathcal{W} is given by (3.107). Then we have*

$$(\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{u}(t) - u(t)) \geq \frac{d}{dt} \frac{1}{2} \left(q_0 (\tilde{u}(t) - u(t))^2 + \int_0^\infty q(r) (\tilde{\psi}(t, r) - \psi(t, r))^2 dr \right). \quad (3.108)$$

Proof. With $\psi = \mathcal{P}[u; \psi_a]$ and $\tilde{\psi} = \mathcal{P}[\tilde{u}; \tilde{\psi}_a]$ we have by Corollary 3.19

$$\dot{\tilde{w}}(t) - \dot{w}(t) = q_0 (\dot{\tilde{u}}(t) - \dot{u}(t)) + \int_0^\infty q(r) (\partial_t \tilde{\psi}(t, r) - \partial_t \psi(t, r)) dr. \quad (3.109)$$

Using (3.105) it follows that, since $q \geq 0$,

$$\begin{aligned} & \int_0^\infty q(r) (\partial_t \tilde{\psi}(t, r) - \partial_t \psi(t, r)) dr \cdot (\tilde{u}(t) - u(t)) \\ &= \int_0^\infty q(r) (\partial_t \tilde{\psi}(t, r) - \partial_t \psi(t, r)) (\tilde{u}(t) - u(t)) dr \\ &\geq \int_0^\infty q(r) \frac{d}{dt} \frac{1}{2} (\tilde{\psi}(t, r) - \psi(t, r))^2 dr \end{aligned}$$

Interchanging the time derivative with the integral as in the proof of Corollary 3.19 and using (3.109) we obtain (3.108). \square

The integral form of (3.108) becomes

$$\int_a^b (\dot{\tilde{w}}(t) - \dot{w}(t))(\tilde{u}(t) - u(t)) dt \geq \frac{1}{2} \left[q_0 (\tilde{u}(t) - u(t))^2 + \int_0^\infty q(r) (\tilde{\psi}(t, r) - \psi(t, r))^2 dr \right]_{t=a}^{t=b}. \quad (3.110)$$

The right side is nonnegative if $\tilde{u}(a) = u(a)$ and $\tilde{\psi}_a = \psi_a$.

In the case $q \leq 0$ the hysteresis loops are traversed clockwise. Here we need a different variant of inequality (3.105) for the play and stop operator (3.103). Using (3.104) we get

$$(\dot{\tilde{u}}(t) - \dot{u}(t))(\tilde{z}(t) - z(t)) \geq (\dot{\tilde{z}}(t) - \dot{z}(t))(\tilde{z}(t) - z(t)) = \frac{1}{2} \frac{d}{dt} (\tilde{z}(t) - z(t))^2. \quad (3.111)$$

It follows that

$$(\dot{\tilde{u}}(t) - \dot{u}(t))(\tilde{w}(t) - w(t)) \leq \frac{1}{2} \frac{d}{dt} (\tilde{u}(t) - u(t))^2 - \frac{1}{2} \frac{d}{dt} (\tilde{z}(t) - z(t))^2. \quad (3.112)$$

On the basis of (3.112) instead of (3.105) we obtain the inequality corresponding to (3.108) for the following variant of the Prandtl-Ishlinskii operator (which includes the classical one). Its proof proceeds along the same lines as that of (3.108) in the proposition above. We omit the details.

Proposition 3.21 *Let $q \in L^1(\mathbb{R}_+)$ and $q_0 \in \mathbb{R}$ with $q \leq 0$ and $q_0 \geq \|q\|_1$. Let $w = \mathcal{W}[u; \psi_a]$ and $\tilde{w} = \mathcal{W}[\tilde{u}; \tilde{\psi}_a]$, with $u, \tilde{u} \in W^{1,1}(a, b)$ and $\psi_a, \tilde{\psi}_a \in \Psi$, where \mathcal{W} is given by (3.107). Then we have*

$$\begin{aligned} & (\dot{\tilde{u}}(t) - \dot{u}(t))(\tilde{w}(t) - w(t)) \\ & \geq \frac{d}{dt} \frac{1}{2} \left((q_0 - \|q\|_1)(\tilde{u}(t) - u(t))^2 + \int_0^\infty q(r)((\tilde{u}(t) - \tilde{\psi}(t, r)) - (u(t) - \psi(t, r)))^2 dr \right). \end{aligned} \quad (3.113)$$

As in (3.110) we can pass to the integral form of (3.113).

For the Preisach model, the situation is somewhat more complicated since it is not a linear superposition of the family of play operators. We first consider an operator \mathcal{W} of Preisach type with

$$Q(\varphi) = \gamma(\varphi(r)), \quad r \geq 0 \text{ fixed}, \quad (3.114)$$

where $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a given function. Thus, \mathcal{W} is the composition of a superposition operator and the play operator \mathcal{P}_r .

The following inequality is due to M. Hilpert [7].

Proposition 3.22 *Let $p = \mathcal{W}[u; \psi_a]$ and $\tilde{p} = \mathcal{W}[\tilde{u}; \tilde{\psi}_a]$, with $u, \tilde{u} \in W^{1,1}(a, b)$ and $\psi_a, \tilde{\psi}_a \in \Psi$, where \mathcal{W} is given by (3.114). We assume that $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and locally Lipschitz continuous. Then*

$$(\dot{\tilde{p}}(t) - \dot{p}(t)) \text{sign}(\tilde{u}(t) - u(t)) \geq \frac{d}{dt} |\tilde{p}(t) - p(t)| \quad \text{a.e. in } (a, b). \quad (3.115)$$

Proof. On the subset $\{t : \tilde{p}(t) = p(t)\}$ of (a, b) , both sides of 3.115 are zero almost everywhere. In the following, all statements are meant to hold for almost all $t \in (a, b)$ with $\tilde{p}(t) \neq p(t)$. Let

$$w = \mathcal{P}_r[u; z_a], \quad \tilde{w} = \mathcal{P}_r[\tilde{u}; \tilde{z}_a], \quad z = \mathcal{S}_r[u; z_a], \quad \tilde{z} = \mathcal{S}_r[\tilde{u}; \tilde{z}_a]$$

(The initial values are chosen such that $w = \psi(\cdot, r)$ and $\tilde{w} = \tilde{\psi}(\cdot, r)$.) We recall from Chapter 2 that $\dot{w}(t)(z(t) - \tilde{z}(t)) \geq 0$ and $\dot{\tilde{w}}(t)(\tilde{z}(t) - z(t)) \geq 0$. Since γ is nondecreasing and locally Lipschitz continuous, we have $\dot{p}(t) = \gamma'(w(t))\dot{w}(t)$ and $\dot{\tilde{p}}(t) = \gamma'(\tilde{w}(t))\dot{\tilde{w}}(t)$ as well as

$$\text{sign}(\dot{p}(t)) = \text{sign}(\dot{w}(t)), \quad \text{sign}(\dot{\tilde{p}}(t)) = \text{sign}(\dot{\tilde{w}}(t)),$$

if $\dot{p}(t) \neq 0$ resp. $\dot{\tilde{p}}(t) \neq 0$. It follows that

$$0 \leq (\dot{\tilde{p}}(t) - \dot{p}(t))(\tilde{z}(t) - z(t)) = (\dot{\tilde{p}}(t) - \dot{p}(t))((\tilde{u}(t) - u(t)) - (\tilde{w}(t) - w(t))).$$

Since the Heaviside function $H = \chi_{\mathbb{R}_+}$ is nondecreasing,

$$0 \leq (\dot{\tilde{p}}(t) - \dot{p}(t))(H(\tilde{u}(t) - u(t)) - H(\tilde{w}(t) - w(t))). \quad (3.116)$$

Interchanging the roles of \tilde{p} and p we get

$$0 \leq (\dot{\tilde{p}}(t) - \dot{p}(t))(H(w(t) - \tilde{w}(t)) - H(u(t) - \tilde{u}(t))). \quad (3.117)$$

As γ is nondecreasing, $\text{sign}(\tilde{p}(t) - p(t)) = \text{sign}(\tilde{w}(t) - w(t))$. Therefore we get, using (3.116) and (3.117) as well as the identity $\text{sign}(x) = H(x) - H(-x)$,

$$\begin{aligned} \frac{d}{dt}|\tilde{p}(t) - p(t)| &= (\dot{\tilde{p}}(t) - \dot{p}(t)) \text{sign}(\tilde{p}(t) - p(t)) = (\dot{\tilde{p}}(t) - \dot{p}(t)) \text{sign}(\tilde{w}(t) - w(t)) \\ &= (\dot{\tilde{p}}(t) - \dot{p}(t))(H(\tilde{w}(t) - w(t)) - H(w(t) - \tilde{w}(t))) \\ &\leq (\dot{\tilde{p}}(t) - \dot{p}(t))(H(\tilde{u}(t) - u(t)) - H(u(t) - \tilde{u}(t))) \\ &= (\dot{\tilde{p}}(t) - \dot{p}(t)) \text{sign}(\tilde{u}(t) - u(t)). \end{aligned}$$

□

The Preisach operator given by

$$Q(\varphi) = \int_0^\infty g(r, \varphi(r)) dr, \quad g(r, s) = 2 \int_0^s \rho(r, \sigma) d\sigma + g_0(r), \quad (3.118)$$

now becomes a linear superposition of operators considered in Proposition 3.22.

Proposition 3.23 *Let $w = \mathcal{W}[u; \psi_a]$ and $\tilde{w} = \mathcal{W}[\tilde{u}; \tilde{\psi}_a]$, with $u, \tilde{u} \in W^{1,1}(a, b)$ and $\psi_a, \tilde{\psi}_a \in \Psi$, where \mathcal{W} is given by (3.118). Let the assumptions of Proposition 3.18(ii) hold. Then*

$$(\dot{\tilde{w}}(t) - \dot{w}(t)) \text{sign}(\tilde{u}(t) - u(t)) \geq \frac{d}{dt} \int_0^\infty |g(r, \tilde{\psi}(t, r)) - g(r, \psi(t, r))| dr \quad (3.119)$$

holds for a.a. $t \in (a, b)$.

Proof. Setting

$$p_r(t) = g(r, \psi(t, r)), \quad \tilde{p}_r(t) = g(r, \tilde{\psi}(t, r)),$$

we apply Proposition 3.22 with $p = p_r$. Using Proposition 3.18(ii) we see that

$$\begin{aligned} (\dot{\tilde{w}}(t) - \dot{w}(t)) \text{sign}(\tilde{u}(t) - u(t)) &= \int_0^\infty (\dot{\tilde{p}}_r(t) - \dot{p}_r(t)) \text{sign}(\tilde{u}(t) - u(t)) dr \\ &\geq \int_0^\infty \frac{d}{dt} |\tilde{p}_r(t) - p_r(t)| dr = \frac{d}{dt} \int_0^\infty |g(r, \tilde{\psi}(t, r)) - g(r, \psi(t, r))| dr. \end{aligned}$$

□

Again, we may pass to the integral form of (3.119),

$$\int_a^b (\dot{w}(t) - \dot{w}(t)) \operatorname{sign}(\tilde{u}(t) - u(t)) dt \geq \left[\int_0^\infty |g(r, \tilde{\psi}(t, r)) - g(r, \psi(t, r))| dr \right]_{t=a}^{t=b}, \quad (3.120)$$

the right side being nonnegative if $\tilde{\psi}_a = \psi_a$ and $\tilde{u}(a) = u(a)$.

A connection to energy and dissipation. The model proposed by Prandtl and Ishlinskii arose from studying the behaviour of mechanical specimens subject to unidirectional tension and compression. At the outset, the input and output functions of this model correspond to the elongation of and the force acting on the specimen, depending on time.

In the field theory of continuum mechanics, the main variables are the displacement vector, the strain tensor and the stress tensor, all of them depending on space and time. They are linked by a system of equations and possibly inequalities which arise from the basic principles of mechanics. This system in particular includes a “constitutive” part (usually called constitutive law) which reflects the specific material of the solid body under consideration. In plasticity, the basic models for the constitutive law relate the stress tensor $\sigma(x, \cdot)$ and strain tensor $\varepsilon(x, \cdot)$ as functions of time at any given space point x .

Under the assumption that the deformations are small, the 1D version of the Prandtl-Reuss law for what is called “perfect plasticity” assumes the additive decomposition of the strain into an elastic and a plastic part

$$\varepsilon = \varepsilon^e + \varepsilon^p, \quad (3.121)$$

and Hooke’s law with the elasticity modulus $E > 0$

$$\sigma = E\varepsilon^e \quad (3.122)$$

for the elastic strain. The behaviour of the plastic strain is described by the variational inequality

$$\begin{aligned} \dot{\varepsilon}^p(t)(\sigma(t) - v) &\geq 0 \quad \text{for all } |v| \leq \sigma_Y, \\ |\sigma(t)| &\leq \sigma_Y, \end{aligned} \quad (3.123)$$

with a given value of the yield stress $\sigma_Y > 0$. The variables $\sigma, \varepsilon, \varepsilon^e$ and ε^p are scalar functions of time. The elastic energy (or strain energy)

$$U = \frac{E}{2}(\varepsilon^e)^2 \quad (3.124)$$

satisfies $\dot{U} = E\varepsilon^e\dot{\varepsilon}^e = \sigma\dot{\varepsilon}^e$. The mechanical power supplied to the system is given by $\sigma\dot{\varepsilon}$. The balance law for energy becomes

$$\sigma\dot{\varepsilon} = \dot{U} + \sigma\dot{\varepsilon}^p. \quad (3.125)$$

The rightmost term $\sigma\dot{\varepsilon}^p$ stands for the rate at which energy is dissipated (that is, changed into a different form, e.g. thermal energy). Accordingly, the variational inequality (3.123) is also called principle of maximal dissipation.

By (3.123),

$$\sigma \dot{\varepsilon} - \dot{U} = \sigma \dot{\varepsilon}^p \geq 0. \quad (3.126)$$

One may view (3.126) as a specific instance of the Clausius-Duhem inequality which is one way of formulating the second principle of thermodynamics.

The system (3.121) – (3.123) can immediately be written in form of a hysteresis operator. Setting $r = \sigma_Y$, $u = E\varepsilon$, $z = \sigma = E\varepsilon^e$ and $w = E\varepsilon^p$ we see that

$$\sigma = \mathcal{S}_r[E\varepsilon; \sigma_a], \quad \varepsilon^p = \frac{1}{E} \mathcal{P}_r[E\varepsilon; \sigma_a] \quad (3.127)$$

for some initial value σ_a . These equations say that the time-dependent functions σ and ε^p can be obtained from ε by applying the stop and the play operator.

We can also formulate the energy balance (3.126) in terms of hysteresis operators. As $u = w + z$, we have

$$\dot{u}z - \dot{z}z = \dot{w}z = r|\dot{w}|. \quad (3.128)$$

We define the hysteresis operators

$$\mathcal{U}[u; z_a] = \frac{1}{2} \mathcal{S}_r[u; z_a]^2, \quad \mathcal{D}[u; z_a] = r \mathcal{P}_r[u; z_a]. \quad (3.129)$$

Then (3.128) becomes (we omit the initial value)

$$\dot{u} \mathcal{S}_r[u] - \frac{d}{dt} \mathcal{U}[u] = \left| \frac{d}{dt} \mathcal{D}[u] \right| \geq 0. \quad (3.130)$$

For the play operator, we obtain from

$$\dot{w}u - \dot{w}w = \dot{w}z = r|\dot{w}| \quad (3.131)$$

in the same manner, setting

$$\mathcal{U}[u; z_a] = \frac{1}{2} \mathcal{P}_r[u; z_a]^2, \quad \mathcal{D}[u; z_a] = r \mathcal{P}_r[u; z_a], \quad (3.132)$$

that

$$u \frac{d}{dt} \mathcal{P}_r[u] - \frac{d}{dt} \mathcal{U}[u] = \left| \frac{d}{dt} \mathcal{D}[u] \right| \geq 0. \quad (3.133)$$

This inequality also holds for the Preisach operator in place of the play operator, provided we define \mathcal{U} and \mathcal{D} in a suitable manner. Let \mathcal{W} be the Preisach operator given as above by

$$Q(\varphi) = \int_0^\infty g(r, \varphi(r)) dr, \quad g(r, s) = 2 \int_0^s \rho(r, \sigma) d\sigma + g_0(r), \quad \rho \geq 0. \quad (3.134)$$

We define the operators \mathcal{U} and \mathcal{D} of Preisach type (they are Preisach operators, too) by

$$\begin{aligned} Q_U(\varphi) &= \int_0^\infty f(r, \varphi(r)) dr, \quad f(r, s) = 2 \int_0^s \sigma \rho(r, \sigma) d\sigma, \\ Q_D(\varphi) &= \int_0^\infty r g(r, \varphi(r)) dr. \end{aligned} \quad (3.135)$$

Let $\psi = \mathcal{P}[u; \psi_a]$ with $u \in W^{1,1}(a, b)$ and $\psi_a \in \Psi$. Since $\psi(t, r) = \mathcal{P}_r[u; z_a](t)$ with a suitable initial value z_a , (3.131) becomes

$$(u(t) - \psi(t, r))\partial_t \psi(t, r) = r|\partial_t \psi(t, r)|. \quad (3.136)$$

We compute, using the formula for the time derivative of Preisach operators from Proposition 3.18(ii),

$$\begin{aligned} u(t) \frac{d}{dt} \mathcal{W}[u; \psi_a](t) - \frac{d}{dt} \mathcal{U}[u; \psi_a](t) &= u(t) \int_0^\infty 2\rho(r, \psi(t, r)) \partial_t \psi(t, r) dr - \int_0^\infty 2\psi(t, r) \rho(r, \psi(t, r)) \partial_t \psi(t, r) dr \\ &= \int_0^\infty 2\rho(r, \psi(t, r)) \partial_t \psi(t, r) (u(t) - \psi(t, r)) dr \\ &= \int_0^\infty 2\rho(r, \psi(t, r)) r |\partial_t \psi(t, r)| dr. \end{aligned}$$

As $\rho \geq 0$ and since the sign of $\partial_t \psi(t, r)$ does not depend on r , we have

$$\left| \frac{d}{dt} \mathcal{D}[u; \psi_a](t) \right| = 2 \int_0^\infty r \rho(r, \psi(t, r)) |\partial_t \psi(t, r)| dr.$$

Thus we obtain for a.a. $t \in [a, b]$

$$u(t) \frac{d}{dt} \mathcal{W}[u; \psi_a](t) - \frac{d}{dt} \mathcal{U}[u; \psi_a](t) = \left| \frac{d}{dt} \mathcal{D}[u; \psi_a](t) \right| \geq 0. \quad (3.137)$$

4 The Vector Stop and Play Operator

In the first three chapters, we considered hysteresis operators for scalar-valued input functions. Their elementary definition was based on min-max operations performed on piecewise monotone functions. Alternatively, variational inequalities were used when discussing input functions of bounded variation, in particular absolutely continuous functions.

For vector-valued input functions, the situation is different. Here, variational inequalities appear to be the basic tool; they arise either explicitly or in the context of projections onto convex sets.

The variational inequality for the vector stop. Let H be a separable Hilbert space, let Z be a closed convex subset of Z . (In Chapter 2, we treated the special case $H = \mathbb{R}$, $Z = [-r, r]$.) By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we denote the scalar product and the norm in H , respectively. Given $u : [a, b] \rightarrow H$ and $z_a \in H$, we look for $z : [a, b] \rightarrow Z$ such that

$$\begin{aligned} \langle \dot{z}(t) - \dot{u}(t), \zeta - z(t) \rangle &\geq 0 \quad \forall \zeta \in Z, \text{ a.e. in } [a, b], \\ z(t) &\in Z \quad \forall t \in [a, b], \quad z(a) = \pi_Z(z_a). \end{aligned} \quad (4.1)$$

Here, $\pi_Z : H \rightarrow Z$ denotes the projection onto Z . The system (4.1) makes sense if $u \in W^{1,1}(a, b; H)$, that is, the weak derivative \dot{u} exists and is an element of $L^1(a, b; H)$, the space of H -valued Bochner integrable functions. These functions have the property that

$$u(t) - u(s) = \int_s^t \dot{u}(\tau) d\tau$$

holds for all $s, t \in [a, b]$.

We assume for the moment that for any given input $u \in W^{1,1}(0, T; H)$ and initial value $z_a \in Z$ there exists a unique solution $z \in W^{1,1}(0, T; H)$ of (4.1). (This will be proved below.) The solution operator $(u, z_a) \mapsto z$ is called the **vector stop operator**. We write

$$z = \mathcal{S}_Z[u; z_a]. \quad (4.2)$$

The corresponding **vector play operator**

$$w = \mathcal{P}_Z[u; z_a] \quad (4.3)$$

should satisfy

$$u(t) = w(t) + z(t), \quad t \in [a, b]. \quad (4.4)$$

We achieve this by simply setting

$$\mathcal{P}_Z[u; z_a] = u - \mathcal{S}_Z[u; z_a].$$

The system

$$\begin{aligned} u(t) &= w(t) + z(t), \quad t \in [a, b], \\ \langle \dot{w}(t), z(t) - \zeta \rangle &\geq 0 \quad \forall \zeta \in Z, \quad \text{for a.a. } t \in (a, b), \\ z(t) &\in Z \quad \forall t \in [a, b], \quad z(a) = \pi_Z(z_a), \end{aligned} \quad (4.5)$$

is obviously equivalent to (4.1).

The geometric meaning of the variational inequality. A subset K of H is called a **cone** if $\lambda v \in K$ for all $v \in K$ and all $\lambda > 0$. Its **polar cone** K^0 is defined by

$$K^0 = \{q : q \in H, \langle q, v \rangle \leq 0 \text{ for all } v \in K\} = \bigcap_{v \in K} \{q \in H : \langle q, v \rangle \leq 0\}. \quad (4.6)$$

Thus K^0 is a closed convex cone in H . For $A \subset H$ we denote by

$$\text{cone}(A) = \{\lambda x : x \in A, \lambda > 0\} \quad (4.7)$$

the cone generated by A .

Let K be a closed convex cone. As K is convex and closed, for $x \in H$ we have $p = \pi_K(x)$ if and only if

$$p \in K, \quad \langle x - p, v - p \rangle \leq 0 \quad \text{for all } v \in K. \quad (4.8)$$

Since K is a closed cone, we may test with $v = 0$ and $v = 2p$ and replace (4.8) by

$$p \in K, \quad \langle x - p, p \rangle = 0, \quad \langle x - p, v \rangle \leq 0 \quad \text{for all } v \in K. \quad (4.9)$$

Proposition 4.1 *Let K be a closed convex cone in a Hilbert space H , let $x \in H$. Then $p = \pi_K(x)$ and $q = \pi_{K^0}(x)$ are the unique elements in K resp. K^0 which satisfy*

$$x = p + q, \quad \langle p, q \rangle = 0. \quad (4.10)$$

Proof. Uniqueness: If $p \in K$ and $q \in K^0$ satisfy (4.10) then

$$\begin{aligned} \langle x - p, v - p \rangle &= \langle q, v - p \rangle = \langle q, v \rangle \leq 0 \quad \text{for all } v \in K, & \text{so } p &= \pi_K(x), \\ \langle x - q, v - q \rangle &= \langle p, v - q \rangle = \langle p, v \rangle \leq 0 \quad \text{for all } v \in K^0, & \text{so } q &= \pi_{K^0}(x). \end{aligned}$$

Now let $p = \pi_K(x)$ and set $q = x - \pi_K(x)$. By (4.9),

$$\langle q, p \rangle = \langle x - p, p \rangle = 0, \quad \langle q, v \rangle = \langle x - p, v \rangle \leq 0 \quad \text{for all } v \in K,$$

so $q \in K^0$ and $\langle p, q \rangle = 0$. Moreover, since $p \in K$,

$$\langle x - q, v - q \rangle = \langle p, v - q \rangle = \langle p, v \rangle \leq 0 \quad \text{for all } v \in K^0,$$

so $q = \pi_{K^0}(x)$. □

For $x \in Z$ we define the **tangent cone** and the **normal cone** to Z at x by

$$\begin{aligned} T_Z(x) &= \overline{\text{cone}(Z - x)} = \overline{\{\lambda(\zeta - x) : \lambda > 0, \zeta \in Z\}} \\ N_Z(x) &= \{q \in H : \langle q, \zeta - x \rangle \leq 0 \text{ for all } \zeta \in Z\} = T_Z(x)^0. \end{aligned} \quad (4.11)$$

The sets $T_Z(x)$ and $N_Z(x)$ are closed convex cones.

Now we assume that $w, z \in W^{1,1}(a, b; H)$ are solutions of (4.5). We then have

$$\dot{w}(t) \in N_Z(z(t)), \quad \dot{z}(t) = \lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} \in T_Z(z(t)) \quad \text{a.e. in } (a, b). \quad (4.12)$$

We apply Proposition 4.1 to the decomposition

$$\dot{u}(t) = \dot{w}(t) + \dot{z}(t), \quad \text{a.e. in } (a, b), \quad (4.13)$$

with $K = T_Z(z(t))$ and $K^0 = N_Z(z(t))$. Since by (4.5)

$$\langle \dot{w}(t), z(t+h) - z(t) \rangle \leq 0$$

for all small enough $h \neq 0$, we obtain that a.e. in (a, b)

$$\begin{aligned} \langle \dot{w}(t), \dot{z}(t) \rangle &= 0. \\ \dot{z}(t) &= \pi_K(\dot{u}(t)), \quad \dot{w}(t) = \pi_{K^0}(\dot{u}(t)). \end{aligned} \quad (4.14)$$

As a further consequence of (4.13) and (4.14), $|\dot{w}|^2 = \langle \dot{u}, \dot{w} \rangle$ as well as $|\dot{z}|^2 = \langle \dot{u}, \dot{z} \rangle$, so

$$|\dot{w}(t)| \leq |\dot{u}(t)|, \quad |\dot{z}(t)| \leq |\dot{u}(t)|, \quad \text{a.e. in } (a, b). \quad (4.15)$$

Uniqueness and basic stability estimate. We obtain this result as usual: we estimate the difference of two solutions against the difference of the data (e.g. the right side) by using each solution as a test function for the inequality satisfied by the other solution.

Proposition 4.2 *Let $z, \tilde{z} \in W^{1,1}(a, b; H)$ be solutions of (4.5) for given functions $u, \tilde{u} \in W^{1,1}(a, b; H)$ and initial values $z_a, \tilde{z}_a \in H$. Then*

$$|z(t) - \tilde{z}(t)| \leq |z_a - \tilde{z}_a| + \int_a^t |\dot{u}(\tau) - \dot{\tilde{u}}(\tau)| d\tau \quad (4.16)$$

for all $t \in [a, b]$. In particular, the solution of (4.5) is unique.

Taking the maximum over t in (4.16) we obtain

$$\|z - \tilde{z}\|_\infty \leq |z_a - \tilde{z}_a| + \text{var}(\dot{u} - \dot{\tilde{u}}). \quad (4.17)$$

Again we see that the maximum norm and the variation norm appears, not the L^2 -norm, as is typical for rate independent evolutions.

Proof: For almost all t we have

$$\begin{aligned} |z(t) - \tilde{z}(t)| \frac{d}{dt} |z(t) - \tilde{z}(t)| &= \frac{d}{dt} \frac{1}{2} |z(t) - \tilde{z}(t)|^2 = \langle \dot{z}(t) - \dot{\tilde{z}}(t), z(t) - \tilde{z}(t) \rangle \\ &= \langle \dot{u}(t) - \dot{\tilde{u}}(t), z(t) - \tilde{z}(t) \rangle - \langle \dot{w}(t) - \dot{\tilde{w}}(t), z(t) - \tilde{z}(t) \rangle \\ &\leq \langle \dot{u}(t) - \dot{\tilde{u}}(t), z(t) - \tilde{z}(t) \rangle \leq |\dot{u}(t) - \dot{\tilde{u}}(t)| |z(t) - \tilde{z}(t)| \end{aligned}$$

due to the variational inequality. Therefore

$$\frac{d}{dt} |z(t) - \tilde{z}(t)| \leq |\dot{u}(t) - \dot{\tilde{u}}(t)|$$

for almost all t . Integrating over $[a, t]$ yields the assertion since $|z(a) - \tilde{z}(a)| \leq |z_a - \tilde{z}_a|$.

□

The discrete vector stop and play. Let a finite or infinite sequence $u^d = (u_0, \dots)$ in H and an initial value $z_a \in Z$ be given. We want to construct sequences $w^d = (w_0, \dots)$ and $z^d = (z_0, \dots)$ in H that define the discrete play and stop operator

$$w^d = \mathcal{P}_Z^d[u^d; z_0], \quad z^d = \mathcal{S}_Z^d[u^d; z_0]. \quad (4.18)$$

The sequences w^d and z^d are constructed iteratively. We begin with

$$z_0 = \pi_Z(z_a), \quad w_0 = u_0 - z_0. \quad (4.19)$$

We introduce the notation

$$\Delta_k u = u_k - u_{k-1}, \quad \Delta_k w = w_k - w_{k-1}, \quad \Delta_k z = z_k - z_{k-1}$$

for the increments; they replace the time derivatives which occur in the continuous setting. (As the process is rate independent, the size of the time step is irrelevant, and there is no reason to introduce time increments.) The discrete analogue of the variational inequality (4.5) is

$$\begin{aligned} u_k &= w_k + z_k \\ \langle \Delta_k w, z_k - \zeta \rangle &\geq 0, \quad \text{for all } \zeta \in Z \\ z_k &\in Z \end{aligned} \quad (4.20)$$

for $k \geq 1$, with the initialization at $k = 0$ given by (4.19).

There are two ways to reformulate (4.20) in terms of the projection π_Z , based on its characterization that for every $x \in H$, $\pi_Z(x)$ is the unique solution in Z of

$$\langle x - \pi_Z(x), \pi_Z(x) - \zeta \rangle \geq 0, \quad \text{for all } \zeta \in Z.$$

Since $\Delta_k w = z_{k-1} + \Delta_k u - z_k$, setting $x = z_{k-1} + \Delta_k u$ just above we see that the system

$$\begin{aligned} z_k &= \pi_Z(z_{k-1} + \Delta_k u) \\ w_k &= u_k - z_k \end{aligned} \quad (4.21)$$

for $k \geq 1$ is equivalent to (4.20). Moreover, since $z_{k-1} + \Delta_k u = u_k - w_{k-1}$, the system

$$\begin{aligned} w_k &= u_k - \pi_Z(u_k - w_{k-1}) \\ z_k &= u_k - w_k \end{aligned} \quad (4.22)$$

for $k \geq 1$ is equivalent to (4.20), too. Both systems (4.21) and (4.22) describe the same iteration step $(u_k, u_{k-1}, w_{k-1}, z_{k-1}) \mapsto (w_k, z_k)$ which yields the unique solution (w^d, z^d) of the discrete system (4.20). Thus, the discrete play and stop operator (4.18) is well-defined by these equivalent systems plus the initialization (4.19).

Bounds for the discrete solutions. Setting $\zeta = z_{k-1}$ in (4.20) we see that for $k \geq 1$

$$\langle \Delta_k w, \Delta_k z \rangle \geq 0. \quad (4.23)$$

Since $\Delta_k u = \Delta_k w + \Delta_k z$, (4.23) implies that $|\Delta_k w|^2 \leq \langle \Delta_k w, \Delta_k u \rangle$. Therefore,

$$\begin{aligned} |\Delta_k w| &\leq |\Delta_k u| \quad \text{for all } k \geq 1, \\ \text{var}(w^d) &= \sum_{k>0} |\Delta_k w| \leq \sum_{k>0} |\Delta_k u| = \text{var}(u^d). \end{aligned} \quad (4.24)$$

Analogously,

$$|\Delta_k z| \leq |\Delta_k u| \quad \text{for all } k \geq 1, \quad \text{var}(z^d) \leq \text{var}(u^d). \quad (4.25)$$

Thus, the variation of u^d bounds the variation of w^d and of z^d .

Concerning the maximum norm, we offer the following remarks. If Z is bounded, say $|\zeta| \leq C$ for all $\zeta \in Z$, then $\|z^d\|_\infty \leq C$ and $\|w^d\|_\infty \leq \|u^d\|_\infty + C$. However, the situation is different if Z is unbounded. Even in $H = \mathbb{R}^2$ one can construct an unbounded closed convex set $Z \subset H$ and a bounded sequence u^d such that w^d is unbounded. (One can do this along the lines of Example 4.1 in [11].) Thus – in contrast to the scalar situation – in that case $\|w^d\|_\infty$ (and, consequently, $\|z^d\|_\infty$) cannot be estimated in terms of $\|u^d\|_\infty$.

Translation. With respect to translation in H , the play and the stop behave in the following way. The projection satisfies

$$\pi_{Z+v}(x+v) = \pi_Z(x) + v, \quad x, v \in H. \quad (4.26)$$

Using this identity for the system (4.21), one checks that

$$\mathcal{S}_{Z+v}^d[u^d + v; z_0 + v] = \mathcal{S}_Z^d[u^d; z_0] + v, \quad \mathcal{P}_{Z+v}^d[u^d + v; z_0 + v] = \mathcal{P}_Z^d[u^d; z_0]. \quad (4.27)$$

Stability estimates for the discrete solution. The basic stability estimates (4.16) and (4.17) for the stop have the following discrete counterpart.

Proposition 4.3 *Let $u_k, \tilde{u}_k \in H$ for $k \geq 0$ and $z_a, \tilde{z}_a \in Z$ be given, set $u^d = (u_0, \dots)$ and $\tilde{u}^d = (\tilde{u}_0, \dots)$. Then $z^d = \mathcal{S}_Z^d[u^d; z_a]$ and $\tilde{z}^d = \mathcal{S}_Z^d[\tilde{u}^d; z_a]$ satisfy*

$$|z_k - \tilde{z}_k| - |z_{k-1} - \tilde{z}_{k-1}| \leq |\Delta_k(u - \tilde{u})|, \quad \text{for all } k > 0, \quad (4.28)$$

and

$$\|z^d - \tilde{z}^d\|_\infty \leq \text{var}(u^d - \tilde{u}^d) + |z_a - \tilde{z}_a|. \quad (4.29)$$

Proof. We have for $k \geq 1$

$$\begin{aligned} |z_k - \tilde{z}_k| (|z_k - \tilde{z}_k| - |z_{k-1} - \tilde{z}_{k-1}|) &\leq |z_k - \tilde{z}_k|^2 - \langle z_k - \tilde{z}_k, z_{k-1} - \tilde{z}_{k-1} \rangle \\ &= \langle \Delta_k z - \Delta_k \tilde{z}, z_k - \tilde{z}_k \rangle \stackrel{(4.20)}{\leq} \langle \Delta_k u - \Delta_k \tilde{u}, z_k - \tilde{z}_k \rangle \\ &\leq |\Delta_k u - \Delta_k \tilde{u}| \cdot |z_k - \tilde{z}_k|. \end{aligned}$$

This proves (4.28). Summing over k yields (4.29). \square

The maximum norm estimate for the scalar play (Proposition 2.4 resp. Lemma 2.14) does not carry over to the vector play. Nevertheless, there is an estimate (Proposition 4.5 below) which serves as an important intermediate step for further results. For this, we first need a lemma concerning a certain property of the scalar product in Hilbert space.

Lemma 4.4 *Let H be a Hilbert space, let $x = (x_0, \dots, x_N)$ with $x_j \in H$ for all j . Then*

$$\frac{1}{2}|x_N|^2 - \frac{1}{2}|x_0|^2 \leq \sum_{j=1}^N \langle x_j - x_{j-1}, x_j \rangle. \quad (4.30)$$

Proof. We have for all $j \geq 1$

$$0 \leq |x_j - x_{j-1}|^2 = \langle x_j - x_{j-1}, x_j \rangle - \langle x_j - x_{j-1}, x_{j-1} \rangle .$$

Therefore,

$$\begin{aligned} \frac{1}{2}|x_N|^2 - \frac{1}{2}|x_0|^2 &= \frac{1}{2} \sum_{j=1}^N (|x_j|^2 - |x_{j-1}|^2) = \frac{1}{2} \sum_{j=1}^N \langle x_j - x_{j-1}, x_j + x_{j-1} \rangle \\ &\leq \sum_{j=1}^N \langle x_j - x_{j-1}, x_j \rangle . \end{aligned}$$

□

The discrete play satisfies the following maximum norm estimate. The essential argument of its proof goes back to [22], Proposition 2g and [10], Theorem 3.1.

Proposition 4.5 *Let $u_k, \tilde{u}_k \in H$ for $k \geq 0$ and $z_a, \tilde{z}_a \in Z$ be given, set $u^d = (u_0, \dots)$ and $\tilde{u}^d = (\tilde{u}_0, \dots)$. Then $w^d = \mathcal{P}_Z^d[u^d; z_a]$ and $\tilde{w}^d = \mathcal{P}_Z^d[\tilde{u}^d; \tilde{z}_a]$ satisfy*

$$\frac{1}{2} \|w^d - \tilde{w}^d\|_\infty^2 \leq \|u^d - \tilde{u}^d\|_\infty (\text{var}(w^d) + \text{var}(\tilde{w}^d)) + \frac{1}{2} |w_0 - \tilde{w}_0|^2 . \quad (4.31)$$

Proof. Let $k > 0$. Applying Lemma 4.4 with $N = k$ and $x_j = w_j - \tilde{w}_j$ we get

$$\frac{1}{2} |w_k - \tilde{w}_k|^2 \leq \frac{1}{2} |w_0 - \tilde{w}_0|^2 + \sum_{j=1}^k \langle \Delta_j w - \Delta_j \tilde{w}, w_j - \tilde{w}_j \rangle . \quad (4.32)$$

By the variational inequality (4.20),

$$\begin{aligned} \langle \Delta_j w, w_j - \tilde{w}_j \rangle &= \langle \Delta_j w, u_j - \tilde{u}_j \rangle - \langle \Delta_j w, z_j - \tilde{z}_j \rangle \leq \langle \Delta_j w, u_j - \tilde{u}_j \rangle \\ &\leq |\Delta_j w| \cdot |u_j - \tilde{u}_j| . \end{aligned}$$

Analogously,

$$\langle \Delta_j \tilde{w}, \tilde{w}_j - w_j \rangle \leq |\Delta_j \tilde{w}| \cdot |\tilde{u}_j - u_j| .$$

Inserting these inequalities into (4.32) we obtain

$$\begin{aligned} \frac{1}{2} |w_k - \tilde{w}_k|^2 &\leq \frac{1}{2} |w_0 - \tilde{w}_0|^2 + \sum_{j=1}^k |u_j - \tilde{u}_j| (|\Delta_j w| + |\Delta_j \tilde{w}|) \\ &\leq \frac{1}{2} |w_0 - \tilde{w}_0|^2 + \|u^d - \tilde{u}^d\|_\infty (\text{var}(w^d) + \text{var}(\tilde{w}^d)) . \end{aligned}$$

As k was arbitrary, (4.31) follows. □

Corollary 4.6 *In the situation of Proposition 4.5 we have*

$$\frac{1}{2} \|w^d - \tilde{w}^d\|_\infty^2 \leq \|u^d - \tilde{u}^d\|_\infty (\text{var}(u^d) + \text{var}(\tilde{u}^d)) + \frac{1}{2} |w_0 - \tilde{w}_0|^2 . \quad (4.33)$$

Proof. This follows immediately from (4.31), since $\text{var}(w^d) \leq \text{var}(u^d)$ and $\text{var}(\tilde{w}^d) \leq \text{var}(\tilde{u}^d)$, see (4.24). \square

In the scalar case we have seen in Proposition 2.12 that $w = \mathcal{P}_r[u; z_a]$ has bounded variation even if u is only continuous. A corresponding result can be obtained in the vector case if Z has nonempty topological interior and if the oscillation of u is small enough. This result goes back to [9].

Proposition 4.7 *We assume that Z contains a ball of radius $\rho > 0$. Let $u_k \in H$ for $k \geq 0$ and $z_a \in H$ be given, set $u^d = (u_0, \dots)$. If*

$$\rho > \alpha := \sup_{k \geq 0} |u_k - u_0| \quad (4.34)$$

then $w^d = \mathcal{P}_r[u^d; z_a]$ and $z^d = \mathcal{S}_r[u^d; z_a]$ satisfy

$$\text{var}(w^d) \leq \frac{1}{2(\rho - \alpha)} |z_0|^2, \quad \|z^d\|_\infty \leq |z_0| + \alpha. \quad (4.35)$$

Proof. Due to the properties of the play and stop under translation of Z given in (4.27) we may assume that Z contains a ball with radius ρ centered at 0.

We test the variational inequality in (4.20) with

$$\zeta = u_k - u_0 + \rho_0 \frac{\Delta_k w}{|\Delta_k w|}, \quad \rho_0 := \rho - \alpha.$$

This is admissible since $|\zeta| \leq |u_k - u_0| + \rho_0 \leq \rho$. From $\langle \Delta_k w, z_k - \zeta \rangle \geq 0$ we obtain

$$\rho_0 |\Delta_k w| \leq \langle \Delta_k w, z_k - u_k + u_0 \rangle = \langle \Delta_k w, u_0 - w_k \rangle.$$

We set $x = u_0 - w_k$, $y = u_0 - w_{k-1}$ and use the inequality

$$\langle y - x, x \rangle \leq -\frac{1}{2}(|x|^2 - |y|^2).$$

This yields

$$\rho_0 |\Delta_k w| \leq -\frac{1}{2}(|w_k - u_0|^2 - |w_{k-1} - u_0|^2).$$

Summing over k we get for all n

$$\rho_0 \sum_{k=1}^n |\Delta_k w| + \frac{1}{2} |w_n - u_0|^2 \leq \frac{1}{2} |w_0 - u_0|^2 = \frac{1}{2} |z_0|^2. \quad (4.36)$$

In particular, $|w_n - u_0| \leq |z_0|$, so

$$|z_n| \leq |z_n + (w_n - u_0)| + |w_n - u_0| \leq |u_n - u_0| + |z_0| \leq \alpha + |z_0|. \quad (4.37)$$

Taking the supremum over n in (4.36) and (4.37), the assertion follows. \square

The vector stop and play operator in continuous time: strategy. As in the discrete case, these operators are obtained as solution operators of a variational inequality. For absolutely continuous input functions u , this variational inequality has already been

stated in a pointwise form in (4.5). For less regular input functions a formulation as a variational integral is adequate. In any case, an integral formulation appears, either as an intermediate step or as the final result. The proofs mainly consist of a limit passage, based on the properties of the discrete version discussed above. For input functions u which are continuous and of bounded variation, this program has been carried out in [11] using linear interpolation of discrete values of u as approximations, the variational integral being a Riemann-Stieltjes integral. For discontinuous input functions, approximation by piecewise linear functions does not work; alternatively one may use piecewise constant approximations of u . Then Stieltjes-type integrals $\int f dg$ arise where f and g may have common discontinuity points. This is outside the scope of the Riemann-Stieltjes integral. Instead, the Young integral has been used in [12] and [5], and the Kurzweil-Stieltjes integral has been used in [13]. In the following we present the approach based on the Kurzweil-Stieltjes integral.

Regulated functions. Let S be a set. The vector space of all bounded functions $f : S \rightarrow H$, equipped with the sup norm

$$\|f\|_\infty = \sup_{t \in S} |f(t)| \quad (4.38)$$

is a Banach space, denoted by $\mathcal{B}(S; H)$.

For $A \subset S$ we denote by χ_A the characteristic function of A which has the value 1 on A and 0 on $S \setminus A$. By

$$\text{osc}(f; A) = \sup_{s, t \in A} |f(t) - f(s)| \quad (4.39)$$

we denote the **oscillation** of f on A .

A function $f : [a, b] \rightarrow H$ is called a **finite step function** or simply a **step function** if it has the form

$$f = \sum_{i=0}^M \hat{c}_i \chi_{\{t_i\}} + \sum_{i=1}^M c_i \chi_{(t_{i-1}, t_i)} \quad (4.40)$$

for some partition $\Delta : a = t_0 < \dots < t_M = b$ with $\hat{c}_i, c_i \in H$. Thus, f is a step function if and only if f has only finitely many different values and only finitely many (or zero) discontinuity points. It is right-continuous (resp. left-continuous) on $[a, b]$ if and only if $\hat{c}_{i-1} = c_i$ (resp. $\hat{c}_i = c_i$) for all $1 \leq i \leq M$.

The space of **regulated functions** $G(a, b; H)$ is defined as the closure of the space of finite step functions in $\mathcal{B}([a, b]; H)$. Thus, $G(a, b; H)$ is a Banach space. The space $G_R(a, b; H)$ is defined as the space of all right-continuous regulated functions. It is a closed subspace of $G(a, b; H)$.

The space $BV(a, b; H)$ is defined as the space of all functions $f : [a, b] \rightarrow H$ of **bounded variation**, that is, functions which satisfy

$$\text{var}(f) = \sup_{\Delta} \sum_{i=1}^M |f(t_i) - f(t_{i-1})| < \infty. \quad (4.41)$$

Here, the supremum ranges over all partitions $\Delta : a = t_0 < \dots < t_M = b$ of $[a, b]$. The space $BV_R(a, b; H)$ is defined as the space of all right-continuous functions in $BV(a, b; H)$. If $I = [c, d]$ is a subinterval of $[a, b]$, we define

$$\text{var}(f; I) = \text{var}(f|I). \quad (4.42)$$

Proposition 4.8 *Let $f : [a, b] \rightarrow H$. The following assertions are equivalent:*

(i) $f \in G(a, b; H)$.

(ii) The right and left limits

$$f(t+) = \lim_{\tau > t, \tau \rightarrow t} f(\tau), \quad f(t-) = \lim_{\tau > t, \tau \rightarrow t} f(\tau)$$

exist for every $t < b$ resp. $t > a$. (By convention, $f(a-) = f(a)$ and $f(b+) = f(b)$.)

(iii) For every $\varepsilon > 0$ there exists a partition $\Delta : a = t_0 < \dots < t_M = b$ such that

$$\text{osc}(f; (t_{i-1}, t_i)) \leq \varepsilon, \quad \text{for all } 1 \leq i \leq M. \quad (4.43)$$

Moreover, if these assertions hold, then for every $\varepsilon > 0$ there exists a step function $f_\varepsilon : [a, b] \rightarrow H$ such that $\|f_\varepsilon - f\|_\infty \leq \varepsilon$ and $\text{var}(f_\varepsilon) \leq \text{var}(f)$. If f is right-continuous on $[a, b]$, then one can choose f_ε to be right-continuous as well.

Proof. “(i) \Rightarrow (ii)” holds, since (ii) is true for step functions and persists under uniform limits.

“(ii) \Rightarrow (iii)”: Let $\varepsilon > 0$. We set $t_0 = a$ and define successively

$$t_i = \sup\{t : t \leq b, |f(\tau) - f(t_{i-1}+)| < \frac{\varepsilon}{2} \text{ for all } \tau \in (t_{i-1}, t)\}. \quad (4.44)$$

If $t_{i-1} < b$ we have $t_i > t_{i-1}$ and $\text{osc}(f; (t_{i-1}, t_i)) \leq \varepsilon$. If $t_M = b$ for some $M \in \mathbb{N}$, (iii) holds. Otherwise, $t_i \uparrow t_*$ for some $t_* \leq b$ as $i \rightarrow \infty$. By (4.44),

$$|f(t_i) - f(t_{i-1}+)| \geq \frac{\varepsilon}{2}, \quad \text{or} \quad |f(t_i+) - f(t_{i-1}+)| \geq \frac{\varepsilon}{2}.$$

Since $t_i \uparrow t_*$, this implies that $\text{osc}(f; (t_* - \delta, t_*)) \geq \varepsilon/2$ for all $\delta > 0$. Thus $f(t_*-)$ does not exist which contradicts (ii). Thus, $t_M = b$ for some finite M .

“(iii) \Rightarrow (i)”: Let $\varepsilon > 0$. We choose Δ according to (iii) and arbitrary $\sigma_i \in (t_{i-1}, t_i)$, and set

$$f_\varepsilon = \sum_{i=0}^M f(t_i) \chi_{\{t_i\}} + \sum_{i=1}^M f(\sigma_i) \chi_{(t_{i-1}, t_i)}.$$

Then f_ε is a step function and $\|f - f_\varepsilon\|_\infty \leq \varepsilon$.

The approximation f_ε just constructed also satisfies

$$\text{var}(f_\varepsilon) = \sum_{i=1}^M |f(\sigma_i) - f(t_{i-1})| + \sum_{i=1}^M |f(t_i) - f(\sigma_i)| \leq \text{var}(f).$$

If f is right-continuous then $\tilde{f}_\varepsilon : [a, b] \rightarrow H$ defined by $\tilde{f}_\varepsilon(t) = f_\varepsilon(t+)$ has the required properties. This proves the final assertion. \square

The Kurzweil-Stieltjes integral. Let $f, g : [a, b] \rightarrow H$. The Kurzweil-Stieltjes integral of f with respect to g , if it exists, is a real number. We denote it by

$$\int_a^b \langle f(\tau), dg(\tau) \rangle, \quad \text{or shortly} \quad \int_a^b \langle f, dg \rangle. \quad (4.45)$$

For its construction we refer to [19] and [13]. Reference [19] systematically and extensively treats the case $H = \mathbb{R}$. For the general case, arguments, proofs and references can be found in [14] and [13].

The Kurzweil-Stieltjes integral (or Kurzweil integral, to be short) is well-defined in particular if f is regulated and g has bounded variation or vice versa. The Kurzweil integral is linear w.r.t. f as well as w.r.t. g . It is additive,

$$\int_a^b \langle f, dg \rangle = \int_a^s \langle f, dg \rangle + \int_s^b \langle f, dg \rangle \quad (4.46)$$

holds for $a \leq s \leq b$ with the convention $\int_a^a \langle f, dg \rangle = 0$. Besides a few other results we mainly need formulas when g is a particular step function. Here are two taken from Lemma 6.3.2 in [19].

Lemma 4.9 *Let $f : [a, b] \rightarrow H$, $v \in H$. Then*

$$\int_a^b \langle f, d(v\chi_{[a,s)}) \rangle = -\langle f(s), v \rangle, \quad a < s \leq b, \quad (4.47)$$

$$\int_a^b \langle f, d(v\chi_{(s,b]}) \rangle = \langle f(s), v \rangle, \quad a \leq s < b, \quad (4.48)$$

$$\int_a^b \langle f, d(v\chi_{\{s\}}) \rangle = \begin{cases} 0, & a < s < b, \\ -\langle f(a), v \rangle, & s = a, \\ \langle f(b), v \rangle, & s = b, \end{cases} \quad \square \quad (4.49)$$

These imply

$$\int_a^b \langle f, d(v\chi_{[r,s)}) \rangle = \langle f(r) - f(s), v \rangle, \quad a < r < s \leq b, \quad (4.50)$$

since $\chi_{[r,s)} = \chi_{[a,s)} - \chi_{[a,r)}$.

Now let g be a right-continuous step function

$$g = \sum_{k=1}^N g_{k-1} \chi_{[t_{k-1}, t_k)} + g_N \chi_{\{t_N\}}, \quad g_k \in H. \quad (4.51)$$

where $\Delta = \{t_k\}$ is a partition of $[a, b]$.

Lemma 4.10 *Let $f, g : [a, b] \rightarrow H$ with g as in (4.51), let $v \in H$. Then*

$$\int_a^b \langle f, dg \rangle = \sum_{k=1}^N \langle f(t_k), g_k - g_{k-1} \rangle. \quad (4.52)$$

Proof. Using (4.47) – (4.50) we compute

$$\begin{aligned}
\int_a^b \langle f, dg \rangle &= \sum_{k=1}^N \int_a^b \langle f, d(g_{k-1} \chi_{[t_{k-1}, t_k]}) \rangle + \int_a^b \langle f, d(g_N \chi_{\{t_N\}}) \rangle \\
&= -\langle f(t_1), g_0 \rangle + \sum_{k=2}^N \langle f(t_{k-1}) - f(t_k), g_{k-1} \rangle + \langle f(t_N), g_N \rangle \\
&= \langle f(t_1), g_1 - g_0 \rangle + \sum_{k=2}^{N-1} \langle f(t_k), g_k - g_{k-1} \rangle + \langle f(t_N), g_N - g_{N-1} \rangle.
\end{aligned}$$

□

Lemma 4.11 *Let $f, g : [a, b] \rightarrow H$, let g be constant on $[a, s]$ and on $(s, b]$ for some $s \in [a, b]$. Then*

$$\int_a^b \langle f, dg \rangle = \langle f(s), g(s+) - g(s-) \rangle. \quad (4.53)$$

(Recall that $g(a-) = g(a)$, $g(b+) = g(b)$.)

Proof. For $s > a$ we have $g = g(s-) \chi_{[a, s)} + g(s) \chi_{\{s\}}$ on $[a, s]$. By Lemma 4.10,

$$\int_a^s \langle f, dg \rangle = \langle f(s), g(s) - g(s-) \rangle.$$

Analogously, for $s < b$ we have $g = g(s) \chi_{\{s\}} + g(s+) \chi_{(s, b]}$ on $[s, b]$. By Lemma 4.10,

$$\int_s^b \langle f, dg \rangle = \langle f(s), g(s+) - g(s) \rangle.$$

For $s = a$ or $s = b$, we are done. For $a < s < b$, adding these two equations yields (4.53).
□

The next result is included in Lemma 6.3.3 of [19].

Lemma 4.12 *Let $g \in G(a, b; H)$, $v \in H$, $s \in [a, b]$. Then*

$$\int_a^b \langle v \chi_{\{s\}}, dg \rangle = \langle v, g(s+) - g(s-) \rangle. \quad (4.54)$$

(Recall that $g(a-) = g(a)$, $g(b+) = g(b)$.)

Lemma 4.13 *Let $f, \tilde{f} : [a, b] \rightarrow H$ with $\tilde{f}(t) = f(t)$ for all $t > a$, let $g \in G(a, b; H)$ be right-continuous at a . Then*

$$\int_a^b \langle \tilde{f}, dg \rangle = \int_a^b \langle f, dg \rangle. \quad (4.55)$$

Proof. This follows from Lemma 4.12, since $\tilde{f} - f = (\tilde{f}(a) - f(a)) \chi_{\{a\}}$ and $g(a+) = g(a)$.
□

The next result is given as Corollary 2.6 in [13].

Proposition 4.14 *Let $f \in G(a, b; H)$ and $g \in BV_R(a, b; H)$. Then*

$$\left| \int_a^b \langle f, dg \rangle \right| \leq \|f\|_\infty \text{var}(g). \quad (4.56)$$

The next result is a specific instance of a partial integration formula, see Corollary A.9 in [14].

Proposition 4.15 *Let $g \in BV_R(a, b; H)$. Then*

$$\int_a^b \langle g, dg \rangle = \frac{1}{2}|g(b)|^2 - \frac{1}{2}|g(a)|^2 + \sum_{t \in [a, b]} |g(t) - g(t-)|^2. \quad (4.57)$$

Finally we quote Theorem A.6 from [14], which below will enable us to pass to the limit when approximating with step functions.

Proposition 4.16 *Let $\{g_n\}$ be a sequence in $BV(a, b; H)$ with $\text{var}(g_n) \leq C$ for some $C > 0$, let $\{f_n\}$ be a sequence in $G(a, b; H)$. If $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly, then $f \in G(a, b; H)$, $g \in BV(a, b; H)$ and*

$$\lim_{n \rightarrow \infty} \int_a^b \langle f_n, dg_n \rangle = \int_a^b \langle f, dg \rangle. \quad (4.58)$$

The vector stop and play operator in continuous time. Let $u : [a, b] \rightarrow H$ be given. Let $\Delta : a = t_0 < \dots < t_N = b$ be a partition of $[a, b]$. Let $u^d = (u_0, \dots, u_N)$, let $u^c : [a, b] \rightarrow \mathbb{R}$ be the right-continuous step function associated with Δ which interpolates u^d ,

$$u^c = \sum_{k=1}^N u_{k-1} \chi_{[t_{k-1}, t_k)} + u_N \chi_{\{t_N\}}. \quad (4.59)$$

Let $z_a \in H$ be given, let $w^d = \mathcal{P}_Z^d[u^d; z_a]$ and $z^d = \mathcal{S}_Z^d[u^d; z_a]$ be the discrete play and stop. Let w^c and $z^c : [a, b] \rightarrow \mathbb{R}$ be the right-continuous step functions which interpolate w^d and z^d respectively,

$$w^c = \sum_{k=1}^N w_{k-1} \chi_{[t_{k-1}, t_k)} + w_N \chi_{\{t_N\}}, \quad z^c = \sum_{k=1}^N z_{k-1} \chi_{[t_{k-1}, t_k)} + z_N \chi_{\{t_N\}}. \quad (4.60)$$

The following lemma shows that the assignment $u^c \mapsto (w^c, z^c)$ does not depend on the choice of Δ and thus yields a well-defined mapping.

Lemma 4.17 *Let $\Delta, \hat{\Delta}$ be partitions of $[a, b]$, let u^c, \hat{u}^c be the step functions associated with $\Delta, \hat{\Delta}$ according to (4.59) for given vectors u^d, \hat{u}^d , let $\hat{u}^c = u^c$. Then $(\hat{w}^c, \hat{z}^c) = (w^c, z^c)$.*

Proof. We first consider the case where $\hat{\Delta}$ arises from Δ by adding a single point, $\hat{\Delta} = \Delta \cup \{t\}$ with $t \in (t_{j-1}, t_j)$ for some j . Then $\hat{u}^d = (u_0, \dots, u_{j-1}, u_{j-1}, u_j, \dots, u_N)$ and

$$\hat{w}^d = (w_0, \dots, w_{j-1}, w_{j-1}, w_j, \dots, w_N), \quad \hat{z}^d = (z_0, \dots, z_{j-1}, z_{j-1}, z_j, \dots, z_N).$$

It follows that $(\hat{w}^c, \hat{z}^c) = (w^c, z^c)$. Since arbitrary partitions Δ and $\hat{\Delta}$ possess a common refinement which is obtained from Δ as well as from $\hat{\Delta}$ by adding finitely many points, the proof is complete. \square

Lemma 4.17 enables us to define the vector play and stop operator for step functions u^c of the form (4.59) and $z_a \in H$, using (4.60), by

$$\mathcal{P}_Z[u^c; z_0] = w^c, \quad \mathcal{S}_Z[u^c; z_0] = z^c. \quad (4.61)$$

Due to the close correspondence of u^c and u^d , the estimate of Proposition 4.5 immediately carries over to step functions.

Lemma 4.18 *Let $u^c, \tilde{u}^c : [a, b] \rightarrow H$ be right-continuous step functions. Then*

$$\frac{1}{2} \|w^c - \tilde{w}^c\|_\infty^2 \leq \|u^c - \tilde{u}^c\|_\infty (\text{var}(w^c) + \text{var}(\tilde{w}^c)) + \frac{1}{2} |w_0 - \tilde{w}_0|^2. \quad (4.62)$$

Proof. Let Δ be a partition of $[a, b]$ such that

$$u^c = \sum_{k=1}^N u_{k-1} \chi_{[t_{k-1}, t_k)} + u_N \chi_{\{t_N\}}, \quad \tilde{u}^c = \sum_{k=1}^N \tilde{u}_{k-1} \chi_{[t_{k-1}, t_k)} + \tilde{u}_N \chi_{\{t_N\}},$$

set $u^d = (u_0, \dots, u_N)$ and $\tilde{u}^d = (\tilde{u}_0, \dots, \tilde{u}_N)$. Then

$$\|u^c - \tilde{u}^c\|_\infty = \|u^d - \tilde{u}^d\|_\infty, \quad \|w^c - \tilde{w}^c\|_\infty = \|w^d - \tilde{w}^d\|_\infty \quad (4.63)$$

$$\text{var}(u^c) = \text{var}(u^d), \quad \text{var}(w^c) = \text{var}(w^d), \quad \text{var}(\tilde{w}^c) = \text{var}(\tilde{w}^d). \quad (4.64)$$

Thus (4.62) follows from (4.31). \square

The discrete variational inequality (4.20) can be directly transformed into a variational inequality for the corresponding step functions.

Lemma 4.19 *Let $u^c : [a, b] \rightarrow H$ be a right-continuous step function, let $z_a \in H$. Then the functions $w^c = \mathcal{P}_Z[u^c; z_a]$ and $z^c = \mathcal{S}_Z[u^c; z_a]$ solve the system*

$$\begin{aligned} u^c(t) &= w^c(t) + z^c(t) \quad \forall t \in [a, b], \\ \int_a^b \langle z^c(\tau) - \zeta(\tau), dw^c(\tau) \rangle &\geq 0 \quad \forall \zeta : [a, b] \rightarrow Z, \\ z^c(t) &\in Z \quad \forall t \in [a, b], \quad z(a) = \pi_Z(z_a). \end{aligned} \quad (4.65)$$

Note that no regularity is needed for the test function ζ .

Proof. Let

$$u^c = \sum_{k=1}^N u_{k-1} \chi_{[t_{k-1}, t_k)} + u_N \chi_{\{t_N\}}, \quad u^d = (u_0, \dots, u_N),$$

let $\zeta : [a, b] \rightarrow Z$. By the construction of \mathcal{P}_Z and \mathcal{S}_Z , for $w^d = \mathcal{P}_Z^d[u^d; z_a]$ and $z^d = \mathcal{S}_Z^d[u^d; z_a]$ we have $w_k = w^c(t_k)$ and $z_k = z^c(t_k)$. It then follows from Lemma 4.10 and the discrete variational inequality (4.20) that

$$\begin{aligned} \int_a^b \langle z^c - \zeta, dw^c \rangle &= \sum_{k=1}^N \langle z^c(t_k) - \zeta(t_k), w^c(t_k) - w^c(t_{k-1}) \rangle = \sum_{k=1}^N \langle z_k - \zeta(t_k), w_k - w_{k-1} \rangle \\ &\geq 0. \end{aligned}$$

Since $z(a) = z_0 = \pi_Z(z_a)$ and $z_k \in Z$ as well as $u_k = w_k + z_k$ for all k , the proof is complete. \square

We want to replace u^c in (4.65) by more general functions u . We consider the system

$$\begin{aligned} u(t) &= w(t) + z(t) \quad \forall t \in [a, b], \\ \int_a^b \langle z(\tau) - \zeta(\tau), dw(\tau) \rangle &\geq 0 \quad \forall \zeta \in G(a, b; Z), \\ z(t) &\in Z \quad \forall t \in [a, b], \quad z(a) = \pi_Z(z_a). \end{aligned} \tag{4.66}$$

We intend to prove that, for given $u \in BV_R(a, b; H)$ and $z_a \in Z$, this system has a unique solution $(w, z) \in BV_R(a, b; H)$. This will give us solution operators

$$\mathcal{P}_Z, \mathcal{S}_Z : BV_R(a, b; H) \times H \rightarrow BV_R(a, b; H). \tag{4.67}$$

At first we show existence.

Proposition 4.20 *Let $u \in BV_R(a, b; H)$ and $z_a \in H$ be given. Then there exist functions $w, z \in BV_R(a, b; H)$ which satisfy (4.66) as well as $\text{var}(w) \leq \text{var}(u)$.*

Proof. Let $u \in BV_R(a, b; H)$. Let $\{u^n\}$ be a sequence of right-continuous step functions of the form

$$u^n = \sum_{k=1}^{N_n} u_{k-1}^n \chi_{[t_{k-1}^n, t_k^n)} + u_{N_n} \chi_{\{b\}}, \tag{4.68}$$

such that $u^n \rightarrow u$ uniformly and $\text{var}(u^n) \leq \text{var}(u)$ for all n , see Proposition 4.8. Let $w^n = \mathcal{P}_Z[u^n; z_a]$ and $z^n = \mathcal{S}_Z[u^n; z_a]$ as defined in (4.61). By Proposition 4.18,

$$\frac{1}{2} \|w^n - w^m\|_\infty^2 \leq \|u^n - u^m\|_\infty (\text{var}(w^n) + \text{var}(w^m)) + \frac{1}{2} |u^n(a) - u^m(a)|^2 \tag{4.69}$$

holds for all $m, n \in \mathbb{N}$ as $w^n(a) - w^m(a) = u^n(a) - u^m(a)$.

We claim that the variation of w^n is uniformly bounded, so that

$$\text{var}(w^n) \leq C \quad \text{for all } n \in \mathbb{N} \tag{4.70}$$

holds for some $C > 0$. This follows from the estimate

$$\text{var}(w^n) = \text{var}(w^{d,n}) \leq \text{var}(u^{d,n}) = \text{var}(u^n) \leq \text{var}(u), \tag{4.71}$$

taking into account the discrete estimate (4.24) as well as (4.64).

Since $u^n \rightarrow u$ uniformly, $\{w^n\}$ is a Cauchy sequence in $G_R(a, b; H)$ by (4.69) and (4.70). Thus, $(w^n, z^n) \rightarrow (w, z)$ uniformly for some $w, z \in G_R(a, b; H)$. Moreover, (4.70) and (4.71) imply that

$$\text{var}(w) \leq \liminf_{n \rightarrow \infty} \text{var}(w^n) \leq \text{var}(u). \quad (4.72)$$

Therefore w and $z = u - w$ belong to $BV_R(a, b; H)$. By Lemma 4.19, the step functions w^n and z^n solve system (4.66) with u^n in place of u . Passing to the limit $n \rightarrow \infty$, we obtain that (w, z) solves (4.66). For the integral inequality this follows from Proposition 4.16. \square

We now prove uniqueness. It holds even if we allow u to belong to the larger space $G_R(a, b; H)$, provided w has bounded variation.

Proposition 4.21 *Let $u \in G_R(a, b; H)$, $z_a \in H$. Then there exists at most one pair (w, z) with $w \in BV_R(a, b; H)$ and $z \in G_R(a, b; H)$ which solves (4.66). For such a pair holds*

$$\int_r^s \langle z(\tau) - \zeta(\tau), dw(\tau) \rangle \geq 0 \quad \forall \zeta \in G(r, s; Z) \quad (4.73)$$

for all $a \leq r < s \leq b$ as well as

$$\langle z(t) - \zeta, w(t) - w(t-) \rangle \geq 0 \quad \forall \zeta \in Z \quad (4.74)$$

for all $t \in [a, b]$.

Proof. We first prove that (4.73) holds for all solutions (w, z) of (4.66). Let $\zeta \in G(r, s; Z)$. As a test function in (4.66) we choose

$$\eta = (\chi_{[a, r]} + \chi_{(s, b]})z + \chi_{(r, s]}\zeta.$$

The function η is regulated and takes values in Z . We have

$$\int_a^r \langle z - \eta, dw \rangle = 0 = \int_s^b \langle z - \eta, dw \rangle,$$

the right equality holds because of Lemma 4.13, since $\eta = z$ on $(s, b]$. Thus

$$0 \leq \int_a^b \langle z - \eta, dw \rangle = \int_r^s \langle z - \eta, dw \rangle = \int_r^s \langle z - \zeta, dw \rangle,$$

again, the rightmost equality holds because of Lemma 4.13.

We now prove uniqueness. Let (\tilde{w}, \tilde{z}) be another solution of (4.66). By what we just have proved, (4.73) also holds for (\tilde{w}, \tilde{z}) in place of (w, z) . Therefore, for every $t > a$

$$\int_a^t \langle z - \tilde{z}, dw \rangle \geq 0, \quad \int_a^t \langle \tilde{z} - z, d\tilde{w} \rangle \geq 0. \quad (4.75)$$

Setting $g = w - \tilde{w}$ and $b = t$ in Proposition 4.15, we obtain, since $w + z = u = \tilde{w} + \tilde{z}$,

$$\frac{1}{2}|(w - \tilde{w})(t)|^2 \leq \int_a^t \langle w - \tilde{w}, d(w - \tilde{w}) \rangle = - \int_a^t \langle z - \tilde{z}, d(w - \tilde{w}) \rangle \leq 0.$$

As t was arbitrary, $w = \tilde{w}$ and consequently $z = \tilde{z}$.

It remains to prove (4.74). Let $\zeta \in Z$. We choose the test function

$$\eta = (\chi_{[a,t]} + \chi_{(t,b]})z + \chi_{\{t\}}\zeta.$$

Therefore,

$$0 \leq \int_a^b \langle z - \eta, dw \rangle = \int_a^t \langle z - \eta, dw \rangle + \int_t^b \langle z - \eta, dw \rangle.$$

We have $z - \eta = \chi_{\{t\}}(z(t) - \zeta)$. Thus, the second integral on the right is zero because of Lemma 4.13, and the first integral equals $\langle z(t) - \zeta, w(t) - w(t-) \rangle$ due to Lemma 4.12. \square

We come back to the original pointwise formulation of the variational system

$$\begin{aligned} u(t) &= w(t) + z(t), \quad t \in [a, b], \\ \langle \dot{w}(t), z(t) - \zeta \rangle &\geq 0 \quad \forall \zeta \in Z, \quad \text{for a.a. } t \in (a, b), \\ z(t) &\in Z \quad \forall t \in [a, b], \quad z(a) = \pi_Z(z_a). \end{aligned} \tag{4.76}$$

Proposition 4.22 *Let $u \in W^{1,1}(a, b; H)$, $z_a \in H$. Then the unique solution (w, z) of (4.66) satisfies $w, z \in W^{1,1}(a, b; H)$ and is the unique solution of (4.76).*

Proof. Let $s, t \in [a, b]$ with $s < t$. Then

$$\text{var}(w; [s, t]) \leq \text{var}(u; [s, t]) = \int_s^t |\dot{u}(\tau)| d\tau \tag{4.77}$$

holds; indeed, in the proof of Proposition 4.20 we only have to choose $\{u^n\}$ such that the points s and t belong to the partitions Δ_n of all the u^n and to apply (4.24) on $[s, t]$. Setting

$$v(t) = \int_a^t |\dot{u}(\tau)| d\tau,$$

it follows from (4.77) that for any finite disjoint collection $\{[s_i, t_i]\}_{i \in I}$ of subintervals of $[a, b]$

$$\sum_{i \in I} |w(t_i) - w(s_i)| \leq \sum_{i \in I} |v(t_i) - v(s_i)|.$$

As v is absolutely continuous, also w and $z = u - w$ are absolutely continuous.

Let now $t \in [a, b]$, $\zeta \in Z$. It follows from (4.73) that for all $s < t$

$$\begin{aligned} 0 &\leq \int_s^t \langle z - \zeta, dw \rangle = \int_s^t \langle z(\tau) - \zeta, \dot{w}(\tau) \rangle d\tau \\ &\leq \langle z(t) - \zeta, \int_s^t \dot{w}(\tau) d\tau \rangle + \max_{s \leq \tau \leq t} |z(\tau) - z(t)| \cdot \int_s^t |\dot{w}(\tau)| d\tau. \end{aligned}$$

Dividing by $t - s$ and passing to the limit $s \uparrow t$ we obtain

$$\langle \dot{w}(t), z(t) - \zeta \rangle \geq 0 \quad \forall \zeta \in Z, \quad \text{for a.a. } t \in (a, b).$$

Since every solution of (4.76) also solves (4.66), the proof is complete. \square

Sweeping processes. The sweeping process provides another way of describing a rate-independent evolution. Its definition and analysis goes back to Moreau [21, 22], see also the monograph [20]. Let H be a Hilbert space, let C_0, C_1, \dots be a finite or infinite sequence of closed convex subsets of H . Given $w_0 \in C_0$, let w_1, \dots be defined by

$$w_k = \pi_{C_k}(w_{k-1}) . \quad (4.78)$$

This iterative scheme is called the **catching-up** algorithm. (4.78) is equivalent to

$$\begin{aligned} \langle w_{k-1} - w_k, \eta - w_k \rangle &\leq 0, \quad \text{for all } \eta \in C_k, \\ w_k &\in C_k . \end{aligned} \quad (4.79)$$

This in turn is equivalent to the inclusion

$$-\Delta_k w \in N_{C_k}(w_k) , \quad (4.80)$$

$N_{C_k}(w_k)$ being the normal cone to C_k at w_k . An important special case arises when the sets C_k are translates of a fixed set, say

$$C_k = u_k - Z , \quad (4.81)$$

where $Z \subset H$ is closed and convex. Then (4.79) is equivalent to

$$\begin{aligned} \langle w_{k-1} - w_k, u_k - w_k - \zeta \rangle &\leq 0, \quad \text{for all } \zeta \in Z, \\ w_k &\in u_k - Z , \end{aligned} \quad (4.82)$$

which is nothing else than the system already considered in (4.20), namely

$$\begin{aligned} u_k &= w_k + z_k \\ \langle \Delta_k w, z_k - \zeta \rangle &\geq 0, \quad \text{for all } \zeta \in Z \\ z_k &\in Z . \end{aligned} \quad (4.83)$$

Thus, in the case of a pure translation (4.81) the catching-up algorithm coincides with the iteration which defines the discrete play operator,

$$w^d = \mathcal{P}_Z^d[u^d; z_a] \quad (4.84)$$

with the initialization $z_0 = \pi_Z(z_a)$ and $w_0 = u_0 - z_0$.

An analogous correspondence arises in continuous time when

$$C(t) = u(t) - Z . \quad (4.85)$$

For $u \in W^{1,1}(a, b; H)$ one checks as above that

$$w = \mathcal{P}_Z[u; z_a] \quad (4.86)$$

coincides with the solution of the differential inclusion

$$-\dot{w}(t) \in N_{C(t)}(w(t)) \quad (4.87)$$

with the initial value $w(0) = u(0) - \pi_Z(z_a)$.

The energetic approach. The energetic approach to rate-independent evolutions has been developed by A. Mielke and several coworkers, see the monograph [18] and the contribution [17]. It is based on two potentials which depend on time, typically via a time-dependent function like an external force; their interaction generates a solution, which is a time-dependent function. This approach also deals with rate-independent problems in a natural manner. It has created a unifying framework for many different problems arising in mechanics.

Let Q be a separable Hilbert space with dual Q^* . We consider an **energy**

$$\mathcal{E} : [a, b] \times Q \rightarrow \mathbb{R} \quad (4.88)$$

and a **dissipation potential**

$$\mathcal{R} : Q \rightarrow [0, +\infty]. \quad (4.89)$$

Here we restrict ourselves to the case where \mathcal{E} is quadratic and coercive and \mathcal{R} is convex. As was described in [17, 18] and will be explained in the following, in that case there is a close relation to the stop and play operator.

More specifically, we assume that

$$\mathcal{E}(t, q) = \frac{1}{2} \langle Aq, q \rangle - \langle u(t), q \rangle. \quad (4.90)$$

Here, $u : [a, b] \rightarrow Q^*$ is the function which drives the evolution. The operator $A : Q \rightarrow Q^*$ is linear, bounded, symmetric and positive definite; in particular, we have

$$\langle Aq, p \rangle = \langle Ap, q \rangle, \quad \alpha_0 |q|^2 \leq \langle Aq, q \rangle \leq \alpha_1 |q|^2, \quad \text{for all } p, q \in Q \quad (4.91)$$

for some numbers $\alpha_0, \alpha_1 > 0$. Here and in the remainder of this subsection, the brackets $\langle \cdot, \cdot \rangle$ denote the duality pairing on $Q^* \times Q$, and $|\cdot|$ denotes the norm on Q . With these assumptions, $A : Q \rightarrow Q^*$ becomes a Hilbert space isomorphism, and

$$|q|_A = \sqrt{\langle Aq, q \rangle} \quad (4.92)$$

defines a norm on Q which by (4.91) is equivalent to $|\cdot|$.

The dissipation potential $\mathcal{R} : Q \rightarrow [0, +\infty]$ is assumed to be lower semicontinuous, convex and positively 1-homogeneous, that is, $\mathcal{R}(\lambda q) = \lambda \mathcal{R}(q)$ for all $\lambda > 0$ and all $q \in Q$.¹ We moreover assume that $\mathcal{R}(0)$ is a finite number; then necessarily $\mathcal{R}(0) = 0$ and $0 \in \partial \mathcal{R}(0)$, so in particular, $\partial \mathcal{R}(0)$ is not empty.

From convex analysis we recall that the subdifferential of \mathcal{R} at $p \in Q$ is defined as the subset of Q^* given by

$$\partial \mathcal{R}(p) = \{ \zeta : \zeta \in Q^*, \mathcal{R}(v) - \mathcal{R}(p) \geq \langle \zeta, v - p \rangle \text{ for all } v \in Q \}. \quad (4.93)$$

Let $q : [a, b] \rightarrow Q$. We consider the **stability condition**

$$(S) \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, v) + \mathcal{R}(v - q(t)), \quad \text{for all } v \in Q, t \in [a, b], \quad (4.94)$$

¹ We remark that for convex functionals on a Hilbert space, the four notions of semicontinuity (weak/strong, sequential/topological) are equivalent. Note also that \mathcal{R} satisfies the triangle inequality, as $\mathcal{R}(p + q) = 2\mathcal{R}((p + q)/2) \leq \mathcal{R}(p) + \mathcal{R}(q)$.

and the **energy balance condition**

$$(E) \quad \mathcal{E}(t, q(t)) + \int_a^t \mathcal{R}(\dot{q}(s)) ds = \mathcal{E}(a, q(a)) + \int_a^t \partial_t \mathcal{E}(s, q(s)) ds \quad (4.95)$$

for all $t \in [a, b]$. This formulation is rather general; if we replace the second integral in (E) by

$$\sup_{\Delta} \sum_j \mathcal{R}(q(t_j) - q(t_{j-1}))$$

where the supremum is taken over all partitions $\Delta = \{t_j\}$ of $[a, t]$, the time derivative of the function q does not appear in conditions (S) and (E).

We restrict ourselves to the case $q \in W^{1,1}(a, b; Q)$. Then q is called an **energetic solution** for $(\mathcal{E}, \mathcal{R})$ if (S) and (E) are satisfied.

Differentiating (4.95) w.r.t. time, we get

$$\frac{d}{dt} \mathcal{E}(t, q(t)) + \mathcal{R}(\dot{q}(t)) = \partial_t \mathcal{E}(t, q(t)) \quad \text{a.e. in } (a, b),$$

which by the chain rule is equivalent to

$$- \langle \partial_q \mathcal{E}(t, q(t)), \dot{q}(t) \rangle = \mathcal{R}(\dot{q}(t)) \quad \text{a.e. in } (a, b). \quad (4.96)$$

Lemma 4.23 *A function $q : [a, b] \rightarrow Q$ satisfies the stability condition (S) if and only if*

$$- \partial_q \mathcal{E}(t, q(t)) \in \partial \mathcal{R}(0) \quad \text{for all } t \in [a, b]. \quad (4.97)$$

Proof. The stability condition (S) is equivalent to

$$g(q(t)) = \min_{v \in Q} g(v), \quad g(v) = \mathcal{E}(t, v) + \mathcal{R}(v - q(t)).$$

Since g is convex, this in turn is equivalent to

$$0 \in \partial g(q(t)) = \partial_q \mathcal{E}(t, q(t)) + \partial \mathcal{R}(0).$$

□

For the quadratic energy \mathcal{E} from (4.90),

$$\partial_q \mathcal{E}(t, q) = Aq - u(t).$$

In view of (4.96) and (4.97) it turns out that $q \in W^{1,1}(a, b; Q)$ is an energetic solution in the quadratic case if and only if

$$u(t) - Aq(t) \in \partial \mathcal{R}(0), \quad (4.98)$$

$$\langle u(t) - Aq(t), \dot{q}(t) \rangle = \mathcal{R}(\dot{q}(t)). \quad (4.99)$$

In order to connect this formula to the stop and play operator, we need the following representation of $\partial \mathcal{R}(0)$ and of \mathcal{R} .

Proposition 4.24 *Let $\mathcal{R} : Q \rightarrow [0, \infty]$ be convex, lower semicontinuous and positively 1-homogeneous with $\mathcal{R}(0) = 0$. Then*

$$\partial\mathcal{R}(0) = \{\zeta : \langle \zeta, v \rangle \leq \mathcal{R}(v) \text{ for all } v \in Q\}. \quad (4.100)$$

The convex conjugate \mathcal{R}^* defined on Q^* by

$$\mathcal{R}^*(\zeta) = \sup_{v \in Q} (\langle \zeta, v \rangle - \mathcal{R}(v)), \quad \zeta \in Q^*, \quad (4.101)$$

satisfies

$$\mathcal{R}^* = I_{\partial\mathcal{R}(0)}^2 \quad (4.102)$$

and we have

$$\mathcal{R}(v) = \sup_{\zeta \in \partial\mathcal{R}(0)} \langle \zeta, v \rangle, \quad \forall v \in Q. \quad (4.103)$$

Proof. Since for $\zeta \in Q^*$ and $v \in Q$ we have $\mathcal{R}(v) - \langle \zeta, v \rangle = \mathcal{R}(v) - \mathcal{R}(0) - \langle \zeta, v - 0 \rangle$, (4.100) follows. Now, setting $Z = \partial\mathcal{R}(0)$,

$$\begin{aligned} \zeta \in Z &\Leftrightarrow 0 = \sup_{v \in Q} (\langle \zeta, v \rangle - \mathcal{R}(v)) = \mathcal{R}^*(\zeta) \\ \zeta \notin Z &\Leftrightarrow \sup_{v \in Q} (\langle \zeta, v \rangle - \mathcal{R}(v)) > 0 \Leftrightarrow \sup_{v \in Q, \lambda > 0} \lambda(\langle \zeta, v \rangle - \mathcal{R}(v)) = +\infty \\ &\Leftrightarrow \mathcal{R}^*(\zeta) = +\infty. \end{aligned}$$

This proves (4.102). Since \mathcal{R} is convex, lower semicontinuous and not identically equal to $+\infty$, we have $\mathcal{R}^{**} = \mathcal{R}$ by a result of convex analysis where $\mathcal{R}^{**} : Q \rightarrow (-\infty, +\infty]$ is defined by

$$\mathcal{R}^{**}(v) = \sup_{\zeta \in Q^*} (\langle \zeta, v \rangle - \mathcal{R}^*(\zeta)).$$

As $\mathcal{R}^* = I_Z$ by (4.102), it follows that

$$\mathcal{R}(v) = \mathcal{R}^{**}(v) = \sup_{\zeta \in Q^*} (\langle \zeta, v \rangle - \mathcal{R}^*(\zeta)) = \sup_{\zeta \in Z} \langle \zeta, v \rangle$$

for all $v \in Q$. □

We define a scalar product in Q^* by

$$\langle \eta, \zeta \rangle_{A^{-1}} = \langle \zeta, A^{-1}\eta \rangle_{Q^*Q}, \quad \eta, \zeta \in Q^*. \quad (4.104)$$

Lemma 4.25 *Let $q \in W^{1,1}(a, b; Q)$. Then q solves (4.98) and (4.99) if and only if*

$$u(t) - Aq(t) \in \partial\mathcal{R}(0), \quad (4.105)$$

$$\langle A\dot{q}(t), u(t) - Aq(t) \rangle_{A^{-1}} = \sup_{\zeta \in \partial\mathcal{R}(0)} \langle A\dot{q}(t), \zeta \rangle_{A^{-1}}. \quad (4.106)$$

²The indicator function I_Z of a set Z is defined to be $I_Z(\zeta) = 0$ for $\zeta \in Z$, and $I_Z(\zeta) = +\infty$ otherwise.

Proof. We have, using (4.103) for the second equality,

$$\begin{aligned}\langle u(t) - Aq(t), \dot{q}(t) \rangle &= \langle A\dot{q}(t), u(t) - Aq(t) \rangle_{A^{-1}} , \\ \mathcal{R}(\dot{q}(t)) &= \sup_{\zeta \in \partial \mathcal{R}(0)} \langle \zeta, \dot{q}(t) \rangle = \sup_{\zeta \in \partial \mathcal{R}(0)} \langle A\dot{q}(t), \zeta \rangle_{A^{-1}} .\end{aligned}$$

□

Now we can characterize the energetic solution in the quadratic case (4.90).

Proposition 4.26 *Let $(\mathcal{E}, \mathcal{R})$ satisfy the assumptions above (4.94), let $u \in W^{1,1}(a, b; Q^*)$ be given. Then for every $q_a \in A^{-1}(u(a) - Z)$ there exists a unique energetic solution $q \in W^{1,1}(a, b; Q)$ for $(\mathcal{E}, \mathcal{R})$ with $q(a) = q_a$, namely*

$$q = A^{-1}\mathcal{P}_Z[u; z_a], \quad Z = \partial \mathcal{R}(0), \quad z_a = u(a) - Aq_a. \quad (4.107)$$

Proof. We set

$$z = \mathcal{S}_Z[u; z_a], \quad w = \mathcal{P}_Z[u; z_a], \quad q = A^{-1}w.$$

Then we have $z = u - Aq$ and $w = Aq$. From the definition of the stop and the play it follows that q satisfies (4.105) and (4.106) which is equivalent to q being an energetic solution. Moreover, $q(a) = q_a$. Conversely, if q is an energetic solution with $q(a) = q_a$, then (4.105) and (4.106) hold. It follows that $Aq = \mathcal{P}_Z[u; z_a]$. □

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