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Behavior of long-range percolation at critical phases

Johannes Bäumler

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Vorsitz:		Prof. Dr. Simone Warzel
Prüfer*innen der Dissertation:	1.	Prof. Dr. Noam Berger Steiger
	2.	Prof. Dr. Marek Biskup
	3.	Prof. Dr. Vincent Tassion

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Introduction

In this thesis, we study different phenomena for long-range percolation. The thesis is based on the following four papers, where one of them is joint work with Noam Berger.

[7] Johannes Bäumler. Distances in $\frac{1}{|x-y|^{2d}}$ percolation models for all dimensions. *arXiv preprint arXiv:2208.04800*, 2022.

[8] Johannes Bäumler. Behavior of the distance exponent for $\frac{1}{|x-y|^{2d}}$ long-range percolation. *arXiv preprint arXiv:2208.04793*, 2022.

[9] Johannes Bäumler and Noam Berger. Isoperimetric lower bounds for critical exponents for long-range percolation. *arXiv preprint arXiv:2204.12410*, 2022.

[10] Johannes Bäumler. Recurrence and transience of symmetric random walks with long-range jumps. *arXiv preprint arXiv:2209.09901*, 2022.

Large parts of the thesis are identical to the papers, where part I contains the material of [7,8], part II contains the results [9], and part III is mostly identical to [10]. Each of the three parts contains a specific introduction to its topic and is mostly self-contained. Before going to the individual parts of the paper, we outline the setup and the main results of this thesis.

In part I and II of the thesis, we will consider independent percolation only. For this, consider a graph $G = (V, E)$ with weighted edge set. Let $J : E \to [0, +\infty]$ be the weight. Let $\beta \geq 0$ be a parameter. An edge $e \in E$ is either open or closed, where the edge e is open with probability $p(\beta, e) = 1 - e^{-\beta J(e)}$, independent of all other edges. We denote the resulting measure by $\mathbb{P}_\beta$. We define $0 \cdot (+\infty) \coloneqq +\infty$, so in particular for all $\beta \geq 0$, an edge e with $J(e) = +\infty$ is open almost surely. With this definition of percolation, one can recover many models of percolation that have been studied a lot over the last decades. For example, for

$$V = \mathbb{Z}^d, E = \left\{ \{x, y\} : x, y \in \mathbb{Z}^d, \|x - y\|_2 = 1 \right\}, \text{ and } J(e) = 1 \text{ for all } e \in E,$$

this gives the model of nearest-neighbor percolation on the d-dimensional integer lattice. Here, an edge is open with probability $1 - e^{-\beta}$. So for $\beta = 0$, all edges are closed almost surely, whereas for $\beta \to \infty$, the probability that an edge is open goes to 1. Another example of percolation is the *Erdős-Rényi* random graph model. For this, define

$$V = \{1, \ldots, n\}, E = \{\{x, y\} : x, y \in V, x \neq y\}, \text{ and } J(e) = 1 \text{ for all } e \in E.$$

Again, an edge is open with probability $p = 1 - e^{-\beta}$ on this graph, so this gives us a reparametrization of the classical Erdős-Rényi model. In this thesis, we are mostly interested in the case where

$$V = \mathbb{Z}^d, E = \{\{x, y\} : x, y \in V, x \neq y\}, \text{ and } J(\{x, y\}) = \Theta\left(\|x - y\|^{-s}\right)$$

for some $s > 0$. Depending on the value of s, there are several different phases. For example for $\beta > 0$, the resulting graph will almost surely be locally finite for $s > d$, whereas each vertex has an infinite degree for $s \leq d$ almost surely. This shows that the value $s = d$ is critical for the local finiteness of the graph. In part I of this thesis we investigate the chemical distances on such random graphs. For two points $x, y \in \mathbb{Z}^d$, the chemical distance, also called graph distance or hop-count distance, is the length of the shortest open path between them; we denote it by $D(x,y)$. Also note that $D(x,y) = +\infty$ is possible, in the case where x and y are not connected by an open path. In order to circumvent this, we will always assume that $p(\beta, \{u,v\}) = 1$ for all $\beta \geq 0$ and all $u, v \in \mathbb{Z}^d$ with $\|u-v\| = 1$. Assume that $p(\beta, \{u,v\}) = \Theta\left(\|u-v\|^{-s}\right)$. It was known before that for $s \in (d, 2d)$ the graph distance $D(x,y)$ between x and y grows polylogarithmically in the Euclidean distance $\|x-y\|$ [18], whereas the graph distance $D(x,y)$ grows linearly in the Euclidean distance $\|x-y\|$ for $s > 2d$ [17]. This shows that the value $s = 2d$ is a critical value for the growth of the chemical distances. In part I of the thesis, we study the chemical distances for $s = 2d$. For this, we will assume that

$$p(\beta, \{u,v\}) = 1 \text{ for } \|u-v\| = 1 \text{ and } p(\beta, \{u,v\}) = \frac{\beta}{\|u-v\|^{2d}} + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right).$$

Let $u \in \mathbb{Z}^d$ be a point with $\|u\|_\infty = n$. We will show that both the graph distance $D(0,u)$ between the origin and u and the diameter of the box $\{0, \ldots, n\}^d$ grow like n^θ, where $\theta = \theta(d,\beta) \in (0,1]$, with $\theta(d,\beta) = 1$ if and only if $\beta = 0$. For fixed dimension d, we will also discuss how the function $\theta(\beta) = \theta(d,\beta)$ depends on β. Here, we determine the asymptotic behavior of $\theta(\beta)$ for large β, we prove that $\theta(\beta)$ is continuous and strictly decreasing in β, and we show that $\theta(\beta) = 1 - \beta + o(\beta)$ for small β in dimension $d = 1$.

Let us assume that the weight J is translation invariant, in the sense that $J(\{u,v\}) = J(\{0, v-u\})$ for all distinct $u, v \in \mathbb{Z}^d$. In this case it is clear that the distribution of the resulting random graph is also invariant under translations. We also write $J(u) = J(\{0,u\})$ for $u \in \mathbb{Z}^d$. When removing the assumption that $J(\{u,v\}) = +\infty$ for nearest-neighbor edges $\{u,v\}$, the resulting open subgraph can have infinite components or not, depending on the weight function J and the value of β. For fixed β and J, this probability will be either 0 or 1, by Kolmogorov's 0-1-law. Whenever $\sum_{u \in \mathbb{Z}^d \setminus \{0\}} J(u) < \infty$, one can show that the resulting open clusters are almost surely finite for $\beta < \left(\sum_{u \in \mathbb{Z}^d \setminus \{0\}} J(u)\right)^{-1}$. On the other hand, when $J(u) = \Theta\left(\|u\|^{-s}\right)$ for some $s \in (1,2]$ in dimension $d = 1$, respectively for some $s > d$ in dimension $d \geq 2$, it is known that for large enough β there will almost surely be an infinite open subgraph [84]. As this property is monotone in β, there exists a critical value β_c at which this change of behavior occurs. We write K_0 for the open cluster containing the origin, and we write $s = d + \alpha$. There are several *critical exponents* which describe the behavior of the random graph for $\beta = \beta_c$. We study two of them in part II of this thesis, namely the critical exponent of the clustersize δ and the two-point function exponent $2 - \eta$. These two exponents are defined by

$$\delta = \lim_{n \to \infty} \frac{-\log(n)}{\log\left(\mathbb{P}_{\beta_c}\left(|K_0| \geq n\right)\right)} \text{ and } 2 - \eta = \lim_{x \to \infty} \frac{\log\left(\mathbb{P}_{\beta_c}\left(0 \leftrightarrow x\right)\right)}{\log(\|x\|)} + d.$$

Provided these exponents exist, we show that

$$\delta \geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)} \quad \text{and} \quad 2 - \eta \geq \alpha \wedge 1.$$

The lower bound on δ is believed to be sharp for $d = 1, \alpha \in \left[\frac{1}{3}, 1\right)$ and for $d = 2, \alpha \in \left[\frac{2}{3}, 1\right]$, whereas the lower bound on $2 - \eta$ is sharp for $d = 1, \alpha \in (0,1)$, and for $\alpha \in (0,1]$ for $d > 1$, and is not believed to be sharp otherwise. Our main tool is a connection between the critical exponents and the isoperimetry of cubes inside $\mathbb{Z}^d$. The reason why $\alpha \wedge 1$ shows up in our lower bounds above is because the value $\alpha = 1$, respectively $s = d + 1$, is the critical value for the isoperimetry of cubes inside $\mathbb{Z}^d$.

In part III of this thesis, we study random walks on percolation clusters, and long-range random walks on the integer lattice. Let $X_1, X_2, \ldots$ be i.i.d. random variables with values in $\mathbb{Z}^d$ satisfying $\mathbb{P}(X_1 = x) = \mathbb{P}(X_1 = -x) = \Theta(\|x\|^{-s})$ for some $s > d$. We show that the long-range random walk defined by $S_n = \sum_{k=1}^{n} X_k$ is recurrent for $d \in \{1, 2\}$ and $s \geq 2d$, and transient otherwise. This also shows that for an electric network in dimension $d \in \{1, 2\}$ the condition $c_{\{x,y\}} \leq C\|x - y\|^{-2d}$ implies recurrence, whereas $c_{\{x,y\}} \geq c\|x-y\|^{-s}$ for some $c > 0$ and $s < 2d$ implies transience. This shows that the value $s = 2d$ is critical for the recurrence of long-range random walks. The underlying reason why the value $s = 2d$ is critical for recurrence of random walks in dimension $d \in \{1, 2\}$ is because the random variables X_i have a mean in dimension $d = 1$ if and only if $s > 2$, and they have a finite variance in dimension $d = 2$ if and only if $s > 4$. This fact about the recurrence and transience of long-range random walks was already previously known, but we give a new proof of it that uses only electric networks. When one considers independent long-range percolation on $\mathbb{Z}^d$ with $J(\{x, y\}) = \frac{1}{\|x-y\|^s}$ and $\beta > \beta_c$, the return properties of the simple random walk have been studied by Berger in [16]. He proved that the simple random walk in dimension $d \in \{1, 2\}$ is recurrent for $s \geq 2d$ and transient for $s \in (d, 2d)$. The same questions can be asked for the *weight-dependent random connection model*, which is a model for *dependent* percolation. Random walks on this model were studied recently by Gracar et al. [49]. We use the results about the long-range random walk S_n to show the recurrence of simple random walks on several new classes of two-dimensional weight-dependent random connection models. For some classes of the random connection model, our newly obtained results show the recurrence for critical cases, whereas for other classes, it is not completely clear yet, what the critical case for recurrence or transience of the simple random walk on such a random graph is.

Contents

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1 Introduction

Consider independent long-range bond percolation on $\mathbb{Z}^d$ where all edges $\{u,v\}$ with $\|u-v\|_\infty = 1$ are open and an edge $\{u,v\}$ with $\|u-v\|_\infty \geq 2$ is open with probability

$$p(\beta,\{u,v\}) \coloneqq 1 - e^{-\beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y},$$

where $\mathcal{C} \coloneqq [0,1)^d$ and $\beta \geq 0$. We call the corresponding probability measure $\mathbb{P}_\beta$ and denote its expectation by $\mathbb{E}_\beta$. The resulting graph is clearly connected and the graph distance $D(u,v)$ between two points $u,v \in \mathbb{Z}^d$ satisfies $D(u,v) \leq \|u-v\|_\infty$. We are interested in the scaling of the typical distance of two points $u,v \in \mathbb{Z}^d$ and the scaling of the diameter of boxes $\{0,\dots,N\}^d$. In [27] it is proven that the typical diameter of some box grows at most polynomially with some power strictly smaller than 1. More precisely, Coppersmith, Gamarnik, and Sviridenko proved that for all $\beta > 0$ there exists an exponent $\theta' = \theta'(\beta) < 1$ such that $\lim_{N\to\infty} \mathbb{P}_\beta \left(\text{Diam}\left(\{0,\dots,N\}^d\right) \leq N^{\theta'}\right) = 1$. However, the authors do not give any polynomial lower bound for dimensions $d \geq 2$. An analogous lower bound was already conjectured in [12, 18], and an exact lower bound was later proven to hold in one dimension: In [33] Ding and Sly showed that for the connection probability $p(\beta,\{u,v\})$ given by $p(\beta,\{u,v\}) = \frac{\beta}{|u-v|^2} \wedge 1$ for $|u-v| \geq 2$ and $p(\beta,\{u,v\}) = 1$ for $|u-v| = 1$ the typical distance between the two points $0, n \in \mathbb{N}$ and the diameter of $\{0,\dots,n\}$ both grow like n^θ for some $\theta \in (0,1]$, where $\theta = 1$ if and only if $\beta = 0$. More precisely, they proved that

$$n^\theta \approx_P D(0,n) \approx_P \text{Diam}\left(\{0,\dots,n\}\right) \approx_P \mathbb{E}\left[D(0,n)\right]$$

where the notation $A(n) \approx_P B(n)$ means that for all $\varepsilon > 0$ there exist $0 < c < C < \infty$ such that $\mathbb{P}\left(cB(n) \leq A(n) \leq CB(n)\right) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. In this thesis, we prove an analogous result for all dimensions.

1.1 Main results

Theorem 1.1. *For all dimensions d and all $\beta > 0$, there exists an exponent $\theta = \theta(d,\beta) \in (0,1)$ such that*

$$\|u\|^\theta \approx_P D(\mathbf{0},u) \approx_P \mathbb{E}_\beta\left[D(\mathbf{0},u)\right] \tag{1}$$

and

$$k^\theta \approx_P \mathit{Diam}\left(\{0,\dots,k\}^d\right) \approx_P \mathbb{E}_\beta\left[\mathit{Diam}\left(\{0,\dots,k\}^d\right)\right]. \tag{2}$$

We write $\mathbf{0}$ for the vector with all entries equal to 0 and the notation $A(u) \approx_P B(u)$ means that for all $\varepsilon > 0$ there exist $0 < c < C < \infty$ such that $\mathbb{P}_\beta\left(cB(u) \leq A(u) \leq CB(u)\right) > 1 - \varepsilon$ for all $u \in \mathbb{Z}^d$. The inclusion probability $p(\beta,\{u,v\}) \coloneqq 1 - e^{-\beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y}$ is only one possible choice for a function which asymptotically grows like $\frac{\beta}{\|u-v\|^{2d}}$. In section 7, we will show the same results for other possible choices of such functions. Examples of inclusion probabilities we consider are $\frac{\beta}{\|u-v\|^{2d}} \wedge 1$ and $1 - e^{-\frac{\beta}{\|u-v\|^{2d}}}$.

The exponent $\theta = \theta(\beta)$ defined in Theorem 1.1 arises through a subadditivity argument (see section 2.2 below) and its precise value is not known to us. However, for fixed dimension d, we determine the asymptotic behavior of the function $\theta(\beta)$ for large β.

Theorem 1.2. *For all dimensions d, there exist constants $0 < c < C < \infty$ such that*

$$\frac{c}{\log(\beta)} \leq \theta(\beta) \leq \frac{C}{\log(\beta)} \tag{3}$$

for all $\beta \geq 2$.

Furthermore, we study several other properties of the dependence of the distance exponent $\theta(\beta)$ on β. For $d = 1$, it is well-known that $\theta(\beta) \geq 1 - \beta$, see for example [27,33]. In section 9, we show that for small β this lower bound is indeed a good approximation for $\theta(\beta)$.

Theorem 1.3. *For $d = 1$, the right-hand derivative of the distance exponent $\frac{\mathrm{d}}{\mathrm{d}\beta}\theta(\beta)$ exists at $\beta = 0$ and furthermore one has $\frac{\mathrm{d}}{\mathrm{d}\beta}\theta(\beta)\Big|_{\beta=0} = -1$. This yields that $\theta(\beta) = 1 - \beta + o(\beta)$ as $\beta \to 0$.*

It is clear that the function $\theta(\beta)$ is monotonically decreasing in β, as for $\beta_1 < \beta_2$ we can couple the respective measures in such a way that the set of edges resulting from $\mathbb{P}_{\beta_1}$ is a subset of the edge-set sampled from $\mathbb{P}_{\beta_2}$. In section 10, we show that $\theta(\beta)$ is even strictly decreasing.

Theorem 1.4. *The distance exponent $\theta : \mathbb{R}_{\geq 0} \to (0, 1]$ is strictly monotonically decreasing.*

The main tool in the proof of Theorems 1.3 and 1.4 is Lemma 8.1, which can be seen as a version of Russo's formula for expectations. Finally, in section 11 we show that $\theta(\beta)$ is a continuous function.

Theorem 1.5. *The distance exponent $\theta : \mathbb{R}_{\geq 0} \to (0, 1]$ is continuous in β.*

So in particular, Theorem 1.5 together with the facts $\lim_{\beta\to 0} \theta(\beta) = \theta(0) = 1$ and $\lim_{\beta\to\infty} \theta(\beta) = 0$ show that $\theta(\beta)$ interpolates continuously between 0 and 1, as β goes from $+\infty$ to 0. The continuity of the distance exponent is also used for the comparison with different inclusion probabilities in section 7.

1.2 The continuous model

For $\beta > 0$, the described discrete percolation model has a self-similarity that comes from a coupling with the underlying continuous model, that we will now describe for any dimension. This will also explain our, at first sight complicated, choice of the connection probability. Consider a Poisson point process $\tilde{\mathcal{E}}$ on $\mathbb{R}^d \times \mathbb{R}^d$ with intensity $\frac{\beta}{2\|t-s\|_2^{2d}}$. Define the symmetrized version $\mathcal{E}$ by $\mathcal{E} := \{(t,s) \in \mathbb{R}^d \times \mathbb{R}^d : (s,t) \in \tilde{\mathcal{E}}\} \cup \tilde{\mathcal{E}}$. For $u, v \in \mathbb{Z}^d$ with $\|u - v\|_\infty \geq 1$ we put an edge between u and v if and only if $((u + \mathcal{C}) \times (v + \mathcal{C})) \cap \mathcal{E} \neq \emptyset$, where we use the notation $\mathcal{C} = [0,1)^d$. The cardinality of $((u + \mathcal{C}) \times (v + \mathcal{C})) \cap \tilde{\mathcal{E}}$ is a random variable with Poisson distribution and parameter $\int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{\beta}{2\|t-s\|^{2d}} \mathrm{d}s\mathrm{d}t$. Thus, by the properties of Poisson processes, the probability that $u \nsim v$ equals

$$\mathbb{P}\left(((u + \mathcal{C}) \times (v + \mathcal{C})) \cap \mathcal{E} = \emptyset\right) = \mathbb{P}\left(((u + \mathcal{C}) \times (v + \mathcal{C})) \cap \tilde{\mathcal{E}} = \emptyset\right)^2$$
$$= \left(e^{-\int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{\beta}{2\|t-s\|^{2d}} \mathrm{d}s\mathrm{d}t}\right)^2 = e^{-\int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s\mathrm{d}t} = 1 - p(\beta, \{u, v\})$$

which is exactly the probability that $u \nsim v$ under the measure $\mathbb{P}_\beta$. Note that for $\{u, v\}$ with $\|u - v\|_\infty = 1$ we have $\int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s\mathrm{d}t = \infty$. So we really get that all edges of the form $\{u,v\}$ with $\|u-v\|_\infty = 1$ are open. The construction with the Poisson process also implies that the presence of different bonds is independent and thus the resulting measure of the random graph constructed above equals $\mathbb{P}_\beta$. The chosen inclusion probabilities have many advantages. First of all, the resulting model is invariant under translation and invariant under the reflection of coordinates, i.e., when we change the i-th component $p_i(x)$ of all $x \in \mathbb{Z}^d$ to $-p_i(x)$. Furthermore, the model has the following self-similarity: For some vector $u = (p_1(u), \ldots, p_d(u)) \in \mathbb{Z}^d$ and $n \in \mathbb{N}_{>0}$ we define the translated boxes $V_u^n := \prod_{i=1}^d \{p_i(u)n, \ldots, (p_i(u)+1)n-1\} = nu + \prod_{i=1}^d \{0,\ldots,n-1\}$. Then for all points $u, v \in \mathbb{Z}^d$, and all $n \in \mathbb{N}_{>0}$ one has

$$\begin{aligned}
\mathbb{P}_\beta\left(V_u^n \nsim V_v^n\right) &= \prod_{x \in V_u^n} \prod_{y \in V_v^n} \mathbb{P}_\beta\left(x \nsim y\right) = \prod_{x \in V_u^n} \prod_{y \in V_v^n} e^{-\int_{x+\mathcal{C}} \int_{y+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s \mathrm{d}t} \\
&= e^{-\sum_{x \in V_u^n} \sum_{y \in V_v^n} \int_{x+\mathcal{C}} \int_{y+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s \mathrm{d}t} = e^{-\int_{nu+[0,n)^d} \int_{nv+[0,n)^d} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s \mathrm{d}t} \\
&= e^{-\int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}s \mathrm{d}t} = \mathbb{P}_\beta\left(u \nsim v\right)
\end{aligned}$$

which shows the self-similarity of the model. Also observe that for any $\alpha \in \mathbb{R}_{>0}$ the process $\alpha\tilde{\mathcal{E}} := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \left(\frac{1}{\alpha}x, \frac{1}{\alpha}y\right) \in \tilde{\mathcal{E}}\right\}$ is again a Poisson point process with intensity $\frac{\beta}{2\|x-y\|^{2d}}$.

1.3 Notation

We use the notation e_i for the i-th standard unit vector in $\mathbb{R}^d$. For a vector $y \in \mathbb{R}^d$, we write $p_i(y)$ for the i-th coordinate of y, i.e., $p_i(y) = \langle e_i, y\rangle$. We also use the notation $\mathbf{0}$ for the vector with all entries equal to 0 and the notation $\mathbf{1}$ for the vector with all entries equal to 1. We use the symbol $\mathcal{C}$ for the box $[0,1)^d$. When we write $\|u\|$ we always mean the 2-norm of the vector u. For $k \in \mathbb{N}$, we define the sets

$$\mathcal{S}_k := \left\{x \in \mathbb{Z}^d : \|x\|_\infty = k\right\} \text{ and } \mathcal{S}_{\geq k} := \left\{x \in \mathbb{Z}^d : \|x\|_\infty \geq k\right\}$$

For the closed ball of radius r around $x \in \mathbb{Z}^d$ in the ∞-norm we use the notation $B_r(x)$, i.e., $B_r(x) = \left\{y \in \mathbb{Z}^d : \|x-y\|_\infty \leq r\right\}$. For a vector $u \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we write

$$V_u^n = n \cdot u + \{0,\ldots,n-1\}^d = \prod_{i=1}^d \{np_i(u), \ldots, np_i(u)+n-1\}$$

for the box of side length n shifted by nu. When we want to emphasize that we work on certain subgraphs $A \subset \mathbb{Z}^d$, we will write $D_A(x,y)$ for the graph distance inside the set A, i.e., when we consider edges with both endpoints inside A only. Whenever we write $\mathrm{Diam}(A)$ for some set $A \subset \mathbb{Z}^d$ we always mean the inside diameter of this set, i.e., $\mathrm{Diam}(A) = \max_{x,y \in A} D_A(x,y)$. The percolation configuration is a random element $\omega \in \{0,1\}^E$, where we say that the edge e exists or is open or present if $\omega(e) = 1$. For $\omega \in \{0,1\}^E$ and $e \in E$ we define the elements ω^{e+}, ω^{e-} by

$$\omega^{e+}(\tilde{e}) = \begin{cases} 1 & \tilde{e} = e \\ \omega(e) & \tilde{e} \neq e \end{cases} \text{ and } \omega^{e-}(\tilde{e}) = \begin{cases} 0 & \tilde{e} = e \\ \omega(e) & \tilde{e} \neq e \end{cases},$$

so this are the edge sets when we insert, respectively delete, the edge e. When we look at a (possibly random) subset of the edges that is defined by some $\omega \in \{0,1\}^E$ we also write $D(u,v;\omega)$ for the graph distance between u and v in the environment represented by ω. For some edge $e=\{u,v\}$ we write $|e|=|\{u,v\}|=\|u-v\|_\infty$ for the distance in the ∞-norm between the endpoints. We use the notation $\log(x)$ for the natural logarithm, i.e., the logarithm to the base e. We define the indirect distance $D^\star(A,B)$ between the sets $A,B\subset\mathbb{Z}^d$ as the graph distance in the environment where we removed all edges between A and B, which is the distance when we only consider paths between A and B that do not use an edge $e=\{u,v\}$ with $u\in A, v\in B$.

1.4 Related work

The scaling of the graph distance, also called chemical distance or hop-count distance, is a central characteristic of a random graph and has also been examined for many different models of percolation, see for example [1, 6, 12, 17–20, 27, 31–33, 36, 45, 54, 59, 82]. For all dimensions d, one can also consider the long-range percolation model with connection probability asymptotic to $\frac{\beta}{\|u-v\|^s}$. When varying the parameter s, there are a total of 5 different regimes, with the transitions happening at $s=d$ and $s=2d$. The value of the first transition $s=d$ is very natural, as the resulting random graph is locally finite if and only if $s>d$. For $s<d$ the graph distance between two points is at most $\lceil\frac{d}{d-s}\rceil$ [14], whereas for $s=d$, the diameter of the box $\{0,\dots,n\}^d$ is of order $\frac{\log(n)}{\log(\log(n))}$ [27, 96]. In [12, 18–20] the authors proved that for $d<s<2d$ the typical distance between two points of Euclidean distance n grows like $\log(n)^\Delta$, where $\Delta^{-1}=\log_2\left(\frac{2d}{s}\right)$. The behavior of the typical distance for long-range percolation on $\mathbb{Z}^d$ also changes at $s=2d$. The reason why $s=2d$ is a critical value is that for $s=2d$ the graph is self-similar, as described in section 1.2. For $s>2d$ the graph distance grows at least linearly in the Euclidean distance of two points, as proven in [17]. In [33] it is shown that the typical distance for $d=1, s=2$ grows like n^θ for some $\theta\in(0,1)$. For $d\geq 2$ and $s=2d$ the authors in [27] proved a polynomial upper bound on the graph distance, but no lower bound. In this thesis, we show a matching polynomial lower bound for all dimensions d, similar to the results of [33] in one dimension.

Another line of research is to investigate what happens when one drops the assumption that $p(\beta,\{u,v\})=1$ for all nearest neighbor edges $\{u,v\}$, but assigns i.i.d. random variables to the nearest neighbor edges instead. For $d=1$, there is a change of behavior at $s=2$. As proven by Aizenman, Newman, and Schulman in [4, 84, 91], an infinite open cluster can not emerge for $s>2$ and for $s=2, \beta\leq 1$, no matter how small $\mathbb{P}(k\nsim k+1)$ is. See also [41] for a new proof of these results. On the other hand, an infinite cluster can emerge for $s<2$ and $s=2,\beta>1$ (see [84]). In [4] the authors proved that there is a discontinuity in the percolation density for $s=2$, contrary to the situation for $s<2$, as proven in [16, 65]. For models, for which an infinite cluster exists the behavior of the percolation model at and near criticality is also a well-studied question (cf. [11, 16, 22, 30, 65, 69, 70]). It is not known up to now how the typical distance in long-range percolation grows for $s=2d$ and $p(\beta,\{u,v\})<1$ for nearest-neighbor edges $\{u,v\}$, but we conjecture also a polynomial growth in the Euclidean distance, whenever an infinite cluster exists.

For $d=1$, the behavior of the mixing time is also a property that exhibits a transition at $s=2$, as proven in [13]. On the line segment $\{0,\dots,n\}$ the mixing time grows quadratic in n for $s>2$ and is of order n^{s-1} for $1<s<2$. The behavior of the mixing time for $s=2$ is still open, but we conjecture a similar behavior to that of the chemical distance, namely that the mixing time interpolates between n and n^2, as β goes from $+\infty$ to 0. A better understanding of the mixing time is useful to study the heat kernel and understand the long-time behavior of the simple random walk on the cluster. In [28, 29] Crawford and Sly give bounds on the heat kernel and prove a scaling limit for the case $s \in (d, d+1)$.

Also the Ising model on the one-dimensional line with interaction energy $J(\{x,y\}) = |x-y|^{-s}$ is a well-studied object. In [42] the author considers the case where $s < 2$, but there are also many results for the critical case $s = 2$, see for example [3, 44, 72]. In particular, the authors in [3] proved a discontinuity of the magnetization.

2 Asymptotic behavior of the distance exponent for large β

In this chapter, we prove Theorem 1.2. On the way, in section 2.1, we prove several elementary bounds on connection probabilities between certain points and boxes in the long-range percolation graph that will also be used in the following sections. In section 2.2, we prove a submultiplicative structure of the expected distance between two points, leading to the existence of a distance exponent, and also to the inverse logarithmic upper bound in Theorem 1.2. In section 2.3, we show that vertices inside a box are not connected to more than one box that is far away, typically. This is necessary in order to prove strict positivity of the distance exponent $\theta(\beta)$ in section 2.4, and the lower bound on $\theta(\beta)$ in Theorem 1.2.

2.1 Bounds on connection probabilities

Lemma 2.1. *For all $\beta \geq 0$, all $n \in \mathbb{N}$, and all $u, v \in \mathbb{Z}^d$ with $\|u-v\|_\infty \geq 2$, one has the upper bound*

$$\mathbb{P}_\beta(u \sim v) = \mathbb{P}_\beta(V_u^n \sim V_v^n) \leq \frac{2^{2d}\beta}{\|u-v\|_\infty^{2d}}, \tag{4}$$

and one has the lower bound

$$\mathbb{P}_\beta(u \sim v) = \mathbb{P}_\beta(V_u^n \sim V_v^n) \geq \frac{(4d)^{-2d}\beta}{\|u-v\|_\infty^{2d}} \wedge \frac{1}{2}. \tag{5}$$

For all $k \geq 2$ one has

$$\mathbb{P}_\beta(\mathbf{0} \sim \mathcal{S}_{\geq k}) \leq \beta 50^d k^{-d}, \tag{6}$$

and for $m \in \mathbb{N}$, any vertex $x \in V_{\mathbf{0}}^m$, and a box V_w^m with $\|w\|_\infty \geq 2$ one has

$$\mathbb{P}_\beta(x \sim V_w^m) \leq \frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}. \tag{7}$$

Proof. The equality $\mathbb{P}_\beta(u \sim v) = \mathbb{P}_\beta(V_u^n \sim V_v^n)$ is clear from the discussion about the underlying continuous model in section 1.2. We start with the proof of (4). Applying the inequalities $1 - e^{-x} \leq x$ and $\|\cdot\|_2 \geq \|\cdot\|_\infty$, we get that for two vertices u, v with $\|u-v\|_\infty \geq 2$

$$\begin{aligned}\mathbb{P}_\beta(u \sim v) &= 1 - e^{-\beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y} \leq \beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y \\ &\leq \beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|_\infty^{2d}} \mathrm{d}x\mathrm{d}y \leq \frac{\beta}{(\|u-v\|_\infty - 1)^{2d}} \leq \frac{2^{2d}\beta}{\|u-v\|_\infty^{2d}}.\end{aligned} \tag{8}$$

In order to bound the connection probability between u and v from below, first observe that $\|x\|_2 \leq \|x\|_1 \leq d\|x\|_\infty$ for all $x \in \mathbb{R}^d$. Thus we have

$$\begin{aligned}&\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{1}{\|t-s\|^{2d}} \mathrm{d}s\mathrm{d}t \geq d^{-2d} \int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{1}{\|t-s\|_\infty^{2d}} \mathrm{d}s\mathrm{d}t \\ &\geq d^{-2d} \int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{1}{(\|u-v\|_\infty+1)^{2d}} \mathrm{d}s\mathrm{d}t \geq (2d)^{-2d} \frac{1}{\|u-v\|_\infty^{2d}}\end{aligned}$$

and this already gives

$$\mathbb{P}_\beta(u \sim v) \geq 1 - e^{-(2d)^{-2d} \frac{\beta}{\|u-v\|_\infty^{2d}}} \geq \frac{(4d)^{-2d}\beta}{\|u-v\|_\infty^{2d}} \wedge \frac{1}{2}$$

as $1 - e^{-x} \geq \frac{x}{2} \wedge \frac{1}{2}$ for all $x \in \mathbb{R}_{\geq 0}$. So we showed (5).

For each point $x \in \mathcal{S}_k = \{z \in \mathbb{Z}^d : \|z\|_\infty = k\}$, at least one of its coordinates $p_i(x)$ equals $-k$ or $+k$. All other coordinates can be any integer between $-k$ and $+k$. Thus we can bound the cardinality of the set $|\mathcal{S}_k|$ by $|\mathcal{S}_k| \leq 2d(2k+1)^{d-1}$. In (8) we showed that $\mathbb{P}_\beta(\mathbf{0} \sim x) \leq \frac{\beta}{(\|x\|_\infty - 1)^{2d}}$. This already implies that for $k \geq 2$

$$\mathbb{P}_\beta(\mathbf{0} \sim \mathcal{S}_k) \leq \sum_{x \in \mathcal{S}_k} \mathbb{P}_\beta(\mathbf{0} \sim x) \overset{(8)}{\leq} \sum_{x \in \mathcal{S}_k} \frac{\beta}{(\|x\|_\infty - 1)^{2d}} \leq 2d(2k+1)^{d-1} \frac{\beta}{(k-1)^{2d}}$$

and thus also

$$\begin{aligned}\mathbb{P}_\beta(\mathbf{0} \sim \mathcal{S}_{\geq k}) &\leq \sum_{k'=k}^\infty \mathbb{P}_\beta(\mathbf{0} \sim \mathcal{S}_{k'}) \leq \sum_{k'=k}^\infty 2d(2k'+1)^{d-1} \frac{\beta}{(k'-1)^{2d}} \\ &\leq \sum_{k'=k}^\infty 2^d 3^d (k')^{d-1} \frac{\beta 2^{2d}}{(k')^{2d}} = \beta 24^d \sum_{k'=k}^\infty (k')^{-d-1} \leq \beta 50^d k^{-d},\end{aligned} \tag{9}$$

which already proves (6). For $m \in \mathbb{N}$, a vertex $x \in V_{\mathbf{0}}^m$, and a box V_w^m with $\|w\|_\infty \geq 2$, we have for all $z \in V_w^m$ that $\|x - z\|_\infty \geq (\|w\|_\infty - 1)m$. This implies

$$\mathbb{P}_\beta(x \sim V_w^m) \overset{(4)}{\leq} \sum_{z \in V_w^m} \frac{2^{2d}\beta}{\|x-z\|_\infty^{2d}} \leq \sum_{z \in V_w^m} \frac{2^{2d}\beta}{((\|w\|_\infty - 1)m)^{2d}} \leq \frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d},$$

which shows (7). □

We will often condition on the event that two blocks V_u^n, V_v^n are connected. So if we write X for the number of edges between them, we condition on the event $X \geq 1$. This conditioning clearly increases the expected number of edges between V_u^n and V_v^n, but by at most $+1$, as shown in the next lemma.

Lemma 2.2. *Let $u, v \in \mathbb{Z}^d$ with $u \neq v$ and let X be the number of edges between the blocks V_u^n and V_v^n. Then for all $\beta > 0$*

$$\mathbb{E}_\beta[X | X \geq 1] \leq 1 + \mathbb{E}_\beta[X]$$

Proof. The random variable X is a sum of independent Bernoulli random variables and we prove the statement for all random variables of this type. We use the notation $X = \sum_{i=1}^m X_i$, where $m \in \mathbb{N}$, and $(X_i)_{i \in \{1,\dots,m\}}$ are independent Bernoulli random variables.

For $i \in \{1, \ldots, m\}$, let A_i be the event that $X_i = 1$ and $X_j = 0$ for all $j \in \{1, \ldots, i-1\}$. As $\{X \geq 1\}$ implies that there is a first index i such that $X_i = 1$, we get that

$$\{X \geq 1\} = \bigsqcup_{i=1}^{m} A_i,$$

where the symbol $\bigsqcup$ means a disjoint union. On the event A_i, we know that all the random variables X_j with $j < i$ equal 0, but we have no information about random variables X_j with $j > i$. Thus we get that

$$\frac{\mathbb{E}_\beta \left[X \mathbb{1}_{A_i}\right]}{\mathbb{P}_\beta \left(A_i\right)} = \mathbb{E}_\beta \left[X | A_i\right] = \mathbb{E}_\beta \left[1 + \sum_{j=i+1}^{m} X_j \middle| A_i\right] = 1 + \mathbb{E}_\beta \left[\sum_{j=i+1}^{m} X_j\right] \leq 1 + \mathbb{E}_\beta \left[X\right].$$

Multiplying by $\mathbb{P}_\beta \left(A_i\right)$ on both sides of this inequality we get that $\mathbb{E}_\beta \left[X \mathbb{1}_{A_i}\right] \leq \mathbb{P}_\beta \left(A_i\right) \left(1 + \mathbb{E}_\beta \left[X\right]\right)$. As the events $(A_i)_{i \in \{1,\ldots,m\}}$ are disjoint, we finally get that

$$\begin{aligned} \mathbb{E}_\beta \left[X | X \geq 1\right] = \frac{\mathbb{E}_\beta \left[X \mathbb{1}_{\{X \geq 1\}}\right]}{\mathbb{P}_\beta \left(X \geq 1\right)} &= \frac{\sum_{i=1}^{m} \mathbb{E}_\beta \left[X \mathbb{1}_{A_i}\right]}{\mathbb{P}_\beta \left(X \geq 1\right)} \\ &\leq \frac{\sum_{i=1}^{m} \mathbb{P}_\beta \left(A_i\right) \left(1 + \mathbb{E}_\beta \left[X\right]\right)}{\mathbb{P}_\beta \left(X \geq 1\right)} = 1 + \mathbb{E}_\beta \left[X\right]. \end{aligned}$$

□

2.2 Submultiplicativity and the upper bound in Theorem 1.2

In this section, we prove the submultiplicative structure in the model in Lemma 2.3. This allows us to define the distance growth exponent $\theta(\beta)$ and also helps to prove the upper bound on $\theta(\beta)$ in Theorem 1.2.

Lemma 2.3. *For all dimensions d and all $\beta \geq 0$ the sequence*

$$\Lambda(n) = \Lambda(n, \beta) := \max_{u,v \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u, v)\right] + 1 \tag{10}$$

is submultiplicative and for all $\beta \geq 0$

$$\theta(\beta) = \inf_{n \geq 2} \frac{\log\left(\Lambda(n, \beta)\right)}{\log(n)} = \lim_{n \to \infty} \frac{\log\left(\Lambda(n, \beta)\right)}{\log(n)}.$$

Proof. We show (10) using a renormalization argument. As before, we define $V_u^n = \prod_{i=1}^{d} \{p_i(u)n, \ldots, (p_i(u)+1)n - 1\}$. The graph G' obtained by identifying all the vertices in V_u^n to one vertex $r(u)$ has the same connection probabilities as the original model. For $x, y \in \{0, \ldots, mn-1\}^d$, say with $x \in V_u^n$ and $y \in V_w^n$, we create a path from x to y as follows. First we consider the shortest path $\mathcal{P} = (r(u_0) = r(u), r(u_1), \ldots, r(u_{l-1}), r(u_l) = r(w))$ from $r(u)$ to $r(w)$ in G', where $l = D_{G'}(r(u), r(w))$ is the distance between $r(u)$ and $r(w)$ in the renormalized model. Inside $V_{u_i}^n$, we first fix two vertices z_i and v_i such that $z_i \sim V_{u_{i-1}}^n$ and $v_i \sim V_{u_{i+1}}^n$; for $i = 0$ set $z_0 = x$ and for $i = l$ set $v_l = y$. In case there are several such vertices z_i and v_i, we choose the one with smallest coordinates, where we weigh the coordinates in decreasing order (any deterministic rule that does not depend on the environment would work here). For each i, there clearly is a path between z_i and v_i that is completely inside $V_{u_i}^n$. As no information has been revealed up to now about the edges with both endpoints inside $V_{u_i}^n$, the expected distance between v_i and z_i inside $V_{u_i}^n$ is at most

$$\max_{a,b \in V_{u_i}^n} \mathbb{E}_\beta \left[D_{V_{u_i}^n}(a,b) \right] = \Lambda(n,\beta) - 1.$$

Now we glue all these paths together to get a path from x to y. To bound the total distance between x and y note that we have $l+1$ sets $V_{u_i}^n$ in which we need to find a path between two vertices. Additionally, we need to add $+l$ for the steps that we make from $V_{u_i}^n$ to $V_{u_{i+1}}^n$ for $i = 0, \ldots, l-1$. Thus we get that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{mn}}(x,y) \mid D_{G'}(r(u), r(w)) = l \right] \leq (l+1) \max_{a,b \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(a,b) \right] + l.$$

Taking expectations on both sides of this inequality yields

$$\begin{aligned}
&\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{mn}}(x,y) \right] \\
&\leq \left(\mathbb{E}_\beta \left[D_{G'}(r(u), r(w)) \right] + 1 \right) \max_{a,b \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(a,b) \right] + \mathbb{E}_\beta \left[D_{G'}(r(u), r(w)) \right] \\
&= \left(\mathbb{E}_\beta \left[D_{G'}(r(u), r(w)) \right] + 1 \right) \left(\max_{a,b \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(a,b) \right] + 1 \right) - 1 \\
&= \left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^m}(u,w) \right] + 1 \right) \left(\max_{a,b \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(a,b) \right] + 1 \right) - 1 \\
&\leq \Lambda(m)\Lambda(n) - 1.
\end{aligned}$$

As $x, y \in \{0, \ldots, nm-1\}^d$ were arbitrary we obtain

$$\Lambda(mn) \leq \Lambda(m)\Lambda(n), \tag{11}$$

and as the sequence is submultiplicative we can define

$$\theta = \theta(\beta) = \lim_{k \to \infty} \frac{\log \left(\Lambda(2^k, \beta) \right)}{\log(2^k)}.$$

Actually, this limit exists not just along dyadic points of the form 2^k, for $k \in \mathbb{N}$, but even when taking a limit along the integers, i.e.,

$$\theta = \theta(\beta) = \lim_{n \to \infty} \frac{\log \left(\Lambda(n, \beta) \right)}{\log(n)},$$

which follows from Lemma 4.1 below. As a next step, we want to show that $\Lambda(n) \geq n^\theta$ for all n. We do this using a proof by contradiction. So assume the contrary, i.e., there exists a natural number N and a $c < 1$ with $\Lambda(N) = cN^\theta$. Using (11) we get that for every integer k

$$\Lambda(N^k) \leq \Lambda(N)^k = c^k N^{\theta k}$$

and thus

$$\theta = \lim_{k \to \infty} \frac{\log \left(\Lambda(N^k, \beta) \right)}{\log(N^k)} \leq \limsup_{k \to \infty} \frac{\log(c^k N^{\theta k})}{\log(N^k)} = \frac{\log(c) + \theta \log(N)}{\log(N)} < \theta$$

which is a contradiction. Knowing this already gives us that for all positive numbers K we have

$$\theta = \lim_{n \to \infty} \frac{\log(\Lambda(n))}{\log(n)} = \inf_{n \geq 2} \frac{\log(\Lambda(n))}{\log(n)} = \inf_{n \geq K} \frac{\log(\Lambda(n))}{\log(n)}. \tag{12}$$

□

This lemma and its proof already have several interesting applications. First, we emphasize that $\Lambda(mn,\beta) \geq \Lambda(n,\beta)$ for all $m,n \in \mathbb{N}_{>0}$. This holds, as for arbitrary $x,y \in V_{\mathbf{0}}^n$, the distance $D_{V_{\mathbf{0}}^{mn}}(u,v)$ between $u \in V_x^m$ and $v \in V_y^m$ is at least the distance between $r(x)$ and $r(y)$ in G'. Using the self-similarity and taking expectations we thus get that

$$\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mn}}(u,v)\right] \geq \mathbb{E}_\beta\left[D_{G'}(r(x),r(y))\right] = \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n}}(x,y)\right]$$

which shows our claim. For $n=3$, we have for all $u,v \in \{0,1,2\}^d$ with $u \neq v$, and for all $\beta > 0$ that

$$\mathbb{E}_\beta\left[D_{[0,2]^d}(u,v)\right] = 1 \cdot \mathbb{P}_\beta(u \sim v) + 2 \cdot \mathbb{P}_\beta(u \nsim v) < 2$$

and this already implies that $\Lambda(3) =: 3^{\theta'} < 3$ for some $\theta' = \theta'(\beta) < 1$. Inductively, with a renormalization at scale 3, we get that

$$\Lambda(3^k N) \leq \Lambda(3)^k \Lambda(N) = 3^{k\theta'}\Lambda(N) \tag{13}$$

for all $k,N \in \mathbb{N}_{>0}$. This inequality already gives the upper bound in expectation for $s=2d$, that was already observed in [27] with a very similar technique. Next, we do a renormalization at scale $\sqrt[2d]{\beta}$ instead of scale 3 in order to get the inverse logarithmic upper bound stated in Theorem 1.2.

Proof of the upper bound in Theorem 1.2. We want to show that for each dimension d there exists a constant $C < \infty$ such that for all $\beta \geq 2$

$$\theta(d,\beta) \leq \frac{C}{\log(\beta)}.$$

As the connection probability $\mathbb{P}_\beta(u \sim v)$ between any two vertices $u,v \in \mathbb{Z}^d$ is increasing in β, the distance exponent $\theta : \mathbb{R}_{\geq 0} \to [0,1]$ is clearly decreasing by the *Harris coupling*, see for example [62]. Thus it suffices to show the upper bound for β large enough with $\sqrt[2d]{\beta} \in \mathbb{N}$. For such a β and all $u,v \in \{0,\dots,\sqrt[2d]{\beta}-1\}^d$, we have for all $y \in u+\mathcal{C}$ and $x \in v+\mathcal{C}$ that

$$\|x-y\|^{2d} \leq d^{2d}\|x-y\|_\infty^{2d} \leq d^{2d}\sqrt[2d]{\beta}^{2d} = d^{2d}\beta \tag{14}$$

and this already implies

$$\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y \geq \frac{1}{d^{2d}\beta}.$$

Inserting this into the definition $p(\beta,\{u,v\})$ and using that $1-e^{-x} \geq \frac{x}{2}$ for all $x \leq 1$ we get that for large enough β that satisfy $\frac{1}{d^{2d}\beta} \leq 1$ we already have

$$\mathbb{P}_\beta(u \sim v) = 1 - e^{-\beta\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} \overset{(14)}{\geq} 1 - e^{-d^{-2d}} \geq \frac{1}{2}d^{-2d} \geq (2d)^{-2d} \tag{15}$$

for all $u,v \in \{0,\dots,\sqrt[2d]{\beta}-1\}^d$. Next, we bound the expected graph distance between u and v. We do this by comparing the distance to a geometric random variable. Let $(u=u_0,u_1,\dots,u_k=v)$ be a deterministic self-avoiding path from u to v inside $V_{\mathbf{0}}^{\sqrt[2d]{\beta}}$, with $k \leq \sqrt[2d]{\beta}$ and $\|u_i - u_{i-1}\|_\infty = 1$ for all $i \in \{1,\dots,k\}$. Starting from this, we build a shorter path between u and v as follows. We start at $u_0 = u$. Then for $i = 0,\dots,k-1$, if $u_i \sim v$, directly go to v. If $u_i \nsim v$, then go to u_{i+1}. This gives a path P between u and v, and for $l \in \{1, \dots, k\}$ this path has length of at least l if and only if $u_i \nsim v$ for all $i \in \{0, \dots, l-2\}$. As the connections between v and different u_i-s are independent we get that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{\sqrt[2d]{\beta}}}(u,v) \right] = \sum_{l=1}^{k} \mathbb{P}_\beta \left(D_{V_{\mathbf{0}}^{\sqrt[2d]{\beta}}}(u,v) \geq l \right) \leq \sum_{l=1}^{k} \mathbb{P}_\beta \left(u_i \nsim v \text{ for all } i \leq l-2 \right)$$
$$\overset{(15)}{\leq} \sum_{l=1}^{k} \left(1 - (2d)^{-2d} \right)^{l-1} \leq \frac{1}{1 - (1-(2d)^{-2d})} = (2d)^{2d}.$$

This already implies that $\Lambda\left(\sqrt[2d]{\beta}, \beta \right) \leq (2d)^{2d} + 1 \leq (3d)^{2d}$. Applying the submultiplicativity of Λ iteratively we get that

$$\theta(\beta) = \lim_{k\to\infty} \frac{\log\left(\Lambda\left(\sqrt[2d]{\beta}^k, \beta \right) \right)}{\log\left(\sqrt[2d]{\beta}^k \right)} \leq \limsup_{k\to\infty} \frac{\log\left(\Lambda\left(\sqrt[2d]{\beta}, \beta \right)^k \right)}{\log\left(\sqrt[2d]{\beta}^k \right)}$$
$$= \frac{\log\left(\Lambda\left(\sqrt[2d]{\beta}, \beta \right) \right)}{\log\left(\sqrt[2d]{\beta} \right)} \leq \frac{\log\left((3d)^{2d} \right)}{\frac{1}{2d}\log(\beta)} = \frac{4d^2 \log(3d)}{\log(\beta)}$$

which finishes the proof. □

2.3 Spacing between points with long bonds

In this section, we investigate certain geometric properties of the cluster inside certain boxes. Mostly, we want to get upper bounds on the probability that a vertex is connected to two different long edges. As we will need it at a later point, namely in section 5.1, we will prove the statements for $\|x-y\|_\infty \leq 1$ instead of $x = y$. This does not cause major difficulties, as for each point $x \in \mathbb{Z}^d$, there are only 3^d many points $y \in \mathbb{Z}^d$ with $\|x-y\|_\infty \leq 1$. We start with showing that the probability that two vertices x, y with $\|x-y\|_\infty \leq 1$ are both connected to far away boxes is very low.

Lemma 2.4. *For blocks V_u^m, V_v^m, V_w^m with $\|u-v\|_\infty, \|v-w\|_\infty \geq 2$, there exists a constant $C_d < \infty$ such that for all $\beta \geq 0$*

$$\mathbb{P}_\beta\left(\exists x,y \in V_v^m : \|x-y\|_\infty \leq 1, x \sim V_u^m, y \sim V_w^m \right) \leq \frac{C_d \beta^2}{\|u-v\|_\infty^{2d} \|w-v\|_\infty^{2d} m^d}.$$

Proof. By translational invariance we can assume that $v = \mathbf{0}$, and thus $\|u\|_\infty, \|w\|_\infty \geq 2$. For each $x \in V_{\mathbf{0}}^m$ there are at most 3^d vertices $y \in V_{\mathbf{0}}^m$ with $\|x-y\|_\infty \leq 1$. For $x, y \in V_{\mathbf{0}}^m$ the probability that $y \sim V_w^m$ is bounded by $\frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}$, and the probability that $x \sim V_u^m$ is bounded by by $\frac{\beta 4^{2d}}{\|u\|_\infty^{2d} m^d}$, by (7). Thus

$$\mathbb{P}_\beta\left(\exists x,y \in V_{\mathbf{0}}^m : \|x-y\|_\infty \leq 1, x \sim V_u^m, y \sim V_w^m \right) \leq \sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m \right) 3^d \frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}$$
$$= \frac{48^d \beta}{\|w\|_\infty^{2d} m^d} \sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m \right) \overset{(7)}{\leq} \frac{48^d \beta}{\|w\|_\infty^{2d} m^d} m^d \frac{\beta 4^{2d}}{\|u\|_\infty^{2d} m^d} \leq \frac{\beta^2 1000^d}{\|w\|_\infty^{2d} \|u\|_\infty^{2d} m^d}$$

which finishes the proof. □

Lemma 2.5. *For blocks V_u^m, V_v^m, V_w^m with $\|v-w\|_\infty \geq 2$ and $\|u-v\|_\infty = 1$, there exists a constant $C_d < \infty$ such that for all $\beta \geq 0$*

$$\mathbb{P}_\beta\left(\exists x, y \in V_v^m : \|x-y\|_\infty \leq 1, x \sim V_u^m, y \sim V_w^m\right) \leq \begin{cases} \frac{C_d \beta \lceil \beta \rceil \log(m)}{\|v-w\|_\infty^{2d} m} & \textit{for } d=1 \\ \frac{C_d \beta \lceil \beta \rceil}{\|v-w\|_\infty^{2d} m} & \textit{for } d \geq 2 \end{cases}.$$

Proof. By translational invariance we can assume that $v = \mathbf{0}$, and thus $\|u\|_\infty = 1, \|w\|_\infty \geq 2$. For each $x \in V_{\mathbf{0}}^m$ there are at most 3^d vertices $y \in V_{\mathbf{0}}^m$ with $\|x-y\|_\infty \leq 1$. For each vertex $y \in V_{\mathbf{0}}^m$ the probability that $y \sim V_w^m$ is bounded by $\frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}$ by (7). Thus

$$\mathbb{P}_\beta\left(\exists x, y \in V_{\mathbf{0}}^m : \|x-y\|_\infty \leq 1, x \sim V_u^m, y \sim V_w^m\right) \leq \sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m\right) 3^d \frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}$$

$$= \frac{48^d \beta}{\|w\|_\infty^{2d} m^d} \sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m\right). \tag{16}$$

As $\|u\|_\infty = 1$ we have $D_\infty(x, V_u^m) \leq m$ for all $x \in V_{\mathbf{0}}^m$, where D_∞ is the distance with respect to the ∞-norm. We furthermore have the inequality

$$\left|\left\{x \in V_{\mathbf{0}}^m : D_\infty\left(x, V_u^m\right) = k\right\}\right| \leq 6^d m^{d-1}$$

for all $k \in \mathbb{N}$. This is clear for $k > m$, as the relevant set is empty in this case. For $k \leq m$ the set $\left\{x \in \mathbb{Z}^d : D_\infty\left(x, V_u^m\right) = k\right\}$ is just the boundary of the box

$$\prod_{i=1}^d \{p_i(u)m - k, \dots, (p_i(u)+1)m - 1 + k\},$$

which is a box of side length $m + 2k \leq 3m$. Thus the boundary has a cardinality of at most $2d(3m)^{d-1} \leq 6^d m^{d-1}$. Using this observation we get that

$$\sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m\right) = \sum_{k=1}^m \sum_{\substack{x \in V_{\mathbf{0}}^m : \\ D_\infty(x, V_u^m) = k}} \mathbb{P}_\beta\left(x \sim V_u^m\right) \leq \sum_{k=1}^m 6^d m^{d-1} \mathbb{P}_\beta\left(x \sim x + \mathcal{S}_{\geq k}\right)$$

$$\overset{(6)}{\leq} 6^d m^{d-1} + 6^d m^{d-1} \sum_{k=2}^m \beta 50^d k^{-d} \leq \begin{cases} m^{d-1} \log(m) \lceil \beta \rceil 400^d & \text{for } d=1 \\ m^{d-1} \lceil \beta \rceil 400^d & \text{for } d \geq 2 \end{cases}. \tag{17}$$

Inserting this into (16), we get that

$$\mathbb{P}_\beta\left(\exists x, y \in V_{\mathbf{0}}^m : \|x-y\|_\infty \leq 1, x \sim V_u^m, y \sim V_w^m\right) \leq \sum_{x \in V_{\mathbf{0}}^m} \mathbb{P}_\beta\left(x \sim V_u^m\right) 3^d \frac{\beta 4^{2d}}{\|w\|_\infty^{2d} m^d}$$

$$\leq \begin{cases} \frac{20000^d \beta \lceil \beta \rceil \log(m)}{\|w\|_\infty^{2d} m} & \text{for } d=1 \\ \frac{20000^d \beta \lceil \beta \rceil}{\|w\|_\infty^{2d} m} & \text{for } d \geq 1 \end{cases}$$

which finishes the proof. □

Lemma 2.6. *Let $m \in \mathbb{N}, l \in \{1, \dots, 3^d - 1\}$, and let $v_0, v_1, \dots, v_{l+1} \in \mathbb{Z}^d$ be distinct with $\|v_{i+1} - v_i\|_\infty = 1$ for all $i \in \{0, \dots, l\}$, $\|v_i - v_0\|_\infty = 1$ for all $i \in \{0, \dots, l\}$ and $\|v_{l+1} - v_0\|_\infty = 2$. Then there exists a constant $C_d < \infty$ such that the two probabilities*

$$\mathbb{P}_\beta\left(\exists i \in \{1, \dots, l\} \exists x, y \in V_{v_i}^m \textit{ with } \|x-y\|_\infty \leq 1, x \sim V_{v_{i-1}}^m, y \sim V_{v_{i+1}}^m \cap \left(y + \mathcal{S}_{\geq \frac{m}{6^d}}\right)\right),$$

$$\mathbb{P}_\beta\left(\exists i \in \{1,\dots,l\} \exists x,y \in V_{v_i}^m \textit{ with } \|x-y\|_\infty \leq 1, x \sim V_{v_{i+1}}^m, y \sim V_{v_{i-1}}^m \cap \left(y + \mathcal{S}_{\geq \frac{m}{6^d}}\right)\right)$$

are both bounded by $\frac{C_d \beta \lceil \beta \rceil \log(m)}{m}$ *for* $d=1$*, respectively by* $\frac{C_d\beta\lceil\beta\rceil}{m}$ *for* $d \geq 2$*.*

Proof. By a union bound we have that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\exists i \in \{1,\dots,l\} \exists x,y \in V_{v_i}^m \text{ with } \|x-y\|_\infty \leq 1, x \sim V_{v_{i-1}}^m, y \sim V_{v_{i+1}}^m \cap \left(y + \mathcal{S}_{\geq \frac{m}{6^d}}\right)\right) \\
&\leq \sum_{i\in\{1,\dots,l\}} \sum_{\substack{x,y\in V_{v_i}^m: \\ \|x-y\|_\infty \leq 1}} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m, y \sim V_{v_{i+1}}^m \cap \left(y + \mathcal{S}_{\geq \frac{m}{6^d}}\right)\right) \\
&\leq \sum_{i\in\{1,\dots,l\}} \sum_{\substack{x,y\in V_{v_i}^m: \\ \|x-y\|_\infty \leq 1}} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m\right) \mathbb{P}_\beta\left(y \sim \left(y + \mathcal{S}_{\geq \frac{m}{6^d}}\right)\right) \\
&\overset{(6)}{\leq} \beta 50^d \left(\frac{m}{6^d}\right)^{-d} \sum_{i\in\{1,\dots,l\}} \sum_{\substack{x,y\in V_{v_i}^m: \\ \|x-y\|_\infty \leq 1}} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m\right) \\
&\leq \beta 150^d \left(\frac{m}{6^d}\right)^{-d} \sum_{i\in\{1,\dots,l\}} \sum_{x\in V_{v_i}^m} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m\right).
\end{aligned}$$

The sum $\sum_{x\in V_{v_i}^m} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m\right)$ was already upper bounded in (17). Using this upper bound, $l \leq 3^d$, and inserting this into the line above we get that

$$\beta 150^d \left(\frac{m}{6^d}\right)^{-d} \sum_{i\in\{1,\dots,l\}} \sum_{x\in V_{v_i}^m} \mathbb{P}_\beta\left(x \sim V_{v_{i-1}}^m\right) \leq \begin{cases} \frac{\beta\lceil\beta\rceil\left(6^d 10^6\right)^d \log(m)}{m} & \text{for } d=1 \\ \frac{\beta\lceil\beta\rceil\left(6^d 10^6\right)^d}{m} & \text{for } d \geq 2 \end{cases}$$

which finishes the proof for the first item in the statement of the lemma. The estimate for the second term works analogously. □

2.4 The lower bound in Theorem 1.2

Finally, we developed all the necessary techniques in order to show the lower bound in Theorem 1.2, i.e., that there for all dimensions d, there exists a constant $c>0$ such that $\theta(\beta) \geq \frac{c}{\log(\beta)}$ for all $\beta \geq 2$.

Proof of the lower bound in Theorem 1.2. Inequality (5) and Lemma 2.4 show that for all dimensions d there exists a constant $C_d < \infty$ such that for all $\beta \geq 2$ and all u, v, w with $\|u-v\|_\infty, \|v-w\|_\infty \geq 2$

$$\mathbb{P}_\beta\left(\exists x \in V_v^m : x \sim V_u^m, x \sim V_w^m \mid V_u^m \sim V_v^m \sim V_w^m\right) \leq \frac{C_d \beta^2}{m^{1/2}}. \tag{18}$$

Analogously, Lemma 2.5 shows that there exists a constant $C_d < \infty$ such that for all $\beta \geq 2$ and all u, v, w with $\|u-v\|_\infty \geq 2$ and $\|v-w\|_\infty = 1$

$$\mathbb{P}_\beta\left(\exists x \in V_v^m : x \sim V_u^m, x \sim V_w^m \mid V_u^m \sim V_v^m \sim V_w^m\right) \leq \frac{C_d \beta^2}{m^{1/2}} \tag{19}$$

where we also used that $\frac{\log(m)}{m} = \mathcal{O}\left(m^{-1/2}\right)$. Lemma 2.6 implies that for every $l \in \{1, \ldots, 3^d - 1\}$ and $v_0, v_1, \ldots, v_{l+1} \in \mathbb{Z}^d$ distinct with $\|v_{i+1} - v_i\|_\infty = 1$ for all $i \in \{0, \ldots, l\}$, $\|v_i - v_0\|_\infty = 1$ for all $i \in \{1, \ldots, l\}$, and $\|v_{l+1} - v_0\|_\infty = 2$, one has the bound

$$\mathbb{P}_\beta\left(\exists x_1, \ldots, x_l : x_i \in V_{v_i}^m, V_{v_0}^m \sim x_1 \sim x_2 \sim \ldots \sim x_l \sim V_{v_{l+1}}^m\right) \leq \frac{C_d \beta^2}{m^{1/2}} \tag{20}$$

as a path from V_{v_0} to $V_{v_{l+1}}$ in $l+1 \leq 3^d$ steps needs to contain at least one edge $\{x_i, x_{i+1}\}$ with $\|x_i - x_{i+1}\|_\infty \geq \frac{m}{3^d}$ and thus $x_i \sim x_i + \mathcal{S}_{\geq \frac{m}{6^d}}$ in this case. We will now show that

$$\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mM}}\left(\mathbf{0}, (mM-1)e_1\right)\right] \geq \left(1 + \frac{1}{3^{d+4}}\right) \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^M}\left(\mathbf{0}, (M-1)e_1\right)\right] \tag{21}$$

for $m \geq \left(2000 \cdot \lceil \beta \rceil^3 3^{5d} C_d\right)^{\left(3^{4d}\right)}$ and all large enough M. We will see later where this condition on m comes from. To see (21), we use a renormalization. For $u \in V_{\mathbf{0}}^M$, we identify the blocks V_u^m to vertices $r(u)$ and call the resulting graph G'. Then we will prove that

$$\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mM}}\left(\mathbf{0}, (mM-1)e_1\right)\right] \geq \left(1 + \frac{1}{3^{d+4}}\right) \mathbb{E}_\beta\left[D_{G'}\left(r(\mathbf{0}), r((M-1)e_1)\right)\right]$$

for large enough M. This implies (21), as the random graphs G' and $V_{\mathbf{0}}^M$ have the same distribution, as shown in section 1.2. Now we condition on the graph G', i.e., we already have the knowledge which blocks of the form V_u^m are connected in the original graph. Let $P' = (r(v_0), \ldots, r(v_k))$ be a self-avoiding path in G' starting at the origin vertex, i.e., $v_0 = \mathbf{0}$. Let $k \geq 3^{d+3}$. Let $l = \left\lfloor \frac{k}{3^{d+1}} \right\rfloor$. For $j \in \{0, \ldots, l\}$, we call the subsequence $R_j := \left(r(v_{2j3^d}), r(v_{2j3^d+1}), \ldots, r(v_{(2j+2)3^d})\right)$ *separated* if there does *not* exist a sequence $(x_i)_{i=2j3^d+1}^{(2j+2)\cdot 3^d - 1}$ such that $x_i \in V_{v_i}^m$ for all $i \in \{2j3^d+1, \ldots, (2j+2)3^d - 1\}$ and

$$V_{v_{j3^d}}^m \sim x_{j3^d+1} \sim x_{j3^d+2} \sim \ldots \sim x_{(j+2)3^d-1} \sim V_{v_{(j+2)3^d}}^m.$$

For a given self-avoiding path $P' \subset G'$ and different values of $j \in \{0, \ldots, l\}$, it is independent whether the subsequences $\left(r(v_{2j3^d}), r(v_{2j3^d+1}), \ldots, r(v_{(2j+2)3^d})\right)$ are separated, and the probability that a specific subsequence $\left(r(v_{2j3^d}), r(v_{2j3^d+1}), \ldots, r(v_{(2j+2)3^d})\right)$ is not separated is bounded by $\frac{C_d \beta^2}{m^{1/2}}$, as for every sequence $\left(v_{2j3^d}, v_{2j3^d+1}, \ldots, v_{(2j+2)3^d}\right)$ at least one of the situations of (18),(19) or (20) holds, as we will argue below. Here we say that the situation of (18) holds if there exists an index $i \in \{2j3^d+1, \ldots, (2j+2)3^d - 1\}$ such that $\|v_i - v_{i+1}\|_\infty, \|v_i - v_{i-1}\|_\infty \geq 2$, the situation of (19) holds if there exists an index $i \in \{2j3^d+1, \ldots, (2j+2)3^d - 1\}$ such that $\|v_i - v_{i+1}\|_\infty = 1, \|v_i - v_{i-1}\|_\infty \geq 2$ or $\|v_i - v_{i-1}\|_\infty = 1, \|v_i - v_{i+1}\|_\infty \geq 2$, and the situation of (20) holds if there exists $l \in \{1, \ldots, 3^d - 1\}$ such that $\|v_{i+1} - v_i\|_\infty = 1$ for all $i \in \{2j3^d, \ldots, 2j3^d + l\}$, $\|v_i - v_{2j3^d}\|_\infty = 1$ for all $i \in \{2j3^d+1, \ldots, 2j3^d + l\}$ and $\|v_{2j3^d+l+1} - v_{2j3^d}\|_\infty = 2$. If none of the situations in (18),(19) holds, then the path makes only nearest neighbor-jumps within the subsequence R_j. However, as that there are only $3^d - 1$ many points $v \in \mathbb{Z}^d$ with $\|v - v_{2j3^d}\|_\infty = 1$, the situation of (20) must occur within the subsequence R_j for some l. So in total we see that

$$\mathbb{P}_\beta\left(R_j \text{ not separated} \mid G'\right) \leq \frac{C_d \beta^2}{m^{1/2}}.$$

The reason why we consider separated subsequences is that in a separated subsequence, the walk on the original graph $V_{\mathbf{0}}^{mM}$ needs to take at least one additional step. For a fixed path $P' \subset G'$ of length k and $l = \lfloor \frac{k}{3^{d+1}} \rfloor$ we have that

$$\begin{aligned}
&\mathbb{P}_\beta \left(|\{j \in \{0,\dots,l\} : R_j \text{ not separated}\}| > \frac{l}{2} \mid G' \right) \\
&= \mathbb{P}_\beta \left(\bigcup_{\substack{U \subset \{0,\dots,l\} \\ |U| > l/2}} \{R_j \text{ not separated for all } j \in U\} \;\Big|\; G' \right) \\
&\leq \sum_{\substack{U \subset \{0,\dots,l\} \\ |U| > l/2}} \mathbb{P}_\beta \left(\{R_j \text{ not separated for all } j \in U\} \;\Big|\; G' \right) \leq 2^l \left(\frac{C_d \beta^2}{m^{1/2}} \right)^{l/2}.
\end{aligned}$$

Next, we want to bound the expected degree of vertices in the long-range percolation graph from above. With the bound on the connection probability $\mathbb{P}_\beta(\mathbf{0} \sim u)$ (4), we get that

$$\begin{aligned}
\mathbb{E}_\beta[\deg(\mathbf{0})] &= \sum_{u \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \mathbb{P}_\beta(\mathbf{0} \sim u) \leq 3^d + \sum_{k=2}^{\infty} \sum_{u \in \mathcal{S}_k} \frac{2^{2d}\beta}{k^{2d}} \leq 3^d + \sum_{k=2}^{\infty} 2d(2k+1)^{d-1} \frac{2^{2d}\beta}{k^{2d}} \\
&\leq 3^d + \beta 2^{3d} 3^d \sum_{k=2}^{\infty} k^{-d-1} \leq 3^d + \beta 3^{4d} \leq \lceil \beta \rceil 3^{5d}. \qquad (22)
\end{aligned}$$

Let $\mathcal{P}_k'$ be the set of self-avoiding paths in G' starting at $r(\mathbf{0})$. With a comparison to the case of a Galton-Watson tree inequality (22) already gives that $\mathbb{E}_\beta[|\mathcal{P}_k'|] \leq \left(\lceil \beta \rceil 3^{5d}\right)^k$. As $\lfloor \frac{k}{3^{d+2}} \rfloor \leq \frac{\lfloor \frac{k}{3^{d+1}} \rfloor}{2}$, we see that

$$\begin{aligned}
&\mathbb{P}_\beta \left(\exists P' \in \mathcal{P}_k' \text{ with less than } \lfloor \frac{k}{3^{d+2}} \rfloor \text{ separated subpaths } R_j \right) \\
&= \mathbb{E}_\beta \left[\mathbb{P}_\beta \left(\exists P' \in \mathcal{P}_k' \text{ with less than } \lfloor \frac{k}{3^{d+2}} \rfloor \text{ separated subpaths } R_j \Big| G' \right) \right] \\
&\leq \mathbb{E}_\beta \left[|\mathcal{P}_k'| \, 2^k \left(\frac{C_d \beta^2}{m^{1/2}} \right)^{\lfloor \frac{k}{3^{d+2}} \rfloor} \right] \leq \left(\lceil \beta \rceil 3^{5d} \right)^k 2^k \left(\frac{C_d \beta^2}{m^{1/2}} \right)^{\lfloor \frac{k}{3^{d+2}} \rfloor} \\
&\leq \left(\lceil \beta \rceil^3 3^{5d} \right)^k 2^k C_d^k \frac{1}{m^{\frac{k}{3^{4d}}}} \leq 0.01^k
\end{aligned}$$

by the choice of $m \geq \left(2000 \cdot 3^{5d} C_d \lceil \beta \rceil^3\right)^{(3^{4d})}$. Next, we want to translate this bound on the probability of certain events to bounds on the expectation of the distances. For this, let $\mathcal{G}_k$ be the event that all self-avoiding paths $P' \subset G'$ starting at the origin and of length $\tilde{k} \geq k$ contain at least $\lfloor \frac{\tilde{k}}{3^{d+2}} \rfloor$ separated subpaths R_j. With the preceding inequality we directly get $\mathbb{P}_\beta(\mathcal{G}_k) \geq 1 - 0.1^k$. On the event $\mathcal{G}_k$, each path $P \subset V_{\mathbf{0}}^{mM}$ starting at the origin, for which the loop-erased projection on G' goes through $\tilde{k}+1$ different blocks of the form V_u^m, needs to have a length of at least $\tilde{k} + \lfloor \frac{\tilde{k}}{3^{d+2}} \rfloor \geq \left(1 + \frac{1}{3^{d+3}}\right) \tilde{k}$. Furthermore, if we have $D_{G'}(r(\mathbf{0}), r((M-1)e_1)) = \tilde{k}$, then every path connecting $\mathbf{0}$ to $(mM-1)e_1$ in the original model $V_{\mathbf{0}}^{mM}$ goes through at least $\tilde{k}+1$ different blocks of the form V_u^m, with

$u \in V_{\mathbf{0}}^M$. So we get that

$$\begin{aligned}
\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mM}}\left(\mathbf{0},(mM-1)e_1\right)\right] &\geq \sum_{k=3^{d+3}}^{\infty} \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mM}}\left(\mathbf{0},(mM-1)e_1\right)\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((M-1)e_1))=k\}}\right] \\
&\geq \sum_{k=3^{d+3}}^{\infty} \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mM}}\left(0,(mM-1)e_1\right)\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((M-1)e_1))=k\}}\mathbb{1}_{\mathcal{G}_k}\right] \\
&\geq \sum_{k=3^{d+3}}^{\infty} \mathbb{E}_\beta\left[k\left(1+\frac{1}{3^{d+3}}\right)\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((M-1)e_1))=k\}}\mathbb{1}_{\mathcal{G}_k}\right] \\
&\geq \sum_{k=3^{d+3}}^{\infty}\left(\mathbb{E}_\beta\left[k\left(1+\frac{1}{3^{d+3}}\right)\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((M-1)e_1))=k\}}\right]-\mathbb{E}_\beta\left[2k\mathbb{1}_{\mathcal{G}_k^C}\right]\right) \\
&\geq \left(1+\frac{1}{3^{d+3}}\right)\sum_{k=3^{d+3}}^{\infty} k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{D_{V_{\mathbf{0}}^M}(\mathbf{0},(M-1)e_1)=k\right\}}\right]-2\sum_{k=1}^{\infty}0.1^k k \\
&\geq \left(1+\frac{1}{3^{d+3}}\right)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^M}\left(\mathbf{0},(M-1)e_1\right)\right]-3^{d+4}\mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^M}\left(\mathbf{0},(M-1)e_1\right)<3^{d+3}\right)-1 \\
&\geq \left(1+\frac{1}{3^{d+4}}\right)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^M}\left(\mathbf{0},(M-1)e_1\right)\right]
\end{aligned}$$

where the last inequality holds for all large enough M, as for all $K \in \mathbb{N}$ the probability of the event $\left\{D_{V_{\mathbf{0}}^M}\left(\mathbf{0},(M-1)e_1\right)<K\right\}$ tends to 0 as $M \to \infty$. Say that it holds for all $M \geq m^N$, where $m = \left\lceil \left(2000 \cdot 3^{5d} C_d \lceil\beta\rceil^3\right)^{\left(3^{4d}\right)}\right\rceil$. The important property about the choice of m is, that its size is polynomial in β. This already implies that

$$\begin{aligned}
\theta(\beta) &\geq \lim_{n\to\infty}\frac{\log\left(\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{m^n}}\left(\mathbf{0},(m^n-1)e_1\right)\right]\right)}{\log\left(m^n\right)} \geq \lim_{n\to\infty}\frac{\log\left(\left(1+\frac{1}{3^{d+4}}\right)^{n-N}\right)}{\log\left(m^n\right)} \\
&= \frac{\log\left(1+\frac{1}{3^{d+4}}\right)}{\log(m)} \geq \frac{c}{\log(\beta)}
\end{aligned}$$

for some small $c > 0$ and all $\beta \geq 2$. □

3 Connected sets in graphs

The expected number of open paths in the long-range percolation model, of length k, and starting at $\mathbf{0}$, grows at most like $\mathbb{E}\left[\deg(\mathbf{0})\right]^k$, which can be easily proven by a comparison with a Galton-Watson tree. However, it is a priori not clear how the number of connected subsets of $\mathbb{Z}^d$ containing the origin grows. In particular, because the maximal degree of vertices is unbounded. In this chapter, we prove several results about the structure of connected sets in the long-range percolation graph. Mostly, we want to prove that with exponentially high probability in k, all connected sets of size k in the graph have not too many edges. First, we need to define what we mean by a connected set. Formally, we define the a connected set as follows. For a graph $G=(V,E)$ we say that a subset $Z \subset V$ is *connected* if the graph (Z,E') with edge set $E'=\{\{x,y\}\in E : x,y \in Z\}$ is connected. As a first step, we bound the expected number of connected sets of certain size in Galton-Watson trees. This counting of connected sets plays an important role in sections 5 and 11 below.

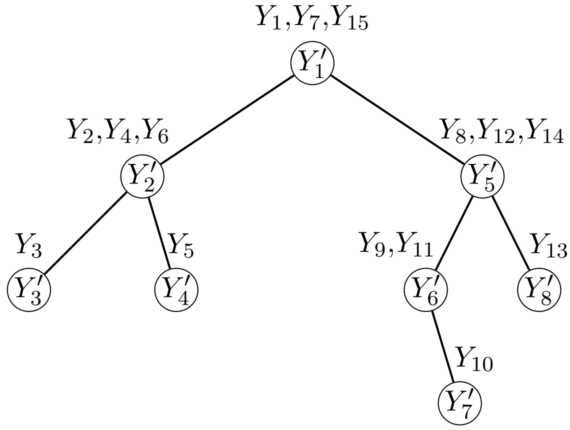


Figure 1: In the above tree, the process $(Y'_i)_{i\in\{1,\dots,9\}}$ is written inside the vertices and the process $(Y_i)_{i\in\{1,\dots,15\}}$ is written above the vertices. For this tree we have $(a_1,\dots,a_{14}) = (d,d,u,d,u,u,d,d,d,d,u,u,d,u,u)$.

Lemma 3.1. *Let $\mathcal{X}$ be a countable set with a total ordering and a minimal element, let X be a countable sum of independent Bernoulli-distributed random variables over this set, i.e., $X = \sum_{i\in\mathcal{X}} X_i$, and let μ be the expectation value of X. Say that $q(k) = \mathbb{P}(X_k = 1)$. Let T be a Galton-Watson tree with offspring distribution $\mathcal{L}(X)$. We denote the set of all subtrees of T of size k containing the origin by $\mathcal{T}_k$. Then*

$$\mathbb{E}\left[|\mathcal{T}_k|\right] \le 4^k \mu^k.$$

Proof. The choice of the set $\mathcal{X}$ and the total ordering on it do not influence the outcome, so we will always work with $\mathcal{X} = \mathbb{N}$ from here on. We can think of the Galton-Watson tree as a model of independent bond percolation on the graph with vertex set $L = \bigcup_{n=0}^{\infty} L_n$, where $L_n = \mathbb{N}^n$, and with edge set $S = \{\{v,(v\ m)\} : v \in L, m \in \mathbb{N}\}$ where some edge of the form $\{v,(v\ m)\}$ is open with probability $q(m)$. Note that the graph $G = (L,S)$ is a tree, so in particular there exists a unique path from the origin $\emptyset$ to every vertex; this tree is also known as the Ulam-Harris tree. For a vertex $v \in L$, the number of open edges of the form $\{v,(v\ m)\}$ has the same law as X and thus we can identify the open cluster connected to the root $\emptyset$ with a Galton-Watson tree with offspring distribution $\mathcal{L}(X)$. So in particular, the expected number of subtrees of the Galton-Watson tree T of size k is the same as the expected number of connected sets of size k in (L,S). For a vertex $v \in L$, we call the vertices of the form $(v\ m)$ that are connected to v by an open bond the *children* of v. Vice versa, we say that v is the *parent* of the vertex $(v\ m)$, if $(v\ m)$ is connected to v. For a connected set $L' \subset L$ of size k, we now describe an exploration process $(Y_i)_{i\in\{1,\dots,2k-1\}}$ of it:

0. Start with $Y_1 = \emptyset$.

1. For $i = 1,\dots,2k-1$

   (a) If there exists $m \in \mathbb{N}$ for which $(Y_i\ m) \in L'$ and $Y_j \ne (Y_i\ m)$ for all $j < i$, let m' be the minimal among those $m \in \mathbb{N}$ and set $Y_{i+1} = (Y_i\ m')$.

   (b) If such an m does not exist, let Y_{i+1} be the parent of Y_i.

An example of this procedure is given in Figure 1. This exploration process traverses every edge exactly twice in opposite directions and starts and ends at the origin of the tree. We also say that the exploration process Y_i goes (one level) down if (a) occurs in the algorithm above and otherwise we say that the process goes (one level) up. We also define a different process $(Y_i')_{i\in\{1,\dots,k\}}$, where Y_i' is the unique point Y_l such that $|\{Y_1,\dots,Y_{l-1}\}|<i$ and $|\{Y_1,\dots,Y_l\}|=i$. So the process $(Y_i')_{i\in\{1,\dots,k\}}$ is like a depth-first search from left to right in the tree. We can encode all information contained in the subtree L' by the two sequences $(a_1,\dots,a_{2k-2})\in\{u,d\}^{2k-2}$ and $(m_1,\dots,m_{k-1})\in\mathbb{N}^{k-1}$. The first sequence $(a_1,\dots,a_{2k-2})$ encodes whether the process Y_i goes one level up or down at a certain point. Here $a_i=u$ if the process goes one level up after Y_i, i.e., if Y_{i+1} is the parent of Y_i. Otherwise we set $a_i=d$, i.e., if Y_{i+1} is a child of Y_i. The sequence $(m_1,\dots,m_{k-1})$ encodes the direction of the process, where the i-th coordinate gives the direction when the walk goes down for the i-th time. This happens when it touches the vertex Y_{i+1}' for the first time. So if v is the parent of Y_{i+1}', then $Y_{i+1}'=(v\ m_i)$.

For fixed $\vec{a}=(a_1,\dots,a_{2k-2})\in\{u,d\}^{2k-2}$, we want to upper bound the expected number of subtrees containing the origin with exactly this up-and-down structure. Assume that the exploration process Y_i visits exactly l children of some vertex Y_j'. Then the expected number of ways to choose these l children among the children of Y_j' in an increasing way is given by

$$\sum_{m_1\in\mathbb{N}} q(m_1) \sum_{\substack{m_2\in\mathbb{N}:\\ m_2>m_1}} q(m_2) \cdots \sum_{\substack{m_l\in\mathbb{N}:\\ m_l>m_{l-1}}} q(m_l) \leq \mu^l .$$

We have this choice for all vertices Y_j' in the tree. The sum over the number of children of all the vertices is $k-1$, as every vertex, except the origin $\emptyset$, is the child of exactly one vertex. Thus the expected number of trees with a specified up-and-down structure can be bounded from above by

$$\sum_{m_1\in\mathbb{N}} \cdots \sum_{m_{k-1}\in\mathbb{N}} \prod_{i=1}^{k-1} q(m_i) = \mu^{k-1}.$$

Up to now, we considered a fixed up-and-down-structure. However, there are at most $\left|\{u,d\}^{2k-2}\right| = 2^{2k-2}$ possible up-and-down structures $(a_1,\dots,a_{2k-2})$ (In fact there are significantly fewer combinations, as one has additional constraints like $a_1=d$). So in total, we get

$$\mathbb{E}\left[|\mathcal{T}_k|\right] \leq \sum_{\vec{a}\in\{u,d\}^{2k-2}} \mu^{k-1} \leq \left(2^{2k-2}\right)\mu^{k-1} \leq 4^k\mu^k .$$

□

We now want to use the above lemma about the Galton-Watson tree in order to get results about the average degree of connected subsets of the long-range percolation graph. For this, we define the average degree of some set finite $Z\subset\mathbb{Z}^d$ by

$$\overline{\deg}(Z) \coloneqq \frac{1}{|Z|}\sum_{v\in Z}\deg(v).$$

One elementary inequality we will use in the following controls the exponential moments of certain random variables. Assume that $(U_i)_{i\in\mathbb{N}}$ are independent Bernoulli random variables and $U = \sum_{i=1}^{\infty} U_i$. Then

$$\mathbb{E}\left[e^U\right] = \mathbb{E}\left[e^{\sum_{i\in\mathbb{N}} U_i}\right] = \prod_{i\in\mathbb{N}} \mathbb{E}\left[e^{U_i}\right] \leq \prod_{i\in\mathbb{N}} \left(1 + e\mathbb{E}\left[U_i\right]\right) \leq \prod_{i\in\mathbb{N}} e^{e\mathbb{E}[U_i]} = e^{e\mathbb{E}[U]} \tag{23}$$

and this already implies, by Markov's inequality, that for any $C > 0$

$$\mathbb{P}\left(U > C\mathbb{E}\left[U\right]\right) = \mathbb{P}\left(e^U > e^{C\mathbb{E}[U]}\right) \leq \mathbb{E}\left[e^U\right] e^{-C\mathbb{E}[U]} \overset{(23)}{\leq} e^{(e-C)\mathbb{E}[U]}. \tag{24}$$

Lemma 3.2. *Let $\mathcal{CS}_k = \mathcal{CS}_k\left(\mathbb{Z}^d\right)$ be all connected subsets of the long-range percolation graph with vertex set $\mathbb{Z}^d$, which are of size k and contain the origin $\mathbf{0}$. We write μ_β for $\mathbb{E}_\beta\left[\deg(\mathbf{0})\right]$. Then for all $\beta > 0$*

$$\mathbb{P}_\beta\left(\exists Z \in \mathcal{CS}_k : \overline{\deg}(Z) \geq 20\mu_\beta\right) \leq e^{-4k\mu_\beta}.$$

Proof. Consider percolation on the tree $L = \bigcup_{n=0}^{\infty} L_n$, where $L_n = \left(\mathbb{Z}^d \setminus \{\mathbf{0}\}\right)^n$, the edge set is given by $S = \left\{\{v, (v\ m)\} : v \in L, m \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\right\}$ and an edge of the form $\{v, (v\ m)\}$ is open with probability $p\left(\beta, \{\mathbf{0}, m\}\right)$. A total ordering on $\mathbb{Z}^d \setminus \{\mathbf{0}\}$ is given by considering an arbitrary deterministic bijection with $\mathbb{N}$. From Lemma 3.1, we know that the expected number of connected sets of size k in L is bounded by $4^k \mu_\beta^k$. We want to project a finite tree $T \subset L$ of size k down to $\mathbb{Z}^d$. Remember the notation $(Y_i')_{i\in\{1,\dots,k\}}$ for the depth-first search from left to right in the tree. The information contained in the structure of the tree can be represented by the vectors $\overrightarrow{a} = (a_1, \dots, a_{2k-2}) \in \{u, d\}^{2k-2}$ and $\overrightarrow{m} = (m_1, \dots, m_{k-1}) \in \left(\mathbb{Z}^d \setminus \{\mathbf{0}\}\right)^{k-1}$. We now define a subgraph $(Z(T), E(T))$ of the integer lattice and an exploration process $(X_i')_{i\in\{1,\dots,k\}}$ as follows:

0. Start with $X_1' = \mathbf{0}, E_1(T) = \emptyset$.

1. For $i = 2, \dots, k$:
   Let $j < i$ be such that $Y_i' = (Y_j'\ m)$ for some $m \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Set $X_i' = X_j' + m$ and $E_i(T) = E_{i-1}(T) \cup \left\{\{X_i', X_j'\}\right\}$

2. $Z(T) = \bigcup_{i=1}^k \{X_i'\}$ and $E(T) = E_k(T)$

See Figure 2 for an example of this projection. The graph $(Z(T), E(T))$ is clearly connected, but it is not necessarily a tree, as there can be $i \neq j$ with $X_i' = X_j'$, in which case there exists a loop containing X_i'. We call both the graph $(Z(T), E(T))$ and the tree T *admissible* if $(Z(T), E(T))$ is a tree. We also write $\mathcal{TA}_k$ for the set of admissible trees $T \subset (L, S)$ of size k. For a tree $T \subset (L, S)$ of size k, the condition $T \in \mathcal{TA}_k$ is equivalent to $|Z(T)| = k$, as every connected graph with k vertices and $k-1$ edges is a tree. Assume that the graph $(Z(T), E(T))$ is admissible. Then the probability that all edges exist in the random graph equals $\prod_{i=1}^{k-1} p(\beta, \{\mathbf{0}, m_i\})$, which is exactly the probability that all edges of the tree T exist inside (L, S). Every connected set $Z \subset \mathbb{Z}^d$ has a spanning tree. Thus there exists a tree $T \subset L$ with $Z = Z(T)$ such that all edges in $E(T)$ exist. This and the result of Lemma 3.1 imply that

$$\begin{aligned}\mathbb{E}_\beta\left[\left|\mathcal{CS}_k(\mathbb{Z}^d)\right|\right] &\leq \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(\text{all edges in } E(T) \text{ exist}\right) = \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(T \in \mathcal{T}_k\right) \\ &\leq \mathbb{E}_\beta\left[|\mathcal{T}_k|\right] \leq 4^k \mu_\beta^k. \end{aligned} \tag{25}$$

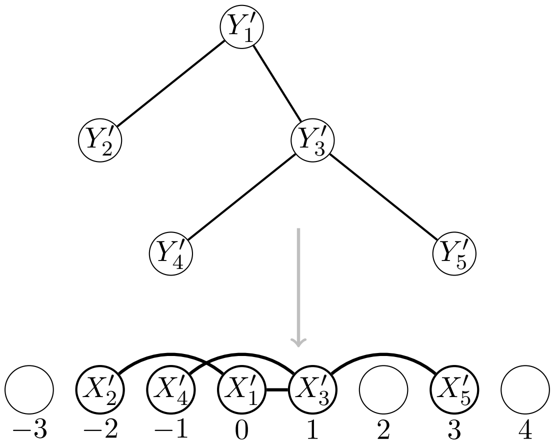


Figure 2: A tree T with 5 vertices, $(a_1, \dots, a_8) = (d, u, d, d, u, d, u, u)$, $(m_1, \dots, m_4) = (-2, 1, -2, 2)$, and its projection on $\mathbb{Z}$. The vertices with thick boundary $\{-2, -1, 0, 1, 3\} \subset \mathbb{Z}$ are the set $Z(T)$ and the thick edges between them are the set $E(T)$. Note that $(Z(T), E(T))$ really is a tree for this example.

For an admissible tree T, the degree of each vertex $v \in Z(T)$ is the sum of an *inside degree* and an *outside degree*, which we will now define. The *inside degree* $\deg_{Z(T)}(v)$ of a vertex $v \in Z(T)$ is defined by

$$\deg_{Z(T)}(v) = \sum_{u \in Z(T)} \mathbb{1}_{\{\{v,u\} \in E(T)\}}$$

which is just the number of edges in $E(T)$ containing v. Note that for a given admissible tree T, the inside degree is purely deterministic and does not depend on the environment. Also note that, by the handshaking lemma,

$$\sum_{v \in Z(T)} \deg_{Z(T)}(v) = 2|E(T)| = 2\left(|Z(T)| - 1\right), \tag{26}$$

where the last equality holds as $(Z(T), E(T))$ is a tree. Now let us turn to the *outside degree* $\deg_{Z(T)^C}(v)$ of a vertex $v \in Z(T)$, which we define by

$$\deg_{Z(T)^C}(v) = \sum_{\substack{u \in \mathbb{Z}^d \setminus \{v\}: \\ \{u,v\} \notin E(T)}} \omega(\{v, u\}).$$

The outside degree depends on the random environment ω and is a non-constant random variable, contrary to $\deg_{Z(T)}(v)$. Now we want to get bounds on the random variable $\sum_{v \in Z(T)} \deg_{Z(T)^C}(v)$. Note that $\{u, v\} \notin E(T)$ does not imply that $u \notin Z(T)$, but only that u and v are not neighbors in the graph induced by T. The random variable $\sum_{v \in Z(T)} \deg_{Z(T)^C}(v)$ is not the sum of independent Bernoulli random variables, as we might count some edges twice. But as one can count every edge at most twice in this sum, one has the bound

$$\frac{1}{2} \sum_{v \in Z(T)} \deg_{Z(T)^C}(v) \leq \sum_{\substack{\{u,v\} \notin E(T): \\ \{u,v\} \cap Z(T) \neq \emptyset}} \omega\left(\{u, v\}\right) \tag{27}$$

where the expression on the right-hand side is a sum of independent Bernoulli random variables with expectation at most $|Z(T)|\mu_\beta$. So for each admissible tree T we always have

$$\sum_{v\in Z(T)} \deg(v) = \sum_{v\in Z(T)} \deg_{Z(T)}(v) + \sum_{v\in Z(T)} \deg_{Z(T)^C}(v) \leq 2|Z(T)| + 2 \sum_{\substack{\{u,v\}\notin E(T):\\ \{u,v\}\cap Z(T)\neq\emptyset}} \omega\left(\{u,v\}\right).$$

We use the notation

$$U = U(T) := \sum_{\substack{\{u,v\}\notin E(T):\\ \{u,v\}\cap Z(T)\neq\emptyset}} \omega\left(\{u,v\}\right).$$

For a given finite admissible tree T, we have that

$$\begin{aligned}\mathbb{P}_\beta\left(\overline{\deg}(Z(T)) \geq 20\mu_\beta\right) &= \mathbb{P}_\beta\left(\sum_{v\in Z(T)} \deg(v) \geq 20|Z(T)|\mu_\beta\right) \leq \mathbb{P}_\beta\left(2U \geq 18|T|\mu_\beta\right)\\ &= \mathbb{P}_\beta\left(U \geq 9|T|\mu_\beta\right) \overset{(24)}{\leq} \mathbb{E}\left[e^U\right] e^{-9|T|\mu_\beta} \leq e^{e|T|\mu_\beta} e^{-9|T|\mu_\beta} \leq e^{-6|T|\mu_\beta}. \end{aligned} \tag{28}$$

So far we only got this bound for a fixed admissible tree $T \subset (L,S)$. Remember that every connected set $Z \in \mathcal{CS}_k$ has a spanning tree and there exists a tree $T \subset (L,S)$ so that $(Z(T), E(T))$ is exactly this spanning tree. Again, we use the notation $\mathcal{TA}_k$ for the set of admissible trees $T \subset (L,S)$ of size k. With the observation from before we get that

$$\begin{aligned}\mathbb{P}_\beta\left(\exists Z \in \mathcal{CS}_k : \overline{\deg}(Z) \geq 20\mu_\beta\right) &\leq \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(\overline{\deg}(Z(T)) \geq 20\mu_\beta, \text{ all edges in } E(T) \text{ exist}\right)\\ &\leq \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(U(T) \geq 9k\mu_\beta, \text{ all edges in } E(T) \text{ exist}\right)\\ &= \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(U(T) \geq 9k\mu_\beta\right) \mathbb{P}_\beta\left(\text{all edges in } E(T) \text{ exist}\right)\\ &\overset{(28)}{\leq} e^{-6k\mu_\beta} \sum_{T\in\mathcal{TA}_k} \mathbb{P}_\beta\left(\text{all edges in } E(T) \text{ exist}\right)\\ &\overset{(25)}{\leq} e^{-6k\mu_\beta} 4^k \mu_\beta^k \leq e^{-6k\mu_\beta} e^{2k} e^{\mu_\beta k} \leq e^{-4k\mu_\beta}\end{aligned}$$

where we used that $\mu_\beta \geq 2$ in the last inequality. This holds for long-range percolation with our parameters, as each vertex is always connected to its nearest neighbors. The final inequality is exactly the result that we wanted to show and thus finishes the proof. □

4 Distances in $V_{\mathbf{0}}^n$

In this section, we give several bounds on the distribution of the graph distances between points, respectively sets, inside of certain boxes. In section 4.1, we determine several different properties of the function $(x,y) \mapsto \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(x,y)\right]$. It is intuitively clear that the expression is large when x,y also have a big Euclidean distance, for example when $x = \mathbf{0}$ and $y = (n-1)\mathbf{1}$. This intuition is made rigorous in Lemma 4.2. In section 4.2, we upper bound the second moment of random variables of the form $D_{V_{\mathbf{0}}^n}(x,y)$. Then, in section 4.3 we use these results in order to bound the distance between certain points and sets in the long-range percolation graph.

4.1 Graph distances of far away points

From the definition of $\Lambda(n,\beta)$ in (10) it is not clear which pair u,v maximizes the expected distance and how the expected graph distances can be compared for different graphs $V_{\mathbf{0}}^{n}$ and $V_{\mathbf{0}}^{n'}$. In Lemma 4.1, we construct a coupling between the long-range percolation graph on $V_{\mathbf{0}}^{n}$ for different n. In Lemma 4.2, we show that, up to a constant factor, the maximum in the definition of $\Lambda(n,\beta)$ gets attained by the pair $\{\mathbf{0},(n-1)e_1\}$ or $\{\mathbf{0},(n-1)\mathbf{1}\}$.

Lemma 4.1. *Let $\beta \geq 0$ and $n',n \in \mathbb{N}_{>0}$ with $n' \leq n$. For $u,v \in V_{\mathbf{0}}^{n}$ define $u' := \lfloor \frac{n'}{n} u \rfloor, v' := \lfloor \frac{n'}{n} v \rfloor$, where the rounding operation is componentwise. There exists a coupling of the random graphs with vertex sets $V_{\mathbf{0}}^{n}$ and $V_{\mathbf{0}}^{n'}$ such that both are distributed according to $\mathbb{P}_\beta$ and*

$$D_{V_{\mathbf{0}}^{n'}}(u',v') \leq 3 D_{V_{\mathbf{0}}^{n}}(u,v) \tag{29}$$

for all $u,v \in V_{\mathbf{0}}^{n}$. The same holds true when one considers the graph $\mathbb{Z}^d$ instead of $V_{\mathbf{0}}^{n}$ and this also implies that

$$Diam\left(V_0^{n'}\right) \leq 3 Diam\left(V_0^{n}\right). \tag{30}$$

Proof. We prove the statement via a coupling with the underlying continuous model. As the claim clearly holds for $\beta = 0$ or for $u = v$, we can assume $\beta > 0$, and $u \neq v$ from here on. Let $\tilde{\mathcal{E}}$ be a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^d$ with intensity $\frac{\beta}{2\|t-s\|^{2d}}$ and define $\mathcal{E} = \left\{(t,s) \in \mathbb{R}^d \times \mathbb{R}^d : (s,t) \in \tilde{\mathcal{E}}\right\} \cup \tilde{\mathcal{E}}$. Remember that this point process has a scaling invariance, namely that for a constant $\alpha > 0$ the set $\alpha\mathcal{E}$ has exactly the same distribution as $\mathcal{E}$. We now define a random graph $G = (V,E)$: For $u,v \in V_{\mathbf{0}}^{n} =: V$ we place an edge between u and v if and only if $(u+\mathcal{C}) \times (v+\mathcal{C}) \cap n\mathcal{E} \neq \emptyset$. We have already seen in section 1.2 about the continuous model that this creates a sample of independent long-range percolation where the connection probability between the vertices u and v is given by $p(\beta,|v-u|) = 1 - e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{\|t-s\|^{2d}} \mathrm{d}t\mathrm{d}s}$. We can do the same procedure for $V' := V_{\mathbf{0}}^{n'}$ and $n'\mathcal{E}$ to get a random graph $G' = (V',E')$. Formally, we place an edge between two vertices $u',v' \in V'$ if and only if $(u'+\mathcal{C}) \times (v'+\mathcal{C}) \cap n'\mathcal{E} \neq \emptyset$. We now claim that for any two vertices $u,v \in V$ with $u \neq v$ and u',v' defined as above one has $D_{G'}(u',v') \leq 2D_G(u,v)+1$, which already implies (29). Assume that $(x_0 = u, x_1, \ldots, x_l = v)$ is the shortest path between u and v in G, where $l = D_G(u,v)$. Then for all $i = 1,\ldots,l$ there are points

$$(y(i,0), y(i,1)) \in (x_{i-1}+\mathcal{C}) \times (x_i+\mathcal{C}) \cap n\mathcal{E}.$$

In particular one has

$$\|y(i-1,1) - y(i,0)\|_\infty < 1$$

for all $i = 2,\ldots,l$, $\|y(1,0) - u\|_\infty < 1$, and $\|y(l,1) - v\|_\infty < 1$. For all $i = 1,\ldots,l$ and $j \in \{0,1\}$ define $y'(i,j) = \frac{n'}{n} y(i,j)$, which implies $(y'(i,0), y'(i,1)) \in n'\mathcal{E}$. With this definition one clearly has

$$\|y'(i-1,1) - y'(i,0)\|_\infty < 1$$

for all $i = 2,\ldots,l$, $\|y'(1,0) - \frac{n'}{n}u\|_\infty < 1$, and $\|y'(l,1) - \frac{n'}{n}v\|_\infty < 1$. So in G' we can use the path from u' to v' that uses all the edges $\{\lfloor y'(i,0)\rfloor, \lfloor y'(i,1)\rfloor\}$ and in the case where $\lfloor y'(i-1,1)\rfloor \neq \lfloor y'(i,0)\rfloor$ holds, respectively the analogous statement for u' or v' holds, we can use the nearest neighbor edge between those vertices, which exists as $\|y'(i-1,1) - y'(i,0)\|_\infty < 1$. So for each vertex that is touched by the shortest path between u and v in G one one needs to make at most one additional step for the path between u' and v' in G', which implies that $D_{G'}(u',v') \leq 2D_G(u,v)+1$. If one does not restrict to the sets $V = V_{\mathbf{0}}^{n}$ and $V' = V_{\mathbf{0}}^{n'}$, but works on the graph with vertex set $\mathbb{Z}^d$ instead, the same proof works. □

Lemma 4.2. *For all $\beta \geq 0$, $n \in \mathbb{N}_{>0}$, and $u, v \in V_{\mathbf{0}}^n$, we have*

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u,v) \right] \leq 6d\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] \tag{31}$$

and

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] \leq 6\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1}) \right] . \tag{32}$$

This lemma already has two interesting implications, that we want to discuss before going to the proof.

Remark 4.3. *Combining (31) and (32) already implies that for $\Lambda(n,\beta) = \max_{u,v \in V_{\mathbf{0}}^n} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u,v) \right] + 1$ one has*

$$\begin{aligned} &\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] + 1 \leq \Lambda(n,\beta) \leq 6d\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] + 1 \quad \textit{and} \\ &\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1}) \right] + 1 \leq \Lambda(n,\beta) \leq 36d\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1}) \right] + 1. \end{aligned}$$

Remark 4.4. *For all bounded sets $K \subset \mathbb{R}_{\geq 0}$ there exists a constant $\theta^\star > 0$ such that for all $\beta \in K$ and all M, N large enough one has*

$$\Lambda(MN,\beta) \geq M^{\theta^\star} \Lambda(N,\beta).$$

Proof. Remark 4.3 together with (21) already show the existence of such an $\theta^\star$ along a subsequence of numbers of the form $M = m^k$. Lemma 4.1 shows the result for all large enough M. □

Proof of Lemma 4.2. Using the triangle inequality and linearity of expectation we get for all $u, v \in V_{\mathbf{0}}^n$ that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u,v) \right] \leq \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u,\mathbf{0}) \right] + \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},v) \right]$$

and thus, in order to prove (31), it suffices to show that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},v) \right] \leq 3d\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] \tag{33}$$

for all $v \in V_{\mathbf{0}}^n$. By symmetry, we can assume that $p_1(v) \geq p_2(v) \geq \ldots \geq p_d(v)$. For $k \in \{0, \ldots, d\}$, we define the vector $v(k) \in V_{\mathbf{0}}^n$ by

$$v(k) = \sum_{i=1}^{k} p_i(v) e_i,$$

i.e., the first k coordinates of $v(k)$ equal the corresponding coordinates of v and all other coordinates are 0. By the triangle inequality and linearity of expectation we clearly have

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},v) \right] \leq \mathbb{E}_\beta \left[\sum_{i=0}^{d-1} D_{V_{\mathbf{0}}^n}(v(i),v(i+1)) \right] = \sum_{i=0}^{d-1} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(v(i),v(i+1)) \right] .$$

So in order to show (33), it suffices to show that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(v(i),v(i+1)) \right] \leq 3\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1) \right] \tag{34}$$

for all $i \in \{0, \ldots, d-1\}$. For each such index i, the cube

$$\mathcal{B}_i = \prod_{j=1}^{i} \{p_j(v) - p_{i+1}(v), \ldots, p_j(v)\} \times \{0, \ldots, p_{i+1}(v)\}^{d-i}$$

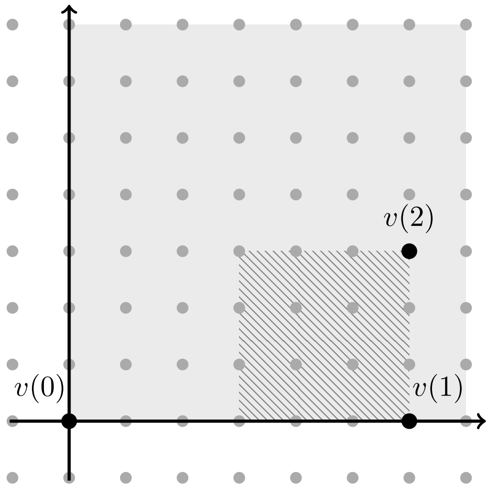


Figure 3: Let $v = (6,3) \in V_{\mathbf{0}}^8$. The points in the gray area are the set $V_{\mathbf{0}}^8$. The points in the hatched area are $\mathcal{B}_1$.

is contained in the cube $V_{\mathbf{0}}^n$ and contains both points $v(i)$ and $v(i+1)$, which lie on adjacent corners of the cube. See figure 3 for an example. Allowing the geodesic to use less edges clearly increases the distance between two points, which implies $D_{V_{\mathbf{0}}^n}(v(i), v(i+1)) \leq D_{\mathcal{B}_i}(v(i), v(i+1))$ as $\mathcal{B}_i \subset V_{\mathbf{0}}^n$. As the model is invariant under changing the coordinates and under the action $e_i \mapsto -e_i$ we already get for all $i \in \{0, \ldots, d-1\}$

$$\mathbb{E}_\beta \left[D_{\mathcal{B}_i}(v(i), v(i+1))\right] = \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{p_{i+1}(v)+1}}(\mathbf{0}, p_{i+1}(v) e_1)\right] \leq 3\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1)\right],$$

where we used Lemma 4.1 for the last inequality. This shows (34) and thus finishes the proof of (31). Now let us go to the proof of (32). Define $y \in \mathbb{Z}^d$ by $p_1(y) = 1, p_i(y) = -1$ for $i \geq 2$ and define the cube $\mathcal{B}$ by $\mathcal{B} = \{n-1, \ldots, 2n-2\} \times \{0, \ldots, n-1\}^{d-1}$. By the triangle inequality we have

$$D_{V_{\mathbf{0}}^{2n-1}}(\mathbf{0}, (2n-2)e_1) \leq D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)\mathbf{1}) + D_{\mathcal{B}}((n-1)\mathbf{1}, (2n-2)e_1). \tag{35}$$

Observe that $(2n-2)e_1 = (n-1)\mathbf{1} + (n-1)y$. The pairs of vertices $\mathbf{0}$ and $(n-1)\mathbf{1}$ lie on opposite corners of the cube $V_{\mathbf{0}}^n$. The vertices $(n-1)\mathbf{1}$ and $(2n-2)e_1$ also lie on opposite corners of the cube $\mathcal{B}$. The two cubes $V_{\mathbf{0}}^n$ and $\mathcal{B}$ differ by a translation only; in particular, they have the same side length. As the long-range percolation model is invariant under translation and reflection of any coordinate the two terms in the sum (35) have the same distribution which implies that

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2n-1}}(\mathbf{0}, (2n-2)e_1)\right] \leq 2\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)\mathbf{1})\right].$$

Using Lemma 4.1, we finally get

$$\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1)\right] \leq 3\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2n-1}}(\mathbf{0}, (2n-2)e_1)\right] \leq 6\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)\mathbf{1})\right]$$

which shows (32). □

4.2 The second moment bound

The next lemma relates the second moment of the distances to their first moment. We use a technique that has already been used in [33] before in a slightly different form for dimension $d = 1$ only. As we need the result in a uniform dependence on β in section 11 below, we directly prove the uniform statement here. The uniformity does not cause any complications for $d \geq 2$, but it causes minor technical difficulties for $d = 1$. So we give a separate proof for dimension $d = 1$ in section 12 below. The situation for $d \geq 2$ is easier, as there are no cut points, in the sense that for every $u, v \in V_{\mathbf{0}}^n$ there exist two disjoint paths between u and v. For $d = 1$, and in particular for $\beta < 1$, such a statement will typically not be true.

Lemma 4.5. *For all $\beta \geq 0$, there exists a constant $C_\beta < \infty$ such that for all $n \in \mathbb{N}$, all $\varepsilon \in [0, 1]$ and all $u, v \in V_{\mathbf{0}}^n$*

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(u,v)^2\right] \leq C_\beta \Lambda(n, \beta+\varepsilon)^2. \tag{36}$$

Proof of Lemma 4.5 for $d \geq 2$. Fix $\beta \geq 0$. We will prove that for all $\varepsilon \in [0, 1]$, all $m, n \in \mathbb{N}$, and all $u, v \in V_{\mathbf{0}}^{mn}$

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{mn}}(u,v)^2\right] \leq 170 m^4 \Lambda(n,\beta+\varepsilon)^2 + 170 \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)^2\right]. \tag{37}$$

Iterating over this inequality one gets for some large enough N that

$$\begin{aligned}
&\max_{u,v \in V_{\mathbf{0}}^{m^k N}} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{m^k N}}(u,v)^2\right] \\
&\leq 170 m^4 \sum_{i=0}^{k} 170^i \Lambda(m^{k-i}N, \beta+\varepsilon)^2 + 170^k \max_{u,v \in V_{\mathbf{0}}^N} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^N}(u,v)^2\right] \\
&\leq 170 m^4 \sum_{i=0}^{k} 170^i \Lambda(m^{k-i}N, \beta+\varepsilon)^2 + 170^k N^2.
\end{aligned} \tag{38}$$

for all $k \in \mathbb{N}$. By Remark 4.4 there exists $\theta^\star = \theta^\star(\beta) > 0$ such that for all $\varepsilon \in [0, 1]$, and all $m, n \in \mathbb{N}$ large enough one has

$$\begin{aligned}
\Lambda(mn, \beta+\varepsilon) &= \max_{u,v \in V_{\mathbf{0}}^{mn}} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{mn}}(u,v)\right] + 1 \\
&\geq m^{\theta^\star}\left(\max_{u,v \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(u,v)\right] + 1\right) = m^{\theta^\star}\Lambda(n, \beta+\varepsilon).
\end{aligned}$$

Take m large enough so that also $170 m^{-2\theta^\star} < \frac{1}{2}$ is satisfied. Inserting this into (38) gives

$$\begin{aligned}
\max_{u,v \in V_{\mathbf{0}}^{m^k N}} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{m^k N}}(u,v)^2\right] &\leq 170 m^4 \sum_{i=0}^{k} 170^i \Lambda(m^{k-i}N, \beta+\varepsilon)^2 + 170^k N^2 \\
&\leq 170 m^4 \sum_{i=0}^{k} 170^i m^{-2\theta^\star i} \Lambda(m^k N, \beta+\varepsilon)^2 + N^2 \Lambda(m^k N, \beta+\varepsilon) \\
&\leq \left(340 m^4 + N^2\right) \Lambda(m^k N, \beta+\varepsilon)^2
\end{aligned}$$

for large enough N. This shows (36) along the subsequence $N, mN, m^2N, \ldots$. For general $n \in \mathbb{N}$, the desired result follows from Lemma 4.1. So we are left with showing (37). For this, we use an elementary observation, that was already used in [33].

Assume that $X_1, \dots, X_{\tilde{m}}$ are independent non-negative random variables and let $\tau = \arg\max_{i \in \{1,\dots,\tilde{m}\}} (X_i)$. Then

$$\mathbb{E}\left[\left(\max_{i\neq\tau} X_i\right)^2\right] \leq \mathbb{E}\left[\sum_{i=1}^{\tilde{m}} X_i \left(\sum_{j\neq i} X_j\right)\right] = \sum_{i=1}^{\tilde{m}} \sum_{j\neq i} \mathbb{E}[X_i]\mathbb{E}[X_j] \leq \tilde{m}^2 \max_i \mathbb{E}[X_i]^2 . \quad (39)$$

We still need to show inequality (37), i.e., that

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{mn}}(u,v)^2\right] \leq 170 m^4 \Lambda(n,\beta+\varepsilon)^2 + 170 \max_{w,z\in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)^2\right].$$

Let $u, v \in V_{\mathbf{0}}^{mn}$, say with $u \in V_x^n, v \in V_y^n$, where $x, y \in V_{\mathbf{0}}^m$. Inequality (37) clearly holds in the case where $x = y$. For the case $x \neq y$, let $x_0 = x, x_1, \dots, x_l = y$ and $x_0' = x, x_1', \dots, x_{l'}' = y$ be two completely disjoint and deterministic paths between x and y inside $V_{\mathbf{0}}^m$ that are of length at most $m+1$ and use only nearest-neighbor edges, i.e., $\|x_i - x_{i-1}\|_\infty = 1$ and $\|x_i' - x_{i-1}'\|_\infty = 1$ for all suitable indices i. By completely disjoint we mean that $\{x_1, \dots, x_{l-1}\} \cap \{x_1', \dots, x_{l'-1}'\} = \emptyset$; the starting point $x = x_0 = x_0'$ and the end point $y = x_l = x_{l'}'$ already need to agree by the construction. Now we iteratively define sequences $(L_i, R_i)_{i=0}^{l}$ and $(L_i', R_i')_{i=0}^{l'}$ as follows:

0. Set $L_0 = u, R_l = v$.
1. For $i = 1, \dots, l$, choose $R_{i-1} \in V_{x_{i-1}}^n$ and $L_i \in V_{x_i}^n$ such that $\|R_{i-1} - L_i\|_\infty = 1$.

Analogously, we define $(L_i', R_i')_{i=0}^{l'}$ by

0. Set $L_0' = u, R_{l'}' = v$.
1. For $i = 1, \dots, l'$, choose $R_{i-1}' \in V_{x_{i-1}'}^n$ and $L_i' \in V_{x_i'}^n$ such that $\|R_{i-1}' - L_i'\|_\infty = 1$.

The choice of these algorithms in step (1.) is typically not unique. If there are several possibilities, we always choose the vertices with some deterministic rule that does not depend on the environment. By construction we have $L_i, R_i \in V_{x_i}^n$ and $L_i', R_i' \in V_{x_i'}^n$ for all $i \in \{0, \dots, l\}$, respectively $i \in \{0, \dots, l'\}$. Define

$$X_i = D_{V_{x_i}^n}(L_i, R_i) \text{ for } i \in \{1, \dots, l-1\} \text{ and}$$
$$X_i' = D_{V_{x_i'}^n}(L_i', R_i') \text{ for } i \in \{1, \dots, l'-1\}.$$

This are at most $l - 1 + l' - 1 \leq 2m$ random variables and they are independent, as the boxes $V_{x_i'}^n$ and $V_{x_i}^n$ are disjoint. We order the random variables $\{X_i : i \in \{1, \dots, l-1\}\} \cup \{X_i' : i \in \{1, \dots, l'-1\}\}$ in a descending way and call them $Y_1, Y_2, \dots, Y_{l+l'-2}$. The idea in finding a short path between u and v is now to avoid the box where the maximum of the Y_i-s is attained. Assume that the maximum of them is one of the X_i-s, i.e., $X_i = Y_1$ for some $i \in \{1, \dots, l-1\}$. Then we consider the path that goes from $L_0' = v$ to R_0' and from there to L_1', and from there we go successively to $R_{l'}' = v$. Otherwise, we have $X_i' = Y_1$ for some $i \in \{1, \dots, l'-1\}$. In this situation, we consider the path that goes from $L_0 = v$ to R_0, from there to L_1, and successively to $R_l = v$. In both cases we have have constructed a path between u and v. The length of this path is an upper bound on the chemical distance between u and v and thus we get

$$D_{V_{\mathbf{0}}^{mn}}(u,v) \leq D_{V_x^n}(L_0, R_0) + D_{V_x^n}(L_0', R_0') + D_{V_y^n}(L_l, R_l) + D_{V_y^n}(L_{l'}', R_{l'}') + mY_2 + (m+1), \quad (40)$$

where the summand $(m+1)$ arises as one still needs to go from R_i to L_{i+1} for all $i \in \{0, \dots, l-1\}$, or from R'_i to L'_{i+1} for all $i \in \{0, \dots, l'-1\}$. But by assumption one has $l, l' \leq m+1$, so one needs at most $m+1$ additional steps. From (39) we know that

$$\mathbb{E}_{\beta+\varepsilon}\left[Y_2^2\right] \leq 4m^2 \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)\right]^2. \tag{41}$$

For the distance between L_0 and R_0 one clearly has

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_x^n}(L_0, R_0)^2\right] \leq \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)^2\right]$$

and the same statements hold for $D_{V_x^n}(L'_0, R'_0), D_{V_y^n}(L_l, R_l)$, and $D_{V_y^n}(L'_{l'}, R'_{l'})$. Using the elementary inequality $\left(\sum_{i=1}^6 a_i\right)^2 \leq 36 \sum_{i=1}^6 a_i^2$ that holds for any six numbers $a_1, \dots, a_6 \in \mathbb{R}$ for the term in (40), we get that

$$\begin{aligned}
&\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{mn}}(u,v)^2\right] \\
&\leq 36\mathbb{E}_{\beta+\varepsilon}\left[D_{V_x^n}(L_0,R_0)^2 + D_{V_x^n}(L'_0,R'_0)^2 + D_{V_y^n}(L_l,R_l)^2 + D_{V_y^n}(L'_{l'},R'_{l'})^2 + m^2Y_2^2 + (m+1)^2\right] \\
&\leq 4 \cdot 36 \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)^2\right] + 36m^2\mathbb{E}_{\beta+\varepsilon}\left[Y_2^2\right] + 6(m+1)^2 \\
&\overset{(41)}{\leq} 170 \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(w,z)^2\right] + 170m^4\Lambda(n,\beta+\varepsilon)^2
\end{aligned}$$

which shows (37) and thus finishes the proof. □

Corollary 4.6. *Iterating this technique one can show that for all $k \in \mathbb{N}$ of the form $k = 2^l$ and for all $\beta > 0$ there exists a constant $C_\beta < \infty$ such that for all $n \in \mathbb{N}$, and all $u, v \in V_{\mathbf{0}}^n$*

$$\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(u,v)^k\right] \leq C_\beta \Lambda(n,\beta)^k. \tag{42}$$

Then, one can extend this bound to all $k \in \mathbb{R}_{\geq 0}$ with Hölder's inequality.

Proof of Corollary 4.6 for $d \geq 2$. For $r > 0$, define the quantity

$$\Lambda^r(\beta, n) := \max_{x,y \in V_{\mathbf{0}}^n} \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(x,y)^r\right]$$

and assume that $\Lambda^r(\beta,n) \leq C\Lambda(\beta,n)^r$ for some constant C and all $n \in \mathbb{N}$. Using the same notation as in (40) above we get that for any $u, v \in V_{\mathbf{0}}^{mn}$, say with $u \in V_x^n$ and $y \in V_y^n$,

$$D_{V_{\mathbf{0}}^{mn}}(u,v) \leq D_{V_x^n}(L_0,R_0) + D_{V_x^n}(L'_0,R'_0) + D_{V_y^n}(L_l,R_l) + D_{V_y^n}(L'_{l'},R'_{l'}) + mY_2 + (m+1),$$

and thus we also get that

$$\begin{aligned}
D_{V_{\mathbf{0}}^{mn}}(u,v)^{2r} \leq 6^{2r}\Big(&D_{V_x^n}(L_0,R_0)^{2r} + D_{V_x^n}(L'_0,R'_0)^{2r} \\
&+ D_{V_y^n}(L_l,R_l)^{2r} + D_{V_y^n}(L'_{l'},R'_{l'})^{2r} + (mY_2)^{2r} + (m+1)^{2r}\Big).
\end{aligned}$$

We have that

$$\mathbb{E}_\beta\left[(mY_2)^{2r}\right] = m^{2r}\mathbb{E}_\beta\left[\left(Y_2^r\right)^2\right] \leq m^{2r} \max_{w,z \in V_{\mathbf{0}}^n} \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(w,z)^r\right]^2 \leq m^{2r}C^2\Lambda(\beta,n)^r$$

and from here the same proof as in Lemma 4.5 shows that $\Lambda^{2r}(\beta, n) \leq C(r)\Lambda(\beta, n)^{2r}$ for some constant $C(r) < \infty$. Inductively, we thus get that for all $r = 2^k$, with $k \in \mathbb{N}$ one has $\Lambda^r(\beta, n) \leq C(r)\Lambda(\beta, n)^r$. Whenever $r > 0$ is not of the form $r = 2^k$ for some $k \in \mathbb{N}$, let k be large enough so that $r < 2^k$. Then we get that

$$\Lambda^r(\beta, n) = \max_{x,y \in V_{\mathbf{0}}^n} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(x, y)^r \right] \leq \max_{x,y \in V_{\mathbf{0}}^n} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(x, y)^{2^k} \right]^{\frac{r}{2^k}} \leq C\Lambda(\beta, n)^r$$

for some constant C. □

4.3 Graph distances between points and boxes

So far, we only considered distances between two different points in a box. In this section, we investigate the distance between certain points and boxes. For $n \in \mathbb{N}$ and $0 < \iota < \frac{1}{2}$ we define the boxes $L_\iota^n := [0, \iota n]^d$ and $R_\iota^n := [n - 1 - \iota n, n - 1]^d$. This are boxes that lie in opposite corners of the cube $V_{\mathbf{0}}^n$, where L_ι^n lies in the corner containing $\mathbf{0}$ and R_ι^n lies in the corner containing $\mathbf{1}$. The next lemma deals with the graph distance of these two boxes. A similar statement of Lemma 4.7 for the continuous model and $d = 1$, was already proven in [33]. We follow the same strategy for the proof of this lemma. Again, we prove it uniformly for β in some compact intervals, as we will need this uniform statement in section 11. The uniformity does not make any complications in this proof here.

Lemma 4.7. *For all $\beta \geq 0$, there exists an $\iota > 0$ such that uniformly over all $\varepsilon \in [0, 1]$ and $n \in \mathbb{N}$*

$$\mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n} (L_\iota^n, R_\iota^n) \right] \geq \frac{1}{2} \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n} (\mathbf{0}, (n-1)\mathbf{1}) \right], \tag{43}$$

and there exists $c^\star > 0$ such that uniformly over all $\varepsilon \in [0, 1]$ and $n \in \mathbb{N}$

$$\mathbb{P}_{\beta+\varepsilon} \left(D_{V_{\mathbf{0}}^n} (L_\iota^n, R_\iota^n) \geq \frac{1}{4} \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n} (\mathbf{0}, (n-1)\mathbf{1}) \right] \right) \geq c^\star. \tag{44}$$

Proof. The statement clearly holds for small n, so we focus on $n \in \mathbb{N}$ large enough from here on. Let $x \in L_\iota^n$ and $y \in R_\iota^n$ be the minimizers of $D_{V_{\mathbf{0}}^n} (L_\iota^n, R_\iota^n)$, i.e., $D_{V_{\mathbf{0}}^n} (L_\iota^n, R_\iota^n) = D_{V_{\mathbf{0}}^n} (x, y)$. If the minimizers are not unique, pick arbitrary ones in some fixed way not depending on the environment. The choice of x, y, and the distance $D_{V_{\mathbf{0}}^n} (L_\iota^n, R_\iota^n)$ depend only on edges with at least one endpoint in $V_{\mathbf{0}}^n \setminus (L_\iota^n \cup R_\iota^n)$. The distances $D_{L_\iota^n}(\mathbf{0}, x)$, respectively $D_{R_\iota^n}(y, (n-1)\mathbf{1})$, depend only on edges with both endpoints in L_ι^n, respectively R_ι^n. Thus we get that

$$\begin{aligned} \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)\mathbf{1}) \right] &\leq \mathbb{E}_{\beta+\varepsilon} \left[D_{R_\iota^n}(\mathbf{0}, x) \right] + \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n}(L_\iota^n, R_\iota^n) \right] + \mathbb{E}_{\beta+\varepsilon} \left[D_{R_\iota^n}(y, (n-1)\mathbf{1}) \right] \\ &\leq 2\Lambda(\lfloor \iota n \rfloor, \beta + \varepsilon) + \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n}(L_\iota^n, R_\iota^n) \right]. \end{aligned}$$

For ι small enough and n large enough, we get uniformly over $\varepsilon \in [0, 1]$ that

$$\Lambda(n, \beta + \varepsilon) \geq \left(\frac{1}{\iota} \right)^{\theta'} \Lambda(\lfloor \iota n \rfloor, \beta + \varepsilon)$$

for some $\theta' > 0$ by Remark 4.4. So by Lemma 4.2, respectively Remark 4.3, we can choose ι small enough so that uniformly over $n \in \mathbb{N}$ large enough and $\varepsilon \in [0, 1]$

$$2\Lambda(\lfloor \iota n \rfloor, \beta + \varepsilon) \leq \frac{1}{2} \mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)\mathbf{1}) \right],$$

and this implies that

$$\begin{aligned}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(L_\iota^n,R_\iota^n)\right] &\geq \mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]-2\Lambda(\lfloor \iota n\rfloor,\beta+\varepsilon)\\ &\geq \frac{1}{2}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]\end{aligned}$$

which proves (43). For such an ι, define $\mathcal{A}=\left\{D_{V_{\mathbf{0}}^n}\left(L_\iota^n,R_\iota^n\right)\geq \frac{1}{4}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)\mathbf{1}\right)\right]\right\}$. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}&\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]\leq 2\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(L_\iota^n,R_\iota^n)\right]\\ &=2\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(L_\iota^n,R_\iota^n)\mathbb{1}_{\mathcal{A}^C}\right]+2\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(L_\iota^n,R_\iota^n)\mathbb{1}_{\mathcal{A}}\right]\\ &\leq \frac{1}{2}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]+2\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})^2\right]^{1/2}\sqrt{\mathbb{P}_{\beta+\varepsilon}\left(\mathcal{A}\right)}\\ &\leq \frac{1}{2}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]+C'\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right]\sqrt{\mathbb{P}_{\beta+\varepsilon}\left(\mathcal{A}\right)},\end{aligned}$$

where the last inequality holds for some $C'<\infty$, by Lemma 4.2 and Lemma 4.5. Solving the previous line of inequalities for $\mathbb{P}_{\beta+\varepsilon}\left(\mathcal{A}\right)$ shows (44). □

Lemma 4.8. *For all $\beta\geq 0$ and all dimensions d, there exists a constant $c_1>0$ such that uniformly over all $n\in\mathbb{N}$ and all $x\in\mathcal{S}_n$*

$$\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},x)\right]\geq c_1\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)\mathbf{1})\right] \tag{45}$$

and the constant c_1 can be chosen in such a way so that it only depends on the dimension d and the value $\iota>0$ in (43).

Proof. Let $v\in\mathcal{S}_n$ be one of the minimizers of $y\mapsto\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},y)\right]$ among all vertices $y\in\mathcal{S}_n$. By reflection symmetry, we can assume that all coordinates of v are non-negative. With the notation $e_0=e_d$ we define the vectors $v_0,\dots,v_{d-1}$ by

$$\langle e_j,v_i\rangle=\langle e_{i+j \bmod d},v\rangle$$

which are just versions of the vector v, where we cyclically permuted the coordinates. By invariance under changes of coordinates, we have

$$\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},v)\right]=\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},v_i)\right]$$

for all $i\in\{0,\dots,d-1\}$. Define the vertices $u_0,\dots,u_d$ by $u_j=\sum_{i=1}^j v_i$. By our construction we have $u_0=\mathbf{0}$ and $u_d=\sum_{i=1}^d v_i=N\mathbf{1}$ for some integer $N\geq n$. The balls $B_n(u_i)$ are all contained in the cube $\Upsilon=\{-n,\dots,N+n\}^d$ for all $i\in\{0,\dots,d\}$. Thus we have

$$\mathbb{E}_\beta\left[D_\Upsilon(\mathbf{0},N\mathbf{1})\right]\leq\sum_{i=1}^{d-1}\mathbb{E}_\beta\left[D_\Upsilon(u_{i-1},u_{i-1}+v_i\mathbf{1})\right]\leq d\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},v)\right],$$

and by translation invariance we also have for the cube $\Upsilon_1=\{0,\dots,2n+N\}^d$

$$\mathbb{E}_\beta\left[D_{\Upsilon_1}(n\mathbf{1},(n+N)\mathbf{1})\right]\leq d\,\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},v)\right].$$

Using the triangle inequality, we see that for all $k\in\mathbb{N}$ the expected distance between $n\mathbf{1}$ and $(n+kN)\mathbf{1}$ inside the cube $\Upsilon_k=\{0,\dots,2n+kN\}^d$ is upper bounded by

$$\mathbb{E}_\beta\left[D_{\Upsilon_k}(n\mathbf{1},(n+kN)\mathbf{1})\right]\leq k\cdot d\,\mathbb{E}_\beta\left[D_{B_n(\mathbf{0})}(\mathbf{0},v)\right].$$

But Lemma 4.1 also gives that for $s = \frac{n}{kN+2n}$ and $w_1 = \lfloor sn\mathbf{1} \rfloor, w_2 = \lfloor s(n+kN)\mathbf{1} \rfloor$

$$\mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(w_1, w_2) \right] \leq 3k \cdot d \, \mathbb{E}_\beta \left[D_{B_n(\mathbf{0})}(\mathbf{0}, v) \right].$$

As $N \geq n$, for each fixed $\iota > 0$ we can choose k large enough so that $w_1 \in L_\iota^n$ and $w_2 \in R_\iota^n$ and thus $\mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(w_1, w_2) \right] \geq \mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(L_\iota^n, R_\iota^n) \right]$. Then we get by the lower bound on the expected distance between the boxes L_ι^n and R_ι^n (43) that for such a k

$$\begin{aligned} \mathbb{E}_\beta \left[D_{B_n(\mathbf{0})}(\mathbf{0}, v) \right] &\geq \frac{1}{3kd} \mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(w_1, w_2) \right] \geq \frac{1}{3kd} \mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(L_\iota^n, R_\iota^n) \right] \\ &\overset{(43)}{\geq} \frac{1}{6kd} \mathbb{E}_\beta \left[D_{V_\mathbf{0}^n}(\mathbf{0}, (n-1)\mathbf{1}) \right] \end{aligned}$$

which finishes the proof, as $v \in \mathcal{S}_n$ was assumed to minimize the expected distance $\mathbb{E}_\beta \left[D_{B_n(\mathbf{0})}(\mathbf{0}, y) \right]$ among all vertices $y \in \mathcal{S}_n$. ☐

Lemma 4.9. *For all dimensions d and all $\beta > 0$, there exists an $\eta \in \left(0, \frac{1}{2}\right)$ such that uniformly over all $n \in \mathbb{N}$ and all $x \in \mathcal{S}_n$*

$$\mathbb{E}_\beta \left[D_{B_n(\mathbf{0})} \left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x) \right) \right] \geq \frac{c_1}{2} \Lambda(n, \beta) \tag{46}$$

where c_1 is the constant from (45) and there exists a constant c_2 such that

$$\mathbb{P}_\beta \left(D_{B_n(\mathbf{0})} \left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x) \right) \geq \frac{c_1}{4} \Lambda(n, \beta) \right) \geq c_2. \tag{47}$$

Furthermore, for each $\beta \geq 0$ there exist constants $c_3 > 0$ such that

$$\mathbb{P}_\beta \left(D \left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C \right) \geq \frac{c_1}{4} \Lambda(n, \beta) \right) \geq c_3 \tag{48}$$

uniformly over all $n \in \mathbb{N}$.

Note that in the above lemma, for $x \in \mathcal{S}_n$ the box $B_{\eta n}(x)$ is not completely contained inside $B_n(\mathbf{0})$, but from the definition of $D_{B_n(\mathbf{0})}(\cdot, \cdot)$, we only consider the part that intersects $B_n(\mathbf{0})$.

Proof. Given the results of Lemma 4.8, the proof of (46) and (47) works in the same way as the proof of Lemma 4.7 and we omit it. Regarding the statement of (48), we will first prove that for $\eta > 0$ small enough

$$\mathbb{P}_\beta \left(D \left(B_{\eta n}(\mathbf{0}), B_n(\mathbf{0})^C \right) \geq \frac{c_1}{4} \Lambda(n, \beta) \right) \geq c_4 \tag{49}$$

for some constant $c_4 > 0$ and uniformly over all $n \in \mathbb{N}$. For this, we use the *FKG inequality*, see [62, Section 1.3] or [43, 56] for the original papers. We can cover the set $\bigcup_{x \in \mathcal{S}_n} B_{\eta n}(x)$ with uniformly (in n) finitely many sets of the form $B_{\eta n}(x)$. For example, we have

$$\bigcup_{\substack{x \in \mathcal{S}_n: \\ \langle x, e_1 \rangle = n}} B_{\eta n}(x) \subset \bigcup_{x \in F} B_{\eta n}(x)$$

where $F = \left\{ n e_1 + \sum_{i=2}^d k_i e_i : k_i \in \left\{ -\lceil \frac{n}{\lceil \eta n \rceil} \rceil, \ldots, \lceil \frac{n}{\lceil \eta n \rceil} \rceil \right\} \text{ for all } i \in \{2, \ldots, d\} \right\}$, and all other faces of the set $\bigcup_{x \in \mathcal{S}_n} B_{\eta n}(x)$ can be covered in a similar way. Suppose that $A_n' \subset \mathcal{S}_n$ is a sequence of finite sets with $\sup_n |A_n'| =: A' < \infty$ such that

$$\bigcup_{x \in \mathcal{S}_n} B_{\eta n}(x) = \bigcup_{x \in A_n'} B_{\eta n}(x)$$

for all $n \in \mathbb{N}$. So in particular we have

$$\begin{aligned}&\left\{D_{B_n(\mathbf{0})}\left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x)\right) \geq \frac{c_1}{4}\Lambda(n,\beta) \text{ for all } x \in \mathcal{S}_n\right\}\\ =&\left\{D_{B_n(\mathbf{0})}\left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x)\right) \geq \frac{c_1}{4}\Lambda(n,\beta) \text{ for all } x \in A_n'\right\}.\end{aligned}$$

The events $\left\{D_{B_n(\mathbf{0})}\left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x)\right) \geq \frac{c_1}{4}\Lambda(n,\beta)\right\}$ are decreasing for all $x \in \mathcal{S}_n$ in the sense that they are stable under the deletion of edges. Thus the FKG inequality and (47) imply that that

$$\begin{aligned}&\mathbb{P}_\beta\left(D_{B_n(\mathbf{0})}\left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x)\right) \geq \frac{c_1}{4}\Lambda(n,\beta) \text{ for all } x \in \mathcal{S}_n\right)\\ =&\mathbb{P}_\beta\left(D_{B_n(\mathbf{0})}\left(B_{\eta n}(\mathbf{0}), B_{\eta n}(x)\right) \geq \frac{c_1}{4}\Lambda(n,\beta) \text{ for all } x \in A_n'\right) \geq c_2^{|A_n'|} \geq c_2^{A'}.\end{aligned}$$

Assume that there is no direct edge from $[-(n-\eta n),(n-\eta n)]^d$ to $\mathbb{Z}^d \setminus [-n,n]^d$. This has a uniform positive probability in n and is also a decreasing event. Then any path from $B_{\eta n}(\mathbf{0})$ to $B_n(\mathbf{0})^C$ goes through at least one box $B_{\eta n}(x) \cap B_n(\mathbf{0})$ for some $x \in \mathcal{S}_n$. So with another application of the FKG inequality we get that

$$\mathbb{P}_\beta\left(D\left(B_{\eta n}(\mathbf{0}), B_n(\mathbf{0})^C\right) \geq \frac{c_1}{4}\Lambda(n,\beta)\right) \geq c_5$$

for some $c_5 > 0$ and uniformly over all $n \in \mathbb{N}$. Next, let $A_n \subset B_n(\mathbf{0})$ be a sequence of sets such that $\bigcup_{x \in A_n} B_{\eta n}(x) = B_n(\mathbf{0})$ and $\sup_n |A_n| =: A < \infty$. Then $D\left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda(n,\beta)$ already implies that there exists a point $x \in A_n$ such that $D\left(B_{\eta n}(x), B_n(x)^C\right) < \frac{c_1}{4}\Lambda(n,\beta)$. By another application of the FKG inequality we have

$$\begin{aligned}&\mathbb{P}_\beta\left(D\left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C\right) \geq \frac{c_1}{4}\Lambda(n,\beta)\right)\\ &\geq \mathbb{P}_\beta\left(D\left(B_{\eta n}(x), B_n(x)^C\right) \geq \frac{c_1}{4}\Lambda(n,\beta) \text{ for all } x \in A_n\right) \geq c_5^{|A_n|} \geq c_5^{A}\end{aligned}$$

which proves (48). □

Lemma 4.10. *For all $\beta \geq 0$ and all $\varepsilon > 0$, there exist $0 < c < C < \infty$ such that*

$$\mathbb{P}_\beta\left(c\Lambda(n,\beta) \leq D\left(\mathbf{0}, B_n(\mathbf{0})^C\right) \leq C\Lambda(n,\beta)\right) > 1-\varepsilon \tag{50}$$

for all $n \in \mathbb{N}$.

Similar statements for one dimension and the continuous model were already proven in [33]. We follow a similar strategy here.

Proof. By the inequality $D\left(\mathbf{0}, B_n(\mathbf{0})^C\right) \leq D_{V_{\mathbf{0}}^{n+2}}\left(\mathbf{0}, (n+1)\mathbf{1}\right)$ we get that

$$\mathbb{E}_\beta\left[D\left(\mathbf{0}, B_n(\mathbf{0})^C\right)\right] \leq \Lambda(n+2,\beta) \leq \Lambda(n,\beta)+2.$$

Using Markov's inequality we see that

$$\mathbb{P}_\beta\left(D(\mathbf{0}, B_n(\mathbf{0})^C) > C\Lambda(n,\beta)\right) \leq \frac{\Lambda(n,\beta)+2}{C\Lambda(n,\beta)},$$

and thus the probability $\mathbb{P}_\beta\left(D(\mathbf{0}, B_n(\mathbf{0})^C) \leq C\Lambda(n,\beta)\right)$ can be made arbitrarily close to 1 by taking C large enough. We will also refer to this case as the upper bound. The probability of the lower bound $\mathbb{P}_\beta\left(c\Lambda(n,\beta)\right) \leq D(\mathbf{0}, B_n(\mathbf{0})^C)$ can be made arbitrarily close to 1 for small n by taking c small enough. So we will always focus on n large enough from here on. Fix $K, N \in \mathbb{N}_{>1}$ such that the function $i \mapsto \Lambda\left(K^{2i}N, \beta\right)$ is increasing in i. This is possible by Remark 4.4. We now consider boxes of the form $B_{K^{2i}N}(\mathbf{0})$. The probability of a direct edge from $B_{K^{2(i-1)}N}(\mathbf{0})$ to $B_{K^{2i}N}(\mathbf{0})^C$ equals the probability of a direct edge between $\mathbf{0}$ and $B_{K^2}(\mathbf{0})^C$, and is by (6) bounded by $\beta 50^d K^{-2}$. So the probability that there is some $i \in \{1, \ldots, K\}$ for which there is a direct edge from $B_{K^{2(i-1)}N}(\mathbf{0})$ to $B_{K^{2i}N}(\mathbf{0})^C$ is bounded by $\beta 50^d K^{-1}$. We denote the complement of this event by $\mathcal{A}$. Conditioned on the event $\mathcal{A}$, where there exists no edge between $B_{K^{2(i-1)}N}(\mathbf{0})$ and $B_{K^{2i}N}(\mathbf{0})^C$ for all $i \in \{1, \ldots, K\}$, each path from $B_N(\mathbf{0})$ to $B_{K^{2K}N}(\mathbf{0})$ needs to cross all the distances from $B_{K^{2(i-1)}N}(\mathbf{0})$ to $B_{2K^{2(i-1)}N}(\mathbf{0})^C$. For odd i, these distances are independent. Remember that $i \mapsto \Lambda\left(K^{2i}N, \beta\right)$ is increasing in i. So conditioned on the event $\mathcal{A}$ we have the bound

$$\begin{aligned}
&\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda(N,\beta)\Big|\mathcal{A}\right) \\
&\leq \mathbb{P}_\beta\left(D\left(B_{K^{2(i-1)}N}(\mathbf{0}), B_{2K^{2(i-1)}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda\left(K^{2(i-1)}N,\beta\right) \forall i \in \{1,\ldots,K\} \text{ odd}\Big|\mathcal{A}\right) \\
&= \prod_{\substack{i=1:\\ i \text{ odd}}}^{k} \mathbb{P}_\beta\left(D\left(B_{K^{2(i-1)}N}(\mathbf{0}), B_{2K^{2(i-1)}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda\left(K^{2(i-1)}N,\beta\right)\Big|\mathcal{A}\right) \\
&\leq \prod_{\substack{i=1:\\ i \text{ odd}}}^{k} \mathbb{P}_\beta\left(D\left(B_{K^{2(i-1)}N}(\mathbf{0}), B_{2K^{2(i-1)}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda\left(K^{2(i-1)}N,\beta\right)\right) \leq (1-c_3)^{\lfloor \frac{K}{2} \rfloor},
\end{aligned}$$

where the second last inequality holds because of FKG, as events of the form $\{D(\cdot,\cdot) < x\}$ are increasing and $\mathcal{A}$ is decreasing, and where c_3 is the constant from (48). Thus we have that

$$\begin{aligned}
&\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda(N,\beta)\right) \\
&\leq \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda(N,\beta)\Big|\mathcal{A}\right) + \mathbb{P}_\beta\left(\mathcal{A}^C\right) \leq (1-c_3)^{\lfloor \frac{K}{2} \rfloor} + \beta 50^d K^{-1}
\end{aligned}$$

and this quantity can be made arbitrary small by suitable choice of K. To finish the proof, remember that $\Lambda(N,\beta)$ and $\Lambda\left(K^{2K}N,\beta\right)$ are off by a factor of at most K^{2K}, as

$$\Lambda(N,\beta) \leq \Lambda\left(K^{2K}N,\beta\right) \overset{(10)}{\leq} \Lambda\left(K^{2K},\beta\right)\Lambda(N,\beta) \leq K^{2K}\Lambda(N,\beta).$$

Thus we have

$$\begin{aligned}
&\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4K^{2K}}\Lambda(K^{2K}N,\beta)\right) \leq \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4}\Lambda(N,\beta)\right) \\
&\leq (1-c_3)^{\lfloor \frac{K}{2} \rfloor} + \beta 50^d K^{-1}.
\end{aligned}$$

Now, for fixed $\varepsilon > 0$, take K large enough so that $(1-c_3)^{\lfloor \frac{K}{2} \rfloor} + \beta 50^d K^{-1} < \varepsilon$. For $n \in \mathbb{N}$ large enough with $n > K^{2K}$, let N be the largest integer for which $K^{2K}N \leq n$. We know that $K^{2K}N \leq n \leq K^{2K}2N$ and this also yields, by Lemma 4.1, that

$$\Lambda(n,\beta) \leq 3\Lambda\left(K^{2K}2N,\beta\right) \leq 6\Lambda\left(K^{2K}N,\beta\right)$$

which already implies

$$\begin{aligned}
&\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_n(\mathbf{0})^C\right) < \frac{c_1}{24K^{2K}}\Lambda(n,\beta)\right) \leq \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_n(\mathbf{0})^C\right) < \frac{c_1}{4K^{2K}}\Lambda(K^{2K}N,\beta)\right) \\
&\leq \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{K^{2K}N}(\mathbf{0})^C\right) < \frac{c_1}{4K^{2K}}\Lambda(K^{2K}N,\beta)\right) \leq (1-c_3)^{\lfloor \frac{K}{2} \rfloor} + \beta 50^d K^{-1} \leq \varepsilon.
\end{aligned}$$

□

The previous lemma tells us that for fixed $\beta > 0$ all quantiles of $D\left(\mathbf{0}, B_n(\mathbf{0})^C\right)$ are of order $\Lambda(n,\beta)$. We want to prove a similar statement for the quantiles of $D\left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C\right)$. However, an analogous statement can not be true, as there is a uniform positive probability of a direct edge between $B_n(\mathbf{0})$ and $B_{2n}(\mathbf{0})^C$. But if we condition on the event that there is no such direct edge, the statement still holds.

Lemma 4.11. *Let $\mathcal{L}$ be the event that there is no direct edge between $B_n(\mathbf{0})$ and $B_{2n}(\mathbf{0})^C$. For all $\beta > 0$ and all $\varepsilon > 0$, there exist $0 < c < C < \infty$ such that*

$$\mathbb{P}_\beta\left(c\Lambda(n,\beta) \leq D(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C) \leq C\Lambda(n,\beta) \;\middle|\; \mathcal{L}\right) > 1 - \varepsilon \tag{51}$$

for all $n \in \mathbb{N}$.

Proof. From Markov's inequality we know that

$$\begin{aligned}
&\mathbb{E}_\beta\left[D(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C)\right] \leq \mathbb{E}_\beta\left[D(n\mathbf{1}, (2n+1)\mathbf{1})\right] \\
&\leq \mathbb{E}_\beta\left[D_{V_1^n}(n\mathbf{1}, (2n-1)\mathbf{1})\right] + 2 \leq \Lambda(n,\beta) + 1,
\end{aligned}$$

and thus the probability $\mathbb{P}_\beta\left(D(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C) \leq C\Lambda(n,\beta) \;\middle|\; \mathcal{L}\right)$ can be made arbitrarily close to 1 by taking C large enough. For the lower bound, we first consider integers of the form $N_k = M^k N_0$, where we fix $M \in \mathbb{N}$ first. Let M be the smallest natural number such that $M \geq 100$ and $\Lambda(M,\beta) \leq \frac{M}{10}$. The inequality $\Lambda(M,\beta) \leq \frac{M}{10}$ holds for large enough M, as $\Lambda(M,\beta)$ asymptotically grows like a power of M that is strictly less than one, see section 2.2. As β is fixed for the rest of the proof, we simply write $\Lambda(n)$ for $\Lambda(n,\beta)$. We write C_n for the annulus $B_{2n}(\mathbf{0}) \setminus B_n(\mathbf{0})$. Let $\mathcal{A}_\delta$ denote the event that for all vertices $x \in C_{MN}$ for which there exists an edge $e = \{x, y\}$ with $\|x-y\|_\infty \geq N$ one has $D\left(x, C_{MN}^C; \omega^{e-}\right) \geq \delta\Lambda(MN)$. We will now show that the probability of the event $\mathcal{A}_\delta$ converges to 1 as $\delta \to 0$. Remember that $\Lambda(MN)$ and $\Lambda(N)$ differ by a factor of at most M. Let us first consider the event that for some $\delta_1 > 0$ there exists a vertex incident to a long edge in one of the boundary regions of thickness $\delta_1 N$ of C_{MN}, Formally, for $\delta_1 \in \left(0, \frac{1}{2}\right)$, we define the boundary region $\partial^{\delta_1} C_{MN}$ of C_{MN} by

$$\partial^{\delta_1} C_{MN} = \{B_{MN+\delta_1 N}(\mathbf{0}) \setminus B_{MN}(\mathbf{0})\} \cup \{B_{2MN}(\mathbf{0}) \setminus B_{2MN-\delta_1 N}(\mathbf{0})\}.$$

The set $\partial^{\delta_1} C_{MN}$ has a size of at most $4d\delta_1 N\left(5MN\right)^{d-1}$, as one needs to fix one of the coordinates within the interval $(MN, MN+\delta_1 N]$, respectively in the interval $(2MN - \delta_1 N, 2MN]$, or one of the reflections of these intervals, and then has at most $4MN+1$ possibilities for each of the remaining $d-1$ coordinates. Combining this gives

$$\left|\partial^{\delta_1} C_{MN}\right| \leq (4d\delta_1 N)(4MN+1)^{d-1} \leq 4d\delta_1 N(5MN)^{d-1}.$$

The probability that a vertex is incident to some edge of length $\geq N$ is proportional to $\frac{\beta}{N^d}$ as shown in (6). So together with (6) we get that

$$\begin{aligned}
\mathbb{P}_\beta\left(\exists x \in \partial^{\delta_1} C_{MN}, y \in B_{N-1}(x)^C : x \sim y\right) &\leq 4d\delta_1 N(5MN)^{d-1}\mathbb{P}_\beta\left(\mathbf{0} \sim \mathcal{S}_{\geq N}\right) \\
&\leq \delta_1 \cdot 4d(5MN)^d \beta 50^d N^{-d} \leq \delta_1 \cdot \beta\left(10^3 M\right)^d.
\end{aligned}$$

Furthermore, the expected number of points $x \in C_{MN}$ which are incident to a long edge is bounded by

$$\mathbb{E}_\beta\left[\left|\{x \in C_{MN} : x \sim B_{N-1}(x)^C\}\right|\right] \leq \sum_{x \in C_{MN}} \sum_{y \in B_{N-1}(x)^C} \mathbb{P}_\beta(x \sim y)$$

$$\leq |C_{MN}| \sum_{y \in B_{N-1}(\mathbf{0})^C} \mathbb{P}_\beta(\mathbf{0} \sim y) \overset{(9)}{\leq} (5MN)^d \beta 50^d N^{-d} \leq \beta(250M)^d. \tag{52}$$

where the second last inequality holds as $|C_{MN}| \leq (4MN+1)^d \leq (5MN)^d$, and because the sum $\sum_{y \in B_{N-1}(\mathbf{0})^C} \mathbb{P}_\beta(\mathbf{0} \sim y)$ can be upper bounded by $\beta 50^d N^{-d}$ in the exact same way as in (9). As the existence of an edge $\{x,y\}$ with $|\{x,y\}| \geq N$ and the distance $D\left(x, B_{\delta_1 N}(x)^C; \omega^{\{x,y\}-}\right)$ are independent random variables, we get with a union bound that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\exists x \in C_{MN}, y \in B_{N-1}(x)^C : x \sim y, D\left(x, B_{\delta_1 N}(x)^C; \omega^{\{x,y\}-}\right) < \delta\Lambda(MN)\right) \\
&\leq \sum_{x \in C_{MN}} \sum_{y \in B_{N-1}(x)^C} \mathbb{P}_\beta(x \sim y) \mathbb{P}_\beta\left(D\left(x, B_{\delta_1 N}(x)^C\right) < \delta\Lambda(MN)\right) \\
&\leq \beta(250M)^d \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{\delta_1 N}(\mathbf{0})^C\right) < \delta\Lambda(MN)\right)
\end{aligned}$$

where we used (52) for the last inequality. Thus we also get that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\mathcal{A}_\delta^C\right) = \mathbb{P}_\beta\left(\exists x \in C_{MN}, y \in B_{N-1}(x)^C : x \sim y, D\left(x, C_{MN}^C; \omega^{\{x,y\}-}\right) < \delta\Lambda(MN)\right) \\
&\leq \mathbb{P}_\beta\left(\exists x \in \partial^{\delta_1} C_{MN}, y \in B_{N-1}(x)^C : x \sim y\right) \\
&\quad + \mathbb{P}_\beta\left(\exists x \in C_{MN}, y \in B_{N-1}(x)^C : x \sim y, D\left(x, B_{\delta_1 N}(x)^C; \omega^{\{x,y\}-}\right) < \delta\Lambda(MN)\right) \\
&\leq \delta_1 \beta\left(10^3 M\right)^d + \beta(250M)^d \mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{\delta_1 N}(\mathbf{0})^C\right) < \delta\Lambda(MN)\right)
\end{aligned} \tag{53}$$

and this converges to 0 as $\delta \to 0$, for an appropriate choice of $\delta_1(\delta)$, by Lemma 4.10 uniformly over $N \in \mathbb{N}$. We write $f(\delta)$ for the supremum of $\mathbb{P}_\beta\left(\mathcal{A}_\delta^C\right)$ over all $N \in \mathbb{N}$ and for $A, B \subset V$, we write $D^\star(A,B)$ for the indirect distance between A and B, i.e., the length of the shortest path between A and B that does not use a direct edge between A and B. Now assume that $D^\star\left(B_{MN}(\mathbf{0}), B_{2MN}(\mathbf{0})^C\right) < \delta\Lambda(MN)$. We now consider the path between $B_{MN}(\mathbf{0})$ and $B_{2MN}(\mathbf{0})^C$ that achieves this distance. Either this path uses some long edge (of length greater than $N-1$), or it only jumps from one block of the form V_v^N to directly neighboring blocks. The probability that there exists a point $x \in C_{MN}$ and a long edge e incident to it such that $D\left(x, C_{MN}^C; \omega^{e-}\right) < \delta\Lambda(MN)$ is relatively small by (53). Any path that does not use long edges can only do jumps between neighboring blocks of the form V_v^N. Say the path uses the blocks $\left(V_{v_i'}^N\right)_{i=0}^{L'}$. Consider the loop-erased trace of this walk on the blocks, i.e., say that the path uses the blocks $\left(V_{v_i}^N\right)_{i=0}^{L} \subset C_{MN}$ with $\|v_i - v_{i-1}\|_\infty = 1$ and never returns to $V_{v_i}^N$ after going to $V_{v_{i+1}}^N$. There need to be at least $\frac{M}{3}$ transitions between blocks of the form $V_{u_i}^N$ and $V_{w_i}^N$ with $\|u_i - w_i\|_\infty = 2$ and $u_i, w_i \in \{v_0, \ldots, v_L\}$, as the path needs to walk a distance in the infinity-norm of at least MN. So in particular we have

$$\begin{aligned}
\sum_{i=1}^{\lceil M/3 \rceil} D^\star_{B_{2MN}(\mathbf{0})}\left(V_{u_i}^N, V_{w_i}^N\right) &\leq D^\star_{B_{2MN}(\mathbf{0})}\left(B_{MN}(\mathbf{0}), B_{2MN}(\mathbf{0})^C\right) \\
&< \delta\Lambda(MN,\beta) \leq \delta\Lambda(M,\beta)\Lambda(N,\beta) \leq \frac{M}{10}\delta\Lambda(N,\beta)
\end{aligned}$$

where we used our assumption on M for the last step. So in particular there need to be at least two transitions between $V_{u_i}^N$ and $V_{w_i}^N$ that satisfy $D_{B_{2MN}(\mathbf{0})}\left(V_{u_i}^N, V_{w_i}^N\right) < \delta\Lambda(N,\beta)$. In fact, there need to be some linear number in M many such transitions, but two are sufficient for our purposes here. All these transitions need to be disjoint, as shortest paths never use the same edge twice. Thus we get by the *BK inequality* (see [62, Section 1.3] or [15, 87]) that

$$\mathbb{P}_\beta\left(\sum_{i=1}^{\lceil M/3\rceil} D^\star_{B_{2MN}(\mathbf{0})}\left(V^N_{u_i}, V^N_{w_i}\right) \leq \frac{M}{10}\delta\Lambda(N)\right) \leq M^2\left(\min_i \mathbb{P}_\beta\left(D^\star_{B_{2MN}(\mathbf{0})}\left(V^N_{u_i}, V^N_{w_i}\right) < \delta\Lambda(N)\right)\right)^2.$$

For each combination of vectors u_i, w_i with $\|u_i - w_i\|_\infty = 2$, we can translate and rotate the boxes $V^N_{u_i}$ and $V^N_{w_i}$ to boxes $T\left(V^N_{u_i}\right)$ and $T\left(V^N_{w_i}\right)$ in such a way that $T\left(V^N_{u_i}\right) \subset B_N(\mathbf{0})$ and $T\left(V^N_{w_i}\right) \subset B_{2N}(\mathbf{0})^C$. By translational and rotational invariance of our long-range percolation model, this already implies that

$$\min_i \mathbb{P}_\beta\left(D^\star_{B_{2MN}(\mathbf{0})}\left(V^N_{u_i}, V^N_{w_i}\right) < \delta\Lambda(N)\right) \leq \mathbb{P}_\beta\left(D^\star_{B_{2N}(\mathbf{0})}\left(B_N(\mathbf{0}), B_{2N}(\mathbf{0})^C\right) < \delta\Lambda(N)\right).$$

There are at most $\left((5M)^d\right)!$ choices for possible choice of vertices $v_0, v_1, \ldots, v_L$, as there are at most $(5M)^d$ possibilities for v_0 and $(5M)^d - 1$ possibilities for v_1 and so on. Overall we see that the probability that there exists an indirect path between $B_{MN}(\mathbf{0})$ and $B_{2MN}(\mathbf{0})^C$ of length $\delta\Lambda(MN)$, which jumps between neighboring blocks of the form V^N_v only, is bounded by

$$\left(5M^d\right)! M^2 \mathbb{P}_\beta\left(D^\star_{B_{2N}(\mathbf{0})}\left(B_N(\mathbf{0}), B_{2N}(\mathbf{0})^C\right) < \delta\Lambda(N)\right)^2.$$

We write S for the constant $\left(5M^d\right)! M^2$. Thus we get that

$$\begin{aligned}&\mathbb{P}_\beta\left(D^\star_{B_{2MN}(\mathbf{0})}\left(B_{MN}(\mathbf{0}), B_{2MN}(\mathbf{0})^C < \delta\Lambda(MN)\right)\right)\\ &\quad\leq S\mathbb{P}_\beta\left(D^\star_{B_{2N}(\mathbf{0})}\left(B_N(\mathbf{0}), B_{2N}(\mathbf{0})^C\right) < \delta\Lambda(N)\right)^2 + \mathbb{P}_\beta\left(\mathcal{A}^C_\delta\right).\end{aligned}$$

We define the sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_0 = \mathbb{P}_\beta\left(D^\star_{B_{2N}(\mathbf{0})}\left(B_N(\mathbf{0}), B_{2N}(\mathbf{0})^C\right) < \delta\Lambda(N)\right)$$

and $a_{n+1} = Sa_n^2 + f(\delta)$. Inductively it follows that

$$\mathbb{P}_\beta\left(D^\star_{B_{2M^kN}(\mathbf{0})}\left(B_{M^kN}(\mathbf{0}), B_{2M^kN}(\mathbf{0})^C\right) < \delta\Lambda(M^kN)\right) \leq a_k$$

for all $k \in \mathbb{N}$. For $f(\delta) < \frac{1}{4S}$, the equation $a = Sa^2 + f(\delta)$ has the two solutions

$$a_- = \frac{1 - \sqrt{1 - 4Sf(\delta)}}{2S} \text{ and } a_+ = \frac{1 + \sqrt{1 - 4Sf(\delta)}}{2S} > \frac{1}{2S}.$$

For $a_0 \in [0, a_+)$, and thus in particular for $a_0 \in \left[0, \frac{1}{2S}\right]$, the sequence a_n converges to $a_- = \frac{1-\sqrt{1-4Sf(\delta)}}{2S} \approx f(\delta)$ and thus we get

$$\limsup_{k\to\infty} \mathbb{P}_\beta\left(D^\star_{B_{2M^kN}(\mathbf{0})}\left(B_{M^kN}(\mathbf{0}), B_{2M^kN}(\mathbf{0})^C\right) < \delta\Lambda(M^kN)\right) \leq a_-.$$

For fixed $N \in \mathbb{N}$, the requirement

$$a_0 = \mathbb{P}_\beta\left(D^\star_{B_{2N}(\mathbf{0})}\left(B_N(\mathbf{0}), B_{2N}(\mathbf{0})^C\right) < \delta\Lambda(N)\right) \leq \frac{1}{2S}$$

is satisfied for small enough $\delta > 0$ and this shows (51) along the subsequence $N_k = M^kN$. To get the statement for all integer numbers, one can use Lemma 4.1 and the fact that $\Lambda(n) \leq \Lambda(mn) \leq m\Lambda(n)$ for all integers m, n. □

With the same technique as above one can also prove that the indirect distance between $V_{\mathbf{0}}^n$ and the set $B_n\left(V_{\mathbf{0}}^n\right)^C = \left\{x \in \mathbb{Z}^d : D_\infty(x, V_{\mathbf{0}}^n) > n\right\} = \bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2} V_u^n$ scales like $\Lambda(n,\beta)$.

Corollary 4.12. *For all $\beta \geq 0$ and $\varepsilon > 0$ there exist $0 < c_\varepsilon < C_\varepsilon < \infty$ such that*

$$\mathbb{P}_\beta\left(c_\varepsilon \Lambda(n,\beta) \leq D^\star\left(V_{\mathbf{0}}^n, B_n\left(V_{\mathbf{0}}^n\right)^C\right) \leq C_\varepsilon \Lambda(n,\beta)\right) \geq 1 - \varepsilon \ .$$

5 The proof of Theorem 1.1

We first give an outline of the proof of Theorem 1.1. In Lemma 4.10, we showed that $D\left(\mathbf{0}, B_n\left(\mathbf{0}\right)^C\right) \approx_P \Lambda(n,\beta)$, and Lemma 4.2 shows that $\Lambda(n,\beta) \approx \mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1)\right]$, meaning that the ratio of these two expressions is uniformly bounded from below and above by constants $0 < c < C < \infty$. In Lemma 5.5 below we prove supermultiplicativity of $\Lambda(n,\beta)$. Together with the submultiplicativity proven in Lemma 2.3 this shows that for each $\beta \geq 0$ there exists $c_\beta > 0$ such that $c_\beta \Lambda(m,\beta)\Lambda(n,\beta) \leq \Lambda(mn,\beta) \leq \Lambda(m,\beta)\Lambda(n,\beta)$. We define $a_k = \log\left(\Lambda(2^k,\beta)\right)$. The sequence is subadditive and thus

$$\theta(\beta) = \lim_{k\to\infty} \frac{\log(\Lambda(2^k,\beta))}{\log(2^k)} = \lim_{k\to\infty} \frac{a_k}{\log(2)k} = \inf_{k\in\mathbb{N}} \frac{a_k}{\log(2)k}$$

exists, where the last inequality holds because of Fekete's Lemma. On the other hand, the sequence $b_k = \log(c_\beta \Lambda(2^k,\beta))$ satisfies

$$b_{k+l} = \log(c_\beta \Lambda(2^{k+l},\beta)) \geq \log(c_\beta \Lambda(2^k,\beta) c_\beta \Lambda(2^l,\beta)) = b_k + b_l$$

and thus

$$\theta(\beta) = \lim_{k\to\infty} \frac{\log(c_\beta \Lambda(2^k,\beta))}{\log(2^k)} = \lim_{k\to\infty} \frac{b_k}{\log(2)k} = \sup_{k\in\mathbb{N}} \frac{b_k}{\log(2)k}.$$

This already implies that

$$2^{k\theta(\beta)} \leq \Lambda(2^k,\beta) \leq c_\beta^{-1} 2^{k\theta(\beta)}$$

for all $k \in \mathbb{N}$. These two inequalities can be extended from points of the form 2^k to all integers with Lemma 4.1. So there exists a constant $0 < C'_\beta < \infty$ such that for all $n \in \mathbb{N}$

$$\frac{1}{C'_\beta} n^{\theta(\beta)} \leq \Lambda(n,\beta) \leq C'_\beta n^{\theta(\beta)}.$$

which shows (1). So we still need to prove supermultiplicativity of $\Lambda(\cdot,\beta)$ in order to prove the first item in Theorem 1.1. The second item of Theorem 1.1, i.e., the bounds on the diameter of cubes (2), we show in section 5.3.

5.1 Distances between certain points

In this chapter, we examine the typical behavior of distances between points that are connected to long edges. In Lemma 5.1, we consider the infinity distance between such points. Using a coupling argument with the continuous model, we compare the situation to the situations occurring in Lemma 2.4 and Lemma 2.5. Then, in Lemma 5.2 we translate these bounds on the infinity distance into bounds on the typical graph distance between points that are incident to long edges.

Fix the three blocks V_u^n, V_w^n and $V_{\mathbf{0}}^n$ with $\|u\|_\infty \geq 2$. The next lemma deals with the infinity distance between points $x, y \in V_{\mathbf{0}}^n$ with $x \sim V_u^n, y \sim V_w^n$.

Lemma 5.1. *For all* $\frac{1}{n} < \varepsilon \leq \frac{1}{4}$ *and* $u, w \in \mathbb{Z}^d \setminus \{0\}$ *with* $\|u\|_\infty \geq 2$ *one has*

$$\mathbb{P}_\beta\big(\exists x, y \in V_{\mathbf{0}}^n : \|x-y\|_\infty \leq \varepsilon n, x \sim V_u^n, y \sim V_w^n \;\big|\; V_{\mathbf{0}}^n \sim V_u^n, V_{\mathbf{0}}^n \sim V_w^n\big) \leq C_d' \varepsilon^{1/2} \lceil \beta \rceil^2$$

where C_d' *is a constant that depends only on the dimension* d.

Proof. Let $\mathcal{E}$ be the symmetrized Poisson process constructed in subsection 1.2 about the continuous model, i.e., $\tilde{\mathcal{E}}$ is a Poisson process on $\mathbb{R}^d \times \mathbb{R}^d$ with intensity $\frac{\beta}{2\|t-s\|^{2d}}$ and $\mathcal{E}$ is defined by $\mathcal{E} := \left\{(s,t) \in \mathbb{R}^d \times \mathbb{R}^d : (t,s) \in \tilde{\mathcal{E}}\right\} \cup \tilde{\mathcal{E}}$. Now we place an edge between $x, y \in \mathbb{Z}^d$ if and only if

$$(x + \mathcal{C}) \times (y + \mathcal{C}) \cap n\mathcal{E} \neq \emptyset$$

and call this graph $G = (V, E)$. The distribution of the resulting graph is identical to $\mathbb{P}_\beta$ by the dilation invariance of $\mathcal{E}$. We can do the same procedure for $\lfloor \frac{1}{2\varepsilon} - 1 \rfloor \mathcal{E}$, i.e., place an edge between $x', y' \in \mathbb{Z}^d$ if and only if

$$(x' + \mathcal{C}) \times (y' + \mathcal{C}) \cap \left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor \mathcal{E} \neq \emptyset$$

and call the resulting graph $G' = (V', E')$. Now assume that in the graph G there exist $x, y \in V_{\mathbf{0}}^n$ with $\|x-y\|_\infty \leq \varepsilon n$ such that $x \sim V_u^n$ and $y \sim V_w^n$ in G. Then there exist

$$x_c \in x + \mathcal{C}, u_c \in nu + [0,n)^d, y_c \in y + \mathcal{C}, w_c \in nw + [0,n)^d$$

with $(x_c, u_c), (y_c, w_c) \in n\mathcal{E}$. We also have $\|x_c - y_c\|_\infty \leq \varepsilon n + 1 \leq 2\varepsilon n$. Now we rescale the process from size n to size $\lfloor \frac{1}{2\varepsilon} - 1 \rfloor$. For $(\tilde{x_c}, \tilde{u_c}) = \frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n}(x_c, u_c)$ and $(\tilde{y_c}, \tilde{w_c}) = \frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n}(y_c, w_c)$ we have

$$(\tilde{x_c}, \tilde{u_c}) \in \left(\left(\frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n} x + \left[0, \frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n}\right)^d\right) \times \left(\left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor u + \left[0, \left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor\right)^d\right)\right) \cap \left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor \mathcal{E},$$
$$(\tilde{y_c}, \tilde{w_c}) \in \left(\left(\frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n} y + \left[0, \frac{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}{n}\right)^d\right) \times \left(\left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor w + \left[0, \left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor\right)^d\right)\right) \cap \left\lfloor \frac{1}{2\varepsilon} - 1 \right\rfloor \mathcal{E}.$$

From the rescaling we also have $\|\tilde{x_c} - \tilde{y_c}\|_\infty \leq 2\varepsilon \lfloor \frac{1}{2\varepsilon} - 1 \rfloor < 1$. So in particular there are vertices $x', y' \in \left\{0, \ldots, \lfloor \frac{1}{2\varepsilon} - 1 \rfloor - 1\right\}^d$ with $x' \sim V_u^{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}, y' \sim V_w^{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor}$ in G', and $\|x' - y'\|_\infty \leq 1$. Write $N = \lfloor \frac{1}{2\varepsilon} - 1 \rfloor$. From (18) and (19) we get

$$\begin{aligned}
&\mathbb{P}_\beta\left(\exists x, y \in V_{\mathbf{0}}^N : \|x-y\|_\infty \leq 1, x \sim V_u^N, y \sim V_w^N \;\middle|\; V_{\mathbf{0}}^N \sim V_u^N, V_{\mathbf{0}}^N \sim V_w^N\right) \\
&\leq \frac{C_d \lceil \beta \rceil^2}{N^{1/2}} = \frac{C_d \lceil \beta \rceil^2}{\lfloor \frac{1}{2\varepsilon} - 1 \rfloor^{1/2}} \leq C_d' \varepsilon^{1/2} \lceil \beta \rceil^2
\end{aligned}$$

for some $C_d' < \infty$. With the coupling argument from before we thus also get

$$\begin{aligned}
&\mathbb{P}_\beta\left(\exists x, y \in V_{\mathbf{0}}^n : \|x-y\|_\infty \leq \varepsilon n, x \sim V_u^n, y \sim V_w^n \;\middle|\; V_{\mathbf{0}}^n \sim V_u^n, V_{\mathbf{0}}^n \sim V_w^n\right) \\
&\leq \mathbb{P}_\beta\left(\exists x, y \in V_{\mathbf{0}}^N : \|x-y\|_\infty \leq 1, x \sim V_u^N, y \sim V_w^N \;\middle|\; V_{\mathbf{0}}^N \sim V_u^N, V_{\mathbf{0}}^N \sim V_w^N\right) \leq C_d' \varepsilon^{1/2} \lceil \beta \rceil^2.
\end{aligned}$$

□

Lemma 5.2. *For all dimensions d and all $\beta \geq 0$, there exists a function $g_1(\varepsilon)$ with $g_1(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 1$ such that for all $u, w \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ with $\|u\|_\infty \geq 2$ and all large enough $n \geq n(\varepsilon)$*

$$\mathbb{P}_\beta\big(D_{V_\mathbf{0}^n}(x,y) > \varepsilon\Lambda(n,\beta) \text{ for all } x,y \in V_\mathbf{0}^n \text{ with } x \sim V_u^n, y \sim V_w^n \;\big|\; V_u^n \sim V_\mathbf{0}^n \sim V_w^n\big) \geq g_1(\varepsilon).$$

Proof. We write $\mathbb{P}_\beta^{u,w}(\cdot)$ for the conditional probability measure $\mathbb{P}_\beta\big(\cdot \;\big|\; V_u^n \sim V_\mathbf{0}^n \sim V_w^n\big)$. As β is fixed throughout the rest of the proof, we write $\Lambda(n)$ for $\Lambda(n,\beta)$. We define the event

$$\begin{aligned}&\mathcal{A}(K,\varepsilon_1,\varepsilon) = \{\|x-y\|_\infty > \varepsilon_1 n \text{ for all } x,y \in V_\mathbf{0}^n \text{ with } x \sim V_u^n, y \sim V_w^n\}\\ &\cap \left\{D_{V_\mathbf{0}^n}\left(x, B_{\varepsilon_1 n}(x)^C\right) > \varepsilon\Lambda(n) \text{ for all } x \in V_\mathbf{0}^n \text{ with } x \sim V_u^n\right\} \cap \{|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| \leq K\}\end{aligned}$$

and observe that

$$\left\{D_{V_\mathbf{0}^n}(x,y) > \varepsilon\Lambda(n) \text{ for all } x,y \in V_\mathbf{0}^n \text{ with } x \sim V_u^n, y \sim V_w^n\right\} \supset \{\mathcal{A}(K,\varepsilon_1,\varepsilon)\}.$$

Thus it suffices to show that $\mathbb{P}_\beta^{u,w}(\mathcal{A}(K,\varepsilon_1,\varepsilon))$ converges to 1 as $\varepsilon \to 0$ for an appropriate choice of $K = K(\varepsilon), \varepsilon_1 = \varepsilon_1(\varepsilon)$. Respectively, we want to show that $\mathbb{P}_\beta^{u,w}\left(\mathcal{A}(K,\varepsilon_1,\varepsilon)^C\right)$ converges to 0. We have that

$$\begin{aligned}&\mathcal{A}(K,\varepsilon_1,\varepsilon)^C = \{|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| > K\}\\ &\cup \{\|x-y\|_\infty < \varepsilon_1 n \text{ for some } x,y \in V_\mathbf{0}^n \text{ with } x \sim V_u^n, y \sim V_w^n\}\\ &\cup \Big(\left\{D_{V_\mathbf{0}^n}\left(x, B_{\varepsilon_1 n}(x)^C\right) \leq \varepsilon\Lambda(n) \text{ for some } x \in V_\mathbf{0}^n \text{ with } x \sim V_u^n\right\} \cap \{|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| \leq K\}\Big)\end{aligned}$$

and thus we get with Lemma 5.1 that

$$\begin{aligned}&\mathbb{P}_\beta^{u,w}\left(\mathcal{A}(K,\varepsilon_1,\varepsilon)^C\right) \leq \mathbb{P}_\beta^{u,w}\left(|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| > K\right) + C_d'\varepsilon_1^{1/2}\lceil\beta\rceil^2\\ &+ \mathbb{P}_\beta^{u,w}\left(\left\{D_{V_\mathbf{0}^n}\left(x, B_{\varepsilon_1 n}(x)^C\right) \leq \varepsilon\Lambda(n) \text{ for some } x \in V_\mathbf{0}^n \text{ with } x \sim V_u^n\right\} \cap \{|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| \leq K\}\right)\\ &\leq \mathbb{P}_\beta^{u,w}\left(|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| > K\right) + C_d'\varepsilon_1^{1/2}\lceil\beta\rceil^2 + K\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{\varepsilon_1 n}(\mathbf{0})^C\right) \leq \varepsilon\Lambda(n)\right).\end{aligned}$$

The expression $\mathbb{P}_\beta^{u,w}\left(|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}| > K\right)$ converges to 0 for $K \to \infty$, by Markov's inequality, as one has the bound

$$\begin{aligned}\mathbb{E}_\beta\left[|\{x \in V_\mathbf{0}^n, z \in V_u^n : x \sim z\}|\right] &= \sum_{x \in V_\mathbf{0}^n}\sum_{z \in V_u^n}\mathbb{P}_\beta(x \sim z) \overset{(4)}{\leq} \sum_{x \in V_\mathbf{0}^n}\sum_{z \in V_u^n}\frac{\beta}{(\|x-z\|_\infty - 1)^{2d}}\\ &\leq \sum_{x \in V_\mathbf{0}^n}\sum_{z \in V_u^n}\frac{\beta}{((\|u\|_\infty - 1)n)^{2d}} \leq \frac{\beta 2^{2d}}{\|u\|_\infty^{2d}} \leq \beta 2^{2d}.\end{aligned}$$

We need an upper bound on this quantity for expectation with respect to the conditional measure $\mathbb{P}_\beta^{u,w}$. Lemma 2.2 then gives that

$$\mathbb{E}_\beta^{u,w}\left[|\{x \in V_\mathbf{0}^n : x \sim V_u^n\}|\right] \leq \mathbb{E}_\beta^{u,w}\left[|\{x \in V_\mathbf{0}^n, z \in V_u^n : x \sim z\}|\right] \leq \beta 2^{2d} + 1$$

and this upper bound does not depend on n or u. Using Lemma 4.10, we see that for fixed $\varepsilon_1 > 0$ the term $\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{\varepsilon_1 n}(\mathbf{0})^C\right) \leq \varepsilon\Lambda(n)\right)$ converges to 0 as $\varepsilon \to 0$ and thus we can take $K = K(\varepsilon)$ and $\varepsilon_1 = \varepsilon_1(\varepsilon)$ that converge to $+\infty$, respectively 0, slow enough such that $K\mathbb{P}_\beta\left(D\left(\mathbf{0}, B_{\varepsilon_1 n}(\mathbf{0})^C\right) \leq \varepsilon\Lambda(n)\right)$ also converges to 0 for $\varepsilon \to 0$. □

We want a similar function for the indirect distance between boxes. Such a function exists by Corollary 4.12.

Definition 5.3. *Let $g_2(\varepsilon)$ be a function with $g_2(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 1$ such that the indirect distance $D^\star$ between the sets $V_{\mathbf{0}}^n$ and $B_n \left(V_{\mathbf{0}}^n\right)^C$ satisfies*

$$\mathbb{P}_\beta \left(D^\star \left(V_{\mathbf{0}}^n, B_n \left(V_{\mathbf{0}}^n\right)^C \right) > \varepsilon \Lambda(n,\beta) \right) \geq g_2(\varepsilon)$$

for all $n \geq n(\varepsilon)$ large enough.

Consider long-range percolation on $\mathbb{Z}^d$. We split the long-range percolation graph into blocks of the form V_v^n, where $v \in \mathbb{Z}^d$. For each $v \in \mathbb{Z}^d$, we contract the block $V_v^n \subset \mathbb{Z}^d$ into one vertex $r(v)$. We call the graph that results from contracting all these blocks $G' = (V', E')$. For $r(v) \in G'$, we define the neighborhood $\mathcal{N}(r(v))$ by

$$\mathcal{N}(r(v)) = \left\{ r(u) \in G' : \|v-u\|_\infty \leq 1 \right\},$$

and we define the neighborhood-degree of $r(v)$ by

$$\deg^{\mathcal{N}}(r(v)) = \sum_{r(u) \in \mathcal{N}(r(v))} \deg(r(u)).$$

We also define these quantities in the same way when we start with long-range percolation on the graph $V_{\mathbf{0}}^{mn}$, and contract the box V_v^n for all $v \in V_{\mathbf{0}}^m$. The next lemma concerns the indirect distance between two sets, conditioned on the graph G'.

Lemma 5.4. *Let $\mathcal{W}(\varepsilon)$ be the event*

$$\mathcal{W}(\varepsilon) := \left\{ D^\star \left(V_v^n, \bigcup_{u \in \mathbb{Z}^d : \|u-v\|_\infty \geq 2} V_u^n \right) > \varepsilon \Lambda(n,\beta) \right\}.$$

For all large enough $n \geq n(\varepsilon)$ one has

$$\mathbb{P}_\beta \left(\mathcal{W}(\varepsilon)^C \mid G' \right) \leq 3^d \deg^{\mathcal{N}}(r(\mathbf{0})) (1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)).$$

Proof. By translation invariance we can assume $v = \mathbf{0}$. We define the set $T = V_{\mathbf{0}}^n \cup \bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2} V_u^n$, and we define the events $\mathcal{W}_1(\varepsilon)$ and $\mathcal{W}_2(\varepsilon)$ by

$$\mathcal{W}_1(\varepsilon) = \Big\{ \exists a,b,x,y \in \mathbb{Z}^d \text{ with } \|a\|_\infty = 1, \|a-b\|_\infty \geq 2, x \in V_a^n, y \in V_b^n : \\ e = \{x,y\} \text{ open}, D(x,T;\omega^{e-}) \leq \varepsilon\Lambda(n), D(y,T;\omega^{e-}) \leq \varepsilon\Lambda(n) \Big\}$$

and

$$\mathcal{W}_2(\varepsilon) = \Big\{ \text{There is an open path } P \text{ of length at most } \varepsilon\Lambda(n) \text{ from } V_{\mathbf{0}}^n \text{ to } \bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2} V_u^n : \\ \forall \{x,y\} \in P \text{ there exist } a,b \in \mathbb{Z}^d \text{ with } x \in V_a^n, y \in V_b^n, \|a-b\|_\infty \leq 1 \Big\}.$$

We will now show that $\mathcal{W}(\varepsilon)^C \subset \mathcal{W}_1(\varepsilon) \cup \mathcal{W}_2(\varepsilon)$. Assuming that $\mathcal{W}(\varepsilon)^C$ holds, there exists an open path P from $V_{\mathbf{0}}^n$ to $\bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2}$ with length $\leq \varepsilon\Lambda(n)$, and this path does not use a direct edge between these two sets. The path P can either be of the form as described in the event $\mathcal{W}_2(\varepsilon)$, or it contains an edge $e = \{x,y\}$ such that $x \in V_a^n, y \in V_b^n$ with $\|a\|_\infty = 1, \|a-b\|_\infty \geq 2$. Let us assume that this path P is not of the form as described in the event $\mathcal{W}_2(\varepsilon)$. As the length of the path is at most $\varepsilon\Lambda(n)$, the distance from the endpoints x, y of such an edge to the set T is at most $\varepsilon\Lambda(n)$, even when the edge $\{x, y\}$ is removed. This holds, as the path P starts at $V_{\mathbf{0}}^n$, then uses the edge e, and then arrives in the set $\bigcup_{u\in\mathbb{Z}^d:\|u\|_\infty\geq 2} V_u^n$. Also note that $y \in T$ is possible, in which case the distance between y and T equals 0. However, we see that $\mathcal{W}_1(\varepsilon)$ holds. Combined, we showed that $\mathcal{W}(\varepsilon)^C \subset \mathcal{W}_1(\varepsilon) \cup \mathcal{W}_2(\varepsilon)$.

The event $\mathcal{W}_2(\varepsilon)$ is independent of G', which implies that $\mathbb{P}_\beta(\mathcal{W}_2(\varepsilon)|G') = \mathbb{P}_\beta(\mathcal{W}_2(\varepsilon)) \leq 1 - g_2(\varepsilon)$. Suppose that $\|a\|_\infty = 1$ and $\|a-b\|_\infty \geq 2$, with $V_a^n \sim V_b^n$. Assume that there exists a path P from $V_{\mathbf{0}}^n$ to $\bigcup_{u\in\mathbb{Z}^d:\|u\|_\infty\geq 2}$ with length $\leq \varepsilon\Lambda(n)$, that uses an edge $e = \{x, y\}$ with $x \in V_a^n, y \in V_b^n$. The path needs to get to x, and it enters the box V_a^n from some box V_e^n with $\|e\|_\infty \leq 1$. Say that the path enters the box V_a^n through the vertex $z \in V_a^n$ with $z \sim V_e^n$. The chemical distance between x and z can be at most $\varepsilon\Lambda(n,\beta)$. There are 3^d such vectors e, so the probability that there exists such a path is bounded by $3^d(1-g_1(\varepsilon))$, as $\|a-b\|_\infty \geq 2$. With a union bound we get that

$$\begin{aligned}\mathbb{P}_\beta(\mathcal{W}_1(\varepsilon)|G') &\leq \sum_{a:\|a\|_\infty=1}\ \sum_{b:\|a-b\|\geq 2, V_a^n\sim V_b^n} \mathbb{P}_\beta\left(\exists x \in V_a^n, y \in V_b^n : x \sim y, D(x, T; \omega^{e-}) < \varepsilon\Lambda(n)\right)\\ &\leq \sum_{a:\|a\|_\infty=1} \deg(r(a))3^d(1-g_1(\varepsilon)) \leq \deg^{\mathcal{N}}(r(\mathbf{0}))3^d(1-g_1(\varepsilon))\end{aligned}$$

and thus we finally get that

$$\mathbb{P}_\beta\left(\mathcal{W}(\varepsilon)^C\right) \leq \mathbb{P}_\beta(\mathcal{W}_1(\varepsilon)) + \mathbb{P}_\beta(\mathcal{W}_2(\varepsilon)) \leq \deg^{\mathcal{N}}(r(\mathbf{0}))3^d(1-g_1(\varepsilon)) + (1-g_2(\varepsilon)).$$

□

5.2 Supermultiplicativity of $\Lambda(n,\beta)$

In this section, we prove the supermultiplicativity of $\Lambda(n,\beta)$. Our main tools for this are the results of the previous section and Lemma 3.2. We also use the the same notation as in Lemma 3.2, i.e., $\mu_\beta = \mathbb{E}_\beta[\deg(\mathbf{0})]$ and $\overline{\deg}(Z) = \frac{1}{|Z|}\sum_{v\in Z}\deg(v)$.

Lemma 5.5. *For all $\beta > 0$, there exists a constant $c > 0$ such that for all $m, n \in \mathbb{N}$*

$$\Lambda(mn,\beta) \geq c\Lambda(n,\beta)\Lambda(m,\beta). \tag{54}$$

Proof. Inequality (54) holds for all small m or $n \in \mathbb{N}$ for some $c > 0$, so it suffices to consider m and n large enough. We split the graph $V_{\mathbf{0}}^{mn}$ into blocks of the form V_v^n, where $v \in V_{\mathbf{0}}^m$. For each $v \in V_{\mathbf{0}}^m$, we contract the block $V_v^n \subset V_{\mathbf{0}}^{mn}$ into one vertex. We call the graph that results from contracting all these blocks $G' = (V', E')$. The graph G' has the same distribution as long range percolation on $V_{\mathbf{0}}^m$ under the measure $\mathbb{P}_\beta$. By $r(v)$, we denote the vertex in G' that results from contracting the box V_v^n. We also define an analogy of the infinity-distance on G' by $\|r(u)-r(v)\|_\infty = \|u-v\|_\infty$. Our goal is to bound the expected distance between the vertices $\mathbf{0}$ and $(mn-1)e_1$ from below, conditioned on the graph G'. For this, we consider all loop-erased walks $P' = (r(v_0), r(v_1), \ldots, r(v_k))$ between $r(\mathbf{0})$ and $r((m-1)e_1)$ in G'. In the following we always work on a certain event $\mathcal{H}_t$, which is defined by

$$\mathcal{H}_t = \bigcap_{k\geq t}\left\{\left|\mathcal{CS}_k\left(G'\right)\right| \leq 10^k\mu_\beta^k\right\} \cap \bigcap_{k\geq t}\left\{\overline{\deg}(Z) \leq 20\mu_\beta \forall Z \in \mathcal{CS}_k\left(G'\right)\right\}.$$

Note that, by Lemma 3.2, (25), and Markov's inequality one has

$$\mathbb{P}_\beta\left(\mathcal{H}_t^C\right) \leq \sum_{k=t}^\infty \mathbb{P}_\beta\left(\left|\mathcal{CS}_k\left(G'\right)\right| > 10^k\mu_\beta^k\right) + \sum_{k=t}^\infty \mathbb{P}_\beta\left(\exists Z \in \mathcal{CS}_k\left(G'\right) : \overline{\deg}(Z) > 20\mu_\beta\right)$$

$$\leq \sum_{k=t}^{\infty} 0.4^k + \sum_{k=t}^{\infty} e^{-4\mu_\beta k} \leq \sum_{k=t}^{\infty} 0.5^k = 2 \cdot 2^{-t}. \tag{55}$$

Let $P' = (r(v_0), \dots, r(v_k))$ be a self-avoiding path in G' starting at the origin vertex, i.e., $v_0 = \mathbf{0}$. Assume that k is large enough (which will be specified later) and let ε be small enough such that

$$\left(27^d 50\mu_\beta(1 - g_1(\varepsilon)) + 2(1 - g_2(\varepsilon))\right)^{\frac{1}{30^d 200\mu_\beta}} \leq \frac{1}{20\mu_\beta}. \tag{56}$$

We will see later on, where this condition on ε comes from. We will now describe what it means for a block $V_{v_i}^n$ to be *separated*; we will also say that the vertex $r(v_i) \in G'$ is separated in this case. Intuitively, a block being separated ensures that a path in the original model that passes through this block needs to walk a distance of at least $\varepsilon\Lambda(n, \beta)$. Formally, let P be a path in the original graph $V_{\mathbf{0}}^{mn}$ between $\mathbf{0}$ and $(mn - 1)e_1$, such that this path goes through the blocks corresponding to $r(u_0), r(u_1), \dots, r(u_K)$ in this order. Let $P' = (r(v_0), \dots, r(v_k))$ be the loop-erasure of the path $(r(u_0), r(u_1), \dots, r(u_K))$. So in particular, P' is self-avoiding. Suppose that $\|v_i - v_{i+1}\|_\infty \geq 2$. Then we call the block $r(v_i)$ separated if

$$D_{V_{v_i}^n}(x, y) \geq \varepsilon\Lambda(n, \beta) \text{ for all } x, y \in V_{v_i}^n \text{ with } x \sim V_{v_{i+1}}^n, y \sim V_w^n, w \notin \{v_i, v_{i+1}\}.$$

If $\|v_i - v_{i+1}\|_\infty = 1$, we call the block $r(v_i)$ separated if

$$D^\star_{V_{\mathbf{0}}^{mn}}\left(V_{v_i}^n, \bigcup_{r(w) \in G' : \|w - v_i\|_\infty \geq 2} V_w^n\right) \geq \varepsilon\Lambda(n, \beta).$$

Next, we want to upper bound the probability that a block $r(v_i)$ is not separated, given the graph G'. Assume that $\|v_i - v_{i+1}\|_\infty \geq 2$. Conditioned on the graph G', the probability that $r(v_i)$ is not separated is bounded by $\deg(r(v_i))(1 - g_1(\varepsilon))$ for large enough n. Assume that $\|v_i - v_{i+1}\|_\infty = 1$. Given the graph G', we have that

$$\mathbb{P}\left(D^\star_{V_{\mathbf{0}}^{mn}}\left(V_{v_i}^n, \bigcup_{r(v) : \|v - v_i\|_\infty \geq 2} V_v^n\right) < \varepsilon\Lambda(n, \beta) \mid G'\right)$$
$$\leq 3^d \deg^{\mathcal{N}}(r(v_i))(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon))$$

for all large enough n, by Lemma 5.4. No matter whether $\|v_i - v_{i+1}\|_\infty = 1$ or $\|v_i - v_{i+1}\|_\infty > 1$, in both cases we have that

$$\mathbb{P}_\beta\left(r(v_i) \text{ not separated} \mid G'\right) \leq 3^d \deg^{\mathcal{N}}(r(v_i))(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)).$$

We define the set

$$\tilde{R}_k = \bigcup_{i=0}^{k-1} \mathcal{N}(r(v_i)).$$

The set $\tilde{R}_k$ is a connected set in G', containing the origin $r(v_0)$, and its size is bounded from above and below by

$$k \leq |\tilde{R}_k| \leq 3^d k.$$

Assuming that the event $\mathcal{H}_k$ holds, we get that the average degree of the set $\tilde{R}_k$ is bounded by $20\mu_\beta k$. A vertex $r(v)$ can be included in several sets $\mathcal{N}(r(v_i))$ for different i, but in at most 3^d many. So in particular we have

$$\sum_{i=0}^{k-1} \deg^{\mathcal{N}}(r(v_i)) \leq 3^d |\tilde{R}_k| 20\mu_\beta k \leq 9^d 20\mu_\beta$$

and thus there can be at most $\frac{k}{2}$ many indices $i \in \{0, \dots, k-1\}$ with $\deg^{\mathcal{N}}(r(v_i)) > 9^d 50\mu_\beta$. We now define a set of special indices $\mathcal{IND}(P') \subset \{1, \dots, k-1\}$ via the algorithm below. For abbreviation, we will mostly just write $\mathcal{IND}$ for $\mathcal{IND}(P')$, but one should remember that the indices really depend on the chosen path P'.

0. Start with $\mathcal{IND}_0 = \emptyset$.
1. For $i = 1, \dots, k-1$:
   If $\deg^{\mathcal{N}}(r(v_i)) \leq 9^d 50\mu_\beta$ and $\mathcal{N}(r(v_i)) \nsim \bigcup_{j \in \mathcal{IND}_{i-1}} \mathcal{N}(r(v_j))$, then define $\mathcal{IND}_i = \mathcal{IND}_{i-1} \cup \{i\}$. Otherwise set $\mathcal{IND}_i = \mathcal{IND}_{i-1}$.
2. Set $\mathcal{IND} := \mathcal{IND}_{k-1}$.

So in particular we have that for an index $i \in \mathcal{IND}$ it always holds that

$$\begin{aligned}
&\mathbb{P}_\beta\left(r(v_i) \text{ not separated} \mid G'\right) \leq 3^d \deg^{\mathcal{N}}(r(v_i))\left(1 - g_1(\varepsilon)\right) + \left(1 - g_2(\varepsilon)\right) \\
&\quad \leq 27^d 50\mu_\beta(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon)) =: g'(\varepsilon)
\end{aligned}$$

On the event $\mathcal{H}_k$, there are at least $\frac{k}{2}-1$ many indices $i \in \{1, \dots, k-1\}$ with $\deg^{\mathcal{N}}(r(v_i)) \leq 9^d 50\mu_\beta$. Suppose that V_v^n is a block with $V_v^n \sim \bigcup_{w \in \mathcal{N}(r(v_i))} V_w^n$. (Note that all boxes V_v^n with $r(v) \in \mathcal{N}(r(v_i))$ are by definition adjacent to $\bigcup_{w \in \mathcal{N}(r(v_i))} V_w^n$.) When we include the index i to the set $\mathcal{IND}$, we can block all the indices $j > i$ with $r(v) \in \mathcal{N}(r(v_j))$. But for fixed v, there can be at most 3^d indices $j > i$ with $r(v) \in \mathcal{N}(r(v_j))$. So including one index i with $\deg^{\mathcal{N}}(r(v_i)) \leq 9^d 50\mu_\beta$ to the set $\mathcal{IND}$, can block at most $3^d 9^d 50\mu_\beta$ other indices. Thus we get that on the event $\mathcal{H}_k$ one has for large enough k that

$$|\mathcal{IND}| \geq \frac{\frac{k}{2} - 1}{27^d 50\mu_\beta + 1} \geq \frac{k}{30^d 100\mu_\beta}.$$

Whether a block $V_{v_i}^n$ is separated in the path P' depends only on the edges with at least one end $\mathcal{N}(r(v_i))$. So in particular for different indices $i \in \mathcal{IND}$, it is independent whether the underlying blocks $V_{v_i}^n$ are separated. Thus we get that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\left|\left\{i \in \mathcal{IND}(P') : r(v_i) \text{ separated }\right\}\right| \leq \frac{k}{30^d 200\mu_\beta} \;\middle|\; G'\right) \\
&\leq 2^{|\mathcal{IND}(P')|}\left(g'(\varepsilon)\right)^{\frac{|\mathcal{IND}(P')|}{2}} \leq 2^k \left(g'(\varepsilon)\right)^{\frac{k}{30^d 200\mu_\beta}} \leq (20\mu_\beta)^{-k}
\end{aligned}$$

where the last inequality holds because of our assumption on ε (56). With another union bound we get that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\exists P' \text{ in } G' \text{ of length } k \text{ s.t.} \left|\left\{i \in \mathcal{IND}(P') : r(v_i) \text{ separated }\right\}\right| \leq \frac{k}{30^d 200\mu_\beta} \;\middle|\; \mathcal{H}_k\right) \\
&\leq (10\mu_\beta)^k (20\mu_\beta)^{-k} = 2^{-k},
\end{aligned}$$

where we say P' in G' if a path P' starts at $r(\mathbf{0})$ and is contained in the graph G'. Using that $\mathbb{P}_\beta\left(\mathcal{H}_k^C\right) \leq 2\cdot 2^{-k}$, we thus get that

$$\mathbb{P}_\beta\left(\exists P' \text{ in } G' \text{ of length } k \text{ s.t. } \left|\left\{i\in\mathcal{IND}(P') : r(v_i) \text{ separated }\right\}\right| \leq \frac{k}{30^d 200\mu_\beta}\right) \leq 3\cdot 2^{-k}.$$

For abbreviation, we define the event $\mathcal{G}_k$ by

$$\mathcal{G}_k^C = \left\{\exists P' \text{ in } G' \text{ of length } k \text{ s.t. } \left|\left\{i\in\mathcal{IND}(P') : r(v_i) \text{ separated }\right\}\right| \leq \frac{k}{30^d 200\mu_\beta}\right\}.$$

Assuming that the events $\mathcal{G}_k$ and $D_{G'}\left(r(\mathbf{0}), r((m-1)e_1)\right) = k$ both hold, we get that for large enough k one has $D_{V_{\mathbf{0}}^{mn}}(\mathbf{0},(mn-1)e_1) \geq \frac{k\varepsilon\Lambda(n,\beta)}{30^d 200\mu_\beta}$. So in total we get that for some large enough k'

$$\begin{aligned}
&\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mn}}(\mathbf{0},(mn-1)e_1)\right] \geq \sum_{k=k'}^{\infty}\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{mn}}(\mathbf{0},(mn-1)e_1)\mathbb{1}_{\mathcal{G}_k}\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((m-1)e_1))=k\}}\right]\\
&\geq \frac{\varepsilon\Lambda(n,\beta)}{30^d 200\mu_\beta}\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\mathcal{G}_k}\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((m-1)e_1))=k\}}\right], \qquad (57)
\end{aligned}$$

and we can further bound the last sum by

$$\begin{aligned}
&\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\mathcal{G}_k}\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((m-1)e_1))=k\}}\right]\\
&=\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((m-1)e_1))=k\}}\right]-\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{\mathcal{G}_k^C\right\}}\mathbb{1}_{\{D_{G'}(r(\mathbf{0}),r((m-1)e_1))=k\}}\right]\\
&\geq\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{D_{V_{\mathbf{0}}^m}(\mathbf{0},(m-1)e_1)=k\right\}}\right]-\sum_{k=k'}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{\mathcal{G}_k^C\right\}}\right]\\
&\geq\sum_{k=1}^{\infty}k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{D_{V_{\mathbf{0}}^m}(\mathbf{0},(m-1)e_1)=k\right\}}\right]-\sum_{k=1}^{k'-1}k\mathbb{E}_\beta\left[\mathbb{1}_{\left\{D_{V_{\mathbf{0}}^m}(\mathbf{0},(m-1)e_1)=k\right\}}\right]-3\sum_{k=k'}^{\infty}k2^{-k}\\
&\geq\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^m}(\mathbf{0},(m-1)e_1)\right]-k'-6\geq c'\Lambda(m,\beta)
\end{aligned}$$

for small enough $c'>0$ and m large enough. Inserting this into (57) finishes the proof. □

5.3 The diameter of boxes

In this section, we prove the second item of Theorem 1.1, i.e., that the diameter of the box $\{0,\ldots,n-1\}^d$ and its expectation both grow like n^θ.

Lemma 5.6. *For all $\beta\geq 0$ one has*

$$n^{\theta(\beta)} \approx_P Diam\left(\{0,\ldots,n-1\}^d\right) \approx_P \mathbb{E}_\beta\left[Diam\left(\{0,\ldots,n-1\}^d\right)\right].$$

Proof. By Lemma 4.1, it suffices to consider the case when $n=2^k$ for some $k\in\mathbb{N}$. We have

$$\text{Diam}\left(\{0,\ldots,n-1\}^d\right) \geq D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)e_1\right)$$

and this already implies that for each $\varepsilon > 0$ there exist constants $c, c_\varepsilon > 0$ such that

$$\mathbb{P}_\beta\left(c_\varepsilon n^{\theta(\beta)} < \text{Diam}\left(\{0,\ldots,n-1\}^d\right)\right) > 1-\varepsilon$$

and

$$cn^{\theta(\beta)} \leq \mathbb{E}_\beta\left[\text{Diam}\left(\{0,\ldots,n-1\}^d\right)\right]$$

uniformly over n. For the upper bound, we make a dyadic decomposition of the box $V_{\mathbf{0}}^n$. Similar ideas were also used in [33] for one dimension. For a constant $S \geq 1$, we say that a box $V_y^{2^l} \subset V_{\mathbf{0}}^{2^k}$ is *S-good* if

$$D_{V_y^{2^l}}\left(2^l y, 2^l y + (2^l-1)e\right) \leq S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta}$$

for all $e \in \{0,1\}^d$, where we simply write θ for $\theta(\beta)$ from here on. We use the notation

$$\Omega_l^S = \bigcap_{y \in V_{\mathbf{0}}^{2^{k-l}}} \left\{V_y^{2^l} \text{ is } S\text{-good}\right\} \quad \text{and} \quad \Omega^S = \bigcap_{l=1}^{k} \Omega_l^S.$$

On the event Ω^S, we can bound the graph distance between $\mathbf{0}$ and any $y \in V_{\mathbf{0}}^{2^k}$ by considering a path that goes along the boxes in a dyadic decomposition. Let $y_0, \ldots, y_k \in \mathbb{Z}^d$ be such that $y \in V_{y_i}^{2^i}$ for all i. So in particular $y_0 = y$ and $y_k = \mathbf{0}$. We also have that $V_{y_0}^{2^0} \subset V_{y_1}^{2^1} \subset \ldots \subset V_{y_k}^{2^k}$ and thus also $2^{i-1}y_{i-1} \in V_{y_i}^{2^i}$ for all $i \geq 1$. This implies that $2^{i-1}y_{i-1} = 2^i y_i + 2^{i-1}e$ for some $e \in \{0,1\}^d$. As all the boxes inside $V_{\mathbf{0}}^{2^k}$ were assumed to be S-good we have

$$\begin{aligned} &D_{V_{\mathbf{0}}^{2^k}}\left(2^i y_i, 2^{i-1}y_{i-1}\right) \leq D_{V_{\mathbf{0}}^{2^k}}\left(2^i y_i, 2^i y_i + (2^{i-1}-1)e\right) + 1 \\ &= D_{V_{2y_i}^{2^{i-1}}}\left(2^{i-1}2y_i, 2^{i-1}2y_i + (2^{i-1}-1)e\right) + 1 \leq S\left(\frac{3}{2}\right)^{(i-1-k)\theta} 2^{k\theta} + 1. \end{aligned}$$

Now we have by the triangle inequality

$$\begin{aligned} D_{V_0^{2^k}}(\mathbf{0}, v) &\leq \sum_{l=1}^{k}\left(S\left(\frac{3}{2}\right)^{(l-1-k)\theta} 2^{k\theta} + 1\right) \leq S2^{k\theta}\sum_{l=1}^{k}\left(\left(\frac{3}{2}\right)^{(l-1-k)\theta} + \frac{1}{2^{k\theta}}\right) \\ &\leq C_\theta S 2^{k\theta} \end{aligned}$$

where the constant C_θ depends only on θ. As $D(u,v) \leq D(\mathbf{0},u) + D(\mathbf{0},v)$ for all $u, v \in V_{\mathbf{0}}^{2^k}$, the previous bound already implies that on the event Ω^S one has

$$\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right) \leq 2C_\theta S 2^{k\theta} \tag{58}$$

and thus it suffices to bound the probability of $\left(\Omega^S\right)^C$. We know from Corollary 4.6 that the r-th moment of $D_{V_y^{2^l}}\left(2^l y, 2^l y + (2^l-1)e\right)$ is of order $2^{rl\theta}$, for all $r \geq 0$. So by a union bound and Markov's inequality we get that for every fixed box $V_y^{2^l}$

$$\mathbb{P}_\beta\left(V_y^{2^l} \text{ is not } S\text{-good}\right) \leq \sum_{e \in \{0,1\}^d} \mathbb{P}_\beta\left(D_{V_y^{2^l}}\left(2^l y, 2^l y + (2^l-1)e\right) > S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta}\right)$$

$$
\begin{aligned}
&= \sum_{e\in\{0,1\}^d} \mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},\mathbf{0}+(2^l-1)e\right)^{\frac{4d}{\theta}} > S^{\frac{4d}{\theta}}\left(\frac{3}{2}\right)^{(l-k)4d} 2^{4dk}\right)\\
&\leq \sum_{e\in\{0,1\}^d} \frac{\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},\mathbf{0}+(2^l-1)e\right)^{\frac{4d}{\theta}}\right]}{S^{\frac{4d}{\theta}}\left(\frac{3}{2}\right)^{(l-k)4d} 2^{4dk}} \leq \frac{C\cdot 2^{l\theta\frac{4d}{\theta}}}{S^{\frac{4d}{\theta}}\left(\frac{3}{2}\right)^{(l-k)4d} 2^{4dk}} = \frac{C\cdot 2^{4dl}}{S^{\frac{4d}{\theta}}\left(\frac{3}{2}\right)^{(l-k)4d} 2^{4dk}}\\
&\leq \frac{C}{S^{\frac{4d}{\theta}}}\left(\frac{2}{3}\right)^{(l-k)4d} 2^{(l-k)4d} = \frac{C}{S^{\frac{4d}{\theta}}}\left(\frac{4}{3}\right)^{(l-k)4d}
\end{aligned}
$$

for some constant $C<\infty$. With another union bound that we get that

$$
\begin{aligned}
\mathbb{P}_\beta\left(\left(\Omega_l^S\right)^C\right) &\leq \sum_{y\in V_{\mathbf{0}}^{2^{k-l}}} \mathbb{P}_\beta\left(V_y^{2^l} \text{ is not } S\text{-good}\right) \leq \sum_{y\in V_{\mathbf{0}}^{2^{k-l}}} \frac{C}{S^{\frac{4d}{\theta}}}\left(\frac{4}{3}\right)^{(l-k)4d}\\
&= \frac{C}{S^{\frac{4d}{\theta}}} 2^{(k-l)d}\left(\frac{4}{3}\right)^{(l-k)4d} = \frac{C}{S^{\frac{4d}{\theta}}}\left(\frac{81}{128}\right)^{(k-l)d}
\end{aligned}
$$

which implies that

$$
\mathbb{P}_\beta\left(\left(\Omega^S\right)^C\right) \leq \sum_{l=1}^{k} \frac{C}{S^{\frac{4d}{\theta}}}\left(\frac{81}{128}\right)^{(k-l)d} \leq \frac{C'}{S^{\frac{4d}{\theta}}} \tag{59}
$$

for some constant $C'<\infty$. Together with (58), this proves that $\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right) \approx_P 2^{k\theta}$. Inequality (58) also implies that

$$
\left\{\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)}{2^{k\theta}} > S\right\} \subset \left(\Omega^{\frac{S}{2C_\theta}}\right)^C
$$

whenever $\frac{S}{2C_\theta} > 1$, and this implies that for some finite $K\in\mathbb{N}$ and all $k\in\mathbb{N}$

$$
\begin{aligned}
\mathbb{E}_\beta\left[\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)}{2^{k\theta}}\right] &\leq K + \sum_{S=K}^{\infty} \mathbb{P}_\beta\left(\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)}{2^{k\theta}} > S\right) \leq K + \sum_{S=K}^{\infty} \mathbb{P}_\beta\left(\left(\Omega^{\frac{S}{2C_\theta}}\right)^C\right)\\
&\leq K + \sum_{S=1}^{\infty} \frac{C'\left(2C_\theta\right)^{\frac{4d}{\theta}}}{S^{\frac{4d}{\theta}}} < \infty
\end{aligned}
$$

where the last term is finite as $\frac{4d}{\theta} > 1$. This also shows that

$$
\mathbb{E}_\beta\left[\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)\right] = \mathcal{O}\left(2^{k\theta}\right)
$$

and thus finishes the proof of Lemma 5.6. □

6 Tail behavior of the distances and diameter

Theorem 1.1 shows that the random variables $\frac{D(\mathbf{0},u)}{\|u\|^{\theta(\beta)}}$ are tight in $(0,\infty)$ under the measure $\mathbb{P}_\beta$. In this section, we give more precise estimates on the tail-behavior of the random variables $\frac{D(\mathbf{0},u)}{\|u\|^{\theta(\beta)}}$. We describe this tail behavior via functions f for which $\sup_{u\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mathbb{E}_\beta\left[f\left(\frac{D(\mathbf{0},u)}{\|u\|^{\theta(\beta)}}\right)\right]$ is finite or infinite. This result is also a useful tool in section 7 and in section 11.

Theorem 6.1. *For all $\eta < \frac{1}{1-\theta(\beta)}$ one has*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta \left[\exp\left(\left(\frac{Diam\left(\{0,\dots,n\}^d\right)}{n^{\theta(\beta)}} \right)^\eta \right) \right] < \infty. \tag{60}$$

For dimension $d = 1$, the bound given by (60) is sharp, as the following lemma shows.

Lemma 6.2. *For all dimensions d and all $\beta > 0$, there exists a constant $t > 0$ such that*

$$\sup_{u\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mathbb{E}_\beta \left[\exp\left(t \left(\frac{D(\mathbf{0},u)}{\|u\|^{\theta(\beta)}} \right)^{\frac{d}{1-\theta(\beta)}} \right) \right] = \infty. \tag{61}$$

Proof. We define the event

$$\mathcal{D}_n = \bigcap_{v\in B_n(\mathbf{0})} \left\{ v \nsim w \text{ for all } w \in \mathbb{Z}^d \text{ with } \|v-w\|_\infty \geq 2 \right\}$$

If $\|u\|_\infty = n$ and the event D_n occurs, the shortest path between $\mathbf{0}$ and u uses nearest-neighbor edges only and thus has a length of $\|u\|_\infty$. Using the FKG-inequality, we get that

$$\mathbb{P}_\beta\left(\mathcal{D}_n\right) \geq \mathbb{P}_\beta(\mathbf{0} \nsim w \text{ for all } w \in \mathbb{Z}^d \text{ with } \|w\|_\infty \geq 2)^{|B_n(\mathbf{0})|} \geq e^{-Cn^d}$$

for some constant $C < \infty$. Thus we see that

$$\mathbb{P}_\beta \left(\frac{D(\mathbf{0},u)}{\|u\|_\infty^{\theta(\beta)}} = \|u\|_\infty^{1-\theta(\beta)} \right) \geq \mathbb{P}_\beta\left(\mathcal{D}_n\right) \geq \exp\left(-C\|u\|^d\right)$$

and from here one can easily verify that (61) holds for t large enough. $\square$

Remark 6.3. *Conditioning on the event that there is no edge longer than $m^{-\frac{1}{1-\theta(\beta)}}n$ open in the box $B_n(\mathbf{0})$, one can actually show that for all $u \in \mathbb{Z}^d$ with $\|u\|_\infty = n$ one has*

$$\mathbb{P}_\beta \left(\frac{D(\mathbf{0},u)}{\|u\|^{\theta(\beta)}} > m \right) \geq \exp\left(-Cm^{-\frac{d}{1-\theta(\beta)}}\right)$$

for some constant $C \in \mathbb{R}_{>0}$, and all large enough n.

For a sequence of positive random variables $(X_n)_{n\in\mathbb{N}}$ and some $\eta > 0$, we have that

$$\begin{aligned} \mathbb{E}\left[\exp\left(X_n^\eta\right)\right] &= \int_0^\infty \mathbb{P}\left(\exp\left(X_n^\eta\right) > s\right) \mathrm{d}s = \int_0^\infty \mathbb{P}\left(X_n^\eta > \log(s)\right) \mathrm{d}s \\ &= 1 + \int_1^\infty \mathbb{P}\left(X_n > \log(s)^{1/\eta}\right) \mathrm{d}s = 1 + \int_0^\infty \mathbb{P}\left(X_n > s\right) \eta s^{\eta-1} \exp\left(s^\eta\right) \mathrm{d}s. \end{aligned}$$

So in particular, if there exist constants $0 < c, C < \infty$ such that

$$\mathbb{P}\left(X_n > s\right) \leq C \exp\left(-cs^{\bar{\eta}}\right), \tag{62}$$

this implies that $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\exp\left(X_n^\eta\right)\right] < \infty$ for all $\eta \in (0, \bar{\eta})$. So in fact we will often show (62) in the following, as this will already imply statements of the form $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\exp\left(X_n^\eta\right)\right] < \infty$, as in (60). Theorem 6.1 directly implies that that for all $\eta < \frac{1}{1-\theta}$ one has

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta \left[\exp\left(\left(\frac{D_{V_0^n}(\mathbf{0},(n-1)e_1)}{n^\theta} \right)^\eta \right) \right] < \infty, \tag{63}$$

whereas (63) does not directly imply any statements about the diameter of boxes as in (60). However, a slightly weaker statement can be deduced from a slight modification of (63), as the next lemma shows.

Lemma 6.4. *Suppose that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta\left[\exp\left(\left(\frac{D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e)}{n^\theta}\right)^\eta\right)\right] < \infty \tag{64}$$

for some $\eta > 0$ and all $e \in \{0,1\}^d$. Then there exist constants $C, C_\theta \in \mathbb{R}_{>0}$ such that

$$\mathbb{P}_\beta\left(\textit{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) > SC_\theta n^\theta \textit{ for some } \bar{n}\in\{0,\dots,n\}\right) \leq C\exp\left(-S^\eta\right),$$

which implies that

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta\left[\exp\left(\left(\frac{\textit{Diam}\left(\{0,\dots,n-1\}^d\right)}{n^\theta}\right)^{\bar{\eta}}\right)\right] < \infty \tag{65}$$

for all $\bar{\eta} \in (0,\eta)$.

Proof. We do the proof for $n = 2^k$ with $k \in \mathbb{N}$. The proof for general $n \in \mathbb{N}$ follows by Lemma 4.1. For $S \geq 1$ and $l \in \{0,\dots,k\}$, define the events

$$\Omega_l^S = \bigcap_{y\in V_{\mathbf{0}}^{2^{k-l}}} \bigcap_{e\in\{0,1\}^d} \left\{ D_{V_y^{2^l}}\left(2^l y, 2^l y + (2^l-1)e\right) \leq S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta}\right\}$$

and

$$\Omega^S = \bigcap_{l=0}^k \Omega_l^S.$$

On the event Ω^S, for all $\bar{n} \leq n$, and for any $y \in V_{\mathbf{0}}^{\bar{n}}$, we can bound the graph distance between $\mathbf{0}$ and y by considering a dyadic path between them, and thus we get that on the event Ω^S

$$D_{V_{\mathbf{0}}^{\bar{n}}}(\mathbf{0},y) \leq \sum_{l=0}^k S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta} + k,$$

and this already implies that

$$\text{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) \leq 2\left(\sum_{l=0}^k S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta} + k\right) \leq C_\theta S 2^{k\theta}$$

for some constant $C_\theta < \infty$ and all $\bar{n} \leq n$. So in particular we see that the event $\left\{\text{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) > SC_\theta n^\theta \text{ for some } \bar{n} \leq n\right\}$ implies that Ω_l^S does not hold for some $l \in \{0,\dots,k\}$. So with a union bound we get that

$$\begin{aligned} &\mathbb{P}_\beta\left(\text{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) > SC_\theta n^\theta \text{ for some } \bar{n} \leq n\right) \\ &\leq \sum_{l=0}^k 2^{(k-l)d} \sum_{e\in\{0,1\}^d} \mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},(2^l-1)e\right) > S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta}\right). \end{aligned} \tag{66}$$

By Markov's inequality we have for any $e \in \{0,1\}^d$

$$\mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},(2^l-1)e\right) > S\left(\frac{3}{2}\right)^{(l-k)\theta} 2^{k\theta}\right) = \mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},(2^l-1)e\right) > S\left(\frac{4}{3}\right)^{(k-l)\theta} 2^{l\theta}\right)$$

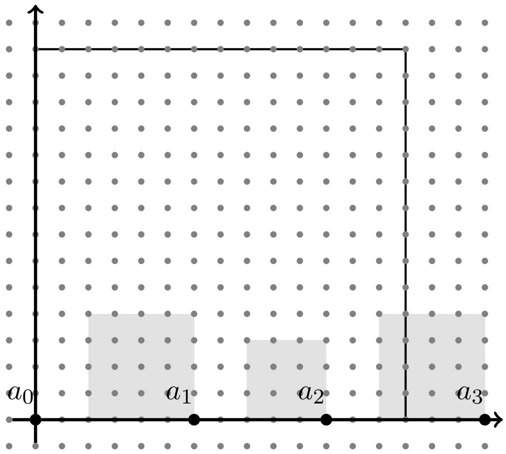


Figure 4: An example of the process $a_0e_1, a_1e_1, a_2e_1, a_3e_1$ for the Graph $V_{\mathbf{0}}^{15}$. We have $K = 3$, as $a_2 < 14 \leq a_3$.

$$= \mathbb{P}_\beta\left(\exp\left(\left(\frac{D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},(2^l-1)e\right)}{2^{l\theta}}\right)^\eta\right) > \exp\left(S^\eta\left(\frac{4}{3}\right)^{(k-l)\theta\eta}\right)\right)$$

$$\leq \mathbb{E}_\beta\left[\exp\left(\left(\frac{D_{V_{\mathbf{0}}^{2^l}}\left(\mathbf{0},(2^l-1)e\right)}{2^{l\theta}}\right)^\eta\right)\right]\exp\left(-S^\eta\left(\frac{4}{3}\right)^{(k-l)\theta\eta}\right) \leq C_\eta \exp\left(-S^\eta\left(\frac{4}{3}\right)^{(k-l)\theta\eta}\right)$$

for some constant $C_\eta < \infty$. Inserting this into (66) shows that

$$\mathbb{P}_\beta\left(\mathrm{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) > SC_\theta n^\theta \text{ for some } \bar{n} \leq n\right) \leq \sum_{l=0}^{k} 2^{(k-l)d} C_\eta \exp\left(-S^\eta\left(\frac{4}{3}\right)^{(k-l)\theta\eta}\right)$$
$$\leq C\exp\left(-S^\eta\right)$$

for some constant $C < \infty$. By taking the constant C large enough we can also guarantee that the above inequality holds for all $S > 0$. This already implies that (65) holds for all $\bar{\eta} \in (0, \eta)$. □

Lemma 6.5. *For all $\beta \geq 0$ and all $e \in \{0,1\}^d$ one has*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta\left[\exp\left(\left(\frac{D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)e\right)}{n^\theta}\right)^{0.5}\right)\right] < \infty. \tag{67}$$

Proof. First, we will consider $e = e_1$ only. We define a process $(a_k(n))_{k\in\mathbb{N}}$. As we will fix n for the rest of the proof we will often simply write a_k for $a_k(n)$. We start with $a_0(n) = 0$ and define $a_k(n)$ inductively by

$$\begin{aligned} a_{k+1}(n) =& (a_k(n)+2) \\ &+ \sup\left\{z \in \mathbb{N}_{>0} : D_{(a_k+2)e_1+\{0,\dots,z\}^d}\left((a_k+2)e_1, (a_k+2)e_1+ze_1\right) \leq n^\theta\right\}. \end{aligned} \tag{68}$$

Given the long-range percolation graph, this sequence can be constructed as follows: for given $a_{k-1}(n)$, we walk along the e_1-axis in positive direction, starting at $(a_{k-1}(n)+2)e_1$.

We do this until the graph distance between $(a_{k-1}(n)+2)e_1$ and $(a_{k-1}(n)+2+z)e_1$ exceeds a certain threshold (n^θ), and then we go one step back, i.e., in negative e_1-direction, and then define this point as $a_k e_1$. This procedure only reveals information about edges with both endpoints in the slice $\{y \in \mathbb{Z}^d : a_{k-1}(n)+2 \leq \langle y, e_1 \rangle \leq a_k(n)+1\}$, so in particular the differences $(a_{k'+1}(n) - a_{k'}(n))$ are independent of $a_k(n)$ for $k' \geq k$. By translation invariance, the differences $(a_{k+1}(n) - a_k(n))_{k \in \mathbb{N}_0}$ are independent and identically distributed random variables. The graph distance between $a_k(n)e_1$ and $a_{k+1}(n)e_1$ is always bounded by $n^\theta + 2$, as we can go from $a_k(n)e_1$ to $(a_k(n)+2)e_1$ in two steps and from there to $a_{k+1}(n)e_1$ in at most n^θ steps. Define

$$K_n = \inf\left\{k \in \mathbb{N} : a_k(n) \geq n\right\}$$

as the index of the first point $a_k(n)e_1$ that lies outside of $V_{\mathbf{0}}^n$. Then one has

$$D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1) \leq K_n n^\theta + 2K_n \leq 3K_n n^\theta$$

as one can walk through the path that goes from $\mathbf{0}$ to $a_1(n)e_1$, from $a_1(n)e_1$ to $a_2(n)e_1$, and from there in the same manner inductively to $a_{K_n-1}(n)e_1$, and from there to $(n-1)e_1$. So our next goal is to show that K_n is typically not too large. We use that for all $\beta \geq 0$ there exists an $\alpha > 0$ such that

$$\mathbb{P}_\beta\left(\frac{a_{k+1}(n) - a_k(n)}{n} \geq \alpha\right) \geq 0.5, \tag{69}$$

which we will prove in Lemma 6.6 below. We define the indices $k_0(n), k_1(n), \ldots$ by $k_0(n) = 0$ and

$$k_{i+1}(n) = \inf\{k > k_i(n) : \frac{a_k(n) - a_{k-1}(n)}{n} \geq \alpha\}.$$

By construction we have $K_n \leq k_{\lceil 1/\alpha \rceil + 1}(n)$. So in particular we have

$$\frac{D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1)}{n^\theta} \leq 3K_n \leq 3k_{\lceil 1/\alpha \rceil + 1}(n) = 3\sum_{i=0}^{\lceil \frac{1}{\alpha} \rceil} k_{i+1}(n) - k_i(n).$$

The differences $(k_{i+1}(n) - k_i(n))_{i \geq 0}$ are independent random variables and are, by (69), dominated by Geometric $\left(\frac{1}{2}\right)$-distributed random variables. This already implies that

$$\begin{aligned} \mathbb{E}_\beta\left[\exp\left(t\frac{D_{V_{\mathbf{0}}^n}(\mathbf{0}, (n-1)e_1)}{n^\theta}\right)\right] &\leq \mathbb{E}_\beta\left[\exp\left(t3\sum_{i=0}^{\lceil \frac{1}{\alpha} \rceil} k_{i+1}(n) - k_i(n)\right)\right] \\ = \prod_{i=0}^{\lceil \frac{1}{\alpha} \rceil} \mathbb{E}_\beta\left[\exp\left(t3(k_{i+1}(n) - k_i(n))\right)\right] &\leq C < \infty \end{aligned} \tag{70}$$

for some $t > 0$ small enough and a uniform constant C that does not depend on n, as the differences $k_{i+1}(n) - k_i(n)$ are dominated by a Geometric $\left(\frac{1}{2}\right)$-distributed random variable. This shows the claim for $e = e_1$. To extend this proof to general $e \in \{0,1\}^d$, we use the same technique as in the proof of Lemma 4.2. For $i \in \{0, \ldots, d\}$, we define $e(i)$ by

$$e(i) = \sum_{j=1}^{i} p_j(e) e_i,$$

and thus we get by the triangle inequality that

$$D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e) \leq \sum_{i=1}^{d} D_{V_{\mathbf{0}}^n}((n-1)e(i-1),(n-1)e(i)).$$

The random variables $D_{V_{\mathbf{0}}^n}((n-1)e(i-1),(n-1)e(i))$ are either equal to 0, when $e(i-1)$ and $e(i)$ coincide, or they have the same distribution as $D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1)$, when $e(i-1)$ and $e(i)$ lie on adjacent corners of the cube $V_{\mathbf{0}}^n$. Hölder's inequality implies that

$$\mathbb{E}_\beta\left[\exp\left(D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e)^{0.5}\right)\right] \leq \mathbb{E}_\beta\left[\exp\left(\sum_{i=1}^{d} D_{V_{\mathbf{0}}^n}((n-1)e(i-1),(n-1)e(i))^{0.5}\right)\right]$$

$$\leq \prod_{i=1}^{d} \mathbb{E}_\beta\left[\exp\left(dD_{V_{\mathbf{0}}^n}((n-1)e(i-1),(n-1)e(i))^{0.5}\right)\right]^{\frac{1}{d}} \leq \mathbb{E}_\beta\left[\exp\left(dD_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1)^{0.5}\right)\right]$$

and the last term is finite uniformly over all $n \in \mathbb{N}$, which follows from (70). □

Lemma 6.6. *For all $\beta > 0$, there exists a constant $\alpha > 0$ such that for all $n \in \mathbb{N}_{>0}$*

$$\mathbb{P}_\beta\left(\frac{a_{k+1}(n)-a_k(n)}{n} \geq \alpha\right) \geq 0.5. \tag{71}$$

Proof. As the differences $(a_{k+1}(n)-a_k(n))_{k\geq 0}$ are identically distributed, it suffices to consider the case $k=0$. The proof uses a dyadic decomposition along the e_1-axis. Let n be large enough so that $\log_2(n) \leq \frac{n^\theta}{2}$; this holds for all n sufficiently large. We can make this assumption, as the statement (71) clearly holds for small n by taking α small enough. Consider $\alpha > 0$ such that $\alpha n = 2^h$ for some $h \in \mathbb{N}$. By our assumption on n we have $h = \log_2(\alpha n) \leq \log_2(n) \leq \frac{n^\theta}{2}$. We define the events

$$\Omega_l = \bigcap_{j=0}^{2^{h-l}-1} \left\{ D_{V_{je_1}^{2^l}}\left(j2^l e_1, (j2^l+2^l-1)e_1\right) \leq \left(2\sum_{i=0}^{\infty}\left(\frac{3}{2}\right)^{-i\theta}\right)^{-1} n^\theta \left(\frac{3}{2}\right)^{(l-h)\theta} \right\}$$

and

$$\Omega = \bigcap_{l=0}^{h} \Omega_l.$$

For an $x \in \{0,\dots,2^h\}$, say $x = \sum_{l=0}^h x_l 2^l$, where $x_l \in \{0,1\}$ for all l, we consider the path that goes from $\mathbf{0}$ to $\left(\sum_{l=h}^h x_l 2^l\right)e_1$, from there to $\left(\sum_{l=h-1}^h x_l 2^l\right)e_1$, and iteratively to $\left(\sum_{l=0}^h x_l 2^l\right)e_1 = xe_1$. Using this path from $\mathbf{0}$ to xe_1 through the dyadic points of the form $2^l e_1$, one gets that on the event Ω one has for all $x \in \{0,\dots,\alpha n\}$

$$D_{V_{\mathbf{0}}^{x+1}}(\mathbf{0},xe_1) \leq \left(2\sum_{i=0}^{\infty}\left(\frac{3}{2}\right)^{-i\theta}\right)^{-1} n^\theta \sum_{l=0}^{h}\left(\frac{3}{2}\right)^{(l-h)\theta} + h < 2^{-1}n^\theta + h \leq n^\theta,$$

where we used that $h \leq \frac{n^\theta}{2}$ in the last step. Now, we want to estimate the probability of the event Ω. Let us write $C(\theta)$ for the constant $\left(2\sum_{i=0}^{\infty}\left(\frac{3}{2}\right)^{-i\theta}\right)^{-1}$ and let $C_{\frac{4}{\theta}}$ be a constant such that

$$\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1)^{4/\theta}\right] \leq C_{\frac{4}{\theta}}\left(n^\theta\right)^{4/\theta} = C_{\frac{4}{\theta}} n^4$$

for all $n \in \mathbb{N}$. Such a constant exists by Corollary 4.6. By an application of Markov's inequality we get that

$$\begin{aligned}
&\mathbb{P}_\beta \left(D_{V_{\mathbf{0}}^{2^l}} \left(\mathbf{0}, (2^l-1)e_1 \right) > C(\theta) n^\theta \left(\frac{3}{2}\right)^{(l-h)\theta} \right) \\
&= \mathbb{P}_\beta \left(D_{V_{\mathbf{0}}^{2^l}} \left(\mathbf{0}, (2^l-1)e_1 \right)^{\frac{4}{\theta}} > C(\theta)^{\frac{4}{\theta}} n^{\theta \frac{4}{\theta}} \left(\frac{3}{2}\right)^{(l-h)\theta\frac{4}{\theta}} \right) \\
&\leq \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^l}} \left(\mathbf{0}, (2^l-1)e_1 \right)^{\frac{4}{\theta}} \right] C(\theta)^{-\frac{4}{\theta}} n^{-4} \left(\frac{3}{2}\right)^{4(h-l)} \leq C(\theta)^{-\frac{4}{\theta}} C_{\frac{4}{\theta}} \left(2^{l\theta}\right)^{\frac{4}{\theta}} n^{-4} \left(\frac{3}{2}\right)^{4(h-l)} \\
&\leq C(\theta)^{-\frac{4}{\theta}} C_{\frac{4}{\theta}} 2^{4l} \alpha^4 2^{-4h} \left(\frac{3}{2}\right)^{4(h-l)} \qquad (72)
\end{aligned}$$

Define $a_k \coloneqq -2$ and define a_{k+1} as in (68). Then one has the line of implications

$$\begin{aligned}
\{\Omega\} &\Rightarrow \left\{ D_{V_{\mathbf{0}}^{x+1}}(\mathbf{0}, xe_1) \leq n^\theta \text{ for all } x \in \{0, \dots, \alpha n\} \right\} \\
&\Leftrightarrow \{a_{k+1}(n) \geq \alpha n\} \Rightarrow \left\{ \frac{a_{k+1}(n) - a_k(n)}{n} > \alpha \right\}.
\end{aligned}$$

This already gives us that

$$\begin{aligned}
&\mathbb{P}_\beta \left(\frac{a_{k+1}(n) - a_k(n)}{n} \leq \alpha \right) \leq \mathbb{P}_\beta \left(\Omega^C \right) \leq \sum_{l=0}^{h} 2^{h-l} \mathbb{P}_\beta \left(D_{V_{\mathbf{0}}^{2^l}} \left(\mathbf{0}, (2^l-1)e_1 \right) > C(\theta) n^\theta \left(\frac{3}{2}\right)^{(l-h)\theta} \right) \\
&\overset{(72)}{\leq} C(\theta)^{-\frac{4}{\theta}} C_{\frac{4}{\theta}} \sum_{l=0}^{h} 2^{h-l} 2^{4l} \alpha^4 2^{-4h} \left(\frac{3}{2}\right)^{4(h-l)} = \alpha^4 C(\theta)^{-\frac{4}{\theta}} C_{\frac{4}{\theta}} \sum_{l=0}^{h} \left(\frac{81}{128}\right)^{h-l} < 0.5
\end{aligned}$$

for some $\alpha > 0$ small enough. So in particular this implies (71). □

Lemma 6.7. *Assume that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}_\beta \left[\exp\left(\left(\frac{Diam\left(\{0, \dots, n-1\}^d\right)}{n^\theta} \right)^\eta \right) \right] < \infty \qquad (73)$$

for some $\eta > 0$. Then

$$\sup_{n \in \mathbb{N}} \mathbb{E}_\beta \left[\exp\left(\left(\frac{Diam\left(\{0, \dots, n-1\}^d\right)}{n^\theta} \right)^{\bar{\eta}} \right) \right] < \infty \qquad (74)$$

for all $\bar{\eta} < 1 + \theta\eta$.

Proof. Assume that (73) holds for some $\eta > 0$. Then Lemma 6.4 implies that

$$\mathbb{P}_\beta \left(\text{Diam}\left(V_{\mathbf{0}}^{\bar{n}}\right) > S C_\theta n^\theta \text{ for some } \bar{n} \in \{0, \dots, n\} \right) \leq C \exp\left(-S^\eta\right) \qquad (75)$$

for some constants $C, C_\theta < \infty$. As before, we define define $a_k(n)$ inductively by $a_0(n) = 0$ and

$$\begin{aligned}
a_{k+1}(n) = &(a_k(n) + 2) \\
&+ \sup\left\{ z \in \mathbb{N}_{>0} : D_{(a_k+2)e_1 + \{0,\dots,z\}^d}\left((a_k(n)+2)e_1, (a_k(n)+2)e_1 + ze_1 \right) \leq n^\theta \right\}.
\end{aligned}$$

The differences $(a_{k+1}(n) - a_k(n))_{k\in\mathbb{N}_0}$ are independent and identically distributed. For $\alpha \in (0,1)$, we have that

$$\begin{aligned}
&\mathbb{P}_\beta\left(\frac{a_1(n)-a_0(n)}{n} \leq \alpha\right) = \mathbb{P}_\beta\left(D_{2e_1+\{0,\dots,z\}^d}\left(2e_1,(2+z)e_1\right) > n^\theta \text{ for some } z\in\{2,\dots,\lfloor \alpha n\rfloor\}\right)\\
&\leq \mathbb{P}_\beta\left(\text{Diam}\left(\{0,\dots,z\}^d\right) > n^\theta \text{ for some } z\in\{0,\dots,\lfloor \alpha n\rfloor\}\right)\\
&= \mathbb{P}_\beta\left(\text{Diam}\left(\{0,\dots,z\}^d\right) > \frac{1}{\alpha^\theta C_\theta} C_\theta (\alpha n)^\theta \text{ for some } z\in\{0,\dots,\lfloor \alpha n\rfloor\}\right)\\
&\overset{(75)}{\leq} C\exp\left(-\left(\frac{1}{\alpha^\theta C_\theta}\right)^\eta\right) = C\exp\left(-C_\theta' \alpha^{-\theta\eta}\right) \qquad (76)
\end{aligned}$$

for a constant $C_\theta' \in \mathbb{R}_{>0}$. Remember that the random variable K_n was defined by

$$K_n = \inf\{k\in\mathbb{N} : a_k(n) \geq n\}$$

and that

$$D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)e_1\right) \leq K_n n^\theta + 2K_n \leq 3K_n n^\theta. \qquad (77)$$

Assume that $K_n > 2L$ for some large integer L. Then there needs to exist at least L indices $i\in\{1,\dots,2L\}$ such that $a_i(n)-a_{i-1}(n) \leq \frac{1}{L}$. Using independence of the random variables $a_i(n)-a_{i-1}(n)$

$$\begin{aligned}
&\mathbb{P}_\beta\left(K_n > 2L\right) \leq \mathbb{P}_\beta\left(\bigcup_{\substack{U\subset\{1,\dots,2L\}:\\ |U|=L}} \left\{a_i(n)-a_{i-1}(n) \leq \frac{1}{L} \text{ for all } i\in U\right\}\right)\\
&\leq \sum_{\substack{U\subset\{1,\dots,2L\}:\\ |U|=L}} \prod_{i\in U} \mathbb{P}_\beta\left(a_i(n)-a_{i-1}(n)\leq \frac{1}{L}\right) \leq 2^{2L}\mathbb{P}_\beta\left(a_1(n)-a_0(n)\leq\frac{1}{L}\right)^L\\
&\overset{(76)}{\leq} 2^{2L} C\exp\left(-C_\theta' L^{\theta\eta}\right)^L \leq \bar{C}\exp\left(-\bar{C}_\theta L^{1+\theta\eta}\right)
\end{aligned}$$

for some constants $\bar{C},\bar{C}_\theta \in \mathbb{R}_{>0}$ and all L large enough. From (77) we have the line of implications

$$\left\{D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)e_1\right) > 6Ln^\theta\right\} \Rightarrow \{K_n > 2L\}$$

and thus we get that for L large enough

$$\mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^n}\left(\mathbf{0},(n-1)e_1\right) > 6Ln^\theta\right) \leq \mathbb{P}_\beta\left(K_n > 2L\right) \leq \bar{C}\exp\left(-\bar{C}_\theta L^{1+\theta\eta}\right),$$

which implies that

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta\left[\exp\left(\left(\frac{D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e_1)}{n^\theta}\right)^{\bar{\eta}}\right)\right] < \infty$$

for all $\bar{\eta} < 1+\theta\eta$. The same technique as in the proof of Lemma 6.5 shows that

$$\sup_{n\in\mathbb{N}} \mathbb{E}_\beta\left[\exp\left(\left(\frac{D_{V_{\mathbf{0}}^n}(\mathbf{0},(n-1)e)}{n^\theta}\right)^{\bar{\eta}}\right)\right] < \infty$$

for all $e \in \{0,1\}^d$ and all $\bar{\eta} < 1 + \theta\eta$. Using Lemma 6.4, we can finally see that this also implies that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_\beta \left[\exp\left(\left(\frac{\text{Diam}\left(\{0,\dots,n-1\}^d\right)}{n^\theta} \right)^{\bar{\eta}} \right) \right] < \infty$$

for all $\bar{\eta} < 1 + \theta\eta$. □

With this, we are ready to go to the proof of Theorem 6.1, which works via a bootstrapping argument.

Proof of Theorem 6.1. Lemma 6.5 and Lemma 6.4 imply that (60) holds for $\bar{\eta} = 0.4$. We define the function $f(x) = 1 + \theta \cdot x$. Lemma 6.7 says that if (60) holds for some $\bar{\eta} > 0$, then it actually holds for all $\eta < f(\bar{\eta})$. Iterating this argument, we see that (60) holds for all $\eta < f^{(k)}(0.4)$, where $k \in \mathbb{N}$ is an arbitrary integer and $f^{(k)}$ is the k-fold iteration of f. Letting k go to infinity, the value $f^{(k)}(0.4)$ converges to the fixed point x_0 of the equation $x = f(x)$, which is given by $x_0 = \frac{1}{1-\theta}$. So in particular we see that (60) holds for all $\eta < \frac{1}{1-\theta}$. □

7 Comparison with different inclusion probabilities

In this section, we compare the graph distances that result from percolation with the measure $\mathbb{P}_\beta$ to the graph distances that result from independent bond percolation on $\mathbb{Z}^d$ where two vertices $u, v \in \mathbb{Z}^d$ are connected with probability $p^\star(\beta, \{u,v\})$, which is assumed to be close enough to $p(\beta, \{u,v\})$. The precise condition required for the function $p^\star(\beta, \{u,v\})$ is that for fixed β it satisfies that

$$p^\star(\beta, \{u,v\}) = 1 \text{ for } \|u-v\| = 1 \text{ and } p^\star(\beta, \{u,v\}) = p(\beta, \{u,v\}) + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right) \tag{78}$$

as $\|u - v\| \to \infty$. An example of such a set of inclusion probabilities $p^\star(\beta, \{u,v\})$ is given by

$$p^\star(\beta, \{u,v\}) = \begin{cases} 1 & \text{for } \|u-v\| = 1 \\ \frac{\beta}{\|u-v\|^{2d}} \wedge 1 & \text{for } \|u-v\| > 1 \end{cases}$$

where we prove in Example 7.2 that (78) is satisfied. These inclusion probabilities were for example also used in [33] for $d = 1$.

We write $\mathbb{P}^\star_\beta$ for the probability measure resulting from independent bond percolation with inclusion probabilities $(p^\star(\beta, \{u,v\}))_{u,v \in \mathbb{Z}^d}$. In the following, we give a proof that both the graph distance $D(\mathbf{0}, x)$ and the diameter of a box $\text{Diam}\left(\{0,\dots,n\}^d\right)$ scale like $\|x\|^{\theta(\beta)}$, respectively $n^{\theta(\beta)}$, under the measure $\mathbb{P}^\star_\beta$.

Theorem 7.1. *For fixed $\beta \geq 0$, suppose that $p^\star(\beta, \{u,v\})$ satisfies (78). Then the graph distance between the origin $\mathbf{0}$ and $x \in \mathbb{Z}^d$ satisfies*

$$\|x\|^{\theta(\beta)} \approx_P D(\mathbf{0}, x) \approx_P \mathbb{E}^\star_\beta[D(\mathbf{0}, x)] \tag{79}$$

under the measure $\mathbb{P}^\star_\beta$. The diameter of cubes satisfies

$$n^{\theta(\beta)} \approx_P Diam\left(\{0,\dots,n\}^d\right) \approx_P \mathbb{E}^\star_\beta\left[Diam\left(\{0,\dots,n\}^d\right)\right] \tag{80}$$

under the measure $\mathbb{P}^\star_\beta$.

For the proof of (79), we follow a technique that was already used in [33] in a similar form for a comparison between the discrete and the continuous model of percolation. The proof of (80) needs more involved methods, and is done in section 7.1.

Proof of (79). We fix the dimension d and β from here on and consider them as constants. We write $E^\star_{u,v}$ for the event when there exists an edge between u and v in the graph sampled with the measure $\mathbb{P}^\star_\beta$, and we write $E_{u,v}$ if there exists an edge between u and v in the graph sampled with the measure $\mathbb{P}_\beta$. With the standard coupling for percolation we can couple the measures $\mathbb{P}_\beta$ and $\mathbb{P}^\star_\beta$ so that uniformly over all $u \in \mathbb{Z}^d$, $v \in \mathbb{Z}^d \setminus \{u\}$

$$\mathbb{P}\left(E^\star_{u,v} \setminus E_{u,v}\right) + \mathbb{P}\left(E_{u,v} \setminus E^\star_{u,v}\right) \leq C_1 \frac{1}{\|u-v\|^{2d+1}}$$

where $C_1 < \infty$ is a constant, and where we write $\mathbb{P}$ for the joint measure. Thus we also get

$$\mathbb{P}\left(\left(E^\star_{u,v}\right)^C \middle| E_{u,v}\right) + \mathbb{P}\left(\left(E_{u,v}\right)^C \middle| E^\star_{u,v}\right) \leq C_2 \frac{1}{\|u-v\|}$$

for some constant $C_2 < \infty$. We write $\omega^\star$ for the percolation configuration sampled by $\mathbb{P}^\star_\beta$ and ω for the percolation configuration sampled by $\mathbb{P}_\beta$. For two points $x, y \in \mathbb{Z}^d$, let P be a geodesic between x and y for the environment ω. We construct a path between x and y in the environment $\omega^\star$ as follows:

- For $\{u,v\} \in P$, if $E^\star_{u,v}$ occurs we use the direct edge between u and v.
- For $\{u,v\} \in P$, if $E^\star_{u,v}$ does not occur go from u to v using $\|u-v\|_1$ many nearest-neighbor edges.

This gives a path $P^\star$ between x and y in the environment $\omega^\star$. The length of this path equals

$$\sum_{\substack{\{u,v\}\in P: \\ E^\star_{u,v}\text{occurs}}} 1 \; + \sum_{\substack{\{u,v\}\in P: \\ (E^\star_{u,v})^C\text{occurs}}} \|u-v\|_1 = \sum_{\{u,v\}\in P} \left(\mathbb{1}_{E^\star_{u,v}} + \|u-v\|_1 \mathbb{1}_{(E^\star_{u,v})^C}\right)$$

and thus we get that

$$\begin{aligned} \mathbb{E}\left[D(x,y;\omega^\star) \;\middle|\; \omega\;\right] &\leq \sum_{\{u,v\}\in P} \mathbb{E}\left[1 + \|u-v\|_1 \mathbb{1}_{(E^\star_{u,v})^C} \;\middle|\; \omega\;\right] \\ &\leq \sum_{\{u,v\}\in P} \left(1 + \|u-v\|_1 C_2 \frac{1}{\|u-v\|}\right) \leq C_3 D(x,y;\omega) \end{aligned} \tag{81}$$

for some constant $C_3 < \infty$. Markov's inequality for the conditional measure $\mathbb{P}\left(\cdot \middle| \omega\right)$ gives that for each $\varepsilon > 0$ there exists a constant C_ε such that

$$\mathbb{P}\left(D(x,y;\omega^\star) \leq C_\varepsilon D(x,y;\omega)\right) \geq 1-\varepsilon \; .$$

Interchanging the roles of ω and $\omega^\star$ one gets that for each $\varepsilon > 0$ there exists a constant $C^\star_\varepsilon$ such that

$$\mathbb{P}\left(D(x,y;\omega) \leq C^\star_\varepsilon D(x,y;\omega^\star)\right) \geq 1-\varepsilon,$$

which shows that $D(x,y;\omega^\star) \approx_P \|x-y\|^{\theta(\beta)}$. Inequality (81), and interchanging the roles of ω and $\omega^\star$, implies that $\mathbb{E}\left[D(x,y;\omega^\star)\right]$ and $\mathbb{E}\left[D(x,y;\omega)\right]$ are at most a constant factor apart. Thus we get that $\|x-y\|^{\theta(\beta)} \approx_P D(x,y;\omega^\star) \approx_P \mathbb{E}\left[D(x,y;\omega^\star)\right]$, which finishes the proof. □

Example 7.2. *The inclusion probabilities given by*

$$p^\star(\beta,\{u,v\}) = \begin{cases} 1 & \text{for } \|u-v\| = 1 \\ \frac{\beta}{\|u-v\|^{2d}} \wedge 1 & \text{for } \|u-v\| > 1 \end{cases}$$

satisfy (78).

Proof. For all $x \in v + \mathcal{C}$ and $y \in u + \mathcal{C}$, we have by the triangle inequality

$$\|u-v\| - \sqrt{d} \le \|x-y\| \le \|u-v\| + \sqrt{d},$$

and this already implies that for $\|u-v\| > \sqrt{d}$

$$\frac{1}{\left(\|u-v\| + \sqrt{d}\right)^{2d}} \le \int_{v+\mathcal{C}} \int_{u+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}y \mathrm{d}x \le \frac{1}{\left(\|u-v\| - \sqrt{d}\right)^{2d}}.$$

With a Taylor expansion we see that

$$\begin{aligned} \frac{1}{\|u-v\| \pm \sqrt{d}} &= \frac{1}{\|u-v\|} \frac{1}{1 \pm \frac{\sqrt{d}}{\|u-v\|}} = \frac{1}{\|u-v\|}\left(1 + \mathcal{O}\left(\frac{1}{\|u-v\|}\right)\right) \\ &= \frac{1}{\|u-v\|} + \mathcal{O}\left(\frac{1}{\|u-v\|^2}\right) \end{aligned}$$

and raising this expression to the $2d$-th power already gives that

$$\int_{v+\mathcal{C}} \int_{u+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}y \mathrm{d}x = \frac{1}{\|u-v\|^{2d}} + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right) \tag{82}$$

for $\|u-v\| \to \infty$. With the Taylor expansion of the exponential function we have $1 - e^{-s} = s + \mathcal{O}(s^2)$ for small s and thus by inserting (82) into the definition of $p(\beta,\{u,v\})$ we get

$$p(\beta,\{u,v\}) = 1 - e^{-\beta \int_{v+\mathcal{C}} \int_{u+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}y \mathrm{d}x} = \frac{\beta}{\|u-v\|^{2d}} + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right) \tag{83}$$

which implies that

$$p^\star(\beta,\{u,v\}) = \frac{\beta}{\|u-v\|^{2d}} \wedge 1 = p(\beta,\{u,v\}) + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right).$$

□

Example 7.3. *The inclusion probabilities given by*

$$\tilde{p}(\beta,\{u,v\}) = \begin{cases} 1 & \text{for } \|u-v\| = 1 \\ 1 - e^{-\frac{\beta}{\|u-v\|^{2d}}} & \text{for } \|u-v\| > 1 \end{cases}.$$

satisfy (78).

Proof. By a Taylor expansion of the exponential function we get

$$1 - e^{-\frac{\beta}{\|u-v\|^{2d}}} = \frac{\beta}{\|u-v\|^{2d}} + \mathcal{O}\left(\frac{1}{\|u-v\|^{4d}}\right) = p^\star(\beta,\{u,v\}) + \mathcal{O}\left(\frac{1}{\|u-v\|^{2d+1}}\right),$$

where $p^\star(\beta,\{u,v\}) = \frac{\beta}{\|u-v\|^{2d}} \wedge 1$ is the function from Example 7.2. We already know from Example 7.2 that $p^\star(\beta,\{u,v\})$ satisfies (78). Thus we directly get that $\tilde{p}(\beta,\{u,v\})$ also satisfies (78). □

7.1 The diameter of boxes

Before going to the proof of (80), we prove a technical lemma that we will use later in this section. It follows directly from the Burkholder-Davis-Gundy-inequality [23].

Lemma 7.4. *Let $X_1, \ldots, X_m$ be independent random variables such that $|\mathbb{E}[X_i]| \leq C$ for all $i \in \{1, \ldots, m\}$. Then for all $p \geq 2$, there exists a constant $C' = C'(p, C)$ such that*

$$\mathbb{E}\left[\left|\sum_{i=1}^m X_i\right|^p\right] \leq C' m^{p/2} \max_i \mathbb{E}[|X_i|^p] + C' m^p.$$

Proof. Define $Y_i = X_i - \mathbb{E}[X_i]$. We clearly have

$$\mathbb{E}\left[\left|\sum_{i=1}^m X_i\right|^p\right] = \mathbb{E}\left[\left|\sum_{i=1}^m Y_i + \sum_{i=1}^m \mathbb{E}[X_i]\right|^p\right] \leq 2^p \mathbb{E}\left[\left|\sum_{i=1}^m Y_i\right|^p\right] + 2^p \mathbb{E}\left[\left|\sum_{i=1}^m \mathbb{E}[X_i]\right|^p\right]$$
$$\leq 2^p \mathbb{E}\left[\left|\sum_{i=1}^m Y_i\right|^p\right] + 2^p |mC|^p. \tag{84}$$

The process $M_t = \sum_{i=1}^t Y_i$ is a martingale and thus we get by the BDG-inequality [23] that there exists a constant C_p such that

$$\mathbb{E}\left[\left|\sum_{i=1}^m Y_i\right|^p\right] \leq C_p \mathbb{E}\left[\left(\sum_{i=1}^m Y_i^2\right)^{p/2}\right] = C_p m^{p/2} \left\|\frac{1}{m}\sum_{i=1}^m Y_i^2\right\|_{p/2}^{p/2}$$
$$\leq C_p m^{p/2} \max_i \left\|Y_i^2\right\|_{p/2}^{p/2} = C_p m^{p/2} \max_i \mathbb{E}[|Y_i|^p]. \tag{85}$$

For $i \in \{1, \ldots, m\}$, we have $\mathbb{E}[|Y_i|^p] \leq 2^p \mathbb{E}[|X_i|^p] + 2^p |\mathbb{E}[X_i]|^p \leq 2^p \mathbb{E}[|X_i|^p] + 2^p |C|^p$. Combining this with (84) and (85), we finally get that

$$\mathbb{E}\left[\left|\sum_{i=1}^m X_i\right|^p\right] \leq 2^p \mathbb{E}\left[\left|\sum_{i=1}^m Y_i\right|^p\right] + 2^p |mC|^p \leq 2^p C_p m^{p/2} \max_i \mathbb{E}[|Y_i|^p] + 2^p |mC|^p$$
$$\leq 2^p C_p m^{p/2} \left(\max_i 2^p \mathbb{E}[|X_i|^p] + 2^p |C|^p\right) + 2^p |mC|^p \leq C' m^{p/2} \max_i \mathbb{E}[|Y_i|^p] + C' m^p$$

for an appropriate constant C' depending on p and C only. □

Assume that $(p^\star(\beta, e))_{e \in E}$ satisfies (78). From the proof about the typical distance above it directly follows that there exists a constant $c > 0$, and for all $\varepsilon > 0$ there exists a $c_\varepsilon > 0$, such that

$$\mathbb{P}_\beta^\star\left(\text{Diam}\left(\{0, \ldots, n\}^d\right) > c_\varepsilon n^{\theta(\beta)}\right) > 1 - \varepsilon \text{ and } \mathbb{E}_\beta^\star\left[\text{Diam}\left(\{0, \ldots, n\}^d\right)\right] > c n^{\theta(\beta)}$$

for all $n \in \mathbb{N}$. So we are left to show that

$$\mathbb{P}_\beta^\star\left(\text{Diam}\left(\{0, \ldots, n\}^d\right) \leq C_\varepsilon n^{\theta(\beta)}\right) > 1 - \varepsilon \text{ and } \mathbb{E}_\beta^\star\left[\text{Diam}\left(\{0, \ldots, n\}^d\right)\right] \leq C n^{\theta(\beta)} \tag{86}$$

uniformly over all $n \in \mathbb{N}$, for appropriate constants C, C_ε. In the following, we will show that

$$\mathbb{P}_\beta^\star\left(D_{V_{\mathbf{0}}^{\bar{n}}}\left(\mathbf{0}, (\bar{n} - 1)e_1\right) \leq \bar{S} n^{\theta(\beta)} \text{ for all } \bar{n} \in \{0, \ldots, n\}\right) \geq 0.25 \tag{87}$$

for some constant $\bar{S}$ and all $n \in \mathbb{N}$. From there one can with the same techniques as in Lemma 6.5 and Lemma 6.4 show that (86) holds. Thus, we will focus on (87) from here on. We will only do the case where $n = 2^k$ for $k \in \mathbb{N}$ large enough. The general case follows with Lemma 4.1. We couple the measures $\mathbb{P}_\beta^\star$ and $\mathbb{P}_\beta$, using the standard Harris coupling for percolation. For an edge $e \in E$, we say that it is *non-regular* if $\omega(e) = 1$, but $\omega^\star(e) = 0$. In words, if the edge is open under the measure $\mathbb{P}_\beta$, but closed under the measure $\mathbb{P}_\beta^\star$. Let C_1 be a constant such that

$$\mathbb{P}\left(e \text{ is non-regular} \mid \omega(e) = 1\right) \leq \tfrac{C_1}{|e|}.$$

Such a constant exists by the assumption (78). We will always use C_1 as this constant in the rest of the chapter. The rough strategy of the proof of (79) above was to fill in the gaps that occurred through non-regularities using edges in the nearest-neighbor lattice. Such an approach does not work for the diameter. Instead, we fill in the gaps using a third percolation configuration ω^-, which is contained in $\omega^\star$. For this, we first choose a list of parameters whose origin will be clear later on. We choose $q = \frac{4}{3\theta(\beta)}$, and we choose $\beta_- \in [0, \beta)$, $\varepsilon > 0$ such that

$$\theta(\beta_-)q - 1 + \varepsilon q = 0.5 \text{ and } \frac{2^{3q(\theta(\beta_-)-\theta(\beta))}}{2.2} < \frac{1}{2.1} \tag{88}$$

which is possible, as the function $\beta \mapsto \theta(\beta)$ is continuous in β by Theorem 1.5. These definitions seem quite arbitrary at the moment, but they are chosen in a way so that the proof works. The third percolation configuration ω^- is distributed according to the measure $\mathbb{P}_{\beta_-}$. So we can couple the three percolation configurations $\omega, \omega^\star$, and ω^- using the standard Harris coupling for percolation. We write $\mathbb{P}$ for the joint measure. We have that $p(\beta_-, e) \leq p^\star(\beta, e)$ for all edges e that are sufficiently long, which follows directly from (78). In the following, we will even assume that $p(\beta_-, e) \leq p^\star(\beta, e)$ for all edges e. Removing this assumption is relatively easy, as all nearest-neighbor edges are open. This already implies that $D(x, y; \omega^\star) \leq D(x, y; \omega^-)$ for all points $x, y \in \mathbb{Z}^d$. With this, we are ready to go to the proof of (87), which already implies (80).

Proof of (87). Define the event $\mathcal{A}$ by

$$\mathcal{A} = \bigcap_{l=0}^{k} \bigcap_{a \in V_{\mathbf{0}}^{2^{k-l}}} \left\{ \text{Diam}\left(V_a^{2^l}; \omega^-\right) \leq 2^{l\theta(\beta_-)} 2^{\varepsilon k} \right\}.$$

For k large enough, we have $\mathbb{P}\left(\mathcal{A}\right) \geq 0.5$, as we will argue now. Using that

$$\sup_{l \in \mathbb{N}} \mathbb{E}\left[\exp\left(\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^l}; \omega^-\right)}{2^{l\theta(\beta_-)}}\right)\right] < \infty$$

by Theorem 6.1, we get that for some constant C

$$\mathbb{P}\left(\mathcal{A}^C\right) = \mathbb{P}\left(\exists l \in \{0, \dots, k\}, a \in V_{\mathbf{0}}^{2^{k-l}} : \text{Diam}\left(V_a^{2^l}; \omega^-\right) > 2^{l\theta(\beta_-)} 2^{\varepsilon k}\right)$$

$$\leq \sum_{l=0}^{k} \sum_{a \in V_{\mathbf{0}}^{2^{k-l}}} \mathbb{P}\left(\text{Diam}\left(V_a^{2^l}; \omega^-\right) > 2^{l\theta(\beta_-)} 2^{\varepsilon k}\right) \leq \sum_{l=0}^{k} 2^{d(k-l)} \mathbb{P}\left(\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^l}; \omega^-\right)}{2^{l\theta(\beta_-)}} > 2^{\varepsilon k}\right)$$

$$\leq \sum_{l=0}^{k} 2^{d(k-l)} Ce^{-2^{\varepsilon k}} = Ce^{-2^{\varepsilon k}} \sum_{l=0}^{k} 2^{dl} < 0.5$$

for k large enough. Assume that $\mathcal{A}$ holds, and let $a \in V_{\mathbf{0}}^{2^{k-l}}$, $u, v \in V_a^{2^l}$. Assume that $2^{m-1} < \|u - v\| \leq 2^m$. Then u, v are either in the same box $V_w^{2^m}$, or in adjacent boxes $V_{w_1}^{2^m}, V_{w_2}^{2^m}$ with $\|w_1 - w_2\|_\infty = 1$. This implies that $D_{V_a^{2^l}}(u, v; \omega^-) \leq 2 \cdot 2^{m\theta(\beta_-)} 2^{\varepsilon k} + 1 \leq 4 \cdot \|u - v\|^{\theta(\beta_-)} 2^{\varepsilon k} + 1$. So if the event $\mathcal{A}$ holds, then for all $u, v \in V_a^{2^l}$

$$D_{V_a^{2^l}}(u, v; \omega^-) \leq 5\|u - v\|^{\theta(\beta_-)} 2^{\varepsilon k}.$$

For $a \in V_{\mathbf{0}}^{2^{k-l}}$, let P be a geodesic between $x = 2^l a$ and $y = 2^l a + (2^l - 1)e_1$ in the set $V_a^{2^l}$ for the environment ω. We construct a path between $2^l a$ and $2^l a + (2^l - 1)e_1$ in the environment $\omega^\star$ as follows:

- For $\{u, v\} \in P$, if $E_{u,v}^\star$ occurs we use the direct edge between u and v.
- For $\{u, v\} \in P$, if $E_{u,v}^\star$ does not occur go from u to v using the shortest path in the set $V_a^{2^l}$ in the environment ω^-.

This gives a path $P^\star$ between $2^l a$ and $2^l a + (2^l - 1)e_1$ in the environment $\omega^\star$, as we assumed that all edges contained in ω^- are also contained in $\omega^\star$. The path $P^\star$ is also contained in $V_a^{2^l}$. Write $X_{\{u,v\}}$ for the distance $D_{V_a^{2^l}}(u, v; \omega^\star)$. The random variable $X_{\{u,v\}}$ is either 1 or $D_{V_a^{2^l}}(u, v; \omega^-)$. We define the random variable $X'_{\{u,v\}}$ by

$$X'_{\{u,v\}} = \begin{cases} 1 & \text{if } X_{\{u,v\}} = 1 \\ \min\left(\|u - v\|, 5\|u - v\|^{\theta(\beta_-)} 2^{\varepsilon k}\right) & \text{else} \end{cases},$$

so in particular we have $X_{\{u,v\}} \leq X'_{\{u,v\}}$ on the event $\mathcal{A}$, and this already implies that

$$D_{V_a^{2^l}}(x, y; \omega^\star) \leq \sum_{\{u,v\} \in P} X'_{\{u,v\}}. \tag{89}$$

The important thing about the random variables X'_e is that they are independent for different edges $e \in P$, as it is independent for different edges whether they are non-regular. Next, we want to estimate the first and the q-th moment of the random variable $X'_{\{u,v\}}$. For the expectation we ge that

$$\mathbb{E}\left[X'_{\{u,v\}} \mid \omega(\{u, v\}) = 1, \mathcal{A}\right] \leq 1 + \|u - v\| \frac{C_1}{\|u - v\|} = 1 + C_1,$$

whereas for the q-th moment we see that

$$\begin{aligned} \mathbb{E}\left[\left(X'_{\{u,v\}}\right)^q \mid \omega(\{u, v\}) = 1, \mathcal{A}\right] &\leq 1 + 5\|u - v\|^{\theta(\beta_-)q} 2^{\varepsilon kq} \frac{C_1}{\|u - v\|} \\ &\leq 1 + 5C_1\|u - v\|^{\theta(\beta_-)q - 1} 2^{\varepsilon kq} \leq C_2 2^{k(\theta(\beta_-)q - 1 + \varepsilon q)}, \end{aligned}$$

for some constant C_2, as $\theta(\beta_-)q > 1$. Using Lemma 7.4, we see that there exists a constant $C < \infty$ such that

$$\mathbb{E}\left[D_{V_a^{2^l}}(x, y; \omega^\star)^q | \mathcal{A}, \omega\right] \leq \mathbb{E}\left[\left(\sum_{\{u,v\} \in P} X'_{\{u,v\}}\right)^q \,\Big|\, \mathcal{A}, \omega\right]$$

$$\leq CD_{V_a^{2^l}}(x,y;\omega)^{q/2} C_2 2^{k(\theta(\beta_-)q-1+\varepsilon q)} + CD_{V_a^{2^l}}(x,y;\omega)^q$$
$$= CD_{V_a^{2^l}}(x,y;\omega)^{q/2} C_2 2^{0.5k} + CD_{V_a^{2^l}}(x,y;\omega)^q,$$

and now taking expectation with respect to ω yields

$$\mathbb{E}\left[D_{V_a^{2^l}}(x,y;\omega^\star)^q|\mathcal{A}\right] \leq \mathbb{E}\left[CD_{V_a^{2^l}}(x,y;\omega)^{q/2} C_2 2^{k(\theta(\beta_-)q-1+\varepsilon)} + CD_{V_a^{2^l}}(x,y;\omega)^q|\mathcal{A}\right]$$
$$\leq \tilde{C}\|x-y\|^{\theta(\beta)q/2} 2^{0.5k} + \tilde{C}\|x-y\|^{\theta(\beta)q}$$

for some constant $\tilde{C}$. Here we also used that $\mathbb{P}(\mathcal{A}) \geq 0.5$, and thus for all $r>0$ the r-th moment of $D_{V_a^{2^l}}(x,y;\omega)$ is of order $\|x-y\|^{r\theta(\beta)}$, under the measure $\mathbb{P}(\cdot|\mathcal{A})$. Assume that $\|x-y\|_\infty = 2^{\gamma k}$ with $\gamma > \frac{3}{4}$. Then we get

$$\mathbb{E}\left[D_{V_a^{2^l}}(x,y;\omega^\star)^q|\mathcal{A}\right] \leq \tilde{C}\|x-y\|^{\theta(\beta)q/2} 2^{k(\theta(\beta_-)q-1+\varepsilon)} + \tilde{C}\|x-y\|^{\theta(\beta)q}$$
$$\leq C'\left(2^{k\left(\frac{\gamma\theta(\beta)q}{2}+0.5\right)} + 2^{\gamma k\theta(\beta)q}\right) = C'\left(2^{k\left(\frac{\gamma 2}{3}+0.5\right)} + 2^{k\frac{\gamma 4}{3}}\right) \leq C'' 2^{k\frac{\gamma 4}{3}} \leq C'''\|x-y\|^{q\theta(\beta)}$$

for some constants $C', C'', C''' < \infty$. The second last inequality holds as $\frac{\gamma 2}{3} + 0.5 < \frac{\gamma 4}{3}$ for $\gamma > \frac{3}{4}$. Using Markov's inequality we see that there exists a constant $C < \infty$ such that for all $l > \frac{3}{4}k$, $a \in V_{\mathbf{0}}^{2^{k-l}}$, and $S \geq 1$

$$\mathbb{P}\left(D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^\star\right) > S 2^{k\theta(\beta)} 1.1^{(l-k)\theta(\beta)} \mid \mathcal{A}\right)$$
$$\leq \mathbb{P}\left(\left(\frac{D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^\star\right)}{2^{\theta(\beta)l}}\right)^q > S^q \left(\frac{2}{1.1}\right)^{(k-l)4/3} \mid \mathcal{A}\right)$$
$$\leq CS^{-q}\left(\frac{2}{1.1}\right)^{-(k-l)4/3} \leq CS^{-q}\left(\frac{1}{2.2}\right)^{k-l}.$$

On the other hand, for $l \leq \frac{3}{4}k$ we have $l \leq 3(k-l)$, which implies that

$$\mathbb{P}\left(D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^\star\right) > S 2^{k\theta(\beta)} 1.1^{(l-k)\theta(\beta)} \mid \mathcal{A}\right)$$
$$\leq \mathbb{P}\left(D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^-\right) > S 2^{k\theta(\beta)-l\theta(\beta)} 2^{l\theta(\beta)} 1.1^{(l-k)\theta(\beta)} \mid \mathcal{A}\right)$$
$$= \mathbb{P}\left(\left(\frac{D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^-\right)}{2^{l\theta(\beta_-)}}\right)^q > S^q \left(\frac{2}{1.1}\right)^{(k-l)\frac{4}{3}} 2^{l(\theta(\beta)-\theta(\beta_-))q} \mid \mathcal{A}\right)$$
$$\leq CS^{-q}\left(\frac{1}{2.2}\right)^{k-l} 2^{ql(\theta(\beta_-)-\theta(\beta))} \leq CS^{-q}\left(\frac{1}{2.2}\right)^{k-l} 2^{q3(k-l)(\theta(\beta_-)-\theta(\beta))} \leq CS^{-q}\left(\frac{1}{2.1}\right)^{k-l}$$

where the last inequality holds because of our assumption on β_- (88). So in total we see that there exists a constant C such that for all $k \in \mathbb{N}$, $l \in \{0,\dots,k\}$, and $a \in V_{\mathbf{0}}^{2^{k-l}}$ one has

$$\mathbb{P}\left(D_{V_a^{2^l}}\left(2^l a, 2^l a + (2^l-1)e_1; \omega^\star\right) > S 2^{k\theta(\beta)} 1.1^{(l-k)\theta(\beta)} \mid \mathcal{A}\right) \leq CS^{-q}\left(\frac{1}{2.1}\right)^{k-l}.$$

Write Ω^S for the event

$$\Omega^S = \bigcap_{l=0}^{k} \bigcap_{j=0}^{2^{k-l}-1} \left\{D_{V_{2^l je_1}^{2^l}}\left(2^l je_1, 2^l je_1 + (2^l-1)e_1; \omega^\star\right) \leq S 2^{k\theta(\beta)} 1.1^{(l-k)\theta(\beta)}\right\}$$

we get with a union bound that

$$\mathbb{P}\left(\left(\Omega^S\right)^C \mid \mathcal{A}\right) \leq \sum_{l=0}^{k} \sum_{j=0}^{2^{k-l}-1} \mathbb{P}\left(D_{V_{2^l je_1}^{2^l}}\left(2^l je_1, 2^l je_1 + (2^l-1)e_1; \omega^\star\right) > S 2^{k\theta(\beta)} 1.1^{(l-k)\theta(\beta)} \;\Big|\; \mathcal{A}\right)$$
$$\leq \sum_{l=0}^{k} 2^{k-l} C S^{-q} \left(\frac{1}{2.1}\right)^{k-l} < 0.5$$

for S large enough. Thus we get that $\mathbb{P}\left(\Omega^S\right) \geq \mathbb{P}\left(\Omega^S \mid \mathcal{A}\right) \mathbb{P}\left(\mathcal{A}\right) > 0.25$. Using a dyadic path between $\mathbf{0}$ and $(\bar{n}-1)e_1$, one can show that on the event Ω^S one has $D_{V_{\mathbf{0}}^{\bar{n}}}(\mathbf{0}, (\bar{n}-1)e_1) \leq C(\theta(\beta)) S n^{\theta(\beta)}$ for some constant $C(\theta(\beta))$, depending on $\theta(\beta)$ only. This shows (87) and thus finishes the proof. □

8 Russo's formula for expectations

In this chapter, we establish one of our main tools in the proofs of Theorem 1.3 and Theorem 1.4, which is a version of Russo's formula. The classical Russo's formula, also called Russo-Margulis lemma, see for example [62, Section 1.3] or [81, 88] for the original papers, is a formula for i.i.d. bond percolation. It states that for any finite graph (V, E) and any increasing event A

$$\frac{\mathrm{d}}{\mathrm{d}p} \mathbb{P}_p\left(A\right) = \sum_{e \in E} \mathbb{P}_p\left(e \text{ is pivotal for } A\right), \tag{90}$$

where we say that an edge e is pivotal for an event A when changing the status of e also changes the occurrence of the event A. Note that it does not depend on the occupation status of the edge e whether e is pivotal for A. Russo's formula (90) tells us how the probability of an event changes for i.i.d. percolation when varying the connection probability p. We modify this formula in two ways. First of all, we adapt it to long-range percolation, where the inclusion probabilities of the edges are not identically distributed. Secondly, we develop a formula that determines the derivative of the expectation of a general function rather than just the probability of a given event.

Lemma 8.1 (Russo's formula for expectations)**.** *Let $G = (V, E)$ be a finite graph with a set of inclusion probabilities $(p(\beta, e))_{e \in E, \beta \geq 0}$, where $\beta \mapsto p(\beta, e)$ is continuously differentiable on $\mathbb{R}_{\geq 0}$ for all $e \in E$. By $\mathbb{P}_\beta$ we denote the Bernoulli product measure on $\{0,1\}^E$ with inclusion probabilities $(p(\beta, e))_{e \in E}$ and its expectation by $\mathbb{E}_\beta$. Let $f : \{0,1\}^E \to \mathbb{R}$ be a function. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \mathbb{E}_\beta\left[f(\omega)\right] = \sum_{e \in E} p'(\beta, e) \mathbb{E}_\beta\left[f(\omega^{e+}) - f(\omega^{e-})\right]. \tag{91}$$

The lemma is stated for any set of continuously differentiable functions $p(\beta, e)$, but one can also always think of the case where $p(\beta, \{u, v\}) = 1 - e^{-\beta \int_{u+\mathcal{C}} \int_{v+\mathcal{C}} \frac{1}{\|x-y\|^{2d}} \mathrm{d}x \mathrm{d}y}$, as we only apply it to this case.

Proof of Lemma 8.1. The proof is similar to the case of the classical Russo's formula, see for example [62]. For a vector $\vec{\beta} = (\beta_e)_{e \in E} \in \mathbb{R}_{\geq 0}^E$, we define the probability measure $\mathbb{P}_{\vec{\beta}}$ on $\{0,1\}^E$ by

$$\mathbb{P}_{\vec{\beta}}(\omega) = \prod_{e : \omega(e) = 1} p(\beta_e, e) \prod_{e : \omega(e) = 0} \left(1 - p(\beta_e, e)\right)$$

so that each component $\omega(e)$ is Bernoulli distributed with expectation $p(\beta_e, e)$, and all components are independent. Under this measure, a function $f : \{0,1\}^E \to \mathbb{R}$ has the expectation

$$\mathbb{E}_{\vec{\beta}}\left[f(\omega)\right] = \sum_{\omega\in\{0,1\}^E} f(\omega) \prod_{e:\omega(e)=1} p(\beta_e, e) \prod_{e:\omega(e)=0} (1 - p(\beta_e, e)).$$

For an edge $f \in E$, differentiation with respect to β_f gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta_f}\mathbb{E}_{\vec{\beta}}\left[f(\omega)\right] = \sum_{\omega\in\{0,1\}^E} f(\omega)\frac{\mathrm{d}}{\mathrm{d}\beta_f}\left(\prod_{e:\omega(e)=1} p(\beta_e, e) \prod_{e:\omega(e)=0} (1 - p(\beta_e, e))\right) \\
&= \sum_{\omega\in\{0,1\}^E} f(\omega)p'(\beta_f, f)\left(\mathbb{1}_{\omega(f)=1} - \mathbb{1}_{\omega(f)=0}\right) \prod_{\substack{e\in E\setminus\{f\}:\\ \omega(e)=1}} p(\beta_e, e) \prod_{\substack{e\in E\setminus\{f\}:\\ \omega(e)=0}} (1 - p(\beta_e, e)) \\
&= p'(\beta_f, f) \sum_{\omega\in\{0,1\}^E:\omega(f)=1} f(\omega) \prod_{e\in E\setminus\{f\}:\omega(e)=1} p(\beta_e, e) \prod_{e\in E\setminus\{f\}:\omega(e)=0} (1 - p(\beta_e, e)) \\
&- p'(\beta_f, f) \sum_{\omega\in\{0,1\}^E:\omega(f)=0} f(\omega) \prod_{e\in E\setminus\{f\}:\omega(e)=1} p(\beta_e, e) \prod_{e\in E\setminus\{f\}:\omega(e)=0} (1 - p(\beta_e, e)) \\
&= p'(\beta_f, f)\mathbb{E}_{\vec{\beta}}\left[f(\omega^{f+})\right] - p'(\beta_f, f)\mathbb{E}_{\vec{\beta}}\left[f(\omega^{f-})\right] = p'(\beta_f, f)\mathbb{E}_{\vec{\beta}}\left[f(\omega^{f+}) - f(\omega^{f-})\right].
\end{aligned}$$

To conclude, consider the mapping $\phi : \mathbb{R} \mapsto \mathbb{R}^E$ defined by $\phi(\beta) = (\beta, \dots, \beta)$. With this and the chain rule we finally get

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}_\beta\left[f(\omega)\right] = \frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}_{\phi(\beta)}\left[f(\omega)\right] = \sum_{e\in E}\frac{\mathrm{d}}{\mathrm{d}\beta_e}\mathbb{E}_\beta\left[f(\omega)\right] = \sum_{e\in E} p'(\beta, e)\mathbb{E}_\beta\left[f(\omega^{e+}) - f(\omega^{e-})\right].$$

□

We now consider the case where $p(\beta, \{u,v\}) = 1 - e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y}$. Note that $p(\beta, e)$ decays like $\frac{\beta}{|e|^{2d}}$ as $|e|$ tends to infinity. By the triangle inequality we have for all $x \in u + \mathcal{C}, y \in v + \mathcal{C}$

$$\|u - v\| - \sqrt{d} \leq \|x - y\| \leq \|u - v\| + \sqrt{d}$$

and thus, for $\|u - v\| \geq \sqrt{d}$ we can bound the integral in the exponent from above and below by

$$\frac{1}{\left(\|u - v\| + \sqrt{d}\right)^{2d}} \leq \int_{v+\mathcal{C}}\int_{u+\mathcal{C}} \frac{1}{\|x - y\|^{2d}}\mathrm{d}y\mathrm{d}x \leq \frac{1}{\left(\|u - v\| - \sqrt{d}\right)^{2d}}. \tag{92}$$

Also note that we have for all edges $\{u, v\}$ with $\|u - v\|_\infty \geq 2$ that

$$1 \geq \int_{v+\mathcal{C}}\int_{u+\mathcal{C}} \frac{1}{\|x - y\|^{2d}}\mathrm{d}y\mathrm{d}x \tag{93}$$

as the integrand is bounded by 1. Next, we consider the derivative of $p(\beta, e)$ for non-nearest neighbor edges $e = \{u, v\}$. By the chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta}p(\beta, e) = \int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{1}{\|x - y\|^{2d}}\mathrm{d}x\mathrm{d}y\ e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y}$$

and using (92) and (93) we see that for edges $\{u,v\}$ with $\|u-v\|_\infty > 1$ and $\|u-v\| \geq \sqrt{d}$

$$p'(\beta,\{u,v\}) = \begin{cases} \leq & \frac{1}{\left(\|u-v\|-\sqrt{d}\right)^{2d}} \\ \geq & \frac{e^{-\beta}}{\left(\|u-v\|+\sqrt{d}\right)^{2d}} \end{cases} \tag{94}$$

and thus in particular for fixed $\beta > 0$ we have $p'(\beta,\{u,v\}) = \Theta\left(\frac{1}{\|u-v\|^{2d}}\right)$ for $\|u-v\| \to \infty$. For every non-nearest-neighbor edge e the probability that e exists is upper bounded by $1-e^{-\beta}$. This implies that

$$\left|\mathbb{E}_\beta\left[f(\omega^{e+}) - f(\omega)\right]\right| \leq \left|\mathbb{E}_\beta\left[f(\omega^{e+}) - f(\omega^{e-})\right]\right| \leq \frac{1}{e^{-\beta}}\left|\mathbb{E}_\beta\left[f(\omega^{e+}) - f(\omega)\right]\right|$$

for all functions $f : \{0,1\}^E \to \mathbb{R}$ and all edges $e \in E$ with $|e| \geq 2$. Above we bounded the derivative $p'(\beta,e)$ from above and below. We also want to bound the connection probability $p(\beta,e)$. As $1-e^{-s} \geq \frac{s}{2} \wedge \frac{1}{2}$ for all $s \geq 0$ one has that

$$p(\beta,\{u,v\}) \geq \int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{2\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y \wedge \frac{1}{2} \overset{(92)}{\geq} \frac{\beta}{2\left(\|u-v\|+\sqrt{d}\right)^{2d}} \wedge \frac{1}{2}. \tag{95}$$

On the other hand one has $1-e^{-x} \leq x$ and thus one can upper bound the connection probability by

$$p(\beta,\{u,v\}) \leq \int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{\|x-y\|^{2d}} \mathrm{d}x\mathrm{d}y \overset{(92)}{\leq} \frac{\beta}{\left(\|u-v\|-\sqrt{d}\right)^{2d}} \tag{96}$$

for $\|u-v\| \geq \sqrt{d}$.

9 Asymptotic behavior of $\theta(\beta)$ for small β and $d=1$

In this section, we prove Theorem 1.3, i.e., that $\theta(\beta) = 1-\beta+o(\beta)$ for $\beta \to 0$ for $d=1$. Determining the asymptotic behavior of $\theta(\beta)$ for dimension two or higher for $\beta \to 0$ is more difficult for several reasons. First, there is no lower bound on $\theta(\beta)$ that arises from considering cut points or something similar. The notion of cut points and its implication on the distance exponent $\theta(\beta)$ in dimension $d=1$ will be explained below. Secondly, it is not clear which pair of vertices $x,y \in V_{\mathbf{0}}^n$ minimizes the expected distance $\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^n}(x,y)\right]$ in dimension two or higher, i.e., whether a similar statement of equation (101) holds for $d \geq 2$. However, for all dimensions d there exists a constant $c > 0$ such that $\theta(\beta) \leq 1 - c\beta$ for β small enough. This can already be shown with the exact same technique that was used in [27].

But now let us consider dimension $d=1$ again. Here we have $\theta(0)=1$ and it is well known that $\theta(\beta) \geq 1-\beta$ (see [27,33]). So we get that $\liminf_{\beta\to 0} \frac{\theta(\beta)-\theta(0)}{\beta} \geq -1$. Thus it suffices to show that $\limsup_{\beta\to 0} \frac{\theta(\beta)-\theta(0)}{\beta} \leq -1$. For the sake of completeness, we give a short sketch of the proof of the lower bound $\theta(\beta) \geq 1-\beta$. For this, we define the notion of a cut point. We say that the vertex $w \in \{1,\ldots,n-2\}$ is a cut point if there exists no edge $\{u,v\}$ with $0 \leq u < w < v \leq n-1$. We have

$$\mathbb{P}_\beta\left(w \text{ is a cut point}\right) = \prod_{0\leq u<w}\prod_{w<v\leq n-1} e^{-\beta\int_u^{u+1}\int_v^{v+1}\frac{1}{|x-y|^2}\mathrm{d}x\mathrm{d}y} = e^{-\beta\int_0^w\int_{w+1}^n\frac{1}{|x-y|^2}\mathrm{d}x\mathrm{d}y}$$

$$\geq e^{-\beta \int_0^w \int_{w+1}^\infty \frac{1}{|x-y|^2} \mathrm{d}x\mathrm{d}y} = e^{-\beta \int_0^w \frac{1}{w+1-y} \mathrm{d}y} = e^{-\beta \log(w+1)} \geq n^{-\beta}. \tag{97}$$

As the distance between 0 and $n-1$ is lower bounded by the number of cut points between 0 and $n-1$ we get, by linearity of expectation, that

$$\mathbb{E}_\beta \left[D_{[0,n-1]}(0,n-1) \right] \geq \mathbb{E}_\beta \left[|\{w : w \text{ is a cut point}\}| \right] \geq (n-2)n^{-\beta} = \Omega(n^{1-\beta}) \tag{98}$$

which shows that $\theta(\beta) \geq 1-\beta$. As a first step towards the proof of Theorem 1.3, we remind ourselves about the submultiplicativity of the expected distance, which was proven in Lemma 2.3. For all dimensions d and all $\beta \geq 0$, the sequence

$$\Lambda(n) = \Lambda(n,\beta) := \max_{u,v \in \{0,\ldots,n-1\}^d} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^n}(u,v) \right] + 1$$

is submultiplicative and furthermore one has

$$\theta(\beta) = \inf_{n \geq 2} \frac{\log(\Lambda(n,\beta))}{\log(n)}. \tag{99}$$

Now we are prepared to prove Theorem 1.3. Our main tools for this are Lemma 2.3 and Russo's formula for expectations (91).

Proof of Theorem 1.3. Note that $\Lambda(n,0) = n$ and thus $\frac{\log(\Lambda(n,0))}{\log(n)} = 1$. Using this and (99) we obtain

$$\begin{aligned}
\limsup_{\beta \searrow 0} \frac{\theta(\beta) - \theta(0)}{\beta} &= \limsup_{\beta \searrow 0} \inf_{n \geq 2} \frac{\log(\Lambda(n,\beta)) - \log(\Lambda(n,0))}{\beta \log(n)} \\
&\leq \inf_{n \geq 2} \limsup_{\beta \searrow 0} \frac{\log(\Lambda(n,\beta)) - \log(\Lambda(n,0))}{\beta \log(n)} \\
&= \inf_{n \geq 2} \frac{1}{\log(n)} \frac{\mathrm{d}}{\mathrm{d}\beta} \log(\Lambda(n,\beta)) \Big|_{\beta=0} \\
&= \inf_{n \geq 2} \frac{1}{\log(n)\Lambda(n,0)} \frac{\mathrm{d}}{\mathrm{d}\beta} \Lambda(n,\beta) \Big|_{\beta=0}
\end{aligned} \tag{100}$$

and this works, as for fixed n the function $\Lambda(n,\beta)$ is differentiable at $\beta = 0$, as the inclusion probabilities $p(\beta, \{u,v\})$ are. Now we want to calculate $\frac{\mathrm{d}}{\mathrm{d}\beta} \Lambda(n,\beta)\Big|_{\beta=0}$. For this, let E be the set of all edges of length at least 2 in the graph with vertex set $\{0,\ldots,n-1\}$. For $e \in E$, let ω^{e+} be the environment, where we added the edge e (or do nothing in case it already existed before). For β very small compared to $\frac{1}{n}$ we have that $\max_{u,v \in \{0,\ldots,n-1\}} \mathbb{E}_\beta \left[D_{[0,n-1]}(u,v) \right] = \mathbb{E}_\beta \left[D_{[0,n-1]}(0,n-1) \right]$. To see this, note that on the one hand for any $u,v \in \{0,\ldots,n-1\}$ we have

$$\mathbb{E}_\beta \left[D_{[0,n-1]}(u,v) \right] \leq |u-v|,$$

whereas on the other hand we have

$$\mathbb{E}_\beta \left[D_{[0,n-1]}(0,n-1) \right] \geq (n-1) \mathbb{P}_\beta \left(\bigcap_{e \in E} \{e \text{ closed}\} \right).$$

As the probability of the event $\bigcap_{e \in E} \{e \text{ closed}\}$ tends to 1 for $\beta \to 0$ we see that

$$\mathbb{E}_\beta \left[D_{[0,n-1]}(0,n-1) \right] = \max_{u,v \in \{0,\ldots,n-1\}} \mathbb{E}_\beta \left[D_{[0,n-1]}(u,v) \right] \tag{101}$$

for small enough β, where small enough of course depends on n. Using this observation we see that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(n,\beta)\Big|_{\beta=0} = \lim_{\beta\searrow 0}\frac{\Lambda(n,\beta)-\Lambda(n,0)}{\beta} = \lim_{\beta\searrow 0}\frac{\mathbb{E}_\beta\left[D_{[0,n-1]}(0,n-1)\right]-\mathbb{E}_0\left[D_{[0,n-1]}(0,n-1)\right]}{\beta}$$
$$= \sum_{e\in E} p'(0,e)\mathbb{E}_0\left[D_{[0,n-1]}(0,n-1;\omega^{e+})-(n-1)\right]. \tag{102}$$

In the environment ω^{e+} sampled by $\mathbb{P}_0$, where only the nearest-neighbor edges and the edge e are present, the shortest path from 0 to $n-1$ will also take the edge e. By taking the edge e, the distance between 0 and $n-1$ decreases by $|e|-1$, and thus equals $n-1-(|e|-1)=n-|e|$. For $d=1$, we get from (94) that $p'(0,\{u,v\}) \geq \frac{1}{(|u-v|+1)^2}$. With this we can upper bound the derivative computed in (102) and obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(n,\beta)\Big|_{\beta=0} = \sum_{e\in E} p'(0,e)\mathbb{E}_0\left[D_{[0,n-1]}(0,n-1;\omega^{e+})-(n-1)\right] = -\sum_{e\in E} p'(0,e)(|e|-1)$$
$$\leq -\sum_{e\in E}\frac{1}{(|e|+1)^2}(|e|-1) = \sum_{k=0}^{n-3}\sum_{j=k+2}^{n-1}\frac{1-|j-k|}{(j-k+1)^2} = \sum_{k=0}^{n-3}\sum_{l=2}^{n-1-k}\frac{1-l}{(l+1)^2}. \tag{103}$$

For $l\in\mathbb{N}$, we have $\frac{-l}{(l+1)^2}\leq\frac{2}{l^2}-\frac{1}{l}$, as we will show now. One has

$$\frac{-l}{(l+1)^2}\leq\frac{2}{l^2}-\frac{1}{l} \Leftrightarrow -l^3 \leq (l+1)^2(2-l) = (l^2+2l+1)(2-l)$$
$$\Leftrightarrow l^3 \geq (l^2+2l+1)(l-2) = l^3-2l^2+2l^2-4l+l-2 \Leftrightarrow\ 0\geq -3l-2$$

and the last line is clearly true. Using that $\frac{-l}{(l+1)^2}\leq\frac{2}{l^2}-\frac{1}{l}$ we also get that $\frac{1-l}{(l+1)^2}\leq \frac{1}{(l+1)^2}+\frac{2}{l^2}-\frac{1}{l}\leq\frac{3}{l^2}-\frac{1}{l}$. Inserting this into (103) we get that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(n,\beta)\Big|_{\beta=0} \leq \sum_{k=0}^{n-3}\sum_{l=2}^{n-1-k}\frac{3}{l^2}-\frac{1}{l} \leq \sum_{k=0}^{n-3}\sum_{l=2}^{\infty}\frac{3}{l^2}+\sum_{k=0}^{n-3}\sum_{l=2}^{n-1-k}\frac{-1}{l}$$
$$\leq 3n+\sum_{k=0}^{n-3}\sum_{l=2}^{n-1-k}\frac{-1}{l} \leq 4n+\sum_{k=0}^{n-3}\int_1^{n-k}\frac{-1}{s}\mathrm{d}s = 4n-\sum_{k=0}^{n-3}\log(n-k)$$
$$= 4n-\sum_{k=3}^{n}\log(k) \leq 4n-\int_2^n\log(s)\mathrm{d}s = 4n-\Big[-s+s\log(s)\Big]_2^n$$
$$\leq 5n+4-n\log(n).$$

Inserting this into (100) gives

$$\limsup_{\beta\searrow 0}\frac{\theta(\beta)-\theta(0)}{\beta} \leq \inf_{n\geq 2}\frac{1}{\Lambda(n,0)\log(n)}\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(n,\beta)\Big|_{\beta=0}$$
$$= \inf_{n\geq 2}\frac{1}{n\log(n)}\frac{\mathrm{d}}{\mathrm{d}\beta}\Lambda(n,\beta)\Big|_{\beta=0} \leq \inf_{n\geq 2}\frac{5n+4-n\log(n)}{n\log(n)} \leq -1$$

where the infimum is achieved when taking $n\to\infty$. As $\theta(\beta)\geq 1-\beta$, and thus $\liminf_{\beta\searrow 0}\frac{\theta(\beta)-\theta(0)}{\beta}\geq -1$, this finishes the proof of Theorem 1.3. □

10 Strict monotonicity of the distance exponent

In this chapter, we prove Theorem 1.4, i.e., that the function $\theta(\beta)$ is strictly decreasing in β. It was known before, see [27,33] and section 2.2, that $\theta(\beta)$ is strictly decreasing at $\beta = 0$, which is equivalent to saying that $\theta(\beta) < 1 = \theta(0)$ for all $\beta > 0$. With the *Harris coupling* (cf. [62]) one can see that the function $\theta(\beta)$ is non-increasing. For this coupling, let $(U_e)_{e\in E}$ be a collection of i.i.d. random variables that are uniformly distributed on $[0,1]$ and then set $\omega(e) := \mathbb{1}_{\{U_e \leq p(\beta,e)\}}$. Then ω is distributed according to the law of $\mathbb{P}_\beta$ and for $\beta \leq \beta'$ one has $\omega(e) = \mathbb{1}_{\{U_e \leq p(\beta,e)\}} \leq \mathbb{1}_{\{U_e \leq p(\beta',e)\}} = \omega'(e)$. So in particular the environment defined by ω' contains all edges defined by ω, and thus $D(u,v;\omega') \leq D(u,v;\omega)$ for all $u,v \in \mathbb{Z}^d$. Taking expectations on both sides of this inequality and letting $\|u-v\| \to \infty$ already shows that $\theta(\cdot)$ is non-increasing.

Before going into the details of the proof of the strict monotonicity, we want to show the main idea. One of the main tools is again Russo's formula for expectations (91). We know that $\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})] = \Theta\left(n^{\theta(\beta)}\right)$, as proven in Theorem 1.1, and thus $\theta(\beta) = \lim_{n\to\infty} \frac{\log\left(\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})]\right)}{\log(n)}$. But for fixed n we can calculate the derivative of $\frac{\log\left(\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})]\right)}{\log(n)}$ with Lemma 8.1 and get that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\log\left(\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1})\right]\right)}{\log(n)} &= \frac{1}{\log(n)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1})\right]}\frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1})\right]\\ &= \frac{1}{\log(n)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1})\right]}\sum_{e\in E}p'(\beta,e)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1};\omega^{e+}) - D_{V_{\mathbf{0}}^{n+1}}(\mathbf{0},n\mathbf{1};\omega^{e-})\right]\end{aligned} \tag{104}$$

where E is the set of edges with both endpoints in $V_{\mathbf{0}}^{n+1}$. For ease of notation, we drop the subscript of $V_{\mathbf{0}}^{n+1}$ in the paragraph below and will implicitly always think of this graph as the underlying graph. A fully formal proof is given in section 10.2. Our goal is to show that for each $\beta > 0$, there exists a $c(\beta) < 0$ such that $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\log\left(\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})]\right)}{\log(n)} < c(\beta)$ uniformly over n. For this, it clearly suffices to consider n large enough, as the bound clearly holds for small n. If we prove this we get that

$$\begin{aligned}\theta(\beta+\varepsilon) - \theta(\beta) &= \lim_{n\to\infty}\left\{\frac{\log\left(\mathbb{E}_{\beta+\varepsilon}\left[D(\mathbf{0},n\mathbf{1})\right]\right)}{\log(n)} - \frac{\log\left(\mathbb{E}_{\beta}\left[D(\mathbf{0},n\mathbf{1})\right]\right)}{\log(n)}\right\}\\ &= \lim_{n\to\infty}\int_\beta^{\beta+\varepsilon}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\log\left(\mathbb{E}_{s}\left[D(\mathbf{0},n\mathbf{1})\right]\right)}{\log(n)}\mathrm{d}s < 0\end{aligned}$$

as we will show in the end of section 10.2 in detail. This implies strict monotonicity of the function $\beta \mapsto \theta(\beta)$. In order to show $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\log\left(\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})]\right)}{\log(n)} < c(\beta)$, we divide the graph into several levels, and assume $n = 2^k - 1$ for some $k \in \mathbb{N}$. The i-th level consists of all edges for which $c2^i < |e| \leq C2^i$ for some constants $0 < c < C < \infty$. Note that an edge can be in several levels, but at most in finitely many. By $\mathcal{G}(\mathbf{0},n\mathbf{1})$ we denote the union of all geodesics between $\mathbf{0}$ and $n\mathbf{1}$. So the occupation status of edges outside $\mathcal{G}(\mathbf{0},n\mathbf{1})$ does not change the distance between $\mathbf{0}$ and $n\mathbf{1}$ which implies that for all edges e one has

$$\mathbb{E}_\beta\left[\left(D(\mathbf{0},n\mathbf{1};\omega) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right)\mathbb{1}_{\{e\in\mathcal{G}(\mathbf{0},n\mathbf{1})\}}\right] = \mathbb{E}_\beta\left[D(\mathbf{0},n\mathbf{1};\omega) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right].$$

For fixed $\beta > 0$, we have by (94) that $p'(\beta,e) = \Theta\left(\mathbb{P}_\beta(\omega(e)=1)\right) = \Theta\left(\frac{1}{|e|^{2d}}\right)$ as $|e| \to \infty$. Thus we have uniformly over all edges of length at least 2 (but not uniformly over β) that

$$p'(\beta,e)\mathbb{E}_\beta\left[D(\mathbf{0},n\mathbf{1};\omega^{e+}) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right] = \Theta\left(\mathbb{E}_\beta\left[D(\mathbf{0},n\mathbf{1};\omega) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right]\right)$$

$$= \Theta\left(\mathbb{E}_\beta\left[\left(D(\mathbf{0},n\mathbf{1};\omega) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right)\mathbb{1}_{\{e\in\mathcal{G}(\mathbf{0},n\mathbf{1})\}}\right]\right).$$

So in order to show that there exists a $c(\beta) < 0$ such that $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\log\left(\mathbb{E}_\beta[D(\mathbf{0},n\mathbf{1})]\right)}{\log(n)} < c(\beta)$ uniformly over n it suffices to show that

$$\sum_{e\in E: c2^i<|e|\leq C2^i} \mathbb{E}_\beta\left[\left(D(\mathbf{0},n\mathbf{1};\omega) - D(\mathbf{0},n\mathbf{1};\omega^{e-})\right)\mathbb{1}_{\{e\in\mathcal{G}(\mathbf{0},n\mathbf{1})\}}\right] < c'(\beta)\mathbb{E}_\beta\left[D(\mathbf{0},n\mathbf{1})\right]$$

for some $c'(\beta) < 0$ and a positive fraction of the levels $i \in \{1,\ldots,k\}$. One needs this for a positive fraction of the levels in order to cancel the logarithm in the denominator of (104).

10.1 The geometry inside blocks

For the proof of Theorem 1.4, we remind ourselves of a few results from the previous chapters:
From Lemma 2.4 and Lemma 2.5 we know that for all $\frac{1}{n} < \varepsilon \leq \frac{1}{4}$ and $u,w \in \mathbb{Z}^d\setminus\{0\}$ with $\|u\|_\infty \geq 2$ one has

$$\mathbb{P}_\beta\left(\exists x,y \in V_{\mathbf{0}}^n : \|x-y\|_\infty \leq \varepsilon n, x\sim V_u^n, y\sim V_w^n \;\middle|\; V_{\mathbf{0}}^n \sim V_u^n, V_{\mathbf{0}}^n\sim V_w^n\right) \leq C_d'\varepsilon^{1/2}\lceil\beta\rceil^2$$

where C_d' is a constant that depends only on the dimension d. This tells us that for a block $V_{\mathbf{0}}^n$ the vertices $x,y\in V_{\mathbf{0}}^n$ that are connected to different boxes $x\sim V_u^n, y\sim V_w^n$ are typically far apart in terms of Euclidean distance, whenever $\|u\|_\infty \geq 2$. However, the same result is also true for the chemical distance, as proven in Lemma 5.2. There we proved that for all dimensions d and all $\beta \geq 0$, there exists a function $g_1(\varepsilon)$ with $g_1(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 1$ such that for all $u,w\in\mathbb{Z}^d\setminus\{\mathbf{0}\}$ with $\|u\|_\infty\geq 2$ and all large enough $n \geq n(\varepsilon)$

$$\mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^n}(x,y) > \varepsilon\Lambda(n,\beta) \text{ for all } x,y\in V_{\mathbf{0}}^n \text{ with } x\sim V_u^n, y\sim V_w^n \;\middle|\; V_u^n\sim V_{\mathbf{0}}^n \sim V_w^n\right) \geq g_1(\varepsilon).$$

Theorem 1.1 shows that

$$D\left(\mathbf{0}, B_n(\mathbf{0})^C\right) \approx_P n^{\theta(\beta)}$$

for some $\theta(\beta) \in (0,1)$. One can ask whether the same statement is true for the distance between two sets that are separated by a euclidean distance of n, for example $D\left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C\right)$. However, a similar statement can never be true, as there is a uniform (in n) positive probability of a direct edge between the sets $B_n(\mathbf{0})$ and $B_{2n}(\mathbf{0})^C$. But if we condition on the event that there is no direct edge, then we can get such a result, as proven in Lemma 4.11 and Corollary 4.12. Formally, let $\mathcal{L}$ be the event that there is no direct edge between $B_n(\mathbf{0})$ and $B_{2n}(\mathbf{0})^C$. For all $\beta\geq 0$ and all $\varepsilon > 0$, there exist $0<c<C<\infty$ such that

$$\mathbb{P}_\beta\left(c\Lambda(n,\beta) \leq D\left(B_n(\mathbf{0}), B_{2n}(\mathbf{0})^C\right) \leq C\Lambda(n,\beta) \;\middle|\; \mathcal{L}\right) > 1-\varepsilon$$

for all $n\in\mathbb{N}$. Let $\mathcal{L}'$ be the event that there is no direct edge between $V_{\mathbf{0}}^n$ and $\bigcup_{u\in\mathbb{Z}^d:\|u\|_\infty\geq 2} V_u^n$. For all $\beta\geq 0$ and all $\varepsilon>0$, there exist $0<c<C<\infty$ such that

$$\mathbb{P}_\beta\left(c\Lambda(n,\beta) \leq D\left(V_{\mathbf{0}}^n, \bigcup_{u\in\mathbb{Z}^d:\|u\|_\infty\geq 2} V_u^n\right) \leq C\Lambda(n,\beta) \;\middle|\; \mathcal{L}'\right) > 1-\varepsilon$$

for all $n \in \mathbb{N}$. So in particular there exists a function g with $\lim_{\varepsilon \to 0} g_2(\varepsilon) = 1$ such that

$$\mathbb{P}_\beta \left(\varepsilon \Lambda(n, \beta) < D \left(V_{\mathbf{0}}^n, \bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2} V_u^n \right) \;\middle|\; \mathcal{L}' \right) \geq g_2(\varepsilon) \tag{105}$$

for all large enough $n \geq n(\varepsilon)$.

Lemma 11.2 below implies that for all $k \in \mathbb{N}$ and $\beta > 0$ there exists a constant C such that for all $n \in \mathbb{N}$

$$\mathbb{E}_\beta \left[\text{Diam} \left(V_{\mathbf{0}}^n \right)^k \right] \leq C n^{k\theta(\beta)}. \tag{106}$$

Let $\delta \in (0,1)$. We define a family of sets $\mathcal{CO}_n^\delta \subset V_{\mathbf{0}}^n$ with the following two properties:

- $\bigcup_{x \in \mathcal{CO}_n^\delta} B_{\delta n}(x) = V_{\mathbf{0}}^n$, and
- $\left| \mathcal{CO}_n^\delta \right| \leq C_{\mathcal{CO}} \delta^{-d}$ for all δ,

where $C_{\mathcal{CO}}$ is a constant that depends only on the dimension d, but non on δ. The abbreviation $\mathcal{CO}$ stands for cover. Such a cover can be constructed by choosing the points in $\mathcal{CO}_n^\delta$ at a distance of approximately δn.

Lemma 10.1. *For $\varepsilon \in (0,1)$, let $\mathcal{DL}(\varepsilon)$ be the event*

$$\mathcal{DL}(\varepsilon) = \bigcap_{x \in \mathcal{CO}_n^{\varepsilon^2}} \left\{ \mathit{Diam} \left(B_{\varepsilon^2 n}(x) \right) < \frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right\}.$$

Then there exists a function $h_1(\varepsilon)$ with $\lim_{\varepsilon \to 0} h_1(\varepsilon) = 1$ such that

$$\mathbb{P}_\beta \left(\mathcal{DL}(\varepsilon) \right) \geq h_1(\varepsilon)$$

for all $n \geq n(\varepsilon)$ large enough. If the event $\mathcal{DL}(\varepsilon)$ holds, we say that $V_{\mathbf{0}}^n$ is ε-near.

Proof. By a union bound we have that

$$\begin{aligned} \mathbb{P}_\beta \left(\mathcal{DL}(\varepsilon)^C \right) &\leq \sum_{x \in \mathcal{CO}_n^{\varepsilon^2}} \mathbb{P}_\beta \left(\text{Diam} \left(B_{\varepsilon^2 n}(x) \right) \geq \frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right) \\ &\leq C_{\mathcal{CO}} \varepsilon^{-2d} \mathbb{P}_\beta \left(\text{Diam} \left(B_{\varepsilon^2 n}(\mathbf{0}) \right) \geq \frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right). \end{aligned} \tag{107}$$

From Markov's inequality we know that for any $k \in \mathbb{N}$ and $n \geq \varepsilon^{-2}$

$$\mathbb{P}_\beta \left(\text{Diam} \left(B_{\varepsilon^2 n}(\mathbf{0}) \right) \geq \frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right) = \mathbb{P}_\beta \left(\text{Diam} \left(B_{\varepsilon^2 n}(\mathbf{0}) \right)^k \geq \left(\frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right)^k \right)$$

$$\leq \mathbb{E}_\beta \left[\text{Diam} \left(B_{\varepsilon^2 n}(\mathbf{0}) \right)^k \right] \left(\frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right)^{-k} \overset{(106)}{\leq} C \left(2\varepsilon^2 n + 1 \right)^{k\theta} \left(\frac{\left(\varepsilon^{1.5} n \right)^\theta}{3} \right)^{-k} \leq C'(k) \varepsilon^{0.5k\theta}$$

for some constant $C'(k) < \infty$. So using $k = 6d \lceil \theta^{-1} \rceil$ and inserting this into (107) we get that

$$\mathbb{P}_\beta \left(\mathcal{DL}(\varepsilon)^C \right) \leq \tilde{C} \varepsilon^{-2d} \varepsilon^{0.5 \cdot 6d \lceil \theta^{-1} \rceil \theta} \leq \tilde{C} \varepsilon^d$$

for some constant $\tilde{C} < \infty$, which finishes the proof. ∎

Consider long-range percolation on $\mathbb{Z}^d$. We split the long-range percolation graph into blocks of the form V_v^n, where $v \in \mathbb{Z}^d$. For each $v \in \mathbb{Z}^d$, we contract the block $V_v^n \subset \mathbb{Z}^d$ into one vertex $r(v)$. We call the graph that results from contracting all these blocks $G' = (V', E')$. For $r(v) \in G'$, we define the neighborhood $\mathcal{N}(r(v))$ by

$$\mathcal{N}(r(v)) = \left\{ r(u) \in G' : \|v - u\|_\infty \leq 1 \right\},$$

and we define the neighborhood-degree of $r(v)$ by

$$\deg^{\mathcal{N}}(r(v)) = \sum_{r(u) \in \mathcal{N}(r(v))} \deg(r(u)). \tag{108}$$

We also define these quantities in the same way when we start with long-range percolation on the graph $V_{\mathbf{0}}^{mn}$, and contract the box V_v^n for all $v \in V_{\mathbf{0}}^m$. Remember that by Lemma 5.4 we have for the event $\mathcal{W}(\varepsilon)$ defined by

$$\mathcal{W}(\varepsilon) := \left\{ D^\star \left(V_v^n, \bigcup_{u \in \mathbb{Z}^d : \|u-v\|_\infty \geq 2} V_u^n \right) > \varepsilon \Lambda(n, \beta) \right\}.$$

that for all large enough $n \geq n(\varepsilon)$ one has

$$\mathbb{P}_\beta \left(\mathcal{W}(\varepsilon)^C \mid G' \right) \leq 3^d \deg^{\mathcal{N}}(r(v)) \left(1 - g_1(\varepsilon)\right) + \left(1 - g_2(\varepsilon)\right), \tag{109}$$

where g_1 and g_2 were defined in Lemma 5.2, respectively (105). Furthermore, before going to the proof of Theorem 1.4, we remind ourselves about the main results of section 3; In particular about Lemma 3.2 and (25). For a finite set Z, we defined its average degree by $\overline{\deg}(Z) = \frac{1}{|Z|} \sum_{v \in Z} \deg(z)$. Let $\mathcal{CS}_k = \mathcal{CS}_k\left(\mathbb{Z}^d\right)$ be all connected subsets of the long-range percolation graph with vertex set $\mathbb{Z}^d$ of size k that contain the origin $\mathbf{0}$. We write μ_β for $\mathbb{E}_\beta\left[\deg(\mathbf{0})\right]$. Then for all $\beta > 0$

$$\mathbb{P}_\beta \left(\exists Z \in \mathcal{CS}_k : \overline{\deg}(Z) \geq 20 \mu_\beta \right) \leq e^{-4k\mu_\beta}$$

and

$$\mathbb{E}_\beta \left[\left| \mathcal{CS}_k \left(\mathbb{Z}^d \right) \right| \right] \leq 4^k \mu_\beta^k.$$

10.2 The proof of Theorem 1.4

With the knowledge from the previous subsections, we are now ready to go to the proof of Theorem 1.4. The proof consists out of three main parts: First, we define a notion of good paths in a renormalized graph. Then we show that every long enough path is good, with high probability. Finally, we argue how this implies strict monotonicity of the distance exponent.

Proof of Theorem 1.4. Consider the graph $V_{\mathbf{0}}^{2^n}$. For $k \leq n$, define the graph G' by contracting all blocks of the form $V_u^{2^k}$. We define $r(u) \in G'$ as the vertex that results from contracting $V_u^{2^k}$. In analogy to $\mathbb{Z}^d$, we call the vertex $r(\mathbf{0})$ the origin of G'. We define a metric on G' by $\|r(u) - r(v)\|_\infty = \|u - v\|_\infty$. Now consider a self-avoiding path $P = (r(u_0), r(u_1), \ldots, r(u_t)) \subset G'$, where $u_0 = \mathbf{0}$, and t is very large (depending on d and β). We divide the path into blocks of length $K = 3^d + 1$: For $j \leq \lfloor \frac{t}{K} \rfloor - 1$, we define $R_j = \left(r(u_{jK}), \ldots, r(u_{jK+3^d}) \right)$. For each such j and R_j, we define a set $\tilde{R}_j$ as follows:

If there exist $(r(u_i), r(u_{i+1})) \subset R_j$ with $\|u_i - u_{i+1}\|_\infty \geq 2$, we simply set $\tilde{R}_j = R_j$. If $\|r(u_i) - r(u_{i+1})\|_\infty = 1$ for all $i \in \{jK, \dots, jK + 3^d\}$, then we set $\tilde{R}_j = R_j \cup \mathcal{N}(r(u_{jK}))$. The set $\bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j$ is a connected set and its cardinality is bounded from below by

$$\left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \geq \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} R_j \right| \geq K \left\lfloor \frac{t}{K} \right\rfloor \geq \frac{t}{2},$$

and bounded from above by

$$\left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \leq 3^d \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} R_j \right| \leq 3^d t.$$

From now on we will always work on the event

$$\mathcal{H}_t \coloneqq \left\{ \overline{\deg}(Z) < 20\mu_\beta \text{ for all } Z \in \mathcal{CS}_{\geq t/2}\left(G'\right) \right\}.$$

Note that

$$\mathbb{P}_\beta\left(\mathcal{H}_t^C\right) \leq e^{-2t\mu_\beta} \leq 2^{-t} \tag{110}$$

by Lemma 3.2. We define the degree of $\tilde{R}_j$ by

$$\deg\left(\tilde{R}_j\right) = \sum_{r(u) \in \tilde{R}_j} \deg(r(u)).$$

Note that we do not necessarily have

$$\sum_{i=0}^{\lfloor \frac{t}{K} \rfloor - 1} \deg\left(\tilde{R}_j\right) = \sum_{r(u) \in \bigcup_{i=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j} \deg(r(u)),$$

as some vertices $r(u_i)$ might be included in more than one of the sets $\tilde{R}_j$. However, each vertex $r(u_i)$ can be included in at most 3^d sets $\tilde{R}_j$ and thus we have

$$\sum_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \deg\left(\tilde{R}_j\right) \leq 3^d \sum_{r(u) \in \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j} \deg(r(u)) \leq 3^d 20\mu_\beta \left| \bigcup_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \tilde{R}_j \right| \leq 9^d 20\mu_\beta t.$$

There are $\lfloor \frac{t}{K} \rfloor \geq \frac{t}{2K} \geq \frac{t}{8^d}$ indices j, and thus we have

$$\frac{1}{\lfloor t/K \rfloor} \sum_{j=0}^{\lfloor \frac{t}{K} \rfloor - 1} \deg\left(\tilde{R}_j\right) \leq \frac{8^d}{t} 9^d 20\mu_\beta t = 72^d 20\mu_\beta,$$

which implies that there are at least $\lceil \frac{t}{20^d} \rceil$ many indices $j \in \{0, \dots, \lfloor t/K \rfloor - 1\}$ with

$$\deg\left(\tilde{R}_j\right) \leq 80^{d+1}\mu_\beta.$$

Say that $j_1, \dots, j_{\lceil \frac{t}{20^d} \rceil}$ are the first such indices. We define a further subset $\mathcal{IND} = \mathcal{IND}(P)$ of these indices by starting with $\mathcal{IND}_0 = \emptyset$ and then iteratively define $(\mathcal{IND}_i)_{i=1}^{\lceil \frac{t}{20^d} \rceil}$ by

$$\mathcal{IND}_i := \begin{cases} \mathcal{IND}_{i-1} \cup \{j_i\} & \text{if } \tilde{R}_{j_i} \nsim \bigcup_{l \in \mathcal{IND}_{i-1}} \tilde{R}_{j_l} \\ \mathcal{IND}_{i-1} & \text{else} \end{cases}.$$

So in particular there is no edge between $\tilde{R}_j$ and $\tilde{R}_{j'}$ for different $j, j' \in \mathcal{IND} := \mathcal{IND}_{\lceil \frac{t}{20^d} \rceil}$. The set $\mathcal{IND}$ has a cardinality of at least $\frac{1}{80^{d+1}\mu_\beta + 1} \lceil \frac{t}{20^d} \rceil$, as for $j_i \in \mathcal{IND}$, the set $\tilde{R}_{j_i}$ has a degree of at most $80^{d+1}\mu_\beta$ and can thus block at most $80^{d+1}\mu_\beta$ many other elements from getting included. So in particular we also have

$$|\mathcal{IND}| \geq \frac{1}{80^{d+1}\mu_\beta + 1} \left\lceil \frac{t}{20^d} \right\rceil \geq \frac{t}{2000^{d+1}\mu_\beta}. \tag{111}$$

If $R_i = \left(r(u_{iK}), \dots, r(u_{iK+3^d})\right)$ is a path of length $K = 3^d + 1$ in the graph G' with $\deg\left(\tilde{R}_i\right) \leq 80^{d+1}\mu_\beta$, we want to investigate the typical minimal length of a path in the original model that goes through the blocks $\left(V_{u_{iK}}^{2^k}, \dots, V_{u_{iK+3^d}}^{2^k}\right)$. If there exists $j \in \{iK, \dots, iK + 3^d - 1\}$ with $\|u_{j+1} - u_j\|_\infty \geq 2$, let j be the smallest such index. The probability that there exist $x, y \in V_{u_j}^{2^k}$ such that $x \sim V_{u_{j+1}}^{2^k}$, $y \sim V_w^{2^k}$, where $w \notin \{u_j, u_{j+1}\}$, and $D_{V_{u_j}^{2^k}}(x,y) \leq \varepsilon\Lambda(2^k, \beta)$ is bounded by $\deg\left(V_{u_j}^{2^k}\right)(1 - g_1(\varepsilon)) \leq 80^{d+1}\mu_\beta (1 - g_1(\varepsilon))$, by Lemma 5.2. If $D_{V_{u_j}^{2^k}}(x,y) > \varepsilon\Lambda(2^k, \beta)$ for all $x, y \in V_{u_j}^{2^k}$ such that $x \sim V_{u_{j+1}}^{2^k}$, $y \sim V_w^{2^k}$, where $w \notin \{u_j, u_{j+1}\}$, we say that the block R_i is ε-separated.

Now suppose that $\|u_j - u_{j+1}\|_\infty = 1$ for all $j \in \{iK, \dots, iK + 3^d - 1\}$. There exists an index $j \in \{iK + 1, \dots, iK + 3^d\}$ with $\|u_{iK} - u_j\|_\infty \geq 2$, as there are only $3^d - 1$ many points $w \in \mathbb{Z}^d$ with $\|u_{iK} - w\|_\infty = 1$. When the path exits the cube $V_{u_{iK}}^{2^k}$ for the last time, it goes to $V_{u_{iK+1}}^{2^k}$, so in particular the walk does not use a long edge from $V_{u_{iK}}$ to $\bigcup_{w : \|w - u_{iK}\|_\infty \geq 2} V_w^{2^k}$ for the last exit. If the indirect distance between the sets $V_{u_{iK}}^{2^k}$ and the set $\bigcup_{r(w) \in G' : \|w - u_{iK}\|_\infty \geq 2} V_w^{2^k}$ is at least $\varepsilon\Lambda\left(2^k, \beta\right)$, i.e, if

$$D^\star_{V_{\mathbf{0}}^{2^n}}\left(V_{u_{iK}}^{2^k}, \bigcup_{r(w) \in G' : \|w - u_{iK}\|_\infty \geq 2} V_w^{2^k}\right) \geq \varepsilon\Lambda\left(2^k, \beta\right),$$

we also say that the subpath R_i and the set $\tilde{R}_i$ are ε-separated. As $\deg^{\mathcal{N}}(r(u_i)) \leq 80^{d+1}\mu_\beta$, the probability that there is a path of length at most $\varepsilon\Lambda(2^k, \beta)$ that goes through $V_{u_{iK}}^{2^k}, \dots, V_{u_{iK+3^d}}^{2^k}$ is bounded by $3^d 80^{d+1}(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon))$, by (109). So we see that in all cases, with probability at least

$$1 - \left(3^d 80^{d+1}(1 - g_1(\varepsilon)) + (1 - g_2(\varepsilon))\right)$$

the original path needs to walk a distance of at least $\varepsilon\Lambda(2^k, \beta)$ inside the sets $V_{u_{iK+1}}^{2^k}$, $V_{u_{iK+2}}^{2^k}, \dots, V_{u_{iK+3^d-1}}^{2^k}$, and this distance can be witnessed from the set of edges with at least one end in $\tilde{R}_i$. Note that we have two notions of ε-separated: one for subpaths that make a jump of size at least 2 and one for subpaths that do not make such a jump.

However, the idea is in both cases that a path that walks through the set R_i needs to walk a distance of at least $\varepsilon\Lambda\left(2^k,\beta\right)$ in the original model.

We say that a sequence R_i is *ε-influential*, if $\tilde{R}_i$ is ε-separated and all boxes $V_{u_{iK}}^{2^k},\dots,V_{u_{iK+3^d}}^{2^k}$ are $\varepsilon^{1/\theta}$-near (see Lemma 10.1 for the definition of ε-near). For a block R_i with $i\in\mathcal{IND}$, we can bound the probability that a sequence R_i is not ε-influential by

$$\mathbb{P}_\beta\left(R_i \text{ is not } \varepsilon\text{-influential}\right)\leq 3^d 80^{d+1}(1-g_1(\varepsilon))+(1-g_2(\varepsilon))+(3^d+1)\left(1-h_1\left(\varepsilon^{1/\theta}\right)\right).$$

Note that it only depends on edges with at least one endpoint inside $\tilde{R}_i$, whether R_i is ε-influential. For different values of different $j_1,\dots,j_l\in\mathcal{IND}$, the sets $\left(\tilde{R}_{j_i}\right)_{i\in\{1,\dots,l\}}$ are not connected, and thus it is independent whether these blocks are ε-influential. Next, let ε be small enough such that

$$\left(3^d 80^{d+1}(1-g_1(\varepsilon))+(1-g_2(\varepsilon))+(3^d+1)\left(1-h_1\left(\varepsilon^{1/\theta}\right)\right)\right)^{\frac{1}{2\cdot 2000^{d+1}\mu_\beta}}\leq\frac{1}{100\mu_\beta^2}. \quad (112)$$

Let $\mathcal{INF}=\mathcal{INF}(P)\subset\mathcal{IND}(P)$ be all indices $i\in\mathcal{IND}$, for which R_i is ε-influential. If $\mathcal{H}_t$ holds we want to get bounds on the cardinality of the set $\mathcal{INF}$ for a fixed path $P\subset G'$ of length t. Remember that we have

$$|\mathcal{IND}|\geq\frac{t}{2000^{d+1}\mu_\beta},$$

as shown in (111). For a path $P=(r(u_0),r(u_1),\dots,r(u_t))\subset G'$ one thus has

$$\begin{aligned}
&\mathbb{P}_\beta\left(|\mathcal{INF}|<\frac{t}{2\cdot 2000^{d+1}\mu_\beta}\;\Big|\;G'\right)=\mathbb{P}_\beta\left(\bigcup_{U\subset\mathcal{IND}:|U|\geq\mathcal{IND}/2}\{R_i\text{ not }\varepsilon\text{-influential }\forall i\in U\}\;\Big|\;G'\right)\\
&\leq 2^{|\mathcal{IND}|}\left(3^d 80^{d+1}(1-g_1(\varepsilon))+(1-g_2(\varepsilon))+(3^d+1)\left(1-h_1\left(\varepsilon^{1/\theta}\right)\right)\right)^{\frac{t}{2\cdot 2000^{d+1}\mu_\beta}}\\
&\leq 2^t\left(\frac{1}{100\mu_\beta^2}\right)^t=\frac{1}{50^t\mu_\beta^{2t}}
\end{aligned}$$

where used the assumption on ε (112) for the last inequality. This shows that a specific path P is satisfies $|\mathcal{INF}(P)|\geq\frac{t}{2\cdot 2000^{d+1}\mu_\beta}$ with high probability. Next, we want to show that all paths $P\subset G'$ of length t starting at the origin $r(\mathbf{0})$ satisfy $|\mathcal{INF}(P)|\geq\frac{t}{2\cdot 2000^{d+1}\mu_\beta}$ with high probability in t. Let $\mathcal{P}_t$ be the set of all paths in G' of length t starting at $r(\mathbf{0})$. We call the previously mentioned event $\mathcal{G}_t$, i.e.,

$$\mathcal{G}_t=\left\{|\mathcal{INF}(P)|\geq\frac{t}{2\cdot 2000^{d+1}\mu_\beta}\text{ for all }P\in\mathcal{P}_t\right\}.$$

By a comparison with a Galton-Watson tree we get that $\mathbb{E}_\beta\left[|\mathcal{P}_t|\right]\leq\mu_\beta^t$. Thus we have, by a union bound

$$\begin{aligned}
\mathbb{P}_\beta\left(\mathcal{G}_t^C\right)&\leq\mathbb{P}_\beta\left(\mathcal{H}_t^C\right)+\mathbb{P}_\beta\left(|\mathcal{P}_t|>2^t\mu_\beta^t\right)+\mathbb{P}_\beta\left(\mathcal{G}_t^C\;\middle|\mathcal{H}_t,|\mathcal{P}_t|\leq 2^t\mu_\beta^t\right)\\
&\leq\mathbb{P}_\beta\left(\mathcal{H}_t^C\right)+\mathbb{P}_\beta\left(|\mathcal{P}_t|>2^t\mu_\beta^t\right)+2^t\mu_\beta^t\frac{1}{50^t\mu_\beta^{2t}}\overset{(110)}{\leq}2^{-t}+\frac{\mathbb{E}\left[|\mathcal{P}_t|\right]}{2^t\mu_\beta^t}+\frac{1}{25^t}\leq 3\cdot 2^{-t}.
\end{aligned}$$

and with another union bound we get for the event $\mathcal{G}_{\geq t} \coloneqq \bigcap_{t'=t}^{\infty} \mathcal{G}_{t'}$ that

$$\mathbb{P}_\beta \left(\mathcal{G}_{\geq t}^C \right) \leq 6 \cdot 2^{-t}. \tag{113}$$

So we see that all paths of length at least t contain at least $\frac{t}{2 \cdot 2000^{d+1} \mu_\beta}$ many ε-influential subpaths R_i with very high probability in t, for ε small enough as in (112). Now, let $\hat{P}$ be a geodesic between $\mathbf{0}$ and $(2^n - 1)\mathbf{1}$ in the graph with vertex set $V_{\mathbf{0}}^{2^n}$. Let $\tilde{P}$ be the projection of this path onto G', and let P be the loop-erased version of it. Whenever the path P crosses an influential subset $R_i = \left(r(u_{iK}), \dots, r(u_{iK+3^d})\right) \subset P$, let $l = l(i) \in \{iK, \dots, iK + 3^d - 1\}$ be the first index for which $\|r(u_l) - r(u_{l+1})\|_\infty \geq 2$ if such an index exists. Respectively let $l = l(i) \in \{iK, \dots, iK + 3^d\}$ be the first index for which $\|r(u_l) - r(u_{iK})\|_\infty \geq 2$, if there does not exist such an index with $\|r(u_l) - r(u_{l+1})\|_\infty \geq 2$. Whenever the path P crosses the set R_i, it enters $V_{u_{iK}}^{2^k}$ through some vertex x_L and it leaves $V_{u_l}^{2^k}$ to $V_{u_{l+1}}^{2^k}$ through some vertex x_R. As the boxes $V_{u_{iK}}^{2^k}$ and $V_{u_l}^{2^k}$ are $\varepsilon^{1/\theta}$-near, there exist cubes B_L, B_R of side length at least $2\varepsilon^{2/\theta} 2^k$ such that

$$x_L \in B_L \subset V_{u_{iK}}^{2^k}, x_R \in B_R \subset V_{u_l}^{2^k}, \text{ and}$$

$$\operatorname{Diam}(B_L), \operatorname{Diam}(B_R) < \frac{\left(\varepsilon^{\frac{1.5}{\theta}} 2^k\right)^\theta}{3} = \frac{\varepsilon^{1.5}}{3} 2^{k\theta}.$$

The graph distance between x_L and x_R is at least $\varepsilon\Lambda\left(2^k, \beta\right)$, as we will argue now. If there exists an index $l \in \{iK, \dots, iK + 3^d - 1\}$ for which $\|r(u_l) - r(u_{l+1})\|_\infty \geq 2$, then we know that the box $V_{u_l}^{2^k}$ is ε-separated. At the last visit of the box $V_{u_l}^{2^k}$, the geodesic $\hat{P}$ enters the box $V_{u_l}^{2^k}$ through some point $z \in V_{u_l}^{2^k}$ with $z \sim V_w^{2^k}$, for some $w \in V_{\mathbf{0}}^{2^{n-k}} \setminus \{u_l, u_{l+1}\}$. We have $w \neq u_{l+1}$, as the loop-erased projection P is self-avoiding. As $D_{V_{u_l}^{2^k}}(x_R, z) \geq \varepsilon\Lambda\left(2^k, \beta\right)$ for all $z \in V_{u_l}^{2^k}$ with $z \sim V_w^{2^k}$ for $w \notin \{u_l, u_{l+1}\}$, we automatically get that $D(x_L, x_R) \geq \varepsilon\Lambda\left(2^k, \beta\right)$, as either $x_L \in V_{u_l}^{2^k}$ with $x_L \sim V_{u_{l-1}}^{2^k}$, or $x_L \notin V_{u_l}^{2^k}$. If there does not exist an index $l \in \{iK, \dots, iK + 3^d - 1\}$ for which $\|r(u_l) - r(u_{l+1})\|_\infty \geq 2$, then we know that $\|u_l - u_{iK}\|_\infty = 2$ and the geodesic between x_L and x_R walks through the set $\bigcup_{r(w) \in G' : \|w - u_{iK}\|_\infty = 1} V_w^{2^k}$, and thus its length is at least

$$D_{V_{\mathbf{0}}^{2^n}}(x_L, x_R) \geq D^\star_{V_{\mathbf{0}}^{2^n}}\left(V_{u_{iK}}^{2^k}, \bigcup_{r(w) \in G' : \|w - u_{iK}\|_\infty \geq 2} V_w^{2^k}\right) \geq \varepsilon\Lambda\left(2^k, \beta\right)$$

where the last inequality holds, as the subpath R_i was assumed to be ε-separated.

When we insert an edge between the boxes B_L and B_R, the distance between x_L and x_R is at most $2\frac{\varepsilon^{1.5}}{3} 2^{k\theta} + 1$. Remember that $\Lambda(2^k, \beta) \geq 2^{k\theta}$. Thus we have for all edges $e \in B_L \times B_R$

$$D_{V_{\mathbf{0}}^{2^n}}(x_L, x_R; \omega) - D_{V_{\mathbf{0}}^{2^n}}\left(x_L, x_R; \omega^{e+}\right) \geq \varepsilon\Lambda\left(2^k, \beta\right) - 2\frac{\varepsilon^{1.5}}{3} 2^{k\theta} - 1 \geq \frac{\varepsilon\Lambda(2^k, \beta)}{4}$$

where the last inequality holds for k large enough. The boxes B_L and B_R are of side length at least $2\varepsilon^{2/\theta} 2^k$ and are disjoint, as $D_{V_{\mathbf{0}}^{2^n}}(x_L, x_R) > \operatorname{Diam}(B_L) + \operatorname{Diam}(B_R)$. Thus there are at least $\left(\varepsilon^{2/\theta} 2^k\right)^d \cdot \left(\varepsilon^{2/\theta} 2^k\right)^d$ pairs of vertices $(a, b) \in B_L \times B_R$ for which $|\{a, b\}| \geq \varepsilon^{2/\theta} 2^k$. On the other hand, we also have $|\{a, b\}| \leq (3^d + 1) 2^k \leq 6^d 2^k$ for all pairs $(a, b) \in B_L \times B_R$, as $\|r(u_{iK}) - r(u_l)\|_\infty \leq 3^d$. So in particular we have

$$\sum_{\substack{e \in B_L \times B_R: \\ \varepsilon^{2/\theta} 2^k \leq |e| \leq 6^d 2^k}} p'(\beta, e) \left(D_{V_{\mathbf{0}}^{2^n}}(x_L, x_R; \omega) - D_{V_{\mathbf{0}}^{2^n}}\left(x_L, x_R; \omega^{e+}\right) \right)$$

$$\geq \sum_{\substack{e\in B_L\times B_R:\\ \varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k}} p'(\beta,e)\frac{\varepsilon\Lambda(2^k,\beta)}{4} \overset{(94)}{\geq} \sum_{\substack{e\in B_L\times B_R:\\ \varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k}} \frac{e^{-\beta}}{\left(|e|+\sqrt{d}\right)^{2d}}\frac{\varepsilon\Lambda(2^k,\beta)}{4} \geq c\Lambda(2^k,\beta)$$

with a constant $c>0$, that depends on β,ε,θ, and d, and for k large enough. For the two points $\mathbf{0}$ and $(2^n-1)\mathbf{1}$, and points x_L and x_R which are in a geodesic between $\mathbf{0}$ and $(2^n-1)\mathbf{1}$ in this order, and any edge e we have

$$D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) = D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},x_L;\omega\right) + D_{V_{\mathbf{0}}^{2^n}}\left(x_L,x_R;\omega\right) + D_{V_{\mathbf{0}}^{2^n}}\left(x_R,(2^n-1)\mathbf{1};\omega\right), \text{ and}$$
$$D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right) \leq D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},x_L;\omega\right) + D_{V_{\mathbf{0}}^{2^n}}\left(x_L,x_R;\omega^{e+}\right) + D_{V_{\mathbf{0}}^{2^n}}\left(x_R,(2^n-1)\mathbf{1};\omega\right).$$

Subtracting these two (in)equalities already yields that

$$D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right) \geq D_{V_{\mathbf{0}}^{2^n}}\left(x_L,x_R;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(x_L,x_R;\omega^{e+}\right),$$

so in particular we also have

$$\sum_{\substack{e\in B_L\times B_R:\\ \varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k}} p'(\beta,e)\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right)\right) \geq c\Lambda\left(2^k,\beta\right).$$

The above inequality holds for fixed $B_L\subset V_{u_{iK}}^{2^k}, B_R\subset V_{u_l}^{2^k}$. However, such boxes exist for all indices $i\in\mathcal{INF}(P)$. Thus, assuming that $D_{G'}\left(\mathbf{0},(2^{n-k}-1)\mathbf{1}\right)=t$ and $\mathcal{G}_{\geq t}$ holds for large enough $t\geq T$, we have for large enough k

$$\sum_{e:\varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k} p'(\beta,e)\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right)\right) \geq |\mathcal{INF}(P)|\, c\Lambda\left(2^k,\beta\right)$$
$$\geq \frac{t}{2\cdot 2000^{d+1}\mu_\beta}c\Lambda\left(2^k,\beta\right) =: c't\Lambda\left(2^k,\beta\right).$$

So far, we always worked on the event $\mathcal{G}_{\geq t}$. Now, we want to get a similar bounds in expectation, not conditioning on $\mathcal{G}_{\geq t}$. Writing E_k for the set of edges e with $\varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k$ we get that there exists a large enough $T<\infty$ so that

$$\sum_{e:\varepsilon^{2/\theta}2^k\leq|e|\leq 6^d2^k} p'(\beta,e)\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right)\right]$$
$$\geq \sum_{t=T}^{\infty}\sum_{e\in E_k} p'(\beta,e)\mathbb{E}_\beta\Big[\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1};\omega^{e+}\right)\right)$$
$$\cdot \mathbb{1}_{\left\{D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)\mathbf{1})\right)=t\right\}}\mathbb{1}_{\{\mathcal{G}_{\geq t}\}}\Big]$$
$$\geq \sum_{t=T}^{\infty} c't\Lambda\left(2^k,\beta\right)\mathbb{E}_\beta\left[\mathbb{1}_{\left\{D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)\mathbf{1})\right)=t\right\}}\mathbb{1}_{\{\mathcal{G}_{\geq t}\}}\right]$$
$$\geq c'\Lambda\left(2^k,\beta\right)\sum_{t=T}^{\infty} t\left(\mathbb{P}_\beta\left(D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)\mathbf{1})\right)=t\right) - \mathbb{P}_\beta\left(\mathcal{G}_{\geq t}^C\right)\right)$$
$$\overset{(113)}{\geq} c'\Lambda\left(2^k,\beta\right)\left(\sum_{t=T}^{\infty} t\left(\mathbb{P}_\beta\left(D_{V_{\mathbf{0}}^{2^{n-k}}}\left(\mathbf{0},(2^{n-k}-1)\mathbf{1}\right)=t\right)\right) - \sum_{t=T}^{\infty}6t2^{-t}\right)$$
$$\geq c'\Lambda\left(2^k,\beta\right)\left(\mathbb{E}_\beta\left[D_{V_{\mathbf{0}}^{2^{n-k}}}\left(\mathbf{0},(2^{n-k}-1)\mathbf{1}\right)\right]-6-T\right) \geq \tilde{c}\Lambda\left(2^k,\beta\right)\Lambda\left(2^{n-k},\beta\right) \geq \tilde{c}\Lambda\left(2^n,\beta\right)$$

for some $\tilde{c} > 0$ and all $k, n-k$ large enough. For each edge e, there are only finitely many levels k for which $\varepsilon^{2/\theta} 2^k \leq |e| \leq 6^d 2^k$. Thus we get that

$$\begin{aligned}
&\sum_e p'(\beta, e) \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega\right) - D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e+}\right) \right] \\
&\geq c_1 \sum_{k=1}^{n} \sum_{e \in E_k} p'(\beta, e) \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega\right) - D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e+}\right) \right] \\
&\geq c_2 \sum_{k=1}^{n} \Lambda\left(2^n, \beta\right) \geq c_3 \log\left(2^n\right) \Lambda\left(2^n, \beta\right)
\end{aligned}$$

for constants $c_1, c_2, c_3 > 0$ and n large enough. This already implies that

$$\begin{aligned}
&\sum_e p'(\beta, e) \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e-}\right) - D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e+}\right) \right] \\
&\geq \sum_e p'(\beta, e) \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega\right) - D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e+}\right) \right] \geq c_3 \log\left(2^n\right) \Lambda\left(2^n, \beta\right).
\end{aligned} \tag{114}$$

Now, let us see how this bound implies strict monotonicity of the distance exponent $\theta(\beta)$. We know that

$$\theta(\beta) = \lim_{n \to \infty} \frac{\log\left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right]\right)}{\log(2^n)}$$

and that for fixed n the function

$$\beta \mapsto \frac{\log\left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right]\right)}{\log(2^n)}$$

is, by Russo's formula for expectations (91), differentiable. So we can calculate the derivative and bound it from above by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta} \frac{\log\left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right]\right)}{\log(2^n)} \\
&= \frac{1}{\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right] \log(2^n)} \frac{\mathrm{d}}{\mathrm{d}\beta} \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right] \\
&= \frac{\sum_{e \in E} p'(\beta, e) \mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e+}\right) - D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}; \omega^{e-}\right) \right]}{\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right] \log(2^n)} \\
&\overset{(114)}{\leq} \frac{-c_3 \Lambda\left(2^n, \beta\right) \log(2^n)}{\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right] \log(2^n)} \leq -c_3 =: c(\beta)
\end{aligned}$$

for some $c(\beta) < 0$ and this holds for all $n \in \mathbb{N}_{>0}$ large enough. Now fix $0 < \beta_1 < \beta_2 < \infty$. We want to show that $\theta(\beta_1) > \theta(\beta_2)$. For each fixed $\beta \in [\beta_1, \beta_2]$ there exists $n(\beta) < \infty$ such that for all $n \geq n(\beta)$

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \frac{\log\left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n-1)\mathbf{1}\right) \right]\right)}{\log\left(2^n\right)} \leq \frac{c(\beta)}{2} \tag{115}$$

holds. So in particular we can take N large enough, and $c<0$ with $|c|$ small enough so that the set of $\beta \in [\beta_1,\beta_2]$ which satisfy $\frac{c(\beta)}{2} < c$, and which satisfy (115) for all $n \geq N$, has Lebesgue measure of at least $\frac{\beta_2-\beta_1}{2}$. Thus we get

$$\begin{aligned}\theta(\beta_2)-\theta(\beta_1) &= \lim_{n\to\infty}\left(\frac{\log\left(\mathbb{E}_{\beta_2}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1}\right)\right]\right)}{\log(2^n)} - \frac{\log\left(\mathbb{E}_{\beta_1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1}\right)\right]\right)}{\log(2^n)}\right)\\ &= \lim_{n\to\infty}\int_{\beta_1}^{\beta_2}\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\log\left(\mathbb{E}_{\beta}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)\mathbf{1}\right)\right]\right)}{\log(2^n)}\mathrm{d}\beta\\ &\leq \frac{\beta_2-\beta_1}{2}c < 0,\end{aligned}$$

which finishes the proof of the strict monotonicity ☐

11 Continuity of the distance exponent

In this section, we show that the distance exponent is continuous in β. This result is also useful for comparing different percolation models with each other, as shown in section 7. With the tools that we have developed so far, we can already prove continuity from the left:

Lemma 11.1. *The distance exponent $\theta(\beta)$ is continuous from the left.*

Proof. Remember that

$$\theta(\beta) = \inf_{n\geq 2}\frac{\log\left(\Lambda(n,\beta)\right)}{\log(n)}$$

which is stated in Lemma 2.3. For fixed n, the function $\beta \mapsto \Lambda(n,\beta)$ is continuous and decreasing in β. The continuity holds, as the inclusion probabilities $p(\beta,e)$ are continuous in β for all edges e, and we only consider the finitely many edges with both endpoints in $V_{\mathbf{0}}^n$. As the functions $p(\beta,e)$ are also increasing in β for all edges e, one can see with the Harris coupling that the function $\beta \mapsto \Lambda(n,\beta)$ is also decreasing. So we get that for all $\beta>0$

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\theta(\beta-\varepsilon) &= \inf_{\varepsilon>0}\theta(\beta-\varepsilon) = \inf_{\varepsilon>0}\inf_{n\geq 2}\frac{\log\left(\Lambda(n,\beta-\varepsilon)\right)}{\log(n)} = \inf_{n\geq 2}\inf_{\varepsilon>0}\frac{\log\left(\Lambda(n,\beta-\varepsilon)\right)}{\log(n)}\\ &= \inf_{n\geq 2}\frac{\log\left(\Lambda(n,\beta)\right)}{\log(n)} = \theta(\beta),\end{aligned}$$

and this shows continuity from the left. ☐

The proof of continuity from the right is more difficult. We consider independent bond percolation on the complete graph with vertex set $V = V_{\mathbf{0}}^{2^n}$ and edge set $E = \{\{x,y\} : x,y \in V_{\mathbf{0}}^{2^n}, x\neq y\}$. For $k\in\{1,\ldots,n\}$ and $\beta_1,\beta_2>0$, we denote by $\mathbb{P}_{\beta_1\leq k}^{\beta_2>k}$ the product probability measure on the $\{0,1\}^E$ where edges $e=\{u,v\}$ are open with the following probabilities:

$$\mathbb{P}_{\beta_1\leq k}^{\beta_2>k}\left(\omega(\{u,v\})=1\right) = \begin{cases}1-e^{-\beta_1\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} & \text{if } 1<|\{u,v\}|\leq 2^k-1\\ 1-e^{-\beta_2\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} & \text{if } |\{u,v\}|\geq 2^k\\ 1 & \text{if } |\{u,v\}|=1\end{cases},$$

so in particular the measure $\mathbb{P}_{\beta_1 \leq 1}^{\beta_2 > 1}$ is identical to the measure $\mathbb{P}_{\beta_2}$, and the measure $\mathbb{P}_{\beta_1 \leq n}^{\beta_2 > n}$ on the graph with vertex set $V_{\mathbf{0}}^{2^n}$ is identical to the measure $\mathbb{P}_{\beta_1}$. For $k \in \{2, \ldots, n-1\}$, we think of the measure $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$ as an interpolation between the probability measures $\mathbb{P}_{\beta_1}$ and $\mathbb{P}_{\beta_2}$ on the graph with vertex set $V_{\mathbf{0}}^{2^n}$. We will mostly work on this graph in this chapter and the distances should be considered as the graph distances inside this graph. We denote by $\mathbb{E}_{\beta_1 \leq k}^{\beta_2 > k}$ the expectation under $\mathbb{P}_{\beta_1 \leq k}^{\beta_2 > k}$. Our main strategy of the proof of Theorem 1.5 is as follows: We know that

$$\theta(\beta) = \lim_{n \to \infty} \frac{\log \left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right)}{\log (2^n - 1)} = \lim_{n \to \infty} \frac{\log \left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right)}{\log(2) n}$$

and thus we also have

$$\begin{aligned}
&\theta(\beta) - \theta(\beta + \varepsilon) \\
&= \lim_{n \to \infty} \frac{1}{\log(2) n} \left(\log \left(\mathbb{E}_\beta \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) - \log \left(\mathbb{E}_{\beta+\varepsilon} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) \right) \\
&= \lim_{n \to \infty} \frac{1}{\log(2) n} \left(\log \left(\mathbb{E}_{\beta \leq n}^{\beta+\varepsilon > n} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) - \log \left(\mathbb{E}_{\beta \leq 1}^{\beta+\varepsilon > 1} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) \right) \\
&= \frac{1}{\log(2)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \left(\log \left(\mathbb{E}_{\beta \leq k}^{\beta+\varepsilon > k} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) - \log \left(\mathbb{E}_{\beta \leq k-1}^{\beta+\varepsilon > k-1} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right] \right) \right) \\
&= \frac{1}{\log(2)} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \log \left(\frac{\mathbb{E}_{\beta \leq k}^{\beta+\varepsilon > k} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]}{\mathbb{E}_{\beta \leq k-1}^{\beta+\varepsilon > k-1} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]} \right). \qquad (116)
\end{aligned}$$

So in order to show that $\lim_{\varepsilon \to 0} \theta(\beta + \varepsilon) = \theta(\beta)$ it suffices to show that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \sum_{k=2}^{n} \log \left(\frac{\mathbb{E}_{\beta \leq k}^{\beta+\varepsilon > k} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]}{\mathbb{E}_{\beta \leq k-1}^{\beta+\varepsilon > k-1} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]} \right) = 0, \qquad (117)$$

and in order to show this, it is sufficient to show that the terms

$$\log \left(\frac{\mathbb{E}_{\beta \leq k}^{\beta+\varepsilon > k} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]}{\mathbb{E}_{\beta \leq k-1}^{\beta+\varepsilon > k-1} \left[D_{V_{\mathbf{0}}^{2^n}} \left(\mathbf{0}, (2^n - 1)\mathbf{1} \right) \right]} \right), \ k \in \{2, \ldots, n\},$$

are bounded uniformly and converge to 0, as $\varepsilon \to 0, k, n-k \to \infty$. Before going to the proof, we need to prove several technical results. In Lemma 11.2, we investigate the exponential moments of $\frac{\text{Diam}\left(V_{\mathbf{0}}^m\right)}{m^\theta}$, uniformly over m. In subsection 11.1, we derive several inequalities for the mixed measure $\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}$ that we need later in the proof. Then, in subsection 11.2 we show how this implies (117) and thus continuity of the distance exponent θ.

Lemma 11.2. *For all $\beta \geq 0$, there exists a constant $C_1 < \infty$ such that for all $s \geq 1$, and all $m \in \mathbb{N}_{>0}$*

$$\mathbb{E}_\beta \left[e^{s \frac{Diam\left(V_{\mathbf{0}}^m\right)}{m^{\theta(\beta)}}} \right] < e^{C_1 s^{C_1}}. \qquad (118)$$

Proof. Define $Y_m := \frac{\mathrm{Diam}\left(V_{\mathbf{0}}^m\right)}{m^{\theta(\beta)}}$. In Theorem 6.1, we proved that for each $\beta \geq 0$ there exists an $\eta > 1$, and a $C < \infty$ such that

$$\mathbb{E}_\beta\left[e^{Y_m^\eta}\right] \leq C \tag{119}$$

for all $m \in \mathbb{N}$. For all $y, s > 0$ one has

$$sy \leq sy\mathbb{1}_{\{s<y^{\eta-1}\}} + sy\mathbb{1}_{\{s\geq y^{\eta-1}\}} \leq y^{\eta-1}y\mathbb{1}_{\{s<y^{\eta-1}\}} + sy\mathbb{1}_{\left\{s^{\frac{1}{\eta-1}}\geq y\right\}} \leq y^\eta + s^{\frac{\eta}{\eta-1}}.$$

Inserting this into (119), we get that for all $s \geq 1$

$$\mathbb{E}_\beta\left[e^{sY_m}\right] \leq \mathbb{E}_\beta\left[e^{Y_m^\eta + s^{\frac{\eta}{\eta-1}}}\right] \leq Ce^{s^{\frac{\eta}{\eta-1}}} \leq e^{C_1 s^{C_1}}$$

for some $C_1 < \infty$. □

11.1 Uniform bounds for the mixed measure

In this chapter, we give several bounds for the measure $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$ that hold uniformly over $\varepsilon \in [0,1]$ and $k \leq n$. These bounds were partially already proven in the previous sections or in [33] for fixed β and $\varepsilon = 0$. One can couple the measures $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$ for different ε with the *Harris coupling*. For some set $V \subset \mathbb{Z}^d$ and $E = \{\{u,v\} : u,v \in V, u \neq v\}$, let $(U_e)_{e\in E}$ be independent random variables with uniform distribution on the interval $[0,1]$. Define the function $p(\beta, \beta+\varepsilon, k, \{u,v\}) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N} \times E \to [0,1]$ by

$$p(\beta,\beta+\varepsilon,k,\{u,v\}) = \begin{cases} 1 - e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} & 1 < |\{u,v\}| \leq 2^k - 1 \\ 1 - e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}} \frac{\beta+\varepsilon}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} & |\{u,v\}| \geq 2^k \\ 1 & |\{u,v\}| = 1 \end{cases}.$$

Define the environment $\omega^\varepsilon \in \{0,1\}^E$ by $\omega(e) = \mathbb{1}_{\{U_e \leq p(\beta,\beta+\varepsilon,k,e)\}}$. Then ω^ε is distributed according to the measure $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$. For $0 \leq \varepsilon_1 < \varepsilon_2$, this construction couples the measures $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon_1>k}$ and $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon_2>k}$ in such a way that all edges contained in the environment defined by ω^{ε_1} are also contained in the environment defined by ω^{ε_2}. The next two lemmas deal with the graph distance of certain points in boxes, that have direct edges to other far away blocks.

Lemma 11.3. *Let $V_u^{2^k}$ be a block with side length 2^k that is connected to $V_{\mathbf{0}}^{2^k}$ and let $\|u\|_\infty \geq 2$. Let $\mathcal{B}_u(\delta)$ be the following event:*

$$\mathcal{B}_u(\delta) = \bigcap_{\substack{x,y\in V_{\mathbf{0}}^{2^k}: \\ x,y\sim V_u^{2^k}, x\neq y}} \left\{D_{V_{\mathbf{0}}^{2^k}}(x,y) \geq \delta 2^{k\theta(\beta)}\right\}.$$

For every $\beta > 0$, there exists a function $f_1(\delta)$ with $f_1(\delta) \underset{\delta\to 0}{\longrightarrow} 1$ such that for all large enough $k \geq k(\delta)$, all $u \in \mathbb{Z}^d$ with $\|u\|_\infty \geq 2$, and all $\varepsilon \in [0,1]$

$$\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\mathcal{B}_u(\delta) \mid V_{\mathbf{0}}^{2^k} \sim V_u^{2^k}\right) \geq f_1(\delta).$$

Lemma 11.4. *Let $V_u^{2^k}, V_v^{2^k}$ be two blocks of side length 2^k that are connected to $V_{\mathbf{0}}^{2^k}$, with $u \neq v \neq \mathbf{0}$ and $\|u\|_\infty \geq 2$. Let $\mathcal{A}_{u,v}(\delta)$ be the following event:*

$$\mathcal{A}_{u,v}(\delta) = \bigcap_{\substack{x \in V_{\mathbf{0}}^{2^k}: \\ x \sim V_u^{2^k}}} \bigcap_{\substack{y \in V_{\mathbf{0}}^{2^k}: \\ y \sim V_v^{2^k}}} \left\{ D_{V_{\mathbf{0}}^{2^k}}(x,y) \geq \delta 2^{k\theta(\beta)} \right\}.$$

For every $\beta > 0$, there exists a function $f_2(\delta)$ with $f_2(\delta) \underset{\delta \to 0}{\longrightarrow} 1$ such that for all large enough $k \geq k(\delta)$, all $u, v \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ with $\|u\|_\infty \geq 2$, and all $\varepsilon \in [0,1]$

$$\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}\left(\mathcal{A}_{u,v}(\delta) \mid V_u^{2^k} \sim V_{\mathbf{0}}^{2^k} \sim V_v^{2^k} \right) \geq f_2(\delta).$$

Proof of Lemma 11.4 and Lemma 11.3. By the Harris coupling, it suffices to consider the case $\varepsilon = 1$. From here on, the proof is analogous to the proofs of Lemma 5.1 and Lemma 5.2. The spacing in terms of infinity distance between distinct points $x, y \in V_{\mathbf{0}}^{2^k}$ with $x \sim V_u^{2^k}, y \sim V_u^{2^k}$ can be bounded in the same way as in Lemma 5.1. As the structure inside $V_{\mathbf{0}}^{2^k}$ is not affected by any change of ε, the graph distance between such points x, y can be bounded as in Lemma 5.2. □

In the following lemma, we define the graph G' as the graph, in which we contract boxes of the form $V_u^{2^k}$ for $u \in \mathbb{Z}^d$. The vertex that results from contracting the box $V_u^{2^k}$ is called $r(u)$.

Lemma 11.5. *Let $\mathcal{B}(\delta)$ be the event*

$$\mathcal{B}(\delta) := \left\{ D^\star \left(V_{\mathbf{0}}^{2^k}, \bigcup_{u \in \mathbb{Z}^d : \|u\|_\infty \geq 2} V_u^{2^k} \right) \geq \delta 2^{k\theta(\beta)} \right\}.$$

For every $\beta > 0$ there exists a function $f_3(\delta)$ with $f_3(\delta) \underset{\delta \to 0}{\longrightarrow} 1$ such that for all large enough $k \geq k(\delta)$, all $\varepsilon \in [0,1]$, and all realizations of G' with $\deg^{\mathcal{N}}(r(\mathbf{0})) \leq 9^d 100 \mu_\beta$

$$\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}\left(\mathcal{B}(\delta) \mid G' \right) \geq f_3(\delta).$$

The proof of this lemma is similar to the proof of Lemma 5.4, and we omit it. From here on we also use the notation $f(\delta) = \min\{f_1(\delta), f_2(\delta), f_3(\delta)\}$.

Lemma 11.6. *For all $\beta > 0$, there exist constants $0 < c_\beta < C_\beta < \infty$ such that uniformly over all $\varepsilon \in [0,1]$*

$$\begin{aligned} c_\beta \Lambda\left(2^k, \beta\right) \Lambda\left(2^{n-k}, \beta+\varepsilon\right) &\leq \mathbb{E}_{\beta \leq k}^{\beta+\varepsilon > k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0}, (2^n - 1)e_1\right) \right] \\ &\leq C_\beta \Lambda\left(2^k, \beta\right) \Lambda\left(2^{n-k}, \beta+\varepsilon\right) \end{aligned} \tag{120}$$

The proof is completely analogous to the proofs of Lemma 2.3 and Lemma 5.5, so we omit it here. The proof of the first inequality is analogous as the proof of Lemma 5.5, and the proof of the second inequality is analogous to the proof of Lemma 2.3. We want to get similar bounds on the second moment of distances $D_{V_{\mathbf{0}}^{2^n}}$ under the measure $\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}$. For this, remember that Lemma 4.5 tells us that for all $\beta \geq 0$, there exists a constant $C_\beta < \infty$ such that for all $n \in \mathbb{N}$, all $\varepsilon \in [0,1]$ and all $x, y \in V_{\mathbf{0}}^n$

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^n}(x,y)^2 \right] \leq C_\beta \Lambda(n, \beta+\varepsilon)^2. \tag{121}$$

Having this inequality uniformly over $\varepsilon \in [0,1]$ allows us to prove a uniform bound on the second moment of distances under the measure $\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}$.

Lemma 11.7. *For all $\beta \geq 0$, there exists a constant $C_\beta < \infty$ such that uniformly over all $\varepsilon \in [0,1]$, all $k \leq n$, and all $x, y \in V_{\mathbf{0}}^{2^n}$*

$$\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}(x,y)^2\right] \leq C_\beta \Lambda(2^k,\beta)^2 \Lambda(2^{n-k},\beta+\varepsilon)^2. \tag{122}$$

Proof. We use a renormalization structure for this proof. We first define the renormalized graph G' where we contract all vertices of the set V_u^k to one vertex $r(u)$ and do this for all $u \in V_{\mathbf{0}}^{2^{n-k}}$. In the graph G', there is an edge between $r(u)$ and $r(v)$ if and only if there is an edge between $V_u^{2^k}$ and $V_v^{2^k}$. Now, let $x, y \in V_{\mathbf{0}}^{2^n}$ be arbitrary, say with $x \in V_u^{2^k}$ and $y \in V_v^{2^k}$. The claim is clear in the case where $u = v$, so we will assume $u \neq v$ from here on. Consider the shortest path between $r(u)$ and $r(v)$. Say that $(r(u_0), \ldots, r(u_K))$ is this path, where $K = D_{G'}(u,v)$, $u_0 = u$, and $u_K = v$. There is a path between x and y that uses only edges in or between the sets $V_{u_i}^{2^k}$ for $i = 0, \ldots, K$. Thus we have an upper bound on the graph distance between x and y given by

$$D_{V_{\mathbf{0}}^{2^n}}(x,y) \leq \sum_{i=0}^{K}\left(\text{Diam}\left(V_{u_i}^{2^k}\right)+1\right). \tag{123}$$

The random variables $\text{Diam}\left(V_{u_i}^{2^k}\right)$ and $K = D_{G'}(u,v)$ are independent, as the diameters $\text{Diam}\left(V_{u_i}^{2^k}\right)$ depend only on edges with both endpoints inside $V_{u_i}^{2^k}$, whereas the distance $K = D_{G'}(u,v)$ depends only on edges that are between two different boxes. For $(X_i)_{i\in\mathbb{N}}$ i.i.d. random variables that are furthermore independent of an integer-valued random variable $\tilde{K}$ one has

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{\tilde{K}} X_i\right)^2\right] &\leq \mathbb{E}\left[\left(\sum_{i=1}^{\infty}\mathbb{1}_{\{i\leq\tilde{K}\}}X_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{1}_{\{i\leq\tilde{K}\}}\mathbb{1}_{\{j\leq\tilde{K}\}}X_iX_j\right]\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{\{i\leq\tilde{K}\}}\mathbb{1}_{\{j\leq\tilde{K}\}}\right]\mathbb{E}\left[X_iX_j\right] \leq \mathbb{E}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathbb{1}_{\{i\leq\tilde{K}\}}\mathbb{1}_{\{j\leq\tilde{K}\}}\right]\mathbb{E}\left[X_1^2\right]\\
&= \mathbb{E}\left[\tilde{K}^2\right]\mathbb{E}\left[X_1^2\right].
\end{aligned}$$

We know that $\mathbb{E}_\beta\left[\text{Diam}\left(V_{l_i}^{2^k}\right)^2\right] \leq C'_\beta \Lambda\left(2^k,\beta\right)^2$ for some $C'_\beta < \infty$, which follows from Theorem 6.1. The distance $D_{G'}(r(u), r(v))$ only depends on the occupation status of edges with both ends in $V_{\mathbf{0}}^{2^n}$ that have a length of at least 2^k. Thus $D_{G'}(r(u), r(v))$ has exactly the same distribution as $D_{V_{\mathbf{0}}^{2^{n-k}}}(u,v)$ under the measure $\mathbb{P}_{\beta+\varepsilon}$. The previous observations together with (123) imply that

$$\begin{aligned}
&\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}(x,y)^2\right] \leq \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left(\sum_{i=0}^{K}\left(\text{Diam}\left(V_{u_i}^{2^k}\right)+1\right)\right)^2\right]\\
&\leq \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left(D_{G'}(r(u),r(v))+1\right)^2\right]\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left(\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)+1\right)^2\right]\\
&\leq \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left(2D_{G'}(r(u),r(v))\right)^2\right]\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left(2\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)\right)^2\right]\\
&\leq 4\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{2^{n-k}}}(u,v)^2\right]4C'_\beta\Lambda(2^k,\beta)^2 \leq 16C_\beta C'_\beta \Lambda(2^{n-k},\beta+\varepsilon)^2\Lambda(2^k,\beta)^2,
\end{aligned}$$

where we used (121) for the last inequality. □

11.2 The proof of Theorem 1.5

In order to prove Theorem 1.5, we use a coupling between the measures $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$ and $\mathbb{P}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}$. Let $\omega \in \{0,1\}^E$ be distributed according to $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$. Let $\chi \in \{0,1\}^E$ be a random vector that is independent of ω and has independent coordinates such that

$$\mathbb{P}\left(\chi\left(\{u,v\}\right)=1\right) = \begin{cases} 1-e^{-\varepsilon\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} & \text{if} 2^{k-1}\leq |\{u,v\}| \leq 2^k-1 \\ 0 & \text{else} \end{cases} . \tag{124}$$

Then set $\omega'(e) = \omega(e) \vee \chi(e) = \max\{\omega(e), \chi(e)\}$ for all edges $e \in E$. The coordinates of ω' are independent and for $e=\{u,v\} \in E$ with $2^{k-1} \leq |e| \leq 2^k - 1$ we have

$$\begin{aligned}\mathbb{P}\left(\omega'(e)=0\right) &= \mathbb{P}\left(\omega(e)=0\right)\mathbb{P}\left(\chi(e)=0\right) = e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{\beta}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y}e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{\varepsilon}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} \\ &= e^{-\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{\beta+\varepsilon}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} = 1-p\left(\beta+\varepsilon,\{u,v\}\right)\end{aligned}$$

and thus ω' is distributed according to the measure $\mathbb{P}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}$. For a block $V_u^{2^k} = \prod_{i=1}^d \left\{p_i(u)2^k,\ldots,(p_i(u)+1)2^k-1\right\}$ of side length 2^k and every vertex $v \in V_u^{2^k}$, there are at most $\left(2(2^k-1)+1\right)^d \leq 2^{(k+1)d}$ vertices w with $2^{k-1} \leq |\{v,w\}| \leq 2^k-1$. As χ can only be $+1$ on edges e with $2^{k-1} \leq |e| \leq 2^k-1$, we have

$$\begin{aligned}&\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists v \in V_u^{2^k}, w \in \mathbb{Z}^d \text{ with } \chi\left(\{v,w\}\right)=1\right) \\ &\leq 2^{kd}\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists w \in \mathbb{Z}^d \text{ with } \chi\left(\{\mathbf{0},w\}\right)=1\right) \\ &\leq 2^{kd}\sum_{w\in\mathbb{Z}^d:\|w\|_\infty\in\left[2^{k-1},2^k-1\right]}\left(1-e^{-\int_{\mathbf{0}+\mathcal{C}}\int_{w+\mathcal{C}}\frac{\varepsilon}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y}\right) \\ &\overset{(126)}{\leq} 2^{kd}\sum_{w\in\mathbb{Z}^d:\|w\|_\infty\in\left[2^{k-1},2^k-1\right]}\frac{2^{2d}\varepsilon}{\|w\|_\infty^{2d}} \leq 2^{kd}2^{(k+1)d}\frac{2^{4d}\varepsilon}{2^{2kd}} = 2^{5d}\varepsilon,\end{aligned} \tag{125}$$

where we used that

$$1-e^{-\varepsilon\int_{u+\mathcal{C}}\int_{v+\mathcal{C}}\frac{1}{\|x-y\|^{2d}}\mathrm{d}x\mathrm{d}y} = \mathbb{P}_\varepsilon\left(u\sim v\right) \leq \frac{2^{2d}\varepsilon}{\|u-v\|_\infty^{2d}} \tag{126}$$

for all $\varepsilon \geq 0$, all $n \in \mathbb{N}$, and all $u,v \in \mathbb{Z}^d$ with $\|u-v\|_\infty \geq 2$. This was proven in Lemma 2.1.

Next, we define a notion of good sets inside the graph with vertex set $V_{\mathbf{0}}^{2^n}$. For $w \in V_{\mathbf{0}}^{2^{n-k}}$, we contract the box $V_w^{2^k} \subset V_{\mathbf{0}}^{2^n}$ to vertices $r(w)$ and call the resulting graph G'. Remember the definition of the events $\mathcal{B}(\delta), \mathcal{B}_u(\delta)$, and $\mathcal{A}_{u,v}(\delta)$ from Lemmas 11.4, 11.3, and 11.5. For a small $\delta > 0$ (that will be defined in (131) below), we call a vertex $r(w)$ and the underlying block $V_w^{2^k}$ *δ-good*, if all the translated events of $\mathcal{B}(\delta), \mathcal{B}_u(\delta)$, and $\mathcal{A}_{u,v}(\delta)$ occur, i.e., if

$$\bigcap_{\substack{x\in V_w^{2^k}:\\ x\sim V_u^{2^k}}}\bigcap_{\substack{y\in V_w^{2^k}:\\ y\sim V_v^{2^k}}}\left\{D_{V_w^{2^k}}\left(x,y\right) \geq \delta 2^{k\theta(\beta)}\right\} \tag{127}$$

for all $u \neq v$ for which $w \neq v$, $V_u^{2^k} \sim V_w^{2^k} \sim V_v^{2^k}$ and $\|u-w\|_\infty \geq 2$, and if

$$\bigcap_{\substack{x,y \in V_w^{2^k}: \\ x,y \sim V_u^{2^k}, x \neq y}} \left\{ D_{V_w^{2^k}}(x,y) \geq \delta 2^{k\theta(\beta)} \right\} \tag{128}$$

for all u with $\|u-w\|_\infty \geq 2$ and $V_u^{2^k} \sim V_w^{2^k}$, and if

$$D^\star \left(V_w^{2^k}, \bigcup_{u \in \mathbb{Z}^d : \|u-w\|_\infty \geq 2} V_u^{2^k} \right) \geq \delta 2^{k\theta(\beta)}. \tag{129}$$

Suppose that a path crosses a good set $V_w^{2^k}$, in the sense that it starts somewhere outside of the set $\bigcup_{u \in \mathbb{Z}^d : \|u-w\|_\infty \leq 1} V_u^{2^k}$, then goes to the set $V_w^{2^k}$, and then leaves the set $\bigcup_{u \in \mathbb{Z}^d : \|u-w\|_\infty \leq 1} V_u^{2^k}$ again. When the path enters the set $V_w^{2^k}$ at the vertex x, coming from some a block $V_u^{2^k}$ with $\|u-w\| \geq 2$, the path needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ to reach a vertex $y \in V_w^{2^k}$ that is connected to the complement of $V_w^{2^k}$, because of (127) and (128). When the path enters the set $V_w^{2^k}$ from a block $V_v^{2^k}$ with $\|v-w\|_\infty = 1$, then the path crosses the annulus between $V_w^{2^k}$ and $\bigcup_{u \in \mathbb{Z}^d : \|u-w\|_\infty \geq 2} V_u^{2^k}$. So in particular it needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ in order to cross this annulus, because of (129). Overall, we see that the path needs to walk a distance of at least $\delta 2^{k\theta(\beta)}$ within the set $\bigcup_{u \in \mathbb{Z}^d : \|u-w\|_\infty \leq 1} V_u^{2^k}$ in order to cross the set $V_w^{2^k}$. Let δ be small enough such that

$$9^{2d} 5000 \mu_{\beta+1}^2 (1 - f(\delta)) \leq (32 \mu_{\beta+1})^{-9^d 400 \mu_{\beta+1}}.$$

Such a $\delta > 0$ exists, as $f(\delta) = \min\{f_1(\delta), f_2(\delta), f_3(\delta)\}$ tends to 1 for $\delta \to 0$. From here on we call a block $V_w^{2^k}$ good if it is δ-good for this specific choice of δ, and we call a vertex $r(w) \in G'$ good if the underlying block $V_w^{2^k}$ is good. For a connected set $Z \subset G'$, we are interested in the number of *separated good* vertices inside this set, that are good vertices $r(v)$ such that the sets $\mathcal{N}(r(v))$ are not connected by a direct edge.

Lemma 11.8. *Let $\varepsilon \in [0,1]$, let $G = \left(V_{\mathbf{0}}^{2^n}, E\right)$ be sampled according to the measure $\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k}$, and let G' be the graph that results from contracting boxes of the form $V_w^{2^k}$. Then for large enough K one has*

$$\mathbb{P}_{\beta \leq k}^{\beta+\varepsilon > k} \left(\exists Z \in \mathcal{CS}_K(G') \text{ with less than } \frac{K}{9^d 400 \mu_{\beta+1}} \text{ separated good vertices} \right) \leq 3 \cdot 2^{-K}. \tag{130}$$

Proof. Let $\hat{Z} = \{r(v_1), \ldots, r(v_K)\}$ be a connected set in G'. Let $\preceq$ be a fixed total ordering of $\mathbb{Z}^d$, where we write $\prec$ for strict inequalities. Such an ordering can be obtained by considering a bijection $f: \mathbb{N} \to \mathbb{Z}^d$ and defining $u \preceq v \Leftrightarrow f^{-1}(u) \leq f^{-1}(v)$. So we can assume that $\hat{Z} = \{r(v_1), \ldots, r(v_K)\}$, where $v_1 \prec v_2 \prec \ldots \prec v_K$. For such a set, we add the nearest neighbors to it. Formally, we define the set

$$\hat{Z}^{\mathcal{N}} = \bigcup_{r(v) \in \hat{Z}} \mathcal{N}(r(v))$$

which is still a connected set and satisfies $K \leq |\hat{Z}^{\mathcal{N}}| \leq 3^d K$. A vertex $r(u) \in G'$ can be included into the set $\hat{Z}^{\mathcal{N}}$ in more than one way, meaning that there can be different vertices $r(v), r(\tilde{v}) \in \hat{Z}$ such that $r(u) \in \mathcal{N}(r(v))$ and $r(u) \in \mathcal{N}(r(\tilde{v}))$. However, each vertex $r(u) \in G'$ can be included into the set $\hat{Z}^{\mathcal{N}}$ in at most 3^d many different ways. So in particular we have

$$\sum_{i=1}^{K} \deg^{\mathcal{N}}(r(v_i)) \leq 3^d \sum_{r(v)\in \hat{Z}^{\mathcal{N}}} \deg(r(v)),$$

where the neighborhood-degree of a vertex $\deg^{\mathcal{N}}(r(u))$ was defined in (108). Next, we iteratively define a set $\mathbb{LI} = \mathbb{LI}(\hat{Z}) = \mathbb{LI}_K \subset \hat{Z}$ as follows:

0. Start with $\mathbb{LI}_0 = \emptyset$.
1. For $i = 1, \ldots, K$: If $\deg^{\mathcal{N}}(r(v_i)) \leq 9^d 50\mu_{\beta+1}$ and $\mathcal{N}(r(v_i)) \nsim \mathbb{LI}_{i-1}$, then set $\mathbb{LI}_i = \mathbb{LI}_{i-1} \cup r(v_i)$; else set $\mathbb{LI}_i = \mathbb{LI}_{i-1}$.

On the event where $\overline{\deg}(Z) \leq 20\mu_{\beta+1}$, for all $Z \in \mathcal{CS}_{\geq K}(G')$, we have

$$\sum_{i=1}^{K} \deg^{\mathcal{N}}(r(v_i)) \leq 3^d \sum_{r(v)\in \hat{Z}^{\mathcal{N}}} \deg(r(v)) \leq 3^d 20\mu_{\beta+1} \left|\hat{Z}^{\mathcal{N}}\right| \leq 9^d 20\mu_{\beta+1} K$$

and thus there can be at most $\frac{K}{2}$ many vertices $r(v_i)$ with $\deg^{\mathcal{N}}(r(v_i)) > 9^d 50\mu_{\beta+1}$, which implies that there are at least $\frac{K}{2}$ many vertices with $\deg^{\mathcal{N}}(r(v_i)) \leq 9^d 50\mu_{\beta}$. Whenever we include such a vertex in the set $\mathbb{LI}$, we can block at most $9^d 50\mu_{\beta+1}$ different vertices, which already implies

$$|\mathbb{LI}| \geq \frac{K}{2(9^d 50\mu_{\beta+1} + 1)} \geq \frac{K}{9^d 200\mu_{\beta+1}}.$$

The event where $\overline{\deg}(Z) \leq 20\mu_{\beta+1}$ for all $Z \in \mathcal{CS}_{\geq K}(G')$ is very likely for large K, by Lemma 3.2.

Conditioned on the degree of the block $V_w^{2^k}$, and assuming that $\deg^{\mathcal{N}}(r(w)) \leq 9^d 50\mu_{\beta+1}$, the probability that the block $V_w^{2^k}$ is not δ-good is bounded by

$$\deg(r(w))^2 (1 - f_2(\delta)) + \deg^{\mathcal{N}}(r(w)) (1 - f_1(\delta)) + (1 - f_3(\delta)) \leq 9^{2d} 5000\mu_{\beta+1}^2 (1 - f(\delta)),$$

where f was defined by $f(\delta) = \min\{f_1(\delta), f_2(\delta), f_3(\delta)\}$. Remember that we chose $\delta > 0$ small enough so that

$$\tilde{f}(\delta) := 9^{2d} 5000\mu_{\beta+1}^2 (1 - f(\delta)) \leq (32\mu_{\beta+1})^{-9^d 400\mu_{\beta+1}}. \tag{131}$$

We now claim that the set $\mathbb{LI}$ contains at least $\frac{K}{9^d 400\mu_{\beta+1}}$ many separated good vertices with high probability. Given the graph G', it is independent whether different vertices in $\mathbb{LI}$ are good or not, as we will argue now. For a vertex $r(u)$, it depends only on edges with at least one end in the set $\bigcup_{r(v)\in\mathcal{N}(r(u))} V_v^{2^k}$ whether the vertex $r(u)$ is good or not. But for different vertices $r(u), r(u') \in \mathbb{LI}$ there are no edges with one end in $\bigcup_{r(v)\in\mathcal{N}(r(u))} V_v^{2^k}$ and the other end in $\bigcup_{r(v)\in\mathcal{N}(r(u'))} V_v^{2^k}$, as $\mathcal{N}(r(u)) \nsim \mathcal{N}(r(u'))$. Thus, it is independent whether different vertices in $\mathbb{LI}$ are good. So in particular, the probability that there are $\frac{|\mathbb{LI}|}{2}$ or more vertices in the set $\mathbb{LI}$ that are not good is bounded by

$$2^{|\mathbb{LI}|} \tilde{f}(\delta)^{\frac{|\mathbb{LI}|}{2}} \leq 2^K \tilde{f}(\delta)^{\frac{K}{9^d 400\mu_{\beta+1}}} \leq 2^K (32\mu_{\beta+1})^{-K} = (16\mu_{\beta+1})^{-K}$$

and thus the set $\mathbb{LI}$ (and also the set $\hat{Z}$) contains at least $\frac{K}{9^d 400\mu_{\beta+1}}$ good vertices with very high probability. Furthermore, the separation property directly follows from the construction. Next, we want to translate such a bound from one connected set to all connected sets simultaneously. Using Lemma 3.2, we get that

$$\begin{aligned}
&\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists Z\in\mathcal{CS}_K\left(G'\right) \text{ with less than } \frac{K}{9^d 400\mu_{\beta+1}} \text{ separated good vertices}\right)\\
&\leq \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists Z\in\mathcal{CS}_K\left(G'\right) : \left|\{r(v)\in\mathbb{LI}(Z) : r(v) \text{ good}\}\right| \leq \frac{K}{9^d 400\mu_{\beta+1}}\right)\\
&\leq \mathbb{P}_{\beta+\varepsilon}\left(\exists Z\in\mathcal{CS}_K\left(G'\right) : \overline{\deg}(Z) > 20\mu_{\beta+1}\right) + \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\left|\mathcal{CS}_K(G')\right| > 8^K\mu_{\beta+1}^K\right)\\
&+ 8^K\mu_{\beta+1}^K\left(16\mu_{\beta+1}\right)^{-K} \leq 3\cdot 2^{-K}
\end{aligned}$$

which finishes the proof. □

With this we can now go to the proof of Theorem 1.5.

Proof of Theorem 1.5. We want to show that for all $\beta \geq 0$ the difference $\theta(\beta) - \theta(\beta+\varepsilon)$ converges to 0 as $\varepsilon \to 0$. At the beginning of section 11, we already showed that the function $\theta(\cdot)$ is continuous from the left, so it suffices to consider $\varepsilon > 0$ now. We have also seen in (116) that

$$\theta(\beta) - \theta(\beta+\varepsilon) = \frac{1}{\log(2)}\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\log\left(\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}\right). \tag{132}$$

Each of the summands in (132) is bounded, which follows directly from the results of Lemma 11.6. So in order to show that $\theta(\beta) = \lim_{\varepsilon\to 0}\theta(\beta+\varepsilon)$, it suffices to show that the summands converge to 0, for large $k, n-k$, as $\varepsilon \to 0$. Showing the convergence of a summand in (132) to 0 is equivalent to proving that the expression inside the logarithm converges to 1, which is equivalent to showing that

$$\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right] - \mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}$$

converges to 0 , as $\varepsilon \to 0$. Again, we write G' for the graph where we contracted boxes of the form $V_v^{2^k}$ into vertices $r(v)$. So each vertex $r(v)$ in G' corresponds to the set $V_v^{2^k}$. We write $\text{Diam}(r(v))$ for $\text{Diam}\left(V_v^{2^k}\right)$. Next, we want to investigate the sum of diameters in connected sets. We claim that there exists a constant $1 < C' < \infty$ such that $\sum_{r(v)\in Z}\text{Diam}(r(v)) \leq C'|Z|2^{k\theta(\beta)}$ for all connected sets Z of some size with high probability. Let Z be a fixed set in G'. Under the measure $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$, the diameter of the box corresponding to some vertex $r(v)\in G'$ always has the same distribution, not depending on ε. By Markov's inequality we have

$$\begin{aligned}
&\mathbb{P}_\beta\left(\sum_{r(v)\in Z}\text{Diam}(r(v)) > C'|Z|2^{k\theta(\beta)}\right) = \mathbb{P}_\beta\left(e^{\sum_{r(v)\in Z}\frac{\text{Diam}(r(v))}{2^{k\theta(\beta)}}} > e^{C'|Z|}\right)\\
&\leq \mathbb{E}_\beta\left[e^{\frac{\text{Diam}(r(v))}{2^{k\theta(\beta)}}}\right]^{|Z|} e^{-C'|Z|} \leq \left(8\mu_{\beta+1}\right)^{-|Z|}
\end{aligned}$$

for C' large enough, as $\frac{\text{Diam}(r(v))}{2^{k\theta(\beta)}}$ has uniform exponential moments (see for example Lemma 11.2). As the diameter of some vertex v is independent of the edges in G' we get by a union bound that

$$\begin{aligned}\mathbb{P}_{\beta+\varepsilon}&\left(\exists Z \in \mathcal{CS}_K\left(G'\right) : \sum_{r(v)\in Z} \text{Diam}(r(v)) > C'|Z|2^{k\theta(\beta)}\right)\\ &\leq \mathbb{E}_{\beta+\varepsilon}\left[\left|\mathcal{CS}_K\left(G'\right)\right|\right]\left(8\mu_{\beta+1}\right)^{-K} \leq 2^{-K}. \end{aligned} \tag{133}$$

Let $\mathcal{Z}_K$ be the event that every connected set $Z \in \mathcal{CS}_K(G')$ satisfies $\sum_{v\in Z}\text{Diam}(v) \leq C'K2^{k\theta(\beta)}$ and that every connected set $Z \in \mathcal{CS}_K(G')$ contains at least $\frac{K}{9^d 400\mu_{\beta+1}}$ separated good vertices. We also define $\mathcal{Z}_{\geq K} \coloneqq \bigcap_{t=K}^{\infty}\mathcal{Z}_t$. By (130), (133) and a union bound over all $t \geq K$ we know that

$$\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\mathcal{Z}_{\geq K}^C\right) \leq \sum_{t=K}^{\infty}\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\mathcal{Z}_t^C\right) \leq 10\cdot 2^{-K} \tag{134}$$

for all large enough K. Now assume that the event $\mathcal{Z}_{\geq K}$ holds and that $D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right) = K$. So it is possible to walk from $\mathbf{0}$ to $(2^n-1)e_1$ and to touch only $K+1$ boxes of the form $V_w^{2^k}$, by going along the shortest path between $r(\mathbf{0})$ and $r\left((2^{n-k}-1)e_1\right)$ in G'. This path is also a connected set in G'. Between these boxes, one needs to take on additional step. Thus we have that

$$D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right) \leq C'(K+1)2^{k\theta(\beta)} + K \leq 2C'K2^{k\theta(\beta)}. \tag{135}$$

On the other hand, let P be a path from $\mathbf{0}$ to $(2^n-1)e_1$ and let $\hat{P}$ be its projection onto G'. Then the projection $\hat{P}$ goes through at least K blocks of the form $V_w^{2^k}$ and the projection $\hat{P}$ is a connected set in G'. Thus, the set $\hat{P}$ contains at least $\frac{K}{9^d 400\mu_{\beta+1}}$ separated good vertices. Now consider the situation where the path P crosses a good block $V_w^{2^k}$. In this case, the path P already needs to make at least $\delta 2^{k\theta(\beta)}$ steps inside the set $\bigcup_{u\in\mathbb{Z}^d:\|u-w\|_\infty\leq 1}V_u^{2^k}$. The sets $\bigcup_{u\in\mathbb{Z}^d:\|u-w\|_\infty\leq 1}V_u^{2^k}$ are not directly connected for different separated good vertices $r(w)$ inside $\hat{P}$. The path P crosses at least $\frac{K}{9^d 400\mu_{\beta+1}}-2$ separated good boxes, where the subtraction of two is necessary because the path touches boxes at the beginning/end without crossing them. This already implies that

$$\text{length}(P) \geq \left(\frac{K}{9^d 400\mu_{\beta+1}} - 2\right)\delta 2^{k\theta(\beta)}. \tag{136}$$

Next, we want to investigate how this helps us to bound the difference

$$\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right] - \mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right].$$

We use the same notation as in the beginning of this chapter, i.e., we assume that ω is distributed according to $\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}$ and χ is independent of ω and distributed as described in (124). Then $\omega' \coloneqq \omega \vee \chi$ has law $\mathbb{P}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}$. The structure of the graph G', in which we contracted blocks of side length 2^k, does not change, as the edges inserted are either inside the blocks $V_w^{2^k}$ or between neighboring blocks. The probability that a block $V_w^{2^k}$ is adjacent to a bond in ω' that did not exist in ω is bounded by $2^{5d}\varepsilon$, see (125). For a connected set $Z \subset G'$, we write Z_χ for the set of vertices $r(w) \in Z$ for which there exists an edge e that is adjacent to $V_w^{2^k}$ and satisfies $\chi(e) = 1$. For a fixed set Z we expect that $|Z_\chi|$ is of order $\varepsilon |Z|$ but it is not clear how to show such a statement for all connected sets of some size. Instead, we show that with high probability for all connected sets Z of size K the set Z_χ is not larger than $\frac{16\mu_{\beta+1}K}{\log(1/\varepsilon)}$. For a fixed connected set $Z \in \mathcal{CS}_K(G')$, define the set Z_χ^l by the vertices $r(w) \in Z_\chi$ for which an edge e with $\chi(e) = 1$ exists, so that e has both endpoints in $V_w^{2^k}$, or one endpoint is in $V_w^{2^k}$ and one endpoint is in $V_u^{2^k}$ with $r(u) \notin Z$, or one endpoint in $V_w^{2^k}$ and one in $V_u^{2^k}$ with $w \prec u$ and $r(u) \in Z$. For different vertices $r(u) \in Z$, it is independent whether they are in the set Z_χ^l or not. Hence the size of the set Z_χ^l is stochastically dominated by $\sum_{i=1}^K X_i$, where X_i are independent Bernoulli-distributed random variables with parameter $2^{5d}\varepsilon$. Furthermore, one has $|Z_\chi| \leq 2|Z_\chi^l|$, as each edge e with $1 - \omega(e) = \omega'(e) = 1$, that creates a vertex in Z_χ^l, can add at most two vertices to Z_χ. As the structure of the graph G' and the sets Z_χ are independent, we get for small enough $\varepsilon > 0$ that

$$\begin{aligned}
&\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists Z \in \mathcal{CS}_K\left(G'\right) : |Z_\chi| > \frac{\mu_{\beta+1}16}{\log(1/\varepsilon)}K\right)\\
&\leq \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\left|\mathcal{CS}_K\left(G'\right)\right|\right]\mathbb{P}\left(2\sum_{i=1}^K X_i > \frac{\mu_{\beta+1}16}{\log(1/\varepsilon)}K\right)\\
&\leq 4^K\mu_{\beta+1}^K\mathbb{P}\left(\sum_{i=1}^K X_i > \frac{\mu_{\beta+1}8}{\log(1/\varepsilon)}K\right) \leq 4^K\mu_{\beta+1}^K 2^K\left(2^{5d}\varepsilon\right)^{\frac{\mu_{\beta+1}8}{\log(1/\varepsilon)}K}\\
&= \left(2^{5d}\right)^{\frac{\mu_{\beta+1}8}{\log(1/\varepsilon)}K} 8^K\mu_{\beta+1}^K e^{-\mu_{\beta+1}8K} \leq 2^{-K}
\end{aligned}$$

where the last inequality holds for small enough ε. Next, let us see how the sums of the inside diameters of the sets Z_χ grow. Let $C_1 \in (0,\infty)$ be a the constant such that $\mathbb{E}_\beta\left[e^{s\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)}{2^{k\theta(\beta)}}}\right] < e^{C_1 s^{C_1}}$ for all $s \geq 1$ and $k \in \mathbb{N}$. Such a constant exists by Lemma 11.2. We define the functions $r(\varepsilon) = \log(1/\varepsilon)^{-\frac{1}{2C_1}}$ and $s(\varepsilon) = \left(\frac{\log(1/\varepsilon)}{\mu_{\beta+1}16C_1}\right)^{\frac{1}{C_1}}$. Let $Z' \subset G'$ be a fixed set of size at most $\frac{\mu_{\beta+1}16}{\log(1/\varepsilon)}K$. Then we have for all small enough ε that

$$\begin{aligned}
&\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\sum_{r(v)\in Z'}\text{Diam}\left(r(v)\right) > r(\varepsilon)2^{k\theta(\beta)}K\right) = \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(e^{s(\varepsilon)\sum_{r(v)\in Z'}\frac{\text{Diam}(r(v))}{2^{k\theta(\beta)}}} > e^{s(\varepsilon)r(\varepsilon)K}\right)\\
&\leq \mathbb{E}_\beta\left[\exp\left(s(\varepsilon)\frac{\text{Diam}\left(V_{\mathbf{0}}^{2^k}\right)}{2^{k\theta(\beta)}}\right)\right]^{|Z'|} e^{-s(\varepsilon)r(\varepsilon)K} \leq e^{\frac{\mu_{\beta+1}16K}{\log(1/\varepsilon)}C_1 s(\varepsilon)^{C_1}} e^{-s(\varepsilon)r(\varepsilon)K}\\
&= e^K\exp\left(-\frac{\log(1/\varepsilon)^{\frac{1}{2C_1}}}{\left(\mu_{\beta+1}16C_1\right)^{\frac{1}{C_1}}}K\right) \leq 16^{-K}\mu_{\beta+1}^{-K}
\end{aligned}$$

where the last inequality holds for ε small enough. As the inside structure of blocks of the form $V_v^{2^k}$ in the graph defined by ω, the sets Z_χ, and the connections inside the graph G' are independent, we get that for $\varepsilon > 0$ small enough

$$\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists Z \in \mathcal{CS}_K(G') : \sum_{r(v)\in Z_\chi}\text{Diam}\left(r(v)\right) > r(\varepsilon)2^{k\theta(\beta)}K\right)$$

$$\leq \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\left|\mathcal{CS}_K(G')\right| > 8^K \mu_{\beta+1}^K\right) + \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\exists Z \in \mathcal{CS}_K(G') : |Z_\chi| > \frac{\mu_{\beta+1}16}{\log(1/\varepsilon)} K\right)$$
$$+ 8^K \mu_{\beta+1}^K 16^{-K} \mu_{\beta+1}^{-K} \leq 3 \cdot 2^{-K}. \qquad (137)$$

Let $\mathcal{D}_K$ be the event that $\sum_{v \in Z_\chi} \text{Diam}\left(r(v)\right) \leq r(\varepsilon) 2^{k\theta(\beta)} K$ for all $Z \in \mathcal{CS}_K(G')$, and let $\mathcal{D}_{\geq K} = \bigcap_{t=K}^{\infty} \mathcal{D}_t$. From (134) and (137) we get that for K large enough

$$\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\left(\mathcal{Z}_{\geq K} \cap \mathcal{D}_{\geq K}\right)^C\right) \leq \sum_{t=K}^{\infty}\left(\mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\mathcal{Z}_t^C\right) + \mathbb{P}_{\beta\leq k}^{\beta+\varepsilon>k}\left(\mathcal{D}_t^C\right)\right) \leq 20 \cdot 2^{-K}.$$

Now assume that $D_{G'}\left(r(\mathbf{0}), r\left((2^{n-k}-1)e_1\right)\right) = K$ and the events $\mathcal{Z}_{\geq K}$ and $\mathcal{D}_{\geq K}$ both hold; Consider a path P between $\mathbf{0}$ and $(2^n - 1)\, e_1$ in the environment $\omega' = \omega \vee \chi$ and its projection $\hat{P}$ on G'. Assume that the events $\mathcal{D}_{\geq K}$ and $\mathcal{Z}_{\geq K}$ both hold, and that $K = D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right)$ is large enough. The path $\hat{P}$ is a (not necessarily self-avoiding) walk on G' between $r(\mathbf{0})$ and $r\left((2^{n-k}-1)e_1\right)$. In the environment ω for $\left|\hat{P}\right|$ large enough, every path that touches $\left|\hat{P}\right|$ distinct 2^k-blocks has length at least $\left(\frac{|\hat{P}|}{9^d 400 \mu_{\beta+1}} - 2\right) \delta 2^{k\theta(\beta)}$ by (136). In the environment $\omega \vee \chi$ such a path may be shorter, but by at most $\sum_{r(v)\in\hat{P}_\chi} \text{Diam}(r(v))$. So we get that

$$\begin{aligned}\text{length}(P) &\geq \left(\frac{\left|\hat{P}\right|}{9^d 400 \mu_{\beta+1}} - 2\right) \delta 2^{k\theta(\beta)} - \sum_{r(v)\in\hat{P}_\chi} \text{Diam}(r(v)) \\ &\geq \left(\frac{\left|\hat{P}\right|}{9^d 400 \mu_{\beta+1}} - 2\right) \delta 2^{k\theta(\beta)} - r(\varepsilon)\left|\hat{P}\right| 2^{k\theta(\beta)} \geq c_1 2^{k\theta(\beta)}\left|\hat{P}\right| \qquad (138)\end{aligned}$$

for some small $c_1 > 0$, ε small enough, and $\left|\hat{P}\right|$ large enough. Now consider the shortest path P between $\mathbf{0}$ and $(2^n - 1)\, e_1$ in the environment ω'. Combining the inequalities (135) and (138) we get that for $K = D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right)$

$$2C'K 2^{k\theta(\beta)} \geq D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0}, (2^n-1)e_1\right) = \text{length}(P) \geq c_1 \left|\hat{P}\right| 2^{k\theta(\beta)}$$

and thus

$$\left|\hat{P}\right| \leq \frac{2C'}{c_1} D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right) =: C_w D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right).$$

So the shortest path P between $\mathbf{0}$ and $(2^n-1)e_1$ in the environment $\omega' = \omega \vee \chi$ does not touch more than $C_w D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right)$ blocks with side length 2^k. This is an interesting observation, as the path also needs to touch at least $D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right)$ many blocks with side length 2^k.

Now let us bound the difference $D\left(\mathbf{0}, (2^n-1)e_1; \omega\right) - D\left(\mathbf{0}, (2^n-1)e_1; \omega'\right)$. Let $(x_0, \ldots, x_s)$ be the shortest path between $x_0 = \mathbf{0}$ and $x_s = (2^n-1)e_1$ in the environment ω'. Then we build a path $(y_0, \ldots, y_{\tilde{s}})$ between $\mathbf{0}$ and $(2^n-1)e_1$ in the environment ω as follows. As long as $\omega\left(\{x_i, x_{i+1}\}\right) = 1$, we follow the path P. If $\omega\left(\{x_i, x_{i+1}\}\right) = 0$, say with $x_i \in V_u^{2^k}$ and $x_{i+1} \in V_w^{2^k}$, we take the shortest path from x_i to $x_{i'}$ where $i' = \max\{s' : x_{s'} \in V_w^{2^k}\}$. That means, we go to the point $x_{i'}$ where the path $(x_0, \ldots, x_s)$ leaves the box $V_w^{2^k}$ for the last time. As ω and ω' can only differ at edges with length in $\left[2^{k-1}, 2^k - 1\right]$, we already have $\|u-w\|_\infty \leq 1$, and thus the distance between x_i and $x_{i'}$ is at most $\text{Diam}(r(u)) + \text{Diam}(r(w)) + 1$. So the length of the path constructed by this procedure is at most $\text{Diam}(r(u)) + \text{Diam}(r(w))$ longer compared to the original path. When at $x_{i'}$, we follow the path P again until there appears again an edge $e = \{x_j, x_{j+1}\}$ with $\omega'(e) = 1 = 1-\omega(e)$ and do the same procedure as before. This construction gives a path in the environment ω of length at most $s + 2\sum_{v\in\hat{P}_\chi} \text{Diam}(v)$ and this already implies

$$\begin{aligned}
&D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right) = D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - s \\
&\leq 2\sum_{r(v)\in\hat{P}_\chi} \text{Diam}(r(v)) \leq 2C_w r(\varepsilon) D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right) 2^{k\theta(\beta)} \qquad (139)
\end{aligned}$$

for small enough ε and when $\mathcal{D}_{\geq K}\cap\mathcal{Z}_{\geq K}$ and $D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right) \geq K$ hold for large enough K. So this gives us a bound on the difference of $D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right)$ and $D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right)$ that goes to 0, as $\varepsilon\to 0$. This bound only holds on the previously mentioned event, but we can also choose K, depending on $n-k$, in such a way such that the probability of this event goes to 1 as $n-k\to\infty$. The residual terms, where the previously mentioned events do not hold, can be estimated with the Cauchy-Schwarz inequality. For small enough $\varepsilon>0$ and large enough $k, n-k$ we have

$$\begin{aligned}
&\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right] - \mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right] \\
&= \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right)\right] \\
&= \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\Big[\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right)\right) \\
&\qquad\qquad \cdot \mathbb{1}_{\{\mathcal{D}_{\geq n-k}\cap\mathcal{Z}_{\geq n-k}\}}\mathbb{1}_{\{D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)e_1)\right)\geq n-k\}}\Big] \\
&\quad + \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\Big[\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right)\right) \\
&\qquad\qquad \cdot \mathbb{1}_{\{\mathcal{D}_{\geq n-k}\cap\mathcal{Z}_{\geq n-k}\}}\mathbb{1}_{\{D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)e_1)\right)< n-k\}}\Big] \\
&\quad + \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\Big[\left(D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega\right) - D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1;\omega'\right)\right)\mathbb{1}_{\left\{\left(\mathcal{D}_{\geq n-k}\cap\mathcal{Z}_{\geq n-k}\right)^C\right\}}\Big] \\
&\overset{(139)}{\leq} \mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{G'}\left(r(\mathbf{0}), r((2^{n-k}-1)e_1)\right)2C_w r(\varepsilon)2^{k\theta(\beta)}\right] \\
&\quad + \sqrt{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)^2\right]}\sqrt{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\mathbb{1}^2_{\{D_{G'}\left(r(\mathbf{0}),r((2^{n-k}-1)e_1)\right)<n-k\}}\right]} \\
&\quad + \sqrt{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)^2\right]}\sqrt{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[\mathbb{1}^2_{\left\{\left(\mathcal{D}_{\geq n-k}\cap\mathcal{Z}_{\geq n-k}\right)^C\right\}}\right]} \\
&\leq 2C_w r(\varepsilon)2^{k\theta(\beta)}\mathbb{E}_{\beta+\varepsilon}\left[D_{V_{\mathbf{0}}^{2^{n-k}}}\left(\mathbf{0},(2^{n-k}-1)e_1\right)\right] + \Big(\sqrt{C_\beta}\Lambda(2^k,\beta) \\
&\quad \cdot\Lambda(2^{n-k},\beta+\varepsilon)\mathbb{P}_{\beta+1}\left(D_{V_{\mathbf{0}}^{2^{n-k}}}\left(\mathbf{0},(2^{n-k}-1)e_1\right)<n-k\right)^{1/2}\Big) \\
&\quad + \sqrt{C_\beta}\Lambda(2^k,\beta)\Lambda(2^{n-k},\beta+\varepsilon)20\cdot 2^{-\frac{n-k}{2}} \\
&\leq \Lambda(2^{n-k},\beta+\varepsilon)\Big(2C_w r(\varepsilon)2^{k\theta(\beta)} + \sqrt{C_\beta}\Lambda(2^k,\beta)\mathbb{P}_{\beta+1}\left(D_{V_{\mathbf{0}}^{2^{n-k}}}\left(\mathbf{0},(2^{n-k}-1)e_1\right)<n-k\right)^{1/2} \\
&\quad + \sqrt{C_\beta}\Lambda(2^k,\beta)20\cdot 2^{-\frac{n-k}{2}}\Big)
\end{aligned}$$

where we used Lemma 11.7 for the second inequality and the Cauchy-Schwarz inequality and (139) for the first inequality. Using Lemma 11.6, we get that

$$\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]-\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}$$

$$\leq \frac{\Lambda(2^{n-k},\beta+\varepsilon)}{c_\beta\Lambda(2^{n-k+1},\beta+\varepsilon)\Lambda(2^{k-1},\beta)}\cdot\Bigg(2C_w r(\varepsilon)2^{k\theta(\beta)}$$

$$+\sqrt{C_\beta}\Lambda(2^k,\beta)\left(20\cdot 2^{-\frac{n-k}{2}}+\mathbb{P}_{\beta+1}\left(D_{V_{\mathbf{0}}^{2^{n-k}}}(\mathbf{0},(2^{n-k}-1)e_1)<n-k\right)^{1/2}\right)\Bigg)$$

$$\leq C_f\left(r(\varepsilon)+20\cdot 2^{-\frac{n-k}{2}}+\mathbb{P}_{\beta+1}\left(D_{V_{\mathbf{0}}^{2^{n-k}}}(\mathbf{0},(2^{n-k}-1)e_1)<n-k\right)^{1/2}\right)$$

for some finite constant $C_f<\infty$. This is true, as both fractions

$$\frac{\Lambda(2^{n-k},\beta+\varepsilon)\Lambda(2^k,\beta)}{c_\beta\Lambda(2^{n-k+1},\beta+\varepsilon)\Lambda(2^{k-1},\beta)} \text{ and } \frac{\Lambda(2^{n-k},\beta+\varepsilon)2^{k\theta(\beta)}}{c_\beta\Lambda(2^{n-k+1},\beta+\varepsilon)\Lambda(2^{k-1},\beta)}$$

are bounded uniformly over all $\varepsilon\in[0,1]$, $k\leq n\in\mathbb{N}$. The last term in the above calculation is the probability $\mathbb{P}_{\beta+1}\left(D_{V_{\mathbf{0}}^{2^{n-k}}}(\mathbf{0},(2^{n-k}-1)e_1)<n-k\right)$, which tends to 0 as $n-k$ goes to infinity, as the graph distance between $\mathbf{0}$ and $(2^{n-k}-1)e_1$ is of order $2^{(n-k)\theta(\beta+1)}\gg n-k$ under the measure $\mathbb{P}_{\beta+1}$. In particular this implies that for large enough k

$$\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]-\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}$$

converges to zero, as $\varepsilon\to 0$ and $n-k\to\infty$. Thus

$$\log\left(\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}\right)$$

converges to 0 as $\varepsilon\to 0$ and $k,n-k\to\infty$. As all terms of this form are bounded uniformly over k,n and $\varepsilon\in(0,1)$, by Lemma 11.6, it already follows that

$$\lim_{\varepsilon\searrow 0}\theta(\beta)-\theta(\beta+\varepsilon)=\lim_{\varepsilon\searrow 0}\frac{1}{\log(2)}\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\log\left(\frac{\mathbb{E}_{\beta\leq k}^{\beta+\varepsilon>k}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}{\mathbb{E}_{\beta\leq k-1}^{\beta+\varepsilon>k-1}\left[D_{V_{\mathbf{0}}^{2^n}}\left(\mathbf{0},(2^n-1)e_1\right)\right]}\right)=0$$

which shows continuity from the right of the distance exponent $\theta(\cdot)$ and thus finishes the proof of Theorem 1.5. □

12 Proofs for $d=1$

In this section, we show a few lemmas for $d=1$, where slightly different techniques compared to $d\geq 2$ are needed. It is well-known that for fixed $\beta<1$ one has $\mathbb{E}_\beta\left[D(0,n)\right]=\Omega\left(n^{1-\beta}\right)$. The next lemma gives a more uniform bound on the growth of $\mathbb{E}_\beta\left[D(0,n)\right]$ that holds for all $\beta\in[0,1]$ simultaneously.

Lemma 12.1. *There exists a $c>0$ such that for all $M,n\in\mathbb{N}$ and $\beta\in[0,1]$*

$$\mathbb{E}_\beta\left[D_{[0,Mn-1]}\left(0,Mn-1\right)\right]\geq cM^{1-\beta}\mathbb{E}_\beta\left[D_{[0,n-1]}\left(0,n-1\right)\right]. \qquad (140)$$

Proof. First note that the proof of (43) does not depend on a uniform bound on the second moment and works as written above. So we can safely apply it in our argumentation here. By (43) we can choose $\iota > 0$ small enough so that

$$\mathbb{E}_\beta\left[D_{[0,n-1]}\left([0,\iota n],[n-\iota n-1,n-1]\right)\right] \geq \frac{1}{2}\mathbb{E}_\beta\left[D_{[0,n-1]}(0,n-1)\right]$$

uniformly over $\beta \in [0,1]$. This implies the existence of a $c' > 0$ such that

$$\mathbb{E}_\beta\left[D_{[-n,2n-1]}\left(V_{-1}^n, V_1^n\right)\right] \geq c'\mathbb{E}_\beta\left[D_{[0,n-1]}(0,n-1)\right] \tag{141}$$

uniformly over $\beta \in [0,1]$ and $n \in \mathbb{N}$ large enough, as we will argue now. For fixed $\iota > 0$ there is a uniform positive probability (in $\beta \in [0,1]$ and $n \in \mathbb{N}$) that the rightmost vertex incident to V_{-1}^n lies inside $[0,\iota n]$ and that the leftmost vertex incident to V_1^n lies inside $[n-\iota n-1,n-1]$. Call this event A. Whenever the event A holds, one already has

$$D_{[-n,2n-1]}\left(V_{-1}^n, V_1^n\right) \geq D_{[0,n-1]}\left([0,\iota n],[n-\iota n-1,n-1]\right),$$

and as both the event A and the distance $D_{[0,n-1]}\left([0,\iota n],[n-\iota n-1,n-1]\right)$ are decreasing one has by the FKG inequality

$$\begin{aligned}
&\mathbb{E}_\beta\left[D_{[-n,2n-1]}\left(V_{-1}^n, V_1^n\right)\right] \geq \mathbb{E}_\beta\left[D_{[-n,2n-1]}\left(V_{-1}^n, V_1^n\right)\mathbb{1}_A\right]\\
&\geq \mathbb{E}_\beta\left[D_{[0,n-1]}\left([0,\iota n],[n-\iota n-1,n-1]\right)\mathbb{1}_A\right]\\
&\geq \mathbb{E}_\beta\left[D_{[0,n-1]}\left([0,\iota n],[n-\iota n-1,n-1]\right)\right]\mathbb{P}_\beta(A) \geq \frac{\mathbb{P}_\beta(A)}{2}\mathbb{E}_\beta\left[D_{[0,n-1]}(0,n-1)\right],
\end{aligned}$$

which shows (141). For long-range percolation on the line segment $\{0,\ldots,M-1\}$, we call an odd point $w \in \{1,\ldots,M-2\}$ a *separation point* if $w \nsim \{0,\ldots,w-2\}, w \nsim \{w+2,\ldots,M-1\}$, and $\{0,\ldots,w-1\} \nsim \{w+1,\ldots,M-1\}$; See Figure 5 for an illustration. Even points can simply never be separation points with our definition. These three events are independent and we can bound the probability of the first event by

$$\mathbb{P}_\beta\left(w \nsim \{0,\ldots,w-2\}\right) \geq e^{-\beta\int_{-\infty}^0\int_1^2 \frac{1}{|t-s|^2}\mathrm{d}t\mathrm{d}s} \geq e^{-1}.$$

The same calculation also works for the second event and shows that $\mathbb{P}_\beta\left(w \nsim \{w+2,\ldots,M-1\}\right) \geq e^{-1}$ for all $\beta \in [0,1]$. The probability of the event $\{0,\ldots,w-1\} \nsim \{w+1,\ldots,M-1\}$ can be bounded from below by

$$\begin{aligned}
&\prod_{0\leq u<w}\ \prod_{w<v\leq M-1} e^{-\beta\int_u^{u+1}\int_v^{v+1}\frac{1}{|x-y|^2}\mathrm{d}x\mathrm{d}y} = e^{-\beta\int_0^w\int_{w+1}^M\frac{1}{|x-y|^2}\mathrm{d}x\mathrm{d}y}\\
&\geq e^{-\beta\int_0^w\int_{w+1}^\infty\frac{1}{|x-y|^2}\mathrm{d}x\mathrm{d}y} = e^{-\beta\int_0^w\frac{1}{w+1-y}\mathrm{d}y} = e^{-\beta\log(w+1)} \geq M^{-\beta}.
\end{aligned}$$

uniformly over $\beta \in [0,1]$. Using the independence of the three relevant events, we get that

$$\begin{aligned}
\mathbb{P}_\beta\left(w \text{ is a separation point}\right) &= \mathbb{P}_\beta\left(w \nsim \{0,\ldots,w-2\}\right)\cdot\mathbb{P}_\beta\left(w \nsim \{w+2,\ldots,M-1\}\right)\\
&\cdot\mathbb{P}_\beta\left(\{0,\ldots,w-1\} \nsim \{w+1,\ldots,M-1\}\right) \geq e^{-2}M^{-\beta} \geq 0.1M^{-\beta}.
\end{aligned}$$

For w odd, we call the set V_w^n a *separation interval* if $V_w^n \nsim [0,(w-1)n-1], V_w^n \nsim [(w+2)n, Mn-1]$, and $\{0,\ldots,wn-1\} \nsim \{(w+1)n,\ldots,Mn-1\}$. Again, an even w can never define a separation interval. By the scaling invariance of the underlying continuous model, the probability that V_w^n is a separation interval is exactly the probability that w is a separation point for the line segment $\{0,\ldots,M-1\}$, and this probability is bounded from

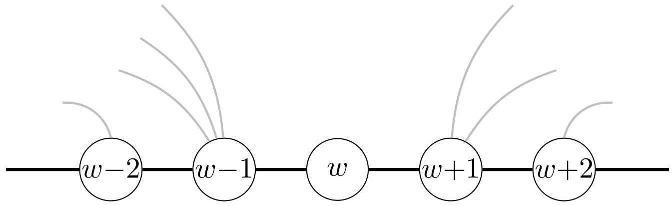


Figure 5: The vertex w is a separation point if all edges e with $|e| \geq 2$ are either strictly to the left or right of w, as above.

below by $0.1M^{-\beta}$. Let $w_1, \ldots, w_l \in \{1, \ldots, M-2\}$ be integers such that $V_{w_i}^n$ is a separation interval for all i. Then each path between 0 and $Mn-1$ in the graph $\{0, \ldots, Mn-1\}$ needs to cross all separation intervals of this form and in particular

$$D_{[0,Mn-1]}\left(0, Mn-1\right) \geq \sum_{i=1}^{l} D_{[(w_i-1)n,(w_i+2)n-1]}\left(V_{w_i-1}^n, V_{w_i+1}^n\right).$$

The fact that V_w^n is a separation interval reveals no information about the edges with both endpoints in $\{(w-1)n, \ldots, (w+2)n-1\}$, except that there is no direct edge from $\{(w-1)n, \ldots, wn-1\}$ to $\{(w+1)n, \ldots, (w+2)n-1\}$. Thus, by taking expectations in the above inequality and using that both the event $\{V_w^n \text{ is a sep. int.}\}$ and the random distance $D_{[(w-1)n,(w+2)n-1]}\left(V_{w-1}^n, V_{w+1}^n\right)$ are decreasing, we get by the FKG-inequality

$$\begin{aligned}
&\mathbb{E}_\beta\left[D_{[0,Mn-1]}\left(0, Mn-1\right)\right] \geq \mathbb{E}_\beta\left[\sum_{w=1}^{M-2} \mathbb{1}_{\{V_w^n \text{ is a sep. int.}\}} D_{[(w-1)n,(w+2)n-1]}\left(V_{w-1}^n, V_{w+1}^n\right)\right] \\
&\geq \sum_{w=1}^{M-2} \mathbb{E}_\beta\left[\mathbb{1}_{\{V_w^n \text{ is a sep. int.}\}}\right] \mathbb{E}_\beta\left[D_{[-n,2n-1]}\left(V_{-1}^n, V_1^n\right)\right] \\
&\overset{(141)}{\geq} \sum_{\substack{w \in \{1,\ldots,M-2\}: \\ w \text{ odd}}} 0.1M^{-\beta} c' \mathbb{E}_\beta\left[D_{[0,n-1]}(0, n-1)\right] \geq cM^{1-\beta} \mathbb{E}_\beta\left[D_{[0,n-1]}(0, n-1)\right]
\end{aligned}$$

for some small constant $c > 0$ and M large enough. For M small, one can take c small enough so that (140) holds for such M, by Lemma 4.1. □

With this we are now ready to go to the proof of Lemma 4.5 for $d = 1$.

Proof of Lemma 4.5 for $d = 1$. We say that the vertex $w \in \{1, \ldots, m-2\}$ is a cut point (for the interval $\{0, \ldots, m-1\}$) if there exists no edge of the form $\{u, v\}$ with $0 \leq u < w < v \leq m-1$. For $w < \frac{m}{2}$ and $\beta \leq 2$ we have

$$\begin{aligned}
\mathbb{P}_\beta\left(w \text{ is a cut point}\right) &= \prod_{0 \leq u < w} \prod_{w < v \leq m-1} e^{-\beta \int_u^{u+1} \int_v^{v+1} \frac{1}{|x-y|^2} \mathrm{d}x\mathrm{d}y} = e^{-\beta \int_0^w \int_{w+1}^m \frac{1}{|x-y|^2} \mathrm{d}x\mathrm{d}y} \\
&\leq e^{-\beta \int_0^w \int_{w+1}^{2w+1} \frac{1}{|x-y|^2} \mathrm{d}x\mathrm{d}y} = e^{-\beta \int_0^w \frac{1}{w+1-y} - \frac{1}{2w+1-y} \mathrm{d}y} \\
&= e^{-\beta(-\log(1)+2\log(w+1)-\log(2w+1))} = e^{-\beta \log\left(\frac{(w+1)^2}{2w+1}\right)} \leq e^{-\beta \log\left(\frac{w+1}{2}\right)} \\
&\leq e^{-\beta \log(w+1)} e^{\beta \log(2)} \leq 4w^{-\beta}
\end{aligned}$$

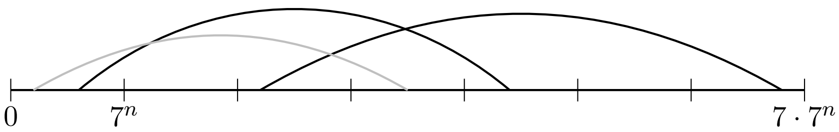


Figure 6: The long edges inside the box $\{0, \dots, 7^{n+1}-1\}$. The set $\mathcal{B}$ are the two bold black edges.

and with this we get, by linearity of expectation and symmetry of the process, that

$$\begin{aligned}\mathbb{E}_\beta\left[|\{w \in \{1, \dots, m-2\} : w \text{ is a cut point}\}|\right] &\leq 1 + 2\sum_{w=1}^{\lfloor m/2 \rfloor} \mathbb{P}_\beta\left(w \text{ is a cut point}\right)\\ \leq 1 + 8\sum_{w=1}^{\lfloor m/2 \rfloor} w^{-\beta} \leq 10 + 8\int_1^m w^{-\beta} \mathrm{d}w &= \begin{cases} 10 + 8\left[\frac{w^{-\beta+1}}{-\beta+1}\right]_1^m & \beta \in [0,2] \setminus \{1\} \\ 10 + 8\log(m) & \beta = 1 \end{cases}.\end{aligned}$$

As the expected number of cut points is monotone decreasing in β, we get that for the function $f(\beta, m) \coloneqq \mathbb{E}_\beta\left[|\{w \in \{1, \dots, m-2\} : w \text{ is a cut point}\}|\right]$ we have the upper bound

$$f(\beta, m) \leq \begin{cases} \frac{20}{1-\beta} m^{1-\beta} & \beta < 1 \\ 10 + 8\log(m) & 1 \leq \beta \leq 2 \\ 20 & \beta > 2 \end{cases}. \tag{142}$$

We now use a method (that was already used in [33] in a similar form for the continuous model) in order to bound the second moment of $D_{V_0^{m^{n+1}}}\left(0, m^{n+1}-1\right)$. We say that an interval $V_k^{m^n}$ is *unbridged* if there exists no edge $\{u, v\}$ with both endpoints in $\left\{0, \dots, m^{n+1}-1\right\}$ and $u < km^n, v \geq (k+1)m^n$; Contrary, if there exists such an edge we say that the interval is *bridged*. In this case, we also say that the interval is bridged by the edge $\{u, v\}$. So clearly the intervals $V_0^{m^n}, V_{m-1}^{m^n}$ are unbridged, and the probability that $V_w^{m^n}$ is unbridged for $w \in \{1, \dots, m-2\}$ is exactly the probability that w is a cut point for the interval $\{0, \dots, m-1\}$. We now define a set of edges $\mathcal{B}$ as follows: Let $i < j \in \{0, \dots, m-1\}$ with $|i-j| > 1$ satisfy $V_i^{m^n} \sim V_j^{m^n}$ and $V_{i-l_1}^{m^n} \nsim V_{j+l_2}^{m^n}$ for all $(l_1, l_2) \in \{0, \dots, i\} \times \{0, \dots, m-1-j\} \setminus \{(0,0)\}$. In this situation, we add one edge between $V_i^{m^n}$ and $V_j^{m^n}$ to $\mathcal{B}$. If there are several edges between $V_i^{m^n}$ and $V_j^{m^n}$ we choose the left-most shortest such edge (this rule is arbitrary, any deterministic rule would work here). An example of this construction is given in Figure 6. So the set $\mathcal{B}$ is the set of possible bridges where we already delete edges that are furthermore bridged by even longer edges. With this construction, we get $|\mathcal{B}| \leq m$, as each interval $V_j^{m^n}$ can be adjacent to at most two edges in $\mathcal{B}$, and each edge in $\mathcal{B}$ touches two intervals. Furthermore, if an interval $V_j^{m^n}$ is bridged, then there exists an edge $e \in \mathcal{B}$ so that $V_j^{m^n}$ is bridged by e. Let $\mathcal{U}'$ be the set of endpoints of edges in $\mathcal{B}$ and let

$$\mathcal{U} \coloneqq \mathcal{U}' \cup \{0, m^n, \dots, (m-1)m^n\} \cup \left\{m^n - 1, 2m^n - 1, \dots, m^{n+1} - 1\right\}.$$

Let $\mathcal{U} = \{x_0, x_1, \dots, x_u\}$, where $x_0 < \ldots < x_u$. By the construction we have $|\mathcal{U}| \leq 4m$ and $|x_{i-1} - x_i| \leq m^n - 1$. For x_{i-1}, x_i with $(x_{i-1}, x_i) \neq (km^n - 1, km^n)$ for all k, we say that $[x_{i-1}, x_i]$ is bridged, if there exists an edge $\{u, v\} \in \mathcal{B}$ with $u \leq x_{i-1} < x_i \leq v$. Assume we have (x_{i-1}, x_i) which is not of the form $(km^n - 1, km^n)$, say with $[x_{i-1}, x_i] \subset V_j^{m^n}$ for some $j \in \{0, \ldots, m-1\}$, and $[x_{i-1}, x_i]$ is not bridged. Then also $V_j^{m^n}$ is not bridged. On the other hand, if $[x_{i-1}, x_i]$ is bridged, then also $V_j^{m^n}$ is bridged by some edge in $\mathcal{B}$. In each interval $V_j^{m^n}$ there are at most two points in $V_j^{m^n} \cap \mathcal{U}$ that come from endpoints of edges in $\mathcal{B}$; Furthermore, the two endpoints of the interval are also in $V_j^{m^n} \cap \mathcal{U}$. So in total there are at most 4 points in $V_j^{m^n} \cap \mathcal{U}$ for all $j \in \{0, \ldots, m-1\}$, and thus there are at most three intervals of the form $[x_{i-1}, x_i]$ inside each $V_j^{m^n}$. This already implies that

$$|\{i \in \{1, \ldots, u\} : [x_{i-1}, x_i] \text{ is not bridged}\}| \leq 3 \left|\{j \in \{0, \ldots, m-1\} : V_j^{m^n} \text{ is not bridged}\}\right|. \tag{143}$$

Now we want to construct a path between 0 and $m^{n+1}-1$. Let

$$\tau = \arg \max_{i \in \{1, \ldots, u\}} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i).$$

If there are multiple maximizers, we pick one with $x_i \neq km^n$ for all k, and with minimal x_i among those maximizers. So in particular $[x_{i-1}, x_i]$ always lies inside some interval $V_j^{m^n}$. If $[x_{\tau-1}, x_\tau]$ is bridged by some edge $e = \{x_{\tau_1}, x_{\tau_2}\} \in \mathcal{B}$, say with $x_{\tau_1} < x_{\tau_2}$, then we consider the path that goes from $0 = x_0$ to x_{τ_1}, then directly jumps to x_{τ_2} and from there goes to $x_u = m^{n+1} - 1$. This implies that

$$\begin{aligned} D_{[0, m^{n+1}-1]}\left(0, m^{n+1}-1\right) &\leq \sum_{i=1}^{\tau_1} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) + 1 + \sum_{i=\tau_2+1}^{u} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) \\ &\leq u \max_{i \neq \tau} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) \leq 4m \max_{i \neq \tau} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) \end{aligned}$$

in this case. For the case where $[x_{\tau-1}, x_\tau]$ is not bridged, we consider the path that goes iteratively from x_0 to x_u. Here we have

$$\begin{aligned} &D_{[0, m^{n+1}-1]}\left(0, m^{n+1}-1\right) \\ &\leq \sum_{i=1}^{\tau-1} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) + D_{[x_{\tau-1}, x_\tau]}(x_{\tau-1}, x_\tau) + \sum_{i=\tau+1}^{u} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) \\ &\leq 4m \max_{i \neq \tau} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i) + \max_{[x_{i-1}, x_i] \text{ not bridged}} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i), \end{aligned}$$

and thus we have in both cases that

$$\begin{aligned} &\left(D_{[0, m^{n+1}-1]}\left(0, m^{n+1}-1\right)\right)^2 \\ &\leq 2\left(4m \max_{i \neq \tau} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i)\right)^2 + 2\left(\max_{[x_{i-1}, x_i] \text{ not bridged}} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i)\right)^2 \\ &\leq 32m^2\left(\max_{i \neq \tau} D_{[x_{i-1}, x_i]}(x_{i-1}, x_i)\right)^2 + 2 \sum_{[x_{i-1}, x_i] \text{ not bridged}} \left(D_{[x_{i-1}, x_i]}(x_{i-1}, x_i)\right)^2. \end{aligned} \tag{144}$$

Next, we want to bound both terms in the above sum in expectation. To bound the first term, we use the following observation: If $X_1, \ldots, X_{\tilde{m}}$ are independent non-negative random variables and $\tau = \arg\max_{i \in \{1, \ldots, \tilde{m}\}}$, then

$$\mathbb{E}\left[\left(\max_{i \neq \tau} X_i\right)^2\right] \leq \mathbb{E}\left[\sum_{i=1}^{\tilde{m}} X_i \left(\sum_{j \neq i} X_j\right)\right] = \sum_{i=1}^{\tilde{m}} \sum_{j \neq i} \mathbb{E}[X_i] \mathbb{E}[X_j] \leq \tilde{m}^2 \max_i \mathbb{E}[X_i]^2. \tag{145}$$

Conditioned on $\mathcal{U}$, the random variables $\left(D_{[x_{i-1},x_i]}(x_{i-1},x_i)\right)_{i=1}^{u}$ are independent and by Lemma 4.1 their expectation is bounded by the expectation of $\left(D_{V_0^{m^n}}(0,m^n-1)\right)$, up to a factor of 3. As $u \leq 4m$, we get with (145) and Lemma 4.1 that

$$\begin{aligned}\mathbb{E}_\beta\left[\max_{i\neq\tau} D_{[x_{i-1},x_i]}(x_{i-1},x_i)^2\right] &= \mathbb{E}_\beta\left[\mathbb{E}\left[\max_{i\neq\tau} D_{[x_{i-1},x_i]}(x_{i-1},x_i)^2 \mid \mathcal{U}\right]\right]\\ &\leq \mathbb{E}_\beta\left[16m^2\max_i \mathbb{E}\left[D_{[x_{i-1},x_i]}(x_{i-1},x_i)\mid\mathcal{U}\right]^2\right] \leq 144m^2\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)\right]^2.\end{aligned} \tag{146}$$

In order to bound the second summand in (144) in expectation, we use the bound on the number of unbridged segments (143). Also note that the second moment of $D_{[x_{i-1},x_i]}(x_{i-1},x_i)$ is, by Lemma 4.1, bounded by the second moment of $\left(D_{V_0^{m^n}}(0,m^n-1)\right)$, up to a factor of $9=3^2$. With this we get that

$$\begin{aligned}&\mathbb{E}_\beta\left[\sum_{[x_{i-1},x_i]\text{ not bridged}}\left(D_{[x_{i-1},x_i]}(x_{i-1},x_i)\right)^2\right]\\ &= \mathbb{E}_\beta\left[\mathbb{E}_\beta\left[\sum_{[x_{i-1},x_i]\text{ not bridged}}\left(D_{[x_{i-1},x_i]}(x_{i-1},x_i)\right)^2 \,\Big|\, \mathcal{U}\right]\right]\\ &\leq 9\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)^2\right]\mathbb{E}_\beta\left[\sum_{[x_{i-1},x_i]\text{ not bridged}} 1\right]\\ &\leq 27\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)^2\right]\mathbb{E}_\beta\left[\left|\left\{j\in\{0,\dots,m-1\}: V_j^{m^n}\text{ unbridged}\right\}\right|\right]\\ &\leq 27\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)^2\right](2+f(\beta,m)) =: \mathbb{E}_\beta\left[D(0,m^n-1)^2\right]\tilde{f}(\beta,m),\end{aligned} \tag{147}$$

where $\tilde{f}(\beta,m) = 27(2+f(\beta,m))$. Combining (146) and (147), and taking expectations in (144), we obtain that

$$\begin{aligned}\mathbb{E}_\beta\left[D_{V_0^{m^{n+1}}}(0,m^{n+1}-1)^2\right] &\leq 5000m^4\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)\right]^2\\ &\quad + 2\tilde{f}(\beta,m)\mathbb{E}_\beta\left[D_{V_0^{m^n}}(0,m^n-1)^2\right].\end{aligned}$$

Iterating this inequality over all $k=1,\dots,n$, we get

$$\mathbb{E}_\beta\left[D_{V_0^{m^{n+1}}}(0,m^{n+1}-1)^2\right] \leq 5000m^4\sum_{k=1}^{n}\left(2\tilde{f}(\beta,m)\right)^{n+1-k}\mathbb{E}_\beta\left[D_{V_0^{m^k}}\left(0,m^k-1\right)\right]^2. \tag{148}$$

Using the bounds on $f(\beta,m)$ from (142), we see that function $\tilde{f}(\beta,m)$ satisfies

$$\tilde{f}(\beta,m) = 27\left(2+f(\beta,m)\right) \leq \begin{cases}\frac{600}{1-\beta}m^{1-\beta} & \beta<1\\ 600\left(1+\log(m)\right) & 1\leq\beta\leq 2\\ 600 & \beta>2\end{cases}. \tag{149}$$

By compactness of each interval $[\beta,\beta+1]$, it suffices to show that the uniform bound on the second moment (36) holds for all $\beta>0$ and $\varepsilon\in(-c_\beta,c_\beta)$ for some $c_\beta>0$ small enough, respectively for $\beta = 0$ and $\varepsilon \in [0, c_\beta)$. To extend the inequality from open sets to compact intervals, one can cover each compact interval with finitely many open sets and then take the largest among these finitely many constants that arose from this procedure. So we are left to show that for all $\beta \geq 0$, there exist a constants $c_\beta > 0$ and $C_\beta < \infty$ such that for all $n \in \mathbb{N}$, all $\varepsilon \in (-c_\beta, c_\beta)$, respectively all $\varepsilon \in [0, c_\beta)$ for $\beta = 0$, and all $u, v \in V_0^n$

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^n}(u,v)^2\right] \leq C_\beta \Lambda(n, \beta+\varepsilon)^2. \tag{150}$$

We start with the case $\beta \geq 1$. By Remark 4.4, there exists a $\theta' = \theta'(\beta) > 0$ such that

$$\mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^{k+1}}}\left(0, m \cdot m^k - 1\right)\right] \geq m^{\theta'(\beta)} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^k}}\left(0, m^k - 1\right)\right] \tag{151}$$

for all $k \in \mathbb{N}$, m large enough, and $|\varepsilon| \leq \frac{1}{2}$. Inserting this into (148), we get

$$\begin{aligned}&\mathbb{E}_\beta\left[D_{V_0^{m^{n+1}}}\left(0, m^{n+1} - 1\right)^2\right] \\ &\leq 5000 m^4 \sum_{k=1}^{n} \left(2\tilde{f}(\beta, m)\right)^{n+1-k} \left(m^{-2\theta'}\right)^{n-k} \mathbb{E}_\beta\left[D_{V_0^{m^n}}\left(0, m^n - 1\right)\right]^2.\end{aligned}$$

Now choose $m \in \mathbb{N}$ large enough and $c_\beta \in (0, 0.1)$ small enough so that $2\tilde{f}(\beta+\varepsilon, m) m^{-2\theta'(\beta)} \leq 0.5$ for all $\varepsilon \in (-c_\beta, c_\beta)$. This is clearly possible for $\beta > 1$. For $\beta = 1$, we can choose c_β small enough so that $c_\beta < \theta'(1)$, where $\theta'(1)$ is the one defined in (151). By monotonicity in the first argument of the function $\tilde{f}(\cdot, \cdot)$ one then has $\tilde{f}(1+\varepsilon, m) \leq \frac{600}{c_\beta} m^{c_\beta}$ for all $\varepsilon \in (-c_\beta, c_\beta)$, which shows that one can find m, c_β so that $2\tilde{f}(1+\varepsilon, m) m^{-2\theta'(1)} \leq 0.5$ for all $\varepsilon \in (-c_\beta, c_\beta)$. This then gives that

$$\begin{aligned}&\mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^{n+1}}}\left(0, m^{n+1} - 1\right)^2\right] \leq 10000 \tilde{f}(\beta - c_\beta, m) m^4 \sum_{k=1}^{n} 0.5^{n-k}\, \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}\left(0, m^n - 1\right)\right]^2 \\ &\leq 20000 \tilde{f}(\beta - c_\beta, m) m^4 \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}\left(0, m^n - 1\right)\right]^2 \leq 20000 \tilde{f}(\beta - c_\beta, m) m^4 \Lambda\left(m^n, \beta+\varepsilon\right)^2\end{aligned}$$

for all $\varepsilon \in (-c_\beta, c_\beta)$. This shows (150) along the subsequence $m, m^2, m^3, \ldots$ To extend inequality (150) from this subsequence to all integers, use Lemma 4.1.

Next, we consider the case where $\beta \in (0,1)$. Using Lemma 12.1, we know that there is a constant $c \in (0,1)$ such that

$$\begin{aligned}\mathbb{E}_\beta\left[D_{V_0^{m^n}}\left(0, m^n - 1\right)\right] &\geq c m^{(n-k)(1-\beta)} \mathbb{E}_\beta\left[D_{V_0^{m^{n-k}}}\left(0, m^{n-k} - 1\right)\right] \\ &\geq \left(c m^{1-\beta}\right)^{n-k} \mathbb{E}_\beta\left[D_{V_0^{m^{n-k}}}\left(0, m^{n-k} - 1\right)\right]\end{aligned}$$

for all $n \geq k$ and $m \in \mathbb{N}$. Now take m large enough and c_β small enough so that $\frac{1200\ m^{\beta+\varepsilon-1}}{c(1-\beta-\varepsilon)} < 0.5$ for all $\varepsilon \in (-c_\beta, c_\beta)$, and that $0 < \beta - c_\beta < \beta + c_\beta < 1$. Using (148) we get that for such m and $\varepsilon \in (-c_\beta, c_\beta)$

$$\begin{aligned}&\mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^{n+1}}}\left(0, m^{n+1} - 1\right)^2\right] \leq 5000 m^4 \sum_{k=1}^{n} \left(2\tilde{f}(\beta, m)\right)^{n+1-k} \mathbb{E}_\beta\left[D_{V_0^{m^k}}\left(0, m^k - 1\right)\right]^2 \\ &\leq 5000 m^4 \sum_{k=1}^{n} \left(2\tilde{f}(\beta+\varepsilon, m)\right)^{n+1-k} \left(c m^{1-\beta-\varepsilon}\right)^{2(k-n)} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}\left(0, m^n - 1\right)\right]^2\end{aligned}$$

$$\overset{(149)}{\leq} 10000\tilde{f}(\beta - c_\beta, m)m^4 \sum_{k=1}^{n} \left(\frac{1200\ m^{\beta+\varepsilon-1}}{c(1-\beta-\varepsilon)} \right)^{n-k} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}(0, m^n-1) \right]^2$$

$$\leq 10000\tilde{f}(\beta - c_\beta, m)m^4 \sum_{k=1}^{n} 0.5^{n-k} \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}(0, m^n-1) \right]^2$$

$$\leq 20000\tilde{f}(\beta - c_\beta, m)m^4 \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}(0, m^n-1) \right]^2 \leq 10^6 m^5 \mathbb{E}_{\beta+\varepsilon}\left[D_{V_0^{m^n}}(0, m^n-1) \right]^2,$$

which shows (150) for numbers of the form $m, m^2, m^3, \ldots$ Here, we used that $f(\beta, m) \leq m-2$ for all $\beta \in \mathbb{R}_{\geq 0}$, and thus $\tilde{f}(\beta, m) = 27(2 + f(\beta, m)) \leq 27m$ for the last inequality. To extend inequality (150) from this subsequence to all integers, use Lemma 4.1. The proof for $\beta = 0$ works analogous to the case $\beta \in (0,1)$, and we omit it. □

Proof of Corollary 4.6 for $d=1$. We use the same notation as in the proof of Lemma 4.5 for $d=1$ above. We have that

$$D_{[0,m^{n+1}-1]}\left(0, m^{n+1}-1\right) \leq 4m \max_{i \neq \tau} D_{[x_{i-1},x_i]}(x_{i-1}, x_i) + \max_{[x_{i-1},x_i] \text{ not bridged}} D_{[x_{i-1},x_i]}(x_{i-1}, x_i)$$

and this implies that for any $r > 0$

$$\left(D_{[0,m^{n+1}-1]}\left(0, m^{n+1}-1\right)\right)^{2r}$$
$$\leq 2^{2r} 4^{2r} m^{2r} \left(\max_{i \neq \tau} D_{[x_{i-1},x_i]}(x_{i-1}, x_i) \right)^{2r} + 2^r \sum_{[x_{i-1},x_i] \text{ not bridged}} \left(D_{[x_{i-1},x_i]}(x_{i-1}, x_i)\right)^{2r}.$$

Taking expectations and the same arguments as in the proof of Lemma 4.5 yield

$$\mathbb{E}_\beta\left[\left(\max_{i \neq \tau} D_{[x_{i-1},x_i]}(x_{i-1}, x_i)^r \right)^2 \right] \leq \mathbb{E}_\beta\left[16m^2 \max_i \mathbb{E}_\beta\left[D_{[x_{i-1},x_i]}(x_{i-1}, x_i)^r \,\middle|\mathcal{U}\right]^2 \right]$$
$$\leq 16m^2 3^{2r} \mathbb{E}_\beta\left[D_{[0,m^n-1]}(0, m^n-1)^r \right].$$

From here, the same proof as before gives that $\mathbb{E}_\beta\left[D_{[0,n]}(0,n)^r\right] \leq C(r)\mathbb{E}_\beta\left[D_{[0,n]}(0,n)\right]^r$ for a constant $C(r)$, and r of the form $r = 2^k$ with natural k. Extending this to all $r > 0$ works with Hölder's inequality. □

Critical exponents

13 Introduction

Consider Bernoulli bond percolation on $\mathbb{Z}^d$ where we include an edge between the vertices $x, y \in \mathbb{Z}^d$ with probability $1 - e^{-\beta J(x,y)}$ and independent of all other edges. The function $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a kernel that is symmetric, i.e., $J(x, y) = J(y, x)$ for all $x, y \in \mathbb{Z}^d$. We denote the resulting probability measure by $\mathbb{P}_\beta$ and its expectation by $\mathbb{E}_\beta$. Edges that are included are also referred to as open. We are interested in the case where the kernel is also translation invariant and integrable, meaning that $J(x, y) = J(0, y - x)$ for all $x, y \in \mathbb{Z}^d$ and $\sum_{x\in\mathbb{Z}^d} J(0, x) < \infty$. The integrability condition guarantees that the resulting graph is almost surely locally finite. This procedure creates certain clusters, which are the connected components in the resulting random graph. Write K_x for the cluster containing the vertex $x \in \mathbb{Z}^d$. A major question in percolation theory is the emergence of infinite clusters, for which we define the critical parameter β_c by

$$\beta_c = \inf \{\beta \geq 0 : \mathbb{P}_\beta (|K_0| = \infty) > 0\} .$$

A comparison with a Galton-Watson tree shows that there are no infinite clusters for $\beta < \left(\sum_{x\in\mathbb{Z}^d} J(0, x)\right)^{-1}$, which shows $\beta_c > 0$. For $d > 1$ and $J \neq 0$ it is well-known that $\beta_c < \infty$, whereas for $d = 1$ it is known that $\beta_c < \infty$ in the case where $J(x, y) \simeq \|x-y\|^{-1-\alpha}$ for $\alpha \leq 1$ [41,84], whereas $\beta_c = \infty$ for $\alpha > 1$. Long-range percolation mostly deals with the case where $J(x, y) \simeq \|x - y\|^{-d-\alpha}$ for some $\alpha > 0$, where we write $J(x, y) \simeq \|x - y\|^{-d-\alpha}$ if the ratio between them satisfies $\varepsilon < \frac{J(x,y)}{\|x-y\|^{-d-\alpha}} < \varepsilon^{-1}$ for a small enough $\varepsilon > 0$ and $\|x - y\|$ large enough. In general it is expected that for $\alpha > d$ the resulting graph looks similar to nearest-neighbor percolation, is very well connected for $\alpha < d$, and shows a self-similar behavior for $\alpha = d$. See [12, 17–19, 33] and the part I of this thesis for results pointing in this direction. From the definition of β_c and the standard Harris coupling [62] we see that $\mathbb{P}_\beta (|K_0| = \infty) > 0$ for $\beta > \beta_c$ and $\mathbb{P}_\beta (|K_0| = \infty) = 0$ for $\beta < \beta_c$, but it is not clear what happens at $\beta = \beta_c$. For $J(x, y) \simeq \|x - y\|^{-d-\alpha}$ with $\alpha \in (0, d)$ and all $d \in \mathbb{N}_{>0}$ Berger showed that $\mathbb{P}_{\beta_c} (|K_0| = \infty) = 0$ [16, Theorem 1.5], whereas for $d = 1$ and $J(x, y) \simeq \|x - y\|^{-2}$ it is a result by Aizenman and Newman that $\mathbb{P}_{\beta_c} (|K_0| = \infty) > 0$ [4]. For $d \geq 2$ and $\alpha \geq d$ it is also expected that $\mathbb{P}_{\beta_c} (|K_0| = \infty) = 0$, but there is no full proof known at the moment. Whenever there is no infinite cluster at the critical value, it is a central question how fast the tail of the cluster at criticality $\mathbb{P}_{\beta_c} (|K_0| \geq n)$ and the two-point function $\mathbb{P}_{\beta_c} (0 \leftrightarrow x)$ tend to 0 as n, respectively $\|x\|$, grow. Here we write $x \leftrightarrow y$ if there exists an open path from x to y. It is conjectured that

$$\mathbb{P}_{\beta_c} (|K_0| \geq n) \approx n^{-1/\delta} \qquad \text{as } n \to \infty, \tag{152}$$

$$\mathbb{P}_{\beta_c} (0 \leftrightarrow x) \approx \|x\|^{-d+2-\eta} \qquad \text{as } \|x\| \to \infty \tag{153}$$

for certain numbers η, δ depending on d and α, but not on the precise details of the kernel J. Here, we write $f(n) \approx n^c$ if $f(n) = n^{c+o(1)}$. Even the existence of the exponents is not clear and it is still open, whether the limits $\lim_{n\to\infty} \frac{\log\left(\mathbb{P}_{\beta_c}(|K_0|\geq n)\right)}{\log(n)}$ and $\lim_{\|x\|\to\infty} \frac{\log\left(\mathbb{P}_{\beta_c}(0\leftrightarrow x)\right)}{\log(\|x\|)}$ exist. The widely accepted conjecture is that they exist. This has been for example proven for other models of percolation like two-dimensional percolation on the triangular lattice [77, 92, 93] or percolation for high enough dimension d, or for small enough α [61]. Recently, Hutchcroft proved the upper bounds $\delta \leq \frac{2d}{d-\alpha}$ and $2 - \eta \leq \alpha$ [70], improving his previous result $\delta \leq \frac{2d+\alpha}{d-\alpha}$ [65] which is, to our knowledge, the first rigorous proof of a power-law decay of $\mathbb{P}_{\beta_c} (|K_0| \geq n)$ for long-range percolation.

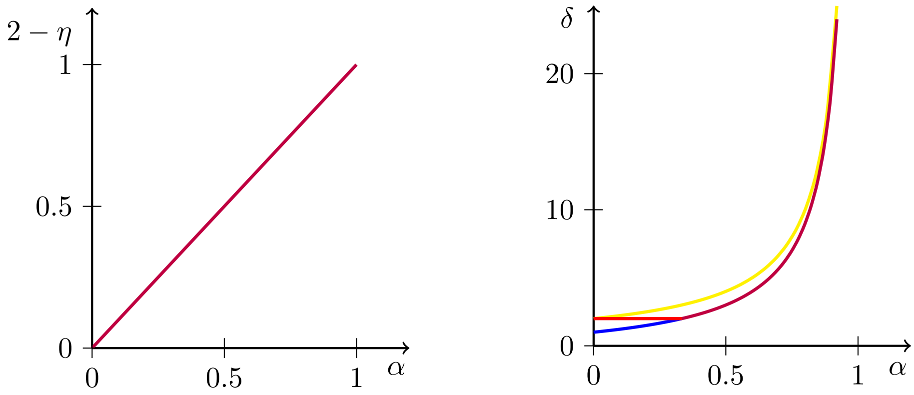


Figure 7: The critical exponents $2-\eta$ and δ for $d=1$. On the left: The purple line is the conjectured true value, our lower bound, and the upper bound proven in [70]. On the right: The yellow curve is the upper bound on δ shown in [70], the red curve is the conjectured true value of δ, and the blue curve is our lower bound. The part where the lower bound and the conjectured true value agree $\left(\alpha \in \left[\frac{1}{3}, 1\right)\right)$ is purple.

Our results In this part of the thesis, we give lower bounds on the exponents δ and $2-\eta$. We will always assume an upper bound on the kernel J of the form $J(x,y) \leq C_1 \|x-y\|^{-d-\alpha}$ for some constant $C_1 < \infty$.

Theorem 13.1. *Let $\alpha \in (0,1)$ for $d=1$, respectively $\alpha > 0$ for $d > 1$. Suppose that $J(x,y) \leq C_1 \|x-y\|^{-d-\alpha}$ and the exponent δ defined in (152) exists. Then*

$$\delta \geq \frac{d + (\alpha \wedge 1)}{d - (\alpha \wedge 1)}.$$

Theorem 13.1 is an immediate consequence of Proposition 14.7. It is only of interest in dimension $d \in \{1,2\}$ and for $\alpha > \frac{d}{3}$, as it is known in wider generality that $\delta \geq 2$ [2,53, Proposition 10.29]. For the case where $d=1$ and $\alpha \in \left[\frac{1}{3}, 1\right)$, respectively where $d=2$ and $\alpha \in \left[\frac{2}{3}, 1\right]$, our lower bound coincides with the conjectured true value of δ.
In particular, Theorem 13.1 shows that for $d \in \{1,2\}$ and $\alpha > \frac{d}{3}$ the model does not exhibit the so called 'mean-field behavior'. The notion of 'mean-field behavior' is a notion that comes from physics, and roughly means that all the critical exponents are the same as in models of infinite dimension, such as Erdös-Rényi graphs (in the $n \to \infty$ limit) or the binary tree. There are several ways of precisely defining this notion, but applied to our case all of them imply, among other things, that the exponents δ and $2-\eta$ exist and take the values $\delta = 2$ and $2 - \eta = 2 \wedge \alpha$. In a major breakthrough by Hara and Slade [55] mean-field behavior was established for high dimensional nearest-neighbour percolation. It was later also established for long-range percolation with $d > 6$ or $\alpha < \frac{d}{3}$ [61]. The lower bounds in Theorem 13.1 rule out the mean-field behavior for $d \in \{1,2\}$ and $\alpha > \frac{d}{3}$, as they imply that $\delta > 2$ in this regime.

Theorem 13.2. *Let $\alpha \in (0,1)$ for $d=1$, respectively $\alpha > 0$ for $d > 1$. Suppose that $J(x,y) \leq C_1 \|x-y\|^{-d-\alpha}$ and the exponent $2-\eta$ defined in (153) exists. Then*

$$2 - \eta \geq \alpha \wedge 1.$$

A graphical representation of our results, previously known results, and the conjectured behavior can be found in Figure 7 for dimension $d=1$ and in Figure 8 for dimension

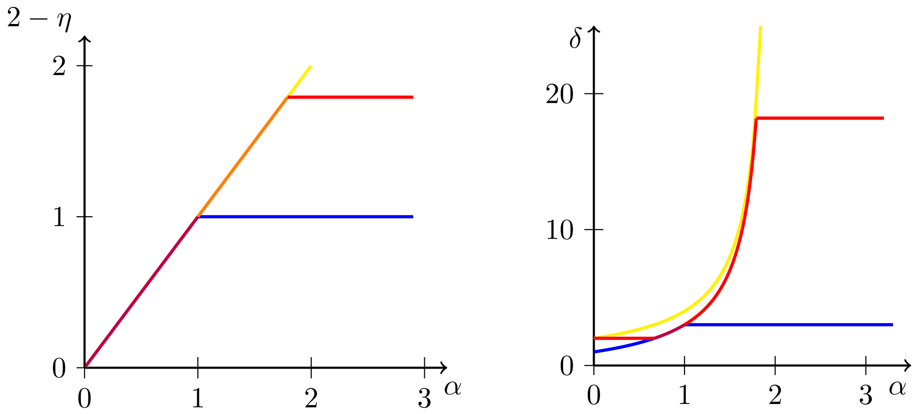


Figure 8: The critical exponents $2-\eta$ and δ for $d=2$. On the left: The blue line is our lower bound, the yellow line is the upper bound proven in [70], and the red line is the conjectured true value. The part where all three of them agree ($\alpha \in (0,1]$) is purple and the part where the upper bound and the conjectured true value agree $\left(\alpha \in \left(1, \frac{43}{24}\right]\right)$ is orange. On the right: The yellow curve is the upper bound on δ shown in [70], the red curve is the conjectured true value of δ, and the blue curve is our lower bound. The part where the lower bound and the conjectured true value agree $\left(\alpha \in \left[\frac{2}{3}, 1\right]\right)$ is purple.

$d=2$. Theorem 13.2 is an immediate consequence of Proposition 14.6. In the case where $J(x,y) \simeq \|x-y\|^{-d-\alpha}$, Theorem 13.2 shows together with Hutchcroft's result [70] that $2-\eta = \alpha$ for $\alpha \leq 1$, respectively $\alpha < 1$ for $d=1$, provided the exponent $2-\eta$ defined in (153) exists. This also gives a partial solution to [70, Problem 4.1], which asks for conditions under which the upper bound $2-\eta \leq \alpha$ has a matching lower bound. Provided that the conjectured picture described in (154) below holds, our proof also shows that the *crossover value* $\alpha_c(d)$ defined in (154) below satisfies $\alpha_c(d) \geq 1$ for all dimensions $d \geq 2$. We could alternatively define the exponent $2-\eta$ by $\sum_{x \in \Lambda_n} \mathbb{P}_{\beta_c}(0 \leftrightarrow x) \approx n^{2-\eta}$. For $\alpha < 1$ the results of [70] together with Proposition 14.6 show that the exponent $2-\eta$ defined like this exists and equals α. See also the discussion after Proposition 14.6 for more details.

Our proofs only assume an upper bound on the kernel J, so in particular the results are still valid for nearest-neighbor percolation. However, the bound $2-\eta \geq 1$ observed in this situation already follows from the proof of sharpness of the phase transition of Duminil-Copin and Tassion (162), and the lower bound $\delta \geq 3$ observed for $d=2$ follows from $2-\eta \geq 1$ and the hyperscaling inequality $(2-\eta)(\delta+1) \leq d(\delta-1)$ proven by Hutchcroft [65]. This hyperscaling inequality can be rearranged to $\delta \geq \frac{d+2-\eta}{d-(2-\eta)}$ and using $d=2, 2-\eta \geq 1$ shows $\delta \geq 3$. But our proof still shows $\delta \geq 3$ without this machinery and without assuming the existence of the exponent $2-\eta$. Our main tool for the proofs of Theorem 13.1 and Theorem 13.2 (respectively Proposition 14.7 and Proposition 14.6) is a connection between the critical exponents and the isoperimetry of the boxes $\Lambda_n = \{-n, \dots, n\}^d$ in section 14.2.

Related work The critical behavior of percolating systems is typically a difficult problem. There has been considerable progress on the understanding of percolation on various graphs at and near criticality over the last years, see for example [30,37–40,57,58,65–70,82].

The physics prediction for the critical exponent $2-\eta$ is given by

$$2-\eta(d,\alpha)=\begin{cases}\alpha & \text{for } \alpha\leq 2-\eta_{\mathrm{SR}}(d)\\ 2-\eta_{\mathrm{SR}} & \text{for } \alpha> 2-\eta_{\mathrm{SR}}(d)\end{cases}$$

where $2-\eta_{\mathrm{SR}}(d)$ is the corresponding exponent for short-range percolation on $\mathbb{Z}^d$. The prediction for the exponent δ is given by

$$\delta(d,\alpha)=\begin{cases}2 & \text{for } \alpha\leq \frac{d}{3}\\ \frac{d+\alpha}{d-\alpha} & \text{for } \alpha\in\left[\frac{d}{3},\alpha_c(d)\right]\\ \delta_{\mathrm{SR}}(d) & \text{for } \alpha\geq \alpha_c(d)\end{cases} \tag{154}$$

where $\delta_{\mathrm{SR}}(d)$ is the corresponding exponent for short-range percolation and $\delta_{\mathrm{SR}}(d)$ and the *crossover value* $\alpha_c(d)$ are such that the function $\delta(d,\alpha)$ is continuous in α. See also [65, section 1.3] or [53, section 9 and 10] for a broader overview of these predictions and references to the physics literature. The critical exponents are typically better understood in high dimension or for $\alpha<\frac{d}{3}$, where the triangle condition holds and methods involving the lace expansion can be used [11, 22, 25, 55, 61]. Also for dimension $d=2$, and in particular for the triangular lattice, the situation is much better understood, due to works of Kesten, Smirnov and Werner [75, 77, 92, 93]. Here one knows that $\delta_{\mathrm{SR}}(2)=\frac{91}{5}$. This also explains the conjectured pictures in Figure 8 and shows that the crossover value $\alpha_c(2)$ is expected to be $\frac{43}{24}$. Also for the hierarchical lattice the phase transition is better understood, due to recent results of Hutchcroft [69]. The lower bound $\delta\geq\frac{d+\alpha}{d-\alpha}$ proven for the hierarchical lattice is similar to our lower bound for $d=1$ and also shows absence of mean-field behavior for $\alpha>\frac{d}{3}$ on the hierarchical lattice.

14 Proofs

Before going to the proofs, we want to introduce a theorem that deals with the universal tightness of the maximum open cluster inside a random graph. It is a subset of [65, Theorem 2.2], which turned out to be extremely useful in various models of random graphs. We write $|K_{\max}(\Lambda)|$ for the cardinality of the largest open cluster in Λ. Note that $K_{\max}(\Lambda)$ is in general not well-defined as a subset of Λ, since there can be distinct clusters with the same cardinality. But this will not cause any problems in the following. We define the typical value of $|K_{\max}(\Lambda)|$ by

$$M_\beta(\Lambda)=\min\left\{n\geq 0:\mathbb{P}_\beta\left(|K_{\max}(\Lambda)|\geq n\right)\leq e^{-1}\right\}. \tag{155}$$

The theorem deals with general *weighted graphs* $G=(V,E,J)$, where $J:E\to[0,\infty)$ is a function that gives weights to the edges. Now edges are open or closed independent of each other and an edge $e\in E$ is open with probability $1-e^{-\beta J(e)}$, where $\beta\geq 0$ is a parameter. In particular, long-range percolation on the integer lattice can be modelled as a weighted random graph with the weight function $J(\{x,y\})=J(x-y)$.

Theorem 14.1 (Universal tightness of the maximum cluster size)**.** *Let $G=(V,E,J)$ be a countable weighted graph and let $\Lambda\subseteq V$ be finite and non-empty. Then the inequalities*

$$\mathbb{P}_\beta\left(|K_{max}(\Lambda)|\geq\alpha M_\beta(\Lambda)\right)\leq e^{-\frac{\alpha}{9}} \tag{156}$$

$$\text{and } \mathbb{P}_\beta\left(|K_u\cap\Lambda|\geq\alpha M_\beta(\Lambda)\right)\leq e\cdot\mathbb{P}_\beta\left(|K_u\cap\Lambda|\geq M_\beta(\Lambda)\right)e^{-\frac{\alpha}{9}} \tag{157}$$

hold for every $\beta\geq 0,\alpha\geq 1,$ and $u\in V$.

We will use this theorem at many points in this chapter. For the lower bound on δ we define $\theta := \frac{1}{\delta}$. In the following we will always assume that

$$\sum_{k=1}^{n} \mathbb{P}_\beta\left(|K_0| \geq k\right) \leq C n^{1-\theta} \tag{158}$$

holds for some constant $C < \infty$. Note that this already implies that $\mathbb{P}_\beta\left(|K_0| \geq n\right) \leq n^{-1}\sum_{k=1}^{n} \mathbb{P}_\beta\left(|K_0| \geq k\right) \leq C n^{-\theta}$. Furthermore, for $\theta < 1$ the bound $\mathbb{P}_\beta\left(|K_0| \geq k\right) \leq C k^{-\theta}$ for all $k \in \{1, \ldots, n\}$ also implies (158) with a different constant C' depending on C and θ.

For the lower bound on the exponent of the two-point function $2-\eta$ we define $\Lambda_n = \{-n, \ldots, n\}^d$ and assume that

$$\frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_\beta\left(0 \leftrightarrow x\right) \leq C n^{-d+2-\eta} \tag{159}$$

holds for some constant $C < \infty$. From this definition we directly see that we can always assume that $-d+2-\eta \leq 0$, as the statement is trivially true otherwise.

14.1 Moments of the cluster size inside boxes

In this section, we give bounds on the expected size of the cluster inside boxes, i.e., $\mathbb{E}_\beta\left[|K_0(\Lambda_n)|\right]$, given the upper bounds on the tail of the cluster (158) or the two-point function (159). For $\Lambda \subset \mathbb{Z}^d$ and $x \in \Lambda$ we use the notation $K_x(\Lambda)$ for the set of vertices $y \in \Lambda$ that are connected to x through an open path that lies entirely within Λ. The next lemma translates bounds of the tail of the cluster size into bounds of the typical largest cluster inside boxes of size n. The proof of such a statement has already been done for many different models of percolation [65, 82]. We give a short proof for completeness.

Lemma 14.2. *Assume that* (158) *holds for some constant* $1 \leq C < \infty$. *Let* $\Lambda \subset \mathbb{Z}^d$ *be a finite set of size* n. *Then one has*

$$M_\beta(\Lambda) \leq 3Cn^{\frac{1}{1+\theta}} \tag{160}$$

Proof. For $x \in \Lambda$, let $K_x(\Lambda)$ be the cluster of x inside Λ. We use the notation $\tilde{C} = 3C$ and get that

$$\begin{aligned}
&\mathbb{E}_\beta\left[\left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right\}\right|\right] = \sum_{x \in \Lambda} \mathbb{P}_\beta\left(|K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right) \\
&\leq \sum_{x \in \Lambda} \mathbb{P}_\beta\left(|K_x| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right) \leq \sum_{x \in \Lambda} C\tilde{C}^{-\theta} n^{-\frac{\theta}{1+\theta}} = C\tilde{C}^{-\theta} n n^{-\frac{\theta}{1+\theta}} = C\tilde{C}^{-\theta} n^{\frac{1}{1+\theta}}.
\end{aligned}$$

If there is one $x \in \Lambda$ such that $|K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}$, then there are at least $\tilde{C}n^{\frac{1}{1+\theta}}$ many such $x \in \Lambda$. So in particular, if $|K_{\max}(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}$, then there are at least $\tilde{C}n^{\frac{1}{1+\theta}}$ many vertices $x \in \Lambda$ with $|K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}$. This implies that

$$\mathbb{1}_{\left\{|K_{\max}(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right\}} \leq \frac{1}{\tilde{C}n^{\frac{1}{1+\theta}}} \left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right\}\right|$$

and taking expectations on both sides yields that

$$\mathbb{P}_\beta\left(|K_{\max}(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right) \leq \frac{1}{\tilde{C}n^{\frac{1}{1+\theta}}} \mathbb{E}_\beta\left[\left|\left\{x \in \Lambda : |K_x(\Lambda)| \geq \tilde{C}n^{\frac{1}{1+\theta}}\right\}\right|\right]$$

$$\leq \frac{1}{\tilde{C} n^{\frac{1}{1+\theta}}} C \tilde{C}^{-\theta} n^{\frac{1}{1+\theta}} = C \tilde{C}^{-1-\theta} = C(3C)^{-1-\theta} < \frac{1}{3} < \frac{1}{e}$$

which shows that $M_\beta(\Lambda) \leq 3Cn^{\frac{1}{1+\theta}}$.

□

Lemma 14.3. *Assume that (158) holds. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of size n. Then there exists a constant $C_2 = C_2(C, \theta)$ such that*

$$\mathbb{E}_\beta \left[|K_0(\Lambda)| \right] \leq C_2 n^{\frac{1-\theta}{1+\theta}}. \tag{161}$$

Proof. The proof is heavily based on the use of Theorem 14.1. For abbreviation, we simply write $M = M_\beta(\Lambda)$. Thus we get that

$$\begin{aligned}
\mathbb{E}_\beta \left[|K_0(\Lambda)| \right] &= \sum_{k=1}^{\infty} \mathbb{P}_\beta \left(|K_0(\Lambda)| \geq k \right) = \sum_{l=0}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta \left(|K_0(\Lambda)| \geq lM + k \right) \\
&= \sum_{k=1}^{M} \mathbb{P}_\beta \left(|K_0(\Lambda)| \geq k \right) + \sum_{l=1}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta \left(|K_x(\Lambda)| \geq lM + k \right) \\
&\leq CM^{1-\theta} + \sum_{l=1}^{\infty} \sum_{k=1}^{M} \mathbb{P}_\beta \left(|K_0(\Lambda)| \geq lM \right) \\
&\overset{(157)}{\leq} CM^{1-\theta} + M \sum_{l=1}^{\infty} e \mathbb{P}_\beta \left(|K_0(\Lambda)| \geq M \right) e^{-\frac{l}{9}} \\
&\leq CM^{1-\theta} + eCM^{1-\theta} \sum_{l=1}^{\infty} e^{-\frac{l}{9}} \leq C' M^{1-\theta} \leq C_2 n^{\frac{1-\theta}{1+\theta}}
\end{aligned}$$

for some constants $C', C_2 < \infty$. Here we used the result of Lemma 14.2 for the last inequality. □

The next Lemma translates the average bound on the two-point function (159) into bounds on the restricted cluster size. For two sets $A, B \subset \mathbb{Z}^d$ we introduce the notation $A \overset{\Lambda_n}{\longleftrightarrow} B$, meaning that there exists a path from A to B that uses edges with both endpoints in Λ_n only.

Lemma 14.4. *Assume that (159) holds. Then one has*

$$\mathbb{E}_\beta \left[|K_0(\Lambda_n)| \right] \leq 3^d C n^{2-\eta}.$$

for all $x \in \Lambda_n$.

Proof. The ∞-distance between different 0 and $x \in \Lambda_n$ is at most n. We have that $|\Lambda_n| = (2n+1)^d$. Thus linearity of expectation gives that

$$\begin{aligned}
&\mathbb{E}_\beta \left[|K_0(\Lambda_n)| \right] = \sum_{x \in \Lambda_n} \mathbb{P}_\beta \left(0 \overset{\Lambda_n}{\longleftrightarrow} x \right) \leq |\Lambda_n| \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathbb{P}_\beta \left(0 \leftrightarrow x \right) \\
&\leq (2n+1)^d C n^{-d+2-\eta} \leq 3^d C n^{2-\eta}.
\end{aligned}$$

□

14.2 Isoperimetric inequalities in expectation

In this section, we use the isoperimetry of the box $\Lambda_n = \{-n, \ldots, n\}^d$ in order to bound the expected number of edges at the boundary of the box, for which the end inside the box is connected to 0. For long-range percolation with a kernel $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ satisfying $J(x,y) \simeq \|x-y\|^{-d-\alpha}$ the isoperimetry of the box Λ_n changes at $\alpha = 1$. More precisely, if we denote by $\partial \Lambda_n$ the set of open edges with exactly one endpoint in Λ_n, we have that

$$\mathbb{E}_\beta \left[|\partial \Lambda_n| \right] \simeq \begin{cases} n^{d-\alpha} & \text{if } \alpha < 1 \\ n^{d-1} \log(n) & \text{if } \alpha = 1 \\ n^{d-1} & \text{if } \alpha > 1 \end{cases} .$$

Consequently, we see that for $\alpha < 1$ long-range effects determine the isoperimetry of the box, whereas for $\alpha \geq 1$ the short-range effects dominate, with logarithmic corrections at $\alpha = 1$. In particular, a point $x \in \Lambda_n$ that is chosen uniformly at random will have of order $n^{-(\alpha \wedge 1) + o(1)}$ neighbors outside of the box. This is also the reason, why the term $\alpha \wedge 1$ pops up in the statements of Theorem 13.1 and Theorem 13.2. In the following, for two sets $A, B \subset \mathbb{Z}^d$ we use the notation $A \sim B$ if there exists a direct edge from A to B. We also use a statement that was shown by Duminil-Copin and Tassion in [39, 40]. There it is shown that for $\beta \geq \beta_c$ and all finite sets $S \subset \mathbb{Z}^d$ containing the origin 0 one has

$$\phi_\beta (S) \coloneqq \sum_{x \in S} \sum_{y \notin S} \left(1 - e^{-\beta J(x,y)} \right) \mathbb{P}_\beta \left(0 \overset{S}{\longleftrightarrow} x \right) \geq 1. \tag{162}$$

Moreover, they also showed the reverse direction, i.e., that $\phi_\beta(S) \geq 1$ for all finite sets $S \subset \mathbb{Z}^d$ with $0 \in S$ implies $\beta \geq \beta_c$, but we will not use this statement in our proof. Similar results to the result in (162) were already shown previously, see for example [76, Lemma 3.1] or [4, Lemma 5.1].

Lemma 14.5. *We write $K_0(\Lambda_k)$ for the set of vertices $y \in \Lambda_k$ that are connected to 0 through an open path that lies entirely within Λ_k. Let $n \in \mathbb{N}$ be arbitrary and fixed. For $d = 1$ and all $\alpha \in (0,1)$, respectively for $d > 1$ and all $\alpha > 0$, and all $\beta > 0$, there exists a constant $C_3 = C_3(\alpha, \beta, d)$ that does not depend on n, so that there exists a $k \in \{1, \ldots, n\}$ with*

$$\phi_\beta (\Lambda_k) = \sum_{x \in \Lambda_k} \sum_{y \notin \Lambda_k} \left(1 - e^{-\beta J(x,y)} \right) \mathbb{P}_\beta \left(0 \overset{\Lambda_k}{\longleftrightarrow} x \right) \leq C_3 \mathbb{E}_\beta \left[|K_0 (\Lambda_n)| \right] f(n, \alpha) \tag{163}$$

where the function $f(n, \alpha)$ is defined by

$$f(n, \alpha) = \begin{cases} n^{-\alpha} & \textit{if } \alpha < 1 \\ n^{-1} \log(n) & \textit{if } \alpha = 1 \\ n^{-1} & \textit{if } \alpha > 1 \end{cases} . \tag{164}$$

Proof. For $x \in \Lambda_n$ we write $t_x \coloneqq \mathbb{P}_\beta \left(x \overset{\Lambda_n}{\longleftrightarrow} 0 \right)$ and get that

$$\sum_{x \in \Lambda_n} t_x = \sum_{x \in \Lambda_n} \mathbb{P}_\beta \left(x \overset{\Lambda_n}{\longleftrightarrow} 0 \right) = \mathbb{E}_\beta \left[|K_0(\Lambda_n)| \right]. \tag{165}$$

Next, we define X_k as the number of open edges between Λ_k and $(\Lambda_k)^C$ for which one end is connected to 0 within Λ_k. Formally, we define

$$X_k \coloneqq \left| \left\{ e = \{a, b\} \text{ open} : a \in \Lambda_k, b \notin \Lambda_k, \text{ and } 0 \overset{\Lambda_k}{\longleftrightarrow} a \right\} \right| .$$

The occupation status of edges inside Λ_k and of edges with one end outside of Λ_k are independent random variables. So by linearity of expectation one has

$$\mathbb{E}_\beta\left[X_k\right] = \sum_{a\in\Lambda_k}\sum_{b\notin\Lambda_k}\left(1-e^{-\beta J(a,b)}\right)\mathbb{P}_\beta\left(0\xleftrightarrow{\Lambda_k}a\right) = \phi_\beta\left(\Lambda_k\right).$$

Thus, it suffices to bound the expected value of X_k and show that there exists a $k \in \{1,\dots,n\}$ such that the expected value $\mathbb{E}_\beta\left[X_k\right]$ is reasonably small, as in (163). For this, let K be a random variable that is uniformly distributed on $\{1,\dots,n\}$ and is independent of the percolation configuration. We write $\mathbf{P}_\beta$ for the joint distribution of the percolation configuration and K, and $\mathbf{E}_\beta$ for its expectation. Thus we get

$$\begin{aligned}
\mathbf{E}_\beta\left[X_K\right] &= \mathbf{E}_\beta\left[\left|\left\{\{a,b\}\text{ open} : a\in\{-K,\dots,K\}^d, b\notin\{-K,\dots,K\}^d,\text{ and } 0\xleftrightarrow{\Lambda_K}a\right\}\right|\right]\\
&= \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_\beta\left[\left|\left\{\{a,b\}\text{ open} : a\in\{-k,\dots,k\}^d, b\notin\{-k,\dots,k\}^d,\text{ and } 0\xleftrightarrow{\Lambda_k}a\right\}\right|\right]\\
&= \frac{1}{n}\sum_{k=1}^{n}\sum_{a\in\Lambda_n}\sum_{b\in\mathbb{Z}^d}\mathbb{E}_\beta\left[\mathbb{1}_{\{a\in\Lambda_k\}}\mathbb{1}_{\{b\notin\Lambda_k\}}\mathbb{1}_{\left\{0\xleftrightarrow{\Lambda_k}a\right\}}\mathbb{1}_{\{a\sim b\}}\right].
\end{aligned}\tag{166}$$

For fixed k, the events $\{0\xleftrightarrow{\Lambda_k}a\}$ and $\{\{a,b\}\text{ is open}\}$ are independent for $b\notin\Lambda_k$, as the first event depends only on edges with both endpoints inside Λ_k. For fixed $a\in\Lambda_n$, the expression $\mathbb{P}_\beta\left(0\xleftrightarrow{\Lambda_k}a\right)$ can only be positive if $k\geq\|a\|_\infty$. Combining the two previous observations we get that

$$\begin{aligned}
\mathbf{E}_\beta\left[X_K\right] &= \frac{1}{n}\sum_{k=1}^{n}\sum_{a\in\Lambda_k}\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\mathbb{P}_\beta\left(0\xleftrightarrow{\Lambda_k}a\right)\mathbb{P}_\beta\left(a\sim b\right)\\
&= \frac{1}{n}\sum_{a\in\Lambda_n}\sum_{k=1\vee\|a\|_\infty}^{n}\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\mathbb{P}_\beta\left(0\xleftrightarrow{\Lambda_k}a\right)\mathbb{P}_\beta\left(a\sim b\right)\\
&\leq \sum_{a\in\Lambda_n}\mathbb{P}_\beta\left(0\xleftrightarrow{\Lambda_n}a\right)\left(\frac{1}{n}\sum_{k=1\vee\|a\|_\infty}^{n}\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\left(1-e^{-\beta J(a,b)}\right)\right)\\
&\leq \sum_{a\in\Lambda_n}t_a\left(\frac{1}{n}\sum_{k=1\vee\|a\|_\infty}^{n}\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\beta C_1\|a-b\|^{-d-\alpha}\right),
\end{aligned}\tag{167}$$

where we used that $1-e^{-x}\leq x$ for the last inequality. Now, for fixed $a\in\Lambda_n$ and $k\geq\|a\|_\infty$ there exist constants $C_1'=C_1'(C_1,d,\beta)<\infty$ and $C_1''=C_1''(C_1,d,\alpha,\beta)<\infty$ such that

$$\begin{aligned}
&\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\beta C_1\|a-b\|^{-d-\alpha}\leq\sum_{l=k+1-\|a\|_\infty}^{\infty}\sum_{b\in\mathbb{Z}^d:\|b-a\|_\infty=l}\beta C_1\|a-b\|^{-d-\alpha}\\
&=\sum_{l=k+1-\|a\|_\infty}^{\infty}\sum_{b\in\mathbb{Z}^d:\|b\|_\infty=l}\beta C_1\|b\|^{-d-\alpha}\leq\sum_{l=k+1-\|a\|_\infty}^{\infty}C_1'l^{d-1}l^{-d-\alpha}\\
&=C_1'\sum_{l=k+1-\|a\|_\infty}^{\infty}l^{-1-\alpha}\leq C_1''(k+1-\|a\|_\infty)^{-\alpha}.
\end{aligned}\tag{168}$$

Using (168) we see that

$$\frac{1}{n}\sum_{k=1\vee\|a\|_\infty}^{n}\sum_{b\in\mathbb{Z}^d\setminus\Lambda_k}\beta C_1\|a-b\|^{-d-\alpha}\leq\frac{1}{n}\sum_{k=\|a\|_\infty}^{n}C_1''(k+1-\|a\|_\infty)^{-\alpha}$$
$$\leq C_1''\frac{1}{n}\sum_{k=1}^{n+1}k^{-\alpha}\leq\hat{C}_1 f(n,\alpha) \tag{169}$$

for a constant $\hat{C}_1=\hat{C}_1(C_1'',\alpha)<\infty$. Inserting this result into (167) yields

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}_\beta\left[X_k\right]=\mathbf{E}_\beta\left[X_K\right]\leq\sum_{a\in\Lambda_n}t_a\hat{C}_1 f(n,\alpha)\overset{(165)}{=}\mathbb{E}_\beta\left[|K_0\left(\Lambda_n\right)|\right]\hat{C}_1 f(n,\alpha).$$

So in particular there needs to exist at least one $k\in\{1,\ldots,n\}$ for which $\mathbb{E}_\beta\left[X_k\right]\leq\mathbb{E}_\beta\left[|K_0\left(\Lambda_n\right)|\right]\hat{C}_1 f(n,\alpha)$, which finishes the proof. □

14.3 The proof of Theorem 13.1 and Theorem 13.2

Now we are ready to go to the main proofs. Theorem 13.1 is an immediate consequence of Proposition 14.7 and Theorem 13.2 is an immediate consequence of Proposition 14.6. Also remember the definition of the function f defined in (164) which we will use at several points below.

Proposition 14.6. *Let $\alpha\in(0,1)$ for $d=1$, respectively $\alpha>0$ for $d>1$, and assume that there exists a constant $C_1<\infty$ such that $J(x,y)\leq C_1\|x-y\|^{-d-\alpha}$ for all $x,y\in\mathbb{Z}^d$. Provided $\beta_c<\infty$ one has $\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}(0\leftrightarrow x)\geq\frac{1}{C_3}f(n,\alpha)^{-1}$ where C_3 is the same constant as in Lemma 14.5.*

Proof. We will first show that $\mathbb{E}_{\beta_c}\left[|K_0\left(\Lambda_n\right)|\right]\geq\frac{1}{C_3}f(n,\alpha)^{-1}$. Assume the contrary, i.e., $\mathbb{E}_{\beta_c}\left[|K_0\left(\Lambda_n\right)|\right]<\frac{1}{C_3}f(n,\alpha)^{-1}$. Then by Lemma 14.5 there exists a $k\in\{1,\ldots,n\}$ with

$$\phi_{\beta_c}\left(\Lambda_k\right)\leq C_3\mathbb{E}_{\beta_c}\left[|K_0\left(\Lambda_n\right)|\right]f(n,\alpha)<1$$

which is a contradiction to (162). Now, by linearity of expectation we have that

$$\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}\left(0\leftrightarrow x\right)\geq\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}\left(0\xleftrightarrow{\Lambda_n}x\right)=\mathbb{E}_{\beta_c}\left[|K_0\left(\Lambda_n\right)|\right]\geq\frac{1}{C_3}f(n,\alpha)^{-1}. \tag{170}$$

□

Proposition 14.6 shows in particular that for a small enough constant $c>0$ we have

$$\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}\left(0\leftrightarrow x\right)\geq cn^{-d}f(n,\alpha)^{-1}=\begin{cases}cn^{-d+\alpha} & \text{for } \alpha<1\\ cn^{-d+1}\log(n)^{-1} & \text{for } \alpha=1\\ cn^{-d+1} & \text{for } \alpha>1\end{cases}$$

which shows that the exponent $2-\eta$ defined in (153) satisfies $2-\eta\geq\alpha\wedge 1$, provided the exponent $2-\eta$ exists. In [70] it is shown that $\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}\left(0\leftrightarrow x\right)=\mathcal{O}\left(n^{-d+\alpha}\right)$. Combining this with Proposition 14.6 we get that for $\alpha<1$ and a kernel J satisfying $J(x,y)\simeq\|x-y\|^{-d-\alpha}$ one has

$$\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}\left(0\leftrightarrow x\right)\simeq n^{-d+\alpha}.$$

So when we alternatively define the two-point critical exponent $2-\eta$ by the averaged version $\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\mathbb{P}_{\beta_c}(0\leftrightarrow x)\approx n^{-d+2-\eta}$, then we see that this exponent exists for $\alpha<1$ and equals α. However, it is not clear whether this statements holds without averaging, i.e., if the exponent $2-\eta$ defined as in (153) also exists. See also [70, Problem 4.3] for a related problem. Next, we consider the lower bound on the exponent δ.

Proposition 14.7. *Let $\alpha\in(0,1)$ for $d=1$, respectively $\alpha>0$ for $d>1$, and assume that there exists a constant $C_1<\infty$ such that $J(x,y)\leq C_1\|x-y\|^{-d-\alpha}$ for all $x,y\in\mathbb{Z}^d$. Suppose that $\beta_c<\infty$ and $\sum_{k=1}^{n}\mathbb{P}_{\beta_c}(|K_0|\geq k)\leq Cn^{1-\frac{1}{\delta}}$ for all $n\in\mathbb{N}$. Then $\delta\geq\frac{d+(\alpha\wedge 1)}{d-(\alpha\wedge 1)}$.*

Proof. We write $\theta=\frac{1}{\delta}$ and get that $\sum_{k=1}^{N}\mathbb{P}_{\beta_c}(|K_0|\geq k)\leq CN^{1-\theta}$ for all $N\in\mathbb{N}$. Lemma 14.3 shows that for some constant $C'<\infty$ we have $\mathbb{E}_\beta\left[|K_0(\Lambda_n)|\right]\leq C'n^{d\frac{1-\theta}{1+\theta}}$. Combining this with inequality (170) we get that

$$C'n^{d\frac{1-\theta}{1+\theta}}\geq\mathbb{E}_\beta\left[|K_0(\Lambda_n)|\right]\geq C_3^{-1}f(n,\alpha)^{-1}\approx n^{(\alpha\wedge 1)+o(1)}$$

and this shows that $d\frac{1-\theta}{1+\theta}\geq\alpha\wedge 1$. As we consider $\alpha\in(0,1)$ only for $d=1$, we always have that $\frac{\alpha\wedge 1}{d}<1$. Elementary calculations show that

$$\begin{aligned}
&d\,\frac{1-\theta}{1+\theta}=d\,\frac{\delta-1}{\delta+1}\geq\alpha\wedge 1\Leftrightarrow\delta-1\geq\frac{\alpha\wedge 1}{d}\delta+\frac{\alpha\wedge 1}{d}\\
&\Leftrightarrow\delta-\frac{\alpha\wedge 1}{d}\delta=\delta\left(1-\frac{\alpha\wedge 1}{d}\right)\geq\frac{\alpha\wedge 1}{d}+1\\
&\Leftrightarrow\delta\geq\frac{1+\frac{\alpha\wedge 1}{d}}{1-\frac{\alpha\wedge 1}{d}}=\frac{d+(\alpha\wedge 1)}{d-(\alpha\wedge 1)}
\end{aligned}$$

which finishes the proof. □

Random walks

15 Introduction

Consider independent $\mathbb{Z}^d$-valued random variables $X_1, X_2, ...$ that are symmetric, i.e., they satisfy $\mathbb{P}(X_1 = x) = \mathbb{P}(X_1 = -x)$ for all $x \in \mathbb{Z}^d$. We want to know for which regimes of decay of $\mathbb{P}(X_i = x)$ the associated random walk defined by $S_n = \sum_{k=1}^n X_k$ is recurrent or transient. For this, we first construct an electrical network that is equivalent to this random walk. We do this by giving conductances to all edges $\{a, b\}$ with $a, b \in \mathbb{Z}^d$, allowing self-loops here. For two points $a, b \in \mathbb{Z}^d$ we give a conductance of $c_{\{a,b\}} = \mathbb{P}(X_i = a - b)$ to the edge between them. The symmetry condition $\mathbb{P}(X_i = x) = \mathbb{P}(X_i = -x)$ guarantees that the conductances defined like this are well-defined. Then consider the reversible Markov chain on this network, i.e., the Markov chain defined by $\mathbb{P}(M_{n+1} = y | M_n = x) = \frac{c_{\{x,y\}}}{\sum_{z \in \mathbb{Z}^d} c_{\{x,z\}}} = c_{\{x,y\}}$. The resulting Markov chain has exactly the same distribution as S_n, and thus, we will analyze this Markov chain from here on. It is a classical result of Pólya that the simple random walk on the integer lattice $\mathbb{Z}^d$ is recurrent for $d \in \{1, 2\}$ and transient for $d \geq 3$ [86]. Furthermore, it is a well-known result about electrical networks that transience of the random walk is equivalent to the existence of a unit flow with finite energy from o to infinity, where o is an arbitrary vertex in the graph, or the origin for the integer lattice; see for example [79, Theorem 2.10]. With this characterization of transience, one directly gets that the random walk S_n defined as above is always transient for $d \geq 3$, and recurrent when the X_i-s are bounded symmetric random variables and $d \in \{1, 2\}$. In this part of the thesis, we answer the question whether the random walk is recurrent or transient when $\mathbb{P}(X = x)$ has a power-law decay, i.e., when $\mathbb{P}(X = x) = \mathbb{P}(X = -x) = \Theta(\|x\|^{-s})$, where $s > d$ is a parameter. Note that this question makes no sense for $s \leq d$, as the probabilities $\mathbb{P}(X_i = x)$ need to sum up to 1. This problem has been studied before at several other places, for example in [24] using the recurrence criterion of [94, Section 8]. However, previous proofs used the characteristic function of the random walk

$$\varphi(\theta) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}(X_1 = x) e^{ix \cdot \theta},$$

whereas our proof does not use characteristic functions, but uses the theory of electric networks. The results of the transience/recurrence of Pólya are often humorously paraphrased as "*A drunk man will find his way home, but a drunk bird may get lost forever.*", which goes back to Shizuo Kakutani. So in this chapter, we study the question which kinds of drunk grasshoppers, which tend to make huge jumps, eventually will find their way home and which kinds may get lost forever. The answer is that the random walk is recurrent for $d \in \{1, 2\}$ and $s \geq 2d$, and transient otherwise.

Theorem 15.1. *Let $X_1, X_2, \dots$ be i.i.d. symmetric $\mathbb{Z}^d$-valued random variables satisfying $\mathbb{P}(X_1 = x) = \mathbb{P}(X_1 = -x) \geq c\|x\|^{-s}$ for some $c > 0, s < 2d$, and all x large enough. Then the random walk S_n defined by $S_n = \sum_{k=1}^n X_k$ is transient.*

This result is not surprising, as for $s < 2d$ the total conductance between the two boxes $A = \{0, \dots, n\}^d$ and $B = 2n \cdot e_1 + \{0, \dots, n\}^d$ satisfies $\sum_{x \in A} \sum_{y \in B} c_{\{x,y\}} \approx n^{2d-s} \gg 1$ and this suggests that it is possible to construct a finite-energy flow from the root to infinity. Here e_1 denotes the standard unit vector pointing in the direction of the first coordinate axis. This suggests that the transition from transience to recurrence in dimension $d \in \{1, 2\}$ happens at $s = 2d$. Also many different properties of the long-range percolation graph change at this value; see section 1.4 for more examples of such phenomena. What happens at the critical value $s = 2d$ is treated in the following theorem.

Theorem 15.2. *Let $d \in \{1, 2\}$, and let $X_1, X_2, \ldots$ be i.i.d. symmetric $\mathbb{Z}^d$-valued random variables satisfying $\mathbb{P}(X_1 = x) = \mathbb{P}(X_1 = -x) \leq C\|x\|^{-2d}$ for some constant $C < \infty$ and all $x \neq 0$. Then the random walk S_n defined by $S_n = \sum_{k=1}^n X_k$ is recurrent.*

So in particular Theorem 15.2 shows that for dimension $d \in \{1, 2\}$ and for $\mathbb{P}(X_1 = x) = c\|x\|^{-2d}$ the associated random walk is recurrent, without having a mean in dimension 1, respectively a finite variance in dimension 2. Both cases lie on the exact borderline that separates the transient regime from the recurrent regime. The transience or recurrence of a Markov chain, or of a sum of i.i.d. random variables, is an elementary question that has been extensively studied in many different regimes [26, 89, 90], including results in random environments [95] and on percolation clusters [5, 16, 71, 85]. We also use parts of the techniques developed by Berger in [16], in particular Lemma 16.2.

The random walk $(X_n)_{n\in\mathbb{N}}$ can also be seen to be equivalent to an annealed random walk on a sequence of long-range percolation graphs when the underlying graph of the percolation gets resampled at every time-step. If one does not do this resampling, then one has a simple random walk on a percolation cluster. It is a natural question to ask how the random walk on a graph with long jumps compares to the simple random walk on the associated graph obtained by percolation. Formally, let $G = (V, E)$ be a connected graph with weighted edges $(c_e)_{e\in E} \in \mathbb{R}_{\geq 0}^E$. Assume that for each vertex $v \in V$ one has $0 < \sum_{e:v\in e} c_e < \infty$, and let $(X_n)_{n\in\mathbb{N}}$ be the random walk defined by the transition probabilities

$$\mathbb{P}(X_{n+1} = x | X_n = y) = \frac{c_{\{x,y\}}}{\sum_{e:y\in e} c_e} \tag{171}$$

for all edges $\{x, y\} \in E$. If the random walk $(X_n)_{n\in\mathbb{N}}$ is recurrent almost surely for all possible starting points, we also say that the graph $G = (V, E)$ is recurrent. Let $\tilde{G} = (V, E, \omega)$ be a random graph with vertex set V, where each edge $e \in E$ has a random non-negative weight $\omega(e)$ that satisfies $\mathbb{E}[\omega(e)] \leq c_e$. Note that we do *not* require that these random weights are independent for different edges. In the case where $\omega(e) \in \{0, 1\}$ almost surely for all edges $e \in E$, one can also think of bond percolation on the graph (V, E). Let $(Y_n)_{n\in\mathbb{N}}$ be the random walk on this weighted graph, i.e., the random walk with transition probabilities

$$\mathbb{P}(Y_{n+1} = x | Y_n = y) = \frac{\omega(\{x, y\})}{\sum_{e:y\in e} \omega(e)} \tag{172}$$

for all vertices $y \in V$ and all vertices $x \in V$ for which $\omega(\{x, y\}) > 0$. In the case where $\sum_{e:y\in e} \omega(e) = 0$, i.e., when all edges with y as one of its endpoints have a weight of 0, we simply define Y_n as the random walk that stays constant on y. For two vertices $x, y \in V$ we say that they are connected if there exists a path of edges between them, such that $\omega(e) > 0$ for all edges e in this path. The graph $\tilde{G}$ will not be connected for many examples of percolation, but we say that it is recurrent if all its connected components are recurrent graphs. We prove that if the random walk with the long-range steps $(X_n)_{n\in\mathbb{N}}$ is recurrent, then almost every realization of the corresponding random weighted graph is also recurrent.

Theorem 15.3. *Let $G = (V, E)$ be a graph with weighted edges $(c_e)_{e\in E} \in \mathbb{R}_{\geq 0}^E$ as above. Assume that the random walk $(X_n)_{n\in\mathbb{N}}$ defined by (171) is recurrent. Let $\tilde{G} = (V, E, \omega)$ be* *a graph, where the edges $e \in E$ carry a random weight $\omega(e)$ with*

$$\mathbb{E}\left[\omega(e)\right] \leq c_e$$

for all $e \in E$. Then the random walk on these weights defined by (172) is recurrent almost surely.

The proof of this theorem will be a direct consequence of Lemma 17.2. In section 17 below we will use Theorem 15.2 and Theorem 15.3 in order to extend the results on recurrence of random walks of percolation clusters of Berger [16] to percolation clusters on the one- or two-dimensional integer lattice with dependencies, i.e., when the occupation statuses of different edges are not independent. We will also apply this extension to the *weight-dependent random connection model* and obtain several new results regarding the recurrence of random walks on such models. Readers interested mostly in the new results regarding recurrence of the random connection model might also consider to skip section 16 directly go to section 17. It is also completely self-contained, up to the use of Theorem 15.2.

Random walks on long-range models are a well-studied object, including results on mixing times [13] and scaling limits [21, 28, 29]. However, many results so far focused on independent long-range percolation or needed assumptions on ergodicity. One model of dependent percolation for which the recurrence and transience has been studied recently is the *weight dependent random connection model* [49]. We consider the weight dependent random connection model in dimension $d = 2$. The vertex set of this graph is a Poisson process of unit intensity on $\mathbb{R}^2 \times (0, 1)$. For a vertex (x, s) in the Poisson process, the value $x \in \mathbb{R}^2$ is called the spatial parameter and the value $s \in (0, 1)$ is called the weight parameter. Two vertices (x, s) and (y, t) are connected with probability $\varphi\left((x, s), (y, t)\right)$, where $\varphi : \left(\mathbb{R}^2 \times (0, 1)\right)^2 \to [0, 1]$ is a function. We will always assume that φ is of the form

$$\varphi\left((x, s), (y, t)\right) = \rho\left(g(s, t)\|x - y\|^2\right)$$

where ρ is a function (also called profile function) from $\mathbb{R}_{\geq 0}$ to $[0, 1]$ that is non-increasing and satisfies

$$\lim_{r\to\infty} r^{\delta}\rho(r) = 1 \tag{173}$$

for some $\delta > 1$. The function $g : (0, 1)\times(0, 1) \to \mathbb{R}_{\geq 0}$ is a kernel that is symmetric and non-decreasing in both arguments. We define different kernels depending on two parameters $\gamma \in [0, 1)$ and $\beta > 0$. The parameter γ determines the strength of the influence of the weight parameter. The parameter β corresponds to the density of edges. Different examples of kernels are the *sum kernel*

$$g(s, t) = g^{\text{sum}}(s, t) = \frac{1}{\beta}\left(s^{-\gamma/d} + t^{-\gamma/d}\right)^{-d},$$

the *min kernel*

$$g(s, t) = g^{\min}(s, t) = \frac{1}{\beta}\left(\min(s, t)\right)^{\gamma},$$

the *product kernel*

$$g(s, t) = g^{\text{prod}}(s, t) = \frac{1}{\beta} s^{\gamma} t^{\gamma},$$

and the *preferential attachment kernel*

$$g(s,t) = g^{\mathrm{pa}}(s,t) = \frac{1}{\beta} \min(s,t)^{\gamma} \max(s,t)^{1-\gamma}.$$

As $g^{\mathrm{sum}} \leq g^{\mathrm{min}} \leq 2^d g^{\mathrm{sum}}$, the min kernel and the sum kernel show typically the same qualitative behavior. Depending on the value of β, there might be an infinite connected cluster [50, 51]. The weight-dependent random connection model and other models with scale-free degree distribution have been studied intensively in recent years, including new results on the convergence of such graphs [47, 52, 73], the chemical distances [34, 48, 64, 74], random walks and the contact process evolving on random graphs [46, 49, 63], and the percolation phase transitions [34, 50, 51, 60]. In section 17.1 below we study for which combinations of γ and δ all connected components of the resulting graph are almost surely recurrent. Our main (and only) tool for this is a consequence of Theorem 15.3, which allows to make statements about random walks on dependent percolation clusters. Whenever there is no infinite cluster, then the random walk is clearly recurrent on all finite clusters. The question of recurrence and transience has been studied before by Gracar, Heydenreich, Mönch, and Mörters in [49]. We will generally adapt to their notation. An overview of their results and our newly obtained results can be found in Figure 9. Our results for the weight-dependent random connection model are as follows.

Theorem 15.4. *Consider the weight-dependent random connection model with profile function ρ satisfying (173) in dimension $d = 2$.*

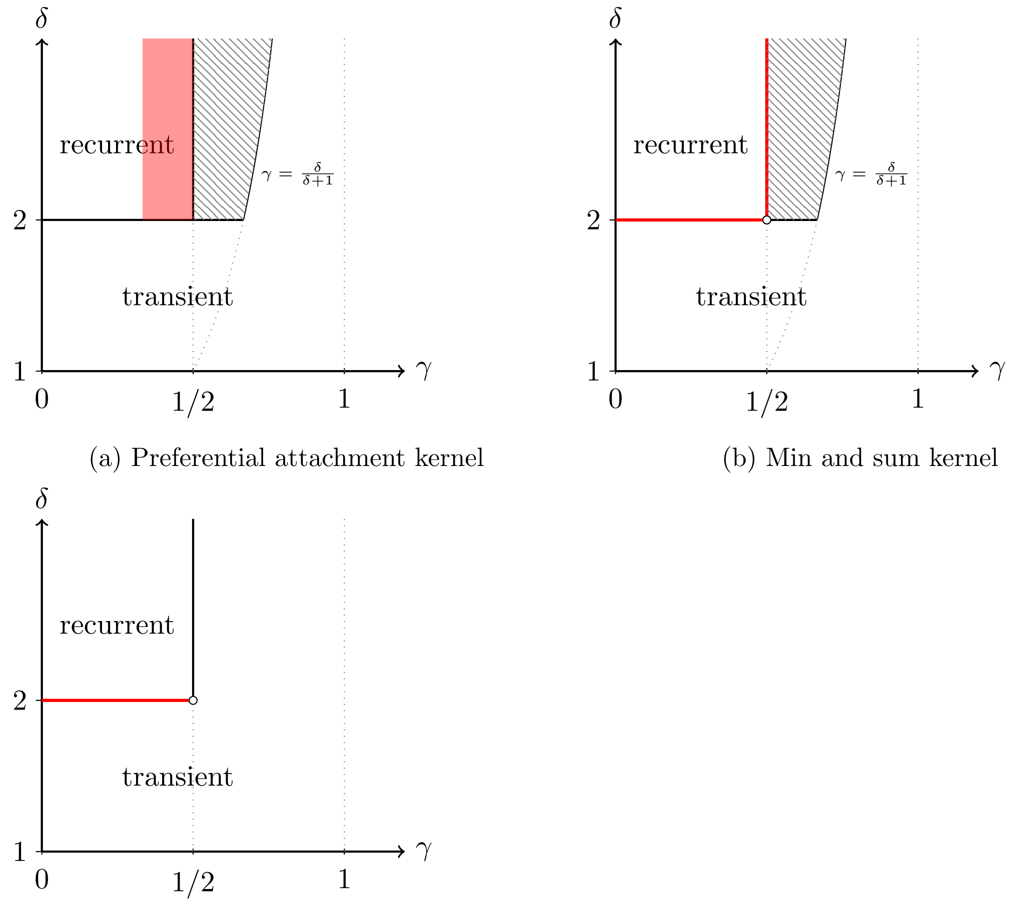
(a) For the preferential attachment kernel, every component is almost surely recurrent if $\delta > 2, \gamma < \frac{1}{2}$.

(b) For the min kernel and the sum kernel, every component is almost surely recurrent if $\delta = 2, \gamma < \frac{1}{2}$ or $\delta > 2, \gamma = \frac{1}{2}$.

(c) For the product kernel, every component is almost surely recurrent if $\delta = 2, \gamma < \frac{1}{2}$.

16 Random walks with large steps

As already shortly discussed in the introduction, we will always study the random walk on an electric network, and this random walk has the same distribution as the sum of random variables $\sum_{k=1}^{n} X_k$. For this, we define the conductances on the edges by $c_{\{x,y\}} = \mathbb{P}(X_1 = x - y)$, which is well-defined as $\mathbb{P}(X_1 = x - y) = \mathbb{P}(X_1 = y - x)$. Now the Markov chain on these conductances has the same distribution as $S_n = \sum_{k=1}^{n} X_k$. We can without loss of generality assume that $\mathbb{P}(X_1 = 0) = 0$, as the steps X_i with $X_i = 0$ have no influence whether a random walk is recurrent or transient. For such a Markov chain, there are well-known criteria for transience/recurrence. A random walk on this network is transient if and only if there exists a unit flow with finite energy from the origin 0 to infinity, see for example [79, Theorem 2.10] or [35, 78, 80]. We use this connection between transience and flows in the proof of Theorem 15.1 and in the proof of Theorem 15.2 for $d = 2$. The use in the proof of Theorem 15.2 for $d = 2$ is more implicit, as it is hidden in the proof of Lemma 16.2. In particular, the proof of Lemma 16.2 uses cutsets [83] and the Nash-Williams criterion in order to show that there can not exist a flow with finite energy from 0 to infinity. Note that the network $\left(c_{\{x,y\}}\right)_{x,y \in \mathbb{Z}^d, x \neq y}$ defined as above is still translation-invariant. The same statements about transience/recurrence of this network can be made without translation invariance, as the following lemma shows.



(a) Preferential attachment kernel

(b) Min and sum kernel

(c) Product kernel

Figure 9: Recurrent and transient regimes for weight-dependent random connection models. The red lines/area is the phase where Theorem 15.4 shows the recurrence of the random walk, and where the recurrence has not been shown by Gracar, Heydenreich, Mönch, and Mörters in [49].

Lemma 16.1. *For an electric network in dimension* $d \in \{1,2\}$ *the condition* $c_{\{x,y\}} \leq C\|x-y\|^{-2d}$ *implies recurrence, whereas* $c_{\{x,y\}} \geq c\|x-y\|^{-s}$ *for some* $c > 0$ *and* $s < 2d$ *implies transience.*

Proof of Lemma 16.1 given Theorem 15.1 and Theorem 15.2. We start with the proof of the recurrence. Let $d \in \{1,2\}$. We have that

$$c_{\{x,y\}} \leq C\|x-y\|^{-2d} =: \tilde{c}_{\{x,y\}}.$$

Thus, using Rayleigh's monotonicity principle [79, Chapter 2.4], it suffices to show that the network defined through the conductances $\left(\tilde{c}_{\{x,y\}}\right)_{x,y\in\mathbb{Z}^d, x\neq y}$ is recurrent. Define $\lambda := \sum_{x\in\mathbb{Z}^d\setminus\{0\}} C\|x\|^{-2d} = \sum_{x\in\mathbb{Z}^d\setminus\{0\}} \tilde{c}_{\{0,x\}}$. Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{P}(X_1 = x) = \lambda^{-1}C\|x\|^{-2d}$ for $x \in \mathbb{Z}^d \setminus \{0\}$. Such random variable exists as

$$\sum_{x\in\mathbb{Z}^d\setminus\{0\}} \lambda^{-1}C\|x\|^{-2d} = 1$$

by the definition of λ. Then the random walk $S_n = \sum_{k=1}^n X_k$ has exactly the same distribution as a random walk started at 0 on the network defined by the conductances

$\left(\tilde{c}_{\{x,y\}}\right)_{x,y}$. Together with Theorem 15.2 this shows that the random walk on the network defined by $\left(\tilde{c}_{\{x,y\}}\right)_{x,y}$ is recurrent and, as argued before, this also shows that the random walk on the network defined by $\left(c_{\{x,y\}}\right)_{x,y}$ is recurrent. The proof of the transience for the case where $c_{\{x,y\}} \geq c\|x-y\|^{-s}$ for some $c > 0$ and $s < 2d$ works analogous and we omit it. $\square$

After seeing the connection between the electrical networks and the random walk $S_n = \sum_{k=1}^{n} X_k$, we are ready to go to the proof of Theorem 15.1.

16.1 The proof of Theorem 15.1

Proof of Theorem 15.1. We iteratively define disjoint boxes $A_0, A_1, \ldots$ as follows. Let $a_0 = b_0 = 0$ and define a_k and b_k iteratively by $a_{k+1} = b_k + 2^{k+1}$, and $b_{k+1} = b_k + 2 \cdot 2^{k+1} - 1 = a_{k+1} + 2^{k+1} - 1$. Then define the box $A_k \coloneqq \{a_k, \ldots, b_k\} \times \{0, \ldots, 2^k - 1\}^{d-1}$. The resulting sets A_k are disjoint for different k, and they are boxes of side length 2^k, thus containing 2^{kd} elements. We now construct a flow between the different boxes as follows. For k large enough, say for $k \geq K$, we have $c_{\{x,y\}} \geq c\|x-y\|^{-s} \geq c' 2^{-ks}$ for all $x \in A_k, y \in A_{k+1}$, where c' is a constant that does not depend on k. So we consider the flow that starts uniformly distributed over A_k and each node $x \in A_k$ distributes its incoming flow uniformly to A_{k+1}, i.e., it sends a flow of strength $\frac{1}{|A_k|}\frac{1}{|A_{k+1}|}$ to each node $y \in A_{k+1}$. The incoming flow in A_{k+1} is again uniformly distributed over the box. As this is only possible for $k \geq K$, we need to send an initial flow to A_K. For this, we simply consider a unit flow 0 to A_K that distributes uniformly over A_K, i.e., each vertex in A_K receives a flow of $\frac{1}{|A_K|}$, and all edges used by this unit flow are in a finite box. Concatenating the described flows clearly gives a unit flow θ from 0 to infinity, from which we now want to estimate the energy. We are only interested in whether its energy is finite or infinite, and thus it suffices to consider the energy that is generated by the flows between A_k and A_{k+1} for large enough k. For one pair of boxes A_k, A_{k+1} with $k \geq K$ there exist constants $C, C' < \infty$ such that

$$\begin{aligned}
\sum_{x \in A_k} \sum_{y \in A_{k+1}} \frac{\theta(x,y)^2}{c_{\{x,y\}}} &\leq \sum_{x \in A_k} \sum_{y \in A_{k+1}} \frac{\left(|A_k| \cdot |A_{k+1}|\right)^{-2}}{c\|x-y\|^{-s}} \\
&\leq \sum_{x \in A_k} \sum_{y \in A_{k+1}} C 2^{-4kd} 2^{ks} \leq C' 2^{-2kd} 2^{ks} = C' 2^{k(s-2d)}.
\end{aligned}$$

Using that $s < 2d$ we can now see that

$$\sum_{k=K}^{\infty} \sum_{x \in A_k} \sum_{y \in A_{k+1}} \frac{\theta(x,y)^2}{c_{\{x,y\}}} \leq \sum_{k=K}^{\infty} C' 2^{k(s-2d)} < \infty$$

which shows that θ is a flow of finite energy and thus shows the transience of the random walk. $\square$

16.2 The proof of Theorem 15.2 for $d = 1$

Proof of Theorem 15.2 for $d = 1$. The main strategy of this proof is to compare the discrete random walk to the sum of independent Cauchy random variables. We assumed that $c_{\{x,y\}} \leq C\|x-y\|^{-2}$ for $x, y \in \mathbb{Z}$. First, we define different weights $\tilde{c}_{\{x,y\}}$ as follows. For

$x = 0$ and $y \neq 0$ we define $\tilde{c}_{\{x,y\}} = \int_{|y|-1}^{|y|} \frac{1}{1+s^2} \mathrm{d}s$. For $x \neq 0$, we define $\tilde{c}_{\{x,y\}}$ accordingly by translation, i.e.,

$$\tilde{c}_{\{x,y\}} = \tilde{c}_{\{0,y-x\}} = \int_{|y-x|-1}^{|y-x|} \frac{1}{1+s^2} \mathrm{d}s.$$

As we started with the assumption $c_{\{x,y\}} \leq C\|x-y\|^{-2}$, we also have that $c_{\{x,y\}} \leq \lambda \tilde{c}_{\{x,y\}}$ for a constant λ large enough and all $x \neq y$. Thus, by Rayleigh's monotonicity principle [79, Chapter 2.4], it suffices to show that the network defined by the conductances $\left(\lambda \tilde{c}_{\{x,y\}}\right)_{x,y \in \mathbb{Z}, x \neq y}$ is recurrent. Multiplying every conductance by a constant factor does not change whether the network is recurrent or transient, and thus it suffices to show that the network defined by the conductances $\left(\tilde{c}_{\{x,y\}}\right)_{x,y \in \mathbb{Z}, x \neq y}$ is recurrent. For this, let $Y_1, Y_2, \ldots$ be i.i.d. Cauchy-random variables and define $X_k' = \operatorname{sgn}(Y_k) \lceil |Y_k| \rceil$. Then X_k' has the distribution of one step of the random walk on the network defined by $\left(\tilde{c}_{\{x,y\}}\right)_{x,y \in \mathbb{Z}, x \neq y}$, and by independence $S_n' = \sum_{k=1}^n X_k'$ has exactly the same distribution as the random walk on the network defined by $\left(\tilde{c}_{\{x,y\}}\right)_{x,y \in \mathbb{Z}}$. Furthermore, we define $R_k = Y_k - X_k'$. Clearly, $R_1, R_2, \ldots$ are i.i.d. random variables that are bounded by 1 and thus we have that

$$\left| \sum_{k=1}^n R_k \right| \leq n. \tag{174}$$

By the stableness of the Cauchy-distribution we furthermore have that

$$\mathbb{P}\left(\left| \sum_{k=1}^n Y_k \right| > 5n \right) = \mathbb{P}\left(|Y_1| > 5\right) = 2 \int_5^\infty \frac{1}{\pi(1+s^2)} \mathrm{d}s \leq \int_5^\infty \frac{1}{s^2} \mathrm{d}s = \frac{1}{5}. \tag{175}$$

Now remember that $S_n' = \sum_{k=1}^n X_k' = \sum_{k=1}^n Y_k - \sum_{k=1}^n R_k$. Combining (174) and (175) gives

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{k=1}^n X_k' \right| \leq 6n \right) &= 1 - \mathbb{P}\left(\left| \sum_{k=1}^n X_k' \right| > 6n \right) \\ &\geq 1 - \mathbb{P}\left(\left| \sum_{k=1}^n R_k \right| > n \right) - \mathbb{P}\left(\left| \sum_{k=1}^n Y_k \right| > 5n \right) \geq 0.8. \end{aligned}$$

Thus, there needs to exist a point $x \in \{-6n, \ldots, 6n\}$ with

$$\mathbb{P}\left(\sum_{k=1}^n X_k' = x \right) \geq \frac{0.8}{|\{-6n, \ldots, 6n\}|} = \frac{0.8}{12n+1}.$$

However, for n even, the $x \in \mathbb{Z}$ that maximizes $\mathbb{P}\left(\sum_{k=1}^n X_k' = x\right)$ is 0. To see this, let ρ be the counting density of $\sum_{k=1}^{n/2} X_k'$. Using the symmetry of ρ (which is inherited from the symmetry of X_i') and a convolution, we see that

$$\begin{aligned} \mathbb{P}\left(\sum_{k=1}^n X_k' = x \right) &= \sum_{k \in \mathbb{Z}} \rho(k)\rho(x-k) \leq \sqrt{\sum_{k \in \mathbb{Z}} \rho(k)^2} \sqrt{\sum_{k \in \mathbb{Z}} \rho(x-k)^2} = \sum_{k \in \mathbb{Z}} \rho(k)^2 \\ &= \sum_{k \in \mathbb{Z}} \rho(k)\rho(-k) = \mathbb{P}\left(\sum_{k=1}^n X_k' = 0 \right) \end{aligned}$$

where we used the Cauchy-Schwarz inequality for the inequality. So in particular, for n even, we have that

$$\mathbb{P}\left(\sum_{k=1}^{n} X_k' = 0\right) \geq \frac{0.8}{12n+1}.$$

Summing this over all even n we get that $\sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{n} X_k' = 0\right) = \infty$, which implies the recurrence of the random walk $S_n' = \sum_{k=1}^{n} X_k'$. As discussed above, this already implies the recurrence of the random walk S_n. □

16.3 The proof of Theorem 15.2 for $d=2$

The proof of Theorem 15.2 for $d=2$ is a direct consequence of Lemma 16.9 and Lemma 16.10 below. But before going to these, we need to introduce several intermediary statements. The first one, Lemma 16.2, is taken from [16, Theorem 3.9]. It has the slight modification that we want that the distribution is the same for all edges with a fixed orientation, whereas [16, Theorem 3.9] does not take into account different orientations (The precise definition of orientation is given in Notation 16.4 below). However, the exact same proof as in [16] also works in our situation and we omit it. We say that a distribution μ *has a Cauchy tail* if there exists a constant C such that

$$\mu\left([Ct, \infty)\right) \leq Ct^{-1} \text{ for all } t > 0. \tag{176}$$

Note that in order to determine whether a distribution μ has a Cauchy tail, it suffices to check that condition (176) holds for all numbers t of the form $C' \cdot 3^j$ with a constant $C' \in \mathbb{R}_{>0}$ and $j \in \mathbb{N}$, instead of all $t > 0$. Our arguments will mostly use the symmetry of the nearest-neighbor bonds with respect to the ∞-norm. Therefore, we will always mean edges $\{x,y\}$ with $\|x-y\|_\infty = 1$ when speaking of nearest-neighbor or short-range edges in the following.

Lemma 16.2. *Let G be a random electrical network on the nearest-neighbor edges of the lattice $\mathbb{Z}^2$, i.e., the edges $\{\{x,y\} : \|x-y\|_\infty = 1\}$. Suppose that all the edges with the same orientation have the same conductance distribution, and this distribution has a Cauchy tail. Then almost all realizations of this random graph G are recurrent graphs.*

Before going to the formal details of the proof of Theorem 15.2, we want to explain the main ideas behind it. Assume that $c_{\{x,y\}}$ are conductances on $\mathbb{Z}^2$ with $c_{\{x,y\}} = \|x-y\|^{-2d}$, where $d=2$. If one has two disjoint boxes A, B of side length 3^k and with distance approximately 3^k, then one has $c_{\{x,y\}} \approx 3^{-4k}$ for all $x \in A$ and $y \in B$. An edge of conductance 3^{-4k} is equivalent to N edges in series with conductance $N \cdot 3^{-4k}$ each, where N is an arbitrary positive integer. In our construction, N will be of order 3^k. So the rough idea is to replace each edge $\{x,y\}$ with $\Theta\left(3^k\right)$ many edges of conductance $\Theta\left(3^{-3k}\right)$. By the parallel law, the conductivity of the network further increases if we erase these $\Theta\left(3^k\right)$ many edges in series of conductance $\Theta\left(3^{-3k}\right)$, and increase the conductances along a path $\gamma_{x,y}^k$ of length $\Theta\left(3^k\right)$ in the nearest-neighbor lattice by $\Theta\left(3^{-3k}\right)$. However, we will not do this independently for all $x \in A, y \in B$, but we want that for different points $x, x' \in A$ and $y, y' \in B$ the paths $\gamma_{x,y}^k$ and $\gamma_{x',y'}^k$ have an overlap that is relatively big. So far, we only looked at fixed $k \in \mathbb{N}$. We will do such a construction for all $k \in \mathbb{N}$. But at each k, we will also look at random, 3^k-periodic shifts of the plane. We use these uniform random shifts so that the distribution of the final conductance is the same for all edges of the same orientation. This construction will then lead to Cauchy tails for the individual conductances of the edges in the nearest-neighbor lattice, and thus, using Lemma 16.2, to the recurrence of the random walk on this network. The environment we started with is completeley deterministic, and the edge-weights arising through our construction are random just because of the random shifts of the plane. This also underlines that it is important for our construction to use random shifts, so that we can apply Lemma 16.2.

Next, we introduce some notation. We do this in order to partition the plane $\mathbb{Z}^2$ into boxes with side length 3^k.

Notation 16.3. *For a point $x = (x_1, x_2) \in \mathbb{Z}^2$ we write*

$$V_x^{3^k} = 3^k x + \{0, \dots, 3^k - 1\}^2 = \{x_1 3^k, \dots, x_1 3^k + 3^k - 1\} \times \{x_2 3^k, \dots, x_2 3^k + 3^k - 1\}$$

for the box with side length 3^k that is translated by $3^k x$. So in particular $\mathbb{Z}^2 = \bigsqcup_{x \in \mathbb{Z}^2} V_x^{3^k}$, where the symbol $\bigsqcup$ stands for a disjoint union. For $l \in \{0, \dots, k\}$, each box of side length 3^k can be written as the disjoint union of $3^{2(k-l)}$ boxes of side length 3^l. This union is simply given by

$$V_x^{3^k} = 3^k x + \{0, \dots, 3^k - 1\} = 3^k x + \bigsqcup_{y \in \{0, \dots, 3^{k-l}-1\}^2} V_y^{3^l}$$
$$= \bigsqcup_{y \in V_0^{3^{k-l}}} \left(3^k x + V_y^{3^l}\right).$$

For each point $x \in \mathbb{Z}^2$, there exists for all $l \geq 0$ a unique $y = y(l, x) \in \mathbb{Z}^2$ with $x \in V_{y(l,x)}^{3^l}$. For a point $x \in \mathbb{Z}^2$, let $m_l(x)$ be the midpoint of $V_{y(l,x)}^{3^l}$, i.e.,

$$m_l(x) = 3^l y(l, x) + \frac{3^l - 1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So in particular we have $m_0(x) = x$ for all $x \in \mathbb{Z}^2$. Also note that $m_l(x)$ and $m_{l+1}(x)$ can be the same point. A point $u \in \mathbb{Z}^2$ for which there exists a point $x \in \mathbb{Z}^2$ with $m_l(x) = u$ is also called a midpoint of the l-th level. Note that a block $V_a^{3^k}$ contains exactly $3^{2(k-l)}$ midpoints of the l-th level, for all $l \in \{0, \dots, k\}$.

Edges of the form $\{x, y\}$ with $x, y \in \mathbb{Z}^2, \|x - y\|_\infty = 1$ can have four different orientations: $\diagdown$, $\diagup$, $|$, and $-$. For an orientation $\vec{\nu} \in \{\diagdown, \diagup, |, -\}$, we write $E_{\vec{\nu}}\left(\mathbb{Z}^2\right)$ for all the short-range edges pointing in this direction in the integer lattice. We also want to make a tiling of $E_{\vec{\nu}}\left(\mathbb{Z}^2\right)$ with a given periodicity. We will simply decide on one tiling now. There are, of course, several other natural options, which come from a different inclusion on the boundary of the blocks V_a^N.

Notation 16.4. *For any $a \in \mathbb{Z}^2, N \in \mathbb{N}$, we define*

$$E_{\diagdown}\left(V_a^N\right) = \left\{\left\{x, x + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} : x \in V_a^N\right\},$$
$$E_{\diagup}\left(V_a^N\right) = \left\{\left\{x, x + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} : x \in V_a^N\right\},$$
$$E_{|}\left(V_a^N\right) = \left\{\left\{x, x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} : x \in V_a^N\right\},$$
$$E_{-}\left(V_a^N\right) = \left\{\left\{x, x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} : x \in V_a^N\right\}.$$

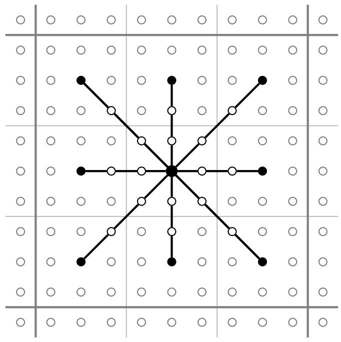


Figure 10: The gray lines between the vertices indicate the partitioning of the plane. The midpoints of the first and the second level are the filled vertices. The canonical shortest paths between the 8 midpoints of the first level and the midpoint of the second level are the 3 bold black edges between these points.

Note that for $x \in \mathbb{Z}^2$ and $l \in \mathbb{N}$, the midpoints $m_l(x)$ and $m_{l+1}(x)$ have either 0 or 3^l as distance in the ∞-metric, i.e., $\|m_l(x) - m_{l+1}(x)\|_\infty \in \{0, 3^l\}$. In the case where $\|m_l(x) - m_{l+1}(x)\|_\infty = 3^l$, there exists a path of length 3^l connecting $m_l(x)$ and $m_{l+1}(x)$ which uses edges $\{u, v\}$ with $\|u - v\|_\infty = 1$ only. Such a path is in general not unique, but it is unique if we make the further restriction that the path uses 3^l edges of the same orientation. So the resulting path, which we refer to as the *canonical shortest path*, is the path that connects $m_l(x)$ and $m_{l+1}(x)$ using the straight line between these two points. Examples of canonical shortest paths are given in Figure 10.

Next, we define a set of paths. We want to define a path $\gamma_{x,y}^k$ for all $x, y \in \mathbb{Z}^2$ for which there exist $a, b \in \mathbb{Z}^2$ with $\|a-b\|_\infty \in \{2, \dots, 7\}$, such that $x \in 3^k a + \{0, \dots, 3^k - 1\}^2 = V_a^{3^k}$ and $y \in 3^k b + \{0, \dots, 3^k - 1\}^2 = V_b^{3^k}$. The path $\gamma_{x,y}^k$ defined below is adopted to the renormalization with scale 3, as it uses this iterative structure. Whenever x, y are not of the form as described above, we simply say that the path $\gamma_{x,y}^k$ does not exist. A picture of our construction is given in Figure 12.

Definition 16.5. *Let $a, b \in \mathbb{Z}^2$ with $\|a - b\|_\infty \in \{2, \dots, 7\}$, and let $x \in 3^k a + \{0, \dots, 3^k - 1\}^2 = V_a^{3^k}$ and $y \in 3^k b + \{0, \dots, 3^k - 1\}^2 = V_b^{3^k}$. We define the path $\gamma_{x,y}^k$ as the path that goes from $x = m_0(x)$ to $m_1(x)$ following the canonical shortest path and from there to $m_2(x)$ following the canonical shortest path and from there, iteratively, following the canonical shortest paths, to $m_k(x)$. From there, the path goes in a deterministic way to $m_k(y)$ and from there iteratively, following the canonical shortest paths, to $m_0(y) = y$. For the path between $m_k(x)$ and $m_k(y)$ we follow the line sketched in Figure 11.*

The paths $\gamma_{x,y}^k$ are no simple paths or shortest paths. In particular, they can go several times over the same edge. Also note that we do *not* have $\gamma_{x,y}^k = \gamma_{y,x}^k$, in general. This is because the path chosen between $m_k(x)$ and $m_k(y)$ is not necessarily the same path, see Figure 11. However, the paths $\gamma_{x,y}^k$ can not be too long. The ∞-distance between the points $m_k(x)$ and $m_k(y)$ is at most $7 \cdot 3^k$, and for $l+1 \leq k$ one has $\|m_l(x) - m_{l+1}(x)\|_\infty \in \{0, 3^l\}$, and the same statement also holds for y instead of x. Writing $|\gamma_{x,y}^k|$ for the length of the

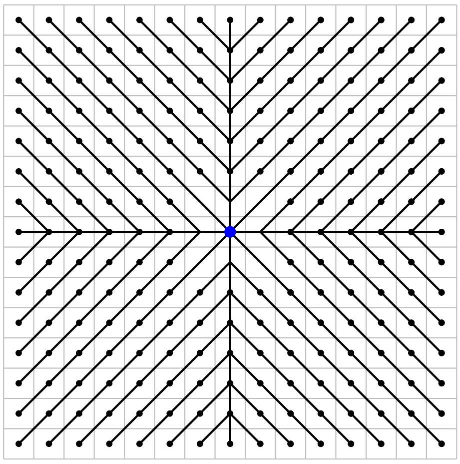


Figure 11: The midpoints of boxes of side length 3^k are the dots. The partition of the lattice into blocks of side length 3^k is marked in gray. The path between the midpoint $m_k(x)$ (the blue dot) and a different midpoint $m_k(y)$ in a different box (a black dot) is obtained by following the black line.

path $\gamma_{x,y}^k$, we thus get that

$$|\gamma_{x,y}^k| \leq 7 \cdot 3^k + 2 \sum_{l=0}^{k-1} 3^l \leq 10 \cdot 3^k. \tag{177}$$

Consider the set of paths $\gamma_{x,y}^k$ over all suitable points $x, y \in \mathbb{Z}^2$. We want to bound the number of edges that lie in N or more paths $\gamma_{x,y}^k$. We say that an edge $e = \{u, v\}$ is in the path $\gamma = (x_0, \ldots, x_n)$, abbreviated by $e \in \gamma$, if $(u, v) = (x_i, x_i + 1)$ or $(v, u) = (x_i, x_i + 1)$ for an $i \in \{0, \ldots, n-1\}$. We first focus on the structure of the paths inside of one box $A = V_a^{3^k} = 3^k a + \{0, \ldots, 3^k - 1\}$. For each $l \in \{0, \ldots, k\}$, there are $3^{2(k-l)}$ midpoints of the l-th level inside A, i.e., points $y \in A$ such that $y = m_l(x)$ for a point $x \in A$. Thus there are $3^{2(k-l-1)}$ midpoints of the form $m_{l+1}(x)$ in A. Each box of side length 3^{l+1} contains 9 boxes of side length 3^l. Thus, there are $8 \cdot 3^l 3^{2(k-l-1)} \leq 3^{2k-l+1}$ edges in A that are on the canonical shortest path between two midpoints of the form $m_l(x)$ and $m_{l+1}(x)$. The factor 8 arises, as for one box of side length 3^{l+1} with midpoint z we only need to consider the $8 = 3^2 - 1$ boxes of side length 3^l that lie inside this box but do not have z as a midpoint. Edges that do not lie on the canonical shortest path between two midpoints of any level are not used in the segments that connect an $x \in A$ to $m(A)$, where $m(A)$ is the midpoint of A. Furthermore, for two boxes $V_a^{3^k}$ and $V_b^{3^k}$ with $\|a - b\|_\infty \leq 7$, there are at most $7 \cdot 3^k$ edges that are on the path between the midpoints of $V_a^{3^k}$ and $V_b^{3^k}$. Many of the edges in this path lie actually outside of both the boxes $V_a^{3^k}$ and $V_b^{3^k}$.

Lemma 16.6. *For each short-range edge e we define the number N_e^k by*

$$N_e^k = \left| \left\{ (x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : e \in \gamma_{x,y}^k \right\} \right|$$

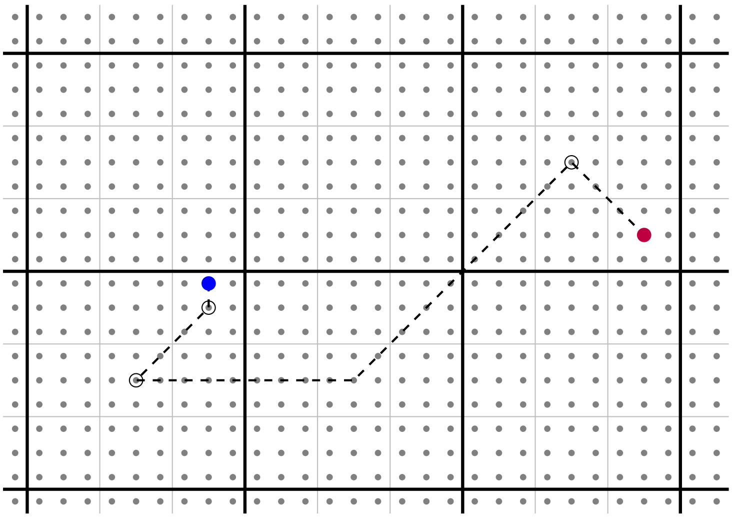


Figure 12: The dashed line is the path $\gamma_{x,y}^2$ between the points x (blue) and y (red). The dots are points in $\mathbb{Z}^2$, the gray lines give the partition of $\mathbb{Z}^2$ into sets of the form V_a^3, and the thick black lines give the partition of $\mathbb{Z}^2$ into sets V_a^9. The encircled points are the points $m_1(x), m_2(x)$, and $m_2(y)$. Note that we have $y = m_0(y) = m_1(y)$ here.

which is just the number of paths of the form $\gamma_{x,y}^k$ that use the edge e. Remember that we defined the path $\gamma_{x,y}^k$ only for points x, y satisfying $x \in V_a^{3^k}, y \in V_b^{3^k}$ for some $a, b \in \mathbb{Z}^2$ with $\|a-b\|_\infty \in \{2, \ldots, 7\}$. So in particular for all edges e we have that $e \notin \gamma_{x,y}^k$ for all points x, y that are not of the form as described above. For a number $r \geq 0$ and an orientation $\overrightarrow{\nu} \in \{\diagdown, \diagup, |, -\}$ we define

$$X_{\geq r}^{k,\overrightarrow{\nu}} = \left| \left\{ e \in E_{\overrightarrow{\nu}} \left(V_0^{3^k} \right) : N_e^k \geq r \right\} \right|$$

which is the number of edges in $E_{\overrightarrow{\nu}}\left(V_0^{3^k}\right)$ that lie in at least r different paths of the form $\gamma_{x,y}^k$. Then for any $l \leq k-1$ one has

$$X_{\geq 50 \cdot 3^{2k+2l}}^{k,\overrightarrow{\nu}} \leq 3^{2k-l+1} + 3^k \leq 3^{2k-l+2} \tag{178}$$

and furthermore, one has

$$X_{\geq 2^{17} \cdot 3^{4k}}^{k,\overrightarrow{\nu}} = 0. \tag{179}$$

Proof. Suppose that an edge e is *not* on the straight line between two midpoints of the l-th level and the $(l+1)$-th level in the set $V_0^{3^k}$, and also *not* on the path between two midpoints $m\left(V_a^{3^k}\right)$ and $m\left(V_b^{3^k}\right)$ for $a, b \in \mathbb{Z}^2$ with $\|a-b\|_\infty \in \{2, \ldots, 7\}$. So the edge e can only be on the straight line between midpoints of the j-th level and the $(j+1)$-th level, for $j \leq l-1$. Thus, there exists a set $V_{f(e)}^{3^{l-1}} \subset V_0^{3^k}$ such that e can only be part of paths of the form $\gamma_{x,y}^k$ where $x \in V_{f(e)}^{3^{l-1}}$ or $y \in V_{f(e)}^{3^{l-1}}$. There are $(2\cdot 7+1)^2 - 9 = 216$ many $a \in \mathbb{Z}^2$ with $2 \leq \|a\|_\infty \leq 7$. Thus, there are at most $216 \cdot 3^{2(l-1)} 3^{2k} < 25 \cdot 3^{2k+2l}$ pairs

(x,y) with $x \in V_{f(e)}^{3^{l-1}}$ and $y \in \bigcup_{a\in\mathbb{Z}^2 : 2 \leq \|a\|_\infty \leq 7} V_a^{3^k}$. Using symmetry between x and y we get that $N_e^k < 50 \cdot 3^{2k+2l}$.

This shows that edges e with $N_e^k \geq 50 \cdot 3^{2k+2l}$ are *either* on the canonical path between two midpoints of the l-th level and the $(l+1)$-th level in the set $V_0^{3^k}$, *or* on the path between two midpoints $m\left(V_a^{3^k}\right)$ and $m\left(V_b^{3^k}\right)$ for $a,b \in \mathbb{Z}^2$ with $\|a-b\|_\infty \in \{2,\dots,7\}$. As discussed before, in the set $V_0^{3^k}$, there are at most 3^{2k-l+1} edges that join a midpoint of the l-th level to a midpoint of the $(l+1)$-th level. For each orientation, there are 3^k edges that are used by paths between different midpoints. For the orientation $\diagup$, for example, this are simply the edges of the form $\left\{ \begin{pmatrix} s \\ s \end{pmatrix}, \begin{pmatrix} s+1 \\ s+1 \end{pmatrix} \right\}$ with $s \in \{0,\dots,3^k-1\}$. Thus we have

$$X^{k,\vec{\nu}}_{\geq 50\cdot 3^{2k+2l}} \leq 3^{2k-l+1} + 3^k \leq 3^{2k-l+2} \tag{180}$$

which shows (178). Note that the last inequality in (180) holds because $l \leq k$. Furthermore, for each edge e there are at most $\left((2\cdot 7+1)^2 3^{2k}\right)^2 < 2^{17} 3^{4k}$ pairs (x,y) such that $\gamma_{x,y}^k$ is defined and for which $e \in \gamma_{x,y}^k$ is possible. This holds, as for every path $\gamma_{x,y}^k$ that uses one of the edges in $E_{\vec{\nu}}\left(V_0^{3^k}\right)$, say for $x \in V_a^{3^k}$ and $y \in V_b^{3^k}$, we already must have $\|a\|_\infty, \|b\|_\infty \leq 7$. This gives us that

$$X^{k,\vec{\nu}}_{\geq 2^{17}\cdot 3^{4k}} = 0 \tag{181}$$

which finishes the proof. □

We are now ready to go to the proof of the recurrence of the network. Remember that we started with conductances $c_{\{x,y\}}$ satisfying $c_{\{x,y\}} \leq C\|x-y\|_\infty^{-4}$ for a uniform constant $0 < C < \infty$. For two networks $\left(c_{\{x,y\}}\right)_{x,y\in\mathbb{Z}^d}$ and $\left(\tilde{c}_{\{x,y\}}\right)_{x,y\in\mathbb{Z}^d}$ we say that the first network has a higher conductivity than the second network if the effective conductances satisfy $\mathscr{C}_{\text{eff}}(A \leftrightarrow B) \geq \tilde{\mathscr{C}}_{\text{eff}}(A \leftrightarrow B)$ for all sets $A, B \subset \mathbb{Z}^d$. Taking $A = \{0\}$ and $B = \mathbb{Z}^d \setminus \{-n,\dots,n\}^d$, and letting n to ∞, this shows that if the network defined by $c_{\{x,y\}}$ is recurrent, then the network defined by $\tilde{c}_{\{x,y\}}$ is also recurrent. By Rayleigh's monotonicity principle [79, Chapter 2.4], the conductivity of the network increases if we increase the conductance of edges. Thus, it suffices to show that the network defined by the conductances $c_{\{x,y\}} = C\|x-y\|_\infty^{-4}$ is recurrent. However, multiplying every conductance of each edge by a constant factor does not change whether the network is recurrent or transient. Thus, we will, from now on, focus on the case where

$$c_{\{x,y\}} = \frac{1}{\|x-y\|_\infty^4} \text{ for all } x,y \in \mathbb{Z}^2, x \neq y.$$

Following an idea of Berger [16], our strategy is that we erase the long edges and give a higher conductance to the short edges instead, in such a way that the total conductivity increases. The way in which this is done in [16] does not work in the situation we are dealing with. The precise way in which we do this is described in Definition 16.7 for edges of length $2,3,\dots,8$, and in Definition 16.8 for edges of length 9 and higher (where the length of an edge is measured in the ∞-distance of its endpoints). Some edges might appear several times, but if we increase the conductances twice for one edge, then it only increases the total conductivity of the network. Before going to these definitions, we need to introduce a bit more notation.

For a path $\gamma = (x_0, x_1, \dots, x_n)$ and a point $r \in \mathbb{Z}^2$, we define the path $r + \gamma = (r + x_0, r + x_1, \dots, r + x_n)$, which is now a path between $r + x_0$ and $r + x_n$. Note that for three points $x, y, r \in \mathbb{Z}^2$, and $k \in \mathbb{N}$, for which the path $\gamma^k_{x+r,y+r}$ exists, the path $-r + \gamma^k_{x+r,y+r}$ is actually a path between x and y. Also remember that we write $E(\mathbb{Z}^2) = \big\{\{x, y\} \subset \mathbb{Z}^2 : \|x - y\|_\infty = 1\big\}$ for the edge set consisting of short edges on $\mathbb{Z}^2$.

Definition 16.7. *For two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{Z}^2$, we define the path $\gamma'_{x,y}$ as the path that goes from x to (x_1, y_2) using $|x_2 - y_2|$ edges of the orientation $|$, and from there to (y_1, y_2) using $|x_1 - y_1|$ edges of the orientation $-$. This path is uniquely defined and has length $\|x-y\|_1 \leq 2\|x-y\|_\infty$. We now define a weight $W : E(\mathbb{Z}^2) \to [0, \infty)$ as follows. Start with $W \equiv 0$. Now, for each pair $(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with $2 \leq \|x - y\|_\infty \leq 8$, increase $W(e)$ for all edges $e \in \gamma'_{x,y}$ by 16. Define W as the limiting object.*

Definition 16.8. *We now define a weight $U_k : E(\mathbb{Z}^2) \to [0, \infty)$ as follows. Start with $U_k \equiv 0$. Choose $r_k \in \{0, \dots, 3^k - 1\}^2$ uniformly at random. Now, for each pair $(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ for which there exist $a, b \in \mathbb{Z}^2$ with $2 \leq \|a-b\|_\infty \leq 7$ with $x+r_k \in V_a^{3^k}, y+r_k \in V_b^{3^k}$, increase $U_k(e)$ for all edges $e \in -r_k + \gamma^k_{x+r_k,y+r_k}$ by $10 \cdot 3^{-3k}$. Define U_k as the limiting object.*

Note that U_k and W are well-defined and do not depend on the order of the exhaustion of $\mathbb{Z}^2 \times \mathbb{Z}^2$, as we only add a non-negative amount at every step, and never subtract anything. Next, we want to show that the nearest-neighbor network $\big(\mathbb{Z}^2, E(\mathbb{Z}^2), U\big)$ defined by $U = W + \sum_{k=1}^\infty U_k$ has a higher conductivity than the original network. Note that we can define $U = W + \sum_{k=1}^\infty U_k$ also directly by increasing the conductances along all suitable paths $\gamma'_{x,y}$ or $\gamma^k_{x,y}$ by the corresponding value and then look at the limiting object.

Lemma 16.9. *The network defined by the weights $U(e) = W(e) + \sum_{k=1}^\infty U_k(e)$ has a higher conductivity than the network defined by the weights*

$$c_{\{x,y\}} = \frac{1}{\|x - y\|_\infty^4} \text{ for all } x, y \in \mathbb{Z}^2, x \neq y. \tag{182}$$

Proof. A non-nearest-neighbor edge $e = \{u, v\}$ is not included in the network defined by U. However, we have increased the conductances along some path connecting u and v, when we consider the sum $W + \sum_{k=1}^\infty U_k$. In the following, we will show that for each edge $e = \{u, v\}$, the conductances indeed were increased at least once along a nearest-neighbor path connecting u and v, and this increase of the conductances of the short edges actually increased the total conductivity of the network. A similar argument for the latter claim was also used in [16]. Assume that $e = \{u, v\}$ is an edge with length at least 9, and let $k \in \{2, 3, \dots\}$ be such that $3^k \leq \|u - v\|_\infty < 3^{k+1}$. Say that $u + r_{k-1} \in V_a^{3^{k-1}}, v+r_{k-1} \in V_b^{3^{k-1}}$. If $2 \leq \|a-b\|_\infty \leq 7$, we deleted the edge $\{u, v\}$ (with conductance $\|u-v\|_\infty^{-4} \leq 3^{-4k}$), but increased the conductance of nearest-neighbor edges along the path $-r_{k-1} + \gamma^{k-1}_{x+r_{k-1},y+r_{k-1}}$ by $10 \cdot 3^{-3(k-1)}$. The path $-r_{k-1} + \gamma^{k-1}_{x+r_{k-1},y+r_{k-1}}$ has a length of at most $10 \cdot 3^{k-1}$ by (177), and thus we increased the total conductivity of the network. To see this, assume we have a nearest-neighbor path of length $N = 10 \cdot 3^{k-1}$ connecting u and v. The edge $\{u, v\}$ is actually equivalent to a string of N edges in series, each with conductance $Nc_{\{u,v\}}$. Identifying the vertices in this string with the vertices in the original path in the nearest-neighbor lattice can only increase the conductivity of the network. Then applying the parallel law with the edges in the original lattice and the newly formed edges is equivalent to adding a conductance of $Nc_{\{u,v\}}$ to each edge in the path connecting u and v. As $Nc_{\{u,v\}} \leq 10 \cdot 3^{k-1} 3^{-4k} \leq 10 \cdot 3^{-3(k-1)}$, this increased the total conductivity of the network.

If u, v with $3^k \leq \|u-v\|_\infty < 3^{k+1}$ are not such that $u+r_{k-1} \in V_a^{3^{k-1}}, v+r_{k-1} \in V_b^{3^{k-1}}$ with $a, b \in \mathbb{Z}^2$ and $2 \leq \|a-b\|_\infty \leq 7$, we already must have that $\|u-v\|_\infty > 6 \cdot 3^{k-1} = 2 \cdot 3^k$. Thus, there exist $a', b' \in \mathbb{Z}^2$ with $2 \leq \|a'-b'\|_\infty \leq 7$ such that $u+r_k \in V_{a'}^{3^k}, v+r_k \in V_{b'}^{3^k}$. The same argument as before shows that we also increased the total conductivity in this case.

For edges $e = \{u, v\}$ with $\|u-v\|_\infty \leq 8$ we increase the conductances of the short edges along the path $\gamma'_{x,y}$ by 16. As $\gamma'_{x,y}$ has a length of $\|x-y\|_1 \leq 16$, we also increased the conductivity of the network for this case. □

Lemma 16.10. *Fix an orientation* $\vec{\nu} \in \{\diagdown, \diagup, |, -\}$. *Then for all edges e of this orientation, $U(e)$ is identically distributed and has a Cauchy tail. Thus, by Lemma 16.2, the random walk on the network $\left(\mathbb{Z}^2, E(\mathbb{Z}^2), U\right)$ is almost surely recurrent.*

Proof. As $W, U_1, U_2, \dots$ are independent, it suffices to show that the distribution of $W(e)$, respectively $U_k(e)$, depends only on the orientation of the edge e. This is clear for W, as the value $W(e)$ depends only on the orientation of the edge e. Remember that we say that $\gamma^k_{x+r_k, y+r_k}$ exists, when $x + r_k \in V_a^{3^k}, y + r_k \in V_b^{3^k}$ for $a, b \in \mathbb{Z}^2$ with $2 \leq \|a-b\|_\infty \leq 7$. For U_k, note that $U_k(e)$ depends only on the number of pairs (x, y) for which $e \in -r_k + \gamma^k_{x+r_k, y+r_k}$, and for which $\gamma^k_{x+r_k, y+r_k}$ exists. More precisely, $U_k(e)$ is simply $10 \cdot 3^{-3k}$ times the number of pairs (x, y) for which $e \in -r_k + \gamma^k_{x+r_k, y+r_k}$, and for which $\gamma^k_{x+r_k, y+r_k}$ exists. However, we have that

$$\left|\left\{(x,y) : e \in -r_k + \gamma^k_{x+r_k, y+r_k}\right\}\right| = \left|\left\{(x,y) : e + r_k \in \gamma^k_{x+r_k, y+r_k}\right\}\right| \\ = \left|\left\{(x,y) : e + r_k \in \gamma^k_{x,y}\right\}\right| = N^k_{e+r_k}, \tag{183}$$

where we write $\{u, v\} + r_k = \{u + r_k, v + r_k\}$ for an edge $e = \{u, v\}$. The quantity N_e^k is clearly 3^k-periodic in both coordinate directions. As r_k is uniformly chosen on $\{0, \dots, 3^k - 1\}^2$, we see that the distribution of $N^k_{e+r_k}$, and thus also of $U_k(e)$, depends only on the orientation of the edge e.

Now let us turn to the tail properties of the random variable $U(e)$. $W(e)$ is uniformly bounded over all e, so we can ignore it from here on. From (179) and (183) we get that there exists a uniform constant $C < \infty$ such that

$$U_k(e) = N^k_{e+r_k} \cdot \left(10 \cdot 3^{-3k}\right) \leq C 3^k$$

and for $l \in \{0, \dots, k-1\}$ we get with (178) that

$$\mathbb{P}\left(U_k(e) \geq 500 \cdot 3^{2l-k}\right) = \mathbb{P}\left(N^k_{e+r_k} \geq 50 \cdot 3^{2l+2k}\right) \leq \frac{3^{2k-l+2}}{3^{2k}} = 3^{-l+2},$$

where we used the uniform distribution of r_k and (178) for the last inequality. Using $j = 2l - k$ and solving this for $l = \frac{k+j}{2}$, we get that there exists a constant $C < \infty$ such that for all $j \in \{-k, -k+2, \dots, k-2\}$

$$\mathbb{P}\left(U_k(e) \geq 500 \cdot 3^j\right) \leq C 3^{-\frac{k+j}{2}}. \tag{184}$$

We want to extend this inequality from $j \in \{-k, -k+2, \dots, k-2\}$ to $j \in \{-k, -k+2, \dots, k-2\}$. The extension from $j \in \{-k, -k+2, \dots, k-2\}$ to $j \in [-k, k]$ is easily doable by increasing the constant C and looking at the nearest integers in the set $\{-k, -k+$

$2,\dots,k-2\}$. For $j<-k$ and $C\geq 1$ there is nothing to show, so (184) holds trivially in this regime. Furthermore one has

$$\mathbb{P}\left(U_k(e) > 2^{17}10\cdot 3^k\right) = \mathbb{P}\left(N_{e+r_k}^k > 2^{17}3^{4k}\right) \overset{(179)}{=} 0$$

which shows that (184) also holds for $j \geq k$ and a large enough constant C. Finally, as inequality (184) holds for all $j \in \mathbb{R}$ with a high enough constant C, by further increasing the constant we can make sure that

$$\mathbb{P}\left(U_k(e) \geq 3^j\right) \leq C 3^{-\frac{k+j}{2}}. \tag{185}$$

for all $j \in \mathbb{R}$. Also note that for $j \ll k$ inequality (185) gives that $\mathbb{P}\left(U_k(e)\geq 3^j\right) \leq C3^{-\frac{k+j}{2}} \ll 3^{-j}$. We want to use this observation in order to show that $\sum_{k=1}^{\infty} U_k(e)$ has a Cauchy tail. Note that if we have $U_k(e) \leq 3^{j+\frac{j-k}{2}}$ for all $k \geq j \in \mathbb{N}$, then we also have that

$$\sum_{k=j}^{\infty} U_k(e) \leq \sum_{k=j}^{\infty} 3^{j+\frac{j-k}{2}} = 3^j \sum_{k=j}^{\infty} 3^{\frac{j-k}{2}} \leq 3^j \sum_{k=0}^{\infty} 3^{\frac{-k}{2}} \leq 3\cdot 3^j.$$

As we furthermore have $U_k(e) \leq C_1 3^k$ for a large enough constant C_1 and all $k \in \mathbb{N}$, we get that

$$\sum_{k=1}^{\infty} U_k(e) = \sum_{k=1}^{j-1} U_k(e) + \sum_{k=j}^{\infty} U_k(e) \leq \sum_{k=1}^{j-1} C_1 3^k + \sum_{k=j}^{\infty} 3^{j+\frac{j-k}{2}} \leq C_1 3^j + 3\cdot 3^j = C_2 3^j$$

for $C_2 = C_1 + 3$. Using the previous arguing in the reverse direction, we see that the event $\left\{\sum_{k=1}^{\infty} U_k(e) > C_2 3^j\right\}$ implies that there exists a $k \geq j$ with $U_k(e) > 3^{j+\frac{j-k}{2}}$. Using this observation and combining it with a union bound, we get that

$$\mathbb{P}\left(\sum_{k=1}^{\infty} U_k(e) > C_2 3^j\right) \leq \mathbb{P}\left(U_k(e) > 3^{j+\frac{j-k}{2}} \text{ for a } k \geq j\right) \leq \sum_{k=j}^{\infty} \mathbb{P}\left(U_k(e) > 3^{j+\frac{j-k}{2}}\right)$$
$$\overset{(185)}{\leq} \sum_{k=j}^{\infty} C 3^{-\frac{k+j+\frac{j-k}{2}}{2}} = C3^{-\frac{3}{4}j} \sum_{k=j}^{\infty} 3^{-\frac{k}{4}} = C3^{-\frac{3}{4}j} 3^{-\frac{j}{4}} \sum_{k=0}^{\infty} 3^{-\frac{k}{4}} \leq 5C\cdot 3^{-j}$$

which shows that $\sum_{k=1}^{\infty} U_k(e)$ has a Cauchy tail and thus finishes the proof. □

Remark 16.11. *Using the definition of U_k, one can easily show that $\mathbb{P}\left(U_k(e) \geq 3^k\right) \approx 3^{-k}$, so (185) is approximately an equality for $k=j$. This already implies that*

$$\mathbb{P}\left(\sum_{k=1}^{\infty} U_k(e) \geq 3^j\right) \geq \mathbb{P}\left(U_j(e) \geq 3^j\right) \approx 3^{-j}$$

which shows together with Lemma 16.10 that the tail of U is approximately that of a Cauchy distribution, i.e., $\mathbb{P}(U(e) > M) \approx M^{-1}$ for M large.

17 Random walks on percolation clusters

In this section, we prove Theorem 15.3, i.e., that random walks on certain percolation clusters are recurrent. In section 17.1 below we apply this result to the weight-dependent random connection model. From Theorem 15.3 we can deduce the following corollary.

Corollary 17.1. *Let $d \in \{1,2\}$ and let $(\mathbb{Z}^d, E, \omega)$ be the complete graph on $\mathbb{Z}^d$ where each edge $\{x,y\} \in E$ carries a random weight $\omega(\{x,y\})$ satsifying $\mathbb{E}[\omega(\{x,y\})] \leq C\|x-y\|^{-2d}$ for a constant $C < \infty$ and all pairs of points $x, y \in \mathbb{Z}^d$. Then the random walk on $(\mathbb{Z}^d, E, \omega)$ is recurrent almost surely.*

For dimension $d \in \{1,2\}$ and for the complete graph on $\mathbb{Z}^d$ with inclusion probabilities $c_{\{x,y\}} = \|x-y\|^{-2d}$ Corollary 17.1 extends a classical result of Berger on recurrence of the random walk on long-range percolation clusters [16, Theorem 1.4]. There are two differences between Corollary 17.1 and [16, Theorem 1.4]. The first is that [16, Theorem 1.4] only deals with the case where $\omega \in \{0,1\}^E$, whereas $\omega \in \mathbb{R}_{\geq 0}^E$ in our situation. The second difference is that Corollary 17.1 does *not* require that the inclusion of edges is independent, whereas [16, Theorem 1.4] requires independence. To deduce this corollary from Theorem 15.3, note that Theorem 15.2 (respectively Lemma 16.1) shows that the random walk on conductances $\left(c_{\{x,y\}}\right)_{x,y \in \mathbb{Z}^d}$ with $c_{\{x,y\}} \leq C\|x-y\|^{-2d}$ is recurrent in dimension $d \in \{1,2\}$. Theorem 15.3 thus implies that the random walk on a percolation cluster with weight distributions $\mathbb{E}[\omega(\{x,y\})] \leq C\|x-y\|^{-2d}$ is recurrent.

Theorem 15.3 will be a direct consequence of Lemma 17.2 below. For two disjoint finite sets $\emptyset \neq A, B \subset V$ we write $\mathscr{C}_{\text{eff}}(A \leftrightarrow B; \omega)$ for the effective conductance between these two sets in the environment ω, which is the environment in which each edge e has the conductance $\omega(e)$. Note that $\mathscr{C}_{\text{eff}}(A \leftrightarrow B; \omega)$ is a random variable that is measurable with respect to ω. We also write $\mathscr{C}_{\text{eff}}(A \leftrightarrow B)$ for the effective conductance between A and B in the environment where each edge e has conductance c_e. For a vertex $a \in V$ we simply write a for the set $\{a\}$. Furthermore, we write $\mathscr{C}_{\text{eff}}(a \leftrightarrow \infty)$ for the limit $\lim_{n\to\infty} \mathscr{C}_{\text{eff}}\left(a \leftrightarrow A_n^C\right)$, where $(A_n)_n$ is a sequence with $a \in A_n$ for all n and $A_n \nearrow V$.

Lemma 17.2. *Let $a \in V$ and let $\Lambda \subset V$ with $a \in \Lambda$ be a finite subset of V. Assume that $\mathbb{E}[\omega(e)] \leq c_e$ for all edges $e \in E$. Then*

$$\mathbb{E}\left[\mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda^C; \omega\right)\right] \leq \mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda^C\right). \tag{186}$$

Let us first see how this implies Theorem 15.3.

Proof of Theorem 15.3 given Lemma 17.2. Let $a \in V$ be a vertex. Our goal is to show that the random walk started at $a \in V$ is recurrent. Let $\varepsilon > 0$ be arbitrary. As the random walk on the conductances $\left(c_{\{x,y\}}\right)_{x,y \in V}$ is recurrent, there exists a finite set $\Lambda_\varepsilon \subset V$ such that $a \in \Lambda_\varepsilon$ and $\mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda_\varepsilon^C\right) < \varepsilon$. Lemma 17.2 already implies that

$$\mathbb{E}\left[\mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda_\varepsilon^C; \omega\right)\right] \leq \mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda_\varepsilon^C\right) < \varepsilon,$$

and as $\mathscr{C}_{\text{eff}}(a \leftrightarrow \infty; \omega) \leq \mathscr{C}_{\text{eff}}\left(a \leftrightarrow \Lambda_\varepsilon^C; \omega\right)$ this already gives that

$$\mathbb{E}\left[\mathscr{C}_{\text{eff}}(a \leftrightarrow \infty; \omega)\right] < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary and $\mathscr{C}_{\text{eff}}(a \leftrightarrow \infty; \omega)$ is a non-negative random variable this already implies that $\mathscr{C}_{\text{eff}}(a \leftrightarrow \infty; \omega) = 0$ almost surely, which is equivalent to saying that the random walk on the weights $(\omega(e))_{e \in E}$ started at $a \in V$ is recurrent almost surely. As $a \in V$ was arbitrary, this finishes the proof. □

Lemma 17.2 shows that the expected conductance always decreases if we say that an edge e with conductance $c_e > 0$ now carries a conductance of $\omega(e)$ with $\mathbb{E}[\omega(e)] \leq c_e$. This inequality might also be strict in many natural examples, despite the fact that the expected conductance over this edge stays the same. The reason why this inequality holds is ultimately linked to the fact that the effective conductance is a concave function over the individual conductances. In the proof of Lemma 17.2 below the concavity is used implicitly, as the infimum over a set of linear functions is a concave function.

Proof of Lemma 17.2. We use Dirichlet's principle for the effective conductance, see for example [79, Exercise 2.13]. It says that for two non-empty disjoint sets $A, B \subset V$ the effective conductance between these two sets can be expressed as

$$\mathscr{C}_{\text{eff}}(A \leftrightarrow B) = \inf_{f \in \mathcal{F}} \sum_{e \in E} c_e \left(df(e)\right)^2 ,$$

where $\mathcal{F}$ is the set of functions f from V to $\mathbb{R}$ that are $+1$ on A and 0 on B. For an edge $e = \{x, y\}$ we write $(df(e))^2 = (f(x) - f(y))^2$ for the squared difference of the values of f at the endpoints of the edge. This is well-defined, even without fixing an orientation for the edge. Dirichlet's principle also holds for $\mathscr{C}_{\text{eff}}(A \leftrightarrow B; \omega)$. Thus we get that

$$\begin{aligned}\mathbb{E}\left[\mathscr{C}_{\text{eff}}(A \leftrightarrow B; \omega)\right] &= \mathbb{E}\left[\inf_{f \in \mathcal{F}} \sum_{e \in E} \omega(e) \left(df(e)\right)^2\right] \leq \inf_{f \in \mathcal{F}} \mathbb{E}\left[\sum_{e \in E} \omega(e) \left(df(e)\right)^2\right] \\ &= \inf_{f \in \mathcal{F}} \sum_{e \in E} \mathbb{E}\left[\omega(e)\right] \left(df(e)\right)^2 \leq \inf_{f \in \mathcal{F}} \sum_{e \in E} c_e \left(df(e)\right)^2 = \mathscr{C}_{\text{eff}}(A \leftrightarrow B)\end{aligned}$$

where we can interchange the sum and the expectation as all summands are non-negative. The change of the infimum and the expectation is always allowed when putting the inequality. Using this inequality for $A = \{a\}$ and $B = \Lambda^C$ finishes the proof. □

17.1 Recurrence for the weight-dependent random connection model

In this section, we prove Theorem 15.4, i.e., different phases of recurrence for the two-dimensional weight-dependent random connection model. Our main tool for proving this is a comparison to *dependent* percolation on the two-dimensional integer lattice in Lemma 17.3 below. A slightly weaker statement was already proven in [49, Lemma 4.1], where the condition (187) needed to hold with $|x-y|^4$ replaced by $|x-y|^\alpha$ for some $\alpha > 4$. This improvement allows us to prove the results of Theorem 15.4. Lemma 17.3 is a direct consequence of Corollary 17.1.

Lemma 17.3. *Let $\mathbf{X}_\infty$ be a unit intensity Poisson process on $\mathbb{R}^2$. Consider a random graph $\mathcal{H}$ on this point process, where points $x, y \in \mathbf{X}_\infty = V(\mathcal{H})$ are joined by an edge with conditional probability $P_{x,y}$, given $\mathbf{X}_\infty$. If*

$$\sup_{x,y} \|x - y\|^4 P_{x,y} < \infty \tag{187}$$

then any infinite component of $\mathcal{H}$ is recurrent.

Note that Lemma 17.3 does not make any assumptions on the independence of different edges. In particular, for the proof of Theorem 15.4, we will also require the statement to hold for dependent percolation models.

Proof. We prove this via a discretization. We construct a weighted graph $G = \left(\mathbb{Z}^2, E, \omega\right)$ as follows. For each $v \in \mathbb{Z}^2$, identify all vertices in $\mathbf{X}_\infty \cap \left(v + [0,1)^2\right)$ to one vertex v, which we also imagine to be at the position $v \in \mathbb{Z}^2$ in space. For some $u, v \in \mathbb{Z}^2$, if there are $m \geq 1$ edges between u and v, replace them by one edge of conductance m, i.e., $\omega(\{u,v\}) = m$. If there is no edge between two vertices $u, v \in \mathbb{Z}^2$ in the graph G, we set $\omega(\{u,v\}) = 0$. Call this new graph G. It is not hard to see that if any connected component of G is recurrent, then also every connected component of $\mathcal{H}$ is recurrent. This holds, as we only contracted vertices of $\mathcal{H}$ and applied the parallel law to parallel edges. So we are left with showing that every connected component of G is recurrent. Assumption (187) implies that there exists a constant $C < \infty$ such that for all $u \neq v$ and for all $x \in u + [0,1)^2, y \in v + [0,1)^2$ one has $P_{x,y} \leq C\|u-v\|^{-4}$. Therefore for each edge $e = \{u,v\} \in E$ one now has

$$\begin{aligned}\mathbb{E}\left[\omega\left(\{u,v\}\right)\right] &= \mathbb{E}\left[\sum_{x \in \mathbf{X}_\infty \cap (u+[0,1)^2)} \sum_{y \in \mathbf{X}_\infty \cap (v+[0,1)^2)} P_{x,y}\right] \\ &\leq \mathbb{E}\left[\sum_{x \in \mathbf{X}_\infty \cap (u+[0,1)^2)} \sum_{y \in \mathbf{X}_\infty \cap (v+[0,1)^2)}\right] C\|u-v\|^{-4} = C\|u-v\|^{-4}\end{aligned}$$

where we used that the Poisson process has a unit intensity in the last equality. This already implies that the random walk on every connected component of G is recurrent, by Corollary 17.1. $\square$

Before going to the proof of Theorem 15.4, we still need to prove a small technical lemma that we will use later.

Lemma 17.4. *Suppose that X is a non-negative random variable satisfying $\mathbb{P}(X \leq \varepsilon) \leq C\varepsilon$ for some constant $C < \infty$ and all $\varepsilon > 0$. Then for $\eta < 1$ one has*

$$\mathbb{E}\left[X^{-\eta}\right] < \infty \tag{188}$$

and for $\eta > 1$ one has

$$\mathbb{E}\left[X^{-\eta} | X \geq \varepsilon\right] = \mathcal{O}\left(\varepsilon^{1-\eta}\right) \tag{189}$$

as ε goes to 0.

Proof. To prove (188) note that

$$\mathbb{E}\left[X^{-\eta}\right] \leq \sum_{n=0}^{\infty} \mathbb{P}\left(X^{-\eta} \geq n\right) = 1 + \sum_{n=1}^{\infty} \mathbb{P}\left(X \leq n^{-\frac{1}{\eta}}\right) \leq 1 + \sum_{n=1}^{\infty} C n^{-\frac{1}{\eta}} < \infty$$

as $\frac{1}{\eta} > 1$. To show (189) note that for small enough ε one has $\mathbb{P}(X \geq \varepsilon) \geq 0.5$ and this implies that for all $\tilde{\varepsilon} \geq \varepsilon$ one has

$$\mathbb{P}\left(X \leq \tilde{\varepsilon} | X \geq \varepsilon\right) = \frac{\mathbb{P}\left(X \leq \tilde{\varepsilon}, X \geq \varepsilon\right)}{\mathbb{P}\left(X \geq \varepsilon\right)} \leq \frac{\mathbb{P}\left(X \leq \tilde{\varepsilon}\right)}{0.5} \leq 2C\tilde{\varepsilon}.$$

For $\tilde{\varepsilon} < \varepsilon$ one obviously has $\mathbb{P}(X \leq \tilde{\varepsilon} | X \geq \varepsilon) = 0$. As $\eta > 1$, this implies that

$$\begin{aligned}\mathbb{E}\left[X^{-\eta} | X \geq \varepsilon\right] &\leq 1 + \sum_{n=1}^{\infty} \mathbb{P}\left(X^{-\eta} \geq n | X \geq \varepsilon\right) = 1 + \sum_{n=1}^{\infty} \mathbb{P}\left(X \leq n^{-\frac{1}{\eta}} | X \geq \varepsilon\right) \\ &= 1 + \sum_{n=1}^{\lceil \varepsilon^{-\eta} \rceil} \mathbb{P}\left(X \leq n^{-\frac{1}{\eta}} | X \geq \varepsilon\right) \leq 1 + \sum_{n=1}^{\lceil \varepsilon^{-\eta} \rceil} 2C n^{-\frac{1}{\eta}} \\ &\leq C' \lceil \varepsilon^{-\eta} \rceil^{1-\frac{1}{\eta}} \leq 2C' \varepsilon^{1-\eta}\end{aligned}$$

for some constant $C' < \infty$ and ε small enough. This shows (189) and thus finishes the proof. $\square$

With this, we are now ready to go the the proof of Theorem 15.4. Remember that the vertex set of the two-dimensional weight-dependent random connection model is a Poisson process of unit intensity on $\mathbb{R}^2 \times (0,1)$. So in particular if we condition that there is a point in this Process with spatial parameter $x \in \mathbb{R}^2$, the weight-parameter of this vertex is still uniformly distributed on the interval $(0,1)$. If we condition that there are two points in the Poisson process with spatial parameters x and y, then the weight-parameters of these points are independent random variables that are uniformly distributed on $(0,1)$.

Proof of Theorem 15.4. Throughout the proof we will always assume that S and T are independent random variables that are uniformly distributed on $(0,1)$. For all cases of random-connection models considered in Theorem 15.4 we will verify that (187) holds. For this we need to show that

$$P_{x,y} = \mathbb{E}\left[\rho\left(g(S,T)\|x-y\|^2\right)\right] = \mathcal{O}\left(\|x-y\|^{-4}\right), \tag{190}$$

as $\|x-y\| \to \infty$. This already implies that all connected components are recurrent by Lemma 17.3. We will only do the case $\gamma > 0$. The case $\gamma = 0$ works analogously or is degenerate. The factor of $\frac{1}{\beta}$ in the kernel $g(S,T)$ does not change whether (190) holds or not, so we will just ignore it from here on and think of $\beta = 1$. We will show (190) for all cases appearing in Theorem 15.4. Assuming that (173) holds we directly get that $\rho(r) \leq Cr^{-\delta}$ for a large enough constant $C < \infty$ and all $r \geq 0$. To strengthen this bound, note that we also have

$$\rho(r) \leq C\left(\mathbb{1}_{[0,1)}(r) + \mathbb{1}_{[1,\infty)}(r) r^{-\delta}\right) \tag{191}$$

for a large enough constant $C < \infty$ and all $r \geq 0$, as $\rho(r) \in [0,1]$ for all $r \in \mathbb{R}_{\geq 0}$. Now let us turn to the individual cases.

(a) (Preferential attachment kernel): For $\gamma < \frac{1}{2}$ we will first determine the limiting behavior near 0 of the distribution of $g(S,T) = \min(S,T)^\gamma \max(S,T)^{1-\gamma}$. For abbreviation we will write $\min = \min(S,T)$, $\max = \max(S,T)$, and $X = \min^\gamma \max^{1-\gamma}$. Let $n \in \mathbb{N}$ be arbitrary. Then we have that

$$\begin{aligned} \mathbb{P}\left(X \leq \frac{1}{2^n}\right) &= \mathbb{P}\left(\min^\gamma \leq \frac{1}{2^n}\right) + \sum_{k=0}^{\infty} \mathbb{P}\left(\frac{1}{2^{n-k}} < \min^\gamma \leq \frac{1}{2^{n-k-1}}, X \leq \frac{1}{2^n}\right) \\ &\leq \mathbb{P}\left(\min^\gamma \leq \frac{1}{2^n}\right) + \sum_{k=0}^{\infty} \mathbb{P}\left(\frac{1}{2^{n-k}} < \min^\gamma \leq \frac{1}{2^{n-k-1}}, \max^{1-\gamma} \leq \frac{1}{2^k}\right) \end{aligned} \tag{192}$$

as $\min^\gamma \max^{1-\gamma} \leq \frac{1}{2^n}$ and $\min^\gamma \geq \frac{1}{2^{n-k}}$ already imply $\max^{1-\gamma} \leq \frac{1}{2^k}$. On the event where $\frac{1}{2^{n-k}} < \min^\gamma \leq \frac{1}{2^{n-k-1}}$ and $\max^{1-\gamma} \leq \frac{1}{2^k}$ we must have that

$$2^{-\frac{n-k}{\gamma}} < \min \leq \max \leq 2^{-\frac{k}{1-\gamma}}$$

which can only hold if $-\frac{n-k}{\gamma} < -\frac{k}{1-\gamma}$, which is equivalent to $k < (1-\gamma)n$. Thus, all addends in the sum (192) are equal to 0 for $k \geq (1-\gamma)n$ and can be ignored. For every two non-negative real numbers a and b we have that

$$\mathbb{P}(\min \leq a, \max \leq b) \leq \mathbb{P}(S \leq a, T \leq b) + \mathbb{P}(T \leq a, S \leq b) \leq 2ab.$$

Inserting the previous observations into (192) we can further calculate that

$$\begin{aligned}\mathbb{P}\left(X \leq \frac{1}{2^n}\right) &\leq \mathbb{P}\left(\min \leq \frac{1}{2^{\frac{n}{\gamma}}}\right) + \sum_{k=0}^{\lfloor(1-\gamma)n\rfloor} \mathbb{P}\left(\min \leq \frac{1}{2^{\frac{n-k-1}{\gamma}}}, \max \leq \frac{1}{2^{\frac{k}{1-\gamma}}}\right) \\ &\leq 2\frac{1}{2^{\frac{n}{\gamma}}} + 2\sum_{k=0}^{\lfloor(1-\gamma)n\rfloor} \frac{1}{2^{\frac{n-k-1}{\gamma}}} \cdot \frac{1}{2^{\frac{k}{1-\gamma}}} = 2\frac{1}{2^{\frac{n}{\gamma}}} + 2^{1+\frac{1}{\gamma}} \frac{1}{2^{\frac{n}{\gamma}}} \sum_{k=0}^{\lfloor(1-\gamma)n\rfloor} 2^{\frac{k}{\gamma}-\frac{k}{1-\gamma}} \\ &\leq C2^{-\frac{n}{\gamma}} + C2^{-\frac{n}{\gamma}} 2^{\frac{(1-\gamma)n}{\gamma} - \frac{(1-\gamma)n}{1-\gamma}} \leq C2^{-\frac{n}{\gamma}} + C2^{-\frac{n}{\gamma}+\frac{(1-\gamma)n}{\gamma}-n} \\ &\leq C2^{-\frac{n}{\gamma}} + C2^{-2n} \leq 2C \cdot 2^{-2n}\end{aligned}$$

for a large enough constant C. We used that $\gamma < \frac{1}{2}$ which implies that $\frac{1}{\gamma} - \frac{1}{1-\gamma} > 0$, and thus the sum $\sum_{k=0}^{\lfloor(1-\gamma)n\rfloor} 2^{\frac{k}{\gamma}-\frac{k}{1-\gamma}}$ is, up to a multiplicative constant, equal to its last addend $2^{\frac{\lfloor(1-\gamma)n\rfloor}{\gamma}-\frac{\lfloor(1-\gamma)n\rfloor}{1-\gamma}}$. This already shows that

$$\mathbb{P}\left(g(S,T) \leq \varepsilon\right) = \mathbb{P}\left(\min(S,T)^{\gamma}\max(S,T)^{1-\gamma} \leq \varepsilon\right) \leq C'\varepsilon^2$$

for some constant $C' < \infty$ and all $\varepsilon > 0$. Taking squares this also implies that

$$\mathbb{P}\left(g(S,T)^2 \leq \varepsilon\right) = \mathbb{P}\left(g(S,T) \leq \sqrt{\varepsilon}\right) \leq C'\varepsilon \tag{193}$$

for all $\varepsilon > 0$. This is useful for us, as we can thus apply Lemma 17.4 to the random variable $g(S,T)^2$. Let ρ be a profile function with $\limsup_{r\to\infty} r^{\delta}\rho(r) < \infty$ for some $\delta > 2$. We still need to show (190). By inequality (191) we can assume that

$$\rho(r) \leq \mathbb{1}_{[0,1)}(r) + \mathbb{1}_{[1,\infty)}(r)r^{-\delta},$$

where we drop the multiplicative constant in (191) for the ease of notation. Using that $\frac{\delta}{2} > 1$ by assumption, we get that for some constant $C < \infty$

$$\begin{aligned}P_{x,y} &= \mathbb{E}\left[\rho\left(g(S,T)\|x-y\|^2\right)\right] \\ &\leq \mathbb{P}\left(g(S,T)\|x-y\|^2 < 1\right) + \mathbb{E}\left[\left(g(S,T)\|x-y\|^2\right)^{-\delta} \;\middle|\; g(S,T)\|x-y\|^2 \geq 1\right] \\ &\leq \mathbb{P}\left(g(S,T) < \frac{1}{\|x-y\|^2}\right) + \|x-y\|^{-2\delta}\mathbb{E}\left[(g(S,T)^2)^{-\frac{\delta}{2}} \;\middle|\; g(S,T)^2 \geq \frac{1}{\|x-y\|^4}\right] \\ &\leq C\frac{1}{\|x-y\|^4} + C\|x-y\|^{-2\delta}\left(\frac{1}{\|x-y\|^4}\right)^{1-\frac{\delta}{2}} = \mathcal{O}\left(\|x-y\|^{-4}\right)\end{aligned}$$

which shows (190) and finishes the proof. The last inequality holds because of Lemma 17.4 and (193).

(*b*) (Min and sum kernel): We show the result for the min kernel. As the sum kernel and the min kernel differ only by a constant, this already implies that (190) also holds for the sum kernel. We start with the case $\delta = 2, \gamma < \frac{1}{2}$. We can assume that $\rho(r) \leq Cr^{-2}$ for a constant $C < \infty$ and thus we get that

$$\begin{aligned}P_{x,y} = \mathbb{E}\left[\rho\left(g(S,T)\|x-y\|^2\right)\right] &\leq C\|x-y\|^{-4}\mathbb{E}\left[\min(S,T)^{-2\gamma}\right] \leq C\|x-y\|^{-4}\mathbb{E}\left[S^{-2\gamma}T^{-2\gamma}\right] \\ &= C\|x-y\|^{-4}\mathbb{E}\left[S^{-2\gamma}\right]\mathbb{E}\left[T^{-2\gamma}\right] = \mathcal{O}\left(\|x-y\|^{-4}\right)\end{aligned}$$

as $2\gamma < 1$ and thus $\mathbb{E}\left[S^{-2\gamma}\right], \mathbb{E}\left[T^{-2\gamma}\right] < \infty$. This finishes the proof for the first case. For the second case $\gamma = \frac{1}{2}, \delta > 2$ we ignore the constant in (191) and will thus assume from here on that

$$\rho(r) \leq \mathbb{1}_{[0,1)}(r) + \mathbb{1}_{[1,\infty)}(r)r^{-\delta}.$$

This implies that

$$\begin{aligned}
P_{x,y} &= \mathbb{E}\left[\rho\left(g(S,T)\|x-y\|^2\right)\right] \\
&\leq \mathbb{P}\left(\min(S,T)^{\frac{1}{2}}\|x-y\|^2 < 1\right) + \mathbb{E}\left[\left(\min(S,T)^{\frac{1}{2}}\|x-y\|^2\right)^{-\delta} \middle| \min(S,T)^{\frac{1}{2}}\|x-y\|^2 \geq 1\right] \\
&= \mathbb{P}\left(\min(S,T) < \frac{1}{\|x-y\|^4}\right) + \|x-y\|^{-2\delta}\mathbb{E}\left[\min(S,T)^{-\frac{\delta}{2}} \middle| \min(S,T) \geq \frac{1}{\|x-y\|^4}\right] \\
&\leq \frac{2}{\|x-y\|^4} + C\|x-y\|^{-2\delta}\left(\frac{1}{\|x-y\|^4}\right)^{1-\frac{\delta}{2}} = \mathcal{O}\left(\frac{1}{\|x-y\|^4}\right)
\end{aligned}$$

for some constant $C < \infty$. The last line holds because of Lemma 17.4, as $\mathbb{P}\left(\min(S,T) \leq \varepsilon\right) \leq 2\varepsilon$ and $\frac{\delta}{2} > 1$. This finishes the proof for the min kernel.

(c) (Product kernel): Now let us turn to the product kernel $g(S,T) = S^\gamma T^\gamma$. Let $\gamma < \frac{1}{2}$ and $\delta = 2$. We can assume that $\rho(r) \leq Cr^{-2}$ and thus we get with the same argument as above that

$$P_{x,y} = \mathbb{E}\left[\rho\left(g(S,T)\|x-y\|^2\right)\right] \leq C\|x-y\|^{-4}\mathbb{E}\left[S^{-2\gamma}T^{-2\gamma}\right] = \mathcal{O}\left(\|x-y\|^{-4}\right)$$

which finishes the proof. □

18 References

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