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Model Reduction for Controller Synthesis of Networks of Cyber-Physical Systems

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Abstract

In recent years, there has been a marked increase in interest in the control and analysis of large-scale interconnected cyber-physical systems (CPSs). This heightened attention is primarily driven by the prevalence of such systems in various contemporary engineering applications, including power systems, transportation networks, and biochemical systems. Due to the challenges posed by the so-called "curse of dimensionality" and the existing limitations in computational performance, the synthesis of controllers for large-scale interconnected systems in order to enforce complex specifications in a reliable and cost-effective manner has become a formidable problem. One approach that has emerged to tackle this challenge is *abstraction* based controller synthesis. In this approach one synthesizes a controller to enforce the complex specifications over a reduced-order model (known as abstraction) instead of the original (concrete) system, and *refines* the controller to that of the concrete system. The error between the output of the concrete system and that of its abstraction is quantified a priori. Therefore, one can ensure that the concrete system also satisfies the specifications (within a priori known error bounds).

Unfortunately, constructing (reduced-order) abstractions for a complex system when viewed monolithically is also a challenging task in itself. One approach to deal with this is to leverage the fact that many large-scale complex systems can be regarded as interconnected systems consisting of smaller *subsystems*. This motivates a *compositional* approach for the construction of the abstractions wherein abstractions of the concrete systems can be constructed by using the abstractions of the smaller subsystems. This dissertation builds on existing related work primarily in three directions:

1. In the first part of the dissertation, we derive conditions under which compositional abstractions of networks of *stochastic hybrid systems* can be constructed using the interconnection topology and joint *dissipativity-type* properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a stochastic hybrid system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process. Moreover, we derive conditions for the construction of abstractions for a class of stochastic hybrid systems involving nonlinearities satisfying an incremental quadratic inequality. In our result, unlike existing results, the stochastic noises and jumps in the concrete subsystem and its abstraction need not to be the same. We provide examples, including a physically motivated case study (electrical network), with numerical simulations to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning for interconnected stochastic hybrid systems.

2. The network topology in many interconnected systems is not fixed, and is either changing dynamically or randomly switching between multiple network topologies. In the second part of the dissertation, techniques for compositional abstractions of networks of control systems under dynamic and randomly switching interconnection topologies are investigated.

3. The state-space of many systems, e.g. configuration spaces of robotic manipulators, are *Riemannian manifolds*, and therefore, their analysis requires techniques from differential geometry. In the third part of the dissertation, a compositional approach for the construction of abstractions for interconnected systems evolving on Riemannian manifolds is presented. This allows for larger classes of systems than the ones considered in existing works defined only over Euclidean spaces.

Zusammenfassung

In den letzten Jahren hat das Interesse an der Steuerung und Analyse großer vernetzter cyber-physischer Systeme (CPS) deutlich zugenommen. Diese verstärkte Aufmerksamkeit ist in erster Linie darauf zurückzuführen, dass solche Systeme in verschiedenen modernen technischen Anwendungen wie Energiesystemen, Verkehrsnetzen und biochemischen Systemen weit verbreitet sind.

Aufgrund der Herausforderungen, die sich aus dem so genannten "Fluch der Dimensionalität" und den bestehenden Beschränkungen der Rechenleistung ergeben, ist die Synthese von Reglern für große vernetzte Systeme zur zuverlässigen und kostengünstigen Durchsetzung komplexer Spezifikationen zu einem gewaltigen Problem geworden. Ein Ansatz, der sich zur Bewältigung dieser Herausforderung herausgebildet hat, ist die auf TextitAbstraktion basierende Controller-Synthese. Bei diesem Ansatz wird ein Controller synthetisiert, um die komplexen Spezifikationen über ein reduziertes Ordnungsmodell (bekannt als Abstraktion) anstelle des ursprünglichen (konkreten) Systems durchzusetzen, und *v*erfeinert den Controller zu dem des konkreten Systems.

Bei diesem Ansatz wird ein Controller synthetisiert, um die komplexen Spezifikationen über ein Modell mit reduzierter Ordnung (bekannt als Abstraktion) anstelle des ursprünglichen (konkreten) Systems durchzusetzen, und der Controller wird auf den des konkreten Systems verfeinert. Der Fehler zwischen der Ausgabe des konkreten Systems und der seiner Abstraktion wird a priori quantifiziert. Daher kann man sicherstellen, dass das konkrete System auch die Spezifikationen erfüllt (innerhalb der a priori bekannten Fehlergrenzen).

Leider ist die Konstruktion von Abstraktionen (reduzierter Ordnung) für ein komplexes System, wenn es monolithisch betrachtet wird, auch eine Herausforderung für sich selbst. Ein Ansatz zur Bewältigung dieses Problems besteht darin, die Tatsache zu nutzen, dass viele große komplexe Systeme als miteinander verbundene Systeme betrachtet werden können, die aus kleineren Teilsystemen bestehen. Dies motiviert einen *compositional*-Ansatz für die Konstruktion der Abstraktionen, bei dem Abstraktionen der konkreten Systeme durch die Verwendung der Abstraktionen kleinerer Subsysteme konstruiert werden können.

Diese Dissertation baut auf bestehenden verwandten Arbeiten hauptsächlich in drei Richtungen auf:

1. Im ersten Teil wird ein bestehender Dissipativitätsansatz zur Konstruktion kompositorischer Abstraktionen kontinuierlicher dynamischer Systeme auf eine Klasse stochastischer Hybridsysteme, nämlich Sprungdiffusionen, erweitert. Im vorgeschlagenen Rahmen kann die Abstraktion, selbst ein stochastisches Hybridsystem (möglicherweise mit einer niedrigeren Dimension), als Ersatz für das ursprüngliche

System im Controller-Designprozess verwendet werden. Darüber hinaus leiten wir Bedingungen für die Konstruktion von Abstraktionen für eine Klasse stochastischer Hybridsysteme ab, die Nichtlinearitäten beinhalten, die eine inkrementelle quadratische Ungleichung erfüllen. In diesem Kapitel müssen die stochastischen Geräusche und Sprünge im konkreten Teilsystem und dessen Abstraktion im Gegensatz zu bestehenden Ergebnissen nicht identisch sein. Wir stellen Beispiele einschließlich einer Fallstudie (elektrisches Netzwerk) mit numerischen Simulationen zur Verfügung, um die Wirksamkeit des vorgeschlagenen kompositorischen Denkens vom dissipativen Typ für miteinander verbundene stochastische Hybridsysteme zu veranschaulichen.

2. Die Netzwerktopologie in vielen miteinander verbundenen Systemen ist nicht festgelegt und ändert sich entweder dynamisch oder wechselt zufällig zwischen mehreren Netzwerktopologien. Im zweiten Teil der Dissertation werden Techniken zur kompositorischen Abstraktion von Netzwerken von Steuerungssystemen unter dynamischen und zufällig wechselnden Verbindungstopologien untersucht.

3. Der Zustandsraum vieler Systeme, z.B. Konfigurationsräume von Robotermanipulatoren, sind Riemannsche Mannigfaltigkeiten und daher erfordert ihre Analyse Techniken aus der Differentialgeometrie. Im dritten Teil der Dissertation wird ein kompositorischer Ansatz zur Konstruktion von Näherungen für vernetzte Systeme vorgestellt, die sich auf *Riemannschen Mannigfaltigkeiten* entwickeln. Dies ermöglicht größere Klassen von Systemen als diejenigen, die in bestehenden Arbeiten berücksichtigt werden, die nur über Euklidischer Räume definiert sind.

Publications by the Author during PhD

1 Journal Papers

- **Asad Ullah Awan** and Majid Zamani. "Formal synthesis of safety controllers for unknown systems using Gaussian Process transfer learning," *to appear in IEEE Control Systems Letters.*

- **Asad Ullah Awan** and Majid Zamani. "Abstractions of networks of stochastic hybrid systems under randomly switched topologies: A compositional approach." Systems & Control Letters 175 (2023): 105512.

- **Asad Ullah Awan** and Majid Zamani. "From dissipativity theory to compositional abstractions of interconnected stochastic hybrid systems." IEEE Transactions on Control of Network Systems 7, no. 1 (2019): 433-445.

2 Conference Papers

- Rameez Wajid, **Asad Ullah Awan**, and Majid Zamani. "Formal synthesis of safety controllers for unknown stochastic control systems using Gaussian Process learning." In Learning for Dynamics and Control Conference, pp. 624-636. PMLR, 2022.

- Sajid Mohamed, **Asad Ullah Awan**, Dip Goswami, and Twan Basten. "Designing image-based control systems considering workload variations." In 2019 IEEE Conference on Decision and Control (CDC), 2019, pp. 3997-4004.

- **Asad Ullah Awan** and Majid Zamani. "Compositional abstraction for interconnected systems over Riemannian manifolds: A small-gain approach." In 2018 IEEE Conference on Decision and Control (CDC), pp. 3789-3794. IEEE, 2018.

- **Asad Ullah Awan**, Samuel Coogan, and Majid Zamani. "Compositional abstraction for interconnected systems over Riemannian manifolds: A dissipativity approach." In 2018 IEEE Conference on Decision and Control (CDC), pp. 3783-3788. IEEE, 2018.

- **Asad Ullah Awan** and Majid Zamani. "Compositional abstractions of networks of stochastic hybrid systems under randomly switched topologies." In 2018 Annual American Control Conference (ACC), pp. 1586-1591. IEEE, 2018.

- **Asad Ullah Awan** and M. Zamani. "Compositional abstraction of interconnected control systems under dynamic interconnection topology." In 2017 IEEE Conference on Decision and Control (CDC), pp. 3543-3550. IEEE, 2017.
- **Asad Ullah Awan** and Majid Zamani. "Compositional abstractions of networks of stochastic hybrid systems: A dissipativity approach." In 2017 IFAC World Congress, IFAC-PapersOnLine 50, no. 1 (2017): 15804-15809.
- **Asad Ullah Awan** and Majid Zamani. "On a notion of estimation entropy for stochastic hybrid systems." In 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 780-785. IEEE, 2016.

3 Papers Under Review

- **Asad Ullah Awan** and Majid Zamani. "Reduced-order Gaussian Processes for partially unknown nonlinear control systems," *under review in IEEE Transactions on Automatic Control.*

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Chapter 1

Introduction

1.1 Motivation

The dissertation is motivated by the need to develop techniques to facilitate computationally efficient controller synthesis for large-scale cyber-physical systems (CPS). CPSs are dynamical systems typically characterized by the complex interaction of computational (cyber) elements and physical systems [Aga19]. CPSs have widespread practical applications; from smart manufacturing and robotics to health care and medicine [BG11; Gun+14; RMR17; TWW17]. Large-scale *networks of CPSs* are also becoming increasingly ubiquitous in the modern world [LS16]. Such interconnected systems appear in many modern-day engineering applications such as power system networks [RSF14; DBS17; YX16], traffic networks [Coo+15; CA15; CAB17], biochemical reaction networks [SAS10; AS08; BG21], and smart water distribution networks [FM20; Pol+23; Guo+22] (see Figure 1.1).

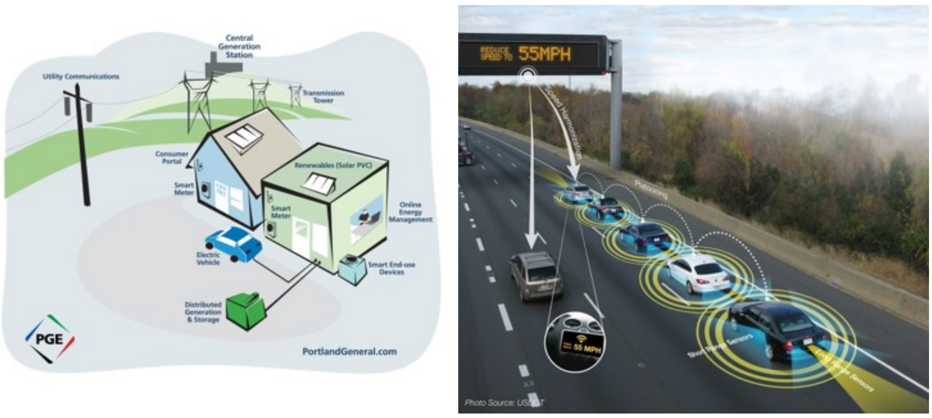

Figure 1.1: Examples of interconnected CPSs

Due to existence of both discrete and continuous elements in such systems, such systems are regarded as *hybrid* in nature [Tab09]. They consist of elements which can be modeled as discrete elements (such as computational units including hardware and software), as well as physical elements (such as energy storage elements like capacitors). In order to ensure that such an interconnected system behaves in a desired way requires synthesizing a controller for the system. Controller synthesis for such large-scale interconnected control systems to achieve *complex specifications* in a cost effective and reliable way is a challenging task. Examples of complex specifications, going beyond conventional objectives (e.g., stability or reference tracking), are those expressed in linear temporal logic (LTL) formulae [Pnu77; Coo+15]. The complexity of the control objectives together with the high dimensionality of such systems pose a difficult challenge for controller synthesis. Existing computational techniques and tool for control synthesis do not scale well to such modern large-scale interconnected systems [AMP16; Vid81].

1.2 Model Order Reduction (Abstraction)

One promising direction to address these issues is a combination of techniques from control theory (e.g., Lyapunov theory) and those of computer science (e.g., formal methods). One approach resulting from this symbiosis is discrete abstractions (also known as *symbolic models*). In this approach, first a finite approximation, called finite abstraction[1], of the concrete (original) system is constructed [PPD16a; Zam+11; Alu+00]. Then, a controller to satisfy complex specifications is synthesized over the finite abstraction using automata-theoretic techniques, which can be refined back to the concrete system [Tab09; RWR16].

Unfortunately, the computational complexity of constructing finite abstractions and synthesizing controllers hinders the applicability of such techniques to large-scale systems. One way to overcome this challenge is to introduce an intermediate step of constructing a continuous low-dimensional abstraction (we refer to this as infinite *abstraction*) of the concrete system (see Figure 1.2). This allows for a potentially easier design of a controller for the abstraction, which can be refined to the one for the original concrete system via a *refinement* map. The error between the output behavior of the concrete system and the one of its abstraction can be quantified a priori in this detour controller synthesis approach, thus ensuring that the concrete system also satisfies the complex specifications within known error bounds. From hereon in, we will simply use the term abstractions for continuous abstractions (reduced-order models) unless stated otherwise.

1.2.1 Compositional Approach to Model Order Reduction (Abstraction)

Constructing abstractions for a complex system when viewed monolithically is a challenging task in itself. Some recent techniques rely on numerically searching for a Lyapunov-like function to construct such abstractions. However, such techniques are only applicable to problems of modest size. One approach to deal with this is to leverage the fact that many large-scale complex systems can be regarded as interconnected systems consisting of smaller subsystems (see Figure 1.3).

This motivates a *compositional* approach for the construction of the abstractions wherein abstractions of the concrete systems can be constructed by using the abstractions of smaller subsystems. In such an approach, rather than treating the interconnected

[1] We call the approximation *infinite* if its set of states is infinite, and *finite* otherwise

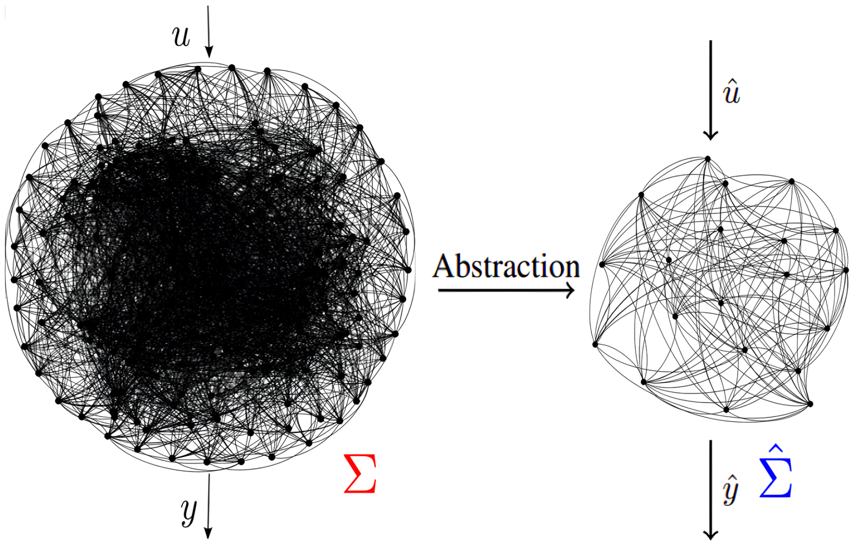

Figure 1.2: Abstraction of interconnected system. Σ represents the original (concrete) network, and $\hat{\Sigma}$ represents the (reduced-order) abstraction. u (resp. $\hat{u}$) represents the input while y (resp. $\hat{y}$) represents the output of the concrete network (resp. abstract network).

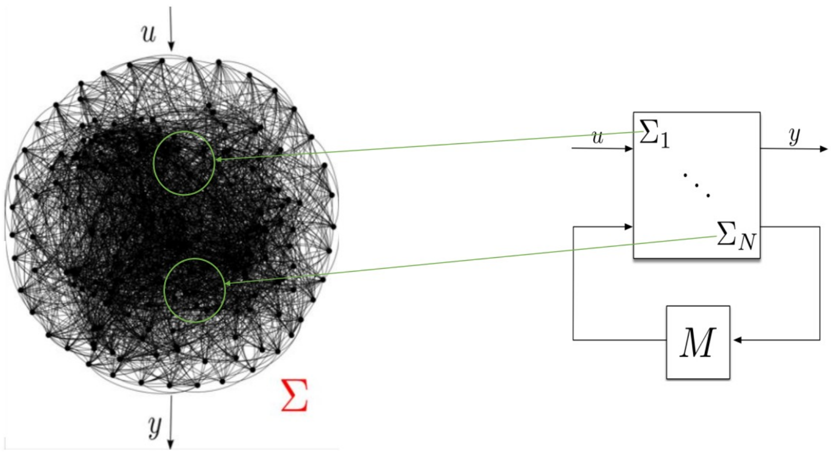

Figure 1.3: Large-scale complex systems can be regarded as interconnected systems consisting of smaller subsystems. Here M represents the coupling between the subsystems of the network.

system in a monolithic manner, compositional schemes provide network-level certifications from main structural properties of the subsystems and their interconnections. To do so, one should first i) divide the overall concrete network into a number of concrete subsystems and construct abstractions of each subsystem individually; ii) then establish a compositional scheme that allows us to construct an abstraction model of the overall network using those of individual subsystems.

In a recent result in [ZA17], a compositional framework utilizing dissipativity theory was introduced for constructing abstractions of networks of control systems. The proposed framework introduced a concept of a storage function, characterizing the joint dissipativity properties of control systems and their abstractions. This concept was then employed to establish compositional conditions, under which a network of abstractions approximate a network of concrete subsystems.

Building upon this work, in Chapter 3 of this dissertation, we expand upon this methodology to encompass a class of stochastic hybrid systems known as jump-diffusions. Stochastic hybrid systems are a general class of systems consisting of continuous and discrete dynamics subject to probabilistic noise and events [LP10]. In the case of jump-diffusions, continuous dynamics are modeled by stochastic differential equations, while discrete jumps are modeled as Poisson processes. We derive conditions under which compositional abstractions of networks of stochastic hybrid systems can be constructed using the interconnection topology and joint dissipativity-type properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a stochastic hybrid system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process.

In more realistic scenarios, the interconnection topology of interconnected systems is not fixed due to various causes, for example loss of communication between the robot agents due to occlusion caused by obstacles [SWX08; OM04; YW10], or failure of switching lines in an electric distribution grid [Cav+19; DKC23; GH21]. To accommodate for this scenario, in Chapter 4, we deal with networks of stochastic hybrid systems wherein the interconnection topology is randomly switching between $\mathcal{P}$ different topologies. We derive compositional conditions for construction of abstractions leveraging the interconnection topology, switching randomly between $\mathcal{P}$ different topologies, and the joint dissipativity-type properties of subsystems and their abstractions. Additionally, we also consider the scenario wherein the interconnection is governed by a dynamical system [Lin84]. In such interconnected systems, the additional dynamics introduced due to the interconnection/interaction system has to be taken into account in the compositional reasoning.

All the aforementioned results in the context of infinite abstractions consider systems evolving over Euclidean spaces. The state-space of many systems are Riemannian manifolds [BL04; LS16; MHP08; Oma+23; AMD17; Ahn+21; Jaq+18; PFA06], and therefore, their analysis requires tools from differential geometry [Tar+13; SB14]. In Chapter 5, we propose techniques for compositional construction of infinite abstractions for interconnected control systems evolving over smooth Riemannian manifolds. To this end, we propose two different approaches. In the first approach, we provide a small-gain type condition that enables the construction of an abstraction for the interconnected control system compositionally. The second approach is based on dissipativity theory wherein we derive sufficient conditions under which compositional abstractions of interconnected systems evolving on Riemannian manifolds can be constructed using the interconnection topology and joint *differential* dissipativity-type properties of subsystems and their abstractions.

We illustrate the efficacy of the proposed techniques using various examples, including physically motivated case studies (network of electric circuits).

1.3 Outline of the Dissertation

This dissertation is divided into 6 chapters, the first of which is the current Introduction. The rest is structured as follows:

Chapter 2 presents mathematical notations, preliminaries and notions from probability theory, manifold theory, and control theory.

Chapter 3 presents techniques for compositional abstraction of networks of stochastic hybrid systems using dissipativity theory. The content of this chapter is based on results published in [AZ17b; AZ19].

Chapter 4 provides compositional construction techniques for networks of dynamical systems with dynamic and randomly switched topologies. The content of this chapter is based on results published in [AZ17a; AZ18b; AZ23].

Chapter 5 discusses techniques for compositional construction of dynamical systems evolving over Riemannian manifolds. The content of this chapter is based on results published in [ACZ18; AZ18a].

Chapter 6 summarizes the results of this dissertation and presents potential directions for future research.

Chapter 2

Preliminaries, Notation and System Definitions

In this chapter, we introduce notations, preliminary concepts in probability theory, control theory, and definitions of systems used for modeling CPS networks that will be used in the subsequent chapters.

2.1 Notation

The sets of non-negative integer and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Those symbols are subscripted to restrict them in the usual way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. The symbol $\mathbb{R}^{n\times m}$ denotes the vector space of real matrices with n rows and m columns. The symbols $\vec{1}_n, \vec{0}_n, I_n, 0_{n\times m}$ denote the vector in $\mathbb{R}^n$ with all its elements to be one, the zero vector, identity, and zero matrices in $\mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times m}$, respectively. For $a, b \in \mathbb{R}$ with $a \leq b$, the closed, open, and half-open intervals in $\mathbb{R}$ are denoted by $[a, b],]a, b[, [a, b[$, and $]a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b],]a; b[, [a; b[$, and $]a; b]$ to denote the corresponding intervals in $\mathbb{N}$. Given $N \in \mathbb{N}_{\geq 1}$, vectors $x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}_{\geq 1}$ and $i \in [1; N]$, we use $x = [x_1; \ldots; x_N]$ to denote the concatenated vector in $\mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$. Similarly, we use $X = [X_1; \ldots; X_N]$ to denote the matrix in $\mathbb{R}^{n\times m}$ with $n = \sum_{i=1}^{N} n_i$, given $N \in \mathbb{N}_{\geq 1}$, matrices $X_i \in \mathbb{R}^{n_i\times m}, n_i \in \mathbb{N}_{\geq 1}$, and $i \in [1; N]$. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the Euclidean norm of x. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the induced 2-norm of M. Given matrices $M_1, \ldots, M_n$, the notation $\mathsf{diag}(M_1, \ldots, M_n)$ represents a block diagonal matrix with diagonal matrix entries $M_1, \ldots, M_n$. Given a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|,\ t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the map $\beta(r, t)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed non zero r, the map $\beta(r, t)$ is decreasing with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$. Given a vector $x = [x_1; x_2; \ldots; x_n] \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n\times n}$, we write $[x_1; x_2; \ldots; x_n]^T A[*]$ to concisely represent the quadratic form $[x_1; x_2; \ldots; x_n]^T A[x_1; x_2; \ldots; x_n]$. Given sets U and $A \subseteq U$, the complement of A with respect to U is defined as $U \backslash A = \{x : x \in U, x \notin A\}$. Given a set S, the cardinality of S is denoted by $\#S$. Given sets S and $A \subseteq S$, $\mathbf{I}_A : S \to \{0, 1\}$ denotes the indicator function defined as $\mathbf{I}_A(x) := 1$ if $x \in A$ and $\mathbf{I}_A(x) := 0$ if $x \notin A$.

2.2 Probability Space

A topological space S is called a Borel space if it is homeomorphic to a Borel subset of a Polish space (i.e., a separable and completely metrizable space) [Sri08]. Here, any Borel space S is assumed to be endowed with a Borel σ-algebra denoted by $\mathbb{B}(S)$. A map $f : X \to Y$ is measurable whenever it is Borel measurable. Moreover, a map $f : X \to Y$ is universally measurable if the inverse image of every Borel set under f is measurable with respect to every complete probability measure on X that measures all Borel subsets of X.

A probability space in this dissertation is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space, $\mathcal{F}$ is a sigma algebra on Ω which consists of subsets of Ω as events, and $\mathbb{P}$ is a complete probability measure. Let $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ be an increasing system of sigma-subalgebras of $\mathcal{F}$, i.e., $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t \leq \infty$. Such a system is called *filteration*, and the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ equipped with this filteration is called a filtered probability space [RY13]. We assume that the filteration $\mathbb{F}$ obeys the usual conditions, i.e., $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$, and the filteration is right-continuous in the sense that $\cap_{s>0}\mathcal{F}_{t+s} = \mathcal{F}_t$ for each $t \geq 0$ [KS12].

The expected value of a measurable function $g(X)$, where X is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is defined by the Lebesgue integral $\mathbb{E}[g(X)] := \int_\Omega g(X(\omega))\mathsf{d}\mathbb{P}(\omega)$, where $\omega \in \Omega$.

2.3 Riemannian Manifolds

An (n-dimensional) manifold $\mathcal{M}_n$ is a pair $(\mathcal{M}_n, \mathcal{A}^+)$ where $\mathcal{M}_n$ is a set and $\mathcal{A}^+$ is a maximal atlas into $\mathbb{R}^n$, such that the topology induced by $\mathcal{A}^+$ is Hausdorff and second countable. We denote the tangent space of $\mathcal{M}_n$ at $x \in \mathcal{M}_n$ by $\mathcal{T}_x\mathcal{M}_n$, and the tangent bundle of $\mathcal{M}_n$ by $\mathcal{T}\mathcal{M}_n = \bigcup_{x\in\mathcal{M}_n}\{x\} \times \mathcal{T}_x\mathcal{M}_n$.

A curve on the manifold is a mapping $\gamma : I \subset \mathbb{R} \to \mathcal{M}_n$. A distance (or metric) $d : \mathcal{M}_n \times \mathcal{M}_n \to \mathbb{R}_{\geq 0}$ on a manifold $\mathcal{M}_n$ is a continuous positive function that satisfies $d(x, y) = 0$ if and only if $x = y$ for each $x, y \in \mathcal{M}_n$, and $d(x, z) \leq d(x, y) + d(y, z)$ for each $x, y, z \in \mathcal{M}_n$. A (pseudo) *Riemannian* metric [FS14] on a smooth manifold $\mathcal{M}_n$ is a smoothly varying inner product on the tangent bundle $\mathcal{T}\mathcal{M}_n$ of manifold $\mathcal{M}_n$.

Given $\mathcal{M}_n$, and a matrix valued map $G : \mathcal{M}_n \to \mathbb{R}^{n\times n}$ such that $G(x)$ is a positive (semi) definite matrix for each $x \in \mathcal{M}_n$, the (pseudo) Riemannian metric corresponding to the (pseudo) Riemannian structure G is given by $\delta x^T G(x)\delta y$ for each $x \in \mathcal{M}_n$, $\delta x \in \mathcal{T}_x\mathcal{M}_n$ and $\delta y \in \mathcal{T}_x\mathcal{M}_n$. Given two points $x, y \in \mathcal{M}_n$, a smooth curve $\gamma : [0, 1] \to \mathcal{M}_n$ such that $\gamma(0) = x$, and $\gamma(1) = y$, and a (pseudo) Riemannian structure G defined on $\mathcal{M}_n$, we define the (pseudo) Riemannian energy functional as $E_G(\gamma) = \int_0^1 \frac{\partial\gamma}{\partial s}^T(s)G(\gamma(s))\frac{\partial\gamma}{\partial s}(s)ds$. The n-dimensional manifold $\mathbb{S}^n$ is defined by $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. For two points $z_1, z_2 \in \mathcal{M}_n$, $\Gamma(z_1, z_2)$ denotes the set of piece-wise continuous curves connecting z_1 and z_2: $\Gamma(z_1, z_2) = \{\gamma : [0, 1] \to \mathcal{M}_n | \gamma \text{ is piece-wise continuous }, \gamma(0) = z_1, \gamma(1) = z_2\}$. Given two points $x, y \in \mathcal{M}_n$, a

Riemannian structure G defined on $\mathcal{M}_n$, $\arg \min_{\gamma \in \Gamma(x,y)} \int_0^1 \sqrt{\frac{\partial \gamma}{\partial s}^T(s) G(\gamma(s)) \frac{\partial \gamma}{\partial s}(s)} ds$ is called a geodesic curve between x and y with respect to G.

In the following section, we introduce the classes of dynamical systems that are used to model networks of CPSs in the dissertation.

2.4 Deterministic Control Systems

Here, we define two classes of control systems which will be used in the modeling of CPS networks.

2.4.1 Control Systems

Definition 2.4.1. *A class of deterministic control systems used in this dissertation is a tuple*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2, h_{u1}, h_{u2}),$$

where

- $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$, $\mathbb{R}^{q_1}$, *and* $\mathbb{R}^{q_2}$ *are the state, external input, internal input, external output, and internal output spaces, respectively;*
- $\mathcal{U}$ *and* $\mathcal{W}$ *are subsets of sets of all measurable functions of time taking values in* $\mathbb{R}^m$ *and* $\mathbb{R}^p$*, respectively;*
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ *is a continuous map satisfying the following Lipschitz assumption: for every compact set* $\mathsf{D} \subset \mathbb{R}^n$*, there exists a constant* $Z \in \mathbb{R}_{>0}$ *such that* $\|f(x,u,w) - f(y,u,w)\| \leq Z\|x - y\|$ *for all* $x, y \in \mathsf{D}$*, all* $u \in \mathbb{R}^m$*, and all* $w \in \mathbb{R}^p$*;*
- $h_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{q_1}$ *is the external output map;*
- $h_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{q_2}$ *is the internal output map.*
- $h_{u1} : \mathbb{R}^m \rightarrow \mathbb{R}^{q_1}$ *is the external feedforward map;*
- $h_{u2} : \mathbb{R}^m \rightarrow \mathbb{R}^{q_2}$ *is the internal feedforward map.*

A control system $\mathfrak{D}$ satisfies

$$\mathfrak{D} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)) + h_{u1}(\upsilon(t)), \\ \zeta_2(t) = h_2(\xi(t)) + h_{u2}(\upsilon(t)), \end{cases} \tag{2.1}$$

for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where a locally absolutely continuous curve $\xi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is called a *state trajectory* of $\mathfrak{D}$, $\zeta_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{q_1}$ is called an external output trajectory of $\mathfrak{D}$, and $\zeta_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{q_2}$ is called an internal output trajectory of Σ. We call the tuple $(\xi, \zeta_1, \zeta_2, \upsilon, \omega)$ a *trajectory* of $\mathfrak{D}$, consisting of a state trajectory ξ, output trajectories ζ_1

and ζ_2, and input trajectories υ and ω, satisfying (2.1). We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the state trajectory at time $t \in \mathbb{R}_{\geq 0}$ under the input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$, where $a \in \mathbb{R}^n$. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to the state trajectory $\xi_{a\upsilon\omega}$.

Remark 2.4.2. *If the control system $\mathfrak{D}$ does not have internal and external feedforward maps, the description of the system defined in Definition 2.4.1 reduces to the tuple*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2).$$

Correspondingly, equation (2.1) *describing the evolution of state and output trajectories reduces to:*

$$\mathfrak{D} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)). \end{cases} \tag{2.2}$$

We will use the notion of control system in (2.2) later to refer to control subsystems in an interconnected system.

Remark 2.4.3. *If the control system $\mathfrak{D}$ does not have internal inputs and outputs, the description of the control system in Definition 2.4.1 reduces to the tuple*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h, h_u).$$

Correspondingly, the equation (2.1) *describing the evolution of state and output trajectories reduces to:*

$$\mathfrak{D} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)) + h_u(\upsilon(t)). \end{cases} \tag{2.3}$$

We will use the notion of control system in (2.3) later to refer to a dynamical interconnection topology in an interconnected system.

Remark 2.4.4. *If the control system does not have internal inputs and outputs, and external feedforward map, the definition of the control system in Definition 2.4.1 reduces to the tuple*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h).$$

Correspondingly, the equation (2.1) *describing the state and output trajectories reduces to:*

$$\mathfrak{D} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{2.4}$$

We will use the notion of control system in (2.4) later to refer to an overall interconnected control system.

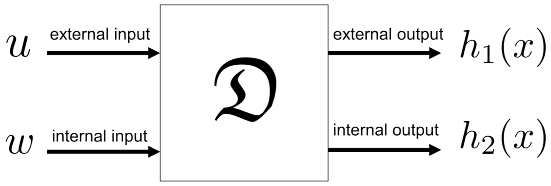

Figure 2.1: Deterministic control system defined in Definition 2.4.1.

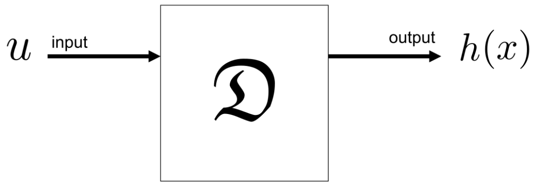

Figure 2.2: Deterministic control system as in Remark (2.4.4).

2.4.2 Control Systems over Riemannian Manifolds

In this subsection, we introduce a class of deterministic control systems whose state space are (smooth) Riemannian manifolds, formally defined as follows.

Definition 2.4.5. *The class of control systems over Riemannian manifolds a tuple*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

where

- $\mathcal{M}_n$ *is an n-dimensional state manifold containing the origin, while* $\mathbb{R}^m, \mathbb{R}^p, \mathbb{R}^{q_1}$*, and* $\mathbb{R}^{q_2}$ *are the external input, internal input, external output, and internal output (Euclidean) spaces of dimension* m, p, q_1*, and* q_2 *respectively;*

- $\mathcal{U}$ *and* $\mathcal{W}$ *are subsets of sets of all measurable functions of time taking values in* $\mathbb{R}^m$ *and* $\mathbb{R}^p$*, respectively;*

- $f : \mathcal{M}_n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{T}\mathcal{M}_n$ *is the continuously differentiable state evolution map. We assume that* $f(0, 0, 0) = 0$*;*

- $h_1 : \mathcal{M}_n \to \mathbb{R}^{q_1}$ *is the continuously differentiable external output map;*

- $h_2 : \mathcal{M}_n \to \mathbb{R}^{q_2}$ *is the continuously differentiable internal output map.*

A control system $\mathfrak{S}$ satisfies

$$\mathfrak{S} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)), \end{cases} \tag{2.5}$$

for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where a locally absolutely continuous curve $\xi : \mathbb{R}_{\geq 0} \to \mathcal{M}_n$ is called a *state trajectory* of $\mathfrak{S}$, $\zeta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_1}$ is called an external output trajectory of $\mathfrak{S}$, and $\zeta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_2}$ is called an internal output trajectory of $\mathfrak{S}$. We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the state trajectory at time $t \in \mathbb{R}_{\geq 0}$ under the input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$, where $a \in \mathcal{M}_n$. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to the state trajectory $\xi_{a\upsilon\omega}$.

Remark 2.4.6. *If the control system $\mathfrak{S}$ does not have internal outputs, the definition of the control system in Definition 2.4.5 reduces to the tuple*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^q, h).$$

Correspondingly, the equation (2.5) *describing the state and output trajectories reduces to:*

$$\mathfrak{S} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t), \omega(t)), \\ \zeta(t) = h(\xi(t)), \end{cases} \tag{2.6}$$

We will use the notions of control system in (2.5) and (2.6) later in Chapter 5 to refer to a subsytem in an interconnected control system over Riemannian manifold.

Remark 2.4.7. *If the control system $\mathfrak{S}$ does not have internal inputs and outputs, the definition of the control system in Definition 2.4.5 reduces to the tuple*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h).$$

Correspondingly, the equation (2.5) *describing the state and output trajectories reduces to:*

$$\mathfrak{S} : \begin{cases} \dot{\xi}(t) = f(\xi(t), \upsilon(t)), \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{2.7}$$

We will use the notion of control system in (2.7) later in Chapter 5 to refer to an overall interconnected control system over Riemannian manifold.

2.5 Stochastic Systems

2.5.1 Stochastic Hybrid Systems

In this section, we formally introduce a class of dynamical systems known as stochastic hybrid systems which will be used in the modeling of CPS networks. Stochastic hybrid systems are dynamical systems which consist of both continuous and discrete dynamics subject to probabilistic noise and events [LP10].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space endowed with a filteration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right continuity [KS12]. Let $(W_s)_{s\geq 0}$ be a b-dimensional $\mathbb{F}$-Brownian motion and $(P_s)_{s\geq 0}$ be an r- dimensional $\mathbb{F}$-Poisson process. We assume that the Poisson process and Brownian motion are independent of each other. The Poisson process $P_s = [P_s^1; \ldots ; P_s^{\mathsf{r}}]$ models r kinds of events whose occurrences are assumed to be independent of each other.

Definition 2.5.1. *A class of stochastic hybrid systems studied in this dissertation is a tuple*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

where

- $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$, $\mathbb{R}^{q_1}$, *and* $\mathbb{R}^{q_2}$ *are the state, external input, internal input, external output, and internal output spaces, respectively;*
- $\mathcal{U}$ *and* $\mathcal{W}$ *are subsets of sets of all* $\mathbb{F}$*-progressively measurable processes taking values in* $\mathbb{R}^m$ *and* $\mathbb{R}^p$*, respectively;*
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ *is the drift term which is globally Lipschitz continuous: there exist Lipschitz constants* $L_x, L_u, L_w \in \mathbb{R}_{\geq 0}$ *such that* $\|f(x,u,w) - f(x',u',w')\| \leq L_x\|x - x'\| + L_u\|u - u'\| + L_w\|w - w'\|$ *for all* $x, x' \in \mathbb{R}^n$*, all* $u, u' \in \mathbb{R}^m$*, and all* $w, w' \in \mathbb{R}^p$*;*
- $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times \mathsf{b}}$ *is the diffusion term which is globally Lipschitz continuous with the Lipschitz constant* L_σ*;*
- $\rho : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times \mathsf{r}}$ *is the reset term which is globally Lipschitz continuous with the Lipschitz constant* L_ρ*;*
- $h_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{q_1}$ *is the external output map;*
- $h_2 : \mathbb{R}^n \rightarrow \mathbb{R}^{q_2}$ *is the internal output map.*

A stochastic hybrid system Σ satisfies

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \upsilon(t), \omega(t))\mathsf{d}t + \sigma(\xi(t))\mathsf{d}W_t + \rho(\xi(t))\mathsf{d}P_t \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)), \end{cases} \tag{2.8}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where stochastic process $\xi : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is called a *solution process* of Σ, stochastic process $\zeta_1 : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{q_1}$ is called an external output trajectory of Σ, and stochastic process $\zeta_2 : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{q_2}$ is called an internal output trajectory of Σ. We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_{\geq 0}$ under input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$ $\mathbb{P}$-a.s., where a is a random variable that is $\mathcal{F}_0$-measurable. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to solution process $\xi_{a\upsilon\omega}$. Here, we assume that the Poisson processes P_s^i, for any $i \in [1; \mathsf{r}]$, have the rates λ_i. We emphasize that the postulated assumptions on f, σ, and ρ ensure existence, uniqueness, and strong Markov property of the solution process [ØS05; Law10].

Remark 2.5.2. *Note that the underlying dynamic considered in* (2.8) *is a class of stochastic hybrid systems in which the drift and diffusion terms model the continuous part and the Poisson process models the discrete jump of the system.*

Remark 2.5.3. *If the stochastic hybrid system Σ does not have internal inputs and outputs, the system defined in Definition 2.5.1 reduces to*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Correspondingly, equation (2.8) *describing the evolution of solution processes reduces to:*

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \upsilon(t))\mathsf{d}t + \sigma(\xi(t))\mathsf{d}W_t + \rho(\xi(t))\mathsf{d}P_t, \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{2.9}$$

2.5.2 Switching Stochastic Hybrid System

We now introduce another class of dynamical systems called switching stochastic hybrid systems to model which will be used to model CPS network with randomly switched topologies.

Definition 2.5.4. *A switching stochastic hybrid system is a tuple*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \mathsf{P}, \mathscr{P}, f, \sigma, \rho, \mathbb{R}^q, h),$$

where

- $\mathbb{R}^n$, $\mathbb{R}^m$, *and* $\mathbb{R}^q$, *are the state, external input and external output spaces, respectively;*
- $\mathcal{U}$ *is a subset of the set of all* $\mathbb{F}$*-progressively measurable processes taking values in* $\mathbb{R}^m$*;*
- $\mathsf{P} = \{1, \ldots, \mathcal{P}\}$ *is a finite set of modes;*
- $\mathscr{P}$ *is a subset of the set of all piece-wise constant càdlàg (i.e. right continuous and with left limits) functions of time from* $\mathbb{R}_{\geq 0}$ *to* P *and characterized by a finite number of discontinuities on every bounded interval in* $\mathbb{R}_{\geq 0}$ *(no Zeno behaviour);*
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathsf{P} \to \mathbb{R}^n$, *is the drift term such that for every fixed* $\mathsf{p} \in \mathsf{P}$, $f(.,.,\mathsf{p})$ *is globally Lipschitz continuous: there exists Lipschitz constants* $L_x, L_u \in \mathbb{R}_{\geq 0}$ *such that* $\|f(x, u, \mathsf{p}) - f(x', u', \mathsf{p})\| \leq L_x\|x - x'\| + L_u\|u - u'\|$ *for all* $x, x' \in \mathbb{R}^n$ *and all* $u, u' \in \mathbb{R}^m$*;*
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times \tilde{p}}$ *is the diffusion term which is globally Lipschitz continuous with the Lipschitz constant* L_σ*;*
- $\rho : \mathbb{R}^n \to \mathbb{R}^{n \times \tilde{r}}$ *is the reset term which is globally Lipschitz continuous with the Lipschitz constant* L_ρ*;*
- $h : \mathbb{R}^n \to \mathbb{R}^q$ *is the external output map.*

The stochastic process $\xi : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ *is a solution process of* Σ *if there exists* $\upsilon \in \mathcal{U}$ *and* $\pi \in \mathscr{P}$ *satisfying*

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \upsilon(t), \pi(t))\mathsf{d}t + \sigma(\xi(t))\mathsf{d}W_t + \rho(\xi(t))\mathsf{d}P_t, \\ \zeta(t) = h(\xi(t)), \end{cases} \tag{2.10}$$

$\mathbb{P}$*-a.s., at each time* $t \in \mathbb{R}_{\geq 0}$*. We denote by* $\xi_{av}(t, \pi(t))$ *the value of the solution process at time* $t \in \mathbb{R}_{\geq 0}$ *under the control input* $\upsilon \in \mathcal{U}$ *and the switching process* $\pi \in \mathscr{P}$*, starting from an initial condition* $\xi_{av}(0, \pi(0)) = a$ $\mathbb{P}$*-a.s., where* a *is measurable in the trivial sigma-algebra* $\mathcal{F}_0$*.*

Remark 2.5.5. *In our definition of switching stochastic hybrid system, we exclude systems which exhibit Zeno behaviour i.e. that an infinite number of mode transitions occur in a finite amount of time [Hes04; Lib03], named after the philosopher Zeno of Elena (500-400 B.C.) [Sim+00].*

Chapter 3

Compositional Abstraction of Stochastic Hybrid Systems

3.1 Introduction

Constructing reduced-order models (i.e., infinite abstractions) for a complex system when viewed monolithically is a challenging task. One approach to deal with this is to leverage the fact that many large-scale complex systems can be regarded as interconnected systems consisting of smaller *subsystems*. This motivates a *compositional* approach for the construction of the abstractions wherein abstractions of the concrete systems can be constructed by using the abstractions of smaller subsystems.

Recently, there have been several results on the compositional construction of infinite abstractions of deterministic control systems including [PPD16b], [TI08], [RZ18], and of a class of stochastic hybrid systems [ZRE17]. These results employ a small-gain type condition for the compositional construction of abstractions. However, as shown in [DK04], this type of condition is a function of the size of the network and can be violated as the number of subsystems grows. Recently in [ZA17], a compositional framework for the construction of infinite abstractions of networks of control systems has been proposed using dissipativity theory. In this result a notion of storage function is proposed which describes joint dissipativity properties of control systems and their abstractions. This notion is used to derive compositional conditions under which a network of abstractions approximate a network of the concrete subsystems. Those conditions can be independent of the number or gains of the subsystems under some properties for the interconnection topologies.

In this chapter of the dissertation, we extend this approach to a class of stochastic hybrid systems, namely, jump-diffusions. Stochastic hybrid systems are a general class of systems consisting of continuous and discrete dynamics subject to probabilistic noise and events. In jump-diffusions, the continuous dynamics are modeled by stochastic differential equations and switches are modeled as Poisson processes. We introduce a notion of so-called stochastic storage functions describing joint dissipativity properties of stochastic hybrid subsystems and their abstractions. Given a network of stochastic hybrid subsystems and the stochastic storage functions between subsystems and their abstractions, we derive conditions based on the interconnection topology, guaranteeing that a network of abstractions quantitatively approximate the network of concrete subsystems. For a class of stochastic hybrid subsystems and using the incremental quadratic inequality for the nonlinearity, we derive a set of matrix (in)equalities facilitating the construction of their abstractions together with the corresponding stochastic storage functions. We illustrate the effectiveness of the proposed results in two examples in which compositionality conditions are satisfied independent of the number or gains of the subsystems.

3.1.1 Contributions

Compositional abstraction for (deterministic) interconnected control systems using dissipativity was introduced in [ZA17]. In [AZ17b], this technique was extended to a class of stochastic hybrid systems. In both works, the joint dissipativity properties are defined with respect to a static map whose input is the (internal) inputs and outputs of the subsystems and their abstractions. In contrast to this, in this chapter we employ a dynamic map based on a similar notion introduced in [TB11]. This allows for a broader class of (stochastic hybrid) subsystems for which one can find (stochastic) storage functions between them and their abstractions (cf. case study in Section 3.5.2). Furthermore, in this work we derive constructive conditions for computing abstractions for a class of stochastic hybrid systems by considering nonlinearities which are more general than the ones considered in [ZA17] and [AZ17b].

Compositional abstractions for jump-diffusions are also introduced in [ZRE17]. However, in [ZRE17] it is assumed that the stochastic noises in a subsystem and its abstraction are the same. This assumption is not realistic in practice, as it requires access to the realization of the noises in the original subsystem in order to refine the constructed controllers for the abstractions to the original subsystems. On the other hand, in our approach concrete subsystems and their abstractions do not share the same stochastic noises. In addition, the results in [ZRE17] use small-gain type conditions for the main compositionality result whereas the proposed approach here uses dissipativity-type conditions which can potentially provide scale-free results under some properties over the interconnection topologies. Although the results in [ZRE17] derive conditions for constructing abstractions for just linear jump-diffusions, here we provide constructive conditions for a class of *nonlinear* jump-diffusions.

3.2 Certificates for Abstraction

In this section, we introduce two notions which we use to formally relate a stochastic hybrid system and its abstraction. The first notion, namely stochastic storage functions, relates a stochastic hybrid system introduced in Definition 2.5.1 and its abstraction. The second notion, namely stochastic simulation functions, relates a stochastic hybrid system without internal inputs and outputs (as in (2.9)) and its abstraction.

3.2.1 Stochastic Storage Function

In this subsection, we introduce a notion of so-called stochastic storage functions, adapted from the notion of storage functions from dissipativity theory [AMP16]. Before introducing the notion of stochastic storage functions, we introduce a linear control system which is given by:

$$\begin{aligned}\dot{\xi}_\theta(t) &= A_\theta \xi_\theta(t) + B_\theta \upsilon_\theta(t)\\ \zeta_\theta(t) &= C_\theta \xi_\theta(t) + D_\theta \upsilon_\theta(t),\end{aligned} \tag{3.1}$$

where $A_\theta \in \mathbb{R}^{l_\theta \times l_\theta}, B_\theta \in \mathbb{R}^{l_\theta \times m_\theta}, C_\theta \in \mathbb{R}^{q_\theta \times l_\theta}$, and $D_\theta \in \mathbb{R}^{q_\theta \times m_\theta}$, where B_θ, and D_θ have the conformal partitions

$$B_\theta = \begin{bmatrix} B_1 & B_2 \end{bmatrix},\ D_\theta = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \tag{3.2}$$

respectively. These conformal partitions will be used later in the chapter. We use the tuple $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ to represent such a linear control system. Now we define the infinitesimal generator of a stochastic process which will be used later to define a notion of stochastic storage functions.

Definition 3.2.1. *Consider two stochastic hybrid systems*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$$

with solution processes ξ and $\hat{\xi}$, respectively. Consider a linear control system $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ satisfying (3.1) *with state trajectory ξ_θ. Consider a twice continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \to \mathbb{R}_{\geq 0}$. The infinitesimal generator of the stochastic process $\Xi = [\xi; \hat{\xi}; \xi_\theta]$, denoted by $\mathcal{L}$, acting on function V is defined as [ØS05]:*

$$\begin{aligned}\mathcal{L}V(x, \hat{x}, \theta) := &\begin{bmatrix} \partial_x V & \partial_{\hat{x}} V & \partial_\theta V \end{bmatrix} \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \\ A_\theta \theta + B_\theta u_\theta \end{bmatrix}\\ &+ \frac{1}{2}\mathsf{Tr}\left(\sigma(x)\sigma^T(x)\partial_{x,x}V\right) + \frac{1}{2}\mathsf{Tr}\left(\hat{\sigma}(\hat{x})\hat{\sigma}^T(\hat{x})\partial_{\hat{x},\hat{x}}V\right)\\ &+ \sum_{j=1}^{\mathsf{r}} \lambda_j (V(x + \rho(x)\mathsf{e}_j^{\mathsf{r}}, \hat{x}) - V(x, \hat{x}))\\ &+ \sum_{j=1}^{\hat{\mathsf{r}}} \hat{\lambda}_j (V(x, \hat{x} + \hat{\rho}(\hat{x})\mathsf{e}_j^{\hat{\mathsf{r}}}) - V(x, \hat{x})),\end{aligned} \tag{3.3}$$

where $\mathsf{e}_j^{\mathsf{r}}$ denotes an r-dimensional vector with 1 on the j-th entry and 0 elsewhere.

Now we have all the ingredients to introduce a notion of stochastic storage functions.

Definition 3.2.2. *Consider two stochastic hybrid systems*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$$

with the same external output space dimension and let $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ *be a linear control system as in* (3.1)*. A twice continuously differentiable function* $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \to \mathbb{R}_{\geq 0}$ *is called a stochastic storage function from* $\hat{\Sigma}$ *to* Σ*, with respect to* Σ_θ*, in the* k-th *moment (SStF-M*$_2$*), where* $k \geq 1$*, if it has polynomial growth rate and there exist convex functions* $\alpha, \eta \in \mathcal{K}_\infty$*, concave function* $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$*, some constant* $\mathsf{c} \in \mathbb{R}_{\geq 0}$*, some matrices* $W, \hat{W}$*, and* H*, and some symmetric matrix* X *of appropriate dimension such that*

$$D_2^T X D_2 \preceq 0, \tag{3.4}$$

where D_2 *is given in* (3.2)*, and* $\forall x \in \mathbb{R}^n$*,* $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$*, and* $\forall \theta \in \mathbb{R}^{l_\theta}$ *one has*

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^k) \leq V(x, \hat{x}, \theta), \tag{3.5}$$

and $\forall x \in \mathbb{R}^n$*,* $\forall \hat{x} \in \mathbb{R}^{\hat{n}}, \forall \hat{u} \in \mathbb{R}^{\hat{m}}\ \exists u \in \mathbb{R}^m$*, such that* $\forall \hat{w} \in \mathbb{R}^{\hat{p}}\ \forall w \in \mathbb{R}^p$*, one obtains*

$$\mathcal{L}V(x, \hat{x}, \theta) \leq -\eta(V(x, \hat{x}, \theta)) + \psi_{\mathsf{ext}}(\|\hat{u}\|^k) + z^T X z + \mathsf{c}, \tag{3.6}$$

where $z = C_d \theta + D_d u_\theta$ *and*

$$u_\theta = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$

The second condition (3.6) implicitly implies the existence of a function $u = k_t(x, \hat{x}, \hat{u})$ to choose u for any $x, \hat{x}$, and $\hat{u}$. We call this function an *interface* function. We use notation $\hat{\Sigma} \preceq \Sigma$ if there exists an SStF-M$_2$ V from $\hat{\Sigma}$ to Σ. The stochastic hybrid system $\hat{\Sigma}$ (possibly with $\hat{n} < n$) is called an abstraction of Σ. This reduction in the dimensionality of the state-space from the concrete system to its abstraction can be viewed as a model-order reduction scheme.

Remark 3.2.3. *If* C_θ *is the zero matrix, and* D_θ *is the identity matrix, then the quadratic term in* (3.6) *reduces to the one in [ZA17; AZ17b], with*

$$z = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$

Remark 3.2.4. *Condition* (3.4) *has also appeared in various forms in the literature as a necessary condition for deriving asymptotic stability from dissipativity properties of a system. See for example [TB11].*

3.2.2 Stochastic Simulation Function

Now, we recall a slightly adapted version of the notion of stochastic simulation function introduced in [ZRE17]. This notion is appropriate for relating interconnected systems without internal inputs and outputs.

Definition 3.2.5. *Let*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h)$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h})$$

be two stochastic hybrid systems. A twice continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \to \mathbb{R}_{\geq 0}$ is called a stochastic simulation function from $\hat{\Sigma}$ to Σ in the k-th moment (SSF-M$_2$), where $k \geq 1$, if there exist convex functions $\alpha, \eta \in \mathcal{K}_\infty$, concave function $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and some constant $\mathsf{c} \in \mathbb{R}_{\geq 0}$, such that $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, and $\forall \theta \in \mathbb{R}^{l_\theta}$, one has

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|^k) \leq V(x, \hat{x}, \theta), \tag{3.7}$$

and $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}, \forall \hat{u} \in \mathbb{R}^{\hat{m}}$ $\exists u \in \mathbb{R}^m$ such that

$$\mathcal{L}V(x, \hat{x}, \theta) \leq -\eta(V(x, \hat{x}, \theta)) + \psi_{\mathsf{ext}}(\|\hat{u}\|^k) + \mathsf{c}. \tag{3.8}$$

We say that a stochastic hybrid system $\hat{\Sigma}$ is approximately simulated by a stochastic hybrid system Σ, denoted by $\hat{\Sigma} \preceq_{AS} \Sigma$, if there exists an SSF-M$_2$ function V from $\hat{\Sigma}$ to Σ. We call $\hat{\Sigma}$ (possibly with lower dimension $\hat{n} < n$) an abstraction of Σ.

The next theorem shows the important of the existence of an SSF-M$_2$ by quantifying the error between the output behaviors of Σ and the ones of its abstractions $\hat{\Sigma}$.

Theorem 3.2.6. *Let*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h})$$

be two stochastic hybrid systems. Suppose V is an SSF-M$_2$ from $\hat{\Sigma}$ to Σ. Then, there exists a $\mathcal{KL}$ function β, a $\mathcal{K}_\infty$ function γ_{ext}, and some constant $\mathsf{c}' \in \mathbb{R}_{\geq 0}$ such that for any $\hat{\upsilon} \in \hat{\mathcal{U}}$, any random variable a and $\hat{a}$ that are $\mathcal{F}_0$-measurable, and any $\theta_0 \in \mathbb{R}^{l_\theta}$, there exists $\upsilon \in \mathcal{U}$ such that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^k] \leq \beta(\mathbb{E}[V(a, \hat{a}, \theta_0)], t) + \gamma_{\mathsf{ext}}(\mathbb{E}[\|\hat{\upsilon}\|_\infty^k]) + \mathsf{c}'. \tag{3.9}$$

Proof. The proof is similar to the one of Theorem 3.5 in [ZRE17]. □

In the next section we first provide a definition of interconnected stochastic hybrid systems. We then provide conditions under which we can construct abstractions of interconnected stochastic hybrid systems in a compositional way.

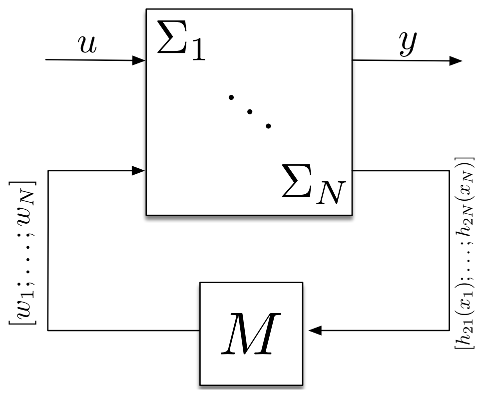

Figure 3.1: Interconnected stochastic hybrid system defined in Definition 3.3.1

3.3 Interconnected Stochastic Hybrid Systems

Next definition provides a notion of interconnection for stochastic hybrid subsystems investigated in this chapter.

Definition 3.3.1. *Consider $N \in \mathbb{N}_{\geq 1}$ stochastic hybrid subsystems*

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1; N]$, and a matrix M (the interconnection matrix) of an appropriate dimension defining the coupling of these subsystems. The interconnected stochastic hybrid system

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, follows by $n = \sum_{i=1}^N n_i, m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{1i}$, and the functions

$$f(x, u) := [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)], \tag{3.10}$$

$$\sigma(x) := [\sigma_1(x_1); \ldots; \sigma_N(x_N)], \tag{3.11}$$

$$\rho(x) := [\rho_1(x_1); \ldots; \rho_N(x_N)], \tag{3.12}$$

$$h(x) := [h_{11}(x_1); \ldots; h_{1N}(x_N)], \tag{3.13}$$

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$ and with internal variables constrained by

$$[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)]. \tag{3.14}$$

Assume we are given N stochastic hybrid subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

together with their corresponding abstractions

$$\hat{\Sigma}_i = (\mathbb{R}^{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{\hat{p}_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \hat{\sigma}_i, \hat{\rho}_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{\hat{q}_{2i}}, \hat{h}_{1i}, \hat{h}_{2i})$$

and with SStF-M$_2$ V_i from $\hat{\Sigma}_i$ to Σ_i. We use α_i, η_i, $\psi_{i\text{ext}}$, A_{θ_i}, B_{θ_i}, C_{θ_i}, D_{θ_i}, H_i, W_i, $\hat{W}_i$, and X_i to denote the corresponding functions, matrices, and their corresponding conformal block partitions appearing in Definition 3.2.2.

The next theorem provides a compositional approach on the construction of abstractions of networks of stochastic hybrid systems.

Theorem 3.3.2. *Consider an interconnected system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic hybrid subsystems Σ_i and the interconnection matrix M. Suppose each subsystem Σ_i admits an abstraction $\hat{\Sigma}_i$ with the corresponding SStF-M$_2$ V_i with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i})$, $i \in [1;N]$. Suppose there exists $\mu_i > 0$, $i \in [1;N]$, symmetric matrix $\tilde{Q} \succeq 0$, and matrix $\hat{M}$ of appropriate dimension such that the matrix (in)equalities*

$$\begin{bmatrix} A_D^T\tilde{Q} + \tilde{Q}A_D & \tilde{Q}B_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \\ \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix}^T S^T B_D^T \tilde{Q} & 0 \end{bmatrix} + \begin{bmatrix} C_D & D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} \mu_1 X_1 & & \\ & \ddots & \\ & & \mu_N X_N \end{bmatrix} \begin{bmatrix} C_D & D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \end{bmatrix} \preceq 0, \tag{3.15}$$

$$WMH = \hat{W}\hat{M}, \tag{3.16}$$

are satisfied, where $\tilde{q} = \sum_{i=1}^N q_{2i}$, and

$$\begin{aligned} W &= \mathsf{diag}(W_1, \ldots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \ldots, \hat{W}_N), \\ H &= \mathsf{diag}(H_1, \ldots, H_N), \end{aligned} \tag{3.17}$$

$$\begin{aligned} A_D &= \mathsf{diag}(A_{\theta_1}, \ldots, A_{\theta_N}), B_D = \mathsf{diag}(B_{\theta_1}, \ldots, B_{\theta_N}), \\ C_D &= \mathsf{diag}(C_{\theta_1}, \ldots, C_{\theta_N}), D_D = \mathsf{diag}(D_{\theta_1}, \ldots, D_{\theta_N}), \end{aligned} \tag{3.18}$$

and S is the following permutation matrix:

$$S = \begin{bmatrix} I_{r_{W_1}} & 0_{r_{W_2}} & \cdots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \cdots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \cdots & 0_{r_{W_N}} & I_{r_{H_1}} & 0_{r_{H_2}} & \cdots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & I_{r_{W_2}} & \cdots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \cdots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \cdots & 0_{r_{W_N}} & 0_{r_{H_1}} & I_{r_{H_2}} & \cdots & 0_{r_{H_N}} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \cdots & I_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \cdots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \cdots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \cdots & I_{r_{H_N}} \end{bmatrix}, \tag{3.19}$$

where, for each $i \in [1;N]$, r_{W_i} and r_{H_i} denote the number of rows in W_i and H_i, respectively. Then

$$V(x, \hat{x}, \theta) := \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T \tilde{Q}\theta,$$

where $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{l_\theta}, l_\theta = \sum_{i=1}^{N} l_{\theta_i}$, *is an SSF-M*$_2$ *from the interconnected system* $\hat{\Sigma} := \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$*, with the coupling matrix* $\hat{M}$*, to* Σ*.*

Proof. The proof is inspired by that of Theorem 4.2 in [ZA17]. First we show that the inequality (3.7) holds for some convex $\mathcal{K}_\infty$ function α. As also argued in the proof of Theorem 4.2 in [ZRE17], for any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, any $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and any $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{l_\theta}$, one gets:

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\|^k &\leq N^{\max\{\frac{k}{2},1\}-1} \sum_{i=1}^{N} \|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|^k \\ &\leq N^{\max\{\frac{k}{2},1\}-1} \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i, \theta_i)) \\ &\leq \underline{\alpha}(V(x, \hat{x}, \theta)),\end{aligned}$$

for any $k \geq 1$, where $\underline{\alpha}$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}(s) := \begin{cases} \max\limits_{\vec{s} \geq 0} & N^{\max\{\frac{k}{2},1\}-1} \sum\limits_{i=1}^{N} \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu^T \vec{s} = s, \end{cases} \tag{3.20}$$

where $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$ and $\mu = [\mu_1; \ldots; \mu_N]$. The function $\underline{\alpha}$ is a concave function as argued in [ZRE17]. By defining the convex function[1] $\alpha(s) = \underline{\alpha}^{-1}(s), \forall s \in \mathbb{R}_{\geq 0}$, one obtains

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^k) \leq V(x, \hat{x}, \theta),$$

satisfying inequality (3.7). Now we prove the inequality (3.8). Consider any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n, \hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $\hat{u} = [\hat{u}_1; \ldots; \hat{u}_N] \in \mathbb{R}^{\hat{m}}$. For any $i \in [1; N]$, there exists $u_i \in \mathbb{R}^{m_i}$, consequently, a vector $u = [u_1; \ldots; u_N] \in \mathbb{R}^m$, satisfying (3.6) for each pair of subsystems Σ_i and $\hat{\Sigma}_i$ with the internal inputs given by $[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)]$ and $[\hat{w}_1; \ldots; \hat{w}_N] = \hat{M}[\hat{h}_{21}(\hat{x}_1); \ldots; \hat{h}_{2N}(\hat{x}_N)]$, respectively. The dynamics of Σ_{θ_i}, $i \in [1; N]$, can be lumped together into a single

[1] The inverse of a strictly increasing concave (resp. convex) function is a strictly increasing convex (resp. concave) function.

auxiliary system as the following:

$$\begin{aligned}
\dot{\theta}(t) &= A_D\theta(t) + B_D S \begin{bmatrix} W_1 w_1 - \hat{W}_1\hat{w}_1 \\ \vdots \\ W_N w_N - \hat{W}_N\hat{w}_N \\ h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \\
&= A_D\theta(t) + B_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix}, \\
z(t) &= C_D\theta(t) + D_D S \begin{bmatrix} W_1 w_1 - \hat{W}_1\hat{w}_1 \\ \vdots \\ W_N w_N - \hat{W}_N\hat{w}_N \\ h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \\
&= C_D\theta(t) + D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix},
\end{aligned} \tag{3.21}$$

where $z = [z_1; \ldots; z_N]$. We now consider the infinitesimal generator of the function V, and employ the previous auxiliary system and conditions (3.15) and (3.16) to derive the chain of inequalities given in (3.22), where $\mathsf{c}' = \sum_{i=1}^N \mu_i \mathsf{c}_i$,

$$\Theta(x,\theta) := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \\ h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix}, \tag{3.23}$$

and the functions $\eta \in \mathcal{K}_\infty$ and $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$ are defined as

$$\eta(s) := \begin{cases} \min\limits_{\vec{s}\geq 0} & \sum_{i=1}^N \mu_i\eta_i(s_i) \\ \text{s.t.} & \mu^T\vec{s} = s, \end{cases} \tag{3.24}$$

$$\psi_{\mathsf{ext}}(s) := \begin{cases} \max\limits_{\vec{s}\geq 0} & \sum_{i=1}^N \mu_i\psi_{i\mathsf{ext}}(s_i) \\ \text{s.t.} & \|\vec{s}\| \leq s. \end{cases} \tag{3.25}$$

$$
\begin{aligned}
\mathcal{L}V(x,\hat{x},\theta) &= \sum_{i=1}^{N} \mu_i \mathcal{L}V_i(x_i,\hat{x}_i,\theta_i) + \dot{\theta}^T \tilde{Q}\theta + \theta^T \tilde{Q}\dot{\theta} \\
&\leq \sum_{i=1}^{N} \mu_i \Big(-\eta_i(V_i(x_i,\hat{x}_i,\theta_i)) + \psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k) + z_i^T X_i z_i + \mathsf{c}_i \Big) + \dot{\theta}^T \tilde{Q}\theta + \theta^T \tilde{Q}\dot{\theta} \\
&= -\sum_{i=1}^{N} \mu_i \eta_i(V_i(x_i,\hat{x}_i,\theta_i)) + \sum_{i=1}^{N} \mu_i \psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k) + \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}^T \begin{bmatrix} \mu_1 X_1 & & \\ & \ddots & \\ & & \mu_N X_N \end{bmatrix} \underbrace{\begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}}_{z} \\
&\quad + \Theta(x,\theta)^T \begin{bmatrix} A_D^T \tilde{Q} + \tilde{Q} A_D & \tilde{Q} B_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \\ \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix}^T S^T B_D^T \tilde{Q} & 0 \end{bmatrix} \Theta(x,\theta) + \mathsf{c}' \\
&= -\sum_{i=1}^{N} \mu_i \eta_i(V_i(x_i,\hat{x}_i,\theta_i)) + \sum_{i=1}^{N} \mu_i \psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k) \\
&\quad + \Theta(x,\theta)^T \begin{bmatrix} A_D^T \tilde{Q} + \tilde{Q} A_D & \tilde{Q} B_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \\ \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix}^T S^T B_D^T \tilde{Q} & 0 \end{bmatrix} \Theta(x,\theta) \\
&\quad + \Theta(x,\theta)^T \begin{bmatrix} C_D & D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} \mu_1 X_1 & & \\ & \ddots & \\ & & \mu_N X_N \end{bmatrix} \begin{bmatrix} C_D & D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \end{bmatrix} \Theta(x,\theta) \\
&\quad + \mathsf{c}' \\
&\leq -\eta(V(x,\hat{x},\theta)) + \psi_{\mathsf{ext}}(\|\hat{u}\|^k) + \mathsf{c}', \qquad (3.22)
\end{aligned}
$$

It remains to show that η is a convex function and ψ_{ext} is a concave one. Let us recall that by assumption functions η_i, $\forall i \in [1;N]$, are convex functions. Thus the function η above defines a *perturbation function* which is a convex one; see [BV04] for further details. Again, by assumption $\psi_{i\mathsf{ext}}$, $\forall i \in [1;N]$, are concave functions. By similar reasoning, we conclude that ψ_{ext} is a concave function. Hence, we conclude V is an SSF-M$_2$ function from $\hat{\Sigma}$ to Σ. □

Remark 3.3.3. *If C_{θ_i} is the zero matrix and D_{θ_i} is the identity matrix (i.e. Σ_{θ_i} is a static map), $\forall i \in [1;N]$, then matrix inequality* (3.15) *reduces to matrix inequality (15) in [AZ17b, Theorem 7] (which is a stochastic counterpart of matrix inequality (IV.1) in [ZA17, Theorem 4.2]).*

Remark 3.3.4. *The matrix inequality* (3.15) *is linear with respect to the decision variables $\tilde{Q}$ and $\mu = [\mu_1; \ldots; \mu_N]$, and matrix equality* (3.16) *is linear with respect to the decision variable $\hat{M}$, which can be solved by using readily available software packages such as* `YALMIP` *[Lof04].*

In the next section, we consider a specific class of stochastic hybrid systems Σ, and a specific candidate SStF-M$_2$ function V. We derive conditions facilitating the construction of $\hat{\Sigma}$ as an abstraction of Σ and such that V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ.

3.4 A Class of Stochastic Hybrid Systems

We consider a specific class of stochastic hybrid systems with the drift, diffusion, reset, and output functions given by

$$\begin{aligned}
&\mathsf{d}\xi(t) = (A\xi(t) + Bv(t) + E\varphi(t, F\xi) + D\omega(t))\mathsf{d}t + G\mathsf{d}W_t + \sum_{i=1}^{\mathsf{r}} R_i \mathsf{d}P_t^i,\\
&\zeta_1(t) = C_1\xi(t),\\
&\zeta_2(t) = C_2\xi(t),
\end{aligned} \tag{3.26}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times p}, E \in \mathbb{R}^{n\times l_k}, F \in \mathbb{R}^{l_k\times n}, G \in \mathbb{R}^{n\times 1}, R_i \in \mathbb{R}^n, \forall i \in [1;\mathsf{r}], C_1 \in \mathbb{R}^{q_1\times n}$, and $C_2 \in \mathbb{R}^{q_2\times n}$. The vector R_i and scalar $\lambda_i > 0$ (rate of the Poisson process), $\forall i \in [1;\mathsf{r}]$, parametrize the jumps associated with events i. The time-varying non-linearity is the one considered in [AC11], which satisfies an incremental quadratic inequality: for all $\tilde{M} \in \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is the set of symmetric matrices referred to as incremental multiplier matrices, the following incremental quadratic constraint holds for all $t \in \mathbb{R}_{\geq 0}$, and $k_1, k_2 \in \mathbb{R}^{l_k}$:

$$\begin{bmatrix} k_2 - k_1 \\ \varphi(t, k_2) - \varphi(t, k_1)\end{bmatrix}^T \tilde{M} \begin{bmatrix} k_2 - k_1 \\ \varphi(t, k_2) - \varphi(t, k_1)\end{bmatrix} \geq 0. \tag{3.27}$$

To facilitate subsequent analysis, we write matrix $\tilde{M}$ in the following conformal partitioned form

$$\tilde{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22}\end{bmatrix}. \tag{3.28}$$

We use the tuple

$$\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda),$$

where $\mathsf{R} = \{R_1, \ldots, R_{\mathsf{r}}\}$ and $\lambda = \{\lambda_1, \ldots, \lambda_{\mathsf{r}}\}$, to refer to the class of system of the form (3.26). We now consider a specific candidate function and derive conditions under which it is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ.

3.4.1 Stochastic Storage Function

Here, we consider a candidate SStF-M$_2$ of the form

$$V(x, \hat{x}, \theta) = (x - P\hat{x})^T \widehat{M}(x - P\hat{x}) + \theta^T \Lambda \theta, \tag{3.29}$$

where P, $\widehat{M}^T = \widehat{M} \succ 0$, and $\Lambda = \Lambda^T \succ 0$ are matrices of appropriate dimensions. In order to show that $V(x, \hat{x}, \theta)$ in (3.29) is an SStF-M$_2$ from an abstraction $\hat{\Sigma}$ to the concrete system Σ, with respect to $\Sigma_\theta = (A_d, B_d, C_d, D_d)$, where $B_\theta = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and $D_\theta = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$, we require the following assumptions on the concrete system Σ and on Σ_θ.

Assumption 3.4.1. *Let $\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda)$. There exist matrices $\widehat{M} \succ 0$, K, X, L_1, Z, W, Λ, A_d, C_d, $B_\theta := [B_1\ B_2]$, $D_\theta := [D_1\ D_2]$, and positive constants $\hat{\kappa}$ and $\bar{\kappa}$, such that*

$$D_2^T X D_2 \preceq 0,$$

and the following (in)equalities hold,

$$D = ZW, \tag{3.30}$$

$$\begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1 + E) & C_2^T B_2^T \Lambda \\ Z^T \widehat{M} & 0 & 0 & B_1^T \Lambda \\ (BL_1 + E)^T \widehat{M} & 0 & 0 & 0 \\ \Lambda B_2 C_2 & \Lambda B_1 & 0 & {A_d}^T \Lambda + \Lambda A_d \end{bmatrix}$$
$$\preceq \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^T D_2^T X D_2 C_2 - F^T M_{11} F & C_2^T D_2^T X D_1 & -F^T M_{12} & C_2^T D_2^T X C_\theta \\ D_1^T X D_2 C_2 & D_1^T X D_1 & 0 & D_1^T X C_\theta \\ -M_{12}^T F & 0 & -M_{22} & 0 \\ C_\theta^T X D_2 C_2 & C_\theta^T X D_1 & 0 & {C_d}^T X C_d - \bar{\kappa}\Lambda \end{bmatrix}, \tag{3.31}$$

where

$$\Delta = (A + BK)^T \widehat{M} + \widehat{M}(A + BK). \tag{3.32}$$

An equivalent geometric characterization of (3.30) is given by the following lemma.

Lemma 3.4.2. *Given D and Z, the condition* (3.30) *is satisfied for some matrix W if and only if*

$$\mathrm{im}\ D \subseteq \mathrm{im}\ Z. \tag{3.33}$$

Remark 3.4.3. *Remark that when the non-linearity in* (3.26) *reduces to the one described in [ZA17, Section V] and Σ_θ is a static map, matrix inequality* (3.31) *reduces to (V.5) in [ZA17, Theorem 5.5]. Note also that in the absence of the non-linearity in* (3.26)*, matrix inequality* (3.31) *is feasible if the pair (A, B) is stabilizable and A_θ is Hurwitz.*

Now, we provide one of the main results of this section showing under which conditions V in (3.29) is an SStF-M$_2$.

Theorem 3.4.4. *Let*

$$\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda),$$

and

$$\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{\mathsf{R}}, \varphi, \hat{\lambda}),$$

with the same external output dimension. Suppose Assumption 3.4.1 holds and there exist matrices P, Q, H, $\hat{W}$ and L_2 of appropriate dimensions such that:

$$AP = P\hat{A} - BQ \tag{3.34a}$$

$$C_1P = \hat{C}_1 \tag{3.34b}$$

$$C_2P = H\hat{C}_2 \tag{3.34c}$$

$$FP = \hat{F} \tag{3.34d}$$

$$E = P\hat{E} + B(L_2 - L_1) \tag{3.34e}$$

$$P\hat{D} = Z\hat{W}. \tag{3.34f}$$

Then, function V defined in (3.29) *is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$.*

Proof. We note that from (3.34b), $\forall x \in \mathbb{R}^n$ and $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, we have $\|C_1x - \hat{C}_1\hat{x}\|^2 = (x - P\hat{x})^T C_1^T C_1 (x - P\hat{x})$. It can be readily verified that $\frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^T C_1)}\|C_1x - \hat{C}_1\hat{x}\|^2 \le V(x, \hat{x}, \theta)$ for all $\theta \in \mathbb{R}^{l_\theta}$, implying that inequality (3.5) holds with $\alpha(r) = \frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^T C_1)}r$ for any $r \in \mathbb{R}_{\ge 0}$, which is a convex function. We proceed to prove inequality (3.6). By the definition of V, one has

$$\begin{aligned}
\partial_x V &= 2(x - P\hat{x})^T \widehat{M}, \\
\partial_{\hat{x}} V &= -2(x - P\hat{x})^T \widehat{M} P, \\
\partial_{x,x} V &= 2\widehat{M}, \\
\partial_{\hat{x},\hat{x}} V &= 2P^T \widehat{M} P.
\end{aligned}$$

Following the definition of $\mathcal{L}$, for any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}, \theta \in \mathbb{R}^{l_\theta}$, one obtains:

$$\begin{aligned}
\mathcal{L}V(x, \hat{x}, \theta) = {} & 2(x - P\hat{x})^T \widehat{M}(Ax + E\varphi(Fx) + Bu + Dw) \\
& - 2(x - P\hat{x})^T \widehat{M} P(\hat{A}\hat{x} + \hat{E}\varphi(\hat{F}\hat{x}) + \hat{B}\hat{u} + \hat{D}\hat{w}) + G^T \widehat{M} G \\
& + \hat{G}^T P^T \widehat{M} P \hat{G} + 2(x - P\hat{x})^T \widehat{M} \sum_{i=1}^{\mathsf{r}} \lambda_i R_i + \sum_{i=1}^{\mathsf{r}} \lambda_i R_i^T \widehat{M} R_i \\
& - 2(x - P\hat{x})^T \widehat{M} \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i P \hat{R} + \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i \hat{R}_i^T P^T \widehat{M} P \hat{R}_i \\
& + 2\theta^T \Lambda\Big(A_d\theta + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} Ww - \hat{W}\hat{w} \\ C_2x - H\hat{C}_2\hat{x} \end{bmatrix}\Big).
\end{aligned}$$

Given any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}$, and $\hat{u} \in \mathbb{R}^{\hat{m}}$, we use the following *interface* function to choose $u \in \mathbb{R}^m$:

$$u = K(x - P\hat{x}) + Q\hat{x} + \tilde{R}\hat{u} + L_1\varphi(t, Fx) - L_2\varphi(t, \hat{F}\hat{x}), \tag{3.35}$$

where L_2, Q, and $\tilde{R}$ are matrices of appropriate dimension. Using the interface function in (3.35), and the conditions (3.30), (3.34a), (3.34d), (3.34e), and (3.34f), one obtains:

$$\begin{aligned}
\mathcal{L}V(x, \hat{x}, \theta) = {}& 2(x - P\hat{x})^T \widehat{M}\Big(A(x - P\hat{x}) + BK(x - P\hat{x}) \\
& + ZWw - Z\hat{W}\hat{w} + (B\tilde{R} - P\hat{B})\hat{u} + (BL_1 + E)\delta\varphi\Big) \\
& + G^T\widehat{M}G + \hat{G}^T P^T \widehat{M} P\hat{G} + \sum_{i=1}^{\mathsf{r}} \lambda_i R_i^T \widehat{M} R_i + \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i \hat{R}_i^T P \widehat{M} P \hat{R}_i \\
& + 2(x - P\hat{x})^T \widehat{M}(\sum_{i=1}^{\mathsf{r}} \lambda_i R_i - \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i P\hat{R}_i) + 2\theta^T \Lambda A_d \theta \\
& + 2\theta^T \Lambda B_1(Ww - \hat{W}\hat{w}) + 2\theta^T \Lambda B_2(C_2 x - H\hat{C}_2\hat{x}),
\end{aligned}$$

where $\delta\varphi = \varphi(t, Fx) - \varphi(t, \hat{F}\hat{x})$. Using Young's inequality [You12], Cauchy-Schwarz inequality [MV70], (3.31), and (3.34c), one obtains the upper bound for $\mathcal{L}V(x, \hat{x}, \theta)$ as given in (3.36), where $\pi, \pi' \in \mathbb{R}_{>0}$ satisfy $\pi + \pi' < \hat{\kappa}$, $\tilde{\kappa} = \min\{\hat{\kappa} - \pi - \pi', \bar{\kappa}\}$, and

$$\tilde{\mathsf{c}} = G^T\widehat{M}G + \hat{G}^T P^T \widehat{M} P\hat{G} + \sum_{i=1}^{\mathsf{r}} \lambda_i R_i^T \widehat{M} R_i + \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i \hat{R}_i^T P^T \widehat{M} P \hat{R}_i, \tag{3.37}$$

$$\mathsf{c}' = \frac{\|\sqrt{\widehat{M}}\left(\sum_{i=1}^{\mathsf{r}} \lambda_i R_i - \sum_{i=1}^{\hat{\mathsf{r}}} \hat{\lambda}_i P\hat{R}_i\right)\|^2}{\pi'}. \tag{3.38}$$

Here, we have used the fact that for any $x \in \mathbb{R}^n$ and any $\hat{x} \in \mathbb{R}^{\hat{n}}$, one has [AC11],

$$\begin{bmatrix} x - P\hat{x} \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix}^T \tilde{M} \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix} \begin{bmatrix} x - P\hat{x} \\ \delta\varphi \end{bmatrix} \geq 0. \tag{3.39}$$

Using the upper bound (3.36), the inequality (3.6) is satisfied, implying that V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$, with the convex function $\eta(s) = \tilde{\kappa}s$, concave function $\psi_{\mathsf{ext}}(s) = \frac{\|\sqrt{\widehat{M}}(B\tilde{R} - P\hat{B})\|^2}{\pi}s, \forall s \in \mathbb{R}_{\geq 0}$, matrix X, and $\mathsf{c} = \tilde{\mathsf{c}} + \mathsf{c}'$. $\square$

Remark 3.4.5. *Note that matrix $\tilde{R}$ is a free design parameter in the interface function. As explained in [ZA17] and [GP09], one can choose $\tilde{R}$ to minimize the function ψ_{ext} for V and, hence, lower the upper bound on the error between the output behaviors of Σ and $\hat{\Sigma}$. The choice of $\tilde{R}$ minimizing ψ_{ext} is given by*

$$\tilde{R} = (B^T\widehat{M}B)^{-1}B^T\widehat{M}P\hat{B}. \tag{3.40}$$

$$\begin{aligned}
\mathcal{L}V(x,\hat{x},\theta) &= \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix}^T \begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1+E) & C_2^TB_2^T\Lambda \\ Z^T\widehat{M} & 0 & 0 & B_1^T\Lambda \\ (BL_1+E)^T\widehat{M} & 0 & 0 & 0 \\ \Lambda B_2C_2 & \Lambda B_1 & 0 & {A_d}^T\Lambda+\Lambda A_d \end{bmatrix} \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix} \\
&\quad + 2(x-P\hat{x})^T\widehat{M}(B\tilde{R}-P\hat{B})\hat{u} + 2(x-P\hat{x})^T\widehat{M}\left(\sum_{i=1}^{\mathsf{r}}\lambda_iR_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}_i\right) + \tilde{\mathsf{c}} \\
&\leq \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix}^T \begin{bmatrix} -\hat{\kappa}\widehat{M}+C_2^TD_2^TXD_2C_2-F^TM_{11}F & C_2^TD_2^TXD_1 & -F^TM_{12} & C_2^TD_2^TXC_\theta \\ D_1^TXD_2C_2 & D_1^TXD_1 & 0 & D_1^TXC_\theta \\ -M_{12}^TF & 0 & -M_{22} & 0 \\ C_\theta^TXD_2C_2 & C_\theta^TXD_1 & 0 & {C_d}^TXC_d-\bar{\kappa}\Lambda \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \\
&\quad + 2(x-P\hat{x})^T\widehat{M}(B\tilde{R}-P\hat{B})\hat{u} + 2(x-P\hat{x})^T\widehat{M}\left(\sum_{i=1}^{\mathsf{r}}\lambda_iR_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}_i\right) + \tilde{\mathsf{c}} \\
&\leq -(\hat{\kappa}-\pi-\pi')(x-P\hat{x})^T\widehat{M}(x-P\hat{x}) + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 - \bar{\kappa}\theta^T\Lambda\theta \\
&\quad - 2\begin{bmatrix} x-P\hat{x} \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix}^T \tilde{M} \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix} \begin{bmatrix} x-P\hat{x} \\ \delta\varphi \end{bmatrix} \\
&\quad + \left(C_d\theta + \begin{bmatrix} D_1 & D_2 \end{bmatrix}^T \begin{bmatrix} Ww-\hat{W}\hat{w} \\ C_2x-H\hat{C}_2\hat{x} \end{bmatrix}\right)^T X \left(C_d\theta + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} Ww-\hat{W}\hat{w} \\ C_2x-H\hat{C}_2\hat{x} \end{bmatrix}\right) + \tilde{\mathsf{c}} \\
&\quad + \frac{\|\sqrt{\widehat{M}}\left(\sum_{i=1}^{\mathsf{r}}\lambda_iR_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}_i\right)\|^2}{\pi'} \\
&\leq -(\hat{\kappa}-\pi-\pi')(x-P\hat{x})^T\widehat{M}(x-P\hat{x}) - \bar{\kappa}\theta^T\Lambda\theta + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 + z^TXz + \tilde{\mathsf{c}} + \mathsf{c}' \\
&\leq -\tilde{\kappa}V(x,\hat{x},\theta) + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 + z^TXz + \tilde{\mathsf{c}} + \mathsf{c}' \qquad (3.36)
\end{aligned}$$

Remark 3.4.6. *The constant* c, *can be also minimized, thereby lowering the upper bound on the error between the output behaviours of* Σ *and* $\hat{\Sigma}$. *One can choose* $\hat{G}$ *to be the zero matrix and choose* $\hat{\lambda}$ *and* $\hat{\mathsf{R}}$ *to solve the following optimization problem:*

$$\arg\min_{\hat{\mathsf{R}},\hat{\lambda}>0} \quad \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^TP^T\widehat{M}P\hat{R}_i - \frac{2(\sum_{i=0}^{\mathsf{r}}\lambda_iR_i^T)\widehat{M}P(\sum_{i=0}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i)}{\pi'} + \frac{(\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^T)P^T\widehat{M}P(\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i)}{\pi'}, \qquad (3.41)$$

where $\hat{\lambda} = \{\hat{\lambda}_1,\ldots,\hat{\lambda}_{\hat{\mathsf{r}}}\}$ *and* $\hat{\mathsf{R}} = \{\hat{R}_1,\ldots,\hat{R}_{\hat{\mathsf{r}}}\}$. *This optimization problem is, in general, a non-convex one.*

Remark 3.4.7. *The matrix inequality* (3.31) *is bi-linear in* $\widehat{M}, K, L_1, Z$, *and linear in* X *and* Λ *if we fix* $\hat{\kappa}$, $\bar{\kappa}$, *and the matrices* $A_\theta, B_\theta, C_\theta$, *and* D_θ.

In the following theorem we show that conditions (3.34a), (3.34b), (3.34c), (3.34d), and (3.34e) are not only sufficient, but also necessary for (3.29) to be an SStF-M$_2$ from

$\hat{\Sigma}$ to Σ, provided that the interface function is as in (3.35) for some matrices $K, Q, \tilde{R}, L_1$, and L_2, of appropriate dimensions.

Theorem 3.4.8. *Let*

$$\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda)$$

and

$$\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{\mathsf{R}}, \varphi, \hat{\lambda}),$$

with the same external output space dimension. Assume that $G = \hat{G} = 0$, and $R_i = \hat{R}_i = 0\ \forall i \in [1; \hat{\mathsf{r}}]$, where 0 represents the zero matrices of appropriate dimensions. Suppose that V, defined in (3.29), is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta = (A_d, B_d, C_d, D_d)$, with the interface function given in (3.35). Then equations (3.34a), (3.34b), (3.34c), (3.34d), and (3.34e) hold.

Proof. Since V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, there exists a $\mathcal{K}_\infty$ function α such that $\|C_1 x - \hat{C}_1 \hat{x}\|^2 \leq \alpha^{-1}(V(x, \hat{x}, \theta))$ for any $x \in \mathbb{R}^n$, any $\hat{x} \in \mathbb{R}^{\hat{n}}$, and any $\theta \in \mathbb{R}^{l_\theta}$. From (3.29), it follows that $\|C_1 P \hat{x} - \hat{C}_1 \hat{x}\|^2 \leq \alpha^{-1}(V(P\hat{x}, \hat{x}, 0)) = 0$ holds for all $\hat{x} \in \mathbb{R}^{\hat{n}}$ which implies (3.34b). Let us assume that $D_2^T X D_2 \neq 0$. To prove (3.34c), we consider the inputs $w \equiv 0, \hat{w} \equiv 0, \hat{u} \equiv 0$, and choose $x = P\hat{x}$ and $\theta = 0$ in (3.6). One has:

$$0 \leq (C_2 P \hat{x} - H \hat{C}_2 \hat{x})^T D_2^T X D_2 (C_2 P \hat{x} - H \hat{C}_2 \hat{x}), \tag{3.42}$$

for all $\hat{x} \in \mathbb{R}^{\hat{n}}$. Since $D_2^T X D_2 \preceq 0$, and $D_2^T X D_2 \neq 0$ by assumption, one obtains $C_2 P - H \hat{C}_2 = 0$, which implies (3.34c). Consider the input signals $\hat{\upsilon} \equiv 0, \omega \equiv 0, \hat{\omega} \equiv 0$. It can be easily seen that the subspace $\{(x, \hat{x}, \theta) : x = P\hat{x}, \theta = 0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta}$ is invariant [Kha96], which implies that when $\xi(0) = P\hat{\xi}(0)$ and $\xi_\theta(0) = 0$, one has:

$$\xi(t) = P\hat{\xi}(t), \quad \xi_\theta(t) = 0, \quad \mathsf{d}\xi(t) = P\mathsf{d}\hat{\xi}(t), \tag{3.43}$$

for all $t \in \mathbb{R}_{\geq 0}$, from which we derive that

$$\begin{aligned}
&(AP\hat{\xi}(t) + BQ\hat{\xi}(t) + BL_1\varphi(t, F\xi(t)) - BL_2\varphi(t, \hat{F}\hat{\xi}(t)) + E\varphi(t, FP\hat{\xi}(t)))\mathsf{d}t \\
&= (P\hat{A}\hat{\xi}(t) + P\hat{E}\varphi(t, \hat{F}\hat{\xi}(t)))\mathsf{d}t,
\end{aligned} \tag{3.44}$$

for all $t \in \mathbb{R}_{\geq 0}$, thus implying (3.34a), (3.34d), and (3.34e). □

3.4.2 Geometric Interpretation of Different Conditions

In this section, we provide geometric conditions on matrices appearing on the definition of $\hat{\Sigma}$, of stochastic storage function and its corresponding interface function. The geometric conditions facilitate the construction of the abstraction. First, we recall the following result from [GP09], providing necessary and sufficient conditions for the existence of $\hat{A}$ and Q satisfying (3.34a).

Lemma 3.4.9. *Consider matrices A, B, and P. There exist matrices $\hat{A}$ and Q satisfying* (4.36a) *if and only if*

$$\text{im } AP \subseteq \text{im } P + \text{im } B. \tag{3.45}$$

Similarly, we provide necessary and sufficient conditions for the existence of $\hat{C}_2$ and $\hat{E}$, L_2 satisfying (3.34c) and (3.34e), respectively.

Lemma 3.4.10. *Given P and C_2, there exists matrix $\hat{C}_2$ satisfying* (3.34c) *if and only if*

$$\text{im } C_2 P \subseteq \text{im } H \tag{3.46}$$

for some matrix H of appropriate dimension.

Lemma 3.4.11. *Given P, B, and L_1, there exist matrices $\hat{E}$ and L_2 satisfying* (3.34e) *if and only if*

$$\text{im } E \subseteq \text{im } B + \text{im } P. \tag{3.47}$$

Lemmas 3.4.9, 3.4.10, and 3.4.11 provide sufficient and necessary conditions on P and H, resulting in the construction of matrices $\hat{A}$, $\hat{C}_2$, and $\hat{E}$ and matrices Q and L_2 appearing in the interface function (3.35). The next lemma provides a sufficient and necessary condition on the existence of $\hat{D}$ satisfying (4.36e).

Lemma 3.4.12. *Given Z, there exists matrix $\hat{D}$ satisfying* (3.34f) *if and only if*

$$\text{im } Z\hat{W} \subseteq \text{im } P, \tag{3.48}$$

for some matrix $\hat{W}$ of appropriate dimension.

Although condition (3.48) is readily satisfied by choosing $\hat{W} = 0$, one should preferably aim at finding a nonzero $\hat{W}$ with the highest possible rank to facilitate later the satisfaction of compositionality condition (3.16).

3.4.3 Construction of Abstraction

We summarize the construction of abstraction $\hat{\Sigma}$, stochastic storage function V in (3.29), and its corresponding interface function in (3.35) in Table 3.1.

Remark 3.4.13. *One way to solve the matrix inequality* (3.31) *is as follows: First, we select arbitrary C_θ and $D_\theta = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$, and solve the following bilinear matrix inequality (BMI) for $\hat{\kappa}$, X, $\widehat{M}$, and L_1:*

$$\begin{aligned}
&\begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1 + E) \\ Z^T\widehat{M} & 0 & 0 \\ (BL_1+E)^T\widehat{M} & 0 & 0 \end{bmatrix} \\
&\preceq \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^T D_2^T X D_2 C_2 - F^T M_{11} F & C_2^T D_2^T X D_1 & -FM_{12} \\ D_1^T X D_2 C_2 & D_1^T X D_1 & 0 \\ -M_{12}^T F & 0 & -M_{22} \end{bmatrix}.
\end{aligned} \tag{3.49}$$

1. Choose matrix Z such that (3.33) is satisfied;
2. Choose W such that $D = ZW$;
3. Choose matrices $\widehat{M}, K, L_1, \Lambda, X, A_\theta, C_\theta,\ B_\theta = [B_1\ \ B_2], D_\theta = [D_1\ \ D_2]$, and constants $\hat{\kappa}$, $\bar{\kappa}$ such that (3.31) is satisfied (see Remark 3.4.13);
4. Determine matrix P of lowest rank with $\ker P = 0$ that satisfies (3.45), (3.46), (3.47), and (3.48) (see Remark 3.4.14);
5. Choose $\hat{A}$ and Q according to (3.34a);
6. Choose L_2 and $\hat{E}$ according to (3.34e);
7. Compute $\hat{F} = FP$;
8. Compute $\hat{C}_1 = C_1 P$;
9. Choose $\hat{G} = 0$. Choose $\hat{\mathsf{R}} = \{\hat{R}_1, \ldots, \hat{R}_{\hat{\mathsf{r}}}\}$ and $\hat{\lambda} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{\hat{\mathsf{r}}}\}$ according to (3.41);
10. Choose $\hat{C}_2$ satisfying $H\hat{C}_2 = C_2 P$ for some H;
11. Choose $\hat{D}$ satisfying $P\hat{D} = Z\hat{W}$ for some $\hat{W}$ with the highest possible rank;
12. Choose $\hat{B}$ freely (e.g. $\hat{B} = I_{\hat{n}}$ making $\hat{\Sigma}$ fully actuated);
13. Compute $\tilde{R}$, appearing in (3.35), according to (3.40);

Table 3.1: Construction of $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{\mathsf{R}}, \varphi, \hat{\lambda})$ together with the corresponding stochastic storage function V in (4.31), with $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$, and interface function in (3.35) for a given $\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda)$.

We then solve the following bilinear matrix equation for $\bar{\kappa}$, Λ, and A_θ:

$$A_\theta^T \Lambda + \Lambda A_\theta = C_\theta^T X C_\theta - \bar{\kappa}\Lambda.$$

Finally, we solve the following linear equations for $B_\theta = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$:

$$\Lambda B_1 = C_\theta^T X D_1^T,$$

$$\Lambda B_2 = C_\theta^T X D_2.$$

Remark 3.4.14. *One way to satisfy the geometric conditions* (3.45)-(3.48) *is to start with a scalar abstraction (i.e. $\hat{n} = 1$) and pick P to be an arbitrary column vector, and check if* (3.45)-(3.48) *hold. If not, then increase the state-space dimension of the abstraction by one (i.e. $\hat{n} = 2$), add a linearly independent column vector to P, and check again if* (3.45)-(3.48) *hold. Repeat this process until* (3.45)-(3.48) *are satisfied. Note that in the worst-case scenario, this process will terminate when $\hat{n} = n$ (i.e. the state-space dimension of the concrete subsystem and abstraction are equal).*

In the next section, we provide two examples for compositional construction of abstractions of a network of stochastic hybrid systems using the technique presented in the paper. First, in a physically motivated example, we construct a compositional abstraction of a network of resistor-capacitor (R-C) circuits affected by stochastic noise. In the second example, we illustrate the advantage of using a linear control system Σ_θ over just a static map (which was used in [ZA17; AZ17b]) to conclude the joint dissipativity property of a concrete subsystem and its abstraction.

3.5 Examples

3.5.1 Case Study - Electrical Network

Consider an interconnection of n first order R-C circuits. The i-th R-C circuit has a dynamic given by:

$$\begin{aligned} \mathsf{d}v_{c_i} = & \left(-\frac{1}{R_i C_i} v_{c_i} + \frac{1}{R_i C_i} v_{s_i} + \frac{1}{C_i}\tilde{w}_i\right)\mathsf{d}t + \varpi \mathsf{d}W_t \\ & + \tau \mathsf{d}P_t, \end{aligned} \tag{3.50}$$

where $\varpi \in \mathbb{R}_{>0}, \tau \in \mathbb{R}_{>0}$, $i \in [1;n]$, $v_{s_i} \in \mathbb{R}$ represents the input source voltage (external input), $v_{c_i} \in \mathbb{R}$ is the voltage across capacitor, C_i is the capacitance, R_i is the resistance, and $\tilde{w}_i \in \mathbb{R}$ is the total current inflow from other R-C circuits in the network. The continuous noise and jump terms represent the thermal noise (also known as Johnson-Nyquist noise) and the so-called Shot noise [HHR80], respectively. Assume the rate of the Poisson process P_t is λ. For illustration purposes, in this example we fix $R_i = 1$ Ohm, and $C_i = 1$ Farad $\forall i \in [1;N]$. We consider the above interconnected system as an interconnection of N concrete subsystems Σ_i, $i \in [1;N]$, wherein each subsystem Σ_i is formed by clustering n_i R-C circuits ($n_i \leq n$). We also add a non-linearity belonging to the class of nonlinearities presented in this chapter. Each subsystem, $\Sigma_i = (A_i, B_i, C_{1i}, I_{ni}, D_i, \vec{1}_{ni}, \vec{1}_{ni}^T, \varpi\vec{1}_{ni}, \tau\vec{1}_{ni}, \varphi, \lambda)$, generates a scalar external output:

$$\Sigma_i : \begin{cases} \mathsf{d}\xi_i = (A_i\xi_i + B_iu_i + D_iw_i + \vec{1}_{ni}\varphi(\vec{1}_{ni}^T\xi_i))\mathsf{d}t \\ \qquad\quad +\varpi\vec{1}_{ni}\mathsf{d}W_t + \tau\vec{1}_{ni}\mathsf{d}P_t, \\ \zeta_{1i} = C_{1i}\xi_i, \\ \zeta_{2i} = \xi_i, \end{cases}$$

where $\xi_i = \mathsf{L}_i v$, $v = [v_{c_1}; \ldots; v_{c_n}]$, $\mathsf{L}_i \coloneqq [e_{i1}; \ldots; e_{in_i}]$, $e_{ij} \in \mathbb{R}^{1\times n}$ is a row vector whose k-th element is defined as

$$e_{ij}^{(k)} = \begin{cases} 1 \text{ if } k\text{-th R-C circuit is part of the } i\text{-th cluster} \\ 0 \text{ otherwise,} \end{cases} \tag{3.51}$$

$A_i, B_i, D_i \in \mathbb{R}^{n_i\times n_i}$ are readily obtained from (3.50), $C_{1i} \in \mathbb{R}^{1\times n_i}$, $u_i = \mathsf{L}_i v_s$, $v_s = [v_{s_1}; \ldots; v_{s_n}]$, $w_i = \mathsf{L}_i\tilde{w}$, $\tilde{w} = [\tilde{w}_1; \ldots; \tilde{w}_n]$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\varphi(x) = \sin(x).$$

The interconnection topology in this example is given by

$$M = -\begin{bmatrix} n-1 & -1 & \ldots & \ldots & -1 \\ -1 & n-1 & -1 & \ldots & -1 \\ -1 & -1 & n-1 & \ldots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \ldots & \ldots & -1 & n-1 \end{bmatrix}. \tag{3.52}$$

The interconnection topology represents a fully-connected interconnection topology. We aggregate each Σ_i into a scalar deterministic abstraction $\hat{\Sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_{1i}, 1, 1, 1, 1, 0, 0, \varphi, 0)$ given by the following dynamics

$$\hat{\Sigma}_i : \begin{cases} \mathsf{d}\hat{\xi}_i = (\hat{A}_i\hat{\xi}_i + \hat{B}_i\hat{u}_i + \hat{w}_i + \varphi(\hat{\xi}_i))\mathsf{d}t \\ \hat{\zeta}_{1i} = \hat{C}_{1i}\hat{\xi}_i, \\ \hat{\zeta}_{2i} = \hat{\xi}_i, \end{cases}$$

where $\hat{A}_i$ satisfies $A_i\vec{1}_{n_i} = \vec{1}_{n_i}\hat{A}_i$, $\hat{B}_i$ is chosen arbitrarily (in this example we choose $\hat{B}_i = 1$), $\hat{C}_{1i} = C_{1i}\vec{1}_{n_i}$. The function $V_i(x_i, \hat{x}_i) = (x_i - \vec{1}_{n_i}\hat{x}_i)^T(x_i - \vec{1}_{n_i}\hat{x}_i)$ (i.e. $\widehat{M_i} = I_{n_i}, P_i = \vec{1}_{n_i}, \Lambda_i = 0$) is a SStF-M$_2$ function from $\hat{\Sigma}_i$ to Σ_i, with the following parameters

$$\begin{aligned} &K_i = -\chi I_{n_i}, Z_i = I_{n_i}, W_i = I_{n_i}, X_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ I_{n_i} & 0_{n_i} \end{bmatrix}, \\ &\hat{\kappa}_i = 2\chi - 2\lambda\tau - \varpi^2 - \lambda\tau^2, Q_i = 0_{n_i}, H_i = \hat{W}_i = \vec{1}_{n_i}, \\ &L_{1i} = -\vec{1}_{n_i}, A_{\theta_i} = 0, B_{\theta_i} = 0, C_{\theta_i} = 0, D_{\theta_i} = I_{2n_i}, \bar{\kappa} = 0, \end{aligned} \tag{3.53}$$

where $\chi > \lambda\tau + \frac{\varpi^2}{2} + \frac{\lambda\tau^2}{2}$, and with $\alpha_i(r) = \frac{1}{\lambda_{\max}(C_{1i}^T C_{1i})} r$, $\eta_i(r) = (2\chi - 2\lambda\tau - \varpi^2 - \lambda\tau^2)r, \psi_{i\mathsf{ext}}(r) = 0, \forall r \in \mathbb{R}_{\geq 0}$, and $\mathsf{c}_i = \tau^2 + \varpi^2$. Inputs $u_i \in \mathbb{R}^{n_i}$ is given via the interface function in (3.35) as (i.e. $\tilde{\tilde{R}}_i = \vec{1}_{n_i}, L_{2i} = \vec{1}_{n_i}$)

$$u_i = -\chi(x_i - \vec{1}_{n_i}\hat{x}_i) + \vec{1}_{n_i}\hat{u}_i - \vec{1}_{n_i}\varphi(\vec{1}_{n_i}^T x_i) + \vec{1}_{n_i}\varphi(\hat{x}_i). \tag{3.54}$$

By selecting $\mu_1 = \ldots = \mu_N = 1$, the function $V(x, \hat{x}, \theta) = \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i, \theta_i)$ is an SSF-M$_2$ function from $\hat{\Sigma}$ to Σ, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ with a coupling matrix $\hat{M}$, satisfying condition (3.16) as the following

$$M\mathsf{diag}(\vec{1}_{n_1}, \ldots, \vec{1}_{n_N}) = \mathsf{diag}(\vec{1}_{n_1}, \ldots, \vec{1}_{n_N})\hat{M}. \tag{3.55}$$

A matrix $\hat{M}$ exists satisfying (3.55) if there exist N *equitable partitions* of the graph described by the Laplacian matrix $L = -M$, which is always true here because L represents a fully connected graphs, as explained in [GR13].

It can be easily seen that condition (3.15) reduces to

$$\begin{bmatrix} -L \\ I_n \end{bmatrix}^T \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} -L \\ I_n \end{bmatrix} = -(L + L^T) \preceq 0, \tag{3.56}$$

which always holds since $L = L^T \succeq 0$, which is always true for Laplacian matrices of undirected graphs [GR13].

3.5.1.1 Controller Synthesis

Now, we synthesize a controller for the abstract interconnected system $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ to enforce a specification, and then refine the designed controller to the one for the concrete interconnected system. We fix $n = 9$, $N = 3$, $\tau = 0.2, \varpi = 0.4, \lambda = 1, \chi = 10$ and

$$C_{11} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, C_{13} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

We synthesize a controller using toolbox `SCOTS` [RZ16] to enforce the following linear temporal logic specification [BK08] over the outputs of $\hat{\Sigma}$:

$$\Psi = \Box S \wedge \left(\bigwedge_{i=1}^{5} \Box(\neg O_i) \right) \wedge \Box\Diamond T_1 \wedge \Box\Diamond T_2, \tag{3.57}$$

which can be interpreted as follows: the output trajectory of the closed loop system evolves inside the set S, avoids regions $O_i, i \in [1; 5]$, indicated with blue boxes in Figure 3.2, and visits $T_i, i \in [1; 2]$ infinitely often, indicated with red boxes in Figure 3.2. We use (3.54) to generate the corresponding input enforcing this specification over the original system Σ.

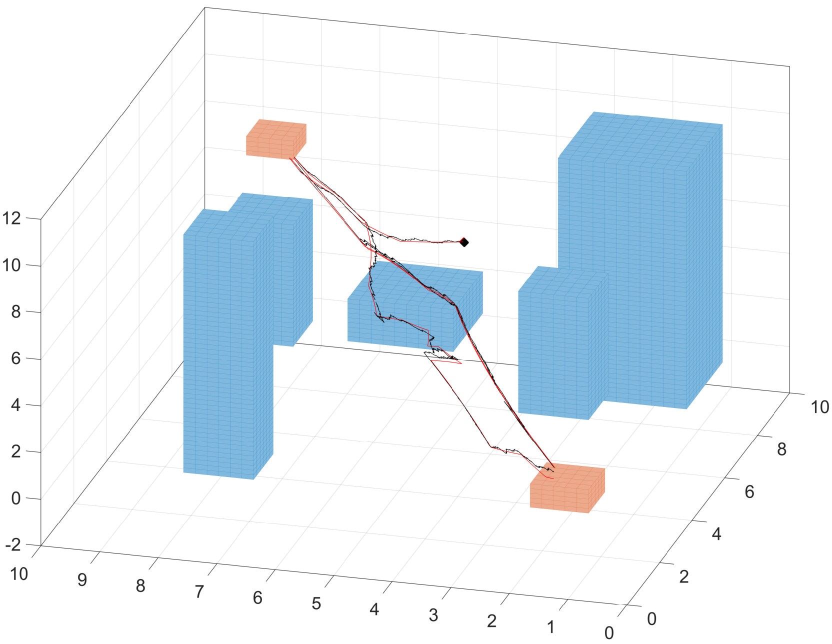

Figure 3.2: The figure shows the output trajectories of the abstract (red) and one realization of the concrete (black) interconnected systems. The initial point of the trajectories is represented by the diamond.

3.5.2 Example 2

In this part, we provide compositional abstractions of a network of subsystems wherein the joint dissipativity property of each concrete subsystem and its abstraction is only concluded with respect to a linear control system Σ_θ rather than a static map. Consider an interconnection of N second order subsystems Σ_i, where each Σ_i is given by

$$\Sigma_i : \begin{cases} \mathsf{d}\xi_i(t) = (A_i\xi_i(t) + B_i\upsilon_i(t) + D_i\omega_i(t))\mathsf{d}t, \\ \zeta_{1i}(t) = C_{1i}\xi_i(t), \\ \zeta_{2i}(t) = \xi_i(t), \end{cases} \tag{3.58}$$

where

$$A_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ -I_{n_i} & -0.5I_{n_i} \end{bmatrix}, B_i = D_i = \begin{bmatrix} 0_{n_i} \\ I_{n_i} \end{bmatrix}, C_{1i} = \begin{bmatrix} 0_{n_i} \\ e_{n_i} \end{bmatrix}^T, \tag{3.59}$$

vector e_{n_i} represents a column vector whose first element is 1 and remaining elements are zero. For the sake of simulation we choose $N = 3$, $n_i = 10$, $\forall i \in [1; N]$. We consider the following abstract system $\hat{\Sigma}_i$,

$$\hat{\Sigma}_i : \begin{cases} \mathsf{d}\hat{\xi}_i(t) = \left(\begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \hat{\xi}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\upsilon}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\omega}_i(t) \right) \mathsf{d}t, \\ \hat{\zeta}_{1i}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{\xi}_i(t), \\ \hat{\zeta}_{2i}(t) = \hat{\xi}_i(t). \end{cases}$$

We restrict K_i for each $i \in [1;N]$ appearing in (3.35) such that the last n_i columns are identically zero. This restriction can appear in practice when for example only some state variables are available to be measured. With this restriction on the structure of K_i, one cannot find a storage function with $C_{\theta_i} = 0$ in this example. Using the guidelines shown in Table 3.1 and the solver package `Yalmip` [Lof04], it can be shown that the function

$$V_i(x_i, \hat{x}_i, \theta_i) = (x_i - P\hat{x}_i)^T \widehat{M}(x_i - P\hat{x}_i) + \theta_i^T \Lambda \theta_i$$

is an SStF-M$_2$ from $\hat{\Sigma}_i$ to Σ_i, with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i})$, $\forall i \in [1;N]$, with the following parameters

$$\widehat{M}_i = \begin{bmatrix} 2I_{n_i} & I_{n_i} \\ I_{n_i} & I_{n_i} \end{bmatrix}, P_i = \begin{bmatrix} \vec{1}_{n_i} & \vec{0}_{n_i} \\ \vec{0}_{n_i} & \vec{1}_{n_i} \end{bmatrix}, K_i = \begin{bmatrix} -0.5I_{n_i} & 0_{n_i} \end{bmatrix},$$

$$\hat{\kappa}_i = 0.1, W_i = I_{n_i}, Q_i = 0, H_i = \hat{W}_i = \vec{1}_{n_i}, L_{1i} = 0, \Lambda = I_{2n_i}, \tag{3.60}$$

$$A_{\theta_i} = -5I_{2n_i}, B_{\theta_i} = \begin{bmatrix} 0_{n_i} & 0.207I_{n_i} \\ 0_{n_i} & -0.573I_{n_i} \end{bmatrix}, C_{\theta_i} = 0.1I_{2n_i}, \tag{3.61}$$

$$D_{\theta_i} = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ 0_{n_i} & I_{n_i} \end{bmatrix}, X_i = \begin{bmatrix} 9.47785I_{n_i} & -7.4055I_{n_i} \\ -7.4055I_{n_i} & 1.6526I_{n_i} \end{bmatrix}, \bar{\kappa}_i = 1, \tag{3.62}$$

with $\alpha_i(r) = \frac{\lambda_{\min}(\widehat{M}_i)}{\lambda_{\max}(C_{1i}^T C_{1i})} r$, $\eta_i(r) = 0.1r$, $\psi_{i\mathsf{ext}} = 0, \forall r \in \mathbb{R}_{\geq 0}$, and $c_i = 0$. Functions $u_i \in \mathbb{R}^{n_i}$ are given via the interface function:

$$u_i = -K_i(x_i - P_i\hat{x}_i) + \vec{1}_{n_i}\hat{u}_i, \tag{3.63}$$

(i.e. $\tilde{R}_i = \vec{1}_{n_i}, L_{2i} = 0$). With the interconnection matrix M given by

$$M = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}. \tag{3.64}$$

and by selecting $\mu_1 = \dots = \mu_N = 1$, it can be verified that the function $V = \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T \theta$, where $\theta = [\theta_i; \dots; \theta_N]$, is an SSF-M$_2$ from $\hat{\Sigma}$ to Σ, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \dots, \hat{\Sigma}_N)$ with the coupling matrix $\hat{M}$ given by

$$\hat{M} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{3.65}$$

satisfying conditions (3.15) and (3.16). In the simulation, the input signal to the abstract system is chosen arbitrarily as $\hat{\upsilon}(t) = [\sin(t); 0.1\mathsf{e}^{-t}; -t]$. Figure 3.3 shows the evolution

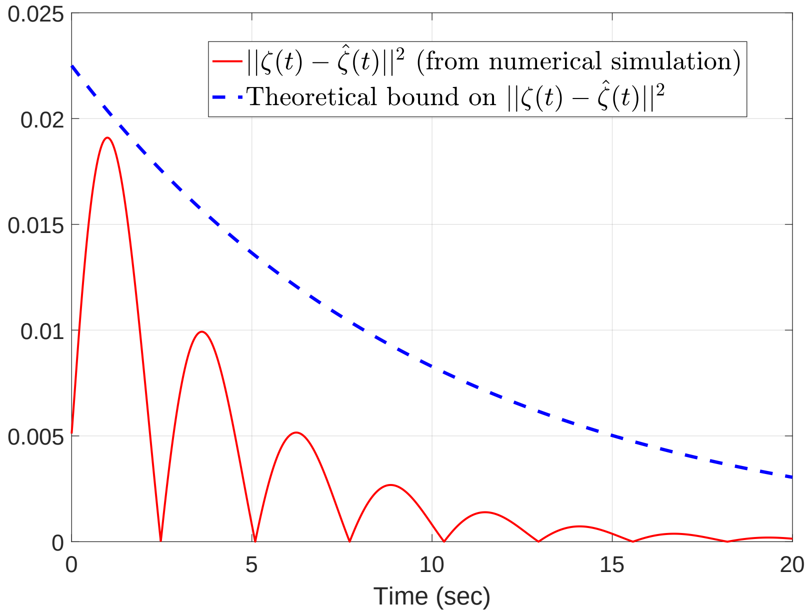

Figure 3.3: The evolution of $\|\zeta(t) - \hat{\zeta}(t)\|^2$, where $\zeta(t) = [\zeta_{11}(t); \ldots; \zeta_{1N}(t)]$, and $\hat{\zeta}(t) = [\hat{\zeta}_{11}(t); \ldots; \hat{\zeta}_{1N}(t)]$, and the theoretical upper bound obtained for this example according to (3.9).

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of the absolute value of the error between the output trajectories of the concrete interconnected system and its abstraction. One can readily verify that the error is always bounded by the computed error bound in Theorem 3.2.6.

3.6 Summary

In this chapter, using tools from stochastic calculus and dissipativity theory, we derived conditions under which abstractions of interconnected stochastic hybrid systems can be constructed compositionally. We offered two examples demonstrating the compositional construction of abstractions for a network of stochastic hybrid systems based on the methodology outlined in the chapter. First we applied this approach to a physically motivated case study, compositionally constructing an abstraction for a network of resistor-capacitor (R-C) circuits influenced by stochastic noise. In the second example, we highlighted the efficacy of employing a linear control system Σ_θ in contrast to a static map in establishing the joint dissipativity property between a subsystem and its abstraction.

Chapter 4

Compositional Abstraction of Interconnected Systems with Variable Topology

4.1 Introduction

In more realistic scenarios, the topology of interconnected systems is not fixed due to various causes, for example loss of communication between the robot agents due to occlusion caused by obstacles [SWX08; OM04], or failure of switching lines in an electric distribution grid [Cav+19].

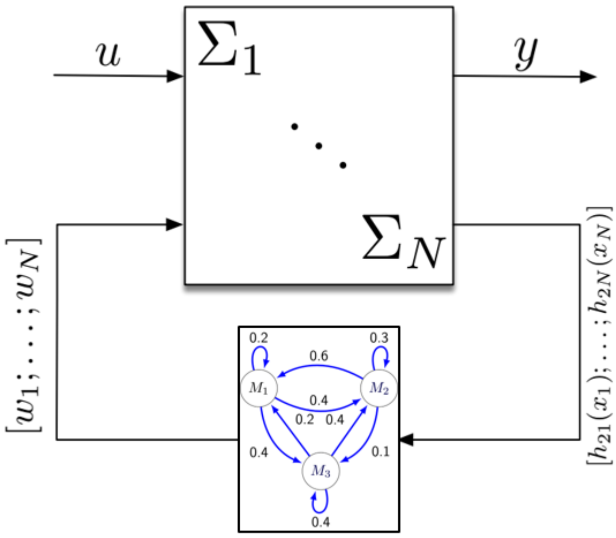

Figure 4.1: Interconnected system consisting of stochastic hybrid subsystems with a switched interconnection topology switching among three matrices (exemplification). The overall system is modeled as a switched stochastic hybrid system.

Contribution: To accommodate for this scenario, in this chapter, we deal with interconnected systems wherein the topology is not fixed. We derive conditions under which compositional abstractions of interconnected systems can be constructed using the varying interconnection topology and joint dissipativity-type properties of subsystems and their abstractions. In this regard, we consider two different cases.

In the first case, we consider a network wherein interconnection topology among the subsystems, modeled as stochastic hybrid systems, is randomly switching between $\mathcal{P}$ different topologies (see Figure 4.1). We derive conditions under which one can construct an abstraction of a given network of stochastic hybrid systems under randomly switched topologies in a compositional way. The random switching is modeled using a continuous-time Markov chain, with each state of the chain representing a different interconnection topology. In addition, we consider a scenario wherein some of these compositionality conditions are not satisfied by all the interconnection topologies (cf. Section 4.2.5). Inspired by a recent result in [WZ17], we show that compositional abstractions of interconnected stochastic hybrid systems can still be achieved under weaker compositionality conditions, provided that an additional condition on the parameters of the Markov chain is satisfied. This additional condition on the parameters of the Markov chain can be interpreted as a probabilistic version of the so-called *dwell time* condition used in the context of deterministic switching systems [Lib12]. We illustrate the effectiveness of the approach by synthesizing a controller to enforce a given specification expressed as a linear temporal logic formula [BK08] over the interconnected abstraction and then refining it back to the original interconnected system.

In the second case, we consider networks of deterministic control systems in which the interconnection topology is governed by a *linear dynamical* system [Lin84]. In such interconnected systems, the additional dynamics introduced due to the interconnection/interaction system has to be taken into account in the compositional reasoning. We derive conditions under which compositional abstractions of networks of control systems, interconnected via some dynamic interconnection topology, can be constructed using the dynamic interconnection and joint dissipativity-type properties of subsystems and their abstractions. We provide an example to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning by reducing a 150-dimensional nonlinear system (electric network) to a 3-dimensional one.

4.2 Interconnected Stochastic Hybrid System with Randomly Switched Topologies

We study the problem of constructing an abstraction of networks of stochastic hybrid systems. The interconnection topology in this network is randomly switching between $\mathcal{P}$ different topologies. Each topology is modeled by an interconnection matrix. Figure 4.1 shows such an interconnected system.

We first define the infinitesimal generator of a stochastic process which we will later use to define notions of certificates for abstractions of subsystems in Section 4.2.3. This definition is similar to the one in Definition 3.2.1.

Definition 4.2.1. *Consider two stochastic hybrid systems*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

as in Definition 2.5.1. Consider a twice continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$. *The infinitesimal generator of the stochastic process* $\Xi = [\xi; \hat{\xi}]$, *denoted by* $\mathcal{L}$, *acting on the function* V *is defined as [ØS05]:*

$$\begin{aligned}\mathcal{L}V(x,\hat{x}) :=& \begin{bmatrix}\partial_x V & \partial_{\hat{x}} V\end{bmatrix} \begin{bmatrix} f(x,u,w) \\ \hat{f}(\hat{x},\hat{u},\hat{w})\end{bmatrix} \\ &+ \frac{1}{2} Tr\left(\begin{bmatrix}\sigma(x) \\ \hat{\sigma}(\hat{x})\end{bmatrix} \begin{bmatrix}\sigma^T(x) & \hat{\sigma}^T(\hat{x})\end{bmatrix} \begin{bmatrix}\partial_{x,x}V & \partial_{x,\hat{x}}V \\ \partial_{\hat{x},x}V & \partial_{\hat{x},\hat{x}}V\end{bmatrix}\right) \\ &+ \sum_{j=1}^{\tilde{r}} \lambda_j (V(x+\rho(x)\mathsf{e}_j, \hat{x}+\hat{\rho}(\hat{x})\mathsf{e}_j) - V(x,\hat{x})),\end{aligned} \tag{4.1}$$

where e_j *denotes a vector with 1 on the* j*-th entry and 0 elsewhere.*

In the next sub-section, we introduce the model of the randomly switching interconnection topology for the interconnected system.

4.2.1 Switching Interconnection Topology

We model the switching interconnection topology using a continuous-time Markov chain defined as follows.

Definition 4.2.2. *A continuous-time Markov chain is a tuple* $\Pi = (\mathsf{P}, \mathsf{Q})$, *where*

- P *is a finite set with cardinality* $\mathcal{P}$, *called the state-space of the Markov-chain;*
- $\mathsf{Q} = \{q_{ij}\} \in \mathbb{R}^{\mathcal{P} \times \mathcal{P}}$ *is called the generator matrix.*

Associated with Π *there is a stochastic process* $\widehat{\pi} : \Omega \times \mathbb{R}_{\geq 0} \to \mathsf{P}$, *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *such that for every fixed* $\omega \in \Omega$, $\pi(\cdot) = \widehat{\pi}(\omega, \cdot) : \mathbb{R}_{\geq 0} \to \mathsf{P}$. *For any* $i, j \in \mathsf{P}$ *and* $t \in \mathbb{R}_{\geq 0}$, *one has*

$$\mathbb{P}(\pi(t+h) = j | \pi(t) = i) = \begin{cases} q_{ij}h + o(h), & i \neq j, \\ 1 + q_{ii}h + o(h), & i = j, \end{cases} \tag{4.2}$$

where $h > 0$, $\lim_{h \to 0} \frac{o(h)}{h} = 0$, $q_{ii} = -\sum_{i \neq j} q_{ij}$, *and* $q_{ij} \geq 0$ *are called the transition jump rates from* i *to* j *if* $i \neq j$. *We denote the value of the solution process at time* $t \in \mathbb{R}_{\geq 0}$ *by* $\pi(t)$, *and refer to* π *as the switching process.*

We assume that Π is ergodic. The ergodicity of Π implies that there exists a unique stationary distribution denoted by $\varrho_s = (\varrho_{s1}, \ldots, \varrho_{s\mathcal{P}})$, such that for any $i, j \in \mathsf{P}$:

$$\lim_{t \to \infty} \mathbb{P}(\pi(t+h) = j | \pi(t) = i) = \varrho_{sj}. \tag{4.3}$$

We also assume that $\widehat{\pi}(\cdot, 0) = \pi(0)$ is measurable in trivial sigma-algebra $\mathcal{F}_0$. Let $\{T_k\}$, $k \in \mathbb{N}$, be the sequence of times at which switching occurs. The sojourn time S_j (the time to stay in a mode) in each state $j \in \mathsf{P}$ is exponentially distributed with mean $\theta_j = \frac{1}{|q_{jj}|}$, i.e. $\mathbb{P}(S_j \leq z) = 1 - \mathsf{e}^{-\frac{z}{\theta_j}}$, for any $z \in \mathbb{R}_{>0}$. In addition, the sequence $\{T_{k+1} - T_k\}$, where $k \in \mathbb{N}$, is a collection of independent random variables and is independent of the switching process π.

4.2.2 Interconnected System

We now show how a switching stochastic hybrid system is induced by the interconnection of stochastic hybrid subsystems via a switched interconnection topology governed by a continuous-time Markov chain as in Definition 4.2.2.

Definition 4.2.3. *Consider $N \in \mathbb{N}_{\geq 1}$ stochastic hybrid subsystems*

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1; N]$. Consider a continuous-time Markov chain $\Pi = (\mathsf{P}, \mathsf{Q})$, as in Definition 4.2.2, with $\mathsf{P} = \{1, \ldots, \mathcal{P}\}$, and switching process π. Consider a set of interconnection matrices $M = \{M_1, \ldots, M_{\mathcal{P}}\}$, where each matrix M_i, $i \in [1; \mathcal{P}]$, defines the coupling of these subsystems. The interconnected switching stochastic hybrid system[1]

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \mathsf{P}, \pi, f, \sigma, \rho, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}_\pi^M(\Sigma_1, \ldots, \Sigma_N)$, follows by $n = \sum_{i=1}^N n_i, m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{1i}$, and the function

$$\begin{aligned} f(x, u, p) &\coloneqq [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)], && (4.4)\\ \sigma(x) &\coloneqq [\sigma_1(x_1); \ldots; \sigma_N(x_N)], && (4.5)\\ \rho(x) &\coloneqq [\rho_1(x_1); \ldots; \rho_N(x_N)], && (4.6)\\ h(x) &\coloneqq [h_{11}(x_1); \ldots; h_{1N}(x_N)], && (4.7) \end{aligned}$$

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$ and the internal variables are constrained by

$$[w_1; \ldots; w_N] = M_p[h_{21}(x_1); \ldots; h_{2N}(x_N)], \tag{4.8}$$

for any $p \in \mathsf{P}$, where p is determined by the switching process π.

In the next section, we address the problem of constructing an abstraction of a switching stochastic hybrid system by introducing a Lyapunov-like function which is used to establish a quantitative bound between the output of the abstraction and the concrete system.

[1] see Definition 2.5.4 for the definition

4.2.3 Certificates for Abstraction for Networks with Randomly Switched Topologies

In the following subsection, we introduce the notion of stochastic simulation function used to relate two switching stochastic hybrid systems. This notion is similar to the one introduced in Definition 3.2.5, but adapted for switching stochastic hybrid systems.

4.2.3.1 Stochastic Simulation Function

Definition 4.2.4. *Let* $\Pi = (\mathsf{P}, \mathsf{Q})$ *be a continuous-time Markov chain with switching process* π. *Let*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \mathsf{P}, \pi, f, \sigma, \rho, \mathbb{R}^q, h),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \mathsf{P}, \pi, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h}),$$

be two switching stochastic hybrid systems as in Definition 2.5.4. A function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathsf{P} \to \mathbb{R}_{\geq 0}$ *is called a stochastic simulation function in the second moment*[2] *(SSF-M*$_2$*), from* $\hat{\Sigma}$ *to* Σ *if* $V(\cdot, \cdot, j) : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ *is twice continuously differentiable, it has a polynomial growth rate* $\forall j \in \mathsf{P}$, *and there exist a convex function* $\alpha \in \mathcal{K}_\infty$, *concave function* $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, *and positive constant* κ, *such that* $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, *and* $\forall j \in \mathsf{P}$, *one has*

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|^2) \leq V(x, \hat{x}, j), \tag{4.9}$$

and $\forall j \in \mathsf{P}$, $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, *and* $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$ $\exists u \in \mathbb{R}^m$ *such that:*

$$\mathcal{L}V(x, \hat{x}, j) \leq -\kappa V(x, \hat{x}, j) + \psi_{\text{ext}}(\|\hat{u}\|^2). \tag{4.10}$$

We say that a switching stochastic hybrid system $\hat{\Sigma}$ is approximately simulated by a switching stochastic hybrid system Σ if there exists an SSF-M$_2$ V from $\hat{\Sigma}$ to Σ. We call $\hat{\Sigma}$ (possibly with $\hat{n} < n$) an abstraction of Σ.

The next theorem shows the importance of the existence of an SSF-M$_2$ by quantifying the error between the output trajectories of Σ and those of its abstraction $\hat{\Sigma}$.

Theorem 4.2.5. *Let* $\Pi = (\mathsf{P}, \mathsf{Q})$ *be a continuous-time Markov chain with switching process* π. *Let us consider two switching stochastic hybrid systems*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \mathsf{P}, \pi, f, \sigma, \rho, \mathbb{R}^q, h),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \mathsf{P}, \pi, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h}).$$

Suppose V *is an SSF-M*$_2$ *from* $\hat{\Sigma}$ *to* Σ. *Then, there exists a* $\mathcal{KL}$ *function* β *and a function* $\gamma_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ *such that for any random variables* a *and* $\hat{a}$ *that are* $\mathcal{F}_0$*-measurable,*

[2] We use the term *second moment* because the stochastic simulation function is used to quantify the *square* of the norm of the error between output trajectories of Σ and $\hat{\Sigma}$ (see Theorem 4.2.5).

and for any $\hat{\upsilon} \in \hat{\mathcal{U}}$ *there exists* $\upsilon \in \mathcal{U}$ *such that the following inequality holds for any* $t \in \mathbb{R}_{\geq 0}$*:*

$$\mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^2] \leq \beta(\mathbb{E}[V(a, \hat{a}, \pi(0))], t) + \gamma_{\text{ext}}(\mathbb{E}[\|\hat{\upsilon}\|_\infty^2]). \tag{4.11}$$

Proof. Note that inequality (4.10) can be written in the so-called implication form, i.e., $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}, \forall \hat{u} \in \mathbb{R}^{\hat{m}}, \exists u \in \mathbb{R}^m$ such that

$$\mathcal{L}V(x, \hat{x}, j) \leq -\lambda_0 V(x, \hat{x}, j), \quad \text{whenever } V(x, \hat{x}, j) > \tfrac{\psi_{\text{ext}}(\|\hat{u}\|^2)}{\epsilon_0}, \tag{4.12}$$

for some $0 < \epsilon_0 < \kappa$, where $\lambda_0 = \kappa - \epsilon_0$.

For any $r \in \mathbb{R}_{\geq 0}$, we define the following set

$$\bar{B}(r) := \left\{ [x; \hat{x}] \in \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \;\middle|\; \begin{array}{l} \|[x; \hat{x}]\| \leq \alpha^* \\ \text{where} \quad \alpha^* \quad = \\ \sup\limits_{j \in \mathsf{P}} \sup\limits_{V(y, \hat{y}, j) = r} \|[y; \hat{y}]\| \end{array} \right\}. \tag{4.13}$$

We define the following functions

$$c_1(r) := \sup_{(x, \hat{x}, j) \in \bar{B}(r) \times \mathsf{P}} \mathcal{L}V(x, \hat{x}, j), \tag{4.14}$$

and

$$c_2(r) := \lambda_0 \left(\sup_{(x, \hat{x}, j) \in \bar{B}(r) \times \mathsf{P}} V(x, \hat{x}, j) \right). \tag{4.15}$$

When $[x; \hat{x}] \in \bar{B}(\rho(\|\hat{u}\|^2))$, where $\rho(r) := \frac{\psi_{\text{ext}}(r)}{\epsilon_0}$, it follows that

$$\mathcal{L}V(x, \hat{x}, j) \leq c_1(\rho(\|\hat{u}\|^2)), \tag{4.16}$$

and therefore one can write

$$\mathcal{L}V(x, \hat{x}, j) \mathbf{I}_{\bar{B}(\rho(\|\hat{u}\|^2))}([x; \hat{x}]) \leq \left(c_1(\rho(\|\hat{u}\|^2))\right) \mathbf{I}_{\bar{B}(\rho(\|\hat{u}\|^2))}([x; \hat{x}]), \tag{4.17}$$

for any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}, j \in \mathsf{P}$. From (4.12), one can readily write

$$\mathcal{L}V(x, \hat{x}, j) \left(1 - \mathbf{I}_{\bar{B}(\rho(\|\hat{u}\|^2))}([x; \hat{x}])\right) \leq -\lambda_0 V(x, \hat{x}, j) \left(1 - \mathbf{I}_{\bar{B}(\rho(\|\hat{u}\|^2))}([x; \hat{x}])\right). \tag{4.18}$$

Therefore, one has

$$\mathcal{L}V(x, \hat{x}, j) \leq -\lambda_0 V(x, \hat{x}, j) + C(\rho(\|\hat{u}\|^2)) \mathbf{I}_{\bar{B}(\rho(\|\hat{u}\|^2))}([x; \hat{x}]), \tag{4.19}$$

for any $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}, j \in \mathsf{P}$, where $C(r) := c_1(r) + c_2(r)$. Inequality (4.19) has a form similar to inequality (3.52) in [Cha07]. Following the subsequent arguments in the proof of Theorem 3.29 in[Cha07], one can use Ito's formula, Doob's optional sampling theorem [GS20], the monotone convergence theorem [KS88], and Fatou's lemma [Rud87] to state that there exists a concave function $\vartheta \in \mathcal{K}_\infty$ such that for any $\hat{\upsilon} \in \hat{\mathcal{U}}$ there exists $\upsilon \in \mathcal{U}$ such that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\mathbb{E}[V(\xi(t), \hat{\xi}(t), \pi(t))] \leq \mathsf{e}^{-\lambda_0 t} V(\xi(0), \hat{\xi}(0), \pi(0)) + \frac{\vartheta(\rho(\|\hat{\upsilon}\|_\infty^2))}{\lambda_0}. \tag{4.20}$$

Using the convexity of α in inequality (4.9) and Jensen's inequality [Rud87], we obtain the following inequality which holds for any $t \in \mathbb{R}_{\geq 0}$

$$\mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^2] \leq \beta(\mathbb{E}[V(a, \hat{a}, \pi(0))], t) + \gamma_{\text{ext}}(\mathbb{E}[\|\hat{\upsilon}\|_\infty^2]), \tag{4.21}$$

where the functions β and γ_{ext} are defined as

$$\begin{aligned} \beta(r, s) &:= \alpha^{-1}(2r\mathsf{e}^{-\lambda_0 s}), \\ \gamma_{\text{ext}}(r) &:= \alpha^{-1}\left(\frac{2}{\lambda_0}\vartheta\left(\frac{\psi_{\text{ext}}(r)}{\epsilon_0}\right)\right). \end{aligned} \tag{4.22}$$

□

Finding such a stochastic simulation function between a high-dimensional switching stochastic hybrid system (e.g. one induced by the interconnection of large number of subsystems) and a candidate abstraction can be difficult. To circumvent this problem, we adopt a bottom-up approach by utilizing a local certificate, namely *stochastic storage function* similar to the one defined in Definition 2.5.1, to relate each (stochastic hybrid) subsystem with its respective abstraction. Later, in Section 4.2.4, we derive conditions under which one can compositionally construct a stochastic simulation function between the concrete interconnected system and the interconnection of abstractions of subsystems from these local certificates.

4.2.3.2 Stochastic Storage Function

To relate two stochastic hybrid subsystems, we recall the notion of stochastic storage function from Definition 2.5.1, adapted from the notion of storage functions from dissipativity theory [AMP16]. The notion we introduce here is similar to the one defined in Definition 2.5.1, with the restriction that the joint dissipativity properties are defined with respect to a static map, instead of a dynamic map, whose input is the (internal) inputs and outputs of the subsystems and their abstractions.

Definition 4.2.6. *Let*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

be two stochastic hybrid systems as in Definition 2.5.1 with the same external output space dimension, and with solution processes ξ *and* $\hat{\xi}$*, respectively. A twice continuously differentiable function* $\mathcal{S} : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ *is called a stochastic storage function from* $\hat{\Sigma}$ *to* Σ *in the second moment (SStF-M*$_2$*), if it has polynomial growth rate and if there exist convex function* $\alpha \in \mathcal{K}_\infty$*, concave function* $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$*, some positive constant* κ*, some matrices* $W, \hat{W}$*, and* H *of appropriate dimensions, and some symmetric matrix* X *of appropriate dimension with conformal block partitions* $X^{ij}, i, j \in [1;2]$*, with* $X^{22} \preceq 0$*, such that* $\forall x \in \mathbb{R}^n$ *and* $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$*, one has*

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^2) \leq \mathcal{S}(x, \hat{x}), \tag{4.23}$$

and for any $x \in \mathbb{R}^n$*,* $\hat{x} \in \mathbb{R}^{\hat{n}}$*, and* $\hat{u} \in \mathbb{R}^{\hat{m}}$ *there exists* $u \in \mathbb{R}^m$ *such that for any* $\hat{w} \in \mathbb{R}^{\hat{p}}$ *and any* $w \in \mathbb{R}^p$*, one obtains*

$$\begin{aligned}\mathcal{L}\mathcal{S}(x, \hat{x}) \leq &-\kappa \mathcal{S}(x, \hat{x}) + \psi_{\text{ext}}(\|\hat{u}\|^2) \\ &+ \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.\end{aligned} \tag{4.24}$$

Condition (4.24) implies the existence of a function $u = k(x, \hat{x}, \hat{u})$ to choose u for any $x, \hat{x}$, and $\hat{u}$. We call this function an *interface* function. The stochastic hybrid system $\hat{\Sigma}$ (possibly with $\hat{n} < n$) is called an abstraction of Σ.

4.2.4 Compositionality Result

The next theorem, inspired by Theorem 7 in [AZ17a] and adapted to account for randomly switched topologies, provides a compositional approach on the construction of abstractions of networks of stochastic hybrid systems under randomly switched topologies.

Assume we are given N stochastic hybrid subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i})$$

together with their corresponding abstractions

$$\hat{\Sigma}_i = (\mathbb{R}^{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{\hat{p}_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \hat{\sigma}_i, \hat{\rho}_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{\hat{q}_{2i}}, \hat{h}_{1i}, \hat{h}_{2i}),$$

and with SStF-M$_2$ $\mathcal{S}_i$ from $\hat{\Sigma}_i$ to Σ. We use W_i, $\hat{W}_i$, H_i, and X_i to denote the corresponding matrices appearing in Definition 4.2.6.

Theorem 4.2.7. *Consider an interconnected switching stochastic hybrid system* $\Sigma = \mathcal{I}_\pi^M(\Sigma_1, \ldots, \Sigma_N)$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *stochastic hybrid subsystems* Σ_i*, a set of interconnection matrices* $M = \{M_1, \ldots, M_{\mathcal{P}}\}$*, and a continuous-time Markov chain* $\Pi = (\mathsf{P}, \mathsf{Q})$ *governing the switching between the interconnection topologies with associated stochastic process* π*. More specifically, the interconnection topology at any time* $t \in \mathbb{R}_{\geq 0}$ *is given by* $M_{\pi(t)}$*. Suppose each subsystem* Σ_i *admits an abstraction* $\hat{\Sigma}_i$ *with the corresponding SStF-M*$_2$ $\mathcal{S}_i$*. If there exists a finite set of matrices*

$\hat{M} = \{\hat{M}_1, \ldots, \hat{M}_{\mathcal{P}}\}$ of appropriate dimension such that for each $j \in [1;\mathcal{P}]$ the matrix (in)equalities

$$\begin{bmatrix} WM_j \\ I_{\tilde{q}} \end{bmatrix}^T X(\mu_1^j X_1, \ldots, \mu_N^j X_N) \begin{bmatrix} WM_j \\ I_{\tilde{q}} \end{bmatrix} \preceq 0, \tag{4.25}$$

$$WM_j H = \hat{W}\hat{M}_j, \tag{4.26}$$

are satisfied for some $\mu_i^j > 0$, $i \in [1;N]$, where $\tilde{q} = \sum_{i=1}^N q_{2i}$ and

$$\begin{aligned} &W = \mathsf{diag}(W_1, \ldots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \ldots, \hat{W}_N),\\ &H = \mathsf{diag}(H_1, \ldots, H_N),\\ &X(\mu_1^j X_1, \ldots, \mu_N^j X_N) =\\ &\begin{bmatrix} \mu_1^j X_1^{11} & & & \mu_1^j X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N^j X_N^{11} & & & \mu_N^j X_N^{12} \\ \mu_1^j X_1^{21} & & & \mu_1^j X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N^j X_N^{21} & & & \mu_N^j X_N^{22} \end{bmatrix}, \end{aligned} \tag{4.27}$$

then

$$V(x, \hat{x}, j) := \sum_{i=1}^N \mu_i^j \mathcal{S}_i(x_i, \hat{x}_i),$$

is an SSF-M$_2$ from the interconnected switching stochastic hybrid system $\hat{\Sigma} := \mathcal{I}_\pi^{\hat{M}}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$, with the interconnection topology at time $t \in \mathbb{R}_{\geq 0}$ given by $\hat{M}_{\pi(t)}$, to Σ.

Proof. First we show that inequality (4.9) holds for some convex $\mathcal{K}_\infty$ function α. For any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $j \in \mathsf{P}$, one gets:

$$\begin{aligned} \|h(x) - \hat{h}(\hat{x})\|^2 &\leq \sum_{i=1}^N \|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|^2 \\ &\leq \sum_{i=1}^N \alpha_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i)) \leq \underline{\alpha}_j(V(x, \hat{x}, j)), \end{aligned}$$

where $\underline{\alpha}_j$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}_j(s) := \begin{cases} \max\limits_{\vec{s} \geq 0} & \sum_{i=1}^N \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu_j^T \vec{s} = s, \end{cases} \tag{4.28}$$

where $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$ and $\mu_j = [\mu_1^j; \ldots; \mu_N^j]$. Since $\underline{\alpha}_j \in \mathcal{K}_\infty$ are concave functions as argued in [ZRE17], there exists a concave function $\underline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}_j \leq \underline{\alpha}$ $\forall j \in \mathsf{P}$. By defining $\alpha = \underline{\alpha}^{-1}$ which is a convex $\mathcal{K}_\infty$ function, one obtains

$$
\begin{aligned}
&\mathcal{L}V(x,\hat{x},j)=\sum_{i=1}^{N}\mu_i^j\mathcal{L}\mathcal{S}_i(x_i,\hat{x}_i)\\
&\leq \sum_{i=1}^{N}\mu_i^j\Bigg(-\kappa_i\mathcal{S}_i(x_i,\hat{x}_i)+\psi_{i\mathrm{ext}}(\|\hat{u}_i\|^2)+\begin{bmatrix}W_iw_i-\hat{W}_i\hat{w}_i\\ h_{2i}(x_i)-H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}^T\begin{bmatrix}X_i^{11} & X_i^{12}\\ X_i^{21} & X_i^{22}\end{bmatrix}\begin{bmatrix}W_iw_i-\hat{W}_i\hat{w}_i\\ h_{2i}(x_i)-H_i\hat{h}_{2i}(\hat{x}_i)\end{bmatrix}\Bigg)\\
&\leq \sum_{i=1}^{N}-\mu_i^j\kappa_i\mathcal{S}_i(x_i,\hat{x}_i)+\sum_{i=1}^{N}\mu_i^j\psi_{i\mathrm{ext}}(\|\hat{u}_i\|^2)\\
&\quad+\begin{bmatrix}W\begin{bmatrix}w_1\\ \vdots\\ w_N\end{bmatrix}-\hat{W}\begin{bmatrix}\hat{w}_1\\ \vdots\\ \hat{w}_N\end{bmatrix}\\ h_1(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}^T X(\mu_1^jX_1,\dots,\mu_N^jX_N)\begin{bmatrix}W\begin{bmatrix}w_1\\ \vdots\\ w_N\end{bmatrix}-\hat{W}\begin{bmatrix}\hat{w}_1\\ \vdots\\ \hat{w}_N\end{bmatrix}\\ h_1(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\\
&\leq \sum_{i=1}^{N}-\mu_i^j\kappa_i\mathcal{S}_i(x_i,\hat{x}_i)+\sum_{i=1}^{N}\mu_i^j\psi_{i\mathrm{ext}}(\|\hat{u}_i\|^2)\\
&\quad+\begin{bmatrix}h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}^T\begin{bmatrix}WM_j\\ I_{\tilde{q}}\end{bmatrix}^T X(\mu_1^jX_1,\dots,\mu_N^jX_N)\begin{bmatrix}WM_j\\ I_{\tilde{q}}\end{bmatrix}\begin{bmatrix}h_{21}(x_1)-H_1\hat{h}_{21}(\hat{x}_1)\\ \vdots\\ h_{2N}(x_N)-H_N\hat{h}_{2N}(\hat{x}_N)\end{bmatrix}\\
&\leq -\kappa V(x,\hat{x},j)+\psi_{\mathrm{ext}}(\|\hat{u}\|^2).
\end{aligned}\tag{4.29}
$$

$$\alpha(\|h_1(x)-\hat{h}_1(\hat{x})\|^2)\leq V(x,\hat{x},j),$$

satisfying inequality (4.9). Now we show inequality (4.10). Consider any $x=[x_1;\dots;x_N]\in\mathbb{R}^n,\hat{x}=[\hat{x}_1;\dots;\hat{x}_N]\in\mathbb{R}^{\hat{n}}$, $\hat{u}=[\hat{u}_1;\dots;\hat{u}_N]\in\mathbb{R}^{\hat{m}}$ and $j\in\mathsf{P}$. For any $i\in[1;N]$, there exists $u_i\in\mathbb{R}^{m_i}$, consequently, a vector $u=[u_1;\dots;u_N]\in\mathbb{R}^m$, satisfying (4.63) for each pair of subsystems Σ_i and $\hat{\Sigma}_i$ with the internal inputs given by $[w_1;\dots;w_N]=M_j[h_{21}(x_1);\dots;h_{2N}(x_N)]$ and $[\hat{w}_1;\dots;\hat{w}_N]=\hat{M}_j[\hat{h}_{21}(\hat{x}_1);\dots;\hat{h}_{2N}(\hat{x}_N)]$. We consider the infinitesimal generator of function V and employ conditions (4.25) and (4.26) which result in the chain of inequalities (4.29). In (4.29) the constant $\kappa=\min\limits_{i\in[1;N]}\kappa_i$ and the function $\psi_{\mathrm{ext}}\in\mathcal{K}_\infty\cup\{0\}$ is defined as the following. Consider $\mathcal{K}_\infty\cup\{0\}$ functions

$$\psi_{\mathrm{ext}}^j(s):=\begin{cases}\max\limits_{\vec{s}\geq 0} & \sum_{i=1}^N\mu_i^j\psi_{i\mathrm{ext}}(s_i)\\ \text{s.t.} & \|\vec{s}\|\leq s.\end{cases}$$

Let us recall that by assumption functions $\psi_{i\mathrm{ext}}$ $\forall i\in[1;N]$ are concave functions. Thus, function ψ_{ext}^j above defines a *perturbation function* which is a concave one; see [BV04] for further details. Since $\psi_{\mathrm{ext}}^j\in\mathcal{K}_\infty\cup\{0\}$ are concave functions, there exists a concave function $\psi_{\mathrm{ext}}\in\mathcal{K}_\infty\cup\{0\}$ such that $\psi_{\mathrm{ext}}^j\leq\psi_{\mathrm{ext}}$ $\forall j\in\mathsf{P}$. Hence, we conclude that V is an SSF-M$_2$ function from $\hat{\Sigma}$ to Σ. □

In the next section, we provide a result for compositional construction of abstractions under compositionality conditions which are weaker than the ones given in Theorem 4.2.7.

4.2.5 Weaker Compositionality Conditions

In this section, we consider an interconnection of a specific class of stochastic hybrid systems called jump linear stochastic systems (JLSS).

4.2.5.1 Jump Linear Stochastic Systems

In a JLSS, the drift, diffusion, reset, and output functions are given as:

$$\begin{aligned}
&\mathsf{d}\xi(t) = (A\xi(t) + B\upsilon(t) + D\omega(t))\mathsf{d}t + E\xi(t)\mathsf{d}W_t + \sum_{i=1}^{\tilde{r}} F_i\xi(t)\mathsf{d}P_t^i,\\
&\zeta_1(t) = C_1\xi(t),\\
&\zeta_2(t) = C_2\xi(t),
\end{aligned} \tag{4.30}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times p}, E \in \mathbb{R}^{n\times n}, F_i \in \mathbb{R}^{n\times n}, \forall i \in [1;\tilde{r}], C_1 \in \mathbb{R}^{q_1\times n}$, and $C_2 \in \mathbb{R}^{q_2\times n}$. We use the tuple

$$\Sigma = (A, B, C_1, C_2, D, E, \mathsf{F}),$$

where $\mathsf{F} = \{F_1, \ldots, F_{\tilde{r}}\}$, to refer to the class of system of the form (4.30).

We now we recall the result from [AZ17a] which shows that the function

$$\mathcal{S}(x, \hat{x}) = (x - P\hat{x})^T\Phi(x - P\hat{x}), \tag{4.31}$$

where $\Phi \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $P \in \mathbb{R}^{n\times n}$, is a SStF-M$_2$ function from an abstraction $\hat{\Sigma}$ to the concrete JLSS Σ under some conditions. We first require the following assumption on the concrete system Σ.

Assumption 4.2.8. *Let $\Sigma = (A, B, C_1, C_2, D, E, \mathsf{F})$. Assume that for some constant $\widehat{\kappa} \in \mathbb{R}_{>0}$, there exist matrices $\Phi \succ 0, K, Z, W, X^{11}, X^{12}, X^{21}$, and $X^{22} \preceq 0$ of appropriate dimensions such that the following matrix (in)equalities hold:*

$$\begin{bmatrix} \Delta & \Phi Z \\ Z^T\Phi & 0 \end{bmatrix} \preceq \begin{bmatrix} -\hat{\kappa}\Phi + C_2^T X^{22} C_2 & C_2^T X^{21} \\ X^{12}C_2 & X^{11} \end{bmatrix}, \tag{4.32}$$

$$D = ZW, \tag{4.33}$$

where

$$\Delta := (A + BK + \sum_{j=1}^{\tilde{r}} \lambda_i F_i)^T\Phi + \Phi(A + BK + \sum_{i=1}^{\tilde{r}} \lambda_i F_i) \tag{4.34}$$

$$+ E^T\Phi E + \sum_{i=1}^{\tilde{r}} \lambda_i F_i^T \Phi F_i. \tag{4.35}$$

We now recall a result from [AZ17a] showing that $\mathcal{S}$ is a SStF-M$_2$ function from an abstraction $\hat{\Sigma}$ to the concrete JLSS Σ

Theorem 4.2.9. *Let $\Sigma = (A, B, C_1, C_2, D, E, \mathsf{F})$, and $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{\mathsf{F}})$ with same external output dimension. Suppose Assumption 4.2.8 holds and there exist matrices Q, H, and $\hat{W}$ of appropriate dimensions such that:*

$$AP = P\hat{A} - BQ \tag{4.36a}$$

$$C_1 P = \hat{C}_1 \tag{4.36b}$$

$$X^{12} C_2 P = X^{12} H \hat{C}_2 \tag{4.36c}$$

$$X^{22} C_2 P = X^{22} H \hat{C}_2 \tag{4.36d}$$

$$P\hat{D} = Z\hat{W} \tag{4.36e}$$

$$EP = P\hat{E} \tag{4.36f}$$

$$F_i P = P\hat{F}_i \quad \forall i \in [1; \tilde{r}]. \tag{4.36g}$$

Then function $\mathcal{S}$ defined in (4.31) *is a SStF-M$_2$ function from $\hat{\Sigma}$ to Σ.*

Proof. The proof can be found in [AZ17a] and is omitted here. □

4.2.5.2 Feasibility of matrix inequality (4.32)

In this subsection, we discuss the feasibility of matrix inequality (4.32) appearing in Assumption 4.2.8. For the restricted case of $X^{12} = X^{21} = 0$, $X^{22} = -I_*$, and $X^{11} = \frac{1}{\tilde{\gamma}^2} I_*$ for some $\tilde{\gamma} > 0$, where 0 denotes the zero matrix and I_* the identity matrix of appropriate dimensions, the matrix inequality (4.32) reduces to

$$\begin{aligned}
&(A + BK + \sum_{j=1}^{\tilde{r}} \lambda_i F_i)^T \Phi + \Phi(A + BK + \sum_{i=1}^{\tilde{r}} \lambda_i F_i) \\
&+ E^T \Phi E + \sum_{i=1}^{\tilde{r}} \lambda_i F_i^T \Phi F_i + \hat{\kappa}\Phi + C_2^T C_2 + \tilde{\gamma}^2 \Phi Z Z^T \Phi \preceq 0.
\end{aligned} \tag{4.37}$$

Suppose we can find a matrix K such that the matrix pair $(A + BK, Z)$ is controllable and the matrix pair $(C_2, A + BK)$ is observable, then by virtue of an extension of the Positive Real Lemma for stochastic systems [RH16] to jump linear stochastic systems, condition (4.37) means that the system

$$\mathsf{d}\xi(t) = (A\xi(t) + B\upsilon(t) + Z\omega(t))\mathsf{d}t + E\xi(t)\mathsf{d}W_t + \sum_{i=1}^{\tilde{r}} F_i \xi(t) \mathsf{d}P_t^i, \tag{4.38}$$

$$\zeta(t) = C_2 \xi(t), \tag{4.39}$$

can be enforced stochastically exponentially nonexpansive (see Section IV in [RH16] for a definition) under a linear control law $\upsilon = K\xi$. Thus the feasibility of the restricted version of the matrix inequality in Assumption 4.2.8 is dual to the problem wherein the system (4.39) is enforced stochastically exponentially nonexpansive by a linear control law. For deterministic systems, the analogue of enforcing stochastic exponential nonexpansivity is the finite L_2 gain assignment problem [AMP16]. In the context of observer design and observer based control, the feasibility of those dual control problems have been investigated for several physical problems in [Sch04], [FA03], and [Arc+03].

4.2.5.3 Weaker Compositionality Conditions for Interconnected JLSSs

We now present the main result in this section which provides sufficient conditions under which compositional abstraction of networks of JLSSs under randomly switched topologies can be constructed with weaker compositionality conditions.

Assume we are given N JLSSs

$$\Sigma_i = (A_i, B_i, C_{1i}, C_{2i}, D_i, E_i, \mathsf{F}_i),$$

where $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}, D_i \in \mathbb{R}^{n_i \times p_i}, E_i \in \mathbb{R}^{n_i \times n_i}$, $\mathsf{F}_i = \{F_{1i}, \ldots, F_{\tilde{r}i}\}, C_{1i} \in \mathbb{R}^{q_{1i} \times n_i}, C_{2i} \in \mathbb{R}^{q_{2i} \times n_i}$ together with their corresponding abstractions

$$\hat{\Sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_{1i}, \hat{C}_{2i}, \hat{D}_i, \hat{E}_i, \hat{\mathsf{F}}_i),$$

where $\hat{A}_i \in \mathbb{R}^{\hat{n}_i \times \hat{n}_i}, \hat{B}_i \in \mathbb{R}^{\hat{n}_i \times \hat{m}_i}, \hat{D}_i \in \mathbb{R}^{\hat{n}_i \times \hat{p}_i}, \hat{E}_i \in \mathbb{R}^{\hat{n}_i \times \hat{n}_i}$, $\hat{\mathsf{F}}_i = \{\hat{F}_{1i}, \ldots, \hat{F}_{\tilde{r}i}\}, \hat{C}_{1i} \in \mathbb{R}^{q_{1i} \times \hat{n}_i}$, $\hat{C}_{2i} \in \mathbb{R}^{\hat{q}_{2i} \times \hat{n}_i}$, and with SStF-M$_2$

$$\mathcal{S}_i(x_i, \hat{x}_i) = (x_i - P_i \hat{x}_i)^T \Phi_i (x_i - P_i \hat{x}_i), \tag{4.40}$$

from $\hat{\Sigma}_i$ to Σ_i, $i \in [1; N]$, where $P_i \in \mathbb{R}^{n_i \times \hat{n}_i}$ and $\Phi_i \in \mathbb{R}^{n_i \times n_i}$. We use α_i, $\psi_{i\text{ext}}$, κ_i, W_i, $\hat{W}_i$, H_i, and X_i to denote the corresponding functions, constants, and matrices appearing in Definition 4.2.6.

We consider an interconnected switching stochastic hybrid system $\Sigma = \mathcal{I}_\pi^M(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ JLSSs Σ_i, a set of interconnection matrices

$$M = \{M_1, \ldots, M_{\mathcal{P}}\},$$

and a continuous-time Markov chain $\Pi = (\mathsf{P}, \mathsf{Q})$ with associated stochastic process π governing the switching of the interconnection topologies. In order to derive the compositionality result, we first require the following two assumptions:

Assumption 4.2.10. *Functions $\mathcal{S}_i$ given in* (4.40) *are SStF-M$_2$ from $\hat{\Sigma}_i$ to Σ_i with $\psi_{i\text{ext}} \equiv 0$ (i.e. the zero function) in* (4.24), $\forall i \in [1; N]$.

Assumption 4.2.10 is not very restrictive and has been shown to be satisfied for various case studies in [AZ19; ZA17; Lav19].

Assumption 4.2.11. *There exist constants $\kappa_{u_j} \in \mathbb{R}_{\geq 0}, \mu_i^j \in \mathbb{R}_{>0}, \forall i \in [1; N], \forall j \in \mathsf{P}$, and matrices $\hat{M} = \{\hat{M}_1, \ldots, \hat{M}_{\mathcal{P}}\}$ of appropriate dimensions such that following (in)equalities hold*

$$C_2^T \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix}^T X(\mu_1^j X_1, \ldots, \mu_N^j X_N) \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix} C_2 \leq \kappa_{u_j} \Phi, \tag{4.41}$$

$$W M_j H = \hat{W} \hat{M}_j, \tag{4.42}$$

where

$$\begin{aligned}
&W = \mathsf{diag}(W_1, \dots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \dots, \hat{W}_N),\\
&H = \mathsf{diag}(H_1, \dots, H_N),\\
&X(\mu_1^j X_1, \dots, \mu_N^j X_N) =\\
&\begin{bmatrix}
\mu_1^j X_1^{11} & & & \mu_1^j X_1^{12} & & \\
& \ddots & & & \ddots & \\
& & \mu_N^j X_N^{11} & & & \mu_N^j X_N^{12} \\
\mu_1^j X_1^{21} & & & \mu_1^j X_1^{22} & & \\
& \ddots & & & \ddots & \\
& & \mu_N^j X_N^{21} & & & \mu_N^j X_N^{22}
\end{bmatrix},
\end{aligned} \tag{4.43}$$

$\tilde{q} = \sum_{i=1}^{N} q_{2i}$, $\Phi = \mathsf{diag}\{\Phi_1, \dots, \Phi_N\}$, and $C_2 = \mathsf{diag}\{C_{21}, \dots, C_{2N}\}$.

Now we present the main result of this section wherein we provide sufficient conditions to quantify the error between the output trajectories of the interconnected system and that of its abstraction under weaker conditions (4.41).

Theorem 4.2.12. *Consider an interconnected switching stochastic hybrid system $\Sigma = \mathcal{I}_\pi^M(\Sigma_1, \dots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ JLSSs Σ_i, a set of interconnection matrices $M = \{M_1, \dots, M_{\mathcal{P}}\}$, and a continuous-time Markov chain $\Pi = (\mathsf{P}, \mathsf{Q})$ with associated stochastic process π governing the switching of the interconnection topologies. Specifically, the interconnection topology at any time $t \in \mathbb{R}_{\geq 0}$ is given by $M_{\pi(t)}$. Suppose Assumptions 4.2.10, 4.2.11, and the following inequality holds:*

$$\tilde{c}\bar{q}\left(\frac{\tilde{\kappa}\hat{\theta}_u}{1-\tilde{\kappa}\hat{\theta}_u}\tilde{d}+1\right)\left(\frac{\tilde{\kappa}\hat{\theta}_u}{1-\tilde{\kappa}\hat{\theta}_u}+1\right) - \tilde{\kappa}_s - \check{q} < 0, \tag{4.44}$$

where $\tilde{\kappa} = \tilde{\kappa}_s + \tilde{\kappa}_u$,

$$\tilde{\kappa}_s := \begin{cases} 0, & \text{if } \#S_s = 0,\\ \min\limits_{j \in S_s}\{\kappa - \kappa_{u_j}\}, & \text{otherwise,} \end{cases}$$

$$\tilde{\kappa}_u := \begin{cases} 0, & \text{if } \#S_u = 0,\\ \max\limits_{j \in S_u}\{\kappa_{u_j} - \kappa\}, & \text{otherwise,} \end{cases}$$

$\kappa = \min\limits_{i\in[1;N]} \kappa_i$, $S_u = \{j \in \mathsf{P} : \kappa_{u_j} - \kappa \geq 0\}$, $S_s = \mathsf{P}\backslash S_u$, $\bar{q} = \max\limits_{i\in\mathsf{P}} |q_{ii}|$, $\tilde{d} = \sum\limits_{i\in S_u} \varrho_{si}$, $\check{q} = \{\max\limits_{i,j\in\mathsf{P}} q_{ij}, i \in S_u, j \in S_s\}$, $\hat{\theta}_u = \max\limits_{i\in S_u} \theta_i$, and

$$\tilde{c} := \max\left\{1+\epsilon, \left(\max_{l,m\in\mathsf{P},i\in[1;N]}\left\{\frac{\mu_i^l}{\mu_i^m}\right\}\right)\right\}, \tag{4.45}$$

for some $\epsilon > 0$. Then, there exists a $\mathcal{KL}$ function β such that for any random variables a and $\hat{a}$ that are $\mathcal{F}_0$-measurable, and for any $\hat{\upsilon} \in \hat{\mathcal{U}}$ there exists $\upsilon \in \mathcal{U}$ such that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^2] \leq \beta(V(a, \hat{a}, \pi(0)), t), \tag{4.46}$$

where

$$V(x, \hat{x}, j) := \sum_{i=1}^{N} \mu_i^j \mathcal{S}_i(x_i, \hat{x}_i). \tag{4.47}$$

Proof. For any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $j \in \mathsf{P}$, one gets:

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\|^2 &\leq \sum_{i=1}^{N}\|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|^2\\ &\leq \sum_{i=1}^{N} \alpha_i^{-1}(\mathcal{S}_i(x_i, \hat{x}_i)) \leq \underline{\alpha}_j(V(x, \hat{x}, j)),\end{aligned}$$

where $\underline{\alpha}_j$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}_j(s) := \begin{cases} \max\limits_{\vec{s}\geq 0} & \sum_{i=1}^{N} \alpha_i^{-1}(s_i)\\ \text{s.t.} & \mu_j^T \vec{s} = s,\end{cases} \tag{4.48}$$

where $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}_{\geq 0}^N$ and $\mu_j = [\mu_1^j; \ldots; \mu_N^j]$. Since $\underline{\alpha}_j \in \mathcal{K}_\infty$ are concave functions due to convexity of α_i as argued in [ZRE17], there exists a concave function $\underline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}_j \leq \underline{\alpha}$ $\forall j \in \mathsf{P}$. By defining $\alpha = \underline{\alpha}^{-1}$ which is a convex $\mathcal{K}_\infty$ function, one obtains

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^2) \leq V(x, \hat{x}, j). \tag{4.49}$$

Similarly, one can show that for any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$ and $j \in \mathsf{P}$, the infinitesimal generator of V satisfies the series of inequalities given in (4.50). Now, for all $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $j, k \in \mathsf{P}$, V satisfies the

$$
\begin{aligned}
&\mathcal{L}V(x,\hat{x},j) = \sum_{i=1}^{N} \mu_i^j \mathcal{L}\mathcal{S}_i(x_i,\hat{x}_i) \\
&\leq \sum_{i=1}^{N} \mu_i^j \Bigg(-\kappa_i \mathcal{S}_i(x_i,\hat{x}_i) + \begin{bmatrix} W_i w_i - \hat{W}_i \hat{w}_i \\ C_{2i} x_i - H_i \hat{C}_{2i} \hat{x}_i \end{bmatrix}^T \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22} \end{bmatrix} \begin{bmatrix} W_i w_i - \hat{W}_i \hat{w}_i \\ C_{2i} x_i - H_i \hat{C}_{2i} \hat{x}_i \end{bmatrix} \Bigg) \\
&= \sum_{i=1}^{N} -\mu_i^j \kappa_i \mathcal{S}_i(x_i,\hat{x}_i) + \begin{bmatrix} W \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} - \hat{W} \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_N \end{bmatrix} \\ C_{21} x_1 - H_1 \hat{C}_{21} \hat{x}_1 \\ \vdots \\ C_{2N} x_N - H_N \hat{C}_{2N} \hat{x}_N \end{bmatrix}^T X(\mu_1^j X_1, \ldots, \mu_N^j X_N) \begin{bmatrix} W \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} - \hat{W} \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_N \end{bmatrix} \\ C_1 x_1 - H_1 \hat{C}_{21} \hat{x}_1 \\ \vdots \\ C_{2N} x_N - H_N \hat{C}_{2N} \hat{x}_N \end{bmatrix} \\
&= \sum_{i=1}^{N} -\mu_i^j \kappa_i \mathcal{S}_i(x_i,\hat{x}_i) \\
&\quad + \begin{bmatrix} C_{21} x_1 - H_1 \hat{C}_{21} \hat{x}_1 \\ \vdots \\ C_{2N} x_N - H_N \hat{C}_{2N} \hat{x}_N \end{bmatrix}^T \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix}^T X(\mu_1^j X_1, \ldots, \mu_N^j X_N) \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix} \begin{bmatrix} C_{21} x_1 - H_1 \hat{C}_{21} \hat{x}_1 \\ \vdots \\ C_{2N} x_N - H_N \hat{C}_{2N} \hat{x}_N \end{bmatrix} \\
&= \sum_{i=1}^{N} -\mu_i^j \kappa_i \mathcal{S}_i(x_i,\hat{x}_i) \\
&\quad + \begin{bmatrix} x_1 - P_1 \hat{x}_1 \\ \vdots \\ x_N - P_N \hat{x}_N \end{bmatrix}^T C_2^T \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix}^T X(\mu_1^j X_1, \ldots, \mu_N^j X_N) \begin{bmatrix} W M_j \\ I_{\tilde{q}} \end{bmatrix} C_2 \begin{bmatrix} x_1 - P_1 \hat{x}_1 \\ \vdots \\ x_N - P_N \hat{x}_N \end{bmatrix} \\
&\leq -\tilde{\kappa}_s V(x,\hat{x},j) \mathbf{I}_{S_s}(j) + \tilde{\kappa}_u V(x,\hat{x},j) \mathbf{I}_{S_u}(j).
\end{aligned} \tag{4.50}
$$

following inequality:

$$
\begin{aligned}
V(x,\hat{x},j) = &\sum_{i=1}^{N} \mu_i^j \mathcal{S}_i(x_i,\hat{x}_i) \\
&\leq \left(\max_{l,m \in \mathsf{P}, i \in [1;N]} \left\{ \frac{\mu_i^l}{\mu_i^m} \right\} \right) \sum_{i=1}^{N} \mu_i^k \mathcal{S}_i(x_i,\hat{x}_i) \\
&= \left(\max_{l,m \in \mathsf{P}, i \in [1;N]} \left\{ \frac{\mu_i^l}{\mu_i^m} \right\} \right) V(x,\hat{x},k) \\
&\leq \tilde{c} V(x,\hat{x},k),
\end{aligned} \tag{4.51}
$$

where $\tilde{c}$ is defined in (4.45). Using Lemmas 3.4 and 3.5 in [WZ17], it can be shown that for any random variables a and $\hat{a}$ that are $\mathcal{F}_0$-measurable, and for any $\hat{\upsilon} \in \hat{\mathcal{U}}$ there exists

$\upsilon \in \mathcal{U}$ such that for any $t \in \mathbb{R}_{\geq 0}$, one has:

$$\mathbb{E}[V(\xi_{a\upsilon}(t), \hat{\xi}_{\hat{a}\hat{\upsilon}}(t), \pi(t))] \leq V(a, \hat{a}, \pi(0))\mathsf{e}^{\chi t}, \tag{4.52}$$

where

$$\chi = \tilde{c}\bar{q}\left(\frac{\tilde{\kappa}\hat{\theta}_u}{1-\tilde{\kappa}\hat{\theta}_u}\tilde{d}+1\right)\left(\frac{\tilde{\kappa}\hat{\theta}_u}{1-\tilde{\kappa}\hat{\theta}_u}+1\right) - \tilde{\kappa}_s - \breve{q}.$$

Using (4.49) and Jensen's inequality [Rud87], one can show that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:

$$\mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^2] \leq \alpha^{-1}\Big(V(a, \hat{a}, \pi(0))\mathsf{e}^{\chi t}\Big). \tag{4.53}$$

Due to the assumption that $\chi < 0$ from (4.44), inequality (4.46) is satisfied with $\beta(r,t) = \alpha^{-1}(r\mathsf{e}^{\chi t}), \forall r \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}_{\geq 0}$, which concludes the proof.

□

Remark 4.2.13. *The matrix inequality* (4.41) *is linear with respect to the decision variables* κ_{u_j} *and* $\mu^j = [\mu_1^j; \ldots; \mu_N^j]$, $j \in \mathsf{P}$. *The matrix equality* (4.42) *is linear in the decision variables* $\hat{M}_j, j \in \mathsf{P}$, *and can readily be solved using optimization tools such as* `Yalmip` *[Lof04].*

Remark 4.2.14. *Condition* (4.41) *is similar to the linear matrix inequality (LMI) in [AMP16] as a compositional stability condition based on dissipativity theory. It is shown in [AMP16] that this condition can hold independently of the number of subsystems in many physical applications with certain interconnection topologies, e.g., skew symmetric interconnection.*

4.2.6 Example

In the following example we consider an interconnected system with randomly switched topologies for which an abstraction cannot be constructed compositionally using the conditions in Theorem 4.2.7, but can be constructed using the weaker conditions in Theorem 4.2.12.

Consider an interconnection of $N \in \mathbb{N}$ JLSSs Σ_i, $i \in [1; N]$, where each Σ_i is given by

$$\Sigma_i = (0_{n_i}, I_{n_i}, C_{1i}, I_{n_i}, I_{n_i}, \varpi I_{n_i}, \tau I_{n_i}),$$

where $\varpi \in \mathbb{R}_{>0}$, $\tau \in \mathbb{R}_{>0}$, and $C_{1i} \in \mathbb{R}^{q_{1i} \times n_i}$. The dynamics of each Σ_i are given by:

$$\Sigma_i : \begin{cases} \mathsf{d}\xi_i(t) = (\omega_i(t) + \upsilon_i(t))\mathsf{d}t + \varpi\xi_i(t)\mathsf{d}W_t + \tau\xi_i(t)\mathsf{d}P_t, \\ \zeta_{1i}(t) = C_{1i}\xi_i(t), \\ \zeta_{2i}(t) = \xi_i(t). \end{cases}$$

Assume the rate of the Poisson process P_t is λ. We consider a set of two interconnection topologies $M = \{M_1, M_2\}$ given by:

$$M_1 = \frac{2}{n}\begin{bmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & & \ddots & \ddots & \vdots \\ 1 & \dots & \dots & 1 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}, \tag{4.54}$$

where $n = \sum_{i=1}^{N} n_i$. We consider a Markov chain $\Pi = (\mathsf{P}, \mathsf{Q})$, with $\mathsf{P} = \{1, 2\}$ and

$$\mathsf{Q} = \begin{bmatrix} -0.1 & 0.1 \\ 0.01 & -0.01 \end{bmatrix},$$

with the switching process π, governing the switching between matrices M_1 and M_2. The interconnected switching stochastic hybrid system is denoted by $\mathcal{I}_\pi^M(\Sigma_1, \dots, \Sigma_N)$. We consider a scalar JLSS abstraction

$$\hat{\Sigma}_i = (0, 1, C_{1i}\vec{1}_{n_i}, 1, 1, \varpi, \tau).$$

The function $\mathcal{S}_i(x_i, \hat{x}_i) = (x_i - \vec{1}_{n_i}\hat{x}_i)^T(x_i - \vec{1}_{n_i}\hat{x}_i)$ is an SStF-M$_2$ from $\hat{\Sigma}_i$ to Σ_i, $\forall i \in [1; N]$, with the following parameters:

$$\kappa_i = 2\tilde{\chi} - 2\lambda\tau - \varpi^2 - \lambda\tau^2, W_i = I_{n_i}, X_i^{11} = 0_{n_i}, \tag{4.55}$$
$$X_i^{22} = 0_{n_i}, X_i^{12} = X_i^{21} = I_{n_i}, H_i = \hat{W}_i = \vec{1}_{n_i}, \tag{4.56}$$

for some $\tilde{\chi} > \lambda\tau + \frac{\varpi^2}{2} + \frac{\lambda\tau^2}{2}$, and with $\alpha_i(r) = \frac{1}{\lambda_{\max}(C_{1i}^T C_{1i})} r$, and $\psi_{i\mathsf{ext}}(r) = 0$, $\forall r \in \mathbb{R}_{\geq 0}$. Input $u_i \in \mathbb{R}^{n_i}$ is given via the interface function

$$u_i = -\tilde{\chi}(x_i - \vec{1}_{n_i}\hat{x}_i) + \vec{1}_{n_i}\hat{u}_i. \tag{4.57}$$

In this example, we choose $N = 3$, $n_i = 50$, $C_{1i} = [1\ 0\ \dots\ 0]$, $\forall i \in [1; N]$, $\varpi = 0.3$, $\tau = 0.03$, $\tilde{\chi} = 0.6$, and $\lambda = 1$. Note that for the interconnection matrix M_1, the condition (4.25) in Theorem 4.2.7 cannot be satisfied for any $\mu_i^1 \in \mathbb{R}_{\geq 0}, i \in [1; N]$. Therefore, we resort to the weaker conditions in Theorem 4.2.12 to solve this example. We select $\mu_1^j = \dots = \mu_N^j = 1$ for every $j \in \{1, 2\}$. We determine the smallest constants $\kappa_{u_j} \geq 0, j \in [1; 2]$, satisfying (4.41), which results in $\kappa_{u_1} = 3.9733$ and $\kappa_{u_2} = 0$. It can

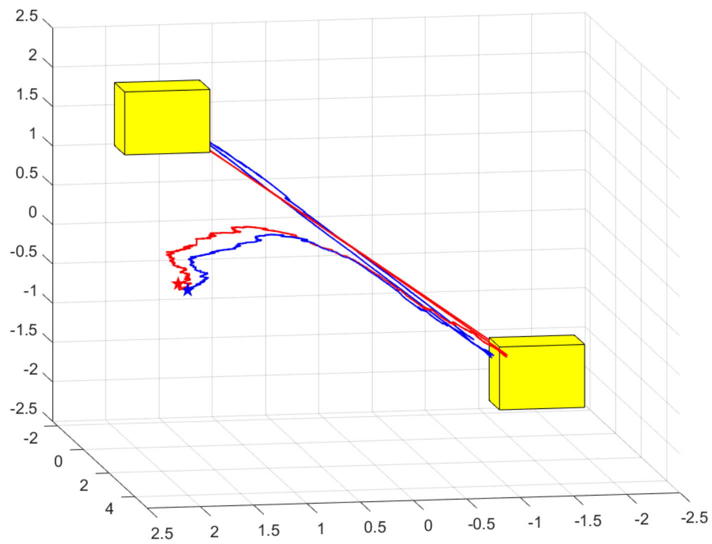

Figure 4.2: A realization of the output trajectories of the concrete (blue) and abstract (red) interconnected stochastic hybrid systems with switched topologies ($\zeta(t)$ and $\hat{\zeta}(t)$ respectively). The yellow boxes indicate the two targets T_1 and T_2. The start points of the trajectories are indicated by the pentagram markers.

be readily verified that conditions (4.42) and (4.44) in Theorem 4.2.12 are satisfied with $\tilde{c} = 1 + \epsilon$, where $\epsilon = 0.01$, and a set of interconnection matrices $\hat{M} = \{\hat{M}_1, \hat{M}_2\}$, where

$$\hat{M}_1 = \frac{2}{n} \begin{bmatrix} 49 & 50 & 50 \\ 50 & 49 & 50 \\ 50 & 50 & 49 \end{bmatrix}, \hat{M}_2 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{4.58}$$

4.2.6.1 Controller synthesis

In this sub-section we synthesize a controller for the abstract interconnected switching stochastic hybrid system $\hat{\Sigma} = \mathcal{I}_\pi^{\hat{M}}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ to enforce a given specification, and then refine the designed controller to the interconnected switching stochastic hybrid system $\Sigma = \mathcal{I}_\pi^{M}(\Sigma_1, \ldots, \Sigma_N)$. First, we consider the randomly switched interconnected system $\tilde{\Sigma} = \mathcal{I}_\pi^{\hat{M}}(\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_N)$, wherein each $\tilde{\Sigma}_i$, $i \in [1; N]$, results from $\hat{\Sigma}_i$ by setting the diffusion and reset terms to zero. It can be shown that the function

$$\tilde{V}(\hat{x}, \tilde{x}) = \begin{bmatrix} \hat{x} \\ \tilde{x} \end{bmatrix}^T \tilde{M} \begin{bmatrix} \hat{x} \\ \tilde{x} \end{bmatrix},$$

is an SSF-M$_2$ from $\tilde{\Sigma}$ to $\hat{\Sigma}$ with the interface function

$$\hat{u} = \tilde{u}, \tag{4.59}$$

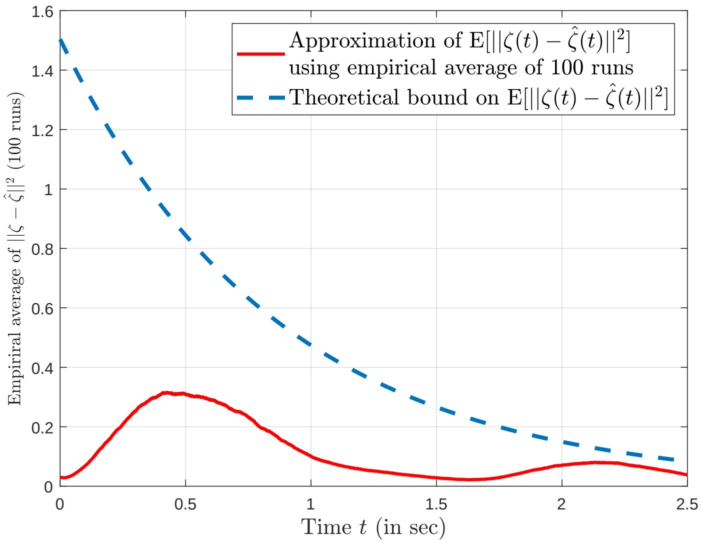

Figure 4.3: An approximation of $\mathbb{E}[\|\zeta(t) - \hat{\zeta}(t)\|^2]$ using empirical mean based on 100 realizations and the theoretical upper bound obtained from (4.46) with $\beta(r,t) = r\mathrm{e}^{-1.1517t}$.

where $\hat{u}=[\hat{u}_1;\cdots;\hat{u}_N]$, $\tilde{u}=[\tilde{u}_1;\cdots;\tilde{u}_N]$, $\hat{x}=[\hat{x}_1;\cdots;\hat{x}_N]$, $\tilde{x}=[\tilde{x}_1;\cdots;\tilde{x}_N]$, and

$$\tilde{M} = \begin{bmatrix} 0.159 & 0.135 & 0.135 & 0.128 & 0.128 & 0.128 \\ 0.135 & 0.159 & 0.135 & 0.128 & 0.128 & 0.128 \\ 0.135 & 0.135 & 0.159 & 0.128 & 0.128 & 0.128 \\ 0.128 & 0.128 & 0.128 & 0.158 & 0.134 & 0.134 \\ 0.128 & 0.128 & 0.128 & 0.134 & 0.158 & 0.134 \\ 0.128 & 0.128 & 0.128 & 0.134 & 0.134 & 0.158 \end{bmatrix}. \tag{4.60}$$

We synthesize a controller using toolbox `SCOTS` [RZ16] to enforce the following linear temporal logic specification [BK08] over the outputs of $\tilde{\Sigma}$:

$$\Psi = \Box S \wedge \Box\Diamond T_1 \wedge \Box\Diamond T_2. \tag{4.61}$$

The property Ψ can be interpreted as follows: the output trajectory $\tilde{\zeta}$ of the closed loop system evolves inside the set S, and visits T_i, $i \in [1;2]$, infinitely often, indicated with yellow boxes in Figure 4.2. The sets S, T_1, and T_2 are given by: $S = [-5,5]^3$, $T_1 = [1.6, 2.4]^3$, and $T_2 = [-2.4, -1.6]^3$. We use (4.57) and (4.59) to generate the corresponding input enforcing this specification over the original system Σ. Figure 4.2 shows a realization of output trajectories Σ and $\hat{\Sigma}$ started from $\zeta(0) = [2.0058; 4.0060; 0.1030]$ and $\hat{\zeta}(0) = [2.1058; 4.1060; 0.2030]$, respectively. Figure 4.3 shows a comparison of the theoretical upper bound and the empirical average (using 100 realizations) of $\mathbb{E}[\|\zeta(t) - \hat{\zeta}(t)\|^2]$.

4.3 Interconnected Control Systems with Dynamic Interconnection Topology

In this section, we derive conditions under which compositional abstractions of networks of deterministic control systems, interconnected via some dynamic interconnection topology, can be constructed using the dynamic interconnection and joint dissipativity-type properties of subsystems and their abstractions. We provide an example to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning by reducing a 150-dimensional nonlinear system to a 3-dimensional one.

4.3.1 Storage Function

In this section, we introduce a notion of so-called storage function to relate two control systems, which is a deterministic analogue of the notion of stochastic storage function introduced in Definition 4.2.6 for stochastic hybrid systems.

Definition 4.3.1. *Let*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\mathfrak{D}} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

be two control subsystems as in (2.2)*, with the same external output space dimension. A continuously differentiable function* $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \rightarrow \mathbb{R}_{\geq 0}$ *is called a storage function from* $\hat{\mathfrak{D}}$ *to* $\mathfrak{D}$*, if there exist functions* $\alpha, \eta \in \mathcal{K}_\infty$*,* $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$*, some matrices* $W, \hat{W}$*, and* H *of appropriate dimensions, and some symmetric matrix* X *of appropriate dimension with conformal block partitions* $X^{ij}, i, j \in [1;2]$*, such that for any* $x \in \mathbb{R}^n$ *and any* $\hat{x} \in \mathbb{R}^{\hat{n}}$*, one has*

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|) \leq V(x, \hat{x}), \tag{4.62}$$

and $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$*,* $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$*,* $\exists u \in \mathbb{R}^m$*, such that* $\forall \hat{w} \in \mathbb{R}^{\hat{p}}$ $\forall w \in \mathbb{R}^p$*, one obtains*

$$\nabla V(x, \hat{x})^T \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix} \leq -\eta(V(x, \hat{x})) + \psi_{\text{ext}}(\|\hat{u}\|) + z^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} z, \tag{4.63}$$

where

$$z = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}. \tag{4.64}$$

Condition (4.63) implies the existence of an interface function $u = k_t(x, \hat{x}, \hat{u})$ to choose u for any $x, \hat{x}$, and $\hat{u}$.

4.3.2 Simulation Function

We now recall the notion of simulation functions introduced in [GP09] used to relate two control systems without internal inputs and outputs. This notion of simulation function which is a deterministic analogue of the notion of stochastic simulation function introduced in Definition 4.2.4.

Definition 4.3.2. *Let $\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ and $\hat{\mathfrak{D}} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$, be two interconnected control systems. A continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}_{\geq 0}$ is called a* simulation function *from $\hat{\mathfrak{D}}$ to $\mathfrak{D}$ if there exist $\alpha, \eta \in \mathcal{K}_\infty$ and $\rho_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ such that for any $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, one has*

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \tag{4.65}$$

and $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$, $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$, $\exists u \in \mathbb{R}^m$ such that

$$\nabla V\,(x, \hat{x})^T \begin{bmatrix} f(x, u) \\ \hat{f}(\hat{x}, \hat{u}) \end{bmatrix} \leq -\eta(V\,(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|). \tag{4.66}$$

The next theorem, borrowed from [ZA17], shows the importance of the existence of a simulation function by quantifying the error between the output behaviours of $\mathfrak{D}$ and the ones of its abstraction $\hat{\mathfrak{D}}$.

Theorem 4.3.3. *Let $\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$, and $\hat{\mathfrak{D}} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$. Suppose V is a simulation function from $\hat{\mathfrak{D}}$ to $\mathfrak{D}$. Then, there exists a $\mathcal{KL}$ function ϑ such that for any $x \in \mathbb{R}^n$, $\hat{x} \in \mathbb{R}^{\hat{n}}$, $\hat{\upsilon} \in \hat{\mathcal{U}}$, there exists $\upsilon \in \mathcal{U}$ such that the following inequality holds for any $t \in \mathbb{R}_{\geq 0}$:*

$$\|\zeta_{x\upsilon}(t) - \hat{\zeta}_{\hat{x}\hat{\upsilon}}(t)\| \leq \alpha^{-1}(2\vartheta\,(V(x, \hat{x}), t)) + \alpha^{-1}(2\eta^{-1}(2\rho_{\text{ext}}(\|\hat{\upsilon}\|_\infty))). \tag{4.67}$$

Remark 4.3.4. *If functions α and η in Definition 4.3.2 satisfy the triangle inequality, then one can drop coefficients 2 in inequality (4.67) to get a less conservative upper bound.*

4.3.3 Interconnected Systems under Dynamic Interconnection Topology

Here, we define interconnected control systems under dynamic interconnection topology.

Definition 4.3.5. *Consider $N \in \mathbb{N}_{\geq 1}$ control subsystems*

$$\mathfrak{D}_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1; N]$, and a so-called interconnection system

$$\mathfrak{D}_o = (\mathbb{R}^{n_o}, \mathbb{R}^{m_o}, \mathcal{U}_o, f_o, \mathbb{R}^{q_o}, h_o, h_{ou}), \tag{4.68}$$

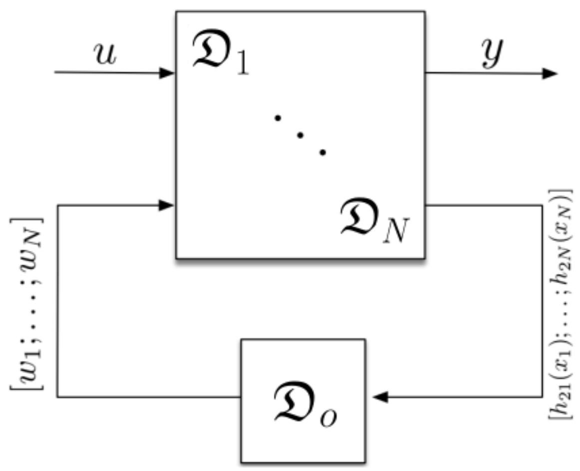

Figure 4.4: Interconnected control system under dynamic interconnection topology

where, for any $x_o \in \mathbb{R}^{n_o}, u_o \in \mathbb{R}^{m_o}$,

$$\begin{aligned} f_o(x_o, u_o) &:= A_o x_o + B_o u_o, \\ h_o(x_o) &:= C_o x_o, \\ h_{ou}(u_o) &:= D_o u_o, \end{aligned} \tag{4.69}$$

for some matrices A_o, B_o, C_o, *and* D_o *of appropriate dimensions,* $q_o = \sum_{i=1}^{N} p_i$, *and* $m_o = \sum_{i=1}^{N} q_{2i}$. *The interconnected control system*

$$\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, \tilde{f}, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}_{\mathfrak{D}_o}(\mathfrak{D}_1, \ldots, \mathfrak{D}_N)$, *follows by* $n = \sum_{i=1}^{N} n_i + n_o, m = \sum_{i=1}^{N} m_i, q = \sum_{i=1}^{N} q_{1i}$, *and the functions*

$$\tilde{f}(x, x_o, u) = \begin{bmatrix} [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)] \\ f_o(x_o, u_o) \end{bmatrix}, \tag{4.70}$$

$$h(x) = [h_{11}(x); \ldots; h_{1N}(x_N)], \tag{4.71}$$

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$, *and with the internal inputs equal to the output of* $\mathfrak{D}_o$, *i.e.* $[w_1; \ldots; w_N] = h_o(x_o) + h_{ou}(u_o)$, *and the input of* $\mathfrak{D}_o$ *equal to the internal outputs, i.e.* $u_o = [h_{21}(x_1); \ldots; h_{2N}(x_N)]$.

Figure 4.4 illustrates such an interconnected system under dynamic interconnection topology.

4.3.4 Compositionality Result

The next theorem provides a compositional approach on the construction of abstractions of dynamically interconnected networks of control systems.

$$\begin{bmatrix} C_o^T C_o A_o + A_o^T C_o^T C_o & C_o^T C_o B_o \\ B_o^T C_o^T C_o & 0_* \end{bmatrix} + \begin{bmatrix} W C_o & W D_o \\ 0_* & I_* \end{bmatrix}^T X(\mu_1 X_1, \ldots, \mu_N X_N) \begin{bmatrix} W C_o & W D_o \\ 0_* & I_* \end{bmatrix}$$
$$\preceq \quad \begin{bmatrix} -\kappa_o C_o^T C_o & 0_* \\ 0_* & 0_* \end{bmatrix} \tag{4.72}$$

Theorem 4.3.6. *Consider an interconnected control system* $\mathfrak{D} = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$, *induced by* $N \in \mathbb{N}$ *control subsystems* $\mathfrak{D}_i$ *as in* (2.2), *and the interconnection system* $\mathfrak{D}_o$ *as in* (4.68). *Suppose each subsystem admits an abstraction* $\hat{\mathfrak{D}}_i$ *with the corresponding storage function* V_i. *If there exist* $\mu_i > 0$, $i \in [1; N]$, *and positive constant* κ_o *such that the inequality*[3] (4.72) *and the following equalities*

$$\begin{aligned} C_o \Pi \hat{A}_o &= C_o A_o \Pi, \\ C_o \Pi \hat{B}_o &= C_o B_o H, \\ \hat{W} \hat{D}_o &= W D_o H, \\ \hat{W} \hat{C}_o &= W C_o \Pi, \end{aligned} \tag{4.73}$$

hold for some matrices $\hat{A}_o$, $\hat{B}_o$, $\hat{C}_o$, $\hat{D}_o$, *and* Π *of appropriate dimensions, where*

$$\begin{aligned} &W = \mathsf{diag}(W_1, \ldots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \ldots, \hat{W}_N), \\ &H = \mathsf{diag}(H_1, \ldots, H_N), \end{aligned} \tag{4.74}$$

$$X(\mu_1 X_1, \ldots, \mu_N X_N) = \begin{bmatrix} \mu_1 X_1^{11} & & & \mu_1 X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{11} & & & \mu_N^{12} X_N^{12} \\ \mu_1 X_1^{21} & & & \mu_1 X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{21} & & & \mu_N^{22} X_N^{22} \end{bmatrix}, \tag{4.75}$$

then

$$V(x, x_o, \hat{x}, \hat{x}_o) = \bar{V}(x, \hat{x}) + V_o(x_o, \hat{x}_o), \tag{4.76}$$

is a simulation function from $\hat{\mathfrak{D}} = \mathcal{I}_{\hat{\mathfrak{D}}_o}(\hat{\mathfrak{D}}_1, \ldots, \hat{\mathfrak{D}}_N)$ *to* $\mathfrak{D} = \mathcal{I}_{\mathfrak{D}_o}(\mathfrak{D}_1, \ldots, \mathfrak{D}_N)$, *where* $\bar{V}(x, \hat{x}) = \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i)$, $V_o(x_o, \hat{x}_o) = (x_o - \Pi \hat{x}_o)^T C_o^T C_o (x_o - \Pi \hat{x}_o)$ *and*

$$\hat{\mathfrak{D}}_o = (\mathbb{R}^{\hat{n}_o}, \mathbb{R}^{\hat{m}_o}, \hat{\mathcal{U}}_o, \hat{f}_o, \mathbb{R}^{\hat{q}_o}, \hat{h}_o, \hat{h}_{ou}),$$

[3] Matrices 0_* and I_* in (4.72) represent zero and identity matrices of appropriate dimensions, respectively.

where, for any $\hat{x}_o \in \mathbb{R}^{\hat{n}_o}$ *and any* $\hat{u}_o \in \mathbb{R}^{\hat{m}_o}$,

$$\hat{f}(\hat{x}_o, \hat{u}_o) := \hat{A}_o\hat{x}_o + \hat{B}_o\hat{u}_o, \tag{4.77}$$

$$\hat{h}_o(\hat{x}_o) := \hat{C}_o\hat{x}_o, \tag{4.78}$$

$$\hat{h}_{ou}(\hat{u}_o) := \hat{D}_o\hat{u}_o. \tag{4.79}$$

Proof. First we show that inequality (4.65) holds for some $\mathcal{K}_\infty$ function α. For any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, and $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, one gets:

$$\begin{aligned}\|h(x) - \hat{h}(\hat{x})\|^2 &\leq \sum_{i=1}^{N} \|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|^2 \\ &\leq \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \leq \underline{\alpha}(\bar{V}(x, \hat{x})),\end{aligned}$$

where $\underline{\alpha}$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}(s) := \begin{cases} \max\limits_{\vec{s}\geq 0} & \sum_{i=1}^{N} \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu^T\vec{s} = s, \end{cases} \tag{4.80}$$

where[4] $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$ and $\mu = [\mu_1; \ldots; \mu_N]$. By defining $\mathcal{K}_\infty$ function $\alpha(s) = \underline{\alpha}^{-1}(s), \forall s \in \mathbb{R}_{\geq 0}$, one obtains $\forall x \in \mathbb{R}^n, \forall \hat{x} \in \mathbb{R}^{\hat{n}}$, $\forall x_o \in \mathbb{R}^{n_o}$, $\forall \hat{x}_o \in \mathbb{R}^{\hat{n}_o}$,

$$\begin{aligned}\alpha(\|h(x) - \hat{h}(\hat{x})\|^2) \leq \bar{V}(x, \hat{x}) &\leq \bar{V}(x, \hat{x}) + V_o(x_o, \hat{x}_o) \\ &= V(x, x_o, \hat{x}, \hat{x}_o),\end{aligned} \tag{4.81}$$

hence satisfying inequality (4.65). Now we prove inequality (4.66). Consider any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n, \hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $\hat{u} = [\hat{u}_1; \ldots; \hat{u}_N] \in \mathbb{R}^{\hat{m}}$. For any $i \in [1; N]$, there exists $u_i \in \mathbb{R}^{m_i}$, consequently, a vector $u = [u_1; \ldots; u_N] \in \mathbb{R}^m$, satisfying (4.63) for each pair of $\mathfrak{D}_i$ and $\hat{\mathfrak{D}}_i$, with the internal inputs given by the outputs of the interconnection systems $\mathfrak{D}_o$ and $\hat{\mathfrak{D}}_o$, respectively, i.e.

$$[w_1; \ldots; w_N] = h_o(x_o) + h_{ou}(u_o),$$

$$[\hat{w}_1; \ldots; \hat{w}_N] = \hat{h}_o(\hat{x}_o) + \hat{h}_{ou}(\hat{u}_o),$$

where the inputs to $\mathfrak{D}_o$ and $\hat{\mathfrak{D}}_o$ are the internal outputs of the subsystems $\mathfrak{D}_i$ and $\hat{\mathfrak{D}}_i$, respectively, i.e.

$$u_o = [h_{21}(x_1); \ldots; h_{2N}(x_N)],$$

$$\hat{u}_o = [\hat{h}_{21}(\hat{x}_1); \ldots; \hat{h}_{2N}(\hat{x}_N)].$$

[4]We interpret inequality $\vec{s} \geq 0$ component-wise.

We employ the conditions in (4.72) and (4.73), which results in the chain of inequalities in (4.84), where the functions $\eta \in \mathcal{K}_\infty$, and $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ are defined as

$$\eta(s) := \begin{cases} \min\limits_{[\vec{s};s_o]\geq 0} & \sum_{i=1}^{N} \mu_i \eta_i(s_i) + \kappa_o s_o \\ \text{s.t.} & \mu^T \vec{s} + s_o = s, \end{cases} \tag{4.82}$$

$$\psi_{\text{ext}}(s) := \begin{cases} \max\limits_{\vec{s}\geq 0} & \sum_{i=1}^{N} \mu_i \psi_{i\text{ext}}(s_i) \\ \text{s.t.} & \mu^T \vec{s} = s. \end{cases} \tag{4.83}$$

Hence we conclude that V is a simulation function from $\hat{\mathfrak{D}}$ to $\mathfrak{D}$.

□

Remark 4.3.7. *Note that the case of static interconnection and its associated conditions presented in [ZA17] can readily be recovered by the results here if C_o is equal to the zero matrix and D_o is equal to the static interconnection matrix (values of A_o and B_o become irrelevant since x_o does not affect the internal input to $\mathfrak{D}$).*

In the next section, we provide a practical example for compositional abstraction of an interconnected system with dynamic interconnection topology.

4.3.5 Case Study - Electrical Network

Consider n first order resistance-capacitance (R-C) circuits, interconnected via resistance-inductance (R-L) series branches. The i-th R-C circuit has the dynamics given by:

$$\dot{v}_{c_i} = -\frac{1}{R_i C_i} v_{c_i} + \frac{1}{R_i C_i} v_{s_i} + \frac{1}{C_i} \tilde{w}_i, \tag{4.85}$$

where $i \in [1;n]$, $v_{s_i} \in \mathbb{R}$ represents the input source voltage (external input), $v_{c_i} \in \mathbb{R}$ the voltage across capacitor, C_i the capacitance, R_i the resistance, and $\tilde{w}_i \in \mathbb{R}$ the total current inflow from other R-L branches connected to the R-C circuit. Assuming identical values of the resistance and inductance in all R-L branches, one can write the dynamics of the total current inflow for the i-th R-C circuit as:

$$\dot{\tilde{w}}_i = a_{o_i} \tilde{w}_i + b_{o_i} v, \tag{4.86}$$

where $v = [v_{c_1}; \ldots; v_{c_n}]$, and $a_{o_i} \in \mathbb{R}$, and $b_{o_i} \in \mathbb{R}^{1\times n}$ represent the parameters of dynamics of the R-L series branch(es) connected to the i-th R-C circuit. We consider the above interconnected system as an interconnection of N concrete subsystems $\mathfrak{D}_i$, $i \in [1;N]$, wherein each subsystem $\mathfrak{D}_i$ is formed by clustering n_i R-C circuits ($n_i \leq n$). Each subsystem, $\mathfrak{D}_i = (A_i, B_i, C_{1i}, I_{ni}, I_{ni}, \vec{1}_{ni}, \vec{1}_{ni}^T, \varphi)$, generates a scalar (external) output. We also add a nonlinearity belonging to the class of nonlinearities characterized by (3.27). We have:

$$\mathfrak{D}_i : \begin{cases} \dot{\xi}_i = A_i \xi_i + B_i u_i + D_i w_i + \vec{1}_{ni} \varphi(\vec{1}_{ni}^T \xi_i), \\ \zeta_{1i} = C_{1i} \xi_i, \\ \zeta_{2i} = \xi_i, \end{cases}$$

where $\xi_i = \mathsf{L}_i v$, $\mathsf{L}_i := [e_{i1}; \ldots; e_{in_i}]$, $e_{ij} \in \mathbb{R}^{1\times n}$ is a row vector whose k-th element is defined as

$$e_{ij}^{(k)} = \begin{cases} 1 \text{ if } k\text{-th R-C circuit is part of the } i\text{-th cluster} \\ 0 \text{ otherwise,} \end{cases} \tag{4.87}$$

$A_i, B_i, D_i \in \mathbb{R}^{n_i \times n_i}$ are readily obtained from (4.85), $C_{1i} \in \mathbb{R}^{1\times n_i}$, $u_i = \mathsf{L}_i v_s$, $v_s = [v_{s_1}; \ldots; v_{s_n}]$, $w_i = \mathsf{L}_i \tilde{w}$, $\tilde{w} = [\tilde{w}_1; \ldots; \tilde{w}_n]$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\varphi(x) = \sin(x).$$

The dynamic of the interconnection topology $\mathfrak{D}_o$ is given by

$$\mathfrak{D}_o : \begin{cases} \dot{x}_o = A_o x_o + B_o v \\ y_o = C_o x_o + D_o v, \end{cases}$$

where $A_o = \mathsf{diag}(a_{o_1}, \ldots, a_{o_n})$, $B_o = [b_{o_1}; \ldots; b_{o_n}]$, $C_o = I_n$, and $D_o = 0_n$. We aggregate each $\mathfrak{D}_i$ into a scalar abstraction $\hat{\mathfrak{D}}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_{1i}, 1, 1, 1, 1, \varphi)$ given by the following dynamics

$$\hat{\mathfrak{D}}_i : \begin{cases} \dot{\hat{\xi}}_i = \hat{A}_i \hat{\xi}_i + \hat{B}_i \hat{u}_i + \hat{w}_i + \varphi(\hat{\xi}_i), \\ \hat{\zeta}_{1i} = \hat{C}_{1i} \hat{\xi}_i, \\ \hat{\zeta}_{2i} = \hat{\xi}_i, \end{cases}$$

where $\hat{A}_i$ satisfies $A_i \vec{1}_{n_i} = \vec{1}_{n_i} \hat{A}_i$, $\hat{B}_i$ is chosen arbitrarily (in this example we choose $\hat{B}_i = 1$), $\hat{C}_{1i} = C_{1i} \vec{1}_{n_i}$. For $R_j = C_j = 1$ in (4.85), $\forall j \in [1; n]$, it can be verified that the function $V_i(x_i, \hat{x}_i) = (x_i - \vec{1}_{n_i} \hat{x}_i)^T (x_i - \vec{1}_{n_i} \hat{x}_i)$ (i.e. $\widehat{M}_i = I_{n_i}, P_i = \vec{1}_{n_i}$) is a storage function from $\hat{\mathfrak{D}}_i$ to $\mathfrak{D}_i$, with the following parameters

$$\begin{aligned} W_i = I_{n_i}, X_i^{11} = 0_{n_i}, X_i^{22} = -2I_{n_i}, \\ X_i^{12} = X_i^{21} = I_{n_i}, H_i = \hat{W}_i = \vec{1}_{n_i}, \end{aligned} \tag{4.88}$$

and with $\alpha_i(r) = \frac{1}{\lambda_{\max}(C_{1i}^T C_{1i})} r^2$, $\eta_i(r) = 2r$, $\psi_{i\text{ext}}(r) = 0, \forall r \in \mathbb{R}_{\geq 0}$. We use the following interface function to select $u_i \in \mathbb{R}^{n_i}$ for any $x_i \in \mathbb{R}^{n_i}, \hat{x}_i \in \mathbb{R}, \hat{u}_i \in \mathbb{R}$:

$$u_i = -2(x_i - \vec{1}_{n_i} \hat{x}_i) + \vec{1}_{n_i} \hat{u}_i - \vec{1}_{n_i} \varphi(\vec{1}_{n_i}^T x_i) + \vec{1}_{n_i} \varphi(\hat{x}_i).$$

By selecting $\mu_1 = \cdots = \mu_N = 1$, the function $V(x, x_o, \hat{x}, \hat{x}_o) = \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i) + (x_o - \Pi \hat{x}_o)^T (x_o - \Pi \hat{x}_o)$, where $\Pi = \mathsf{diag}(\vec{1}_{n_1}, \ldots, \vec{1}_{n_N})$, is a simulation function from $\hat{\mathfrak{D}}$ to $\mathfrak{D}$, where $\hat{\mathfrak{D}}$ is the interconnection of the abstract subsystems with the dynamic interconnection topology $\hat{\mathfrak{D}}_o$ satisfying conditions (4.72) and (4.73). For this example, we choose $C_{1i} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, and the dynamic interconnection system as follows

$$A_o = -3I_n$$
$$B_o = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}. \tag{4.89}$$

For this dynamic interconnection, there always exists $\hat{\mathfrak{D}}_o$ satisfying conditions (4.72) and (4.73) for any even n. Using (4.73), the abstract interconnection system is given by

$$\hat{A}_o = -3I_N, \hat{B}_o = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \hat{C}_o = I_N, \hat{D}_o = 0_N. \tag{4.90}$$

For the sake of simulation, we choose $N = 3, n = 150$, and $n_i = 50, \forall i \in [1; N]$. The simulation results are shown in Figures 4.5 and 4.6. The initial condition of ξ_i is chosen as $\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, while $\hat{\xi}_i$, x_o, and $\hat{x}_o$ are initialized from zero. The input to the abstract interconnected system $\hat{\mathfrak{D}}$ is chosen as $\hat{v}(t) = \begin{bmatrix} 10\cos(t) & 10\sin(2t) & 10\cos(4t) \end{bmatrix}^T$.

4.4 Summary

In this chapter, we provided sufficient conditions to construct compositional abstractions of interconnected systems consisting of stochastic hybrid sub-systems interconnected via randomly switched topologies. We also provided a weaker version of those conditions, which allows for construction of such abstractions for a broader set of interconnected systems. We illustrate the effectiveness of the proposed results by designing a controller enforcing some complex properties over the interconnected abstraction and then refining it back to the original interconnected system.

Additionally, we derived conditions under which compositional abstractions of networks of control systems under dynamic interconnection topologies can be constructed using abstractions of subsystems. We showed the effectiveness of the results on a 150-dimensional network of R-C circuits by reducing it to a 3-dimensional abstraction.

$$
\begin{aligned}
&\dot{V}(x, x_o, \hat{x}, \hat{x}_o) = \sum_{i=1}^{N} \mu_i \dot{V}_i(x_i, \hat{x}_i) + 2(x_o - \Pi\hat{x}_o)^T C_o^T C_o(\dot{x}_o - \Pi\dot{\hat{x}}_o) \\
&\leq \sum_{i=1}^{N} \mu_i \Bigg(-\eta_i(V_i(x_i, \hat{x}_i)) + \psi_{i\mathrm{ext}}(\|\hat{u}_i\|) + \begin{bmatrix} W_i w_i - \hat{W}_i \hat{w}_i \\ h_{2i}(x_i) - H_i \hat{h}_{2i}(\hat{x}_i) \end{bmatrix}^T \begin{bmatrix} X_i^{11} & X_i^{12} \\ X_i^{21} & X_i^{22} \end{bmatrix} \begin{bmatrix} W_i w_i - \hat{W}_i \hat{w}_i \\ h_{2i}(x_i) - H_i \hat{h}_{2i}(\hat{x}_i) \end{bmatrix} \Bigg) \\
&\quad + 2(x_o - \Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - \Pi\hat{A}_o\hat{x}_o - \Pi\hat{B}_o\hat{u}_o) \\
&= \sum_{i=1}^{N} -\mu_i\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{i=1}^{N} \mu_i \psi_{i\mathrm{ext}}(\|\hat{u}_i\|) \\
&\quad + \begin{bmatrix} W \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} - \hat{W} \begin{bmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_N \end{bmatrix} \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \end{bmatrix}^T X(\mu_1 X_1, \dots, \mu_N X_N) \begin{bmatrix} * \end{bmatrix} \\
&\quad + 2(x_o - \Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - A_o\Pi\hat{x}_o - B_o H\hat{u}_o) \\
&= \sum_{i=1}^{N} -\mu_i\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{i=1}^{N} \mu_i \psi_{i\mathrm{ext}}(\|\hat{u}_i\|) \\
&\quad + \begin{bmatrix} W C_o x_o + W D_o u_o - \hat{W}\hat{C}_o\hat{x}_o - \hat{W}\hat{D}_o\hat{u}_o \\ \begin{bmatrix} h_1(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \end{bmatrix}^T X(\mu_1 X_1, \dots, \mu_N X_N) \begin{bmatrix} * \end{bmatrix} \\
&\quad + 2(x_o - \Pi\hat{x}_o)^T C_o^T C_o(A_o x_o + B_o u_o - A_o\Pi\hat{x}_o - B_o H\hat{u}_o) \\
&\leq -\sum_{i=1}^{N} \mu_i\eta_i(V_i(x_i, \hat{x}_i)) + \psi_{\mathrm{ext}}(\|\hat{u}\|) \\
&\quad + \begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} W C_o & W D_o \\ 0_* & I_* \end{bmatrix}^T X(\mu_1 X_1, \dots, \mu_N X_N) \begin{bmatrix} W C_o & W D_o \\ 0_* & I_* \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \\
&\quad + \begin{bmatrix} x_o - \Pi\hat{x}_o \\ \begin{bmatrix} h_{21}(x_1) - H_1\hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N\hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} C_o^T C_o A_o + A_o^T C_o^T C_o & C_o^T C_o B_o \\ B_o^T C_o^T C_o & 0_* \end{bmatrix} \begin{bmatrix} * \end{bmatrix} \\
&\leq -\sum_{i=1}^{N} \mu_i\eta_i(V_i(x_i, \hat{x}_i)) - \kappa_o(x_o - \Pi\hat{x}_o)^T C_o^T C_o(x_o - \Pi\hat{x}_o) + \psi_{\mathrm{ext}}(\|\hat{u}\|) \\
&= -\sum_{i=1}^{N} \mu_i\eta_i(V_i(x_i, \hat{x}_i)) - \kappa_o V_o(x_o, \hat{x}_o) + \psi_{\mathrm{ext}}(\|\hat{u}\|) \\
&\leq -\eta(V(x, x_o, \hat{x}, \hat{x}_o)) + \psi_{\mathrm{ext}}(\|\hat{u}\|)
\end{aligned} \tag{4.84}
$$

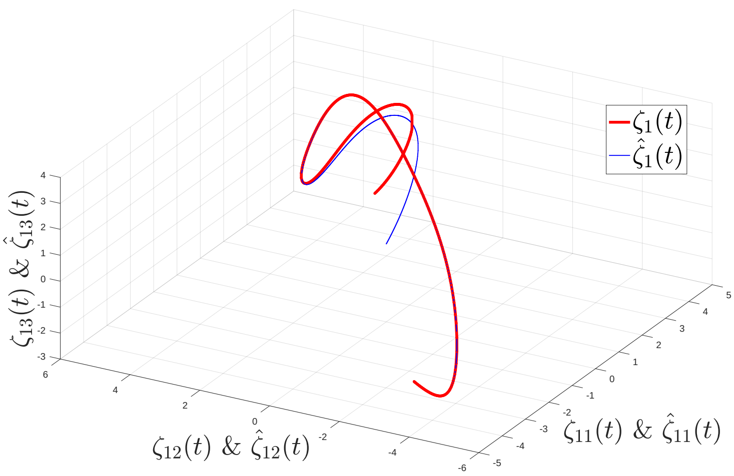

Figure 4.5: Evolution of the external outputs of the concrete and abstract interconnected systems.

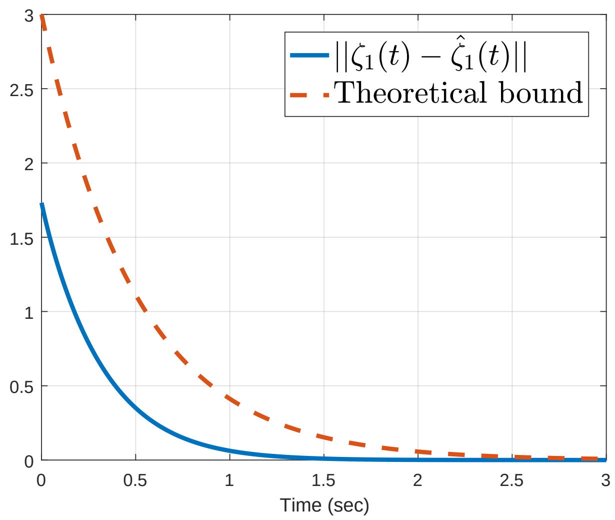

Figure 4.6: The evolution of the norm of error, i.e. $\|\zeta_1(t) - \hat{\zeta}_1(t))\|$, along with the theoretical bound given in (4.67).

Chapter 5

Compositional Abstraction of Interconnected Systems over Riemannian Manifolds

5.1 Introduction

All the results in the previous chapters for constructing (reduced-order) abstractions consider systems evolving over Euclidean spaces. The state-space of many systems constitute Riemannian manifolds [BL04], and consequently their analysis requires techniques from differential geometry [Tar+13]. Robotic manipulators, rotating bodies, rolling disks, etc. are examples of such systems found in many mechanical settings [BL04]. A simple example of Riemannian manifold is illustrated in Figure 5.1. This necessitates a generalization of various notions employed for compositional construction of abstractions for interconnected systems introduced in previous chapters (e.g., notions quantifying closeness of output trajectories between the concrete system and its abstraction) to systems defined over Riemannian manifolds.

5.1.1 Contribution

In this chapter, we propose two techniques for compositional construction of abstractions for interconnected control systems evolving over smooth Riemannian manifolds.

In the first approach, we provide a small-gain type condition that enables the construction of an abstraction for the interconnected control system compositionally. We employ a notion of so-called manifold simulation function, constructed using a (pseudo) Riemannian metric defined over the tangent bundle of the state space, to quantify the error between concrete interconnected control systems and their approximations. Given

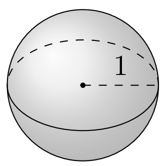

Figure 5.1: Example of a smooth Riemannian manifold - Unit Sphere embedded in $\mathbb{R}^3$: $\mathcal{M}_2 = \mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$

a network of control subsystems and the manifold simulation functions between them and their abstractions, we derive sufficient conditions based on small-gain type reasoning [DRW07], guaranteeing that a network of abstractions quantitatively approximates the original network of concrete subsystems.

In the second approach, we introduce a notion of so-called differential storage functions, adopted from the notion of differential storage functions introduced in the context of differential dissipativity [FS13], describing joint differential dissipativity properties of control subsystems and their abstractions. Given a network of control subsystems and the differential storage functions between them and their abstractions, we derive sufficient conditions based on the interconnection topology, guaranteeing that a network of abstractions quantitatively approximates the original network of concrete subsystems.

5.2 Variational and Augmented Control Systems

In this section, we introduce two notions of which will serve as ingredients to develop abstractions of control systems evolving over Riemannian manifolds.

Definition 5.2.1. *Given any control system over Riemannian manifold*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

*the **variational** control system of $\mathfrak{S}$ is given by the tuple*

$$\delta\mathfrak{S} = (\mathcal{T}\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, \delta f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, \delta h_1, \delta h_2),$$

where for every $[x;\delta x] \in \mathcal{T}\mathcal{M}_n, u \in \mathbb{R}^m, \delta u \in \mathbb{R}^m, w \in \mathbb{R}^p,$ and $\delta w \in \mathbb{R}^p$:

$$\begin{aligned} \delta f(x, \delta x, u, \delta u, w, \delta w) &:= \frac{\partial f}{\partial x}(x,u,w)\delta x + \frac{\partial f}{\partial u}(x,u,w)\delta u + \frac{\partial f}{\partial w}(x,u,w)\delta w \\ \delta h_1(x,\delta x) &:= \frac{\partial h_1}{\partial x}(x)\delta x \\ \delta h_2(x,\delta x) &:= \frac{\partial h_2}{\partial x}(x)\delta x. \end{aligned} \tag{5.1}$$

We can define a similar notion for a control system without internal outputs as follows.

Definition 5.2.2. *Given any*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^q, h),$$

the variational control system of $\mathfrak{S}$ is given by the tuple

$$\delta\mathfrak{S} = (\mathcal{T}\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, \delta f, \mathbb{R}^q, \delta h),$$

where for every $[x;\delta x] \in \mathcal{T}\mathcal{M}_n, u \in \mathbb{R}^m, \delta u \in \mathbb{R}^m, w \in \mathbb{R}^p,$ and $\delta w \in \mathbb{R}^p$:

$$\begin{aligned} \delta f(x, \delta x, u, \delta u, w, \delta w) &:= \frac{\partial f}{\partial x}(x,u,w)\delta x + \frac{\partial f}{\partial u}(x,u,w)\delta u + \frac{\partial f}{\partial w}(x,u,w)\delta w \\ \delta h(x,\delta x) &:= \frac{\partial h}{\partial x}(x)\delta x. \end{aligned} \tag{5.2}$$

The variational control system of $\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ can be defined in a similar manner.

We now introduce the notion of augmented control systems.

Definition 5.2.3. *Let*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

and

$$\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2),$$

be two control subsystems with the same external output space dimension. We define the ***augmented*** *system*

$$\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \tilde{h}_1, \tilde{h}_2),$$

where $\mathcal{M}_{\tilde{n}} = \mathcal{M}_n \times \mathcal{M}_{\hat{n}}$, $\tilde{\mathcal{U}} = \mathcal{U} \times \hat{\mathcal{U}}$, $\tilde{\mathcal{W}} = \mathcal{W} \times \hat{\mathcal{W}}$, $\tilde{m} = m + \hat{m}$, $\tilde{p} = p + \hat{p}$, $\tilde{q}_2 = q_2 + \hat{q}_2$, *and for each* $x \in \mathcal{M}_n, \hat{x} \in \mathcal{M}_{\hat{n}}$, $u \in \mathbb{R}^m, \hat{u} \in \mathbb{R}^{\hat{m}}$, $w \in \mathbb{R}^p$, *and* $\hat{w} \in \mathbb{R}^{\hat{p}}$*:*

$$\begin{aligned} \tilde{f}(\tilde{x}, \tilde{u}, \tilde{w}) &:= \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix}, \\ \tilde{h}_1(\tilde{x}) &:= h_1(x) - \hat{h}_1(\hat{x}), \\ \tilde{h}_2(\tilde{x}) &:= \begin{bmatrix} h_2(x) \\ \hat{h}_2(\hat{x}) \end{bmatrix}, \end{aligned} \tag{5.3}$$

where $\tilde{x} = [x; \hat{x}], \tilde{u} = [u; \hat{u}]$, *and* $\tilde{w} = [w; \hat{w}]$.

For a control system without internal outputs, we can a define a similar notion as follows.

Definition 5.2.4. *Let*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^q, h),$$

and

$$\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^q, \hat{h}),$$

be two control systems with the same output space dimension. We define the augmented system

$$\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^q, \tilde{h}),$$

where $\mathcal{M}_{\tilde{n}} = \mathcal{M}_n \times \mathcal{M}_{\hat{n}}$, $\tilde{\mathcal{U}} = \mathcal{U} \times \hat{\mathcal{U}}$, $\tilde{\mathcal{W}} = \mathcal{W} \times \hat{\mathcal{W}}$, $\tilde{m} = m + \hat{m}$, $\tilde{p} = p + \hat{p}$, *and for each* $x \in \mathcal{M}_n, \hat{x} \in \mathcal{M}_{\hat{n}}$, $u \in \mathbb{R}^m, \hat{u} \in \mathbb{R}^{\hat{m}}$, $w \in \mathbb{R}^p$, *and* $\hat{w} \in \mathbb{R}^{\hat{p}}$*:*

$$\begin{aligned} \tilde{f}(\tilde{x}, \tilde{u}, \tilde{w}) &:= \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix}, \\ \tilde{h}(\tilde{x}) &:= h(x) - \hat{h}(\hat{x}), \end{aligned} \tag{5.4}$$

where $\tilde{x} = [x; \hat{x}], \tilde{u} = [u; \hat{u}]$, *and* $\tilde{w} = [w; \hat{w}]$.

For a control system without internal inputs and outputs, we can a define a similar notion as follows.

Definition 5.2.5. *Let* $\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ *and* $\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$ *be two control systems with the same output space dimension. We define the augmented system* $\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \tilde{f}, \mathbb{R}^q, \tilde{h})$*, where* $\mathcal{M}_{\tilde{n}} = \mathcal{M}_n \times \mathcal{M}_{\hat{n}}$*,* $\tilde{\mathcal{U}} = \mathcal{U} \times \hat{\mathcal{U}}$*,* $\tilde{m} = m + \hat{m}$*, and for each* $x \in \mathcal{M}_n, \hat{x} \in \mathcal{M}_{\hat{n}}$*,* $u \in \mathbb{R}^m$*, and* $\hat{u} \in \mathbb{R}^{\hat{m}}$*:*

$$\begin{aligned} \tilde{f}(\tilde{x}, \tilde{u}) &:= \begin{bmatrix} f(x,u) \\ \hat{f}(\hat{x}, \hat{u}) \end{bmatrix}, \\ \tilde{h}(\tilde{x}) &:= h(x) - \hat{h}(\hat{x}), \end{aligned} \tag{5.5}$$

where $\tilde{x} = [x; \hat{x}]$*, and* $\tilde{u} = [u; \hat{u}]$*.*

5.3 Certificate of Abstraction

In this section, we introduce a notion used to formally relate a control system over Riemannian manifold and its abstraction.

5.3.1 Manifold Simulation Function

In this subsection, we introduce a notion of manifold simulation function, which is used to quantify the closeness of output trajectories of the concrete systems and the ones of their abstractions.

Definition 5.3.1. *Consider two control subsystems*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^q, h)$$

and

$$\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^q, \hat{h})$$

with the same output space dimension and the corresponding augmented system

$$\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^q, \tilde{h})$$

as in Definition 5.2.4. Let

$$\delta\tilde{\mathfrak{S}} = (\mathcal{TM}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \delta\tilde{f}, \mathbb{R}^q, \delta\tilde{h})$$

be the variational control system of $\tilde{\mathfrak{S}}$ *as defined in Definition 5.2.2. Suppose there exists some positive constants* α *and* λ*, a matrix valued function* $G : \mathcal{M}_{\tilde{n}} \to \mathbb{R}^{\tilde{n}\times\tilde{n}}$ *such that* $G(\tilde{x})$ *is a positive (semi) definite matrix for all* $\tilde{x} \in \mathcal{M}_{\tilde{n}}$*, functions* $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$*,* $\psi_{\text{int}} \in \mathcal{K}_\infty \cup \{0\}$ *and a continuously differentiable function*[1] $k : \mathcal{M}_{\tilde{n}} \times \mathbb{R}^{\hat{m}} \times \mathbb{R}^p \to \mathbb{R}^m$ *which satisfies* $k(0,0,0) = 0$*, such that the following two conditions hold:*[2]

[1]We refer to k as the *interface* map.
[2]Here, for brevity, we do not write the arguments of the partial derivatives explicitly.

- *For any* $\tilde{x} \in \mathcal{M}_{\tilde{n}}$*:*

$$G(\tilde{x}) \succeq \alpha \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^T \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right). \tag{5.6}$$

- *For any* $[\tilde{x}; \delta\tilde{x}] \in \mathcal{TM}_{\tilde{n}}$*,* $\hat{u} \in \mathbb{R}^{\hat{m}}$*,* $\delta\hat{u} \in \mathbb{R}^{\hat{m}}$*,* $\hat{w} \in \mathbb{R}^{\hat{p}}, \delta\hat{w} \in \mathbb{R}^{\hat{p}}$*, if we choose* u *using the map* $u = k(\tilde{x}, \hat{u}, \hat{w})$*, then for all* $w \in \mathbb{R}^p, \delta w \in \mathbb{R}^p$*:*

$$\begin{aligned}
&\delta\tilde{x}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G(\tilde{x}) + G(\tilde{x})\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}}\tilde{f}(\tilde{x}, \tilde{u}, \tilde{w})\right)\delta\tilde{x} \\
&+ 2\delta\tilde{w}^T \frac{\partial \tilde{f}}{\partial \tilde{w}}^T G(\tilde{x})\delta\tilde{x} + 2\delta\tilde{u}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G(\tilde{x})\delta\tilde{x} \\
&\leq -\lambda\delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \psi_{\mathrm{int}}(\|\delta w - \delta\hat{w}\|) + \psi_{\mathrm{ext}}(\|\delta\hat{u}\|),
\end{aligned} \tag{5.7}$$

where, $\delta\tilde{u} = [\delta u; \delta\hat{u}]$*,* $\delta u = \frac{\partial k}{\partial \tilde{x}}\delta\tilde{x} + \frac{\partial k}{\partial \hat{u}}\delta\hat{u} + \frac{\partial k}{\partial \hat{w}}\delta\hat{w}$*,* $\tilde{w} = [w; \hat{w}]$*, and* $\delta\tilde{w} = [\delta w; \delta\hat{w}]$*.*

then

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma} \in \Gamma(\tilde{x}, 0)} \int_0^1 \frac{\partial}{\partial s}\tilde{\gamma}(s)^T G(\tilde{\gamma}(s))\frac{\partial}{\partial s}\tilde{\gamma}(s)ds, \tag{5.8}$$

is called a manifold simulation function from $\hat{\mathfrak{S}}$ *to* $\mathfrak{S}$ *with respect to the (pseudo) Riemannian structure* G*. We call* $\hat{\mathfrak{S}}$ *(preferably with* $\hat{n} < n$*) an abstraction of* $\mathfrak{S}$ *if there exists a manifold simulation function from* $\hat{\mathfrak{S}}$ *to* $\mathfrak{S}$*.*

We study interconnected control systems without internal inputs and outputs, resulting from the interconnection of control subsystems having both internal and external signals. Thus we modify the above definition for systems without internal inputs and outputs as follows:

Definition 5.3.2. *Let*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$$

and

$$\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h}),$$

be two control systems without internal inputs and outputs and let

$$\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \tilde{f}, \mathbb{R}^q, \tilde{h})$$

be the corresponding augmented control system as defined in Definition 5.2.5. Let

$$\delta\tilde{\mathfrak{S}} = (\mathcal{TM}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \tilde{\mathcal{U}}, \delta\tilde{f}, \mathbb{R}^q, \delta\tilde{h})$$

be the variational control system of $\tilde{\mathfrak{S}}$*. Suppose there exist some positive constants* α *and* λ*, some function* $\psi_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$*, some matrix valued function* $G : \mathcal{M}_{\tilde{n}} \to \mathbb{R}^{\tilde{n} \times \tilde{n}}$*, where* $G(\tilde{x})$ *is a positive (semi) definite matrix for each* $\tilde{x} \in \mathcal{M}_{\tilde{n}}$*, and a continuously differentiable function* $k : \mathcal{M}_{\tilde{n}} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^m$ *which satisfies* $k(0, 0) = 0$*, such that the following two conditions hold:*

- *For all* $\tilde{x} \in \mathcal{M}_{\tilde{n}}$*:*

$$G(\tilde{x}) \succeq \alpha \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^T \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right). \tag{5.9}$$

- *For any* $[\tilde{x}; \delta\tilde{x}] \in \mathcal{TM}_{\tilde{n}}$*,* $\hat{u} \in \mathbb{R}^{\hat{m}}$*, and* $\delta\hat{u} \in \mathbb{R}^{\hat{m}}$*, if we select* u *using the map* $u = k(\tilde{x}, \hat{u})$*:*

$$\begin{aligned}&\delta\tilde{x}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G(\tilde{x}) + G(\tilde{x})\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}}\tilde{f}(\tilde{x}, \tilde{u})\right)\delta\tilde{x} + 2\delta\tilde{u}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G(\tilde{x})\delta\tilde{x}\\ &\leq -\lambda\delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \psi_{\text{ext}}(\|\delta\hat{u}\|),\end{aligned} \tag{5.10}$$

where $\delta\tilde{u} = [\delta u; \delta\hat{u}]$*, and* $\delta u = \frac{\partial k}{\partial \tilde{x}}\delta\tilde{x} + \frac{\partial k}{\partial \hat{u}}\delta\hat{u}$*,*

then

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma} \in \Gamma(\tilde{x}, 0)} \int_0^1 \frac{\partial}{\partial s}\tilde{\gamma}(s)^T G(\tilde{\gamma}(s))\frac{\partial}{\partial s}\tilde{\gamma}(s)ds, \tag{5.11}$$

is called a manifold simulation function from $\hat{\mathfrak{S}}$ *to* $\mathfrak{S}$ *with respect to the (pseudo) Riemannian structure* G*.*

The next theorem shows the usefulness of the existence of a manifold simulation function in quantifying the closeness of two control subsystems.

Theorem 5.3.3. *Consider two control systems* $\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ *and* $\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \mathbb{R}^q, \hat{h})$*. Suppose* V_G*, associated with the (pseudo) Riemannian structure* G*, is a manifold simulation function from* $\hat{\mathfrak{S}}$ *to* $\mathfrak{S}$*, and* k *is the associated interface map, then there exists* $\beta \in \mathcal{KL}$*, and* $\bar{\psi}_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ *such that for any* $x \in \mathbb{R}^n$*,* $\hat{x} \in \mathbb{R}^{\hat{n}}$*,* $\hat{\upsilon} \in \hat{\mathcal{U}}$*, if we choose* $\upsilon \in \mathcal{U}$ *using the interface map* k*, then the following inequality holds for any* $t \in \mathbb{R}_{\geq 0}$*:*

$$\|\zeta_{x\upsilon}(t) - \hat{\zeta}_{\hat{x}\hat{\upsilon}}(t)\| \leq \beta(V_G(x, \hat{x}), t) + \bar{\psi}_{\text{ext}}(\|\hat{\upsilon}\|_\infty). \tag{5.12}$$

Proof. Consider two points $\tilde{x} = [x; \hat{x}] \in \mathcal{M}_{\tilde{n}}$ and $0 \in \mathcal{M}_{\tilde{n}}$, and a geodesic $\chi : [0, 1] \to \mathbb{R}^{\tilde{n}}$, with respect to the (pseudo) Riemannian structure G, such that $\chi(0) = 0$, and $\chi(1) = \tilde{x}$. The energy functional corresponding to this geodesic is given by

$$V_G(\tilde{x}) = E_G(\tilde{x}, 0) = \int_0^1 \frac{\partial}{\partial s}\chi(s)^T G(\chi(s))\frac{\partial}{\partial s}\chi(s)ds. \tag{5.13}$$

Let $\tilde{\xi}_{\tilde{x}\tilde{\nu}} = [\xi_{x\nu}; \hat{\xi}_{\hat{x}\hat{\nu}}]$ be the solution trajectory of $\tilde{\mathfrak{S}}$ for any initial condition $\tilde{x} \in \mathcal{M}_{\tilde{n}}$, and under the input trajectory $\tilde{\nu} = [\nu; \hat{\nu}]$, where $\nu(t) = k(\tilde{\xi}_{\tilde{x}\tilde{\nu}}(t), \hat{\nu}(t))$, for all $t \in \mathbb{R}_{\geq 0}$, for any $\hat{\nu} \in \hat{\mathcal{U}}$.

For a fixed $t \in \mathbb{R}_{\geq 0}$, consider the straight line $\hat{\eta}(s, t) = s\hat{\nu}(t)$ in s, where $s \in [0, 1]$. For any fixed $t \in \mathbb{R}_{\geq 0}$, the curve $\hat{\eta}(\cdot, t) : [0, 1] \to \mathbb{R}^{\hat{m}}$ is a geodesic, with respect to the Euclidean metric, on $\mathbb{R}^{\hat{m}}$ joining $\hat{\eta}(0, t) = 0$ and $\hat{\eta}(1, t) = \hat{\nu}(t)$.

For any $s \in [0,1]$, let $\tilde{\phi}(s,\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{n}}$ be the solution trajectory of $\tilde{\mathfrak{S}}$ from initial condition $\chi(s)$ under the input $\tilde{\eta}(s,\cdot)$, where $\tilde{\eta}(s,t) = \begin{bmatrix} k(\tilde{\phi}(s,t), \hat{\eta}(s,t)) \\ \hat{\eta}(s,t) \end{bmatrix}, \forall t \in \mathbb{R}_{\geq 0}$. Note that $\tilde{\phi}(0,t) = 0$, and $\tilde{\phi}(1,t) = \tilde{\xi}_{\tilde{x}\tilde{\nu}}(t)$.

For brevity, we denote $\frac{\partial}{\partial s}\tilde{\phi}(s,t) =: \tilde{w}(s,t)$. Note that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\tilde{w}(s,t) = \frac{\partial^2}{\partial t \partial s}\tilde{\phi}(s,t) = \frac{\partial^2}{\partial s \partial t}\tilde{\phi}(s,t) \\
&= \frac{\partial}{\partial s}\tilde{f}(\tilde{\phi}(s,t), \tilde{\eta}(s,t)) = \frac{\partial \tilde{f}}{\partial \tilde{x}}\frac{\partial}{\partial s}\tilde{\phi}(s,t) + \frac{\partial \tilde{f}}{\partial \tilde{u}}\frac{\partial}{\partial s}\tilde{\eta}(s,t) \\
&= \frac{\partial \tilde{f}}{\partial \tilde{x}}\tilde{w}(s,t) + \frac{\partial \tilde{f}}{\partial \tilde{u}}\begin{bmatrix} \frac{\partial k}{\partial \tilde{x}}\tilde{w}(s,t) + \frac{\partial k}{\partial \tilde{u}}\hat{\nu}(t) \\ \hat{\nu}(t) \end{bmatrix}.
\end{aligned} \tag{5.14}
$$

Define

$$
l(t) = \int_0^1 \tilde{w}(s,t)^T G(\phi(s,t))\tilde{w}(s,t)ds, \tag{5.15}
$$

i.e. $l(t)$ is the energy functional of the curve $\tilde{\phi}(\cdot,t)$, with respect to G. We have

$$
\begin{aligned}
\frac{d}{dt}l(t) &= \int_0^1 \frac{\partial}{\partial t}\tilde{w}(s,t)^T G(\tilde{\phi}(s,t))\tilde{w}(s,t)ds \\
&= \int_0^1 \tilde{w}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G + G\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}} f \right) \tilde{w} ds \\
&\quad + 2\int_0^1 \begin{bmatrix} \frac{\partial k}{\partial \tilde{x}}\tilde{w} + \frac{\partial k}{\partial \tilde{u}}\hat{\nu} \\ \hat{\nu} \end{bmatrix}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G\tilde{w}ds,
\end{aligned} \tag{5.16}
$$

where, again, we have dropped explicit arguments for clarity in the last expression. From (5.10), one has:

$$
\begin{aligned}
\frac{d}{dt}l(t) &\leq -\lambda \int_0^1 \tilde{w}(s,t)^T G(\tilde{\phi}(s,t))\tilde{w}(s,t)ds + \int_0^1 \psi_{\text{ext}}\left(\left\| \frac{\partial \hat{\eta}(s,t)}{\partial s} \right\| \right) ds \\
&\leq -\lambda \int_0^1 \tilde{w}(s,t)^T G(\tilde{\phi}(s,t))\tilde{w}(s,t)ds + \psi_{\text{ext}}(\|\hat{\nu}(t)\|) \int_0^1 ds \\
&\leq -\lambda l(t) + \psi_{\text{ext}}(\|\hat{\nu}\|_\infty).
\end{aligned} \tag{5.17}
$$

It follows from the comparison lemma [Kha96] that

$$
l(t) \leq \mathrm{e}^{-\lambda t} l(0) + \frac{1}{\lambda}\psi_{\text{ext}}(\|\hat{\nu}\|_\infty). \tag{5.18}
$$

Note that $l(0) = V_G(\xi_{x\nu}(0), \hat{\xi}_{\hat{x}\hat{\nu}}(0)) = V_G(\tilde{x})$. Now using the fact that for any $t \in \mathbb{R}_{\geq 0}$, $l(t)$ is not necessarily the minimum energy functional corresponding to a geodesic because $\tilde{\phi}(s,t)$ is not necessarily a geodesic, i.e. $V_G(\xi_{x\nu}(t), \hat{\xi}_{\hat{x}\hat{\nu}}(t)) \leq l(t)$, one has:

$$
V_G(\xi_{x\nu}(t), \hat{\xi}_{\hat{x}\hat{\nu}}(t)) \leq \mathrm{e}^{-\lambda t} V_G(\xi_{x\nu}(0), \hat{\xi}_{\hat{x}\hat{\nu}}(0)) + \frac{1}{\lambda}\psi_{\text{ext}}(\|\hat{\nu}\|_\infty). \tag{5.19}
$$

For every $x \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{\hat{n}}$, we use (5.9) and the Schwarz inequality to obtain:

$$\begin{aligned}
&\alpha\|h(x) - \hat{h}(\hat{x})\|^2 = \alpha\|\tilde{h}(\tilde{x})\|^2 \\
&\leq \alpha\left(\int_0^1 \sqrt{\frac{\partial}{\partial s}\chi(s)^T \frac{\partial \tilde{h}}{\partial \tilde{x}}(\chi(s))^T \frac{\partial \tilde{h}}{\partial \tilde{x}}(\chi(s)) \frac{\partial}{\partial s}\chi(s) ds}\right)^2 \\
&\leq \left(\int_0^1 \sqrt{\frac{\partial}{\partial s}\chi(s)^T G(\chi(s)) \frac{\partial}{\partial s}\chi(s) ds}\right)^2 \\
&\leq \int_0^1 \frac{\partial}{\partial s}\chi(s)^T G(\chi(s)) \frac{\partial}{\partial s}\chi(s) ds = V_G(\tilde{x}),
\end{aligned} \tag{5.20}$$

where $\tilde{x} = [x; \hat{x}]$. Combining (5.20) with (5.19), one can conclude that (5.12) is satisfied with $\beta(r, s) = \sqrt{\frac{r}{\alpha}}\mathrm{e}^{-\frac{\lambda}{2}s}$ and $\bar{\psi}_{\text{ext}}(r) = \sqrt{\frac{1}{\alpha\lambda}\psi_{\text{ext}}(r)}, \forall s, r \in \mathbb{R}_{\geq 0}$. $\square$

5.4 Compositional Abstraction: Small-gain Approach

In the next subsection we first provide a definition of interconnected control systems over Riemannian manifolds. We then provide small-gain type conditions under which we can construct abstractions of interconnected control systems in a compositional way.

5.4.1 Interconnected systems

Here, we define the interconnection between the control subsystems by defining the relationship between the outputs and internal inputs. Consider $N \in \mathbb{N}_{\geq 1}$ control subsystems

$$\mathfrak{S}_i = (\mathcal{M}_{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_i}, h_i),$$

$i \in [1; N]$, with partitioned internal inputs and outputs

$$\begin{aligned}
w_i &= [w_{i1}; \ldots; w_{i(i-1)}; w_{i(i+1)}; \ldots; w_{iN}], \quad w_{ij} \in \mathbb{R}^{p_{ij}} \\
y_i &= [y_{i1}; \ldots; y_{i(i-1)}; y_{i(i+1)}; \ldots; y_{iN}], \quad y_{ij} \in \mathbb{R}^{q_{ij}}
\end{aligned} \tag{5.21}$$

and the internal output function

$$h_i = [h_{i1}; \ldots; h_{i(i-1)}; h_{i(i+1)}; \ldots; h_{iN}]. \tag{5.22}$$

We interpret the outputs y_{ii} as *external* ones, whereas the outputs y_{ij} with $i \neq j$ are *internal* ones which are used to define the interconnected control system. In particular, we assume that the dimension of w_{ij} is equal to the dimension of y_{ji} i.e. the following *dimension constraints* hold:

$$p_{ij} = q_{ji}, \quad \forall i, j \in [1; N], i \neq j. \tag{5.23}$$

If there is no connection from the control system $\mathfrak{S}_i$ to $\mathfrak{S}_j$, then we assume that the connecting output function is identically zero for all arguments i.e. $h_{ij} \equiv 0$. Now we provide the definition of the interconnected control system.

Definition 5.4.1. *Consider $N \in \mathbb{N}_{\geq 1}$ control subsystems*

$$\mathfrak{S}_i = (\mathcal{M}_{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_i}, h_i),$$

$i \in [1; N]$, *with the input-output configuration given by* (5.21)*,* (5.22) *and* (5.23)*. The interconnected control system*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}(\mathfrak{S}_1, \ldots, \mathfrak{S}_N)$*, follows by* $\mathcal{M}_n = \Pi_{i=1}^N \mathcal{M}_{n_i}$*,* $m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{ii}$ *and the functions*

$$f(x,u) = \left[f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)\right], \tag{5.24}$$

$$h(x) = [h_1(x); \ldots; h_N(x_N)], \tag{5.25}$$

where $u = [u_1; \ldots; u_N]$*,* $x = [x_1; \ldots; x_N]$*, and the interconnection variables constrained by* $w_{ij} = y_{ji}$*, for all* $i, j \in [1; N], i \neq j$*.*

5.4.2 Small-gain Compositionality Result

In this section we provide sufficient conditions under which an interconnection of abstractions of control systems, is an abstraction of the original interconnected system. We assume that we are given $N \in \mathbb{N}$ control systems

$$\mathfrak{S}_i = (\mathcal{M}_{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_i}, h_i),$$

where $i \in [1; N]$, together with the corresponding abstractions

$$\hat{\mathfrak{S}}_i = (\mathcal{M}_{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{p_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \mathbb{R}^{q_i}, \hat{h}_i),$$

where $i \in [1; N]$ and with manifold simulation function V_{G_i}, associated with the (pseudo) Riemannian structure G_i, from $\hat{\mathfrak{S}}_i$ to $\mathfrak{S}_i$. We use $\alpha_i, \lambda_i, \psi_{i\mathrm{ext}}$, and $\psi_{i\mathrm{int}}$ to denote the corresponding constants and functions appearing in Definition 5.3.1. We require the following assumptions in order to provide the compositionality result:

Assumption 5.4.2. *For any* $i, j \in [1; N]$*,* $i \neq j$*, there exists a positive constant* δ_{ij} *such that for any* $s \in \mathbb{R}_{\geq 0}$*:*

$$h_{ji} \equiv 0 \implies \delta_{ij} = 0 \text{ and} \tag{5.26}$$

$$h_{ji} \not\equiv 0 \implies \psi_{i\mathrm{int}}\left((N-1)\sqrt{\frac{s}{\alpha_j}}\right) \leq \delta_{ij} s. \tag{5.27}$$

For notational simplicity we define the matrix $\Delta \in \mathbb{R}^{N \times N}$ with its components given by $\Delta_{ii} = 0$ for $i \in [1; N]$ and $\Delta_{ij} = \delta_{ij}$ for $i, j \in [1; N]$, $i \neq j$. The next theorem provides a compositionality approach on the construction of abstractions of interconnected control systems and that of the corresponding simulation functions.

Theorem 5.4.3. *Consider the interconnected control system $\mathfrak{S} = \mathcal{I}(\mathfrak{S}_1, \ldots, \mathfrak{S}_N)$, induced by N control subsystems $\mathfrak{S}_i$. Suppose each subsystem $\mathfrak{S}_i$ admits an abstraction $\hat{\mathfrak{S}}_i$ with the corresponding simulation function V_{G_i} with respect to (pseudo) Riemannian metric G_i. If Assumption 5.4.2 holds and there exists a vector $\mu = [\mu_1; \ldots; \mu_N]$, where $\mu_i \geq 1$ $\forall i \in [1; N]$, such that the inequality*

$$\mu^T(-\Lambda + \Delta) < 0 \tag{5.28}$$

is satisfied[3]*, where $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_N\}$, then*

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma} \in \Gamma(\tilde{x}, 0)} \int_0^1 \tilde{\gamma}'(s)^T G(\tilde{\gamma}(s)) \tilde{\gamma}'(s) ds, \tag{5.29}$$

is a manifold simulation function from the interconnected control system $\hat{\mathfrak{S}} = \mathcal{I}(\hat{\mathfrak{S}}_1, \ldots, \hat{\mathfrak{S}}_N)$ to $\mathfrak{S}$, where

$$G(\tilde{x}) = \begin{bmatrix} \mu_1 G_1(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \mu_2 G_2(\tilde{x}_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \mu_N G_N(\tilde{x}_N) \end{bmatrix}, \tag{5.30}$$

$\tilde{x} = [\tilde{x}_1; \ldots; \tilde{x}_N]$, and $\tilde{x}_i = [x_i; \hat{x}_i] \in \mathcal{M}_{n_i} \times \mathcal{M}_{\hat{n}_i}$ $\forall i \in [1; N]$.

Proof. For any $x_i \in \mathcal{M}_{n_i}$, and $\hat{x}_i \in \mathcal{M}_{\hat{n}_i}$, $i \in [1; N]$, define

$$\tilde{h}(\tilde{x}) := \begin{bmatrix} \tilde{h}_1(\tilde{x}_1) \\ \vdots \\ \tilde{h}_N(\tilde{x}_N) \end{bmatrix} := \begin{bmatrix} h_1(x_1) - \hat{h}_1(\hat{x}_1) \\ \vdots \\ h_N(x_N) - \hat{h}_N(\hat{x}_N) \end{bmatrix}, \tag{5.31}$$

where $\tilde{x} = [\tilde{x}_1; \ldots; \tilde{x}_N]$, and $\tilde{x}_i = [x_i; \hat{x}_i]$, $\forall i \in [1; N]$. One has:

$$\frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x}) = \begin{bmatrix} \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \frac{\partial \tilde{h}_2}{\partial \tilde{x}_2}(\tilde{x}_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \frac{\partial \tilde{h}_N}{\partial \tilde{x}_N}(\tilde{x}_N) \end{bmatrix}, \tag{5.32}$$

[3] We interpret the inequality component-wise i.e. for $x \in \mathbb{R}^N$ we have $x < 0$ iff every entry $x_i < 0, i \in \{1, \ldots, N\}$

and

$$\begin{aligned}
&\underline{\alpha}\left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^T \frac{\partial \tilde{h}}{\partial \tilde{x}} \\
&\preceq \operatorname{diag}\left\{\alpha_1\left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}\right)^T \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}, \ldots, \alpha_N\left(\frac{\partial \tilde{h}_N}{\partial \tilde{x}_N}\right)^T \frac{\partial \tilde{h}_N}{\partial \tilde{x}_N}\right\} \\
&\preceq \begin{bmatrix} \mu_1 G_1(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \mu_2 G_2(\tilde{x}_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \mu_N G_N(\tilde{x}_N) \end{bmatrix} = G(\tilde{x}),
\end{aligned} \tag{5.33}$$

where $\underline{\alpha} = \min\{\alpha_1, \ldots, \alpha_N\}$. Thus, the condition (5.9) is satisfied with $\alpha = \underline{\alpha}$. Now we prove inequality (5.10). For all $i \in [1; N]$, consider any $\tilde{x}_i = [x_i; \hat{x}_i] \in \mathcal{M}_{n_i} \times \mathcal{M}_{\hat{n}_i}$, $\delta x_i = [\delta x_i; \delta \hat{x}_i] \in \mathcal{T}_{x_i}\mathcal{M}_{n_i} \times \mathcal{T}_{\hat{x}_i}\mathcal{M}_{\hat{n}_i}$, $\hat{u}_i \in \mathbb{R}^{\hat{m}_i}$, and any $\delta \hat{u}_i \in \mathbb{R}^{\hat{m}_i}$. Under the map $u_i = k_i(\tilde{x}_i, \hat{u}_i, \hat{w}_i)$, (5.10) is satisfied for each pair of subsystems $\mathfrak{S}_i$ and $\hat{\mathfrak{S}}_i$, with the internal inputs given by $w_{ij} = y_{ji} = h_{ji}(x_j)$, and $\hat{w}_{ij} = \hat{y}_{ji} = \hat{h}_{ji}(\hat{x}_j)$. The corresponding differential internal inputs are given $\delta w_{ij} = \delta y_{ji} = \frac{\partial h_{ji}}{\partial x_j}\delta x_j$, and $\delta \hat{w}_{ij} = \delta \hat{y}_{ji} = \frac{\partial \hat{h}_{ji}}{\partial \hat{x}_j}\delta \hat{x}_j$. We consider the time derivative of the function $\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T G(\tilde{x})\delta\tilde{x}$ along the solution trajectory and employ the conditions (5.28) which results in the chain of inequalities (5.34), where we use the triangle inequality and the following inequality [Kel14]

$$\psi_{i\mathrm{int}}(r_1 + \cdots + r_{N-1}) \le \sum_{i=1}^{N-1} \psi_{i\mathrm{int}}((N-1)r_i). \tag{5.35}$$

We define the vector $\delta\hat{u} = [\delta\hat{u}_1; \ldots; \delta\hat{u}_N]$, and the function

$$\psi_{\mathrm{ext}}(s) := \begin{cases} \max\limits_{\vec{s} \ge 0} & \sum_{i=1}^N \mu_i \psi_{i\mathrm{ext}}(s_i) \\ \text{s.t.} & \|\vec{s}\| = s. \end{cases},$$

where $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$, $\psi_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$. Therefore, one has:

$$\dot{S} \le -\lambda \delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \psi_{\mathrm{ext}}(\|\delta\hat{u}\|), \tag{5.36}$$

where λ is the minimum element of the vector

$$-\mu^T(-\Lambda + \Delta),$$

which satisfies inequality (5.10) with $\psi_{\mathrm{int}} \equiv 0$. Hence we conclude that

$$V(x, \hat{x}) = \inf_{\tilde{\gamma} \in \Gamma(\tilde{x}, 0)} \int_0^1 \tilde{\gamma}'(s)^T G(\tilde{\gamma}(s))\tilde{\gamma}'(s) ds, \tag{5.37}$$

is a manifold simulation function from $\hat{\mathfrak{S}}$ to $\mathfrak{S}$. $\square$

$$
\begin{aligned}
\dot{S} &= \frac{d}{dt}\mu_i \sum_{i=1}^{N} \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i \le \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) + \psi_{\mathrm{iint}}(\|\delta w_i - \delta\hat{w}_i\|)\right) \\
&\le \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \psi_{\mathrm{iint}}\Big((N-1)\|\delta w_{ij} - \delta\hat{w}_{ij}\|\Big)\right) + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&= \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \psi_{\mathrm{iint}}\Big((N-1)\|\delta y_{ji} - \delta\hat{y}_{ji}\|\Big)\right) + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&= \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \psi_{\mathrm{iint}}\left((N-1)\left\|\frac{\partial h_j}{\partial x_j}\delta x_j - \frac{\partial \hat{h}_j}{\partial \hat{x}_j}\delta\hat{x}_j\right\|\right)\right) \\
&\quad + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&\le \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \psi_{\mathrm{iint}}\left((N-1)\left\|\frac{\partial \tilde{h}_j}{\partial \tilde{x}_j}\delta\tilde{x}_j\right\|\right)\right) + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&\le \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \psi_{\mathrm{iint}}\left(\frac{1}{\sqrt{\alpha_j}}(N-1)\sqrt{\delta\tilde{x}_j^T G_j(\tilde{x}_j)\delta\tilde{x}_j}\right)\right) \\
&\quad + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&\le \sum_{i=1}^{N} \mu_i \left(-\lambda_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{j=1, j\neq i}^{N} \delta_{ij}\delta\tilde{x}_j^T G_j(\tilde{x}_j)\delta\tilde{x}_j\right) + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \\
&= \mu^T(-\Lambda + \Delta)[\delta\tilde{x}_1^T G(\tilde{x}_1)\delta\tilde{x}_1; \ldots; \delta\tilde{x}_N^T G(\tilde{x}_N)\delta\tilde{x}_N] + \sum_{i=1}^{N} \mu_i \psi_{\mathrm{iext}}(\|\delta\hat{u}_i\|) \qquad (5.34)
\end{aligned}
$$

5.5 Compositional Abstraction: Dissipativity Approach

5.5.1 Differential Storage Function

In this section, we introduce a notion of so-called differential storage functions, adapted from the notion of differential storage function introduced in [FS13] in the context of differential dissipativity.

Definition 5.5.1. *Consider two control subsystems*

$$\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$$

and

$$\hat{\mathfrak{S}} = (\mathcal{M}_{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$$

and the corresponding augmented system

$$\tilde{\mathfrak{S}} = (\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \tilde{h}_1, \tilde{h}_2)$$

as in Definition 5.2.3. Let

$$\delta\tilde{\mathfrak{S}} = (\mathcal{T}\mathcal{M}_{\tilde{n}}, \mathbb{R}^{\tilde{m}}, \mathbb{R}^{\tilde{p}}, \tilde{\mathcal{U}}, \tilde{\mathcal{W}}, \delta\tilde{f}, \mathbb{R}^{q_1}, \mathbb{R}^{\tilde{q}_2}, \delta\tilde{h}_1, \delta\tilde{h}_2)$$

be the variational control system of $\tilde{\mathfrak{S}}$ as defined in Definition 5.2.1. Suppose there exists some positive constants α and λ, a matrix valued function $G : \mathcal{M}_{\tilde{n}} \to \mathbb{R}^{\tilde{n}\times\tilde{n}}$, such that $G(\tilde{x})$ is a positive (semi) definite matrix for all $\tilde{x} \in \mathcal{M}_{\tilde{n}}$, some matrices W, $\hat{W}, X^{ij}, i, j \in [1;2]$, of appropriate dimensions, a function $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$ and a continuously differentiable function $k : \mathcal{M}_{\tilde{n}} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^m$ which satisfies $k(0,0) = 0$, such that the following two conditions hold:

- *For any $\tilde{x} \in \mathcal{M}_{\tilde{n}}$:*

$$G(\tilde{x}) \succeq \alpha \left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}}\right)^T \left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}}\right). \tag{5.38}$$

- *For any $[\tilde{x}; \delta\tilde{x}] \in \mathcal{T}\mathcal{M}_{\tilde{n}}$, $\hat{u} \in \mathbb{R}^{\hat{m}}$, and $\delta\hat{u} \in \mathbb{R}^{\hat{m}}$, if we choose u using the map $u = k(\tilde{x}, \hat{u})$, then for any $\tilde{w} \in \mathbb{R}^{\tilde{p}}$, and any $\delta\tilde{w} \in \mathbb{R}^{\tilde{p}}$:*

$$\begin{aligned}
&\delta\tilde{x}^T \left(\frac{\partial \tilde{f}}{\partial \tilde{x}}^T G(\tilde{x}) + G(\tilde{x})\frac{\partial \tilde{f}}{\partial \tilde{x}} + \frac{\partial G}{\partial \tilde{x}}\tilde{f}(\tilde{x}, \tilde{u}, \tilde{w})\right)\delta\tilde{x} \\
&\quad + 2\delta\tilde{w}^T \frac{\partial \tilde{f}}{\partial \tilde{w}}^T G(\tilde{x})\delta\tilde{x} + 2\delta\tilde{u}^T \frac{\partial \tilde{f}}{\partial \tilde{u}}^T G(\tilde{x})\delta\tilde{x} \\
&\leq -\lambda\delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \begin{bmatrix} W\delta w - \hat{W}\delta\hat{w} \\ \delta y_2 - H\delta\hat{y}_2 \end{bmatrix}^T \begin{bmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{bmatrix} \begin{bmatrix} W\delta w - \hat{W}\delta\hat{w} \\ \delta y_2 - H\delta\hat{y}_2 \end{bmatrix} \\
&\quad + \psi_{\text{ext}}(\|\delta\hat{u}\|),
\end{aligned} \tag{5.39}$$

  where $\delta y_2 = \frac{\partial h_2(x)}{\partial x}\delta x$, $\delta\hat{y}_2 = \frac{\partial \hat{h}_2(\hat{x})}{\partial \hat{x}}\delta\hat{x}$, $\delta\tilde{u} = [\delta u; \delta\hat{u}]$, and $\delta u = \frac{\partial k}{\partial \tilde{x}}\delta\tilde{x} + \frac{\partial k}{\partial \hat{u}}\delta\hat{u}$.

Then $\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T G(\tilde{x})\delta\tilde{x}$ is a differential storage function from $\hat{\mathfrak{S}}$ to $\mathfrak{S}$. We call $\hat{\mathfrak{S}}$ (preferably with $\hat{n} < n$) an abstraction of $\mathfrak{S}$ if there exists a differential storage function from $\hat{\mathfrak{S}}$ to $\mathfrak{S}$.

Remark 5.5.2. *For linear subsystems, one can use the differential storage function given by*

$$\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T \begin{bmatrix} \widehat{M} & -\widehat{M}P \\ -P^T\widehat{M} & P^T\widehat{M}P \end{bmatrix} \delta\tilde{x},$$

where $\widehat{M} \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and $P \in \mathbb{R}^{n\times\hat{n}}$, satisfying the conditions given in [ZA17] together with the associated linear interface map, for the construction of abstractions of subsystems.

5.5.2 Interconnected Systems

Here we define the interconnected system consisting of control subsystems interconnected via a constant interconnection topology.

Definition 5.5.3. *Consider $N \in \mathbb{N}_{\geq 1}$ control subsystems*

$$\mathfrak{S}_i = (\mathcal{M}_{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1; N]$, and an interconnection matrix M of appropriate dimension defining the coupling of these subsystems. The interconnected control system $\mathfrak{S} = (\mathcal{M}_n, \mathbb{R}^m, \mathcal{U}, f, \mathbb{R}^q, h)$ denoted by $\mathcal{I}(\mathfrak{S}_1, \ldots, \mathfrak{S}_N)$, follows by $\mathcal{M}_n = \Pi_{i=1}^N \mathcal{M}_{n_i}$, $m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{1i}$, and the functions

$$f(x, u) = \left[f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)\right], \tag{5.40}$$

$$h(x) = [h_{11}(x); \ldots; h_{1N}(x_N)], \tag{5.41}$$

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$, and with the internal inputs constrained by

$$[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].$$

5.5.3 Compositionality Result

In the next theorem, we derive sufficient conditions under which an interconnection of abstractions of control subsystems, interconnected via another (possibly simpler) interconnection topology, is an abstraction of the original interconnected system.

Theorem 5.5.4. *Consider the interconnected control system $\mathfrak{S} = \mathcal{I}(\mathfrak{S}_1, \ldots, \mathfrak{S}_N)$, induced by N control subsystems and a coupling matrix M. Suppose each subsystem $\mathfrak{S}_i$ admits an abstraction $\hat{\mathfrak{S}}_i$ with a corresponding differential storage function $\mathcal{S}_i$. If there exists $\mu_i \geq 1$ and the matrix $\hat{M}$ such that the following matrix (in)equalities hold:*

$$\begin{bmatrix} WM \\ I_{\bar{q}} \end{bmatrix}^T X(\mu_1 X_1, \ldots, \mu_N X_N) \begin{bmatrix} WM \\ I_{\bar{q}} \end{bmatrix} \preceq 0, \tag{5.42}$$

$$WMH = \hat{W}\hat{M}, \tag{5.43}$$

where $\bar{q} = \sum_{i=1}^N q_{2i}$ and

$$\begin{aligned} &W = \mathsf{diag}(W_1, \ldots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \ldots, \hat{W}_N), \\ &H = \mathsf{diag}(H_1, \ldots, H_N), \end{aligned} \tag{5.44}$$

$$\begin{aligned} &X(\mu_1 X_1, \ldots, \mu_N X_N) = \\ &\begin{bmatrix} \mu_1 X_1^{11} & & & \mu_1 X_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{11} & & & \mu_N X_N^{12} \\ \mu_1 X_1^{21} & & & \mu_1 X_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N X_N^{21} & & & \mu_N X_N^{22} \end{bmatrix}, \end{aligned} \tag{5.45}$$

then

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma}\in\Gamma(\tilde{x},0)} \int_0^1 \frac{\partial}{\partial s}\tilde{\gamma}(s)^T G(\tilde{\gamma}(s)) \frac{\partial}{\partial s}\tilde{\gamma}(s) ds, \tag{5.46}$$

is a manifold simulation function from the interconnected control system $\hat{\mathfrak{S}} = \mathcal{I}(\hat{\mathfrak{S}}_1, \ldots, \hat{\mathfrak{S}}_N)$ *with coupling matrix* $\hat{M}$ *to* $\mathfrak{S}$*, where*

$$G(\tilde{x}) = \begin{bmatrix} \mu_1 G_1(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \mu_2 G_2(\tilde{x}_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \mu_N G_N(\tilde{x}_N) \end{bmatrix}, \tag{5.47}$$

$\tilde{x} = [\tilde{x}_1; \ldots; \tilde{x}_N]$*, and* $\tilde{x}_i = [x_i; \hat{x}_i] \in \mathcal{M}_{n_i} \times \mathcal{M}_{\hat{n}_i}$ $\forall i \in [1; N]$.

Proof. For any $x_i \in \mathcal{M}_{n_i}$, and $\hat{x}_i \in \mathcal{M}_{\hat{n}_i}$, $i \in [1; N]$, define

$$\tilde{h}(\tilde{x}) := \begin{bmatrix} \tilde{h}_1(\tilde{x}_1) \\ \vdots \\ \tilde{h}_N(\tilde{x}_N) \end{bmatrix} := \begin{bmatrix} h_{11}(x_1) - \hat{h}_{11}(\hat{x}_1) \\ \vdots \\ h_{1N}(x_N) - \hat{h}_{1N}(\hat{x}_N) \end{bmatrix}, \tag{5.48}$$

where $\tilde{x} = [\tilde{x}_1; \ldots; \tilde{x}_N]$, and $\tilde{x}_i = [x_i; \hat{x}_i]$ $\forall i \in [1; N]$. One has:

$$\frac{\partial \tilde{h}}{\partial \tilde{x}}(\tilde{x}) = \begin{bmatrix} \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \frac{\partial \tilde{h}_2}{\partial \tilde{x}_2}(\tilde{x}_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \frac{\partial \tilde{h}_N}{\partial \tilde{x}_N}(\tilde{x}_N) \end{bmatrix}, \tag{5.49}$$

and

$$\begin{aligned} &\underline{\alpha} \left(\frac{\partial \tilde{h}}{\partial \tilde{x}}\right)^T \frac{\partial \tilde{h}}{\partial \tilde{x}} \\ &\preceq \mathsf{diag}\left\{ \alpha_1 \left(\frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}\right)^T \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}, \ldots, \alpha_N \left(\frac{\partial \tilde{h}_N}{\partial \tilde{x}_N}\right)^T \frac{\partial \tilde{h}_N}{\partial \tilde{x}_N} \right\} \\ &\preceq \begin{bmatrix} \mu_1 G_1(\tilde{x}_1) & 0 & \ldots & 0 \\ 0 & \mu_2 G_2(\tilde{x}_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \mu_N G_N(\tilde{x}_N) \end{bmatrix} = G(\tilde{x}), \end{aligned} \tag{5.50}$$

where $\underline{\alpha} = \min\{\alpha_1, \ldots, \alpha_N\}$. Thus, the condition (5.9) is satisfied with $\alpha = \underline{\alpha}$. Now we prove inequality (5.10). For all $i \in [1; N]$, consider any $\tilde{x}_i = [x_i; \hat{x}_i] \in \mathcal{M}_{n_i} \times \mathcal{M}_{\hat{n}_i}$, any $\delta x_i = [\delta x_i; \delta \hat{x}_i] \in \mathcal{T}_{x_i}\mathcal{M}_{n_i} \times \mathcal{T}_{\hat{x}_i}\mathcal{M}_{\hat{n}_i}$, any $\hat{u}_i \in \mathbb{R}^{\hat{m}_i}$, and any $\delta \hat{u}_i \in \mathbb{R}^{\hat{m}_i}$. Under the map $u_i = k_i(\tilde{x}_i, \hat{u}_i)$, (5.39) is satisfied for each pair of subsystems $\mathfrak{S}_i$ and $\hat{\mathfrak{S}}_i$, with the internal inputs given by $[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)]$, and $[\hat{w}_1; \ldots; \hat{w}_N] = \hat{M}[\hat{h}_{21}(\hat{x}_1); \ldots; \hat{h}_{2N}(\hat{x}_N)]$. The corresponding differential internal inputs are given by

$$[\delta w_1; \ldots; \delta w_N] = M\left[\frac{\partial h_{21}}{\partial x_1}\delta x_1; \ldots; \frac{\partial h_{2N}}{\partial x_N}\delta x_N\right],$$

and

$$[\delta \hat{w}_1; \ldots; \delta \hat{w}_N] = \hat{M}\left[\frac{\partial \hat{h}_{21}}{\partial \hat{x}_1}\delta \hat{x}_1; \ldots; \frac{\partial \hat{h}_{2N}}{\partial \hat{x}_N}\delta \hat{x}_N\right].$$

We consider the time derivative of the function $\mathcal{S}(\tilde{x}, \delta\tilde{x}) = \delta\tilde{x}^T G(\tilde{x})\delta\tilde{x}$, where $\tilde{x} = [\tilde{x}_1; \ldots; \tilde{x}_N]$ and $\delta\tilde{x} = [\delta\tilde{x}_1; \ldots; \delta\tilde{x}_N]$, along the solution trajectory and employ conditions (4.25) and (4.26) which results in the chain of inequalities (5.55), where $\lambda = \min\{\lambda_1, \ldots, \lambda_N\}$, $\delta y_{2i} = \frac{\partial h_{2i}}{\partial x_i}\delta x_i$, and $\delta \hat{y}_{2i} = \frac{\partial \hat{h}_{2i}}{\partial \hat{x}_i}\delta \hat{x}_i$, $\forall i \in [1; N]$. Using (5.55), and by defining the vector $\delta\hat{u} = [\delta\hat{u}_1; \ldots; \delta\hat{u}_N]$, and the function

$$\psi_{\text{ext}}(s) := \begin{cases} \max\limits_{\vec{s}\geq 0} & \sum_{i=1}^{N} \mu_i \psi_{i\text{ext}}(s_i) \\ \text{s.t.} & \|\vec{s}\| = s \end{cases},$$

where $\vec{s} = [s_1; \ldots; s_N]$ and $\psi_{\text{ext}} \in \mathcal{K}_\infty \cup \{0\}$, we arrive at the following inequality:

$$\dot{S} \leq -\lambda\delta\tilde{x}^T G(\tilde{x})\delta\tilde{x} + \psi_{\text{ext}}(\|\delta\hat{u}\|), \tag{5.51}$$

which satisfies inequality (5.10). Hence we conclude that

$$V_G(\tilde{x}) = \inf_{\tilde{\gamma}\in\Gamma(\tilde{x},0)} \int_0^1 \tilde{\gamma}'(s)^T G(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds, \tag{5.52}$$

is a manifold simulation function from $\hat{\mathfrak{S}}$ to $\mathfrak{S}$, with respect to G. □

In the next section, we provide an example to illustrate the effectiveness of the proposed differential dissipativity-type compositional reasoning for interconnected control systems over Riemannian manifolds.

5.6 Example

Consider an interconnection of $N \in \mathbb{N}$ subsystems $\mathfrak{S}_i$, $i \in [1; N]$, where each $\mathfrak{S}_i$ is given by $\mathfrak{S}_i = (\mathbb{S}^{n_i}, \mathbb{R}^{n_i}, \mathbb{R}^{n_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \mathbb{R}, \mathbb{R}^{n_i}, h_{1i}, h_{2i})$, where for each $\theta_i = [\theta_{i1}; \ldots; \theta_{in_i}] \in \mathbb{S}^{n_i}, u_i \in \mathbb{R}^{n_i}, w_i \in \mathbb{R}^{n_i}$:

$$\begin{aligned} f_i(\theta_i, u_i, w_i) &:= \frac{1}{n_i}\begin{bmatrix} \sum_{k=1}^{n_i} \sin(\theta_{ik} - \theta_{i1}) \\ \vdots \\ \sum_{k=1}^{n_i} \sin(\theta_{ik} - \theta_{in_i}) \end{bmatrix} + w_i + u_i, \\ h_{1i}(\theta_i) &:= \theta_{i1} \\ h_{2i}(\theta_i) &:= \theta_i. \end{aligned} \tag{5.53}$$

$$
\begin{aligned}
\dot{S} &= \frac{d}{dt}\sum_{i=1}^{N} \mu_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i \\
&\le \sum_{i=1}^{N} -\lambda_i\mu_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{i=1}^{N}\mu_i\psi_{i\mathrm{ext}}(\|\delta\hat{u}_i\|) \\
&\quad + \begin{bmatrix} W\begin{bmatrix}\delta w_1\\ \vdots\\ \delta w_N\end{bmatrix} - \hat{W}\begin{bmatrix}\delta\hat{w}_1\\ \vdots\\ \delta\hat{w}_N\end{bmatrix} \\ \delta y_{21} - H_1\delta\hat{y}_{21} \\ \vdots \\ \delta y_{2N} - H_N\delta\hat{y}_{2N}\end{bmatrix}^T X(\mu_1X_1,\dots,\mu_NX_N)\begin{bmatrix} W\begin{bmatrix}\delta w_1\\ \vdots\\ \delta w_N\end{bmatrix} - \hat{W}\begin{bmatrix}\delta\hat{w}_1\\ \vdots\\ \delta\hat{w}_N\end{bmatrix} \\ \delta y_{21} - H_1\delta\hat{y}_{21} \\ \vdots \\ \delta y_{2N} - H_N\delta\hat{y}_{2N}\end{bmatrix} \qquad (5.54) \\
&\le -\lambda\sum_{i=1}^{N} \mu_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{i=1}^{N}\mu_i\psi_{i\mathrm{ext}}(\|\delta\hat{u}_i\|) \\
&\quad + \begin{bmatrix}\delta y_{21} - H_1\delta\hat{y}_{21} \\ \vdots \\ \delta y_{2N} - H_N\delta\hat{y}_{2N}\end{bmatrix}^T \begin{bmatrix} WM \\ I_{\bar{q}}\end{bmatrix}^T X(\mu_1X_1,\dots,\mu_NX_N)\begin{bmatrix} WM \\ I_{\bar{q}}\end{bmatrix}\begin{bmatrix}\delta y_{21} - H_1\delta\hat{y}_{21} \\ \vdots \\ \delta y_{2N} - H_N\delta\hat{y}_{2N}\end{bmatrix} \\
&\le -\lambda\sum_{i=1}^{N} \mu_i \delta\tilde{x}_i^T G_i(\tilde{x}_i)\delta\tilde{x}_i + \sum_{i=1}^{N}\mu_i\psi_{i\mathrm{ext}}(\|\delta\hat{u}_i\|) \qquad (5.55)
\end{aligned}
$$

$$
\begin{aligned}
&\delta f_i(\theta_i,\delta\theta_i,u_i,\delta u_i,w_i,\delta w_i) \\
&\quad := \begin{bmatrix} -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{i1}) & \frac{1}{n_i}\cos(\theta_{i2}-\theta_{i1}) & \dots & \frac{1}{n_i}\cos(\theta_{in_i}-\theta_{i1}) \\ \frac{1}{n_i}\cos(\theta_{i1}-\theta_{i2}) & -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{i2}) & \dots & \\ \vdots & \ddots & & \vdots \\ \frac{1}{n_i}\cos(\theta_{i1}-\theta_{in_i}) & \dots & & -\frac{1}{n_i}\Sigma_{k=1}^{n_i}\cos(\theta_{ik}-\theta_{in_i})\end{bmatrix}\begin{bmatrix}\delta\theta_{i1}\\ \vdots \\ \delta\theta_{in_i}\end{bmatrix} \\
&\qquad + \delta w_i + \delta u_i \\
&\delta h_{1i}(\theta_i,\delta\theta_i) := \delta\theta_{i1} \\
&\delta h_{2i}(\theta_i,\delta\theta_i) := \delta\theta_i \qquad (5.56)
\end{aligned}
$$

The variational control system of $\mathfrak{S}_i$ is given by the tuple

$$\delta\mathfrak{S}_i = (\mathbb{S}^{n_i}\times\mathbb{R}^{n_i}, \mathbb{R}^{n_i}, \mathbb{R}^{n_i}, \mathcal{U}_i, \mathcal{W}_i, \delta f_i, \mathbb{R}, \mathbb{R}^{n_i}, \delta h_{1i}, \delta h_{2i}),$$

where for each $[\theta_i;\delta\theta_i] \in \mathbb{S}^{n_i}\times\mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{n_i}, \delta u_i \in \mathbb{R}^{n_i}$, $w_i \in \mathbb{R}^{n_i}$, and $\delta w_i \in \mathbb{R}^{n_i}$, $\delta f_i, \delta h_{1i}$, and δh_{2i} are defined in (5.56). We assume that the interconnection topology is given by

$$M = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}. \tag{5.57}$$

For each $\mathfrak{S}_i$, we consider that the abstract subsystems are given by the tuple $\hat{\mathfrak{S}}_i = (\mathbb{S}, \mathbb{R}, \mathbb{R}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \mathbb{R}, \mathbb{R}, 1, 1)$, where for each $\hat{\theta}_i \in \mathbb{S}$, $\hat{u}_i \in \mathbb{R}$, and $\hat{w}_i \in \mathbb{R}$:

$$\hat{f}_i(\hat{\theta}_i, \hat{u}_i, \hat{w}_i) = -\frac{1}{n_i} \sin(\hat{\theta}_i) + \hat{w}_i + \hat{u}_i. \tag{5.58}$$

The variational control system of $\hat{\mathfrak{S}}_i$ is given by $\delta\hat{\mathfrak{S}}_i = (\mathbb{S}\times\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \delta\hat{f}_i, \mathbb{R}, \mathbb{R}, \delta\hat{h}_{1i}, \delta\hat{h}_{2i})$, where for each $[\hat{\theta}_i; \delta\hat{\theta}_i] \in \mathbb{S} \times \mathbb{R}$, $\hat{u}_i \in \mathbb{R}, \delta\hat{u}_i \in \mathbb{R}$, $\hat{w}_i \in \mathbb{R}$, and $\delta\hat{w}_i \in \mathbb{R}$, $\delta\hat{f}_i$, $\delta\hat{h}_{1i}$, and $\delta\hat{h}_{2i}$ are given by:

$$\begin{aligned} \delta\hat{f}_i(\hat{\theta}_i, \delta\hat{\theta}_i, \hat{u}_i, \delta\hat{u}_i, \hat{w}_i, \delta\hat{w}_i) &:= -\frac{1}{n_i}\cos(\hat{\theta}_i)\delta\hat{\theta}_i + \delta\hat{w}_i + \delta\hat{u}_i \\ \delta\hat{h}_{1i}(\hat{\theta}_i, \delta\hat{\theta}_i) &:= \delta\hat{\theta}_i \\ \delta\hat{h}_{2i}(\hat{\theta}_i, \delta\hat{\theta}_i) &:= \delta\hat{\theta}_i. \end{aligned} \tag{5.59}$$

Consider the following differential storage function with constant pseudo Riemannian structure:

$$\mathcal{S}_i(\delta\theta_i, \delta\hat{\theta}_i) = \begin{bmatrix} \delta\theta_{i1} & \dots & \delta\theta_{in_i} & \delta\hat{\theta}_i \end{bmatrix} G_i \begin{bmatrix} \delta\theta_{i1} \\ \vdots \\ \delta\theta_{in_i} \\ \delta\hat{\theta}_i \end{bmatrix}, \tag{5.60}$$

where

$$G_i = \begin{bmatrix} 1 & 0 & \dots & -1 \\ 0 & 1 & \dots & -1 \\ \vdots & \ddots & & \\ -1 & -1 & \dots & n_i \end{bmatrix}. \tag{5.61}$$

For each $i \in [1; N]$, we choose $u_i = [u_{i1}; \dots; u_{in_i}] \in \mathbb{R}^{n_i}$ according to the following interface map:

$$\begin{aligned} u_{ij} = &-\frac{1}{n_i}\Sigma_{k=1}^{n_i} \sin(\theta_{ik} - \theta_{ij}) - \frac{1}{2n_i}\theta_{ij} - \frac{1}{n_i}\sin(\hat{\theta}_i) \\ &+ \frac{1}{2n_i}\hat{\theta}_i + \hat{u}_i, \end{aligned} \tag{5.62}$$

where u_{ij} represents the j-th element of the vector u_i, and θ_{ij} represents the j-th element of the vector θ_i, $j = [1; n_i]$. It can be shown that $\mathcal{S}_i$ is a differential storage function from $\hat{\mathfrak{S}}_i$ to $\mathfrak{S}_i$ with the following parameters

$$\begin{aligned} &W_i = I_{n_i}, \hat{W}_i = \vec{1}_{n_i}, H_i = \vec{1}_{n_i}, X_i^{11} = X_i^{22} = 0_{n_i}, \\ &X_i^{12} = X_i^{21} = I_{n_i}, \alpha_i = 1, \lambda_i = \frac{1}{n_i}, \psi_{i\text{ext}} = 0, \end{aligned} \tag{5.63}$$

where 0 represents the zero function. By selecting $\mu_1 = \cdots = \mu_N = 1$, and $\hat{M}$ appropriately, it can be shown that (5.42) and (5.43) are satisfied and therefore one can conclude that

$$V_G(\tilde{\theta}) = \inf_{\tilde{\gamma} \in \Gamma(\tilde{\theta}, 0)} \int_0^1 \frac{\partial}{\partial s} \tilde{\gamma}(s)^T G \frac{\partial}{\partial s} \tilde{\gamma}(s) ds, \tag{5.64}$$

where $\tilde{\theta} = [\theta_1; \hat{\theta}_1; \ldots; \theta_N, \hat{\theta}_N], \theta_i \in \mathbb{S}^{n_i}, \hat{\theta}_i \in \mathbb{S}, \forall i = [1; N]$, and

$$G = \mathsf{diag}(G_1, \ldots, G_N), \tag{5.65}$$

is a simulation function, with respect to G, from $\mathcal{I}(\hat{\mathfrak{S}}_1, \ldots \hat{\mathfrak{S}}_N)$ to $\mathcal{I}(\mathfrak{S}_1, \ldots, \mathfrak{S}_N)$, with the interconnection matrix for $\hat{\mathfrak{S}}$ given by $\hat{M}$. For example, for $N = 3, n_i = 50, \forall i = [1; N]$, (i.e. $M \in \mathbb{R}^{150 \times 150}$), one can choose

$$\hat{M} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}. \tag{5.66}$$

5.7 Summary

In this chapter, using tools from differential geometry, we derived sufficient compositional conditions under which abstractions of interconnected systems evolving on smooth Riemannian manifolds can be constructed. We used two different approaches. In the first approach, we derived sufficient conditions based on small-gain type reasoning under which abstractions of interconnected systems evolving on smooth Riemannian manifolds can be constructed compositionally. In the second approach, we used notions from dissipativity theory to derive conditions for compositional abstraction of interconnected systems over smooth Riemannian manifolds. One advantage of dissipativity approach over the small-gain one is that the interconnection topology of the concrete and abstract networks need not be the same. In the small-gain formulation, the topology of both concrete and abstract networks have to be the same. This provides an additional degree of freedom for compositional construction of abstraction for such networks using the dissipativity approach.

Chapter 6

Conclusion & Outlook

6.1 Conclusion

Constructing reduced-order models (i.e., infinite abstractions) for a complex system, when considered as a monolithic entity, poses a challenging task. In this dissertation, we addressed this challenge by capitalizing on the fact that many large-scale complex systems could be seen as interconnected systems comprised of smaller subsystems. This motivated a compositional approach for constructing abstractions, where abstractions of the concrete systems could be created by utilizing the abstractions of smaller subsystems.

Chapter 3 of the dissertation focused on establishing conditions for constructing compositional abstractions of networks of stochastic hybrid systems. The framework involved using the interconnection topology and joint dissipativity-type properties of subsystems and their abstractions. The resulting abstraction, potentially with a lower dimension, can substitute the original system in controller synthesis.

In Chapter 4, the dissertation addressed scenarios where the interconnection topology in networks is not fixed. This was relevant in realistic situations where the interconnection topology might change due to factors like communication loss between robot agents or failure of switching lines in an electric distribution grid.

The dissertation extended its scope to systems evolving over Riemannian manifolds in Chapter 5. Recognizing that the state-space of many systems exhibited such geometry, the analysis incorporated tools from differential geometry. Two approaches were proposed for the compositional construction of infinite abstractions for interconnected control systems on smooth Riemannian manifolds. The first approach introduced a small-gain type condition facilitating the compositional construction of abstractions for interconnected control systems. The second approach utilized dissipativity theory to derive conditions for constructing compositional abstractions based on the interconnection topology and joint differential dissipativity-type properties of subsystems and their respective abstractions.

6.2 Recommendations for Future Research

In this section, we discuss some topics that could be considered as potential future research directions:

- **Model reduction for *unknown* nonlinear control systems**. All techniques for constructing reduced-order models (i.e., infinite abstractions) in this dissertation require a mathematical model of the concrete subsystem derived from physics-based first principles, which may not be available in practice. One potential future direction is to develop schemes for constructing abstraction of nonlinear control systems with unknown dynamics. In this regard, in a preliminary work in [AZ], we introduced a technique to construct such abstractions for a class of discrete-time control systems with *partially unknown* dynamics. This approach consists of two main ingredients, namely, active subspace identification [CDW14; Con15] and Gaussian Process (GP) regression [WR06]. This technique consists of using data sampled from the concrete system to find a lower dimensional subspace of its state space (which we call the active subspace), and constructing an abstraction candidate using GP regression. We derive sufficient conditions under which the GP candidate is shown to be the abstraction of the original system while quantifying the error bound between the output of the abstraction and that of the concrete system. Extending this preliminary result to a broader class of non-linear systems, stochastic systems and fully unknown systems are also potential directions for future research.

- **Compositional abstraction of networks of cyber-physical systems with unknown interconnection topology**. The techniques presented in this dissertation for compositional abstraction of networks of cyber-physical systems assume full knowledge of the interconnection topology of the network. In many real world scenarios, even if the dynamics of the subsystems of the network is known, the interconnection topology may not be fully known. One potential research direction is to develop techniques for compositional abstraction of interconnected systems with (partially) unknown topologies. In this regard, we propose two potential approaches. In the first approach, one may investigate some recent techniques developed in [DG20; Bla+17b; Bla+17a] in the context of *topology-independent* robust stability of networked systems, and adapt them for deriving topology-independent conditions for constructing abstractions in a compositional manner. Specifically, one may investigate how the compositionality conditions in (3.15) and (3.16) can be replaced by ones that are topology-independent. In the second potential approach, one may utilize machine learning techniques to *learn/identify* the unknown interconnection topology. In this regard, one may utilize data samples generated from the interconnected system to learn the interconnection topology, similar to techniques presented in [Pu+21; Fra+19; CWZ20]. The compositionality conditions in (3.15) and (3.16) will need to be modified to accommodate for the out-of-sample generalization error between the learned topology and the actual one.

- **Relaxed geometric conditions for construction of abstractions of stochastic hybrid systems.** Current approaches to constructing abstractions for stochastic hybrid systems rely on geometric conditions (as in (3.4.9)-(3.4.11)) or, in the case of an interconnected stochastic hybrid system, a condition on the interconnection topology (as in (3.15)-(3.16)). Since these conditions are not always satisfiable, one potential research direction is to relax these restrictions on the choice of abstractions, by opting to select ones which *nearly satisfy* such conditions.

- **Construction of finite abstraction of dynamical systems evolving over Riemannian manifolds**. In order to synthesize a controller to enforce complex specifications (e.g., ones expressed as LTL) over dynamical systems, one promising approach is symbolic control. In this approach, first a finite abstraction of the original system, with discrete state and input sets, is constructed. After constructing the finite abstraction of the original system, one synthesizes a controller for the finite abstraction to satisfy the complex specifications using algorithmic techniques from computer science [BK08], and finally refine the controller to that for the original system. To the best of the author's knowledge, there is no work for the construction of finite abstractions for systems evolving over Riemannian manifolds. Hence, one potential research direction is to develop techniques for construction of finite abstractions for systems evolving over Riemannian manifolds.
- **Compositional abstraction of networks of *stochastic systems* evolving over Riemannian manifolds**. Another potential research directions is extending the techniques in Chapter 5 for compositional abstraction of deterministic system over Riemannian manifolds to *stochastic systems over Riemannian manifolds.* In this regard, one may investigate the methods for analysis of solution flows of stochastic systems, posed as Ito's stochastic differential equations, over a Riemannian manifold identified through a suitably constructed metric [Hsu02; MR22].

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