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Risk limitation and risk sharing in investment and insurance

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Abstract

In this doctoral thesis, we study the optimal investment and risk-sharing problems relevant for decision makers who provide guarantees on the terminal value of their or their clients' portfolios, e.g., insurance companies offering equity-linked products with capital guarantees at the end of the investment horizon. Working in the expected-utility framework, we consider novel portfolio-optimization problems in continuous time, where the financial risk is shared between the involved parties and/or the decision makers have constraints on the terminal portfolio value. We solve the corresponding portfolio-selection problems by applying case-specific optimization methods. In the complete-market setting, we use the martingale approach with suitable transformations of the original problems to tackle risk-sharing mechanisms and constraints on terminal wealth or investment strategies. In the incomplete-market setting due to stochastic volatility as per Heston's model, we apply the stochastic control approach and derive the solution to the wealth-constrained problem by linking it with the solution to the wealth-unconstrained problem in the case of a power-utility function. The portfolio-optimization methodology we develop in this thesis opens the door to solving new previously unsolved problems with other constraints or in other financial markets. In all problems, we complement our theoretical derivations with numerical studies, where we provide economic interpretations as well as implications, and elaborate on the properties of the determined optimal investment and risk-sharing strategies.

Zusammenfassung

In dieser Dissertation befassen wir uns mit Portfoliooptimierungs- und Risikoaufteilungsproblemen, die für Entscheidungsträger relevant sind, die Garantien für den Portfolioendwert beachten müssen. Ein Beispiel wäre ein Versicherungsunternehmen, das ein kapitalmarktgebundenes Produkt mit Kapitalgarantien am Ende des Anlagehorizonts anbietet. Im Rahmen der Erwartungsnutzen-Theorie betrachten wir neuartige Portfoliooptimierungsprobleme in kontinuierlicher Zeit, bei denen das finanzielle Risiko zwischen den beteiligten Parteien geteilt wird und/oder die Entscheidungsträger Nebenbedingungen für den Portfolioendwert haben. Wir lösen die entsprechenden Portfolioselektionsprobleme durch die Anwendung fallspezifischer Optimierungsmethoden. Im Fall von vollständigen Finanzmärkten verwenden wir den Martingal-Ansatz mit geeigneten Transformationen der ursprünglichen Probleme, um die Risikoaufteilungsmechanismen sowie die Nebenbedingungen für das Endvermögen oder die Investitionsstrategien zu berücksichtigen. Im Fall von unvollständigen Finanzmärkten aufgrund stochastischer Volatilität nach dem Modell von Heston wenden wir den Ansatz der stochastischen Prozesskontrolle an und leiten die Lösung des vermögensbeschränkten Problems ab, indem wir diese mit der Lösung des vermögensunbeschränkten Problems verknüpfen, wenn der Investor eine Power-Nutzenfunktion hat. Die in dieser Arbeit entwickelten Portfoliooptimierungsverfahren öffnen die Tür zur Lösung neuer, bisher ungelöster Probleme mit anderen Nebenbedingungen und/oder in anderen Finanzmärkten. Bei allen Problemen ergänzen wir unsere theoretischen Herleitungen durch numerische Studien, in denen wir ökonomische Interpretationen sowie Implikationen liefern und die Eigenschaften der ermittelten optimalen Investitions- und Risikoaufteilungsstrategien näher erläutern.

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Contents

1 Introduction

This chapter consists of three sections. First, in Section 1.1, we provide the motivation for the general research topic and state the objectives we follow in each of the chapters. Second, we describe in Section 1.2 how the dissertation is structured and provide an overview of the content of each following chapter. For each chapter, we summarize the research questions we answer and the core literature we use. Third, we state in Section 1.3 all research papers that were written during the doctoral research of the author of this dissertation and list the corresponding scientific contributions.

1.1 Motivation and objectives

Capital protection plays an important role for people with various levels of income. As a result, proper risk sharing and risk limitation are crucial for the success of the investment strategies that aim at protecting the invested capital. One prominent example is the limitation of risk of the investment portfolios related to pensions, which are managed by pension funds, insurance companies or their asset managers. To limit their risk exposure and be able to fulfill their liabilities, e.g., ensuring that the clients are protected against losing their capital, those institutions can share risk with reinsurance companies. Another example of risk sharing can be observed in the hedge-fund industry. Since hedge funds require a fairly large minimum initial investment ranging from 100 thousand to 2 million US dollars, their clients are predominately high-net-worth individuals. Hedge-fund managers, who decide how to invest money on behalf of the fund's clients, can be contractually obliged to cover potential losses of investors' money. For instance, such contractual agreements are a part of so-called first-loss compensation schemes of hedge-funds managers. Under these compensation schemes, investors as well as hedge-fund managers share financial risk, and managers are better incentivized to deliver positive return on investments for their clients in comparison to the case when they have no obligation to cover any incurred losses.

Even though many researchers have analyzed the problem of optimal asset allocation and risk-sharing in the presence of capital guarantees, there are plenty of research questions that were either not answered or answered in a setting that we believe can be made more realistic. This motivates us to investigate the topic deeper and answer some of those open questions, in particular questions regarding investments in equity-linked insurance products and hedge funds. Thus, in this dissertation, we formulate and solve four novel portfolio optimization problems, where the decision maker has a constraint on the terminal portfolio value and/or the financial risk is shared between the involved parties, i.e., an insurance company and a reinsurance company or a manager and a representative investor of a hedge fund. In each case, we also analyze the obtained solutions from the economic perspective. At this stage, we forego naming the relevant literature, but provide it in the four main chapters of this thesis.

Our objective is to answer the following open questions in the field of risk-sharing and investment strategies with capital guarantees.

First, our goal is to contribute to the literature on the hedge-fund industry by answering the question: Which first-loss compensation schemes can be seen as fair and optimal for both a manager and an investor in a hedge fund, i.e., which management fee, performance fee and first-loss coverage guarantee are mutually preferred by both parties?

Second, we aim at contributing to the actuarial literature by designing a framework for finding optimal investment-reinsurance strategies for equity-linked insurance products with capital guarantees and answering multiple related questions. When is reinsurance needed in the management of such products? What are the optimal (equilibrium) investment strategies, amount of reinsurance and the price of reinsurance? Which potential does reinsurance have in reversing the currently observed trend of falling capital guarantee levels in life insurance market of many countries including Germany?

Third, we strive for extending the class of portfolio optimization problems that can be solved analytically or semi-analytically. Tackling the previously mentioned questions, we model each situation as realistic as we can while maintaining analytical tractability of each problem. As a result, we answer the following questions that push the boundaries of portfolio optimization literature. How to solve portfolio optimization problems with simultaneous Value-at-Risk (VaR) constraint and no-short-selling constraint in a Black-Scholes market with a traded option? How to solve bi-level portfolio optimization problems with a fixed position in a put option and a no-trading constraint? How to derive solutions to portfolio optimization problems with VaR constraints in an incomplete market due to stochastic volatility?

1.2 Structure of the thesis

Next we provide an overview over the structure of the dissertation and give a brief summary of the conducted research and innovations.

Chapter 2 contains results from continuous time finance and optimization theory, which we use in subsequent chapters.

In Chapter 3, we focus on risk sharing in the hedge-fund industry. Specifically, we analyze fee structures of hedge funds with first-loss compensation, according to which the hedge-fund manager charges a management fee as well as a performance fee and guarantees to cover a certain amount of investors' potential losses. The very last element of the compensation scheme is the risk sharing between the involved parties. The research question we answer is which first-loss schemes can be seen as fair and optimal for both parties, i.e., which management fee, performance fee and first-loss coverage guarantee are mutually preferred by both, the manager and the investor. Within standard economics paradigms of rational expectations and utility maximization, we solve the manager's non-concave utility maximization problem, calculate Pareto optimal first-loss schemes and maximize a decision criterion on this set. In the numerical part, we study the impact of parameters of the financial market and of the decision makers' risk aversion on such mutually preferred fee arrangements.

In Chapter 4, we examine both risk sharing and risk limitation in the context of an equity-linked insurance product. The insurer guarantees a certain terminal wealth to the client and can share part of his/her risk with the reinsurer. Risk sharing follows in the form of a continuously traded put option on a portfolio that is acceptable for the reinsurer. This portfolio is different from the individual investment portfolio of the insurer. To model the product feature, we introduce to the asset universe an asset that cannot be reinsured but is correlated with the asset that can be reinsured. We assume that the reinsurer can sell a put option on a constant-mix portfolio containing the reinsurable risky asset and a risk-free asset. The insurer's objective is to maximize its expected utility of terminal wealth by choosing optimally its investment and reinsurance strategies while satisfying two types of constraints. The first constraint is a VaR constraint on terminal wealth ensuring that the insurer's terminal portfolio value is above its liability (capital guarantee to the client) with high probability. The second constraint is a no-short-selling constraint preventing the insurer from shorting its portfolio of assets and reinsurance. We solve the insurer's portfolio optimization problem in three steps. First, we tackle the continuously traded reinsurance similarly to how CPPI-style fund is treated in Hambardzumyan and Korn (2019). Second, we deal with the no-short-selling constraint by means of auxiliary markets as per Cvitanic and Karatzas (1992). Third, we handle the VaR constraint using the methodology of Basak and Shapiro (2001). In the corresponding numerical studies, we parametrize our model in accordance with the German market and analyze the sensitivity of the optimal strategies and the insurer's value function with respect to the market parameters as well as the insurer's risk aversion. We introduce the concept of guarantee-equivalent utility gain and use it to compare life insurance products with and without reinsurance. Our numerical studies indicate that the optimally managed reinsurance allows the insurer to offer significantly higher capital guarantees to clients without any loss in the insurer's expected utility.

In Chapter 5 we model the interaction between the insurer and the reinsurer in a more realistic way than the one in Chapter 4. Here we focus primarily on the risk-sharing aspect and model the interaction between the insurance and the reinsurance company in the form of a Stackelberg game, which is a bi-level optimization problem. In contrast to Chapter 4, we do not consider no-short-selling or terminal-wealth constraints in the main optimization problem in this project to make the overall Stackelberg game more analytically tractable. The reinsurer is the leader in the game and maximizes its expected utility by selecting its optimal investment strategy and a safety loading in the reinsurance contract it offers to the insurer. The reinsurer can assess how the insurer will rationally react on each action of the reinsurer. The insurance company is the follower and maximizes its expected utility by choosing its investment strategy and the amount of reinsurance the company purchases at the price offered by the reinsurer. In this game, we derive the Stackelberg equilibrium for general utility functions by combining and adapting the portfolio optimization techniques from Cvitanic and Karatzas (1992), Desmettre and Seifried (2016), and Korn and Trautmann (1999). For power-utility functions, we calculate the equilibrium explicitly and find that the reinsurer selects the largest reinsurance premium such that the insurer may still buy the maximal amount of reinsurance. Since in the equilibrium the insurer is indifferent in the amount of reinsurance, in practice, the reinsurer should consider charging a smaller reinsurance premium than the equilibrium one. Therefore, we propose several criteria for choosing such a discount rate and investigate its wealth-equivalent impact on the utilities of both parties.

In Chapter 6, we focus on risk limitation in an incomplete financial market based on the Heston model. In that market, we solve the VaR-constrained utility maximization problem. We do that by generalizing to the incomplete market of interest the approach proposed in Kraft and Steffensen (2013), where the authors link the optimal constrained portfolio to the optimal unconstrained one. Our extension of their result relies on the Feynman-Kac formula and Fourier transforms of probability density functions. We demonstrate that the value function in the constrained problem can be represented as an expected modified utility function on a vega-neutral contingent claim on the optimal unconstrained wealth. The optimal wealth and the optimal investment strategy in the constrained problem follow similarly. The case of a power-utility function and a Value-at-Risk constraint is treated in detail. In numerical studies, we investigate the sensitivity of the optimal investment strategy with respect to the model parameters.

In each chapter, we provide at its beginning the research motivation, concrete research questions to be answered, a broad overview of the relevant literature, a more specific description of the methodological part of the chapter, a more detailed summary of our mathematical as well as practical contributions, and a more granular structure of the chapter than it is done above.

1.3 Research articles and scientific contributions

The content of this thesis is based on the following research papers:

- Chapter 3:

  Escobar-Anel, M., Havrylenko, Y. & Zagst, R. Optimal fees in hedge funds with first-loss compensation. *Journal of Banking & Finance*, 2020, 118, 105884.

https://doi.org/10.1016/j.jbankfin.2020.105884

The main scientific contributions of this chapter based on this paper are given below:

1. We are first to analyze first-loss compensation schemes based on the criterion of Pareto optimality and to investigate how hedge fund managers and investors can reasonably select a mutually-preferred Pareto optimal first-loss fee structure.
2. Our methodology yields a preferred Pareto optimal first-loss fee structure that is fair to both parties and decreases significantly the hedge fund's risk in comparison to the traditional fee structure.
3. We find that the common traditional and first-loss fee structures are not Pareto optimal.
4. We give a possible explanation for the current trend of decreasing management fees in hedge funds.

- Chapter 4

Escobar-Anel, M. Havrylenko, Y.; Kschonnek, M. & Zagst, R. Decrease of capital guarantees in life insurance products: Can reinsurance stop it? *Insurance: Mathematics and Economics*, 2022, 105, 14-40.
https://doi.org/10.1016/j.insmatheco.2022.03.009

The main research contributions of this paper-based chapter are provided below:

1. We design a framework for finding optimal investment-reinsurance strategies for equity-linked insurance products with capital guarantees. The framework combines put options, regulatory VaR and no-short-selling constraints, and a separation between reinsurable and non-reinsurable funds.
2. We solve explicitly the portfolio optimization problem with simultaneous VaR and no-short-selling constraints in a financial market with a traded put option.
3. We detect market conditions and the asset manager's proficiency, for which (partial) reinsurance of the capital guarantee is advantageous.
4. We establish that optimal reinsurance significantly increases capital guarantees, while slightly decreasing product costs.

- Chapter 5

Havrylenko, Y., Hinken, M. & Zagst, R. Risk sharing in equity-linked insurance products Stackelberg equilibrium between an insurer and a reinsurer. *Submitted for publication*, 2022.
https://doi.org/10.48550/arXiv.2203.04053

The main research contributions of this chapter based on the above-mentioned paper are as follows:

1. We formulate and analyze a novel Stackelberg game between a reinsurer and an insurer, which is more realistic than Stackelberg games previously studied in the literature
2. We solve the bi-level portfolio-optimization problem that describes the formulated Stackelberg game between a reinsurer and an insurer in the context of an equity-linked product. To the best of our knowledge, this problem has not been considered before in the literature and its challenges include the simultaneous presence of a fixed-term investment opportunity in a put option and a no-trading constraint on one of the risky assets.
3. We find that in the Stackelberg equilibrium the reinsurer selects the largest reinsurance premium such that the insurer may buy the maximal amount of reinsurance and conclude that, in practice, the reinsurer should charge a lower (discounted) safety loading of the reinsurance premium than the equilibrium one in order to secure a deal with the maximal amount of reinsurance.

- Chapter 6

Escobar-Anel, M., Havrylenko, Y. & Zagst, R. Constrained portfolios in incomplete markets: a dynamic programming approach to Heston's model. *Submitted for publication*, 2022.
https://doi.org/10.48550/arXiv.2208.14152

Next we provide the main scientific contributions of this paper-based chapter:

1. We extend to incomplete markets due to stochastic volatility the methodology of Kraft and Steffensen (2013) of solving portfolio optimization problems with terminal-wealth constraints.
2. Our methodology opens the door to solving many further portfolio optimization problems with other incomplete market models and other types of terminal-wealth constraints.

We give an in-depth explanation of theoretical as well as practical contributions and provide references to the corresponding Propositions, Theorems, Corollaries, etc., at the beginning of each chapter. We also summarize all our mathematical contributions in the concluding Chapter 7.

Parts of this thesis are identical with or a reproduction with minor changes of the four articles mentioned above. In addition to these articles, the author of this dissertation wrote another research paper, which is not related to the topic of this thesis. Since that paper was written during the author's time as a doctoral researcher, it is stated here for completeness:

Havrylenko, Y. & Heger, J. Algorithmic detection of interacting variables for generalized linear models via neural networks. *Submitted for publication*, 2022.
https://doi.org/10.48550/arXiv.2209.08030

The main scientific contribution of this paper is a novel methodology we suggest for detecting the next-best pair of strongly interacting variables for generalized linear models via neural networks. This methodology is especially relevant and powerful for big actuarial data sets with dozens of variables.

2 Mathematical preliminaries

This chapter consists of 5 sections. First, we introduce the basic financial market and mathematically describe the economic activities of decision makers in it. Second, we provide the basic results on pricing options, as they play an important role in risk sharing and risk limitation. Third, we define utility functions and address their relevant properties. Fourth, we state a general portfolio optimization problem and give an overview of various methods of solving that problem depending on the type of constraints and on the peculiarities of traded assets. Finally, we conclude this chapter with useful results from calculus, probability theory and deterministic non-linear optimization.

2.1 The basic financial market model

In this section, we introduce the basic financial market model and provide general definitions related to decision makers' activities in it.

We assume that all investment and risk-sharing activities take place on a time horizon $[0, T]$, where $0 < T < +\infty$ is the end of this period. Let $W^{\mathbb{Q}} = (W^{\mathbb{Q}}(t))_{t\in[0,T]}$, $W^{\mathbb{Q}}(t) := \left(W_1^{\mathbb{Q}}(t), \ldots, W_n^{\mathbb{Q}}(t)\right)^\top$, $n \in \mathbb{N}$, be an n-dimensional Wiener process on a filtered complete probability space $\left(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}(t))_{t\in[0,T]}\right)$. Here, Ω is the sample space, $\mathcal{F} = \mathcal{F}(T)$ is a sigma-algebra on Ω, $\mathbb{Q}$ is the real-world probability measure, $\mathcal{F}(t)$ with $t \in [0, T]$ is the natural filtration generated by $W^{\mathbb{Q}}(s)$, $s \in [0, t]$, and augmented by the null sets.

The basic financial market is a Black-Scholes market. It has $n + 1$ assets that are continuously traded without frictions, i.e., without transaction costs, without bid-ask spreads, without taxes, etc. One asset is a risk-free asset whose price process we denote by $S_0(t)$, $t \in [0, T]$. Other assets are risky ones whose price processes at time $t \in [0, T]$ we denote by $S_1(t), \ldots, S_n(t)$.

Under the real-world probability measure $\mathbb{Q}$, the price dynamics of the risk-free asset, also referred to as a bank account, is given by

$$dS_0(t) = rS_0(t)dt, \;\; S_0(0) = 1, \tag{2.1}$$

where $r \in \mathbb{R}$ is a constant interest rate. Equation (2.1) is an ordinary differential equation (ODE). Its solution is well-known and is given by $S_0(t) = \exp(rt)$.

Under $\mathbb{Q}$, the price processes $S_i(t)$, $i = 1, \ldots, n$, evolve according to the following stochastic differential equations (SDEs):

$$dS_i(t) = S_i(t)\left(\mu_i dt + \sigma_i dW^{\mathbb{Q}}(t)\right) = S_i(t)\left(\mu_i dt + \sum_{j=1}^{n} \sigma_{i,j} dW_j^{\mathbb{Q}}(t)\right), \quad S_i(0) = s_i, \quad (2.2)$$

where $\mu := (\mu_1, \ldots, \mu_n)^\top \in \mathbb{R}^n$ with $\mu - r \cdot \mathbf{1}_n > \mathbf{0}_n$ is the constant drift vector, $\mathbf{0}_n := (0, \ldots, 0)^\top \in \mathbb{R}^n$, $\mathbf{1}_n := (1, \ldots, 1)^\top \in \mathbb{R}^n$, $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,n}) \in [0, +\infty)^{1\times n}$ denotes the constant volatility vector of asset $i = 1, \ldots, n$, and $s_i > 0$ is the asset's initial price. The solution to SDEs (2.2) are also well-known:

$$S_i(t) = s_i \exp\left(\left(\mu_i - \frac{1}{2}||\sigma_i||^2\right) t + \sigma_i W^{\mathbb{Q}}(t)\right),$$

where $||\cdot||$ denotes the Euclidean norm operator, i.e., $||\sigma_i||^2 = \sum_{j=1}^{n} \sigma_{i,j}^2$, $i = 1, \ldots, n$.

We denote the volatility matrix by $\sigma = (\sigma_{i,j})_{i,j=1,\ldots,n}$ and the corresponding covariance matrix of log-returns by $\Sigma = \sigma\sigma^\top$, which is assumed to be strongly positive definite, i.e., there exists $K_\Sigma > 0$ such that $\forall x \in \mathbb{R}^n \ x^\top \Sigma x \geq K_\Sigma \cdot ||x||^2$ holds. From Eq. (3.2), p. 45, in Zagst (2002) it follows for all $i = 1, \ldots, n$ that:

$$\sup_{t\in[0,T]} S_i(t) < +\infty \qquad \mathbb{Q}\text{-a.s.}.$$

We denote the set of admissible unconstrained trading strategies by $\mathcal{A}_u^{\varphi}(v)$ and the set of admissible unconstrained relative portfolio processes by $\mathcal{A}_u^{\pi}(v)$. We also denote the corresponding admissibility sets by $\mathcal{A}_u^{\varphi}(t, v)$ and $\mathcal{A}_u^{\pi}(t, v)$ for the time frame $[t, T]$ and given $V(t) = v$.

We denote the market price of risk by $\gamma := \sigma^{-1}(\mu - r\mathbf{1}_n)$. According to Theorem 3.26 on page 147 in Korn (2014), in the aforementioned market there exists a unique risk-neutral probability measure that we denote by $\widetilde{\mathbb{Q}}$:

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}(t)} := Z(t) := \exp\left(-\frac{1}{2}||\gamma||^2 t - \gamma^\top W^{\mathbb{Q}}(t)\right). \quad (2.3)$$

The measure $\widetilde{\mathbb{Q}}$ is also called an equivalent martingale measure (EMM), since the discounted price processes of the basic risky assets, i.e., $\widetilde{S}_i(t) := S_i(t)/S_0(t)$, $t \in [0, T]$, $i = 1, \ldots, n$, are martingales with respect to (w.r.t.) $\widetilde{\mathbb{Q}}$.

Further, we define the associated pricing kernel by $\widetilde{Z} = \left(\widetilde{Z}(t)\right)_{t\in[0,T]}$, which is also known as the state price density or the deflator and is defined as follows:

$$\widetilde{Z}(t) = \exp\left(-\left(r + \frac{1}{2}||\gamma||^2\right)t - \gamma^\top W^{\mathbb{Q}}(t)\right), \quad t \in [0, T]. \tag{2.4}$$

The pricing kernel will play a crucial rule in finding the present value of future cash flows such as the payoff of a put option modeling a reinsurance contract. From (2.4) it follows that $\widetilde{Z}(t)$ satisfies the following SDE:

$$d\widetilde{Z}(t) = -\widetilde{Z}(t)\left(rdt + \gamma^\top dW^{\mathbb{Q}}(t)\right), \quad \widetilde{Z}(0) = 1. \tag{2.5}$$

Next we formally describe trading activities of decision makers in this basic market.

Definition 2.1.1 (Trading strategy, Def. 2.60 a, p. 105, Korn (2014))**.** *A trading strategy φ is an $\mathbb{R}^{n+1}$-valued $(\mathcal{F}(t))_{t\in[0,T]}$-progressively measurable stochastic process $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t))^\top$, $t \in [0, T]$, such that:*

$$\int_0^T |\varphi_0(t)|\, ds < +\infty \quad \text{and} \quad \int_0^T \varphi_i^2(t)\, ds < +\infty \qquad \mathbb{Q}\text{-a.s.}, \forall i = 1, \ldots, n.$$

The value $v := \varphi_0(0)s_0 + \sum_{i=1}^n \varphi_i(0)s_i$ is called the starting value of φ.

Definition 2.1.2 (Wealth process, Def. 2.60 b, p. 105, Korn (2014))**.** *Let φ be a trading strategy with a starting value v. The process:*

$$V(\varphi, t) := \varphi_0(t)S_0(t) + \sum_{i=1}^n \varphi_i(t)S_i(t)$$

is called the wealth process w.r.t. φ with the initial wealth $V(\varphi, 0) = v$.

We consider only self-financing trading strategies, i.e., strategies that satisfy the following property:

$$V(\varphi, t) = V(\varphi, 0) + \int_0^t \varphi_0(s)\, dS_0(s) + \sum_{i=1}^n \int_0^t \varphi_i(s)\, dS_i(s) \qquad \mathbb{Q}\text{-a.s..}$$

This property means that the only source of the change in portfolio value is the change in the trading strategy and the asset prices, i.e., the decision maker does not inject additional capital and does not take out his/her money from the portfolio during the investment period.

The decision maker's actions can be seen from a different angle – relative to the overall portfolio value. We denote the proportion of the decision maker's money invested in the asset S_i at time $t \in [0,T]$ by $\pi_i(t)$, $i = 0,1,\ldots,n$. It holds that $\pi_0(t) = 1 - \sum_{i=1}^n \pi_i(t)\, \forall t \in [0,T]$. The relative portfolio process π is related to the trading strategy φ as follows:

$$\pi_i(t) = \begin{cases} \dfrac{\varphi_i(t) \cdot S_i(t)}{V(\varphi,t)}, & \text{if } V(\varphi,t) \neq 0; \\ 0, & \text{if } V(\varphi,t) = 0, \end{cases} \tag{2.6}$$

for $t \in [0,T]$ and $i = 0,1,\ldots,n$.

For computational purposes and notational convenience, we name the relative portfolio process $\pi = (\pi_1(t),\ldots,\pi_n(t))_{t\in[0,T]}$ only with respect to the risky assets, since $\pi_0(t) = 1 - \sum_{i=1}^n \pi_i(t)\, \forall t \in [0,T]$ holds for a self-financing trading strategy. We will also write $V(t) := V(\varphi,t)$, where it does not lead to confusion. Using (2.6), we can write $V(t)$ in terms of π as follows:

$$dV(t) = V(t)\left(\left(r + \pi(t)^\top(\mu - r\mathbf{1}_n)\right)dt + \pi(t)^\top \sigma dW^{\mathbb{Q}}(t)\right), \qquad V(0) = v. \tag{2.7}$$

When we need to differentiate among wealth processes with different relative portfolio processes we will write the corresponding π in the superscript of V, i.e., V^π. If it is important to keep track of the initial wealth too, we will write $V^{v,\pi}$.

Definition 2.1.3 (Relative portfolio process, Def. 2.65, p. 107, Korn (2014))**.** *The $(\mathcal{F}(t))_{t\in[0,T]}$-progressively measurable stochastic process $\pi(t)$ is called a self-financing relative portfolio process, if (2.7) has a unique solution $V(t)$ such that:*

$$\int_0^T (V(t)\pi_i(t))^2 \, dt < +\infty \qquad \mathbb{Q}\text{-}a.s.,\ \forall i = 1,\ldots,n.$$

Definition 2.1.4 (Admissibility, Def. 2.67, p. 107, Korn (2014))**.** *A self-financing trading strategy φ and the corresponding relative portfolio process π are called admissible for the initial wealth v, if $\forall\, t \in [0,T]$ the following holds:*

$$V(t) \geq 0 \qquad \mathbb{Q}\text{-}a.s.. \tag{2.8}$$

We denote the set of admissible unconstrained trading strategies by $\mathcal{A}_u^\varphi(v)$ and the set of admissible unconstrained relative portfolio processes by $\mathcal{A}_u^\pi(v)$. We also denote the corresponding admissibility sets by $\mathcal{A}_u^\varphi(t,v)$ and $\mathcal{A}_u^\pi(t,v)$ for the time frame $[t,T]$ and given $V(t) = v$. The ability to pick any admissible unconstrained trading strategy allows the decision maker to reach any desired terminal wealth if he/she has a sufficiently high initial wealth. This property of the considered financial market is called market completeness and is formalized in the following theorem.

Theorem 2.1.5 (Market completeness, Th. 2.68 b, p. 109, Korn (2014))**.** *Let $D \geq 0$ be an $\mathcal{F}$-measurable random variable with $v := \mathbb{E}\left[\widetilde{Z}(T)D\right] < +\infty$. Then there exists $\pi \in \mathcal{A}_u^\pi(v)$ such that $V^\pi(T) = D$ $\mathbb{Q}$-a.s..*

Remark to Theorem 2.1.5. $\mathbb{E}[\cdot]$ denotes the expectation under the real-world probability measure $\mathbb{Q}$. When the expectation is taken under a different probability measure $\mathbb{M}$, we indicate that in the superscript $\mathbb{E}^{\mathbb{M}}[\cdot]$. For example, $\mathbb{E}^{\widetilde{\mathbb{Q}}}[\cdot]$ is the expectation operator under some EMM $\widetilde{\mathbb{Q}}$.

Theorem 2.1.5 plays a significant role in pricing financial contracts and in a so-called martingale approach to portfolio optimization, which we will discuss in more detail in Section 2.4. This theorem relies on the fact that the decision maker is not constrained in choosing how to invest. However, in reality, decision makers may have constraints on their actions. We will differentiate two types of constraints: terminal-wealth constraints and allocation constraints. An example of the former type of constraint would be the protection of the initial capital from loss at time T, i.e., $\mathbb{Q}(V(T) \geq V(0)) = 1$. An instance of the allocation constraint would be the prohibition of going short on risky assets, i.e., $\pi_i(t) \geq 0 \, \forall t \in [0, T], i = 1, \ldots, n$. Therefore, we will consider the sets of admissible constrained relative portfolio processes in different chapters, where such constraints play a role. We will denote those admissibility sets by $\mathcal{A}_c^\pi(v, C_V, C_\pi)$, where the set C_V will characterize the constraint on $V(T)$ and C_π will characterize the constraint on π.

We complete this section with a few sentences on the usage of the aforementioned financial market in the following chapters. It is used in Chapter 3, 4, 5. In Chapter 6, we develop a methodology for finding the optimal investment strategies for a decision maker in a financial market with stochastic volatility. In comparison with the basic market model above, we will have $n = 1$ and the volatility σ of the risky asset S_1 will be a mean-reverting stochastic process. In particular, we will assume that the price S_1 of the risky asset follows a Heston's model. In that case, there will be infinitely many EMMs. However, their role will remain the same, i.e., the discounted asset prices will be martingales under each EMM. For each EMM, there will be a corresponding pricing kernel, which will be of the exponential form, similarly to (2.4).

2.2 Basics of relevant financial options

Financial options, options for short, play a prominent role in risk sharing and risk limitation. As we will use options extensively in Chapters 4–6, we provide in this section relevant basic information about options.

An option is a contract that gives to the option holder the right (but not the obligation) of a certain action. We consider European options, which is a class of options where the action can be done (the option can be exercised) only at a specific time in the future, e.g., T. Although there are many types of European options, we use in the sequel only so-called plain vanilla options and digital options, which we define next.

A plain vanilla call option (respectively put option) gives the holder the right to buy (respectively sell) the underlying asset at a date T, also called the expiration date or maturity, for a certain price $K > 0$, known as the exercise price or strike price. We denote the payoff of an option by the name of the option. For example, if the option's underlying is S_1, then:

$$Call(T) = (S_1(T) - K)^+, \qquad Put(T) = (K - S_1(T))^+, \tag{2.9}$$

where $(x)^+ := \max\{x, 0\}$. Analogously, we also write $(x)^- = \max\{-x, 0\}$.

A digital call option (respectively digital put option) gives the holder the right to receive a notional amount $N > 0$ at time T if the value of the underlying asset is above (respectively below) the strike price $K > 0$. Without loss of generality we choose $N = 1$. For instance, if the underlying of a digital option is S_1, then:

$$DigCall(T) = \mathbb{1}_{\{S_1(T) \geq K\}}, \qquad DigPut(T) = \mathbb{1}_{\{S_1(T) \leq K\}}. \tag{2.10}$$

Digital options are also called cash-or-nothing options and are one of the simplest exotic options.

Options are a common type of financial derivatives, i.e., financial instruments whose value is derived from the performance of some underlying asset. A decision maker can buy (have a long position in) and sell (have a short position in) several options at the same time. In this case, the payoff of the overall portfolio of options is more complex than those of plain vanilla or digital options. However, it can still be regarded as a payoff of one complex financial derivative, which we denote by D. The price of any financial derivatives can be calculated using the following theorem.

Theorem 2.2.1 (Price of a financial derivative via pricing kernel, Th. 3.18, p. 135, Korn (2014))**.** *Let $D := D(T) \geq 0$ be an $\mathcal{F}$-measurable random variable with $\mathbb{E}[D^a] < +\infty$ for some $a > 1$. Then the unique value (price) at time $t \in [0, T]$ of the financial derivative with payoff D is given by:*

$$D(t) = \frac{1}{\widetilde{Z}(t)} \mathbb{E}\left[\widetilde{Z}(T) D(T) | \mathcal{F}(t)\right]. \tag{2.11}$$

As mentioned at the end of Section 2.1, we will consider in Chapter 6 an incomplete financial market with infinitely many EMMs. In that case, the price of a financial derivative may not be unique. However, once a specific EMM is chosen, one can define the price of the financial derivative in a way that ensures the absence of so-called arbitrage opportunities, i.e., opportunities in which the investor with zero initial capital can generate money with positive probability and his/her terminal wealth is non-negative with probability 1.

Theorem 2.2.2 (Price of a financial derivative via EMM, Th. 1.42, p. 39, Desmettre and Korn (2018))**.** *Let $\widetilde{\mathbb{Q}}$ be an equivalent martingale measure. Let $D := D(T) \geq 0$ be an $\mathcal{F}$-measurable random variable. Define the value (price) at time $t \in [0, T]$ of the financial derivative with payoff D by:*

$$D^{\widetilde{\mathbb{Q}}}(t) = \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\frac{S_0(t)}{S_0(T)}D(T)|\mathcal{F}(t)\right]. \tag{2.12}$$

Then there are no arbitrage opportunities in the financial market consisting of the original financial market and the financial derivative.

Next we provide closed-form formulas for the prices of plain vanilla as well as digital options, denoting by $\Phi(\cdot)$ the distribution function of the standard normal random variable.

Proposition 2.2.3. *Consider the basic financial market with $n = 1$, $\mu = \mu_1 > r$, $\sigma = \sigma_{1,1} > 0$. Let $K > 0$ be an option's strike. Then the time-t prices of plain vanilla and digital call options and put options are given by:*

$$\begin{aligned} Call(t, S_1(t), K, r, \sigma) &= S_1(t)\Phi(d_1(t, S_1(t), K, r, \sigma)) \\ &\quad - \exp\left(-r(T-t)\right) K\Phi(d_2(t, S_1(t), K, r, \sigma)), \qquad (2.13) \\ Put(t, S_1(t), K, r, \sigma) &= \exp\left(-r(T-t)\right) K\Phi(-d_2(t, S_1(t), K, r, \sigma)) \\ &\quad - S_1(t)\Phi(-d_1(t, S_1(t), K, r, \sigma)), \qquad (2.14) \\ DigCall(t, S_1(t), K, r, \sigma) &= \exp\left(-r(T-t)\right)\Phi(d_2(t, S_1(t), K, r, \sigma)), \qquad (2.15) \\ DigPut(t, S_1(t), K, r, \sigma) &= \exp\left(-r(T-t)\right)\Phi(-d_2(t, S_1(t), K, r, \sigma)), \qquad (2.16) \end{aligned}$$

where

$$d_1(t, S_1(t), K, r, \sigma) := \frac{\ln\left(\frac{S_1(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{2.17}$$

$$d_2(t, S_1(t), K, r, \sigma) := d_1(t, S_1(t), K, r, \sigma) - \sigma\sqrt{T-t}; \tag{2.18}$$

Proof. For the derivation of (2.13) and (2.14) see Corollary 3.21 on page 136 in Korn (2014). For the derivation of (2.15) see Example 3.29 on page 150 in Korn (2014), whereas the formula (2.16) is provided on page 191 in Korn (2014) and is proven analogously to (2.15). □

Remark to Proposition 2.2.3. The option-pricing formulas (2.13)–(2.16) can be applied whenever the terminal value of the option's underlying asset is a log-normal random variable, i.e., it has the following density function:

$$f_{LN(\mu,\sigma^2)}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right),$$

where π is the mathematical constant that is the ratio of a circle's circumference to its diameter, $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ being the distribution parameters. For example, if the relative portfolio process π in (2.7) is a constant-mix strategy, i.e., $\pi(t) = \pi_{CM} \in \mathbb{R}^n$ constant, $\forall t \in [0,T]$, then the corresponding terminal wealth $V^{\pi_{CM}}(T)$ given $\mathcal{F}(t)$ is log-normally distributed with parameters $\mu_{CM} = \left(r + \pi_{CM}^\top(\mu - r\mathbf{1}_n) - ||\gamma||^2/2\right)(T-t)$ and $\sigma_{CM}^2 = ||\pi_{CM}^\top \sigma||^2 \, (T-t)$.

For exotic options or plain vanilla options but in more complex financial market models, it may be very challenging to calculate the expectation of the discounted payoff of the option under the EMM. In such cases, the option-pricing technique based on characteristic functions can be very helpful. Although we provide those results under the measure $\mathbb{Q}$, they also hold for any probability measure.

Definition 2.2.4 (Characteristic function, Sec. 3.3.1, p. 108, Durrett (2019))**.** *Let X be an $\mathcal{F}$-measurable random variable. The characteristic function under the probability measure $\mathbb{Q}$ is defined by:*

$$\phi^{X,\mathbb{Q}}(u) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(i \cdot u \cdot X\right)\right], \tag{2.19}$$

where $i := \sqrt{-1}$ is the imaginary unit.

If X has a density function $f_X^{\mathbb{Q}}(\cdot)$, then (2.19) can be written as:

$$\phi^{X,\mathbb{Q}}(u) = \int_{-\infty}^{+\infty} \exp\left(iux\right) f_X^{\mathbb{Q}}(x)\, dx. \tag{2.20}$$

The characteristic function uniquely determines the distribution of X. The following theorem shows how to recover the distribution function from the characteristic function. We will frequently use it in Chapter 6.

Theorem 2.2.5 (The inversion formula, Theorem 3.3.14, p. 112, Durrett (2019))**.**
If $\int_{-\infty}^{+\infty} \left|\phi^{X,\mathbb{Q}}(u)\right| du < \infty$, then the density function of X is given by :

$$f_X^{\mathbb{Q}}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-iux\right) \phi^{X,\mathbb{Q}}(u)\, du. \tag{2.21}$$

Next we state an important result, which is useful for pricing financial derivatives. For that we will use a more general setting, as we will apply this result both in the basic market in Chapter 4 and in a market with stochastic volatility in Chapter 6.

Definition 2.2.6 (Strong solution of an SDE, Def. 3.34, p. 157, Korn (2014))**.** *If on* $(\Omega, \mathcal{F}, \mathbb{Q})$ *there exists an* m*-dimensional continuous process* $X = (X(t))_{t\geq 0}$ *with*

$$X(0) = x, \quad x \in \mathbb{R}^m \quad \text{constant,} \tag{2.22}$$

$$X_i(t) = x_i + \int_0^t \mu_i(s, X(s))\, ds + \sum_{j=1}^{n} \int_0^t \sigma_{i,j}(s, X(s))\, dW_j^{\mathbb{Q}}(s) \tag{2.23}$$

$\mathbb{Q}$*-a.s. for all* $t \geq 0$, $i \in 1, \ldots, m$, *such that*

$$\int_0^t \left(|\mu_i(s, X(s))| + \sum_{j=1}^{n} (\sigma_{i,j}(s, X(s)))^2 \right) ds < +\infty \tag{2.24}$$

$\mathbb{Q}$*-a.s. for all* $t \geq 0$, $i \in 1, \ldots, m$, *holds, then* X *is called a strong solution of the SDE:*

$$dX(t) = \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW^{\mathbb{Q}}(t); \tag{2.25}$$

$$X(0) = x \tag{2.26}$$

for given functions $\mu : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times n}$.

Definition 2.2.7 (Characteristic operator, Def. 3.39, p. 164, Korn (2014))**.** *Let* X *be the unique solution of SDE* (2.25) *such that* $\mu(t, x)$ *as well as* $\sigma(t, x)$ *are continuous and satisfy the following conditions:*

$$||\mu(t, x) - \mu(t, y)|| + ||\sigma(t, x) - \sigma(t, y)|| \leq K||x - y|| \tag{2.27}$$

$$||\mu(t, x)||^2 + ||\sigma(t, x)||^2 \leq K^2 \left(1 + ||x||^2\right) \tag{2.28}$$

for all $t \geq 0$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ *and a constant* $K > 0$. *For a twice continuously differentiable function* $f : \mathbb{R}^D \to \mathbb{R}$, *the operator* $\mathcal{D}_t$ *defined by*

$$(\mathcal{D}_t f)(x) := \frac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} a_{i,k}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_k}(x) + \sum_{i=1}^{m} \mu_i(t, x) \frac{\partial f}{\partial x_i}(x) \tag{2.29}$$

with

$$a_{i,k}(t, x) := \sum_{j=1}^{n} \sigma_{i,j}(t, x) \sigma_{k,j}(t, x)$$

is called the characteristic operator for X.

Theorem 2.2.8 (Feynman-Kac representation, Th. 3.41, p. 165, Korn (2014))**.** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a twice continuously differentiable function that satisfies the following condition:*

$$|f(x)| \leq L\left(1 + ||x||^{2\lambda}\right) \quad or \quad f(x) \geq 0 \tag{2.30}$$

for $\lambda \geq 1$. Assume that there exists a solution $u(t,x) : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ to the following Cauchy problem:

$$-u_t + ku = \mathcal{D}_t u \quad on \quad [0,T) \times \mathbb{R}^m; \tag{2.31}$$

$$u(T,x) = f(x) \quad for \quad x \in \mathbb{R}^m, \tag{2.32}$$

where $k : [0,T] \times \mathbb{R}^m \to [0,+\infty)$ is a continuous function, $u(t,x)$ is continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. x, $\mathcal{D}_t$ is the characteristic operator for X that is a unique solution to SDE (2.25) *with μ as well as σ being continuous and satisfying conditions* (2.27) *and* (2.28) *respectively. If also $u(t,x)$ satisfies the following condition:*

$$\max_{t\in[0,T]} |u(t,x)| \leq M(1 + ||x||^{2\eta}) \quad for \quad M > 0, \eta \geq 1, \tag{2.33}$$

then $u(t,x)$ has the following representation:

$$u(t,x) = \mathbb{E}^{\mathbb{Q}}\left[f(X(T)) \cdot \exp\left(-\int_t^T k(s,X(s))\right)\Big|X(t) = x\right] \tag{2.34}$$

and is a unique solution to the Cauchy problem (2.31)-(2.32)*, which fulfills condition* (2.33)*.*

2.3 Utility functions

In this section, we provide a general definition of a utility function as well as measures of risk aversion of a decision maker. After that we define a specific class of utility functions that we will frequently use in numerical studies.

We assume that decision makers choose their actions within the expected utility framework. In particular, they maximize the expectation of their utility functions $U(\cdot)$ evaluated at their wealth at the terminal time T. Next we provide a standard definition of a utility function. Using the word "standard" in the previous sentence, we want to emphasize that there can be more general definitions of a utility function, see, e.g., Definition 2.1 in Reichlin (2013).

Definition 2.3.1 (Utility function, adapted from Def. 5.1, p. 254, Korn (2014))**.** *Let $\underline{v} \in \mathbb{R}$. A function $U : (\underline{v}, +\infty) \to \mathbb{R}, v \mapsto U(v)$ that is continuously differentiable,* *strictly concave and satisfies conditions*

$$\lim_{v \downarrow \underline{v}} U'(v) = +\infty \qquad \text{and} \qquad \lim_{v \uparrow +\infty} U'(v) = 0$$

is called a utility function.

The required properties of $U(\cdot)$ imply that a decision maker prefers a higher amount of wealth to a lower one. However, his/her additional utility gained from an additional unit of wealth decreases with the actual wealth amount.

Different utility functions describe different attitudes to risks. In particular, utility functions indicate how risk averse (or risk seeking) a decision maker is. The most common measures of risk aversion were proposed by Pratt (1964) and Arrow (1970).

Definition 2.3.2 (Absolute risk aversion, Eq. (2), p. 39, Korn (1997))**.** *The Arrow-Pratt measure of the absolute risk aversion (ARA) of a utility function $U(\cdot)$ is defined as:*

$$ARA_U(v) = -\frac{U''(v)}{U'(v)} = -\frac{\partial}{\partial v} \ln\left(U'(v)\right). \tag{2.35}$$

Definition 2.3.3 (Relative risk aversion, Eq. (2), p. 39, Korn (1997))**.** *The Arrow-Pratt measure of the relative risk aversion (RRA) of a utility function $U(\cdot)$ is given by*

$$RRA_U(v) = -\frac{vU''(v)}{U'(v)} = vARA_U(v). \tag{2.36}$$

If an Arrow-Pratt measure is positive, then the decision maker is risk-averse. If it is equal to zero, then the decision maker is risk-neutral. Finally, negative Arrow-Pratt measures indicate that the decision maker is risk-seeking . In this thesis, we focus on utility functions with positive ARA and RRA. Although we derive many results for general utility functions, in numerical studies we will consider HARA-utility functions, where HARA stands for **h**yperbolic **a**bsolute **r**isk **a**version. According to Ingersol 1987, this is one of the most commonly used classes of utility functions.

Definition 2.3.4 (HARA-utility function)**.** *A function $U(\cdot)$ is called a HARA-utility function if it admits a representation:*

$$U(v) = \frac{(v+a)^p}{p}$$

for $p < 1$, $p \neq 0$, $v + a > 0$, $a \in \mathbb{R}$.

Remark to Definition 2.3.4. In the academic literature the class of HARA-utility functions can be defined in a more general way. For example, in Definition 3 on p. 40 in Korn (1997) a HARA-utility function is a function that can be written as

$$U(v) = \frac{1-p}{p}\left(\frac{\beta}{1-p}v + \eta\right)^p \tag{2.37}$$

for $p < 1$, $p \neq 0$, $\beta > 0$, $\frac{\beta}{1-p}v + \eta > 0$. Our Definition 2.3.4 can be obtained from (2.37) by choosing $\beta = (1-p)^{1-1/p}$ and $\eta = a(1-p)^{-1/p}$.

The Arrow-Pratt measures for a HARA-utility function from Definition 2.3.4 are:

$$ARA_U(v) = \frac{1-p}{v+a} \qquad \text{and} \qquad RRA_U(v) = \frac{(1-p)v}{v+a}. \tag{2.38}$$

When $a = 0$ in Definition 2.3.4, then the RRA measure is $1-p$ and, thus, constant and independent of v. The corresponding utility function is called a CRRA-utility function, where CRRA stands for **c**onstant **r**elative **r**isk **a**version. It is also known as a Power-utility function:

$$U(v) = \frac{v^p}{p} \tag{2.39}$$

for $p < 1$, $p \neq 0$, $v > 0$.

Knowing his/her utility function[1], the decision maker can find the best strategy that maximizes his/her expected utility of terminal wealth. A natural question that may arise: How good is the optimal strategy in comparison to suboptimal ones? To answer this question, one could look at the difference of the corresponding expected utilities. However, this distance is not stable with respect to positive affine transformations of the utility function and is less economically informative than a comparison in terms of units of wealth, e.g., see Section 5.4 in Munk (2017). Therefore, we will use the concept of a wealth-equivalent utility loss to compare the optimal strategy with suboptimal ones to better understand the monetary benefit of the optimal behavior of the decision maker.

Definition 2.3.5 (Wealth-equivalent utility loss, Eq. (14), p. 271, Larsen and Munk (2012))**.** *Let π^* be the optimal strategy and π_S be an admissible **s**uboptimal strategy for some portfolio optimization problem. Then the wealth-equivalent utility loss (WEUL) is a number that is denoted by $WEUL(\pi^*, \pi_S)$ and solves the following equation:*

$$\mathbb{E}\left[U\left(V^{v\cdot(1-WEUL(\pi^*,\pi_S)),\pi^*}(T)\right)\right) = \mathbb{E}\left[U(V^{v,\pi_S}(T)\right]. \tag{2.40}$$

WEUL represents the proportion of the initial wealth "lost" when a suboptimal strategy π_S is followed instead of the optimal strategy π^*. The larger this number, the more

[1]The determination of the utility function that best characterizes the risk preferences of a decision maker is a difficult question that is beyond the scope of this dissertation.

money the decision maker could "save" if he/she switches from π_S to π^* while maintaining the same as before expected utility level. In Chapter 4 we will calculate the monetary benefit to the insurer if the insurer follows the optimal investment-reinsurance strategy instead of implementing a suboptimal one. In Chapter 5 we will generalize WEUL to a wealth-equivalent utility change (WEUC) that allows for comparing any two admissible strategies.

2.4 Portfolio optimization techniques

In this section, first, we describe a general portfolio optimization problem as a building block of the problems solved in this thesis. Second, we provide an overview of the portfolio optimization techniques that are used to solve those problems. Since the optimization concepts are application-specific, we refrain from covering them in detail here, but do that in the corresponding chapters.

The building block of portfolio optimization problems in this thesis is the following basic problem:

$$\begin{aligned} \max_{\pi} \quad & \mathbb{E}\Big[U(V^{\pi}(T))\Big] \\ \text{s.t.} \quad & \pi \in \Big\{\pi \in \mathcal{A}_c^{\pi}\left(v, C_V, C_{\pi}\right) \;\Big|\; \mathbb{E}\Big[\left(U(V^{\pi}(T))\right)^{-}\Big] < +\infty\Big\}, \end{aligned} \tag{BOP}$$

where π is the control process, $V^{\pi}(T)$ is the controlled process, C_V characterizes the terminal-wealth constraint, C_{π} characterizes the allocation constraint. The additional condition regarding the finiteness of the negative part of the expected utility is needed to ensure that the expectation exists and, thus, the utility-maximization problem is well-defined. This condition is redundant, if the utility function is non-negative. We denote the value function associated with Problem (BOP) as follows:

$$\begin{aligned} \mathcal{V}(t, v) = \max_{\pi}\Big\{\mathbb{E}\left[U(V^{\pi}(T))\right] \;\Big|\; & \pi \in \mathcal{A}_c^{\pi}\left(t, v, C_V, C_{\pi}\right), \\ & \mathbb{E}\Big[\left(U(V^{\pi}(T))\right)^{-}\Big] < +\infty\Big\}. \end{aligned} \tag{2.41}$$

In general, the process of solving (BOP) depends on the types of constraints C_V, C_{π}, availability of continuously traded options, illiquid assets, etc. It often requires a combination of various techniques. Therefore, we do not provide all the details on various portfolio optimization techniques in this section. Instead, we aim at conveying the intuition behind the two core approaches to optimizing portfolios and behind the additional relevant application-specific portfolio optimization techniques. In the following chapters, we will provide application-specific details on solving portfolio optimization problems with those methods.

2.4.1 Martingale approach

The martingale approach (MA) has been originally developed for solving expected-utility maximization problems in complete financial markets without constraints on terminal wealth or allocation. This approach is based on the theory of martingales and stochastic integration. The main idea is to separate the derivation of the optimal terminal wealth and the determination of the corresponding optimal investment strategy leading to this terminal wealth. The former object is found by solving a static optimization problem, whereas the latter object is derived by solving a representation problem. For further details we refer readers to Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), Korn (1997).

The martingale method has been extended in various directions: non-concave utility functions, constraints on allocation, constraints on terminal wealth, continuously trading options, availability of illiquid assets in the market, etc.

To maximize the expectation of a non-concave utility function, Carpenter (2000) considers a modified problem where the original utility is replaced by its so-called concave envelope, which is the smallest concave function that is equal to or larger than the original utility. Under some technical conditions the solution to the modified problem and the solution to the original problem coincide. The modified problem can be solved via the classical MA with a few technical adjustments due to the presence of linear pieces in the concave envelope. For more information, see Chapter 3 and references therein.

When a portfolio optimization problem has terminal-wealth constraints, in the first step of the MA one derives the optimal terminal wealth by solving the constrained static optimization problem via the Lagrange multipliers methodology or the Karush-Kuhn-Tucker methodology, which we cover in the next section. Once the optimal terminal wealth is found, the representation problem is solved. We provide further details and relevant references in Chapter 4.

To tackle problems in the incomplete-market setting due to allocation constraints, Cvitanic and Karatzas (1992) developed a methodology that links the allocation-constrained problem in the original financial market with the allocation-unconstrained problems in auxiliary financial markets. The researchers show that under specific technical conditions the solution to the original problem coincides with the solution to the problem in the optimal auxiliary market. Once the optimal auxiliary market is determined, the corresponding unconstrained portfolio optimization problem can be solved by the standard MA. For more information, see Chapter 4 and references therein.

For decision makers who optimize portfolios of European options in a complete market, Korn and Trautmann (1999) derive the optimal investment strategies by combining the replication approach to option pricing and the martingale approach. As in the classical MA, the first step is finding the optimal terminal wealth by solving a static optimization problem. The second step is calculating the replicating strategy of the optimal terminal wealth in terms of cash and stocks by solving the corresponding representation problem.

The third step is transforming the previously found replicating strategy to the one in terms of cash and options. This is possible due to market completeness. For further details, see Chapter 5 and references therein.

To solve portfolio optimization problems with a fixed-term (illiquid) investment, Desmettre and Seifried (2016) developed a generalized martingale approach. In contrast to the classical MA, the process of finding the optimal terminal wealth is more complex. It has two steps. First, the optimal terminal wealth is found for an arbitrary but fixed position in the fixed-term asset. For that, the expected utility function conditioned on the illiquid investment is maximized. Second, the optimal position in the fixed-term asset is computed by maximizing the corresponding value function. Finally, the investment strategy leading to the previously found optimal terminal wealth is determined. See Chapter 5 and references therein for details.

2.4.2 Stochastic control approach

The stochastic control approach (SCA) to portfolio optimization, also known as Merton's approach or the dynamic programming approach, was introduced in Merton (1969) and Merton (1971). The main idea of this method is to consider the SDE of the wealth process of a decision maker as a controlled diffusion process and to find the optimal control by solving a so-called Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE). This HJB PDE comes from applying the Bellman's principle of optimality to the value function of the portfolio optimization problem. For further readings we refer to Bellman (1957), Merton (1992), and Korn (1997).

In contrast to MA, the stochastic control method can be directly applied to portfolio optimization problems in incomplete markets. However, the inclusion of constraints on allocation or on wealth poses significant challenges to the solution process. To the best of our knowledge, very few papers analytically derive solutions to allocation-constrained portfolio optimization problems via SCA, see, e.g., Pham (2002), Mnif (2007). Moreover, even fewer papers generalize SCA to solving wealth-constrained portfolio-optimization problems. Kraft and Steffensen (2013) do it for a complete Black-Scholes market. In particular, they demonstrate via SCA that for various terminal-wealth constraints the optimal constrained terminal wealth can be represented as a financial derivative on the optimal unconstrained terminal wealth. Thus, solving the terminal-wealth constrained problem requires constructing the proper financial derivative for the constraint of interest. In Chapter 6 we extend this methodology to incomplete markets due to stochastic volatility and provide further details as well as references.

2.5 Selected mathematical tools

In this final section of Chapter 2, we provide several helpful results from calculus, probability theory and non-linear optimization. First, we state the Leibniz integral rule (LIR) and several useful results from calculus. Second, we state a dominated convergence theorem. Third, we present the method of Lagrange multipliers and the method of Karush-Kuhn-Tucker.

2.5.1 Calculus

Theorem 2.5.1 (Leibniz integral rule, Theorem 3, Chapter 8, p. 425, Protter and Morrey (1985))**.** *Let $g(\alpha, x), \partial g(\alpha, x)/\partial\alpha$ be continuous functions and $l(\alpha), m(\alpha)$ be continuously differentiable functions. Then it holds:*

$$\frac{\partial}{\partial\alpha}\int\limits_{l(\alpha)}^{m(\alpha)} g(\alpha, x)\,dx = g(\alpha, m(\alpha))m'(\alpha) - g(\alpha, l(\alpha))l'(\alpha) + \int\limits_{l(\alpha)}^{m(\alpha)} \left(\frac{\partial}{\partial\alpha} g(\alpha, x)\right) dx. \quad \text{(LIR)}$$

Theorem 2.5.2 (Mean-value theorem, Th. 4.12, p. 71, Protter (1998))**.** *Suppose that $f(\cdot)$ is continuous on a closed interval $[a, b] \subset \mathbb{R}$ and has a derivative $\forall\, x \in (a, b)$. Then $\exists c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 2.5.3 (Weierstrass' theorem, Cor. 2.35, p. 40, Aliprantis and Border (2006))**.** *Let $f(\cdot)$ be a continuous real-valued function defined on a closed and bounded set $X \subset \mathbb{R}^n$. Then $f(\cdot)$ achieves its maximum and minimum values, i.e., $\exists c \in X,\ d \in X$ such that*

$$f(c) \le f(x) \le f(d) \qquad \forall x \in X.$$

Theorem 2.5.4 (Berge's maximum theorem, Th. 17.31, p. 570, Aliprantis and Border (2006))**.** *Let $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ be intervals. Let $c : X \to \mathbb{R}$ and $f : Y \times X \to \mathbb{R}$ be continuous functions. Define the function $m : X \to \mathbb{R}$ by*

$$m(x) = \max_{y\in[0,c(x)]} f(y, x).$$

If $m(x)$ is a function, i.e., the maximum is uniquely attained, then the following function is continuous:

$$a(x) = \operatorname*{argmax}_{y\in[0,c(x)]} f(y, x).$$

Remark to Theorem 2.5.4. In Aliprantis and Border (2006), the Berge's maximum theorem is formulated and proven in a more general version for correspondences ("multi-valued functions"). In their formulation the correspondence $a(x)$ is upper hemicontinuous, which is an extension of the concept of continuity from functions to correspondences. By Lemma 17.6 on page 559 in Aliprantis and Border (2006), a singleton-valued correspondence that is upper hemicontinuous is a continuous function.

2.5.2 Probability theory

Theorem 2.5.5 (Dominated convergence theorem, Th. 1.6.7, p. 26, Durrett (2019))**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables such that $X_n \to X$ $\mathbb{Q}$-almost surely, i.e.,*

$$\mathbb{Q}\left(\left\{\omega \in \Omega : \lim_{n\to+\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

If there exist a random variable Y such that $|X_n| \leq Y \, \forall n \in \mathbb{N}$ and $\mathbb{E}[Y] < +\infty$, then $\lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.

2.5.3 Non-linear optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We denote by

$$\nabla f(x) := (\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n)^\top \in \mathbb{R}^n$$

the gradient of $f(\cdot)$. For differentiable $g : \mathbb{R}^n \to \mathbb{R}^m$ we use the following notation:

$$\nabla g(x) := (\nabla g_1(x), \ldots, \nabla g_m(x)) \in \mathbb{R}^{n\times m}.$$

In the remainder of this section, when we write that a vector is equal to zero, it should be understood component-wise, i.e., each component of the vector equals zero.

Now we introduce the method of Lagrange multipliers that is a powerful tool for static optimization with equality constraints. It will be a helpful tool for dynamic portfolio optimization in stochastic volatility environment under terminal-wealth constraints.

Theorem 2.5.6 (Method of Lagrange multipliers. Theorem 1.13, p. 285, Fuente (2000))**.** *Let x^* be the optimal solution of a static optimization problem with equality constraints:*

$$\max_{x\in\mathbb{R}^n} f(x) \qquad s.t. \qquad h(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable functions, and rank $\nabla h(x^*) = l \leq n$. Then there exist unique Langrange multipliers $\lambda^* \in \mathbb{R}^l$ such that

$$\nabla f(x^*) + \nabla h(x^*)\lambda^* = 0. \tag{2.42}$$

In the closing part of this section we provide an important result from convex optimization. It plays a crucial role in static optimization problems with inequality constraints. Consider a convex optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \qquad g(x) \leq 0, \quad h(x) = 0. \tag{COP}$$

with continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^l$, such $f(\cdot)$, $g_i(\cdot)$, $i \in \{1, \ldots, m\}$, are convex and $h(\cdot)$ is affine linear.

Definition 2.5.7 (Feasible region, Definition 15.1, p. 89, Ulbrich and Ulbrich (2012))**.** *The set* $C_x = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$ *is called the feasible region of* (COP).

Definition 2.5.8 (Slater's condition, p. 110, Ulbrich and Ulbrich (2012))**.** *Consider a convex optimization problem (COP). It is said that Slater's condition is satisfied if there exists* $y \in \mathbb{R}^n$ *such that* $g_i(y) < 0$ *for all* $i \in \{1, \ldots, m\}$ *and* $h(y) = 0$.

Slater's condition is a so-called constraint qualification that guarantees that the Karush-Kuhn-Tucker (KKT) conditions stated in the next theorem are necessary and sufficient optimality conditions for convex problems.

Theorem 2.5.9 (KKT conditions for convex problems, Theorem 16.26, p. 101, Ulbrich and Ulbrich (2012))**.** *Let the optimization problem (COP) be convex and satisfy Slater's condition. Then:*

1. *Each local solution of (COP) is also a global solution.*
2. *If* $\bar{x} \in C_x$ *is a global solution of (COP), then the KKT conditions hold at* $\bar{x}$*, i.e., there exist* $\bar{\lambda} \in \mathbb{R}^m$ *and* $\bar{\mu} \in \mathbb{R}^l$ *such that:*
    a) $\nabla f(\bar{x}) + \nabla g(\bar{x})\bar{\lambda} + \nabla h(\bar{x})\bar{\mu} = 0$;
    b) $h(\bar{x}) = 0$;
    c) $\bar{\lambda} \geq 0$, $g(\bar{x}) \leq 0$, $\bar{\lambda}^\top g(\bar{x}) = 0$.
3. *If the KKT conditions hold at* $\bar{x}$*, then* $\bar{x}$ *is a global solution of (COP).*

3 Risk sharing between a hedge-fund manager and an investor

> Being good is easy, what is difficult is being just.
>
> Victor Hugo

In this chapter, we focus on risk sharing in hedge funds. Here risk sharing means sharing the loss of a fund's value between the investor and the fund's manager. We start this chapter, which is a reproduction of Escobar-Anel et al. (2020) with minor changes, by explaining what a hedge fund is and motivating our research thereafter.

A hedge fund is an asset-managing company that oversees pooled investment vehicles and whose clients are solely accredited investors. It faces less regulation than pension funds and mutual funds. Its investment strategies usually exploit rare opportunities and the key determinant of its performance are the skills of the hedge fund managers. Traditionally, hedge funds' managers charged for their service a fixed management fee and a variable performance fee. These are often referred to as a traditional fee structure or a traditional compensation scheme. The size of these fees frequently follows the so-called "2 & 20" rule, i.e., 2% of the assets under management (AUM) was the management fee and 20% of the fund's profit above a set benchmark was the performance fee.

In 2010s, the hedge fund sector has faced a lot of criticism of its high fees and lukewarm performance. According to the Financial Times[1], investors pulled billions of dollars from hedge funds in 2016 and resorted to passive or private equity strategies. As the investors became more dissatisfied, some hedge funds introduced a risk-sharing component to their fee structures, namely a so-called first-loss coverage guarantee, which is a promise by the manager to cover the investor's potential loss up to a specific percentage of the investor's endowment. To be concise, we call a traditional fee structure with a first-loss coverage guarantee a first-loss fee structure or a first-loss scheme.

Let us consider an example of a first-loss coverage guarantee. If the manager offers a 10% first-loss coverage guarantee and the investor's capital in the hedge fund became 15% smaller at the end of the investment horizon than it was at the beginning of the horizon, then the manager refunds the first 10% of the investor's loss, whereas the

[1]See `https://www.ft.com/content/b8ca99da-9782-11e7-a652-cde3f882dd7b`

investor absorbs the remaining 5% loss. As a compensation for this first-loss coverage guarantee, the manager may charge higher management and/or performance fees, e.g., 3%&40%. Since there is a trade-off between the fees and the first-loss coverage guarantee in the first-loss scheme, it is worth investigating what a "fair" or "well-balanced" first-loss fee structure is.

So in this chapter, we answer the following questions:

1. Which first-loss schemes can be seen as fair and optimal for both parties, i.e., which management fee, performance fee and first-loss coverage guarantee are mutually preferred by both the manager and the investor?
2. What is the impact of the financial market parameters and of the decision makers' risk-aversion coefficients on such a mutually preferred fee arrangement?

To solve the manager's non-concave utility maximization problem, where non-concavity arises due to his/her payoff profile, we combine the martingale approach (Karatzas et al. (1987), Cox and Huang (1989)) and the concavification technique. The latter technique is based on the construction of the concave envelope of a utility function. It dates back to Aumann and Perles (1965). This technique was first used in the context of managerial compensation in Carpenter (2000). Later this result was extended to more general managerial compensation schemes (Larsen (2005), Bichuch and Sturm (2014)). Reichlin (2013) proves the existence and several fundamental properties of the solution to unconstrained portfolio optimization problems in a very general framework. However, the author does not provide any specific payoffs. We take advantage of our specific setting to derive explicitly the manager's optimal terminal wealth without the need for convex conjugates and sub-differentials.

The literature on managerial compensation in hedge funds mainly focuses on the traditional fee structure, in particular on the link between risk taking and performance fees. Almost no papers can be found which search for an "equilibrium" fee structure. Carpenter (2000) analyzes the impact of performance fees on the optimal investment strategy of a manager holding an option on the fund's assets and having preferences modeled by a HARA utility function. The author finds that the performance fee causes the manager to reduce the fund's risk. Kouwenberg and Ziemba (2007) analyze how the performance fee and the manager's own investment in the fund influence the risk aversion of a manager whose preferences are modeled using prospect theory. In contrast to Carpenter (2000) the researchers find that performance fees increase the manager's risk appetite. In their broad empirical study, they find though, that for individual hedge funds there is no significant relation between volatility and performance fees. Hodder and Jackwerth (2007) consider a hedge fund manager with a power utility and a traditional compensation scheme. They come to the conclusion that although the manager's risk-taking may change drastically depending on the fund value within a one-year investment period, this effect is moderate over longer investment horizons. Guasoni and Obłój (2016) study hedge funds with traditional compensation schemes where performance fees are high-water marked. The researchers find that such performance fees increase risk-taking for managers with typical levels of risk aversion. Zou (2017) analyzes traditional fee structures and derives optimal investment strategies of a hedge fund manager with a piecewise exponential utility and an S-shaped utility from the Cumulative Prospect Theory (CPT) framework. The researcher concludes that the manager pursues less risky investment strategies when his/her loss aversion, risk aversion, ownership in the fund or the management fee ratio increases. When the performance fee increases, though, the manager acts riskier. Escobar-Anel et al. (2018) is the only paper that studies the question of how the manager and the investor can agree on a single mutually preferred traditional fee structure. They propose two procedures on how to select a fair traditional fee structure for both parties and conclude that for reasonable market parameters the fee arrangement with 0.5% management and 30.7% performance fee stands out as a fair one. We apply their approach based on Pareto optimality and Sharpe ratio maximization in the context of first-loss schemes. The fee structure $(0.5\%, 30.7\%, 0\%)$ is not Pareto optimal in the presence of first-loss coverage. It is almost twice more volatile and yields to a Sharpe ratio of that fund that is about 20% lower in comparison to the preferred first-loss fee structure.

There are two papers that analyze the novel first-loss fee structure, although from different angles. Djerroud et al. (2016) examine the first-loss scheme in the derivative pricing framework, but do not optimize portfolios. They conduct a cost-benefit analysis of particular fee structures and calculate “fair” performance fees, where the investor has a payoff with present value equal to his/her initial cash injection. However, they do not search for an optimal or an “equilibrium” fee structure. He and Kou (2018) consider a hedge fund with a manager whose capital is invested in the fund, i.e., the manager’s and the investor’s money is commingled. The authors refer to the proportion of the fund that belongs to the manager as the managerial ownership ratio and consider the management fee to be part of it. Working in the CPT framework, they conclude that the first-loss scheme with 30% performance fee and 10% managerial ownership ratio used to cover first loss is better for both the investor and the manager than the fee structure with 20% performance fee and 10% managerial ownership ratio that is not used to cover any potential losses. However, the researchers do not investigate if the suggested first-loss fee structure $(0\%, 30\%, 10\%)$ is Pareto optimal for representative managers and investors. As opposed to He and Kou (2018), we consider the investors’ assets being segregated from the managers’ money and follow the goal of finding Pareto optimal fee structures.

Next we summarize the scientific contributions of this chapter. To the best of our knowledge, we are first to analyze first-loss schemes based on the criterion of Pareto optimality and to investigate how hedge-fund managers and investors can reasonably select a single Pareto optimal first-loss fee structure. Conducting extensive numerical studies for HARA-utility functions with our model’s parameter values consistent with current risk appetites in the hedge-fund sector, we find that the common 2% management and 20% performance fees are not Pareto optimal in the traditional scheme. The “closest” Pareto optimal fee structure in the traditional setting has a 0% management fee and a 20%

performance fee, which may explain the current trend of decreasing management fees in hedge funds that use traditional managerial compensation. Furthermore, the first-loss fee structures typically used by hedge funds (with a performance fee around 40% and a first-loss coverage guarantee around 10%, see Djerroud et al. (2016), He and Kou (2018)) are not Pareto optimal either. The Pareto optimal first-loss fee structure that maximizes the hedge-fund's Sharpe ratio has a management fee of 5%, a performance fee about 35% and a first-loss coverage guarantee around 25%. However, the manager might not agree on this fee structure as his/her expected utility is lower than the one for the $(0\%, 20\%)$ traditional fee structure. Using the same criterion for the first-loss fee structure selection but requiring that both parties are better off when the traditional fee structure is replaced by the first-loss one, the decision makers should agree on a management fee of 5%, a performance fee about 48% and a first-loss coverage guarantee around 24%. The methodology we use yields a preferred Pareto optimal first-loss fee structure that is fair to both parties and decreases significantly the hedge fund's risk in comparison to the traditional fee structure. Also to the best of our knowledge, we are the first to study the trade-off between the parameters of the first-loss scheme in the expected utility framework, which can be of great help in the fee-structure negotiation process.

The majority of theoretical results stated in this chapter were obtained by the author of this thesis in Havrylenko (2018). As part of his doctoral research, the author proved the existence of a mutually preferred fee structure in Proposition 3.2.4 below. Furthermore, the author of this dissertation improved the algorithm for finding the preferred fee (Substep 4 a on page 55 below) and conducted more comprehensive numerical studies. In contrast to Havrylenko (2018), in this chapter the fund's optimal terminal value is compared with the fund's terminal values related to suboptimal investment strategies in terms of Sharpe ratio as well as expected utilities. Moreover, a broader sensitivity analysis is conducted on a larger and more realistic grid of model parameters.

The remainder of the chapter is organized as follows. In Section 3.1 we specify the model of a hedge fund with first-loss compensation. In Subsection 3.1.2 we state the portfolio optimization problem of the hedge-fund manager and the problem of the preferred fee-structure selection. Furthermore, we provide theoretical results necessary for solving the former problem and present a rational methodology for determining a single first-loss scheme that both parties can agree on. Considering the manager and the investor equipped with HARA utility functions, we solve the manager's optimization problem in Subsection 3.2.2 and derive the value function of each party. In Section 3.3 we select reasonable model parameters and conduct numerical studies. The proofs of the main results are provided in Appendix A.1. Appendix A.2 contains auxiliary results and their proofs.

3.1 Problem setting

In this section, we explain how we model a hedge fund and the profit-and-loss sharing between the manager and the investor. Second, we present our approach to determining a mutually preferred fee structure that specifies the profit-and-loss sharing.

3.1.1 Hedge-fund model

Let us consider the basic financial market described in Section 2.1. In this market, we set $n = 1$, since our focus is on profit-and-loss sharing in hedge funds in terms of first-loss fee structures, not the investment strategies of the fund. The hedge-fund model we use is the one introduced in Djerroud et al. (2016). There are two parties – the hedge-fund manager and the investor. At time 0 the investor entrusts his/her initial capital $I(0) > 0$ to the hedge fund, so that the initial value of the fund equals $V(0) = I(0) := v_0$. The hedge-fund manager invests the investor's capital and manages the money until the end of the investment period T. At that time, the fund's terminal value is split between the manager and the investor:

$$V(T) = I(V(T)) + M(V(T)),$$

where $M(V(T))$ is the terminal wealth of the manager and $I(V(T))$ is the terminal wealth of the investor.

The manager's payoff is determined by the compensation scheme, i.e., the fee structure. In the first-loss compensation model, the manager

- *charges* a management fee;
- *charges* a performance fee on the investor's net profit, if it is positive;
- *guarantees* to cover incurred loss up to a certain percentage of the initial capital.

There are two basic types of first-loss arrangements:

1. the investor's assets and the manager's deposit are commingled;
2. the investor's assets and the manager's deposit account are segregated.

The first type is usually better for the manager, as it gives the manager shareholder rights. The second type is preferred by investors, as it removes shareholder rights from managers. There are many other considerations dealing with how the deposit account is securitized, they are usually variants of these two basic ones. We analyze the second case, using the model of the first-loss fee structure from Djerroud et al. (2016).

We assume a single payment at the end of a fixed term T, which also implies that the fees are not invested. Later in the case study, we set $T = 1$, which means that the fees are charged at the end of the investment year. The assumption that the fees are charged once at time T is common in the literature (see e.g. Kouwenberg and Ziemba (2007), He and Kou (2018), Zou (2017), Escobar-Anel et al. (2018)).

We denote by $m \in [0,1]$ the fixed share of v_0 that is charged by the hedge fund manager at time $t = T$. We call both m and mv_0 the management fee. It will be clear from the context which term is meant.

In case the wealth generated for the investor is greater than the initial endowment, i.e., $V(T) - mv_0 > v_0$, the manager charges the share $\alpha \in (0,1]$ of the capital surplus[2]. We refer to both α and $\alpha(V(T) - mv_0 - v_0)^+$ as the performance fee. It will be clear from the context which element is meant.

We denote by $c \in [0,1]$ the maximal share of v_0 paid by the manager to the investor if the latter faces a loss at time T. The loss occurs whenever the terminal portfolio value less the management fee is lower than the investor's initial investment. We refer to both c and cv_0 as the manager's first-loss coverage guarantee.

Then the investor's terminal wealth in the first-loss scheme is given by:

$$I(V(T)) = \begin{cases} V(T) + v_0(c - m), & \text{if } V(T) - mv_0 < (1-c)v_0; \\ v_0, & \text{if } (1-c)v_0 \leq V(T) - mv_0 < v_0; \\ V(T) - mv_0 - \alpha(V(T) - (1+m)v_0), & \text{if } V(T) - mv_0 \geq v_0. \end{cases} \tag{3.1}$$

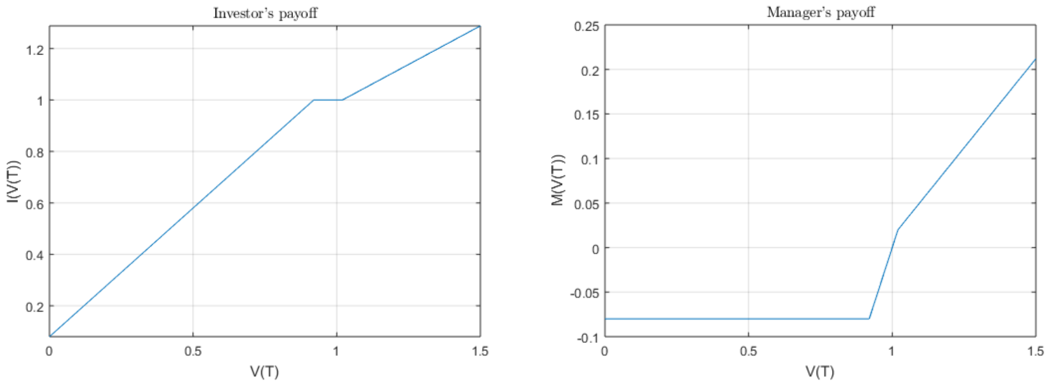
Using (3.1) and the relation $M(V(T)) = V(T) - I(V(T))$, we obtain the following terminal payoff of the manager:

$$M(V(T)) = \begin{cases} v_0(m - c), & \text{if } V(T) - mv_0 < (1-c)v_0; \\ V(T) - v_0, & \text{if } (1-c)v_0 \leq V(T) - mv_0 < v_0; \\ mv_0 + \alpha(V(T) - (1+m)v_0), & \text{if } V(T) - mv_0 \geq v_0. \end{cases} \tag{3.2}$$

When we need to emphasize the dependence of the parties' terminal payoffs on the fee-structure parameters, we write $M(V(T)|m, \alpha, c)$ and $I(V(T)|m, \alpha, c)$. Since we do not want to overcomplicate the model and focus on the fair fee selection process, we do not model the investments of the manager and of the investor outside of the hedge fund. From (3.1) and (3.2) we see that the terminal wealth of each party is a continuous non-decreasing piecewise linear function of the fund's terminal value. Figure 3.1 illustrates the payoffs of both parties.

Note that the terminal wealth of the investor and that of the manager can attain negative values. For example, the manager's payoff may be negative when the fund's terminal

[2] We exclude $\alpha = 0$ for several reasons. First, the performance fee is a distinct feature of the hedge fund industry. Second, $\alpha = 0$ requires special treatment in the derivation of the fund's optimal terminal value, as in this case the manager's utility function is flat for $v \geq (1+m)v_0$. This would make the chapter longer without contributing much to our aims. Third, we need $\alpha > 0$ for proving the existence of first-best Pareto optimal fee structures, see Proposition 3.2.4 in Subsection 3.2.1



(a) Investor's payoff. (b) Manager's payoff.

Figure 3.1: Parties' payoffs for $m = 2\%, \alpha = 40\%, c = 10\%$.

value is sufficiently small and the first-loss guarantee offered by the manager is greater than the charged management fee. The investor's terminal wealth may be negative when the fund's terminal value is sufficiently small and the provided first-loss coverage is less than the management fee the investor paid.

Theoretically, each parameter of the first-loss fee structure can attain values between 0% and 100%. A few hedge funds have experimented with negative management fees to attract clients. In Section 3.3, where we present the results of our numerical studies, we consider $\mathcal{P} = \{(m, \alpha, c) : m \in [0\%, 5\%], \alpha \in [0.1\%, 50\%], c \in [0\%, 30\%]\}$, which is motivated by practical considerations described later. We refer to this set as the set of admissible fee structures.

3.1.2 Portfolio optimization and fee-structure preferences

We assume that the manager's and the investor's preferences are described by utility functions $U_M(\cdot)$ and $U_I(\cdot)$ respectively in the sense of Defintion 2.3.1. Since the manager's minimal terminal payoff equals $(m - c)v_0$ (for all $V(T) \in [0, (1 + m - c)v_0)$), we impose the condition $U_M((m - c)v_0) > -\infty$. Since the investor's minimal terminal payoff is $(c - m)v_0$, we impose the condition $U_I((c - m)v_0) > -\infty$. For instance, these inequalities are fulfilled for exponential utility functions or HARA-utility functions with suitably chosen parameters.

We denote the parties' utility functions as functions of the fund's terminal value by

$$\widetilde{U}_M(V(T)) := U_M(M(V(T))), \qquad \widetilde{U}_I(V(T)) := U_I(I(V(T))).$$

In general, the manager solves the following portfolio optimization problem after fixing his/her first-loss fee structure:

$$\max_{\pi} \mathbb{E}\left[\widetilde{U}_M(V(T))\right];$$
$$\text{s.t.} \quad \pi \in \left\{\pi \in \mathcal{A}_u^{\pi}(v_0) \;\middle|\; \mathbb{E}\left[\left(\widetilde{U}_M(V(T))\right)^{-}\right] < +\infty\right\},$$

where $\mathcal{A}_u^{\pi}(v_0)$ is defined in Definition 2.1.4.

As we focus on the fee-structure selection instead of the manager's optimal investment strategy, we consider only the related terminal portfolio value problem, which is also called a static optimization problem in the martingale approach. This is possible due to the completeness of the considered financial market (see Theorem 2.1.5).

The manager's static optimization problem for finding his/her optimal terminal wealth is given by:

$$\begin{aligned} &\max_{V(T)} \mathbb{E}[\widetilde{U}_M(V(T))]; \\ &\text{s.t.}\, \mathbb{E}[\widetilde{Z}(T)V(T)] \leq v_0; \\ &\quad V(T) \geq 0. \end{aligned} \tag{P_M}$$

In accordance with Chapter 2, we denote by $V^*(T)$ the solution to (P_M) and refer to it as the fund's optimal terminal value. Note that the only influence the investor has on this value is the (first-loss) fee structure the parties negotiate at time 0. However, there are infinitely many admissible fees. The crucial question is how the manager and the investor can agree rationally on a single mutually preferred fee, such that the interests of both parties are taken into account.

To emphasize the dependence of the fund's optimal terminal value on the fee-structure parameters, we adapt the notation $V^*(T|m, \alpha, c)$. Obviously, various fee structures lead to different optimal terminal values of the fund and, hence, different payoffs to the parties'. Therefore, the way how the manager and the investor agree on the fee structure is of high importance. In our view, it should lead to a fee structure that does not favor one party over the other and has a positive impact on the fund's overall performance.

We denote the parties' expected utilities of the terminal payoffs for $V^*(T)$ by

$$\widetilde{\mathcal{V}}_M(m, \alpha, c) := \mathbb{E}\left[\widetilde{U}_M(V^*(T|m, \alpha, c)\right] \text{ and } \widetilde{\mathcal{V}}_I(m, \alpha, c) := \mathbb{E}\left[\widetilde{U}_I(V^*(T|m, \alpha, c)\right],$$

and refer to them as value functions.

We examine optimal first-loss fee structures $(m, \alpha, c) \in \mathcal{P}$ in an analogous way Filipović et al. (2015) investigated optimal investment and premium policies.

Definition 3.1.1 (First-best Pareto optimal fee structures)**.** *A fee structure* (m^*, α^*, c^*) *is first-best Pareto optimal (FBPO) if it solves the optimization problem*

$$\begin{aligned} \max_{m,\alpha,c} \, & \widetilde{\mathcal{V}}_I(m,\alpha,c); \\ s.t. \, & \widetilde{\mathcal{V}}_M(m,\alpha,c) \geq \widetilde{\mathcal{V}}_M^{RUL}; \\ & (m,\alpha,c) \in \mathcal{P}; \end{aligned} \tag{$P_{I|M}$}$$

for some reservation utility level $\widetilde{\mathcal{V}}_M^{RUL} \in \mathbb{R}$ *of the manager. When we need to emphasize the dependence of this problem on the parameter* $\widetilde{\mathcal{V}}_M^{RUL}$*, we write* $P_{I|M}(\widetilde{\mathcal{V}}_M^{RUL})$*.*

We denote the set of all first-best Pareto optimal fee structures by $\mathcal{P}_{FBPO}$ and refer to the set of all pairs $(\widetilde{\mathcal{V}}_M(m^*, \alpha^*, c^*), \widetilde{\mathcal{V}}_I(m^*, \alpha^*, c^*))$ such that $(m^*, \alpha^*, c^*) \in \mathcal{P}_{FBPO}$ as the Pareto frontier.

There are many ways how the manager and the investor can agree on a single FBPO fee structure. We focus on the hedge-fund's Sharpe ratio maximization as the criterion in the fee selection process. The Sharpe ratio is defined as follows:

$$SR^*(m,\alpha,c) := SR(V^*(T|m,\alpha,c)) = \frac{\mathbb{E}\left[R(V^*(T))\right] - r}{\sqrt{\mathbb{V}\text{ar}\left(R(V^*(T))\right)}} = \frac{\mathbb{E}\left[V^*(T)\right] - v_0(1+r)}{\sqrt{\mathbb{V}\text{ar}\left(V^*(T)\right)}},$$

where $R(V^*(T))$ denotes the rate of return of the hedge fund under the optimal investment strategy:

$$R(V^*(T)) = \frac{V^*(T) - v_0}{v_0}.$$

We use this criterion for several reasons: the Sharpe ratio is a popular performance measure that can be observed in the hedge-fund sector, it is not based on utility functions and does not favor exclusively any of the parties. So we consider the fee structure selection as the process of solving the following optimization problem:

$$\begin{aligned} \max_{m,\alpha,c} \; & SR^*(m,\alpha,c); \\ \text{s.t.} \; & (m,\alpha,c) \in \mathcal{P}_{FBPO}. \end{aligned} \tag{P_{SR}}$$

We denote the solution of Problem (P_{SR}) as $(\hat{m}, \hat{\alpha}, \hat{c})$ and call it the preferred fee structure.

3.2 Solution approach

In the first part of this section, we describe theoretical and numerical approaches to finding the preferred fee structure in general. In the second part of this section, we provide specific formulas for the case when decision makers have HARA-utility functions. All proofs can be found in the appendix.

3.2.1 General setting

Using (3.2), we can write the manager's utility function in the following way:

$$\begin{aligned}\widetilde{U}_M(V(T)) :=& U_M(M(V(T))) = \widetilde{U}_{M,1}(V(T))\mathbb{1}_{[0,\widetilde{\mathcal{X}}_1)}(V(T)) \\ &+ \widetilde{U}_{M,2}(V(T))\mathbb{1}_{[\widetilde{\mathcal{X}}_1,\widetilde{\mathcal{X}}_2)}(V(T)) + \widetilde{U}_{M,3}(V(T))\mathbb{1}_{[\widetilde{\mathcal{X}}_2,+\infty)}(V(T)),\end{aligned} \tag{3.3}$$

where $\widetilde{\mathcal{X}}_1 = (1+m-c)v_0, \widetilde{\mathcal{X}}_2 = (1+m)v_0$, $\widetilde{U}_{M,1}(v) = U_M((m-c)v_0))$, $\widetilde{U}_{M,2}(v) = U_M(V(T)-v_0)$, $\widetilde{U}_{M,3}(v) = U_M(mv_0 + \alpha(V(T) - mv_0 - v_0))$.

We denote $\widetilde{U}'_M(\bar{v}-)$ and $\widetilde{U}'_M(\bar{v}+)$ the left- and right-hand derivatives of $\widetilde{U}_M$ at $\bar{v} \in \mathbb{R}$, and $\widetilde{U}_M(\bar{v}-)$ and $\widetilde{U}_M(\bar{v}+)$ for left- and right-hand limits of $\widetilde{U}_M$ at $\bar{v} \in \mathbb{R}$. To avoid any ambiguity, we set $\widetilde{U}_M(v) = -\infty$ for $v < 0$, $\widetilde{U}_M(0) = \widetilde{U}_M(0+)$.

Obviously, $\widetilde{U}_M$ is not concave, whence standard optimization tools cannot be applied to solve Problem (P_M). Therefore, we use the concavification technique (Carpenter (2000), Reichlin (2013), He and Kou (2018)). First, we construct the concave envelope of $\widetilde{U}_M$ and solve the resulting concavified problem defined below. Second, we show that the corresponding fund's optimal terminal value is also the solution of Problem (P_M).

There are several ways the concave envelope of a function may be defined. We follow Reichlin (2013).

Definition 3.2.1. *The concave envelope $\tilde{u}_M(\cdot)$ of $\widetilde{U}_M(\cdot)$ is the smallest concave function $\tilde{u}_M : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $\widetilde{U}_M(v) \le \tilde{u}_M(v)$ for all $v \in \mathbb{R}$.*

The next lemma shows in a constructive way that the concave envelope of $\widetilde{U}_M$ exists and is unique.

Lemma 3.2.2. *Let $\widetilde{U}_M(\cdot)$ be defined according to (3.3). Assume that $\widetilde{U}''_{M,2}(\cdot)$ and $\widetilde{U}''_{M,3}(\cdot)$ exist on $(\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$ and $(\widetilde{\mathcal{X}}_2, +\infty)$ respectively. Then there exists a unique $\tilde{\chi}_1 \ge \widetilde{\mathcal{X}}_1$ such that the function*

$$\tilde{u}_M(v) = \begin{cases} -\infty, & \text{if } v < 0; \\ \widetilde{U}_M(0) + s(\tilde{\chi}_1)v, & \text{if } v \in [0, \tilde{\chi}_1); \\ \widetilde{U}_M(v), & \text{if } v \ge \tilde{\chi}_1. \end{cases} \tag{3.4}$$

is the concave envelope of $\widetilde{U}_M(\cdot)$, where $s(\tilde{\chi}_1) = \frac{\widetilde{U}_M(\tilde{\chi}_1) - \widetilde{U}_M(0)}{\tilde{\chi}_1}$.

Proof. See Appendix A.1. □

For denoting the original function $\widetilde{U}_M(\cdot)$ and its change points $(\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$ we use uppercase letters, whereas we use the same but lowercase letters for the concave envelope $\tilde{u}_M(\cdot)$ and its change points $(\tilde{\chi}_1, \tilde{\chi}_2)$. We resort to such notation, since these functions coincide in some parts but we still need to differentiate between them. Using (3.3) and (3.4), we obtain that the concave envelope is a continuous concave function that be written as:

$$\tilde{u}_M(v) = \begin{cases} -\infty, \text{ if } v < 0; \\ \tilde{u}_{M,1}(v)\mathbb{1}_{[\tilde{\chi}_0,\tilde{\chi}_1)}(v) + \tilde{u}_{M,2}(v)\mathbb{1}_{[\tilde{\chi}_1,\tilde{\chi}_2)}(v) + \tilde{u}_{M,3}(v)\mathbb{1}_{[\tilde{\chi}_2,\tilde{\chi}_3)}(v), \text{ if } v \geq 0, \end{cases} \tag{3.5}$$

where $0 = \tilde{\chi}_0 < \tilde{\chi}_1 \leq \tilde{\chi}_2 < \tilde{\chi}_3 = +\infty$, $\tilde{u}_{M,1}(\cdot)$ is a strictly increasing linear function, $\tilde{u}_{M,i}(\cdot)$, $i \in \{2, 3\}$, are strictly increasing concave functions. In our setting, $\tilde{u}_M(\cdot)$ can have between two or three pieces. According to the proof of Lemma 3.2.2, $\widetilde{\mathcal{X}}_2 \leq \tilde{\chi}_1 = \tilde{\chi}_2$ when $\tilde{u}_M(\cdot)$ consists of two pieces, otherwise $\widetilde{\mathcal{X}}_1 < \tilde{\chi}_1 < \widetilde{\mathcal{X}}_2 = \tilde{\chi}_2$.

Consider now the concavified version of Problem (P_M):

$$\begin{aligned} &\max_{V(T)} \mathbb{E}[\tilde{u}_M(V(T))]; \\ &\text{ s.t. } \mathbb{E}[\widetilde{Z}(T)V(T)] \leq v_0; \\ &\quad V(T) \geq 0. \end{aligned} \tag{P_M^{conc}}$$

In the next theorem, we show how to solve Problem (P_M) via Problem (P_M^{conc}).

Theorem 3.2.3. *Let $v^*(\lambda_v, \tilde{z})$ be the solution to the following pointwise optimization problem for any fixed $\lambda_v > 0, \tilde{z} > 0$:*

$$\max_{v \geq 0} \{\tilde{u}_M(v) - \lambda_v \cdot \tilde{z} \cdot v\}, \tag{3.6}$$

where $\tilde{u}_M(\cdot)$ is defined in (3.5). If the following integrability condition holds

$$h(\lambda_v) := \mathbb{E}\left[\widetilde{Z}(T) \cdot v^*(\lambda_v, \widetilde{Z}(T))\right] < +\infty \ \forall \lambda_v \in (0, +\infty), \tag{3.7}$$

then:

1. *there exists a unique $\lambda_v^* \in (0, +\infty)$ such that $h(\lambda_v^*) = v_0$;*
2. *$V^*(T) = v^*(\lambda_v^*, \widetilde{Z}(T))$ is the $\mathbb{Q}$-a.s. unique optimal terminal value in the concavified Problem (P_M^{conc});*
3. *$V^*(T) = v^*(\lambda_v^*, \widetilde{Z}(T))$ is the $\mathbb{Q}$-a.s. unique optimal terminal value in the original Problem (P_M).*

Proof. See Appendix A.1. □

For many utility functions $\widetilde{U}_M(\cdot)$ common in the literature (e.g., HARA, exponential), the fund's optimal terminal value $V^*(T)$ can be found explicitly. Moreover, $\widetilde{\mathcal{V}}_P(m,\alpha,c) = \mathbb{E}\left[U_P(V^*(T))\right]$ for $P \in \{M, I\}$ can be found in a semi-explicit form. The next step is to obtain the set of the first-best Pareto optimal fee structures by solving the non-linear optimization Problem $(P_{I|M})$. The solution to this optimization problem exists, as proven in the following result.

Proposition 3.2.4. *For any* $\widetilde{\mathcal{V}}_M^{RUL} \in \left[\min_{(m,\alpha,c)\in\mathcal{P}} \widetilde{\mathcal{V}}_M(m,\alpha,c),\ \max_{(m,\alpha,c)\in\mathcal{P}} \widetilde{\mathcal{V}}_M(m,\alpha,c)\right]$ *there exists* (m^*, α^*, c^*) *solving* $(P_{I|M})$.

Proof. See Appendix A.1. □

In Problem $(P_{I|M})$, the objective function $\widetilde{\mathcal{V}}_I(m,\alpha,c)$ and the constraint function $\widetilde{\mathcal{V}}_M(m,\alpha,c)$ are in general quite complex and do not exhibit properties enabling us to solve this problem in closed form. Therefore, we will resort to numerical techniques, in particular, the Sequential Quadratic Programming (SQP) approach. For more information on this optimization technique, the interested reader is referred to Nocedal and Wright (2006).

3.2.2 Explicit solution for HARA-utility functions

In this section, we derive the fund's optimal terminal value as well as the parties' expected utility functions at the fund's optimal terminal value using the methodology from the previous section and assuming that the involved decision makers have HARA-utility functions. Let $U_M(\cdot)$ be given by:

$$U_M(v) = \frac{1}{p_M}\left(v + a_M\right)^{p_M},\ U_M : (-a_M, +\infty) \to \mathbb{R},$$

with $a_M \geq v_0(c-m)$ if $p_M \in (0,1)$ and $a_M > v_0(c-m)$ if $p_M \in (-\infty, 0)$. In this case, we have the following concretization of (3.3):

$$\begin{aligned} \widetilde{U}_M(V(T)) = U_M(M(V(T))) = &\underbrace{\frac{1}{p_M}\left(v_0(m-c) + a_M\right)^{p_M}}_{\widetilde{U}_{M,1}(V(T))} \cdot \mathbb{1}_{[0,(1+m-c)v_0)}(V(T)) \\ &+ \underbrace{\frac{1}{p_M}\left(V(T) - v_0 + a_M\right)^{p_M}}_{\widetilde{U}_{M,2}(V(T))} \cdot \mathbb{1}_{[(1+m-c)v_0,(1+m)v_0)}(V(T)) \\ &+ \underbrace{\frac{1}{p_M}\left(mv_0 + \alpha(V(T) - (1+m)v_0) + a_M\right)^{p_M}}_{\widetilde{U}_{M,3}(V(T))} \cdot \mathbb{1}_{[(1+m)v_0,+\infty)}(V(T)). \end{aligned} \tag{3.8}$$

We apply Lemma 3.2.2 to construct the concave envelope of $\widetilde{U}_M(\cdot)(\cdot)$ and Theorem 3.2.3 to derive the optimal terminal value of the hedge fund.

Corollary 3.2.5 (Fund's optimal terminal value)**.**
Let $\widetilde{U}_M(\cdot)$ be as defined in (3.8). Denote:

$$H = \frac{(mv_0 + a_M)^{p_M} - (v_0(m-c) + a_M)^{p_M}}{p_M(1+m)v_0} \tag{3.9}$$

and

$$\begin{aligned} \mathcal{P}_A &= \Big\{(m, \alpha, c) \in \mathcal{P} : H < \alpha(mv_0 + a_M)^{p_M - 1}\Big\}; \\ \mathcal{P}_B &= \Big\{(m, \alpha, c) \in \mathcal{P} : \alpha(mv_0 + a_M)^{p_M - 1} \leq H \leq (mv_0 + a_M)^{p_M - 1}\Big\}; \\ \mathcal{P}_C &= \Big\{(m, \alpha, c) \in \mathcal{P} : (mv_0 + a_M)^{p_M - 1} < H\Big\}. \end{aligned} \tag{3.10}$$

Then, the fund's optimal terminal value is given by
Case A, $(m, \alpha, c) \in \mathcal{P}_A$:

$$\begin{aligned} V^*(T) = &\left(\alpha^{1/(1-p_M)-1}\left(\lambda_v^* \widetilde{Z}(T)\right)^{-1/(1-p_M)} + (1 + m - \alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \\ &\cdot \mathbb{1}_{\left\{\widetilde{Z}(T) \in \left(0,\, s(\tilde{\chi}_1^A)/\lambda_v^*\right]\right\}}, \end{aligned} \tag{3.11}$$

where $\tilde{\chi}_1^A$ is the unique solution of the following equation w.r.t. v

$$\begin{aligned} (\alpha v + (m - \alpha(1+m))v_0 + a_M)^{p_M - 1}((1-p_M)\alpha v + (m - \alpha(1+m))v_0 + a_M) \\ = (v_0(m-c) + a_M)^{p_M} \end{aligned}$$

on $v \in ((1+m)v_0, +\infty)$, and $s(\tilde{\chi}_1^A) = \alpha(\alpha\tilde{\chi}_1^A + (m - \alpha(1+m))v_0 + a_M)^{p_M - 1}$.
Case B, $(m, \alpha, c) \in \mathcal{P}_B$:

$$\begin{aligned} V^*(T) = &\left(\alpha^{1/(1-p_M)-1}\left(\lambda_v^* \widetilde{Z}(T)\right)^{-1/(1-p_M)} + (1 + m - \alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \\ &\cdot \mathbb{1}_{\left\{\widetilde{Z}(T) \in \left(0,\, \alpha(mv_0 + a_M)^{p_M - 1}/\lambda_v^*\right)\right\}} + (1+m)v_0 \\ &\cdot \mathbb{1}_{\left\{\widetilde{Z}(T) \in \left[\alpha(mv_0 + a_M)^{p_M - 1}/\lambda_v^*,\, s(\tilde{\chi}_1^B)/\lambda_v^*)\right]\right\}}, \end{aligned} \tag{3.12}$$

where $\tilde{\chi}_1^B = (1+m)v_0$ and $s(\tilde{\chi}_1^B) = H$.

Case C, $(m,\alpha,c) \in \mathcal{P}_C$:

$$V^*(T) = \left(\alpha^{1/(1-p_M)-1}\left(\lambda_v^*\widetilde{Z}(T)\right)^{-1/(1-p_M)} + (1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right)$$
$$\cdot \mathbb{1}_{\left\{\widetilde{Z}(T)\in\left(0,\,\alpha(mv_0+a_M)^{p_M-1}/\lambda_v^*\right)\right\}}$$
$$+ (1+m)v_0\mathbb{1}_{\left\{\widetilde{Z}(T)\in\left[\alpha(mv_0+a_M)^{p_M-1}/\lambda_v^*,\,(mv_0+a_M)^{p_M-1}/\lambda_v^*\right)\right]\right\}}$$
$$+ \left(\left(\lambda_v^*\widetilde{Z}(T)\right)^{-1/(1-p_M)} + v_0 - a_M\right)\mathbb{1}_{\left\{\widetilde{Z}(T)\in\left((mv_0+a_M)^{p_M-1}/\lambda_v^*,\,s(\tilde{\chi}_1^C)/\lambda_v^*\right]\right\}}, \tag{3.13}$$

where $\tilde{\chi}_1^C$ is the unique solution of the following equation w.r.t. v

$$(v - v_0 + a_M)^{p_M-1}((1-p_M)v - v_0 + a_M) = (v_0(m-c) + a_M)^{p_M}$$

on $v \in ((1+m-c)v_0, (1+m)v_0)$, and $s(\tilde{\chi}_1^C) = (\tilde{\chi}_1^C - v_0 + a_M)^{p_M-1}$.

In all three cases, $\lambda_v^* > 0$ is the unique solution of the equation $\mathbb{E}\left[\widetilde{Z}(T)V^*(T)\right] = v_0$.

Proof. See Appendix A.1. □

Remark 3.2.6. *Note that $\tilde{\chi}_1^X$, $X \in \{A,B,C\}$, is the rightmost concavification point of $\widetilde{U}_M(\cdot)$ in the corresponding concavification case, i.e., the rightmost point of the linear part of the concave envelope of $\widetilde{U}_M(\cdot)$.*

In the next two propositions we provide the semi-explicit formulas for the parties' value functions. These are needed for computing FBPO fee structures in Problem $(P_{I|M})$.

Proposition 3.2.7 (Manager's value function).
Let the manager's preferences be determined by $\widetilde{U}_M(\cdot)$ as per (3.8). Let λ_v^*, $\tilde{\chi}_1^X$ and $s(\tilde{\chi}_1^X), X \in \{A,B,C\}$, be as defined in Corollary 3.2.5. Define:

$$\xi_1 = \exp\left(p_M(1-p_M)^{-1}\left(r + 0.5\gamma^2\right)T + 0.5p_M^2(1-p_M)^{-2}\gamma^2 T\right).$$

Then the manager's value function $\widetilde{\mathcal{V}}_M(m,\alpha,c)$ is given by
Case A, $(m,\alpha,c) \in \mathcal{P}_A$:

$$\widetilde{\mathcal{V}}_M(m,\alpha,c) = \widetilde{U}_M(0)\Phi\left(d_2^A(\lambda_v^*)\right) + p_M^{-1}(\lambda_v^*)^{-p_M/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1$$
$$\cdot\left(\Phi\left(d_1^A(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right) - \Phi\left(d_2^A(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right)\right),$$

where $d_1^A(\lambda_v^*) = +\infty, d_2^A(\lambda_v^*) = \dfrac{\log\left(\lambda_v^*/s(\tilde{\chi}_1^A)\right) - \left(r+0.5\gamma^2\right)T}{\gamma\sqrt{T}}$;

Case B, $(m, \alpha, c) \in \mathcal{P}_B$*:*

$$\begin{aligned}\widetilde{\mathcal{V}}_M(m,\alpha,c) &= \widetilde{U}_M(0)\Phi\left(d_3^B(\lambda_v^*)\right) + p_M^{-1}(\lambda_v^*)^{-p_M/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1 \\ &\cdot \left(\Phi\left(d_1^B(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right) - \Phi\left(d_2^B(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right)\right) \\ &+ p_M^{-1}(mv_0 + a_M)^{p_M}\left(\Phi\left(d_2^B(\lambda_v^*)\right) - \Phi\left(d_3^B(\lambda_v^*)\right)\right),\end{aligned}$$

$$\begin{aligned}\text{where } d_1^B(\lambda_v^*) = +\infty, \quad d_2^B(\lambda_v^*) &= \frac{\log\left(\lambda_v^*\alpha^{-1}(mv_0 + a_M)^{1-p_M}\right) - \left(r + 0.5\gamma^2\right)T}{\gamma\sqrt{T}}, \\ d_3^B(\lambda_v^*) &= \frac{\log\left(\lambda_v^*/s(\tilde{\chi}_1^B)\right) - \left(r + 0.5\gamma^2\right)T}{\gamma\sqrt{T}};\end{aligned}$$

Case C, $(m, \alpha, c) \in \mathcal{P}_C$*:*

$$\begin{aligned}\widetilde{\mathcal{V}}_M(m,\alpha,c) &= \widetilde{U}_M(0)\Phi\left(d_4^C(\lambda_v^*)\right) + p_M^{-1}(\lambda_v^*)^{-p_M/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1 \\ &\cdot \left(\Phi\left(d_1^C(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right) - \Phi\left(d_2^C(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right)\right) \\ &+ p_M^{-1}(mv_0 + a_M)^{p_M}\left(\Phi\left(d_2^C(\lambda_v^*)\right) - \Phi\left(d_3^C(\lambda_v^*)\right)\right) \\ &+ p_M^{-1}(\lambda_v^*)^{-p_M/(1-p_M)}\xi_1 \\ &\cdot \left(\Phi\left(d_3^C(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right) - \Phi\left(d_4^C(\lambda_v^*) + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right)\right),\end{aligned}$$

$$\begin{aligned}\text{where } d_1^C(\lambda_v^*) = +\infty, \quad d_2^C(\lambda_v^*) &= \frac{\log\left(\lambda_v^*\alpha^{-1}(mv_0 + a_M)^{1-p_M}\right) - \left(r + 0.5\gamma^2\right)T}{\gamma\sqrt{T}}, \\ d_3^C(\lambda_v^*) &= \frac{\log\left(\lambda_v^*(mv_0 + a_M)^{1-p_M}\right) - \left(r + 0.5\gamma^2\right)T}{\gamma\sqrt{T}}, \\ d_4^C(\lambda_v^*) &= \frac{\log\left(\lambda_v^*/s(\tilde{\chi}_1^C)\right) - \left(r + 0.5\gamma^2\right)T}{\gamma\sqrt{T}}.\end{aligned}$$

Proof. See Appendix A.1. □

Let the investor have a HARA-utility function:

$$U_I(v) = \frac{1}{p_I}\left(v + a_I\right)^{p_I}, \ U_I(v) : (-a_I, +\infty) \to \mathbb{R},$$

where $a_I \geq v_0(m-c)$ if $p_I \in (0,1)$ and $a_I > v_0(m-c)$ if $p_I \in (-\infty, 0)$. Such choice of a_I guarantees that the function $\widetilde{U}_I(V(T)) := U_I(I(V(T)))$ is real-valued for any $V(T) \in [0, +\infty)$.

Proposition 3.2.8 (Investor's value function)**.**
Let λ_v^*, $\tilde{\chi}_1^X$ *and* $s(\tilde{\chi}_1^X), X \in \{A, B, C\}$, *be as defined in Corollary 3.2.5. Let the investor's preferences be determined by* $\widetilde{U}_I(\cdot)$ *as defined above. Then the investor's value function is given in*
Case A, $(m, \alpha, c) \in \mathcal{P}_A$*:*

$$\begin{aligned}\widetilde{\mathcal{V}}_I(m,\alpha,c) &= p_I^{-1}\left(v_0(c-m)+a_I\right)^{p_I}\Phi(d_2^A(\lambda_v^*)) \\ &\quad + p_I^{-1}\mathbb{E}\left[\left(k\left(\widetilde{Z}(T)\right)^{-1/(1-p_M)}+l\right)^{p_I}\mathbb{1}_{\left\{\widetilde{Z}(T)\in(0,s(\tilde{\chi}_1)/\lambda_v^*]\right\}}\right];\end{aligned}$$

Case B, $(m, \alpha, c) \in \mathcal{P}_B$*:*

$$\begin{aligned}\widetilde{\mathcal{V}}_I(m,\alpha,c) &= p_I^{-1}\left(v_0(c-m)+a_I\right)^{p_I}\Phi(d_3^B(\lambda_v^*)) \\ &\quad + p_I^{-1}\mathbb{E}\left[\left(k\left(\widetilde{Z}(T)\right)^{-1/(1-p_M)}+l\right)^{p_I}\mathbb{1}_{\left\{\widetilde{Z}(T)\in(0,\alpha(mv_0+a_M)^{p_M-1}/\lambda_v^*)\right\}}\right] \\ &\quad + p_I^{-1}\left(v_0+a_I\right)^{p_I}\left(\Phi\left(d_2^B(\lambda_v^*)\right)-\Phi\left(d_3^B(\lambda_v^*)\right)\right);\end{aligned}$$

Case C, $(m, \alpha, c) \in \mathcal{P}_C$*:*

$$\begin{aligned}\widetilde{\mathcal{V}}_I(m,\alpha,c) &= p_I^{-1}(v_0(c-m)+a_I)^{p_I}\Phi(d_4^C(\lambda_v^*)) \\ &\quad + p_I^{-1}\mathbb{E}\left[\left(k\left(\widetilde{Z}(T)\right)^{-1/(1-p_M)}+l\right)^{p_I}\mathbb{1}_{\left\{\widetilde{Z}(T)\in(0,\alpha(mv_0+a_M)^{p_M-1}/\lambda_v^*)\right\}}\right] \\ &\quad + p_I^{-1}\left(v_0+a_I\right)^{p_I}\left(\Phi\left(d_2^C(\lambda_v^*)\right)-\Phi\left(d_4^C(\lambda_v^*)\right)\right),\end{aligned}$$

where $k = (1-\alpha)\alpha^{1/(1-p_M)-1}(\lambda_v^*)^{-1/(1-p_M)}$, $l = (1+m-\alpha^{-1}m)v_0 + a_M(1-\alpha^{-1}) + a_I$, *and* $d_2^A(\cdot)$, $d_2^B(\cdot)$, $d_3^B(\cdot)$, $d_2^C(\cdot)$, $d_4^C(\cdot)$ *are defined in Proposition 3.2.7.*

Proof. See Appendix A.1. □

In Appendix A.2, we provide several supplementary results related to the fund's optimal terminal value. In Proposition A.2.3 we derive the explicit form of the equations for finding λ_v^*. In Proposition A.2.4 we provide the explicit form of the the first and the second moment of $V^*(T)$. These are needed for computing the hedge-fund's Sharpe ratio in Problem (P_{SR}).

3.3 Numerical studies

3.3.1 Algorithm overview

The selection process for the preferred fee structure is an optimization problem that has several nested optimization subproblems:

$$\begin{aligned} \max_{m,\alpha,c} \quad & SR(V^*(T \mid m,\alpha,c)); \\ \text{s.t.} \quad & V^*(T \mid m,\alpha,c) \text{ solves } (P_M) \\ & \exists \widetilde{\mathcal{V}}_M^{\mathrm{RUL}} \text{ s.t. } (m,\alpha,c) \text{ solves } (P_{I|M}). \end{aligned} \tag{P_{SR}}$$

To solve some subproblems we resort to numerical techniques, as the analytical solution is not available. Here we provide the algorithm we use in the next subsections to compute the solution to Problem (P_{SR}):

1. Initialize:
   a) model parameters $r, \gamma, T, v_0, a_M, p_M, a_I, p_I$;
   b) discretization steps Δm, $\Delta\alpha$, Δc, $\Delta\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$;
   c) set $\mathcal{P}_{FBPO} = \varnothing$.
2. For each $m \in \{0\%, \Delta m, 2\Delta m, \ldots, 5\%\} =: G_m$,
   $\alpha \in \{\Delta\alpha, 2\Delta\alpha, 3\Delta\alpha, \ldots, 50\%\} =: G_\alpha$,
   $c \in \{0\%, \Delta c, 2\Delta c, \ldots, 30\%\} =: G_c$ calculate:
   a) λ_v^* using Proposition A.2.3 from Section A.2 in Appendix A and the bisection method;
   b) $\widetilde{\mathcal{V}}_M(m,\alpha,c)$ using Proposition 3.2.7;
   c) $\widetilde{\mathcal{V}}_I(m,\alpha,c)$ using Proposition 3.2.8.
3. Calculate $\widetilde{\mathcal{V}}_M^{min} = \min_{(m,\alpha,c)\in G}\{\widetilde{\mathcal{V}}_M(m,\alpha,c)\}$, $\widetilde{\mathcal{V}}_M^{max} = \max_{(m,\alpha,c)\in G}\{\widetilde{\mathcal{V}}_M(m,\alpha,c)\}$, where $G := G_m \times G_\alpha \times G_c$.
4. For each $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}} \in \left\{\widetilde{\mathcal{V}}_M^{min}, \widetilde{\mathcal{V}}_M^{min} + \Delta\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}, \widetilde{\mathcal{V}}_M^{min} + 2\Delta\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}, \ldots, \widetilde{\mathcal{V}}_M^{max}\right\}$
   a) find a good initial fee structure (m_0, α_0, c_0) by solving $(P_{I|M})$ on the discrete set G;
   b) calculate $(m^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, \alpha^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, c^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}})$ that solves $(P_{I|M})$ with SQP starting from (m_0, α_0, c_0);
   c) $\mathcal{P}_{FBPO} = \mathcal{P}_{FBPO} \cup (m^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, \alpha^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, c^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}})$.

5. Find $(\hat{m}, \hat{\alpha}, \hat{c}) \in \mathcal{P}_{FBPO}$ that solves (P_{SR}) using Proposition A.2.4 from Section A.2 in Appendix A.

Note: Step 4 (a) speeds up the SQP algorithm and decreases the odds of getting to a local optimum.

3.3.2 Model parametrization

The highest management fee we are aware of equals 5% and was charged at Renaissance Technologies, according to an article published on March 7, 2016, on Investopedia[3]. According to the same source, this hedge fund charged a 44% incentive fee. We find it very improbable that an investor would be willing to pay a management fee of more than 5% of his/her initial endowment or a performance fee greater than 50% of his/her net profit. Therefore, we set the upper bound for m and α to 5% and 50% respectively. Djerroud et al. (2016) examined first-loss coverage guarantees between 1% and 25%. To be a bit more flexible, we consider $c \in [0\%, 30\%]$.

We choose the same values of the financial market parameters as those considered in two papers related to the first-loss scheme. He and Kou (2018) conduct numerical studies for an interest rate of 5%, arguing that such choice is motivated by historical data. However, they also investigate the case $r = 2\%$ due to the low interest-rate environment and conclude that the results for the low interest rate are similar to those obtained for $r = 5\%$. Djerroud et al. (2016) consider $r = 2\%$. To be in line with the mentioned papers and consistent with the current economic conditions in Europe and North America, we set $r = 2\%$. We fix the market price of risk γ at 40%, as it is done in He and Kou (2018). Since the majority of hedge funds charge fees annually, we set $T = 1$. We assume that $v_0 = 1$.

For many years the hedge-fund sector was asking for management fees around 2% and performance fees around 20%. However, due to the investors' concerns that such fees might not be justified by the hedge-funds' performance, the management fees have been decreasing. In May 2018, the average management fee was traded around 1.58%[4], whereas some hedge funds had already canceled their management fees completely. We believe that the hedge-fund sector will increase in transparency and charged fees will approach the optimal ones. In our view, hedge funds that stick to the traditional fee structure will thus gradually end up with a management fee close to 0% and a performance fee around 20%. Therefore, we assume that these values of the traditional fee structure parameters are optimal in the absence of the first-loss coverage guarantee and best represent the risk preferences of an average investor and an average manager of a

[3] https://www.investopedia.com/articles/investing/030716/jim-simons-justifying-5-management-fee.asp

[4] http://docs.preqin.com/newsletters/hf/Preqin-Hedge-Fund-Spotlight-May-2018.pdf

hedge fund. Taking into account the technical conditions

$$a_I \overset{(3.1)}{\geq} \max_{\substack{m \in [0, 0.05] \\ c \in [0, 0.3]}} \{v_0(m-c)\} = 0.05, \qquad a_M \overset{(3.2)}{\geq} \max_{\substack{m \in [0, 0.05] \\ c \in [0, 0.3]}} \{v_0(c-m)\} = 0.3,$$

which guarantee that the utility function evaluated at the minimal terminal wealth of the corresponding party is finite, we find the following parameters for our HARA-utility functions that are consistent with the above mentioned intuition of the traditional fee structure: $a_M = 0.3$, $p_M = 0.35$, $a_I = 0.3$, $p_I = 0.35$. For these parameters the investor's optimal fee structure without any first-loss coverage guarantee is:

$$\underset{\substack{m \in [0\%,\ 5\%] \\ \alpha \in [0\%,\ 50\%]}}{\text{argmax}} \ \widetilde{\mathcal{V}}_I(m, \alpha, 0\%) = (0\%, 20.3\%). \tag{3.14}$$

Note that the investor's value function does not appear in the manager's portfolio optimization problem. $\widetilde{\mathcal{V}}_I(m, \alpha, c)$ comes from Proposition 3.2.8 and equals the investor's expected utility for the terminal portfolio value optimal for the fund manager who solves (P_M) for a certain fee structure $(m, \alpha, c) \in \mathcal{P}$. The obtained risk-aversion parameters are consistent with Holt and Laury (2002). Analyzing power-utility functions, which are a subclass of HARA-utility functions, the authors classify a decision maker as (moderately) risk averse if $p_M(p_I) \in (0.32, 0.59)$. We consider $a_M = a_I = 0.3$ and $p_M = p_I = 0.35$ as base case parameters in our analysis, i.e., the manager and the investor have the same utility function. Later in the study we will vary these parameters for further investigations.

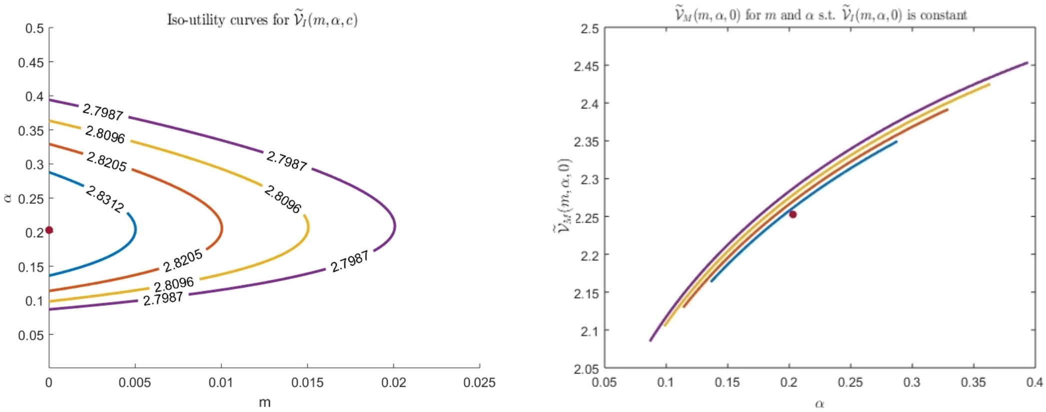
Parameter specifications in the base case are summarized in Table 3.1.

	Market		Investment		Utility functions			
Parameters	r	γ	v_0	T	a_M	p_M	a_I	p_I
Value	2%	40%	1	1	0.3	0.35	0.3	0.35

Table 3.1: Values of the model parameters in the base case.

3.3.3 From traditional to first-loss fee structure

Let us first consider the traditional fee structure. Setting $c = 0\%$ in our model, we plot in Figure 3.2 the investor's indifference curves corresponding to several traditional fee structures and the manager's expected utility along these iso-utility curves of the investor. The lines in Figure 3.2a show all parameter pairs (m, α) that lead to the same expected utility of the investor. The iso-utility values originate from the fees common in the hedge-fund sector: 2% & 20% (violet line), 1.5% & 20% (yellow line), 1% & 20% (red line), 0.5% & 20% (blue line). We use the same coloring structure in Figure 3.2b. The trade-off between the management and the performance fees is also analyzed in Goetzmann et al. (2003). However, the authors work in a valuation framework and consider a hedge fund with two peculiarities – a liquidation boundary and a traditional compensation structure where the performance fee is based on the hedge-fund's high-water mark.



(a) Investor's iso-utility curves in the traditional compensation scheme.

(b) Manager's value function along the investor's iso-utility curves given $c = 0$.

Figure 3.2: Investor's iso-utility curves in the traditional compensation scheme for iso-utility values $\widetilde{\mathcal{V}}_I(2\%, 20\%, 0\%) = 2.7987$, $\widetilde{\mathcal{V}}_I(1.5\%, 20\%, 0\%) = 2.8096$, $\widetilde{\mathcal{V}}_I(1\%, 20\%, 0\%) = 2.8205$, $\widetilde{\mathcal{V}}_I(0.5\%, 20\%, 0\%) = 2.8312$ and the manager's expected utility along them.

In Figure 3.2a, we observe that the lower the management fee, the more the iso-utility curves approach the constrained maximal value of $\widetilde{\mathcal{V}}_I$, which is marked in brown. Note that any change in the fee structure changes $\widetilde{\mathcal{V}}_I$ indirectly: it influences first the manager's optimal portfolio allocation decision, which in turn has an impact on the value function of the investor. Consider, for example, a manager who charged a 2%&20% fee structure and wants to renegotiate the fee structure while keeping the client as happy as before, i.e., the manager should choose a fee structure on the violet line. For performance fees above 20%, we can easily see that the higher the performance fee, the lower the management fee should be to keep the investor's value function constant. For example, an increase of the performance fee from 20% to 30% should be compensated by a decrease of the management fee from 2% to 1.5%, so that the investor's expected utility stays at the same level as it has for the 2% & 20% fee structure. According to Figure 3.2b, a rational manager will not negotiate a lower performance fee while keeping the investor's expected utility level constant, as the manager's expected utility decreases in this case.

In contrast to the investor's value function, the manager's value function is strictly increasing in both m and α. So if an investor insists on decreasing the management fee, then the manager has to increase the performance fee to maintain his/her expected utility constant.

Note that the Pareto optimal fees can be also calculated for the traditional fee structure, namely by solving Problem $(P_{I|M})$ for fixed $c = 0\%$ and different $\widetilde{\mathcal{V}}_M^{\text{RUL}} \in \mathbb{R}$. In the numerical studies done for various combinations of $p_M \in (0,1)$ and $p_I \in (0,1)$, we observed that in the universe of traditional fees each Pareto optimal fee has either $m^* = 0\%$ or $\alpha^* = 50\%$. The fee $(2\%, 20\%)$ is clearly not Pareto optimal and favors the manager more than the investor. This is compatible with the investors' concerns regarding high fees that may not be justified by the hedge-funds' performance and the push for a decrease in management fees given a fixed 20% performance fee.

As already mentioned, in the traditional fee structure the manager always earns at least the management fee, whereas the investor is not provided with any guarantee regarding his/her minimal profit or maximal loss. This asymmetry, along with other reasons such as poor performance of hedge funds in comparison to passive investment vehicles, animated some hedge funds to start offering first-loss coverage guarantees along with management and performance fees. But how much are investors better off under the new fee structure? As market management fees are "converging" to 0%, what is the trade-off between performance fees and the first-loss coverage guarantee levels?

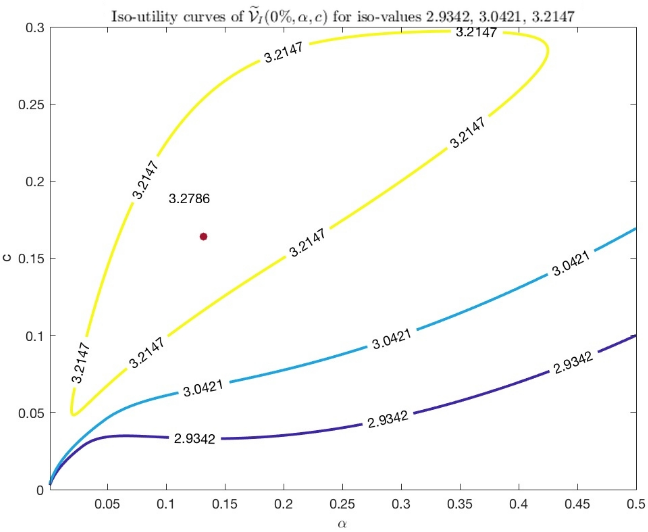


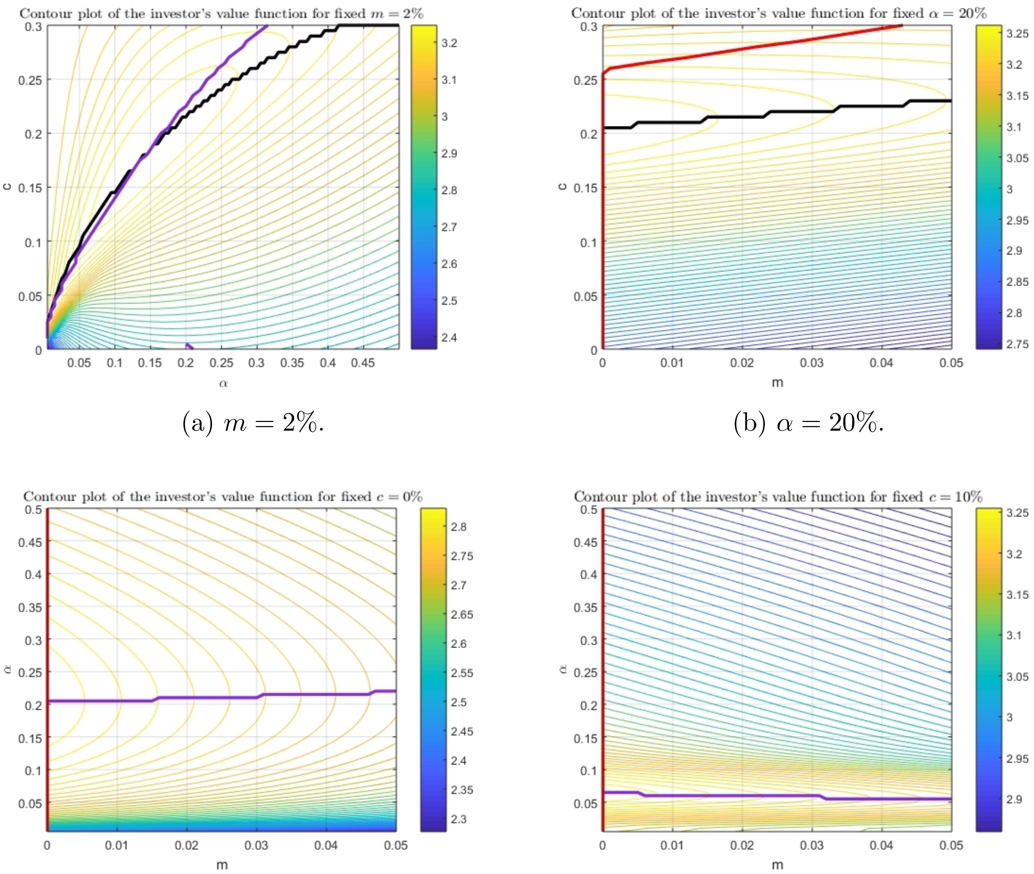
Figure 3.3: Investor's iso-utility curves in the first-loss compensation scheme with $m = 0\%$ for iso-utility values $\widetilde{\mathcal{V}}_I(0\%, 50\%, 10\%) = 2.9342$, $\widetilde{\mathcal{V}}_I(0\%, 30\%, 10\%) = 3.0421$, $\widetilde{\mathcal{V}}_I(0\%, 30\%, 20\%) = 3.2147$.

To get an intuition about answers to these questions, we fix $m = 0\%$ and plot several iso-utility curves of the investor in Figure 3.3. We see that the trade-off between the performance fee and the first-loss coverage guarantee is intuitive – the higher performance fee, the higher the first-loss coverage guarantee should be to maintain the investor's expected utility on the same level. Consider, for example, the violet line in Figure 3.3 which corresponds to a relatively popular first-loss fee structure (0%, 50%, 10%). If the manager decides to decrease the first-loss coverage guarantee from 10% to 5%, he/she should also decrease the performance fee from 50% to 35% in order to maintain the investor's utility level constant. Comparing with Figure 3.2a, we also see that the percentage gain in the investor's value function from switching to this fee from the optimal traditional fee $(0\%, 20\%, 0\%)$ is about 3%. The blue line corresponds to the performance fee 30% and the first-loss coverage guarantee 10%, which are recommended for the first-loss scheme in He and Kou (2018). Under this first-loss fee structure, the investor is further better off, although he/she is still far from his/her maximal expected utility. Further increase in the first-loss coverage guarantee to 20%, while other fee parameters are fixed, results in a significant increase in the investor's value function. The maximum of the investor's value function (marked in brown) is attained at $\alpha = 13.2\%$, $c = 16.4\%$ and equals 3.2786, which is 15% larger than the expected utility for the optimal fee in the traditional setting.

Figure 3.4 illustrates the contour plots of the investor's value function. Each subfigure contains 50 equidistant contours (thin lines). In each subfigure, two thick lines indicate the constrained maximum of the investor's value function w.r.t one of the variables as a function of another variable given that the value of the third variable is fixed as indicated in the legend. These lines are related to the constrained optimal m (red), α (purple) and c (black) respectively. We see in Subfigure 3.4a that the risk-averse investor is willing to pay a performance fee higher than the minimal one. This differs from the findings in He and Kou (2018), who discover that the expected utility of a loss-averse investor is decreasing in the performance fee. If α is low, the manager is not incentivized well enough to generate attractive returns for the hedge fund. Consequently, the investor is worse off. Obviously, high performance fees have also a negative impact on the investor's terminal wealth. All that leads to a moderate optimal α strictly positive and larger than its minimal value.

We also observe in Subfigure 3.4a and in Subfigure 3.4b that the investor would negotiate a moderate first-loss coverage guarantee. Low levels of the first-loss coverage guarantee (or its absence) are good for the manager. For a fixed management fee, the first-loss coverage guarantee is an increasing function of the performance fee and vice versa. See Subfigure 3.4a. However, high levels of the first-loss coverage guarantee motivate the manager to take less risks, as he/she is responsible for potential losses with his/her own money. Lower risks are accompanied by lower returns, which, in turn, decrease the investor's terminal wealth as well.

In contrast to the investor's expected utility, the manager's value function is strictly increasing in m and α, and strictly decreasing in c. So the manager's highest expected utility is attained at $m = 5\%$, $\alpha = 50\%$, $c = 0\%$. To see the trade-off between fee



(a) $m = 2\%$.

(b) $\alpha = 20\%$.

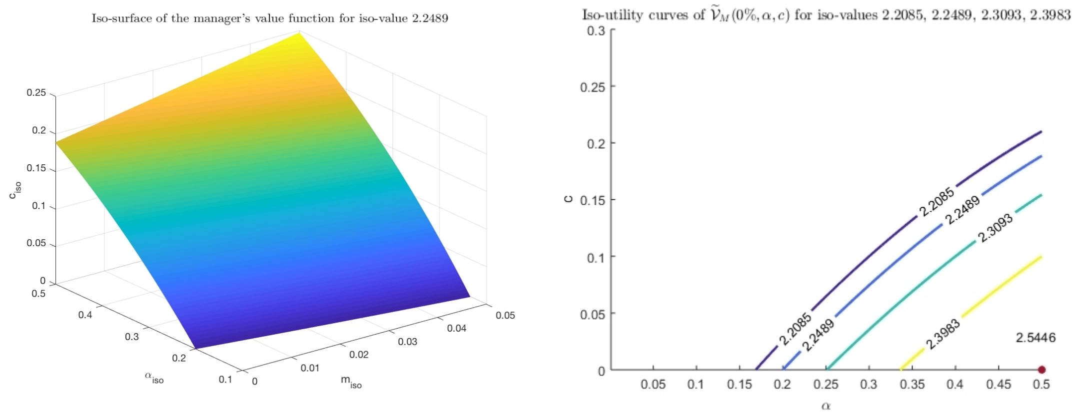
(c) $c = 0\%$.

(d) $c = 10\%$.

Figure 3.4: Investor's value function in the base case.

constituents from the manager's perspective, we illustrate his/her indifference regions in Figure 3.5. In Subfigure 3.5a we plot the set of all first-loss fee structures for which the manager has the same expected utility as he/she does for the fee structure $(0\%, 20\%, 0\%)$. Consider a situation when the investor insists on having a 20% first-loss coverage guarantee. Then the manager has to increase m to 1% and α to 50% to get the same expected utility as he/she has for the traditional fee structure. Subfigure 3.5b shows the iso-utility curves of the manager for fixed $m = 0\%$. The isovalues originate from fee structures mentioned in Djerroud et al. (2016) and He and Kou (2018): $\widetilde{\mathcal{V}}_M(0\%, 30\%, 10\%) = 2.2085$, $\widetilde{\mathcal{V}}_M(0\%, 20\%, 0\%) = 2.2489, \widetilde{\mathcal{V}}_M(0\%, 40\%, 10\%) = 2.3093, \widetilde{\mathcal{V}}_M(0\%, 50\%, 10\%) = 2.3983$. We see that the traditional fee structure $(0\%, 20\%, 0\%)$ yields a higher expected utility to the manager (in the base case) than the fee arrangement $(0\%, 30\%, 10\%)$[5]. However, it yields him/her a lower expected utility than the first-loss fee structures

[5]He and Kou (2018) find that this fee structure often yields higher expected utilities to both loss averse managers and loss averse investors in a hedge fund with parties money being commingled



(a) Manager's iso-utility surface for isovalue $\widetilde{\mathcal{V}}_M(0\%, 20\%, 0\%) = 2.2489$.

(b) Iso-utility curves of $\widetilde{\mathcal{V}}_M(0\%, \alpha, c)$ for various isovalues.

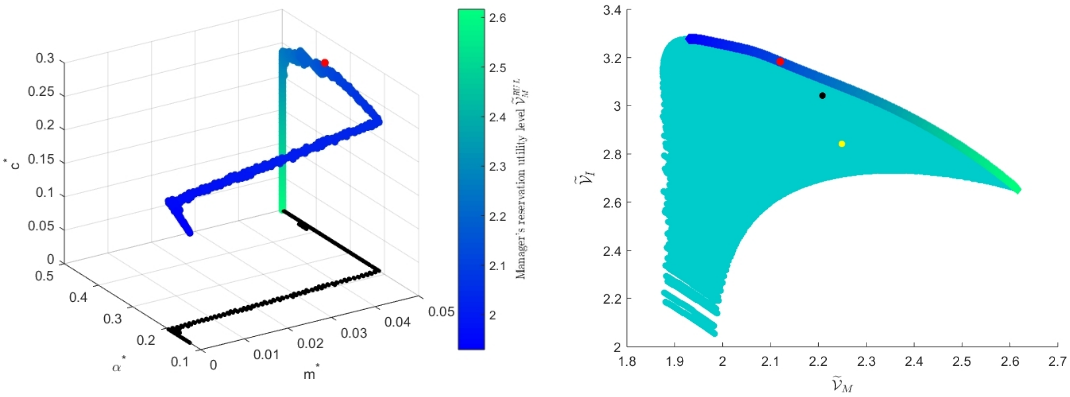
Figure 3.5: Indifference regions of the manager.

$(0\%, 40\%, 10\%)$ and $(0\%, 50\%, 10\%)$. The manager's expected utility can be improved further by charging maximal performance fee and offering minimal first-loss coverage guarantee (see the brown mark). For the manager, the lower the performance fee he/she charges, the lower first-loss coverage guarantee he/she should offer to keep his/her utility level constant. For example, assume that the investor being charged by the fee structure $(0\%, 50\%, 10\%)$ (yellow line) insists on a lower performance fee, for example 40%. The manager, to preserve his/her expected utility, should decrease the first-loss coverage guarantee from 10% to 4%.

3.3.4 First-best Pareto optimal fee structures and fee preferences

In Subsection 3.3.3 we have seen that each party has different preferences regarding fee structures. Next we calculate FBPO fee structures and find the single optimal fee structure that maximizes the hedge-fund's Sharpe ratio. For each admissible $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$, we calculate the corresponding FBPO fee structure using the SQP approach with a good starting point (see Step 4 in the algorithm in Subsection 3.3.1).

Figure 3.6 illustrates the set of FBPO fee structures, the parties' attainable expected utilities and the Pareto frontier in the base case. First, consider Subfigure 3.6a and in particular the plot of the parametric curve $\mathcal{K} := \mathcal{K}\left(\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}\right) = \left(m^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, \alpha^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}, c^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}\right)$ showing FBPO fee structures depending on $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$. The subfigure indicates that $\mathcal{K}(\widetilde{\mathcal{V}}_M^{\mathrm{RUL}})$ is continuous w.r.t. $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ and has four regions where FBPO fee structures behave differently w.r.t. changes in the manager's reservation utility level. In the front end of the plot of $\mathcal{K}$ we see the FBPO fee structure $(0\%, 14\%, 17\%)$ that corresponds to the manager's minimal reservation utility level $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}} = 1.925$. We observe that for small



(a) Set of first-best Pareto optimal first-loss schemes.

(b) Attainable expected utilities and Pareto frontier.

Figure 3.6: First best Pareto optimal fee structures and Pareto frontier in the base case.

$\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$, FBPO fee structures have zero management fee but non-trivial performance fee and first-loss coverage guarantee. As the manager's reservation utility level increases, it is not possible to satisfy his/her participation constraint with zero management fee. Consequently, FBPO fee structures for moderate values of $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ are non-trivial in all components. For high reservation utility levels of the manager, the optimal management fee attains its maximal value of 5%. The optimal performance fee is growing and reaches its maximum as well, whereas the first-loss coverage guarantee is still non-trivial. The largest $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ leads to the FBPO fee structure $(5\%, 50\%, 0\%)$, depicted in the back end of the plot of $\mathcal{K}$.

Let us now analyze the behavior of each component of FBPO fee structures separately. Consider the projection (black dots) of the FBPO fee structures on the (m^*, α^*)–plane. Looking at it from the front left to the rear right, we observe that for increasing $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ the optimal management fee is non-decreasing. In a similar way we verify that the optimal performance fee is also non-decreasing w.r.t. $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$. On the contrary, looking at the whole plot of $\mathcal{K}$, we find that the optimal first-loss coverage guarantee as a function of $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ is not monotonic. It is increasing up to a certain level that is strictly smaller than the maximal possible one ($c^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}} = 26.2\%$ for moderate participation constraint of the manager $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}} \approx 2.2$). After that, $c^*_{\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}}$ decreases with increasing $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$. When $\widetilde{\mathcal{V}}_M^{\mathrm{RUL}}$ is large, the highest management and performance fees along with some positive first-loss coverage guarantee are not satisfactory for the manager. Hence, the only way the investor could appease the manager's appetite in such cases would be decreasing the first-loss coverage guarantee.

We also observe that in the first-loss setting common fees used in the traditional fee structure (i.e., $m \in [0\%, 2\%]$, $\alpha \in [15\%, 25\%]$, $c = 0\%$) are not Pareto optimal. The majority of FBPO fee structures have $c^* \in [15\%, 25\%]$ and they correspond to the situations when the manager's participation condition is moderate, i.e., it is neither too restrictive nor too slack. The point marked red is the the preferred fee structure. It has a management fee of 5%, a performance fee of 37.5% and a first-loss coverage guarantee of 26%. Interestingly, this first-loss coverage guarantee is slightly higher than the first-loss coverage guarantee levels commonly used in the hedge-fund industry ($10\% - 20\%$).

Consider now Subfigure 3.6b showing the parties' attainable expected utilities and the Pareto frontier. The frontier is colored in accordance with Subfigure 3.6a. The point marked yellow originates from the fee structure $(0\%, 20\%, 0\%)$, whereas the point colored in black originates from the fee structure $(0\%, 30\%, 10\%)$, recommended in He and Kou (2018). We observe that both fee structures are not Pareto optimal in our hedge-fund model, although the latter scheme yields the parties' expected utilities much closer to the Pareto frontier. Note that our hedge-fund model differs from the one considered in He and Kou (2018) (the manager's and the investor's money is commingled, the fund has a liquidation boundary), which might explain why the black point is not on the efficient frontier. As before, the red marker corresponds to the preferred fee structure, which the manager and the investor should eventually agree on in the framework we consider.

3.3.5 Sensitivity analysis of preferred fee structures

In this subsection we explore the impact of various model parameters on the preferred fee structures. First, we investigate the influence of the risk-aversion parameters p_M and p_I on $(\hat{m}, \hat{\alpha}, \hat{c})$. The preferred fee structures for p_M and p_I taking values in the set $\{0.45,\ 0.35,\ 0.25,\ -0.25,\ -1.5,\ -4\}$ are shown in Table 3.2. The calculated preferred fee structures show clear patterns.

For a fixed value of p_M and decreasing p_I, we observe that $\hat{m}$ is decreasing, $\hat{\alpha}$ is increasing, $\hat{c}$ is increasing. Recall from (2.38) that $RRA_{U_M}(v) = (1 - p_M)v/(v + a_M)$ and $RRA_{U_I}(v) = (1 - p_I)v/(v + a_I)$. Hence, a more risk-averse investor prefers a lower management fee, a higher performance fee, and a larger first-loss coverage guarantee.

For a fixed value of p_I and decreasing manager's risk-aversion parameter p_M, we observe that $\hat{m}$ has an increasing, $\hat{\alpha}$ has a decreasing, $\hat{c}$ has a decreasing trend. So the more risk-averse the manager is, the higher the management fee, the lower the performance fee and the lower first-loss coverage guarantee tend to be.

The majority of obtained $\hat{m}$ are higher than the management fees usually seen in the industry. The majority of obtained $\hat{\alpha}$ are close to actually traded performance fees of $20\% - 50\%$ (as part of first-loss fee structures, see, e.g., Djerroud et al. (2016), He and Kou (2018)). The computed $\hat{c}$ are usually higher than industry-common first-loss coverage guarantee levels of $10\% - 20\%$[6]. This difference may be explained by practical peculiarities: transaction costs, taxes, or more complex asset prices dynamics.

[6]See http://www.ogcap.com/benefits-first-loss-firlo-capital/

	$p_I = 0.45$	$p_I = 0.35$	$p_I = 0.25$	$p_I = -0.25$	$p_I = -1.5$	$p_I = -4$
$p_M = 0.45$	$(5.0, 36.0, 26.3)$	$(5.0, 40.5, 29.0)$	$(5.0, 43.0, 30.0)$	$(1.6, 49.8, 30.0)$	$(0.1, 50.0, 30.0)$	$(0.1, 50.0, 30.0)$
$p_M = 0.35$	$(5.0, 33.1, 23.8)$	$(5.0, 37.5, 26.0)$	$(5.0, 40.1, 28.2)$	$(4.5, 46.0, 30.0)$	$(1.2, 50.0, 30.0)$	$(1.2, 50.0, 30.0)$
$p_M = 0.25$	$(5.0, 30.0, 21.4)$	$(5.0, 34.5, 24.0)$	$(5.0, 38.8, 26.4)$	$(4.8, 47.5, 30.0)$	$(2.6, 50.0, 30.0)$	$(2.6, 50.0, 30.0)$
$p_M = -0.25$	$(5.0, 19, 14.3)$	$(5, 23.8, 17.0)$	$(5.0, 29.6, 19.4)$	$(5.0, 42.5, 25.0)$	$(5.0, 50.0, 26.5)$	$(5.0, 50.0, 26.5)$
$p_M = -1.5$	$(5.0, 10.0, 7.7)$	$(5.0, 12.5, 9.5)$	$(5.0, 14.0, 10.5)$	$(5.0, 23.0, 15.0)$	$(4.7, 50.0, 30.0)$	$(4.7, 50.0, 30.0)$
$p_M = -4$	$(5.0, 4.8, 3.8)$	$(5.0, 5.5, 4.5)$	$(5.0, 6.6, 5.3)$	$(5.0, 12.6, 8.9)$	$(4.7, 50.0, 27.0)$	$(4.7, 50.0, 27.5)$

Table 3.2: Preferred fee structures $(\hat{m}, \hat{\alpha}, \hat{c})$ (in percent) for various p_M, p_I.

	$r = -2\%$	$r = 0\%$	$r = 2\%$	$r = 4\%$	$r = 6\%$
$p_M = p_I = 0.35$	$(5.00, 38.73, 25.95)$	$(5.00, 35.50, 26.00)$	$(5.00, 37.49, 26.01)$	$(5.00, 37.04, 26.21)$	$(5.00, 37.49, 26.54)$
$p_M = p_I = -1.5$	$(2.91, 50.00, 30.00)$	$(3.81, 50.00, 30.00)$	$(4.70, 50.00, 30.00)$	$(5.00, 50.00, 19.01)$	$(5.00, 50.00, 18.51)$

Table 3.3: Preferred fee structures $(\hat{m}, \hat{\alpha}, \hat{c})$ (in percent) for various interest rates (r).

	$\gamma = 30\%$	$\gamma = 40\%$	$\gamma = 50\%$	$\gamma = 60\%$	$\gamma = 70\%$
$p_M = p_I = 0.35$	$(5.00, 34.00, 26.00)$	$(5.00, 37.49, 26.01)$	$(5.00, 40.92, 25.08)$	$(5.00, 44.41, 23.78)$	$(5.00, 44.04, 22.52)$
$p_M = p_I = -1.5$	$(4.99, 50.00, 30.00)$	$(4.70, 50.00, 30.00)$	$(5.00, 50.00, 20.01)$	$(5.00, 50.00, 19.51)$	$(5.00, 50.00, 20.01)$

Table 3.4: Preferred fee structures $(\hat{m}, \hat{\alpha}, \hat{c})$ (in percent) for various market prices of risk (γ).

The impact of the interest rate r on the preferred fee structures is shown in Table 3.3. Here we consider decision makers with the same level of risk aversion and consider two cases: $p_M = p_I = 0.35 \in (0,1)$ and $p_M = p_I = -1.5 \in (-\infty, 0)$. The former value of the risk-aversion parameter is motivated in Section 3.3. The latter level of the risk-aversion parameter, i.e., -1.5, is a compromise between the following two papers. Koijen (2014) finds that the median level of the RRA coefficient for mutual funds equals 2.43 ($p = -1.43$), and the mean one is equal to 5.72 ($p = -4.72$), under the assumption of power-utility functions. Buraschi et al. (2010) conducts numerical studies with RRA coefficients 1.5 and 2, which corresponds to p being equal to -0.5 and -1. We observe that the preferred management fee tends to increase with the interest rate. The preferred performance fee does not show any monotonic behavior with respect to r and fluctuates between 35% and 50%. The preferred first-loss coverage guarantee exhibits a decreasing trend. Constant γ along with increasing interest rates mean a better risk-return profile of the risky asset S_1[7]. Therefore, with all market parameters being unchanged but increasing interest rates the manager has higher chances to yield a decent return for the investor, whence he/she requires a higher management fee for his/her job. Simultaneously, with high enough interest rates and good enough risk-return profile of the hedge-fund's risky investment opportunity, the investor tends to care less for the first-loss coverage guarantee, which is why $\hat{c}$ decreases.

Table 3.4 illustrates the influence of the market price of risk γ on the preferred fee structures. We observe no trend for the preferred management fee or the preferred performance fee. Interestingly, the preferred first-loss coverage guarantee tends to decrease. So, the investor is willing to cut down on the first-loss coverage guarantee as the financial market offers a higher excess return per unit of risk.

Finally, we also investigate numerically the fund's Sharpe ratio for the manager's optimal strategy corresponding to the preferred first-loss fee structure. In Table 3.5 we compare the fund's Sharpe ratio as well as the parties' expected utilities for the optimal terminal fund value and the terminal values of different funds that follow specific constant-mix strategies. We write $V^{\pi_{CM}}(T), \pi_{CM} \in \{0.25, 0.5, 0.75, 1\}$ for the fund's terminal value if the manager follows a constant-mix strategy, where the constant proportion π_{CM} of the budget is invested in the risky asset. We also write $\hat{\mathcal{V}}_M(V^{\cdot}(T)) := \mathbb{E}\left[U_M\Big(M(V^{\cdot}(T)|\hat{m},\hat{\alpha},\hat{c})\Big)\right]$ and $\hat{\mathcal{V}}_I(V^{\cdot}(T)) := \mathbb{E}\left[U_I\Big(I(V^{\cdot}(T)|\hat{m},\hat{\alpha},\hat{c})\Big)\right]$ for some fund's value $V^{\cdot}(T)$ and the preferred fee structure $(\hat{m},\hat{\alpha},\hat{c})$. In this table, we also use the notation $SR(V^{\cdot}(T))$ to emphasize that the fund's Sharpe ratio explicitly depends on the fund's terminal value but does not explicitly depend on the preferred fee structure. In contrast, the expected utilities of the parties depend on both the fee scheme and the fund's terminal value. We observe that the constant-mix strategies yield a slightly higher Sharpe ratio than the manager's optimal (in the sense of Problem (P_M)) trading strategy. However, as anticipated, due to the distribution of the fund's terminal value between the parties, the expected utility of the manager is much lower

[7] $r \uparrow \Rightarrow \gamma\sigma + r = \mu \uparrow,\ r \uparrow \Rightarrow \gamma^{-1}(\mu - r) = \sigma \downarrow$

for the constant-mix strategies than that of a manager following the optimal trading strategy. Interestingly, when the manager executes a constant-mix strategy, the investor has a slightly lower or a comparable expected utility compared to the investor's expected utility when the manager follows his/her optimal trading strategy.

Parameters	Quantity	$V^*(T)$	$V^1(T)$	$V^{0.75}(T)$	$V^{0.5}(T)$	$V^{0.25}(T)$
$p_M = p_I = 0.35$	$SR(V^\cdot(T))$	34.06%	38.15%	38.73%	39.30%	39.97%
$\Rightarrow (\hat{m}, \hat{\alpha}, \hat{c}) =$	$\hat{\mathcal{V}}_M(V^\cdot(T))$	2.112	1.635	1.690	1.768	1.853
$(5\%, 35.5\%, 26\%)$	$\hat{\mathcal{V}}_I(V^\cdot(T))$	3.190	3.130	3.137	3.138	3.134
$p_M = p_I = -1.5$	$SR(V^\cdot(T))$	37.80%	38.15%	38.72%	39.30%	39.97%
$\Rightarrow (\hat{m}, \hat{\alpha}, \hat{c}) =$	$\hat{\mathcal{V}}_M(V^\cdot(T))$	−3.556	−18.988	−12.513	−6.760	−4.426
$(4.8\%, 50\%, 30\%)$	$\hat{\mathcal{V}}_I(V^\cdot(T))$	−0.449	−0.450	−0.446	−0.447	−0.449

Table 3.5: Comparison of the fund's Sharpe ratios as well as parties' value functions for the optimal trading strategy and constant-mix trading strategies, given the preferred fee structure choice.

The hedge-fund's Sharpe ratio predominantly increases when the manager or the investor becomes more risk-averse. In each considered combination of the risk-aversion parameters in Table 3.2, it is greater than the Sharpe ratio of the hedge fund with the traditional fee structure $(0\%, 20\%, 0\%)$ by about 25 percentage points on average and greater than the Sharpe ratio originating from the fee structure $(0\%, 30\%, 10\%)$ by around 12 percentage points on average. The volatility of the hedge fund with the preferred first-loss scheme is increasing in both p_M and p_I. On average, the volatility of the hedge fund with the preferred first-loss fee structure is about 50% lower than the volatility of the hedge fund with the fee structure $(0\%, 20\%, 0\%)$ and around 17% lower than the volatility of the hedge fund with the fee structure $(0\%, 30\%, 10\%)$. So the preferred first-loss schemes significantly decrease the fund's risk and increase the fund's Sharpe ratio.

Pareto optimality of the preferred fee structures does, however, not ensure that both the manager and the investor are better off when switching a traditional fee structure to a preferred first-loss scheme. In fact, in all cases from Table 3.2 the investor's expected utility is considerably higher than it is for a traditional fee structure $(0\%, 20\%, 0\%)$, whereas the manager's expected utility is slightly worse than it is for a traditional scheme $(0\%, 20\%, 0\%)$. If the hedge-fund manager wants to ensure that he/she only offers first-loss fee structures yielding an expected utility that is not worse than that at the traditional fee structure, we recommend to choose the constrained preferred fee structure in the following way:

$$\begin{aligned} \max_{m,\alpha,c} \; & SR^*(m,\alpha,c); \\ \text{s.t.} \; & (m,\alpha,c) \in \mathcal{P}_{FBPO}^{\widetilde{\mathcal{V}}_M^{trad.}}; \end{aligned} \tag{3.15}$$

where $\mathcal{P}_{FBPO}^{\widetilde{\mathcal{V}}_M^{trad.}} := \left\{(m,\alpha,c) \in \mathcal{P}_{FBPO} : \widetilde{\mathcal{V}}_M(m,\alpha,c) \geq \widetilde{\mathcal{V}}_M^{trad.}\right\}$, $\widetilde{\mathcal{V}}_M^{trad.} := \widetilde{\mathcal{V}}_M(\bar{m},\bar{\alpha},0\%)$ for a fixed management fee $\bar{m}$ and a fixed performance fee $\bar{\alpha}$ that the manager charged in the traditional scheme.

Our numerical studies show that the preferred fee structures in the optimization (3.15) usually have a slightly lower management fee as well as first-loss coverage guarantee and higher performance fee than the corresponding components of the preferred fee structures in the optimization (P_{SR}).

4 Optimal investment under risk limitation and risk sharing in insurance

> Nobody can really guarantee the future. The best we can do is size up the chances, calculate the risks involved, estimate our ability to deal with them and then make our plans with confidence.
>
> Henry Ford II

The focus of this chapter is risk limitation and risk sharing for insurance companies. They have to manage risks of different nature – financial, longevity-related, reputational, operational risks, etc. We consider only financial risks in the sequel. Managing financial risks is of high importance, since insurers have liabilities, which can be long-term and have a significant impact on the financial health of both policyholders and insurance companies. Ensuring that assets match liabilities is a complex task where proper risk limitation and risk sharing is crucial. When it becomes very challenging for insurers to ensure liabilities, insurance companies tend to decrease liabilities in their new contracts. This is what could be observed in equity-linked insurance products in the previous decade. This chapter is a reproduction of Escobar-Anel et al. (2022) with minor changes.

An equity-linked insurance is a product that has features of both a life insurance policy and an investment vehicle, since it gives the buyer an opportunity to benefit from the upside potential of equity markets while being protected against the downside risk. These products usually have a so-called capital guarantee at the maturity of the contract and/or at the time of death of the policyholder. Since the financial crisis of 2007-2008, ensuring capital guarantees in such products has been arduous and problematic for insurance companies due to low interest rates and strict regulations. As a consequence, global insurers began decreasing guarantee levels embedded in their products. For instance, starting in 2021, Allianz provides only 60% to 90% capital guarantee in their new life insurance products[1], although the company maintains old policies that have a non-negative guaranteed rate of return on all client's contributions.

[1] See, e.g., https://www.sueddeutsche.de/wirtschaft/lebensversicherung-allianz-kuenftig-ohne-beitragsgarantie-1.5056917

In theory, the capital guarantee can be achieved via a Constant Proportion Portfolio Insurance (CPPI) strategy or an Option-Based Portfolio Insurance (OBPI) strategy. In a CPPI strategy, the capital guarantee is ensured via investing a large proportion of the portfolio value in bonds. Also, a CPPI-fund can be combined with another riskier fund to improve portfolio performance while still reaching the guarantee. This approach is implemented, e.g., in the so-called Drei-Topf-Hybrid[2] (DTH) products Hambardzumyan and Korn (2019). In contrast to CPPI-investors, OBPI-investors secure the guarantee on the invested capital by holding a put option on their portfolio. OBPI strategies require the managed portfolio and the portfolio underlying the put option to be equal. An example of an OBPI-based product is "ERGO Rente Garantie" launched in early 2010s. Purchasing this equity-linked product, clients can choose a guarantee of either 80% or 100% of their contributions. According to the product description[3], clients' contributions are invested in fixed-income assets as well as a target volatility fund (TVF) and are reinsured by Munich Re[4].

Declining capital guarantees and the scarcity of academic literature analyzing the role of reinsurance in the design of life-insurance products with capital guarantees motivate our research. In this chapter, we answer the following questions in the context of equity-linked insurance products with capital guarantees:

1. When is it beneficial for an insurance company to share its financial risk with a reinsurance company?
2. How to find the optimal investment and reinsurance strategy?
3. What is the impact of the optimal risk-sharing strategy on capital guarantee levels?

When answering these questions, we take into account an important practical aspect of reinsurance – not any risk can be reinsured, but only the risk that is acceptable to the reinsurance company. An insurance company may follow an individual investment strategy that is quite risky, e.g., to deliver higher returns to their customers and shareholders. However, the reinsurance company may not be willing to reinsure the insurer's individual investment strategy due to its high riskiness, because the insurer does not want to disclose its strategy, or because the reinsurer only wants to sell reinsurance based on a standard market index. To model this situation, we assume that reinsurance is possible only for a subset of available financial assets and a subset of admissible investment strategies of the insurer[5]. So the reinsurer sells protection only on specific portfolios or indices whose risks it understands and/or can control sufficiently well. A popular example is when the reinsurance is on a well-known index and the actual investment opportunity for the insurer is based on an own strategic portfolio or on an

[2]In English "Three-Pot-Hybrid"
[3]See Slide 13 in `https://www.yumpu.com/de/document/view/22247401/expertenwissen-zur-ergo-rente-garantie`
[4]See `https://www.focus.de/finanzen/steuern/ergo-ergo-rente-garantie_id_3550999.html`
[5]This is different from an OBPI strategy where the put option's underlying is the actually managed portfolio

exchange-traded fund that is not equal to the index. In another example, the insurer would invest part of its money beyond the reinsured portfolio in a riskier fund to have more upside potential.

For simplicity, we consider one insurance company and one reinsurance company. The insurer can invest clients' money in a riskless asset as well as a risky asset and can purchase a reinsurance contract. The reinsurance contract is modelled as a put option on a constant-mix portfolio consisting of a portfolio of bonds and a broad market index that is highly correlated with but different from the individual portfolio of the insurer. We assume that the reinsurance contract is continuously traded and liquid, as it is written on a liquid asset like futures on a broad market index, not on the insurer's individual portfolio or investment strategy. This is more realistic than the assumption of a continuously traded reinsurance on the portfolio of the insurer as assumed in most other papers studying dynamic investment-reinsurance problems (e.g., Bai and Guo (2010), Liang et al. (2011), Guan and Liang (2016), and others).

The insurer maximizes its expected utility from terminal wealth given two constraints. First, the probability that the terminal portfolio value is below the capital guarantee to the client must be less than or equal to some threshold probability, i.e., a Value-at-Risk (VaR) constraint. Second, the insurer cannot have negative positions in risky assets and reinsurance, i.e., a no-short-selling constraint.

The above-mentioned portfolio optimization problem has three aspects different from standard settings: a traded put option (reinsurance), a no-short-selling constraint and a VaR-constraint. We solve our optimization problem in three steps. First, we link the problem in the original market with a reinsurance contract to a problem in the market containing only basic assets without optional features. This approach is inspired by Korn and Trautmann (1999), where it is applied to the optimal control of a portfolio including stock options, and Hambardzumyan and Korn (2019), where it is used in the optimal control of a DTH product. Second, we transform the optimization problem with both VaR- and allocation constraints in the financial market with basic assets to the one with only VaR-constraint but in an auxiliary financial market, following Cvitanic and Karatzas (1992). Third, we transform that VaR-constrained problem to a VaR-unconstrained one and solve the latter using ideas from Basak and Shapiro (2001) and Kraft and Steffensen (2013).

After solving the original investment-reinsurance problem in a semi-closed form, we show analytically when reinsurance is needed, i.e., when the investment in reinsurance of the optimal strategy is positive. Next, we calibrate our model to the German market and conduct a suboptimality analysis. There, we analyze the optimal investment-reinsurance strategy, the optimal investment strategy without reinsurance and a constant-mix strategy representative for an average life insurance company. First, we compare these strategies with respect to wealth-equivalent utility loss studied in Larsen and Munk (2012). Second, we introduce a novel suboptimality measure — the guarantee-equivalent utility gain — and compare strategies with respect to it. After the suboptimality analysis, we investigate the impact of model parameters on the investment strategies and also give insights into measuring the level of reinsurance in the insurer's portfolio.

In the following overview of relevant literature sources, we distinguish between two streams – more applied sources with focus on insurance and more theoretical ones with focus on portfolio optimization. In the former stream of literature, Müller (1985) derived the optimal investment-reinsurance strategy of a pension fund in a static portfolio-optimization framework with an exponential utility function and no constraints on terminal wealth or allocation. In a dynamic portfolio optimization setting, the literature on reinsurance is mainly focused on the insurers' overall surplus processes and not on specific products or investment portfolios (e.g., Luo et al. (2008), Bai and Guo (2010), Liang et al. (2011), Li et al. (2014)). To the best of our knowledge, there are no publications on the optimal dynamic investment-reinsurance strategies for life insurance products with capital guarantees. Our research fills this gap.

Several papers study optimal investment strategies for life insurance products with capital guarantees in financial markets without reinsurance opportunities. We mention only the most recent. Chen et al. (2019) analyze optimal investment strategies when the capital guarantee is embedded in a piece-wise linear payoff of the decision maker such that the whole payoff is fairly priced. Dong and Zheng (2019) and Dong and Zheng (2020) consider loss-averse insurers and derive optimal investment strategies under both no-short-selling and terminal portfolio value constraints. In the former paper, the capital guarantee is modelled as a hard lower bound on the terminal portfolio value, whereas in the latter paper the guaranteed amount is part of the VaR-constraint. Hambardzumyan and Korn (2019) is centered around DTH products in the expected utility framework. The researchers consider the insurer who can invest in a risk-free bank account, a CPPI-fund as well as a free fund. Imposing also a capital guarantee as a hard lower bound on the terminal portfolio value, the authors derive the optimal trading strategies. In our framework, the insurer can purchase reinsurance instead of a CPPI fund, has a no-short-selling constraint and a VaR-constraint, which is more general than the strict lower bound constraint.

Within the literature focusing on portfolio optimization, Merton (1969) and Merton (1971) are seminal papers where the classic continuous-time dynamic portfolio optimization problem of a utility-maximizing investor in a Black-Scholes market was first considered and solved in a setting without constraints on terminal wealth and investment strategies. Since then those papers have been extended in a myriad of different ways. A fruitful branch is the addition of constraints restricting an investor's portfolio choice. One of the most natural constraints is a strict lower bound on the (terminal) portfolio wealth, which was considered, e.g., in Teplá (2001) and Korn (2005) or a probabilistic lower bound, also called a VaR constraint, which was considered, e.g., in Basak and Shapiro (2001) and Boyle and Tian (2007). A combination of these two constraints has been analyzed in Chen et al. (2018a), where the authors derive the optimal investment strategies for a utility maximization problem under a VaR constraint and a minimum insurance constraint, relevant for life insurance companies. Regulations and product peculiarities in the insurance industry have motivated scientists to analyze other practically relevant and mathematically interesting portfolio optimization problems with terminal wealth constraints. For example, a utility maximization problem with multiple VaR constraints has been solved in Chen et al. (2018b). Chen et al. (2019) derive the optimal investment strategies for a portfolio optimization problem under a fair-pricing constraint. This constraint ensures that neither the issuer of a life insurance contract (a life insurance company maximizing its expected utility) nor the holder of a life insurance contract (policyholder) systematically benefits from the contract of interest. All sources on constrained portfolio optimization, mentioned in this paragraph, do not consider allocation constraints but only terminal wealth constraints.

Constraints on the terminal wealth can be linked to a corresponding unconstrained portfolio optimization problem through the addition of a suitable Lagrange multiplier to the terminal utility, which is solvable via the martingale approach. This link was investigated by Kraft and Steffensen (2013) using Hamilton-Jacobi-Bellman (HJB) equations and Black-Scholes partial differential equations (PDEs), which allow them to express the optimal terminal wealth as a function of the optimal unconstrained terminal wealth. The methodology developed in Kraft and Steffensen (2013) is not designed for constraints on portfolio allocation such as no-short-selling constraints, non-traded asset constraints, no-borrowing constraints or other general convex constraints on the allocation. However, by embedding an allocation-constrained portfolio optimization problem into a family of unconstrained portfolio optimization problems in different auxiliary markets, Cvitanic and Karatzas (1992) were able to derive closed-form solutions for constant relative risk aversion (CRRA) utility functions by solving the HJB PDE for an associated dual optimization problem.

Thus far, combinations of constraints on terminal wealth and portfolio allocation have rarely been studied in the existing literature due to the increased complexity of the associated HJB PDEs. Exceptions to this can be found in the work of Bardhan (1994), Dong and Zheng (2019), Escobar et al. (2019) as well as Dong and Zheng (2020). Specifically, in Dong and Zheng (2020) the authors consider a portfolio optimization problem of a pension fund manager with S-shaped utility, convex cone allocation constraints as well as VaR constraints motivated by a defined contribution pensions plan. They remove the VaR constraint by adding a Lagrange multiplier to the utility function and concavify the resulting function. For every Lagrange multiplier, the resulting portfolio optimization problem is allocation-constrained and wealth-unconstrained and can thus be solved via the HJB PDE of the associated dual problem, as shown in Bian et al. (2011). Finally, the authors of Dong and Zheng (2020) show that the optimal Lagrange-multiplier exists such that the optimal solution for the original VaR-constrained problem can be obtained by this methodology. The similarity of their and our optimization problems lies in the simultaneous VaR constraint and allocation constraint, whereas the difference lies in utility functions[6] and traded assets. Although we consider a similar optimization

[6]Our solution approach is also applicable to an S-shaped utility used in Dong and Zheng (2020). In this case, an additional step would be necessary – the construction of the concave envelope (see Definition 3.2.1) of the S-shaped utility. For more information on concave envelopes, see, e.g., Reichlin (2012).

problem as in Dong and Zheng (2020), our solution approach is different, providing a new path to solving these challenging problems. We build on the results in Basak and Shapiro (2001) to show that in a class of allocation-unconstrained, but VaR-constrained portfolio optimization problems the optimal solution can be expressed as a function of the VaR-unconstrained optimal terminal wealth, which is in the spirit of the ideas presented in Kraft and Steffensen (2013). Then, we show that the same auxiliary market as in Cvitanic and Karatzas (1992) admits the optimal solution to the original problem. This way we avoid determining the solution to the dual HJB PDE explicitly, which in some cases may not be viable. Further, we demonstrate that the optimal terminal portfolio value of the allocation- and VaR-constrained portfolio optimization problem can be expressed as a function of the optimal terminal wealth of the allocation-constrained and wealth-unconstrained portfolio optimization problem, i.e., the derivative structure proposed in Kraft and Steffensen (2013) is preserved under allocation constraints.

Next we summarize the scientific contributions of this chapter to the existing literature. This summary is organized in two parts: contributions related to the actuarial strand of literature and ones related to theoretical portfolio-optimization literature. In the realm of actuarial literature, first, we design in Section 4.1 a realistic and workable framework for finding optimal investment-reinsurance strategies for equity-linked insurance products with capital guarantees. The framework combines put options, regulatory VaR and no-short-selling constraints, and a separation between insurable and reinsurable funds. Second, we detect in Proposition 4.2.7 market conditions and the asset manager's proficiency, for which (partial) reinsurance of the capital guarantee is advantageous. In particular, we find that reinsurance is optimal when the risky asset in the insurer's investment portfolio has a high correlation with and has a higher Sharpe ratio than the reinsurable asset. Third, we establish in Subsection 4.3.2 that optimal reinsurance significantly increases capital guarantees, while slightly decreasing product costs. In a typical example, a 10-year equity-linked insurance product that follows an optimal investment strategy, uses no reinsurance and offers a terminal guarantee of 100% (equivalent to 0% annualized guaranteed return) of the client's initial endowment can be replaced by a product that follows an optimal investment-reinsurance strategy with a terminal guarantee of 110% (i.e., 0.96% annualized guaranteed return) of the initial capital. This is with no loss in the insurer's expected utility. Moreover, our proposal allows the insurer to guarantee 128% (equivalent to approximately 2.5% annualized guaranteed return) of the client's initial endowment without any decrease in the insurer's expected utility in comparison to the one obtained by a constant-mix 85% bonds and 15% stocks strategy, which is representative of investment strategies of life insurers.

As for our contribution to portfolio-optimization literature, we explicitly solve the portfolio optimization problem with simultaneous VaR-constraint and no-short-selling constraint in a financial market with a traded put option on a well-known portfolio. In particular, we show in Propositions 4.2.1 and 4.2.2 how to deal with a traded put option and in Proposition 4.2.6 how to solve the VaR-constrained and allocation-constrained portfolio-optimization problem. The methodology we present can be adapted to other types of terminal-wealth constraints (e.g., an expected shortfall constraint), other types of allocation constraints (e.g., specific assets are not allowed to be traded), and other derivative-like traded assets (e.g., a call option, an OBPI fund). We also propose in Equation (4.27) an intuitive way of quantifying the benefit of following the optimal strategy instead of suboptimal ones in terms of the capital guarantee.

The remainder of this chapter is organized as follows. In Section 4.1 we describe formally the optimization problem of the insurer selecting its investment and risk-sharing strategies. The solution to this problem is presented in Section 4.2. In Section 4.3 we calibrate our model parameters to the German market and conduct numerical studies. First, we conduct a suboptimality analysis of the strategies with respect to the initial wealth as well as the capital guarantee. Second, we calculate numerically the sensitivity of the optimal investment-reinsurance strategy with respect to the model parameters. In Appendix B.1, we provide the proofs of the main theoretical results. Appendix B.2 contains auxiliary theoretical results and their proofs.

4.1 Problem setting

We consider the basic market model that was introduced in Section 2.1 and set $n = 2$. For convenience, we explicitly state the price dynamics of the basic assets:

$$\begin{aligned} dS_0(t) &= S_0(t)rdt && \text{bank-account} \\ dS_1(t) &= S_1(t)(\mu_1 dt + \sigma_1 dW_1(t)) && \text{non-reinsurable fund} \\ dS_2(t) &= S_2(t)(\mu_2 dt + \sigma_2(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t)) && \text{reinsurable fund} \end{aligned} \tag{4.1}$$

with $\rho \in (-1, 1)$ and other parameters satisfying conditions stated in Section 2.1. In this market, S_1 models a risky fund in the individual investment strategy of an insurance company, whereas S_2 models a risky fund that a reinsurance company can sell protection on. As mentioned in the motivation section, the fund in the insurer's individual strategy is not equal to what the reinsurer is willing to or is able to reinsure, e.g., because in practice the insurer's strategy can be considered too risky by the reinsurer.

We refer to this market as a linear market, since the value of any portfolio consisting of S_0, S_1, S_2 is linear with respect to the prices of basic assets.

Consider three parties in the market: a client, an insurance company (insurer), and a reinsurance company (reinsurer). In this chapter, the main focus is on the insurer, and the other two parties are considered implicitly without equipping them with utility functions, investment strategies, etc. The insurance company receives at $t = 0$ an initial endowment by $v_0 > 0$ from a representative client, invests this money on the client's behalf, and promises to pay back to the client at time T at least the capital guarantee $G_T > 0$.

We assume that only constant-mix (CM) portfolios can be reinsured. This choice is motivated by the equivalence of constant-mix strategies and target-volatility strategies in a Black-Scholes market. Recall that the reinsurable risky portfolio in "ERGO Rente Garantie" is a target volatility fund. For CM strategies, the reinsurer can evaluate sufficiently well the potential loss in advance and can easily price it. So for $\pi_B^{CM} \in [0,1]$, we denote by $\pi_B(t) = (0, \pi_B^{CM})^\top$, $t \in [0,T]$, the relative portfolio process related to the risky assets in the CM strategy that the reinsurer can reinsure via a put option. Under this strategy, the proportion of wealth invested in S_0 equals $1 - \pi_B^{CM}$, $t \in [0,T]$, whereas the proportion of wealth invested in S_2 equals π_B^{CM}, $t \in [0,T]$. The dynamics of the corresponding CM portfolio value is given by:

$$\begin{aligned} dV^{v_0,\pi_B}(t) &= (1-\pi_B^{CM})V^{v_0,\pi_B}(t)\frac{dS_0(t)}{S_0(t)} + \pi_B^{CM}V^{v_0,\pi_B}(t)\frac{dS_2(t)}{S_2(t)} \\ &= V^{v_0,\pi_B}(t)((r + \pi_B^{CM}(\mu_2 - r))dt \\ &\quad + \pi_B^{CM}\sigma_2(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t)). \end{aligned} \tag{4.2}$$

Let $Put(t)$ be the fair price at time t, $t \in [0,T]$, of a put option with the payoff $(G_T - V^{v_0,\pi_B}(T))^+$. Since the financial market of basic assets is complete and the price of a put option is once continuously differentiable w.r.t. time and twice continuously differentiable w.r.t. the initial price of the underlying asset, we can apply Theorem 2.6 in Korn and Trautmann (1999) and get that the SDE describing the price dynamics of the put option:

$$dPut(t) = \underbrace{\frac{\partial Put(t)}{\partial V^{v_0,\pi_B}(t)}}_{\text{delta hedge}} dV^{v_0,\pi_B}(t) + \underbrace{\left(\frac{Put(t)}{S_0(t)} - \frac{V^{v_0,\pi_B}(t)\frac{\partial Put(t)}{\partial V^{v_0,\pi_B}(t)}}{S_0(t)}\right)}_{\text{money left}} dS_0(t). \tag{4.3}$$

The delta-hedge of the put option on the CM portfolio is well-know and given by:

$$\frac{\partial Put(t)}{\partial V^{v_0,\pi_B}(t)} = \Phi(d_+) - 1, \tag{4.4}$$

where

$$d_+ := d_1(t, V^{v_0,\pi_B}(t), G_T, r, \pi_B^{CM}\sigma_2)$$

with $d_1(\cdot)$ defined in (2.17).

Using (4.2), (4.3) and (4.4), we get the dynamics of the value of the reinsurance (put option) in terms of its basic underlying assets:

$$
\begin{aligned}
dPut(t) &= (\Phi(d_+) - 1)V^{v_0,\pi_B}(t)((r + \pi_B^{CM}(\mu_2 - r))dt + \pi_B^{CM}\sigma_2(\rho dW_1^{\mathbb{Q}}(t) \\
&\quad + \sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t))) \\
&\quad + \left(\frac{Put(t)}{S_0(t)} - \frac{V^{v_0,\pi_B}(t)(\Phi(d_+) - 1)}{S_0(t)}\right) S_0(t) r dt \\
&= \big((\Phi(d_+) - 1)V^{v_0,\pi_B}(t)\pi_B^{CM}(\mu_2 - r) + rPut(t)\big)\, dt \\
&\quad + (\Phi(d_+) - 1)V^{v_0,\pi_B}(t)\pi_B^{CM}\sigma_2\rho dW_1^{\mathbb{Q}}(t) \\
&\quad + (\Phi(d_+) - 1)V^{v_0,\pi_B}(t)\pi_B^{CM}\sigma_2\sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t).
\end{aligned}
$$

To sum up, the insurer can invest in a risk-free asset S_0, a risky asset S_1, and a reinsurance (put option) $Put(t)$ with the underlying $V^{v_0,\pi_B}(t)$ and strike G_T. We refer to the market consisting of S_0, S_1, Put as a non-linear market, since herein the portfolio value is in general a non-linear function w.r.t. the prices of the basic assets S_0, S_1, S_2.

Let $\bar{\pi}(t) = (\bar{\pi}_1(t), \bar{\pi}_2(t))^\top$, $t \in [0,T]$, be the insurer's relative portfolio process with respect to assets $S_1(t)$, $Put(t)$, with $\bar{\pi}_0(t) = 1 - \bar{\pi}_1(t) - \bar{\pi}_2(t)$, $t \in [0,T]$. Let $\bar{\varphi}(t)$ be the corresponding trading strategy at $t \in [0,T]$, i.e., number of bonds, shares or reinsurance contracts. The portfolio value has the following dynamics:

$$
d\bar{V}^{v_0,\bar{\pi}}(t) = (1-\bar{\pi}_1(t)-\bar{\pi}_2(t))\frac{\bar{V}^{v_0,\bar{\pi}}(t)}{S_0(t)}dS_0(t) + \bar{\pi}_1(t)\frac{\bar{V}^{v_0,\bar{\pi}}(t)}{S_1(t)}dS_1(t) + \bar{\pi}_2(t)\frac{\bar{V}^{v_0,\bar{\pi}}(t)}{Put(t)}dPut(t)
$$

with $\bar{V}^{v_0,\bar{\pi}}(0) = v_0$. Note that this SDE is different from the SDE (2.7), because the put option is continuously traded instead of S_2. The bar in the expressions is used to indicate this.

Similarly to the relation (2.6), the relative portfolio process and the trading strategy in the insurer's investment-reinsurance universe are linked in the following way:

$$
\bar{\varphi}_0(t) = \frac{\bar{\pi}_0(t)\bar{V}^{v_0,\bar{\pi}}(t)}{S_0(t)}, \quad \bar{\varphi}_1(t) = \frac{\bar{\pi}_1(t)\bar{V}^{v_0,\bar{\pi}}(t)}{S_1(t)}, \quad \bar{\varphi}_2(t) = \frac{\bar{\pi}_2(t)\bar{V}^{v_0,\bar{\pi}}(t)}{Put(t)}, \tag{4.5}
$$

where $\bar{V}^{v_0,\bar{\pi}}(t)$ is the value of the insurer's portfolio at time $t \in [0,T]$.

Analogously, let $\pi(t) = (\pi_1(t), \pi_2(t))^\top$, $t \in [0,T]$, be the relative portfolio processes w.r.t. assets $S_1(t), S_2(t)$, with $\pi_0(t) = 1 - \pi_1(t) - \pi_2(t)$, $t \in [0,T]$, and $\varphi(t)$, $t \in [0,T]$, be the corresponding trading strategy. Then it follows from (2.7) that the portfolio value in the financial market consisting of S_0, S_1, S_2 has the following dynamics:

$$
dV^{v_0,\pi}(t) = (1-\pi_1(t)-\pi_2(t))\frac{V^{v_0,\pi}(t)}{S_0(t)}dS_0(t) + \pi_1(t)\frac{V^{v_0,\pi}(t)}{S_1(t)}dS_1(t) + \pi_2(t)\frac{V^{v_0,\pi}(t)}{S_2(t)}dS_2(t)
$$

with $V^{v_0,\pi}(0) = v_0$.

We assume that the insurer has a power-utility function as per (2.39). The insurer has to fulfill a Value-at-Risk (VaR) constraint, which is widely used in the life insurance literature (see, e.g., Dong and Zheng (2020), Guan and Liang (2016), Nguyen and Stadje (2020)) and is motivated by solvency regulations and the management rules of insurance companies. We denote by $\varepsilon \in [0,1]$ the probability of not achieving a guarantee by the insurer.

We also add a no-short-selling constraint to the insurer's optimization problem. The motivation for it is twofold. First, shorting reinsurance is against the nature of the reinsurance business. The no-short-selling constraint on the reinsurance prevents the insurer from using reinsurance for speculation purposes. Second, shorting assets is quite uncommon for insurance companies due to regulations (see Dong and Zheng (2020)).

Define:

$$\begin{aligned}
\bar{\mathcal{A}}_u^{\bar{\pi}}(v_0) := \Bigg\{ \bar{\pi} = \Big((\bar{\pi}_1(t), \bar{\pi}_2(t))^\top \Big)_{t\in[0,T]} \Bigg| \, & \bar{\pi} \text{ is prog. meas. and self-financing,} \\
& \bar{V}^{v_0,\bar{\pi}}(t) \geq 0 \ \mathbb{Q}\text{-a.s.} \forall t \in [0,T], \ \int_0^T ||\bar{\pi}(t)\bar{V}^{v_0,\bar{\pi}}(t)||^2 \, dt < \infty \, \mathbb{Q}\text{-a.s.} \Bigg\}; \\
\bar{C}_{\bar{V}}(\varepsilon) := \{ \bar{V}^{v_0,\bar{\pi}}(T) | \mathbb{Q} & \left(\bar{V}^{v_0,\bar{\pi}}(T) < G_T \right) \leq \varepsilon \}; \\
\bar{C}_{\bar{\pi}} := [0, +\infty) & \times [0, +\infty),
\end{aligned}$$

where "prog. meas." stands for "progressively measurable". Then the set of the insurer's admissible constrained relative-portfolio processes w.r.t. S_1 and Put is given by:

$$\bar{\mathcal{A}}_c^{\bar{\pi}} \left(v_0, \bar{C}_{\bar{V}}(\varepsilon), \bar{C}_{\bar{\pi}} \right) := \left\{ \bar{\pi} \in \bar{\mathcal{A}}_u^{\bar{\pi}}(v_0) \, | \, \bar{V}^{v_0,\bar{\pi}}(T) \in \bar{C}_{\bar{V}}(\varepsilon), \, \bar{\pi}(t) \in \bar{C}_{\bar{\pi}} \ \mathbb{Q}\text{-a.s.} \ \forall \, t \in [0,T] \right\}.$$

Note that $\bar{\mathcal{A}}_c^{\bar{\pi}} \left(v_0, \bar{C}_{\bar{V}}(1), \mathbb{R}^2 \right) = \bar{\mathcal{A}}_u^{\bar{\pi}}(v_0)$. The optimization problem of the insurer under the no-short-selling constraint and VaR constraint is as follows:

$$\max_{\bar{\pi}} \mathbb{E}\left[U(\bar{V}^{v_0,\bar{\pi}}(T)) \right] \quad \text{s.t.} \quad \bar{\pi} \in \bar{\mathcal{A}}_c^{\bar{\pi}} \left(v_0, \bar{C}_{\bar{V}}(\varepsilon), \bar{C}_{\bar{\pi}} \right). \qquad (\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$$

The notation $\left(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}} \right)$ indicates that control variables in this optimization problem are relative portfolio processes w.r.t. assets S_0, S_1, Put, there is a VaR-type terminal wealth constraint with probability ε and there is an investment strategy constraint $\bar{\pi} \in \bar{C}_{\bar{\pi}}$. Special cases of this notation are $\left(\bar{P}_{1,\bar{C}_{\bar{\pi}}} \right)$, i.e., the optimization problem does not have a terminal wealth constraint, and $\left(\bar{P}_{0,\bar{C}_{\bar{\pi}}} \right)$, i.e., the optimization problem has a hard lower bound, also known as the portfolio insurance constraint.

4.2 Solution to the optimization problem

In this section, we first provide an overview of our approach to solving $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$. In the solution procedure, we make several transformations of the problem eventually linking the solution to the original problem and the solution to a simpler problem that has neither terminal-wealth constraints nor allocation constraints. After the general overview, we describe the transformations in detail, each addressing a specific challenge of the optimization problem: reinsurance, no-short-selling constraint, VaR constraint. At the end of this section, we answer one of the main questions of our chapter – in which situations is it optimal for the insurer to buy reinsurance?

We solve Problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$ as follows. First, we deal with reinsurance by transforming the original problem with traded reinsurance to an allocation-constrained VaR-constrained problem in the financial market with basic assets S_0, S_1, S_2. Second, we tackle the no-short-selling constraint by transforming the problem from the first step to an allocation-unconstrained VaR-constrained problem in an auxiliary financial market of basic assets. Third, we solve the allocation-unconstrained VaR-constrained problem from the second step and use it to recover the solution to the original problem. Figure 4.1 schematically illustrates our approach[7].

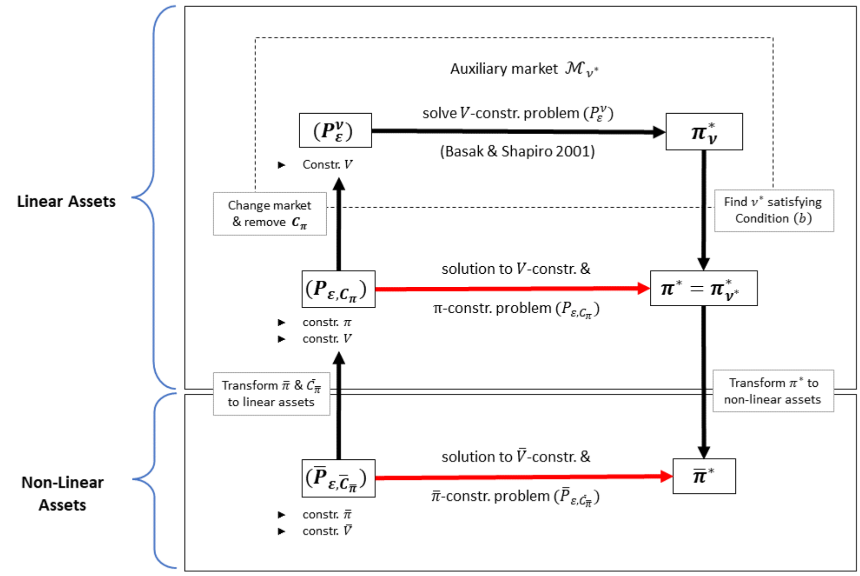


Figure 4.1: Schematic representation of the solution procedure for Problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$.

[7]New elements of notation are explained in the corresponding subsections of Section 4.2

4.2.1 Relation between portfolios in non-linear and linear markets

The next proposition links the dynamics of a portfolio that consists of S_0, S_1, Put to the dynamics of a portfolio of the basic assets S_0, S_1, S_2.

Proposition 4.2.1. *If π and $\bar{\pi}$ satisfy the following relation:*

$$\begin{cases} \bar{\pi}_1(t) = \pi_1(t); \\ \bar{\pi}_2(t) = \pi_2(t) \dfrac{Put(t)}{\pi_B^{CM} V^{v_0,\pi_B}(t)(\Phi(d_+) - 1)}, \end{cases} \tag{4.6}$$

then:

$$\bar{V}^{v_0,\bar{\pi}}(t) = V^{v_0,\pi}(t) \qquad \forall t \in [0,T] \ \mathbb{Q} - a.s.. \tag{4.7}$$

Proof. See Appendix B.1. □

For convenience, we denote:

$$\Psi(t) := \begin{pmatrix} 1 & 0 \\ 0 & \frac{Put(t)}{\pi_B^{CM} V^{v_0,\pi_B}(t)(\Phi(d_+)-1)} \end{pmatrix} \quad , \qquad t \in [0,T]. \tag{4.8}$$

Observe that:

$$\begin{aligned} &\forall\, t \in [0,T],\ \forall \begin{pmatrix} \bar{\pi}_1(t) \\ \bar{\pi}_2(t) \end{pmatrix} \in \bar{C}_{\bar{\pi}} : \\ &\Psi^{-1}(t) \begin{pmatrix} \bar{\pi}_1(t) \\ \bar{\pi}_2(t) \end{pmatrix} = \begin{pmatrix} \bar{\pi}_1(t) \\ \frac{\bar{\pi}_2(t) \pi_B^{CM} V^{v_0,\pi_B}(t)(\Phi(d_+)-1)}{Put(t)} \end{pmatrix} \in [0,\infty) \times (-\infty, 0]. \end{aligned} \tag{4.9}$$

Therefore, we define:

$$\begin{aligned} &C_V(\varepsilon) := \{V^{v_0,\pi}(T) \,|\, \mathbb{Q}\left(V^{v_0,\pi}(T) < G_T\right) \le \varepsilon\}; \\ &C_\pi := [0,\infty) \times (-\infty, 0]; \\ &\mathcal{A}_c^\pi\left(v_0, C_V(\varepsilon), C_\pi\right) := \left\{\pi \in \mathcal{A}_u^\pi(v_0) \,|\, V^{v_0,\pi}(T) \in C_V(\varepsilon),\ \pi(t) \in C_\pi\ \mathbb{Q}\text{-a.s.}\ \forall\, t \in [0,T]\right\}, \end{aligned}$$

where $\mathcal{A}_u^\pi(v_0)$ is the set of admissible unconstrained relative-portfolio processes for the initial wealth v_0, as per Definition 2.1.4. Consider the following optimization problem in the market with basic assets S_0, S_1, S_2:

$$\max_\pi \mathbb{E}\Big[U(V^{v_0,\pi}(T))\Big] \quad \text{s.t.} \quad \pi \in \mathcal{A}_c^\pi\left(v_0, C_V(\varepsilon), C_\pi\right) \tag{P_{ε,C_π}}$$

The next proposition links the solution to the original Problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$ and the solution to the transformed Problem[8] (P_{ε,C_π}).

Proposition 4.2.2. *Let π^* be the optimal solution to Problem (P_{ε,C_π}). Then the portfolio process:*

$$\bar{\pi}^*(t) := \Psi(t)\pi^*(t)$$

is the solution to Problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$, where $\Psi(t)$ is the transformation matrix from (4.8).

Proof. See Appendix B.1. □

Due to Proposition 4.2.2, we focus on solving (P_{ε,C_π}) in the sequel of this chapter.

4.2.2 Solving the transformed problem in the market with basic assets

We deal with the additional allocation constraints and VaR constraints on terminal wealth by borrowing and combining two popular approaches from the literature: the auxiliary market approach from Cvitanic and Karatzas (1992) as well as the idea of using option-like terminal payoffs on the unconstrained terminal wealth to eliminate terminal wealth constraints from Kraft and Steffensen (2013). This is how we are going to proceed. First, we extend the auxiliary market framework from Cvitanic and Karatzas (1992) to our setting and derive an optimality condition, which links the solutions of a family of wealth-constrained and allocation-unconstrained portfolio optimization problems (P_ε^ν) to the solution of the wealth- and allocation-constrained portfolio optimization problem (P_{ε,C_π}). Second, we use the results from Basak and Shapiro (2001) to derive the solutions to the wealth-constrained and allocation-unconstrained portfolio optimization problems (P_ε^ν). Third, we find the optimal auxiliary market $\mathcal{M}_{\nu^*}$, compute the corresponding optimal portfolio and verify its optimality for the primal problem (P_{ε,C_π}) via the previously derived optimality condition.

Auxiliary market with VaR- and allocation constraints

In the literature, the classic approach to portfolio optimization under the presence of allocation constraints is the auxiliary market approach from Cvitanic and Karatzas (1992). Despite the presence of the additional VaR-constraints, the concept of auxiliary market proves to be vital in solving (P_{ε,C_π}). For setting up the auxiliary markets, we need to introduce the support function

$$\delta(x) = -\inf_{y\in C_\pi}\big(x^\top y\big). \tag{4.10}$$

[8] For other allocation constraints $\bar{C}_{\bar{\pi}}$ in the original Problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$, e.g., $\bar{C}_{\bar{\pi}} = [0,a]\times[0,a]$ with fixed $a>0$, the allocation constraint C_π in the transformed Problem (P_{ε,C_π}) may become stochastic.

In our setting, with $C_\pi = [0,\infty) \times (-\infty, 0]$, the infimum in (4.10) is attained by $y = 0$, if $x \in C_\pi$, and is $-\infty$ otherwise. Hence, $\delta(x) = 0 \ \forall x \in C_\pi$.

Further, we introduce the class of $\mathbb{R}^2$-valued dual processes $\mathcal{D}$:

$$\mathcal{D} := \Bigg\{ \nu = \big((\nu_1(t), \nu_2(t))^\top \big)_{t\in[0,T]} \Big| \nu \text{ progressively measurable,} \\ \mathbb{E}\left[\int_0^T \|\nu(t)\|^2 \, dt\right] < \infty, \, \mathbb{E}\left[\int_0^T \delta(\nu(t)) dt\right] < \infty \Bigg\}.$$

The second integrability condition implies that $\nu(t) \in C_\pi \ \mathbb{Q}\otimes\mathcal{L}[0,T]$-a.s., $\mathcal{L}[0,T]$ denoting the Lebesgue measure on $t \in [0,T]$. In particular, every constant process with value in C_π, from now on referred to as dual vector, is contained in $\mathcal{D}$.

For each $\nu \in \mathcal{D}$, we define the auxiliary market $\mathcal{M}_\nu$, where the assets S_0^ν, S_1^ν and S_2^ν follow the dynamics

$$\begin{aligned} dS_0^\nu(t) &:= S_0^\nu(t)\big(r + \delta(\nu(t))\big)dt = S_0^\nu(t) r dt; \\ dS_1^\nu(t) &:= S_1^\nu(t)\big(\big(\mu_1 + \nu_1(t) + \delta(\nu(t))\big)dt + \sigma_1 dW_1^{\mathbb{Q}}(t)\big) \\ &= S_1^\nu(t)\big((\mu_1 + \nu_1(t))dt + \sigma_1 dW_1^{\mathbb{Q}}(t)\big); \\ dS_2^\nu(t) &:= S_2^\nu(t)\big(\big(\mu_2 + \nu_2(t) + \delta(\nu(t))\big)dt + \sigma_2(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2})dW_2^{\mathbb{Q}}(t)\big) \\ &= S_2^\nu(t)\big((\mu_2 + \nu_2(t))dt + \sigma_2(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2})dW_2^{\mathbb{Q}}(t)\big). \end{aligned} \tag{4.11}$$

In $\mathcal{M}_\nu$, the market price of risk and the pricing kernel are stochastic processes given by:

$$\gamma_\nu(t) = \sigma^{-1}(\mu + \nu(t) - r\mathbf{1}_2), \, t \in [0,T]; \tag{4.12}$$

$$\widetilde{Z}_\nu(t) = \exp\left(-rt - 0.5\int_0^t ||\gamma_\nu(s)|^2 \, ds - \int_0^t \gamma_\nu^\top(s) \, dW^{\mathbb{Q}}(s)\right), \, t \in [0,T]. \tag{4.13}$$

As we will see later, it is sufficient for our problem to consider dual vectors $\nu \in C_\pi$[9]. For such cases the market price of risk and the pricing kernel are simplified to

$$\gamma_\nu = \sigma^{-1}(\mu + \nu - r\mathbf{1}_2), \, \widetilde{Z}_\nu(t) = \exp\Big(-\big(r + 0.5||\gamma_\nu||^2\big)\, t - \gamma_\nu^\top \, W(t)\Big), \, t \in [0,T] \tag{4.14}$$

[9] Note that for any dual vector $\nu \in C_\pi$ the market $\mathcal{M}_\nu$ is a two-dimensional Black-Scholes market with deterministic market coefficients. In particular, $\mathcal{M}_{(0,0)^\top}$ is the standard market with assets (S_0, S_1, S_2).

and $\mathcal{M}_\nu$ admits a unique risk-neutral probability measure $\widetilde{\mathbb{Q}}_\nu$[10] with density

$$\left.\frac{d\widetilde{\mathbb{Q}}_\nu}{d\mathbb{Q}}\right|_{\mathcal{F}(T)} := Z_\nu(T) := \exp\left(-0.5||\gamma_\nu||^2 T - \gamma_\nu^\top W^{\mathbb{Q}}(T)\right).$$

The asset S_0^ν represents the bank account, whereas the assets S_1^ν and S_2^ν represent the fund and market index from our original setting but with partially changed drift coefficients. Clearly, changing the drift coefficients of the basic assets in $\mathcal{M}_\nu$ has an effect on the wealth process of an investor trading in $\mathcal{M}_\nu$. Indeed, it is straightforward to show that the wealth process $V_\nu^{v_0,\pi}(T)$, corresponding to trading in $\mathcal{M}_\nu$ according to π with initial wealth v_0, satisfies the SDE

$$\begin{aligned} dV_\nu^{v_0,\pi}(t) &= (1-\pi(t)^\top \mathbf{1}_2)\frac{V_\nu^{v_0,\pi}(t)}{S_0^\nu(t)}dS_0^\nu(t) + \pi_1(t)\frac{V_\nu^{v_0,\pi}(t)}{S_1^\nu(t)}dS_1^\nu(t) + \pi_2(t)\frac{V_\nu^{v_0,\pi}(t)}{S_2^\nu(t)}dS_2^\nu(t) \\ &= (1-\pi(t)^\top \mathbf{1}_2)\frac{V_\nu^{v_0,\pi}(t)}{S_0(t)}dS_0(t) + \pi_1(t)\frac{V_\nu^{v_0,\pi}(t)}{S_1(t)}dS_1(t) + \pi_2(t)\frac{V_\nu^{v_0,\pi}(t)}{S_2(t)}dS_2(t) \\ &\quad + V_\nu^{v_0,\pi}(t)\underbrace{\left(\nu(t)^\top\pi(t)\right)}_{\geq 0,\ \text{if } \pi(t)\in C_\pi} dt, \end{aligned} \tag{4.15}$$

which is the same SDE as in the original market, but with an additional drift term. Due to $\nu(t) \in [0,\infty)\times(-\infty,0] = C_\pi$ $\mathbb{Q}\otimes\mathcal{L}[0,T]$-a.e., the additional drift of the wealth process $V_\nu^{v_0,\pi}$ in $\mathcal{M}_\nu$ is guaranteed to be non-negative if $\pi(t) \in C_\pi$ $\mathbb{Q}$-a.s. $\forall t \in [0,T]$. Hence, the insurer always performs at least as good in $\mathcal{M}_\nu$ as it would have in the original market, i.e., $V_\nu^{v_0,\pi}(T) \geq V^{v_0,\pi}(T)$, provided that it abides by the allocation constraints C_π. The two wealth processes $V_\nu^{v_0,\pi}$ and $V^{v_0,\pi}$ coincide if and only if $\nu(t)^\top\pi(t) = 0$ $\mathbb{Q}\otimes\mathcal{L}[0,T]$-a.e..

Since we have changed the dynamics of the wealth process $V_\nu^{v_0,\pi}$ of an investor trading in $\mathcal{M}_\nu$, we need to adjust the class of admissible portfolio processes as well. For this purpose we define for every $\nu \in \mathcal{D}$

$$\begin{aligned} \mathcal{A}_{u,\nu}^\pi(v_0) &:= \Bigg\{\pi = \left((\pi_1(t),\pi_2(t))^\top\right)_{t\in[0,T]} \Bigg| \pi \text{ is prog. meas. and self-financing}, \\ &\qquad V_\nu^{v_0,\pi}(t) \geq 0 \ \mathbb{Q}\text{-a.s.}\forall t\in[0,T],\ \int_0^T ||\pi(t)V_\nu^{v_0,\pi}(t)||^2\,dt < \infty\,\mathbb{Q}\text{-a.s.}\Bigg\}; \\ C_{V_\nu}(\varepsilon) &:= \{V_\nu^{v_0,\pi}(T)|\mathbb{Q}\left(V_\nu^{v_0,\pi}(T) < G_T\right) \leq \varepsilon\}; \\ \mathcal{A}_{c,\nu}^\pi\left(v_0, C_{V_\nu}(\varepsilon)\right) &:= \left\{\pi \in \mathcal{A}_{u,\nu}^\pi(v_0)\,|\,V_\nu^{v_0,\pi}(T) \in C_{V_\nu}(\varepsilon)\right\}. \end{aligned}$$

[10]One obtains the risk-neutral measure for the original market $\mathcal{M}$ as $\widetilde{\mathbb{Q}}_{(0,0)^\top} = \widetilde{\mathbb{Q}}$.

Consider now the following portfolio optimization problem, which is an allocation-unconstrained formulation in $\mathcal{M}_\nu$:

$$\max_{\pi} \mathbb{E}\Big[U(V_\nu^{v_0,\pi}(T))\Big] \quad \text{s.t.}\, \pi \in \mathcal{A}_{c,\nu}^{\pi}\left(v_0, C_{V_\nu}(\varepsilon)\right). \tag{P_ε^ν}$$

For any fixed dual control process $\nu \in \mathcal{D}$ we see that every portfolio process admissible for the original problem (P_{ε,C_π}) is also admissible for the optimization problem (P_ε^ν) in the auxiliary market and yields at least the same terminal wealth (or expected utility) as in the original problem. This leads to the following condition, which can be used to verify that the optimal portfolio processes for $(P_\varepsilon^{\nu^*})$ and (P_{ε,C_π}) coincide for a particular dual process $\nu^* \in \mathcal{D}$.

Lemma 4.2.3. *Let $\nu \in \mathcal{D}$, π_ν be the optimal portfolio process for (P_ε^ν) in $\mathcal{M}_\nu$ and $V_\nu^{v_0,\pi_\nu}$ be the corresponding wealth process. If*

$$\pi_\nu(t)^\top \nu(t) = 0 \text{ and } \pi_\nu(t) \in C_\pi \quad \mathbb{Q}\text{-a.s., } \forall t \in [0,T], \tag{4.16}$$

then $V_\nu^{v_0,\pi_\nu}(t) = V^{v_0,\pi_\nu}(t)$ $\mathbb{Q}$-a.s. $\forall t \in [0,T]$ and π_ν is admissible and optimal for the original problem (P_{ε,C_π}).

Proof. See Appendix B.1 □

In the following, a dual process ν^* and the corresponding auxiliary market $\mathcal{M}_{\nu^*}$ are referred to as optimal, if they satisfy (4.16). Unfortunately, Lemma 4.2.3 only provides a convenient condition to verify optimality for a candidate dual process $\nu^* \in \mathcal{D}$, but not a constructive way of finding such a ν^*.

For a setting without additional terminal wealth constraints, i.e., for $\varepsilon = 1$, Cvitanic and Karatzas (1992) were able to prove several equivalent optimality conditions that offer a way of computing the optimal ν^* for the case of an investor following a power-utility function. They also derived an explicit form for ν^* using stochastic control methods. However, to solve (P_{ε,C_π}), we do not need to prove similar equivalencies, but we will have the opportunity to use a selection of their results from the wealth-unconstrained setting (P_{1,C_π}), when solving (P_{ε,C_π}). As a matter of fact, we show that the optimal ν^* is the same for both settings. Interestingly, due to the market coefficients μ, r, and σ being constant, it is sufficient to only consider constant dual vectors $\nu \in C_\pi$ from the start, as the optimal ν^* is constant.

The required results from Cvitanic and Karatzas (1992) are summarized in the following corollary.

Corollary 4.2.4. *Consider the optimization problems* (P_{1,C_π}) *and* (P_1^ν)*. Furthermore, set*

$$\nu^* := \underset{x \in C_\pi}{\operatorname{argmin}} \|\gamma + \sigma^{-1} x\|^2. \tag{4.17}$$

Then the optimal portfolio process $\pi^*_{u,\nu}$ *for* (P_1^ν)*, for any dual vector* $\nu \in C_\pi$*, is given as*

$$\pi^*_{u,\nu}(t) := \pi^*_{u,\nu} := \frac{1}{1-p} \Sigma^{-1}(\mu + \nu - r\mathbf{1}_2) \quad \textit{with } \Sigma := \sigma \cdot \sigma^\top. \tag{4.18}$$

Furthermore, for the particular dual vector $\nu^* \in C_\pi$*,* (P_{1,C_π}) *and* $(P_1^{\nu^*})$ *have the same optimal relative portfolio process* π^*_{u,ν^*}*, which satisfies* (4.16)*.*

Proof. See Appendix B.1. □

Remark

The optimal wealth corresponding to $\pi^*_{u,\nu}$ solving the VaR-unconstrained problem (P_1^ν) equals

$$V_\nu^{v_0,\pi^*_{u,\nu}}(t) = v_0 \exp\left(\left(r + (\mu + \nu - r\mathbf{1}_2)^\top \pi^*_{u,\nu} - \frac{1}{2}\|\sigma^\top \pi^*_{u,\nu}\|^2\right)t + \left(\pi^*_{u,\nu}\right)^\top \sigma W(t) \right),$$

where $t \in [0, T]$. If this optimal wealth satisfies

$$\mathbb{Q}(V_\nu^{v_0,\pi^*_{u,\nu}}(T) < G_T) \leq \varepsilon, \tag{4.19}$$

then $\pi^*_{u,\nu}$ also solves the VaR-constrained problem (P_ε^ν), i.e., the VaR constraint is non-binding.

VaR-constrained and allocation-unconstrained portfolio optimization

As noted in Kraft and Steffensen (2013), the introduction of terminal wealth constraints on a portfolio optimization problem with initial wealth $v_0 > 0$ frequently results in an optimal terminal portfolio value that is a derivative on the unconstrained optimal portfolio value with a possibly lower initial capital v_D, i.e., $0 < v_D \leq v_0$. In other words, we can express the optimal terminal portfolio value in the constrained problem as some financial derivative with payoff $D(\cdot)$ and the terminal optimal unconstrained portfolio value V^{v_D,π^*_u} as the underlying, where π^*_u is the corresponding optimal unconstrained relative portfolio process. Notable examples include

- Lower bound G_T on terminal wealth (Teplá (2001), Korn (2005)):

$$D(V^{v_D,\pi_u^*}(T)) = G_T + (V^{v_D,\pi_u^*}(T) - G_T)^+;$$

- VaR constraint with boundary G_T and level of confidence ε (Basak and Shapiro (2001)):

$$D(V^{v_D,\pi_u^*}(T)) = V^{v_D,\pi_u^*}(T) + (G_T - V^{v_D,\pi_u^*}(T))\mathbb{1}_{[k^\varepsilon,G_T]}(V^{v_D,\pi_u^*}(T))$$

  for appropriate parameters $0 \le k^\varepsilon \le G_T$, determined by the budget constraint and the confidence level.

As long as the derivative payoff $D(\cdot)$ satisfies sufficient regularity conditions, we can determine the optimal portfolio process via delta-hedging (see Lemma B.2.1 in Appendix B.2 for details). Below we give a closed-form expression for the optimal portfolio process π_ν^* corresponding to (P_ε^ν) for a dual vector $\nu \in C_\pi$, provided that the solution exists and the VaR-unconstrained solution does not satisfy the VaR constraint (4.19), i.e., the VaR constraint is binding. The result below is a corollary from Propositions 1 and 3 in Basak and Shapiro (2001). In Section 4.3, we use these sufficiency results for solution optimality to solve examples of interest as well as analyze the role of reinsurance.

Corollary 4.2.5 (Solution to VaR-constrained allocation-unconstrained (P_ε^ν))**.** *Consider the Black-Scholes market $\mathcal{M}_\nu$ for a dual vector $\nu \in C_\pi$ and the portfolio optimization problem (P_ε^ν) under the binding VaR constraint, i.e., the VaR-unconstrained solution to (P_1^ν) violates Condition* (4.19)*. Assume that $v_0 > \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_\nu(T) G_T \mathbb{1}_{\{\widetilde{Z}_\nu(T) < \bar{z}_\nu^\varepsilon\}}\right]$, where $\bar{z}_\nu^\varepsilon$ solves $\mathbb{Q}\left(\widetilde{Z}_\nu(T) > \bar{z}_\nu^\varepsilon\right) = \varepsilon$. Let $\pi_{u,\nu}^*$ be the optimal portfolio process for (P_1^ν) as defined in* (4.18)*. Let the parameters $0 < k_\nu^\varepsilon < G_T$, $0 < v_{D_\nu} < v_0$ be determined such that for*

$$D(V) := V + \big(G_T - V\big)\mathbb{1}_{[k_\nu^\varepsilon, G_T]}(V),$$

the following system of equations with respect to $(v_{D_\nu}, k_\nu^\varepsilon)$ is satisfied

$$\begin{aligned} \mathbb{E}^{\widetilde{\mathbb{Q}}_\nu}[\exp(-rT)\, D(V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(T))] &= v_0; \\ \mathbb{Q}(D(V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(T)) < G_T) &= \varepsilon. \end{aligned}$$

Then, $D\left(V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(T)\right)$ is the optimal terminal wealth for (P_ε^ν).

The time-t value, $t \in [0,T]$, of $D(V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(T))$ given $V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(t) = V$ can be expressed as

$$D_\nu(t,V) = V - \left[V\Phi(-d_1^\nu(G_T,V,t)) - G_T\exp\left(-r(T-t)\right)\Phi(-d_2^\nu(G_T,V,t))\right] + \left[V\Phi(-d_1^\nu(k_\nu^\varepsilon,V,t)) - G_T\exp\left(-r(T-t)\right)\Phi(-d_2^\nu(k_\nu^\varepsilon,V,t))\right], \tag{4.20}$$

where

$$\begin{aligned} \Gamma_\nu(t) &= \frac{p}{1-p}\left(r+\frac{\|\gamma_\nu\|^2}{2}\right)(T-t) + \left(\frac{p}{1-p}\right)^2\frac{\|\gamma_\nu\|^2}{2}(T-t), \\ d_2^\nu(x,V,t) &= \frac{(p-1)\ln\left(\frac{x}{V}\right)+(p-1)\Gamma_\nu(t)+\left(r-\frac{\|\gamma_\nu\|^2}{2}\right)(T-t)}{\|\gamma_\nu\|\sqrt{T-t}}, \\ d_1^\nu(x,V,t) &= d_2^\nu(x,V,t)+\frac{1}{1-p}\|\gamma_\nu\|\sqrt{T-t}. \end{aligned} \tag{4.21}$$

Lastly, $D\left(V_\nu^{v_{D_\nu},\pi^*_{u,\nu}}(T)\right)$ *is attained by the portfolio process*

$$\pi^*_\nu(t) := \pi^*_\nu(t, V_\nu^{v_{D_\nu},\pi^*_{u,\nu}}(t)) = \beta_\nu^D(t, V^{v_{D_\nu},\pi^*_{u,\nu}}(t))\cdot\pi^*_{u,\nu},$$

with

$$\begin{aligned} \beta_\nu^D(t,V) &= 1-\frac{G_T\exp\left(-r(T-t)\right)\left(\Phi(-d_2^\nu(G_T,V,t))-\Phi(-d_2^\nu(k_\nu^\varepsilon,V,t))\right)}{D_\nu(t,V)} \\ &\quad + \frac{(1-p)(G_T-k_\nu^\varepsilon)\exp\left(-r(T-t)\right)\phi(d_2^\nu(k_\nu^\varepsilon,V,t))}{D_\nu(t,V)\ \|\gamma_\nu\|\sqrt{T-t}} \geq 0 \end{aligned} \tag{4.22}$$

Proof. See Appendix B.1. □

Remarks to Corollary 4.2.5

Feasibility of the VaR constraint. As indicated in Footnote 5 in Basak and Shapiro (2001), the VaR constraint is feasible in (P_ε^ν), i.e., there exists a solution that satisfies the VaR constraint, if the following condition holds[11]:

[11] The correspondence between our notation and notation in Basak and Shapiro (2001): $G_T \leftrightarrow \underline{W}$, $\varepsilon \leftrightarrow \alpha$, $v_0 \leftrightarrow W(0)$, $\widetilde{Z}_\nu(t) \leftrightarrow \xi(t)$, $\overline{z}_\nu^\varepsilon \leftrightarrow \overline{\xi}$, $\mathbb{Q} \leftrightarrow \mathbb{P}$. We denote by $\widetilde{\mathbb{Q}}$ the risk-neutral probability measure and also use that $\mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_\nu(T)\mathbb{1}_{\{\widetilde{Z}_\nu(T)<\overline{z}_\nu^\varepsilon\}}\right] = \exp(-rT)\,\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_{\{\widetilde{Z}_\nu(T)<\overline{z}_\nu^\varepsilon\}}\right] = \exp(-rT)\,\widetilde{\mathbb{Q}}\left(\widetilde{Z}_\nu(T)<\overline{z}_\nu^\varepsilon\right)$

$$v_0 \geq v_0^{min} := \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_\nu(T) G_T \mathbb{1}_{\left\{\widetilde{Z}_\nu(T) < \bar{z}_\nu^\varepsilon\right\}}\right] = G_T \exp\left(-rT\right) \widetilde{\mathbb{Q}}_\nu \left(\widetilde{Z}_\nu(T) < \bar{z}_\nu^\varepsilon\right),$$

where $\bar{z}_\nu^\varepsilon$ solves $\mathbb{Q}\left(\widetilde{Z}_\nu(T) > \bar{z}_\nu^\varepsilon\right) = \varepsilon$. When the initial capital equals the minimal capital that ensures the existence of a solution to (P_ε^ν), i.e., $v_0 = v_0^{min}$, then the corresponding optimal terminal portfolio value is given by $G_T \mathbb{1}_{\left\{\widetilde{Z}_\nu(T) < \bar{z}_\nu^\varepsilon\right\}}$. As indicated in Remark 2 in Chen et al. (2018a), it can be seen as a limiting value of the case when $v_0 > v_0^{min}$ by sending the Lagrange multiplier corresponding to the budget constraint to $+\infty$. This is equivalent to sending v_{D_ν} to 0, in which case also $k_\nu^\varepsilon \to 0$. When $v_0 < v_0^{min}$, then (P_ε^ν) does not admit a solution.

Bindingness of the VaR constraint. The assumption that the solution to (P_1^ν) does not satisfy Condition (4.19) effectively means that the VaR constraint is binding in (P_ε^ν).

Existence of $(v_{D_\nu}, k_\nu^\varepsilon)$. The existence of $(v_{D_\nu}, k_\nu^\varepsilon)$ solving the budget constraint and the binding VaR constraint is related to the existence of the optimal Lagrange multiplier corresponding to the budget constraint. This multiplier is denoted by y in Basak and Shapiro (2001).

First, observe that:

$$D(V^{v_{D_\nu}, \pi_{u,\nu}^*}(T)) < G_T \Leftrightarrow V^{v_{D_\nu}, \pi_{u,\nu}^*}(T) < k_\nu^\varepsilon \overset{(B.9)}{\Leftrightarrow} v_{D_\nu} \exp\left(-\Gamma_\nu(0)\right) \left(\widetilde{Z}_\nu(T)\right)^{\frac{1}{p-1}} < k_\nu^\varepsilon$$
$$\Leftrightarrow \widetilde{Z}_\nu(T) > \left(\frac{k_\nu^\varepsilon}{v_{D_\nu}} \exp\left(\Gamma_\nu(0)\right)\right)^{p-1},$$

for any fixed $v_{D_\nu} > 0$. Thus, the equation related to the binding VaR constraint yields:

$$\varepsilon = \mathbb{Q}\Big(D(V^{v_{D_\nu}, \pi_{u,\nu}^*}(T)) < G_T\Big) = \mathbb{Q}\left(\widetilde{Z}_\nu(T) > \left(\frac{k_\nu^\varepsilon}{v_{D_\nu}} \exp\left(\Gamma_\nu(0)\right)\right)^{p-1}\right).$$

However, $\bar{z}_\nu^\varepsilon$ is by definition the unique solution to $\mathbb{Q}\left(\widetilde{Z}_\nu(T) > \bar{z}_\nu^\varepsilon\right) = \varepsilon$. Thus, for each given v_{D_ν}, there is a unique value of k_ν^ε such that the VaR constraint is satisfied with equality:

$$k_\nu^\varepsilon = v_{D_\nu} \exp\left(-\Gamma_\nu(0)\right) \left(\bar{z}_\nu^\varepsilon\right)^{\frac{1}{p-1}}.$$

The optimal v_{D_ν} is found from the budget equation. As it can be seen in the proof of Corollary 4.2.5, v_{D_ν} is linked to the Lagrange multiplier y via the following continuous bijection:

$$y = \left(v_{D_\nu} \exp\left(\Gamma_\nu(0)\right)\right)^{p-1}.$$

Thus, the optimal v_{D_ν} exists if and only if the optimal Lagrange multiplier y exists. The existence of the optimal y is guaranteed by Lemma A.3 Point 3 in Chen et al. (2018a) with $l = 0$ in the authors' notation[12]. The assumptions of that lemma obviously holds, i.e., $\mathbb{E}\left[\widetilde{Z}_\nu(T)\left(y\widetilde{Z}_\nu(T)\right)^{\frac{1}{p-1}}\right] < +\infty\ \forall y > 0$, which follows from direct calculation.

Readers interested in the details of proving the existence of the optimal Lagrange multiplier in portfolio optimization problems with VaR constraints are referred to Chen et al. (2018a), Dong and Zheng (2020), Nguyen and Stadje (2020).

VaR-constrained and allocation-constrained portfolio optimization

This section concludes the previous derivations by combining the results from Cvitanic and Karatzas (1992) and Basak and Shapiro (2001) to solve (P_{ε,C_π}). In short, we prove that the optimal payoff for (P_{ε,C_π}) is a derivative of the optimal payoff for (P_{1,C_π}) with some initial wealth $v_{D_{\nu^*}} \leq v_0$, by verifying (4.16) for $\nu = \nu^*$, as in Corollary 4.2.4.

Proposition 4.2.6. *Set*

$$\nu^* := \underset{x \in C_\pi}{\operatorname{argmin}} \|\gamma + \sigma^{-1}x\|.$$

Denote by π^*_{u,ν^*} the optimal portfolio process for $(P_1^{\nu^*})$, which is defined in (4.18). Let the parameters $0 \leq k^\varepsilon_{\nu^*} < G_T$, $0 < v_{D_{\nu^*}} \leq v_0$ be determined so that the financial derivative on the optimal terminal wealth for (P_{1,C_π}) with payoff

$$D(V) := V + (G_T - V)\mathbb{1}_{[k^\varepsilon_{\nu^*}, G_T]}(V)$$

satisfies the system of equations with respect to $(v_{D_{\nu^*}}, k^\varepsilon_{\nu^*})$

$$\begin{aligned} \exp(-rT)\,\mathbb{E}^{\widetilde{\mathbb{Q}}_{\nu^*}}\left[D\left(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\right] &= v_0; \\ \mathbb{Q}(D(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)) < G_T) &= \varepsilon. \end{aligned} \tag{4.23}$$

Then, $D\left(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)$ is the optimal terminal wealth for (P_{ε,C_π}). The corresponding optimal portfolio process π^* for (P_{ε,C_π}) is given by

$$\pi^*(t) := \pi^*(t, V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(t)) = \beta^D_{\nu^*}(t, V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(t)) \cdot \pi^*_{u,\nu^*}, \tag{4.24}$$

with $\beta^D_{\nu^*} > 0$ as in Corollary 4.2.5.

Proof. See Appendix B.1. □

The proof of Proposition 4.2.6 uses three facts:

[12]In Chen et al. (2018a), the Lagrange multiplier is denoted by ν. It should not be confused with our notation of ν denoting the dual process characterizing an auxiliary market.

- $\pi^*_{u,\nu^*} \in C_\pi$, according to Cvitanic and Karatzas (1992);
- $\pi^*_{\nu^*} = \beta^D_{\nu^*} \cdot \pi^*_{u,\nu^*}$ according to Basak and Shapiro (2001);
- $\beta^D_{\nu^*} \geq 0$.

As it can be seen in Proposition 5 in Basak and Shapiro (2001), all of these three facts are also true for a portfolio optimization problem with an expected shortfall constraint[13]. Hence, our methodology can also be used to calculate the optimal portfolio process for such an investor with a no-short-selling constraint.

Having obtained π^* that solves (P_{ε,C_π}), we apply Proposition 4.2.2 to get the solution to the original problem $(\bar{P}_{\varepsilon,\bar{C}_{\bar{\pi}}})$: $\bar{\pi}^* := (\Psi(t)\pi^*(t))_{t\in[0,T]}$, where $\Psi(t)$ is defined in (4.8).

4.2.3 Reinsurance optimality

We now want to answer the question when it is optimal for an insurer to buy reinsurance in the product under consideration.

We denote by

$$SR^\nu_i = \frac{\mu_i + \nu_i - r}{\sigma_i}$$

the Sharpe ratio of the corresponding asset in the auxiliary market $\mathcal{M}_\nu$ with $SR_i := SR^{(0,0)^\top}_i$, $i \in \{1, 2\}$.

Proposition 4.2.7. *It is optimal for the insurer to buy partial reinsurance if and only if:*

$$SR^{\nu^*}_2 < \rho \cdot SR^{\nu^*}_1, \tag{4.25}$$

where ν^* is given by (4.17) *in Corollary 4.2.4.*

Proof. See Appendix B.1. □

Condition (4.25) holds, e.g., when the correlation between the basic risky assets is sufficiently high and the asset that is not reinsurable is performance seeking, i.e., has a higher Sharpe ratio than the Sharpe ratio of the reinsurable risky asset.

Remark. Condition (4.25) indicates when the optimal unconstrained investment strategy $\pi^*_{u,\nu^*,2}$ w.r.t. $S^{\nu^*}_2$ (reinsurable risky asset in the market $\mathcal{M}_\nu$) is negative. Analogously, the condition $SR^{\nu^*}_1 < \rho \cdot SR^{\nu^*}_2$ indicates when $\pi^*_{u,\nu^*,1} < 0$. The optimal VaR-constrained

[13] Considering an expected shortfall constraint with a threshold G_T and a tolerance level ε, we would have $D(V^{v_D,\pi^*_u}(T)) = \frac{G_T}{k^\varepsilon} V^{v_D,\pi^*_u}(T)\mathbb{1}_{[0,k^\varepsilon]}(V) + (G_T - V^{v_D,\pi^*_u}(T))\mathbb{1}_{(k^\varepsilon,G_T]}(V^{v_D,\pi^*_u}(T)) + V^{v_D,\pi^*_u}(T)\mathbb{1}_{(G_T,\infty)}(V^{v_D,\pi^*_u}(T))$ and k^ε being determined through budget constraint and confidence level.

investment strategy π^* is obtained via multiplying π^*_{u,ν^*} by a positive factor $\beta(\cdot)$, see (4.24). So in general, π^* can have both short and long positions. In case $\pi^* \notin C_\pi$, the allocation constraint is binding and $\nu^* \neq (0,0)^\top$.

Next we provide two explicit numerical examples – one example for which the allocation constraint is binding and one example for which it is not.

Example 1 – base parameterization.

Using the parameter values from Table 4.1, we calculate:

$$SR_1 = 69.74\%,\ \rho \cdot SR_1 = 55.88\%,\ SR_2 = 51.64\%,\ \rho \cdot SR_2 = 41.37\%.$$

Then the optimal unconstrained solution in the market (S_0, S_1, S_2) is, as expected with respect to its signs, given by:

$$\pi^*_{u,1} = 27.84\%\,(SR_1 > \rho \cdot SR_2),\ \pi^*_{u,2} = -5.38\%\,(SR_2 < \rho \cdot SR_1).$$

The resulting optimal VaR-constrained solution satisfies the modified allocation constraint, i.e., $\beta(0) \cdot \pi^*_u(0) \in C_\pi = [0, +\infty) \times (-\infty, 0]$. Hence, $\nu^* = (0,0)^\top$, i.e., the (optimal) auxiliary market coincides with the original market. Multiplying the VaR-constrained solution in the auxiliary market with the matrix $\Psi(0)$ defined in (4.8), we get the corresponding VaR-constrained solution in the market (S_0, S_1, Put):

$$\bar{\pi}^*_1(0) = 33.48\%,\ \bar{\pi}^*_2(0) = 2.57\%$$

with obviously $\bar{\pi}^*(0) \in \bar{C}_{\bar{\pi}} = [0, +\infty) \times [0, +\infty)$. Hence, it is optimal to buy partial reinsurance.

Example 2 – base parametrization with $\rho = 70\%$.

In this case, we have the same Sharpe ratios as in Example 1 but:

$$\rho \cdot SR_1 = 48.82\%,\ \rho \cdot SR_2 = 36.15\%.$$

Then the optimal unconstrained solution in the market (S_0, S_1, S_2) is, as expected with respect to its signs, given by:

$$\pi^*_{u,1} = 27.84\%\,(SR_1 > \rho \cdot SR_2),\ \pi^*_{u,2} = 2.52\%\,(SR_2 > \rho \cdot SR_1).$$

The resulting optimal VaR-constrained solution does not satisfy the modified allocation constraint, i.e., $\beta(0) \cdot \pi^*_u(0) \notin C_\pi = [0, +\infty) \times (-\infty, 0]$. Hence, the allocation constraint is binding and requires a transition from (S_0, S_1, S_2) to the (optimal) auxiliary market $(S_0^{\nu^*}, S_1^{\nu^*}, S_2^{\nu^*})$. The auxiliary market is characterized by $\nu^* = (0, 0.0062)^\top$ and yields:

$$\pi^*_{u,\nu^*} = (29.47\%,\ 0\%)^\top.$$

Multiplying the unconstrained solution by $\beta(0) > 0$, we get the optimal VaR-constrained investment strategy $\pi^*(0)$ in the auxiliary market. Multiplying this strategy by the matrix $\Psi(0)$ defined in (4.8), we get the corresponding VaR-constrained solution in the market (S_0, S_1, Put):

$$\bar{\pi}_1^*(0) = 29.47\%, \ \bar{\pi}_2^*(0) = 0\%$$

with obviously $\bar{\pi}^*(0) \in \bar{C}_{\bar{\pi}} = [0, +\infty) \times [0, +\infty)$. Hence, it is not optimal to buy partial reinsurance. Note that (4.25) is violated in this case, as $SR_2^{\nu^*} = 54.46\%$ and $\rho \cdot SR_1^{\nu^*} = 48.82\%$.

4.3 Numerical studies

First, we explain how we choose the model parameters. Second, we analyze the potential benefits of reinsurance. In particular, we calculate how much capital can be saved and how much higher a guarantee can be offered to the client when reinsurance is used in the design of insurance products with capital guarantees. Finally, we address the question of measuring how much of the insurer's loss is covered by reinsurance and perform a sensitivity analysis of this measure as well as the optimal investment-reinsurance strategy w.r.t. the model parameters.

4.3.1 Model parametrization and numerical algorithms

We estimate our model parameters in accordance with the European market. We choose the estimation period from January 1, 2003, till June 8, 2020, to include both bearish (financial 2008-2009 crisis, COVID-19 pandemic in 2020) and bullish markets.

To estimate the risk-free rate we use Euro OverNight Index Average (EONIA) daily quotes. Parameters of S_1 are calibrated to the TecDAX daily data, whereas parameters of S_2 are estimated using DAX daily data. In this way we model the following situation:

1. the asset manager of a German insurer overweights the technological sector, the corresponding portfolio is more performance-seeking than the overall market;
2. the reinsurer agrees to sell protection only on the overall market index, represented by the DAX in our study.

In general, the asset manager of the insurer focuses on subindustries of its expertise. For the reinsurer these industries may be too risky or it does not have enough expertise in those areas. Therefore, it does not reinsure the specific portfolio of the insurer. Note that in the US market a comparable example would be the S&P 500 Health Care Index or the S&P 500 Consumer Discretionary Index as S_1 and the S&P 500 Index as S_2. For estimating the risk-free rate in the US market, one could use the Effective Federal Funds Rate (EFFR).

In Table 4.1 we summarize model parametrization.

Parameter	Value	Explanation
r	1.02%	EONIA
μ_1	17.52%	TecDAX rate of return
μ_2	12.37%	DAX rate of return
σ_1	23.66%	TecDAX volatility
σ_2	21.98%	DAX volatility
ρ	80.12%	TecDAX and DAX correlation
$S_0(0)$	1	For convenience
$S_1(0)$	1	For convenience
$S_2(0)$	1	For convenience
v_0	100	For convenience
T	10	Long-term investment
G_T	100	Representative guarantee in the German market
ε	0.5%	High client's confidence in the guarantee
p	-9	Corresponds to an RRA coefficient of 10
π_B^{CM}	29.47%	Optimal initial proportion of money invested in the risky asset in the case of no reinsurance

Table 4.1: Model parametrization summary.

We choose the capital guarantee G_T as 100% of the initial investment to reflect the current situation in the "German Market". In the past decades, insurers offered a positive guaranteed rate of return on clients' paid contributions. Due to a low interest-rate environment and other challenges, insurers have recently started offering products with a full guarantee on paid capital but without any positive rate of return. In some products only a partial guarantee is embedded, e.g., the product ERGO Rente Guarantee allows a customer to choose between 80% and 100% of the invested capital. Allianz offers policyholders a choice of guarantee levels between 60% and 90% of clients' contributions.

Our choice of $p = -9$ leads to an insurer's relative risk aversion (RRA) coefficient of 10, which is motivated by several aspects. In general, the RRA coefficient is a compromise between common RRA coefficients in theoretical research on long-term portfolio optimization and empirical evidence on RRA of mutual funds. On the one hand, Broeders et al. (2011) and Chen et al. (2018a), investigating longer-term investment strategies in continuous time, set the RRA to 3 in the base case. Brandt et al. (2005), Garlappi and Skoulakis (2010) and Cong and Oosterlee (2017), analyzing optimal asset allocation in discrete time, consider higher RRA coefficients in their numerical studies, namely from 5 to 15. On the other hand, empirical research shows that the median and mean RRA coefficient of mutual fund managers are 5.75 and 2.43 respectively, see Table I in Koijen (2014). Since mutual funds have less restrictive regulatory constraints than insurers do, it is reasonable to assume that RRA of the latter will be higher. Therefore, we set the RRA coefficient to 10 in the base case and investigate a range of RRA coefficients from

5 to 15 in the sensitivity analysis section. Note that the insurer's optimal 1-year investment strategy without reinsurance — the solution to $(\bar{P}_{0.5\%,[0,+\infty)\times\{0\}})$ for $T = 1$ — has approximately 15% of portfolio value invested in the risky asset. This value belongs to the range $10\% - 15\%$, which is a representative range for the proportion of wealth insurance companies invest in the risky assets such as listed and private equity according to Gründl et al. (2016).

Appendix B.2 contains some propositions relevant for the numerical studies. In Proposition B.2.4 we provide the explicit formula of the left-hand side of the system of non-linear equations (SNLE) from Corollary 4.2.5. In Proposition B.2.5 we calculate explicitly the insurer's value function, which is needed for Subsection 4.3.2. These two propositions use auxiliary Lemma B.2.2 and Lemma B.2.3, which are also provided for completeness.

For solving the SNLE, we convert it to a minimization problem and apply the Sequential Quadratic Programming approach. For finding the roots of standalone non-linear equations appearing in the welfare loss and the guarantee gain analysis, we use the bisection method.

4.3.2 Monetary benefits of reinsurance

The first natural question is whether the insurer needs reinsurance at all. We assume that the insurer chooses the constant-mix strategy $\pi_B = (0, \pi_B^{CM})^\top = (0, \pi^*_{DN,1}(0))^\top$ that has the same proportion of wealth invested in S_2 as the proportion of wealth invested in S_1 in the insurer's optimal investment strategy under the no-reinsurance constraint. Here π^*_{DN} solves $(\bar{P}_{0.5\%,[0,+\infty)\times\{0\}})$ and DN stands for **d**ynamic (strategy with) **n**o reinsurance. In the base case, this leads to $\pi_B^{CM} = \pi^*_{DN,1}(0) = 29.47\%$. The insurer's optimal relative portfolio process at time $t = 0$ is given by:

$$\bar{\pi}_0(0) = 63.95\%, \qquad \bar{\pi}_1(0) = 33.48\%, \qquad \bar{\pi}_2(0) = 2.57\%.$$

The optimal initial investment in terms of asset units is given by:

$$\bar{\varphi}_0(0) = 65.85, \qquad \bar{\varphi}_1(0) = 33.48, \qquad \bar{\varphi}_2(0) = 0.67,$$

and the price of one put is approximately equal to 3.85. We see that it is optimal for the insurer to buy partial reinsurance, which costs about 2.5% of the initial portfolio value. Interestingly, the optimal initial reinsured proportion of the benchmark portfolio is 67%. As we will see in Section 4.3.3, this partial reinsurance still leads to a high level of the insurer's expected loss covered by the reinsurer.

Welfare loss analysis

In this subsection, we calculate the monetary benefit to the insurer if the insurer follows the optimal investment-reinsurance strategy instead of implementing a suboptimal one. We determine the wealth-equivalent utility loss (WEUL) as per Definition 2.3.5. It is denoted by $WEUL(\bar{\pi}^*, \pi_{SS})$ and represents the proportion of the initial wealth "lost" when a suboptimal strategy π_S instead of the optimal strategy $\bar{\pi}^*$ is followed. In particular, $WEUL(\bar{\pi}^*, \pi_S)$ is the solution to the following equation:

$$\mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0(1-WEUL(\bar{\pi}^*,\pi_S)),\bar{\pi}^*}(T)\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\pi_S}(T)\right)\right]. \tag{4.26}$$

So if the insurance company would follow an optimal investment-reinsurance strategy, the company would have needed $100 \cdot WEUL(\bar{\pi}^*, \pi_S)\%$ less initial capital to match the expected utility from the suboptimal strategy π_S. If the expected utility from a suboptimal strategy is acceptable for both the insurer and the client, then switching to the optimal strategy may decrease product costs due to the saved $100 \cdot WEUL(\bar{\pi}^*, \pi_S)\%$ of the initial investment.

We consider the following suboptimal strategies π_S:

1. the optimal dynamic strategy of the insurer under the no-reinsurance constraint, i.e., π^*_{DN} that solves $(\bar{P}_{0.5\%,[0,+\infty)\times\{0\}})$. If the VaR-constraint is non-binding, then this is the optimal unconstrained investment strategy.
2. the $(15\%, 0\%)^\top$ constant-mix strategy that approximates the long-term investment strategy of an average life insurer according to Gründl et al. (2016), which we denote by π_{CN} where CN stands for **c**onstant-mix (strategy with) **n**o reinsurance.

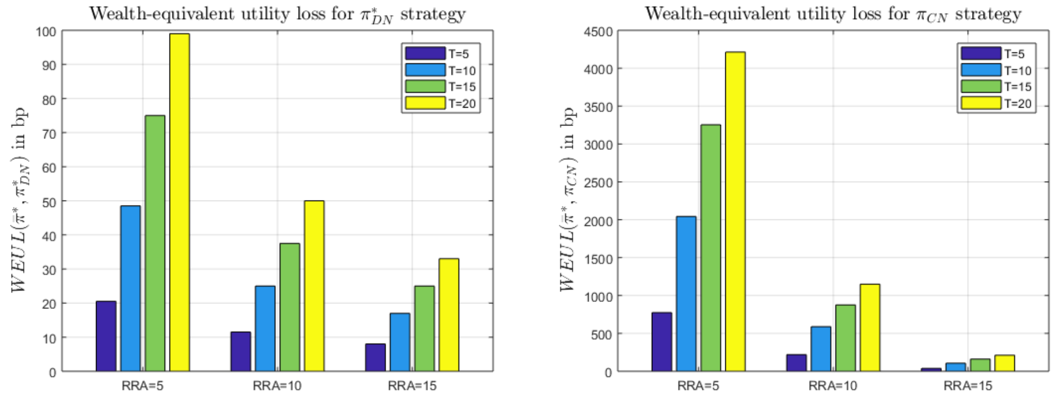
We obtain the following WEULs:

1. $WEUL(\bar{\pi}^*, \pi^*_{DN}) = 25\text{bp}$[14], i.e., replacing a product with the optimal no-reinsurance strategy with a product with the optimal investment-reinsurance strategy can make the product 25bp cheaper to the customer without any loss in the insurer's expected utility;
2. $WEUL(\bar{\pi}^*, \pi^*_{CN}) = 588\text{bp}$, i.e., a product with optimal investment-reinsurance strategy requires 5.88% less initial capital to reach the same expected utility as the suboptimal constant-mix $(15\%, 0\%)^\top$ strategy yields.

In Figure 4.2, we show the impact of the insurer's risk aversion and investment horizon on WEUL. The more risk averse the insurer, the less WEUL. For π^*_{DN}, this measure exhibits roughly linear dependence on RRA, whereas for π_{CN} it shows a rather convex behaviour w.r.t RRA. The longer the investment period, the larger the WEUL. For both considered suboptimal strategies, WEUL shows approximately linear dependence on T. For very risk-averse insurers and short investment horizons, product costs that can be saved by using optimal reinsurance are relatively low. However, less risk-averse insurers offering mid to long-term equity-linked products with capital guarantees can decrease the corresponding product costs significantly, especially in comparison to products with

[14] bp stands for "basis point", 1bp= 0.01% = 0.0001

underlying strategy π_{CN}. For example, for an insurer with $RRA = 5$ and $T = 15$ the cost reduction is about 32%.



(a) WEUL w.r.t. π^*_{DN}.

(b) WEUL w.r.t. π_{CN}.

Figure 4.2: Impact of risk aversion and investment horizon on WEUL.

Overall, the results in this section indicate that the inclusion of dynamic reinsurance in the design of equity-linked insurance products with a capital guarantee decreases the product costs for the clients. The actual "loss" of capital from investing suboptimally in practice may be different, as there are transaction costs, safety loadings in pricing reinsurance contracts, discrete trading times, jumps in asset prices, etc.

For a broader view, we also provide for each of the these three strategies the corresponding risk-return profile and the probability that the terminal portfolio value falls below the guarantee G_T in Table 4.2.

	$\bar{\pi}^*$	π^*_{DN}	π_{CN}
Annualized return	6.11%	6.06%	3.56%
Annualized standard deviation of return	12.85%	12.71%	5.05%
Probability of not reaching G_T	0.5%	0.5%	0.0011%

Table 4.2: Strategies' risk-return profiles and probabilities of not reaching G_T

We see that the optimal dynamic investment strategy with reinsurance and the one without reinsurance have very similar risk-return profiles and fully use the available risk budget in the optimization problem, i.e., the corresponding underperformance probabilities are equal to the VaR probability 0.5%. The CN strategy, on the other side, does not fully use the available risk budget and thus loses more than 2.5% in performance (annualized return).

Guarantee gain analysis

Here we measure the benefit of the optimal investment-reinsurance strategy in terms of a potential increase in the capital guarantee. We calculate the guarantee-equivalent utility gain (GEUG), denoted by $GEUG(\bar{\pi}^*, \pi_S)$, that indicates the proportion by which the terminal guarantee G_T to the client can be increased such that the expected utility of the insurer following $\bar{\pi}^*$ and the correspondingly higher guarantee is equal to the expected utility of the insurer following the suboptimal strategy π_S with the original guarantee G_T. Denote by $\bar{V}^{v_0,\bar{\pi}^*}(T|G_T)$ the portfolio value at time T with the initial capital v_0, relative portfolio process $\bar{\pi}^*$ and guarantee G_T. Then, $GEUG(\bar{\pi}^*, \pi_S)$ is the solution to the following equation:

$$\mathbb{E}^{\mathbb{Q}}\left[U(\bar{V}^{v_0,\bar{\pi}^*}(T|(1+GEUG(\bar{\pi}^*,\pi_S))\cdot G_T))\right] = \mathbb{E}^{\mathbb{Q}}\left[U(\bar{V}^{v_0,\pi_S}(T|G_T))\right]. \tag{4.27}$$

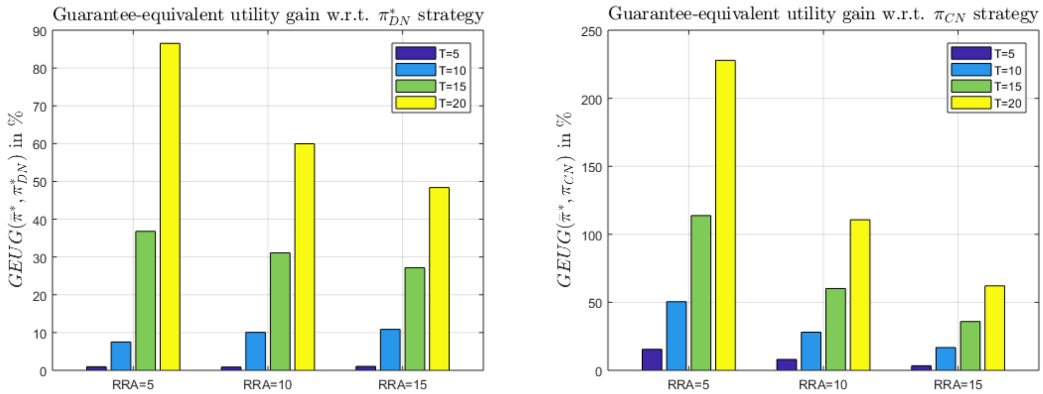
We obtain the following GEUGs:

1. $GEUG(\bar{\pi}^*, \pi^*_{DN}) = 10.08\%$, i.e., a product with optimal investment strategy without reinsurance and with a guarantee of 100% of the client's initial endowment at product maturity (0% annualized guaranteed return) can be replaced — without any loss in the insurer's expected utility — by a product with optimal investment-reinsurance strategy and a guarantee of 110% of the client's initial endowment (0.96% annualized guaranteed return);
2. $GEUG(\bar{\pi}^*, \pi^*_{CN}) = 28.09\%$, i.e., a product with a constant-mix $(15\%, 0\%)^\top$ investment strategy without reinsurance and a guarantee of 100% of the initial endowment at product maturity can be replaced — without any loss in the insurer's expected utility — by a product with optimal investment-reinsurance strategy and a guarantee of 128% of the client's initial contribution (2.5% annualized guaranteed return).

Figure 4.3 illustrates how the insurer's risk aversion and investment horizon influence GEUG. The more risk averse the insurer, the less GEUG. With increasing risk aversion, the optimally behaving insurer invests more in bonds and less in stocks and reinsurance, as it will be shown in Section 4.3.4. Since a risk-free investment has a comparably low rate of return, GEUG decreases. We also observe that GEUG is convex w.r.t. RRA in both cases, π^*_{DN} and π_{CN}. The longer the investment period, the larger the GEUG. This dependence also illustrates convexity w.r.t. T for both considered suboptimal strategies. We see that even very risk-averse insurers with short to mid-term equity-linked products can significantly increase their guarantee levels without any loss in expected utility. For $RRA = 15$ and $T = 5$, the guarantee in the product following the optimal investment-reinsurance strategy is increased from 100% to 108% (1.55% guaranteed annualized return) in comparison to a product with the strategy π_{CN}[15]. For less risk-averse insurers and products with longer investment horizons, GEUG is even more prominent. For

[15]For $T = 5$, $\mathbb{Q}\left(\bar{V}^{v_0,\pi_{CN}}(T) < G_T\right) \approx 1.5\%$.

example, for $RRA = 5$ and $T = 15$, the insurer with the optimally managed reinsurance can guarantee that the client's terminal payoff equals at least 135% of the initial contribution (about 2% annualized guaranteed return) and achieve the same expected utility as for a product with the underlying strategy π^*_{DN} and a guarantee of only 100% of the initial contribution.



(a) GEUG w.r.t. π^*_{DN}.

(b) GEUG w.r.t. π_{CN}.

Figure 4.3: Impact of risk aversion and investment horizon on GEUG.

Overall, the inclusion of dynamic reinsurance in the design of equity-linked insurance products with capital guarantees can lead to substantial increase in the guarantee levels that insurance companies can offer to their clients. The actual guarantee gain in practice may be different due to reasons mentioned at the end of the previous subsection.

4.3.3 Reinsurance proportion

In this section we briefly address a natural question about reinsurance: how much protection against the insurer's loss does the reinsurance provide? The precise measurement of the reinsurance level/proportion is challenging as:

1. the underlying portfolio in the reinsurable portfolio is not the same as the insurer's actual portfolio due to different assets ($S_1 \neq S_2$);

2. the corresponding relative portfolio processes are different ($\pi_B^{CM} \neq \bar{\pi}_1^*(0)/(1 - \bar{\pi}_2^*(0))$), where the former term is the proportion of money invested in S_1 in the benchmark portfolio and the latter term is the proportion of investment in S_1 in the insurer's portfolio after subtracting money spent on reinsurance;

3. the initial capital of the reinsured benchmark portfolio is slightly higher than the capital invested in S_0 and S_1 by the insurer due to the purchase of reinsurance.

The first insight into the reinsurance level can be captured by the number of put options in the insurer's portfolio $\bar{\varphi}_2^*(0)$, i.e., 1 put approximately hedges the portfolio of the insurer. Approximately because of the above-mentioned points 1 to 3. We could also look at the number of puts adjusted by the correlation between the insurer's portfolio and the reinsured portfolio $\rho\bar{\varphi}_2^*(0)$.

In the literature on reinsurance, two types of reinsurance are differentiated: proportional reinsurance and excess-of-loss reinsurance. In the former reinsurance type, the insurer's total loss is shared proportionally between the insurer and the reinsurer. However, the reinsurance is written on the exact portfolio the insurer has. Motivated by it, we consider the proportion of expected loss coverage (PELC), which we define as follows:

$$PELC_t = \frac{\textit{Amount of reinsurance at } t \times \textit{Expected coverage from } 1 \textit{ reinsurance contract}}{\textit{Expected total loss of insurer}}$$
$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\bar{\varphi}_2^*(t)\left(G_T - V^{v_0,\pi_B}(T)\right)^+ |\mathcal{F}(t)\right]}{\mathbb{E}^{\mathbb{Q}}\left[\left(G_T - V^{v_0,\bar{\pi}^*}(T)\right)^+ |\mathcal{F}(t)\right]} = \frac{\bar{\varphi}_2^*(t)\mathbb{E}^{\mathbb{Q}}\left[\left(G_T - V^{v_0,\pi_B}(T)\right)^+ |\mathcal{F}(t)\right]}{\mathbb{E}^{\mathbb{Q}}\left[\left(G_T - V^{v_0,\bar{\pi}^*}(T)\right)^+ |\mathcal{F}(t)\right]}.$$

The calculation of PELC requires Monte-Carlo simulations due to the sophisticated optimal investment-reinsurance strategy of the insurer, which is needed to estimate the denominator of PELC.

For the base case we have:

$$\bar{\varphi}_2^*(0) = 0.67, \quad \rho\bar{\varphi}_2^*(0) = 0.54, \quad PELC_0 = 138.52\%, \quad \rho PELC_0 = 110.82\%.$$

We see that the initial optimal reinsurance strategy implies buying 67% reinsurance, which leads to a correlation-corrected $PELC_0$ slightly higher than 100%. Taking into account the above-mentioned challenges 1–3, we find these numbers reasonable.

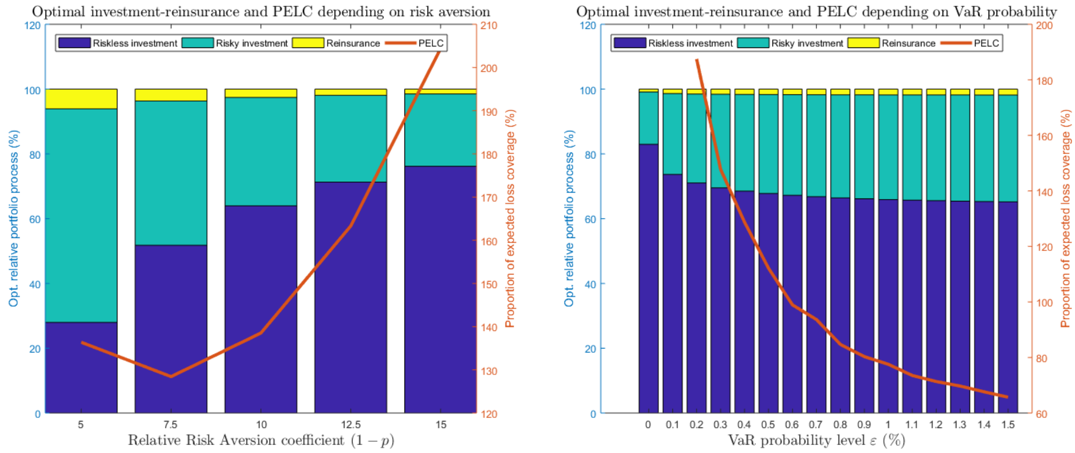
4.3.4 Sensitivity analysis of optimal investment-reinsurance strategies

In this subsection, we summarize the impact of changes in model parameters on the optimal investment-reinsurance strategy. Not to make the chapter unnecessarily longer, we provide figures depicting the sensitivity analysis results only for the risk-aversion parameter and the VaR probability threshold.

As mentioned in Subsection 4.3.1, we explore the risk-aversion parameter values $1-p = RRA \in \{5, 7.5, 10, 12.5, 15\}$. The higher the RRA coefficient, the less the optimally behaving insurer invests in the risky assets and the less money is spent on reinsurance. However, $PELC_0$ increases as the put option becomes cheaper due to the decreasing riskiness of the benchmark portfolio. This is illustrated in Subfigure 4.4a.

For the VaR probability $\varepsilon \in \{0\%, 0.1\%, 0.2\%, ..., 1.5\%\}$, we observe that the higher ε, the more money is invested in both the risky asset and the reinsurance. However, the $PELC_0$

gradually decreases due to the increasing riskiness of the insurer's optimal investment strategy and the insurer's inability to hedge out all the residual risk arising due to a less risky reinsurable benchmark portfolio. Since for shorter investment horizons the influence of the VaR constraint is more prominent, we illustrate for $T = 5$ the sensitivity of $\bar{\pi}^*$ and $PELC_0$ w.r.t. ε in Subfigure 4.4b.



(a) Sensitivity of $\bar{\pi}^*$ and $PELC_0$ w.r.t. p. (b) Sensitivity of $\bar{\pi}^*$ and $PELC_0$ w.r.t. ε.

Figure 4.4: Sensitivity of the optimal strategy w.r.t. risk aversion and VaR probability.

Varying the weight of the risky asset in the benchmark portfolio $\pi_B^{CM} \in \{\pi_{DN,1}^*(0) - 15\%, \pi_{DN,1}^*(0) - 10\%, ..., \pi_{DN,1}^*(0) + 15\%\}$, we find no change in the optimal investment strategy with respect to the risky asset. However, more money is invested in reinsurance and the $PELC_0$ increases.

The higher the interest rate $r \in \{-2\%, -1\%, 0\%, 1\%, 2\%\}$, the less money the optimally behaving insurer invests in the risky asset. However, more money is invested in reinsurance, which, in conjunction with a decreasing price for the reinsurance, leads to an increase of $PELC_0$.

For an increasing investment horizon $T \in \{1, 5, 10, 15, 20\}$ we observe that both the optimal initial investment in the risky assets as well as the proportion of initial wealth invested in reinsurance increase. The $PELC_0$ gradually increases too.

When the terminal capital guarantee $G_T \in \{0.7 \cdot v_0\,, 0.8 \cdot v_0, \ldots, 1.1 \cdot v_0\}$ increases, the insurer's optimal investment in stocks slightly decreases. Simultaneously, the insurer invests more money in reinsurance, even though the price of the reinsurance contract surges. The $PELC_0$ gradually decreases.

In all numerical studies we also observe that the inclusion of reinsurance in the product design increases the insurer's optional initial risky asset exposure by up to 10% in comparison to the optimal investment strategy in the no-reinsurance case.

5 Optimal risk sharing between an insurer and a reinsurer

> For what you do to others, you do to yourself.
>
> Eckhart Tolle

In the previous chapter, we learned how an insurance company should optimally invest and share with a reinsurance company the risk of not achieving the capital guarantee in the context of an equity-linked insurance product. We had two assumptions on the risk-sharing process, namely that the reinsurance can be continuously adjusted and it is fairly priced. Even though the former assumption is common in the academic literature on the optimal investment-reinsurance strategies, a reinsurance contract in practice is not dynamically traded or adjusted. The latter assumption meant that the price of reinsurance was equal to the fair price of the corresponding put option, i.e., the reinsurer did not charge anything extra. However, reinsurance companies frequently include in the reinsurance premium a so-called safety loading, in addition to the expected loss (see, e.g., page 219 in Albrecher et al. (2017)). Therefore, in this chapter, which is a reproduction of Havrylenko et al. (2022) with minor changes, we revisit the problem from the previous chapter and derive the optimal investment and risk-sharing strategies for an insurer and a reinsurer in a model without the above-mentioned assumptions. Moreover, we model the interaction between the parties in a more realistic way such that the actions of one company directly influence the actions of the other. Thus, we mainly concentrate on the optimal risk sharing between an insurer and a reinsurer in the context of equity-linked insurance products.

Next we describe in more detail the practical problem we solve in this chapter. An insurance company sells an equity-linked product with a capital guarantee to a customer. The insurer is willing to execute its individual investment strategy in some sub-universe of a global liquid market. It does not want to disclose its investment strategy to a third party like other insurers or reinsurers. Focusing on its own investment strategy, the insurer can buy (partial) reinsurance on a simple investment strategy in the global liquid market or even another liquid sub-universe of the global market. To get a reasonable reinsurance contract, the "simple" investment strategy to be reinsured should be transparent, easy to understand as well as to implement and it should be sufficiently correlated with the individual investment strategy of the insurer. A typical candidate is a constant-mix strategy, as also used in Chapter 4. However, in contrast to that chapter, we model the reinsurance contract as a long-maturity put option that is not continuously traded in the market. Instead, it is bought from a reinsurance company at product inception[1]. Due to the long maturity of the put option, which models the reinsurance contract, and the impossibility to trade it on exchanges, the reinsurance company can charge an additional safety loading, i.e., a price margin above the expected discounted loss of the contract.

Since a reinsurance contract is an agreement between a primary insurance company and a reinsurance company, it implies an interaction (negotiation) between the parties of the contract. The number of insurance companies in the world is significantly larger than then number of reinsurance companies (e.g., see Albrecher et al. (2017), Chen and Shen (2018)). Moreover, reinsurance companies are usually larger than primary insurance companies and act internationally, whereas insurance companies often act on a national level. Therefore, the reinsurance company has a stronger position in the negotiation process about the terms of the reinsurance agreement (e.g., see Chen and Shen (2018), Bai et al. (2022)), is likely to have more investment opportunities and is able to assess the reaction of the insurance company to various terms of a reinsurance contract, in particular, reinsurance premium. Due to the above aspects of the insurer-reinsurer interaction, a hierarchical game is a reasonable framework to model risk-sharing through reinsurance contracts.

The above-mentioned situation suits well to the concept of a Stackelberg game. Originally, it was introduced in Stackelberg (1934) in the context of two manufacturing companies competing on the quantity of their product. Later this concept found applications in many other economic situations with a leader-follower relationship between economic agents. Stackelberg games have a hierarchical structure, where the leader "dominates" the follower. This means that the leader moves first and selects its strategy knowing the future optimal response of the follower, whereas the follower moves afterwards and chooses its strategy depending on the choice of the leader. Therefore, in this chapter, we set up and solve a Stackelberg game between an insurance company and a reinsurance company in the context of an equity-linked insurance product with a capital guarantee. The aim of each party is to maximize its expected utility of the total terminal wealth, where the total terminal wealth consists of the terminal portfolio value including the reinsurance payoff. However, the reinsurer (leader) moves first, knowing the optimal response of the insurer (follower), and chooses its investment strategy as well as the price of reinsurance, which consists of the fair price (pure reinsurance premium) of the put option and a safety loading. Afterwards, the insurer (follower) selects its investment strategy and the amount of reinsurance it buys, knowing the action of the reinsurer (leader). The solution to a Stackelberg game is called the Stackelberg equilibrium. Sometimes it is also called the Bowley solution, since a sequential game between a manufacturer and a supplier of the material was considered in Bowley (1928).

[1] In practice, the reinsurance contract can be adjusted at regular intervals, e.g., annually. To solve the corresponding problem with several discrete adjustments, our model can be applied sequentially.

Intuitively, if the reinsurer takes into consideration only its own interests when negotiating the terms of a reinsurance contract, the insurer may find the reinsurance premium unfairly high and, therefore, may not buy as much reinsurance as the reinsurer expects. On the other hand, if the reinsurance premium is too low, the insurer may share more risk via reinsurance, but the profitability of this deal may be suboptimal for the reinsurer due to the low price. Hence, the result for the reinsurer may be worse than expected.

Within the above-described framework considered in this chapter, we answer the following research questions:

1. How can we analytically find the Stackelberg equilibrium, i.e., derive the optimal investment and risk-sharing strategy of each party in the game?
2. What happens with expected utilities of parties in case they deviate from the Stackelberg equilibrium, e.g., the insurance company does not buy reinsurance or the reinsurance company offers a reinsurance contract at a price lower than the equilibrium one?
3. Which impact do model parameters have on the Stackelberg equilibrium?

Next we provide an overview of the relevant literature and organize it in two groups. The first stream of literature is related to modeling the investment and risk-sharing dynamics for insurance and reinsurance companies. The second stream of literature is related to advanced portfolio optimization techniques, which we combine to solve the novel Stackelberg game considered in this chapter.

In actuarial literature, there are different ways of modeling financial risk sharing between a reinsurer and an insurer in the presence of investment opportunities. For instance, Li et al. (2016) considers a general insurance group that holds shares of a primary insurance company and a reinsurance company. Maximizing the expected exponential utility of the weighted sum of the terminal wealth of each company, the authors find the optimal investment as well as proportional-reinsurance strategies via SCA. Gu et al. (2020) formulate the interaction between the reinsurer and the insurer as a principal-agent problem. In this problem, the reinsurer maximizes the expected utility of its terminal wealth assuming the worst-case scenario that depends on the insurer's choice of its retention level in the excess-of-loss reinsurance. The researchers derive the optimal investment and risk-sharing strategies via SCA. To the best of our knowledge, Chen and Shen (2018) is the first paper that formulated and solved a Stackelberg game between the insurer and the reinsurer in the context of a reinsurance contract and with the presence of investment opportunities. The researchers assume that the reinsurer offers reinsurance on the whole claim process of the insurer and the contract specifications, i.e., the reinsurance premium and the amount of reinsurance, can be dynamically adjusted. Assuming the leadership of the reinsurer and the followership of the insurer, the researchers derive the solution to the game by means of SCA and backward SDEs. Further papers on Stackelberg games in the context of the insurer-reinsurer interaction are Chen et al. (2019), Bai et al. (2022), Chen et al. (2020), Yuan et al. (2021), Yang et al. (2021), and Bai et al. (2021). In these literature sources, the authors assume that the whole portfolio of aggregated insurance obligations is reinsured and that the specifications of the reinsurance contract can be continuously adjusted over the investment horizon. Chen and Shen (2018), Chen et al. (2019) and Chen et al. (2020) assume that the surplus process of each party is fully invested in a risk-free asset. In contrast to them, Bai et al. (2022) extend the investment universe of the parties by adding one risky asset.

Next we provide a few more remote literature sources in the area of Stackelberg games in the insurance-reinsurance context. Gavagan et al. (2022) considers a Stackelberg game between a reinsurer and an insurer with model uncertainty. Assuming that the reinsurer minimizes a so-called Range-Value-at-Risk[2] of its terminal payoff and the insurer maximizes its expected utility of its terminal payoff, the researchers derive a Stackelberg equilibrium consisting of the optimal static safety loading, the indemnity function[3] and the distribution function of the insurer's loss, which comes from model uncertainty. Assuming that the preferences of the reinsurer and the insurer are described by monotone risk measures, Boonen and Ghossoub (2023) derive the optimal static reinsurance premium and indemnity function in a Stackelberg equilibrium, which they call the Bowley solution. Asmussen et al. (2019) sets up and solves a Stackelberg game that models the situation when two insurance companies compete for clients. Assuming that each party maximizes its expected present discounted wealth and that the surplus processes of each company is fully invested in a risk-free asset, the researchers derive the optimal insurance premiums.

We would like to emphasize that we are not aware of any academic literature sources that study optimal investment-reinsurance problems in the context of a Stackelberg game where reinsurance is static and offered within an equity-linked insurance product. The research results stated in this chapter fill this gap.

Since we have provided the overview of portfolio optimization literature in the previous chapters, we mention now only three articles including the methods we use to derive the equilibrium of the considered Stackelberg game. This game is bi-level and consists of two utility-maximization sub-problems whose peculiarities do not allow direct usage of standard solution methods.

In particular, when solving the insurer's optimization problem, we are dealing with two peculiarities. First, there is an additional static control variable, which represents the fixed long position in the put option. Second, the put option is not spanned by the risky assets that the insurer can continuously trade.

To solve such an optimization problem, we combine the ideas of Desmettre and Seifried (2016) and Cvitanic and Karatzas (1992). Cvitanic and Karatzas (1992) solve a constrained optimization problem, where the constraint on the investor's relative portfolio process is given by a convex set. As also discussed in the previous chapter, their

[2] It is a risk measure that includes Value-at-Risk as a special case.

[3] An indemnity function is a function that quantifies the part of the insurer's loss transferred to a reinsurer.

methodology is based on constructing a family of unconstrained optimization problems in different auxiliary markets and on finding the unconstrained optimization problem that models the required portfolio constraint in the original market. In our case, an auxiliary market is the global market with changed drift coefficients, where the reinsurance contract can be replicated by continuously traded assets. This way we make the reinsurance contract spanned and derive closed-form solutions via the methodology introduced in Desmettre and Seifried (2016). In Desmettre and Seifried (2016), the authors generalize MA to solve optimization problems where the investor can additionally invest in a fixed-term security (the reinsurance contract in our setting) at the beginning of the investment period. In the first step of the generalized MA, one has to derive the optimal investment in the risky asset for a fixed position in the fixed-term security, which requires the inversion of conditional random utility functions[4]. In the second step of the generalized MA, one has to determine the optimal investment in the fixed-term security given the investment in the liquid risky assets by maximizing the corresponding value function with respect to a fixed-term position.

When solving the optimization problem of the reinsurer, we face the following two challenges. First, the reinsurer has a fixed short position in the reinsurance contract (put option), which is predetermined by the insurer. Second, the reinsurance safety loading is an additional static control variable in the problem. In contrast to the insurer's case, we assume that the reinsurer can hedge its static position in the reinsurance contract because the reinsurer is larger than the insurer and invests in the global market, as motivated at the beginning of this chapter. Therefore, we use the concept of replicating strategies in portfolio optimization with options, introduced in Korn and Trautmann (1999). Using the financial market completeness, the authors establish the one-to-one correspondence between the optimal portfolio in a market with options and the optimal portfolio in a market without options by means of option-replicating strategies. Therefore, one can apply the classic MA to solve the optimization problem in the latter market and use the established correspondence to derive the optimal portfolio with options. To find the optimal static safety loading, we follow a two-step procedure analogous to the one we use for deriving the optimal amount of reinsurance in the insurer's optimization problem.

Now we summarize the scientific contributions of this chapter to the above-mentioned literature sources. As in Chapter 4, we organize this summary in two parts: contributions to actuarial literature and contributions to the literature on portfolio optimization. As for the scientific novelties we bring to the actuarial literature, first, we formulate in Section 5.1 a novel Stackelberg game (SG) between a reinsurer and an insurer, which is more realistic than Stackelberg games previously studied in the literature. In contrast to them, in our model the reinsurance is written on potential losses within an insurance

[4]The process of inverting conditional random utility functions rarely leads to closed-form solutions when the fixed-term investment is not spanned by continuously traded assets. We tackle this challenge by treating the insurer's problem as an allocation-constrained problem and transforming it to the allocation-unconstrained one in an auxiliary market where the reinsurance contract is spanned by continuously traded risky assets.

product rather than the whole surplus process of an insurer, and is purchased at the beginning of the investment period, not dynamically traded. Second, we establish in Corollary 5.3.2 that in the Stackelberg equilibrium the power-utility maximizing reinsurer selects the largest reinsurance premium such that purchasing the maximal amount of reinsurance is in the set of the insurer's best responses to the reinsurer's action. However, as we established in Corollary 5.3.1, the set of the insurer's best responses may contain other choices of reinsurance amount. Therefore, in practice, the reinsurer should charge a lower (discounted) safety loading of the reinsurance premium than the equilibrium one in order to secure a deal with the maximal amount of reinsurance. Third, we provide in Subsection 5.4.3 different ways of how the reinsurer can determine a reasonable discount on the equilibrium safety loading. In this case the optimally acting insurer may significantly reduce its product costs without decreasing its expected utility, while the reinsurer still profits from selling reinsurance.

As for our research contributions in the area of portfolio-optimization literature, first, we solve in Proposition 5.2.1 and Proposition 5.2.2 a portfolio optimization problem (P_I) with an allocation constraint and a fixed-term investment in a put option. We combine the concept of auxiliary markets from Cvitanic and Karatzas (1992) and the generalized martingale approach from Desmettre and Seifried (2016). Although we consider a specific no-trading constraint, the methodology can be extended to other types of allocation constraints considered in Cvitanic and Karatzas (1992). Second, we solve in Proposition 5.2.4 and Proposition 5.2.5 a portfolio optimization problem $(P_R^{\pi_R,\eta_R})$ with a fixed position in a put option. We use the idea of replicating strategies in utility maximization introduced in Korn and Trautmann (1999). The author's of that paper show in their Theorem 5.1 how to optimize a portfolio of options given that the investor holds fixed positions in stocks and claim in Remark 5.2 that the reverse problem of optimizing a portfolio of stocks given fixed positions in options can be tackled similarly, which is done in Subsection 5.2.2 of this chapter.

Finally, we give an overview of the remaining parts of this chapter. In Section 5.1 we explain the general concept of a Stackelberg game and formulate a specific game between the reinsurer and the insurer. The optimal solution to the Stackelberg game for utility functions as per Definition 2.3.1 is derived in Section 5.2. It is divided into the subsections devoted to the insurer's optimization problem and the reinsurer's optimization problem, since the solution methods are different. In Section 5.3 we explicitly derive the Stackelberg equilibrium when the parties' preferences are described by power-utility functions. In Section 5.4 we choose the values of model parameters in accordance with the German market, as in Chapter 4, and conduct numerical studies. First, we investigate the sensitivity of the Stackelberg equilibrium w.r.t. the parties' risk-aversion coefficients and w.r.t. the model parameters influencing the fair price of the put option. Second, we study the impact of deviating from the Stackelberg equilibrium on the expected utilities of each party. In Appendix C.1 we provide the proofs of the results from Section 5.2. Appendix C.2 contains the proofs of the results from Section 5.3.

5.1 Problem setting

We start by providing the definition of a Stackelberg game in a general sense. Afterwards we will specify the concrete Stackelberg game that models the interaction between the insurer and the reinsurer.

A Stackelberg game is a game with two players and a hierarchical structure. One player is a called leader (L) and another one is called a follower (F). It is assumed that each of them optimizes the corresponding objective function $J_i(a_L, a_F) : \Lambda_L \times \Lambda_F \to \mathbb{R}$ for $i \in \{L, F\}$, respectively, where a_L is an action of the leader chosen from the set of admissible actions Λ_L and a_F an action of the follower chosen from the set of admissible actions Λ_F. Without loss of generality, it is assumed that optimization here means maximization. The aim of each player is the maximization of his/her objective function with respect to his/her action. The hierarchical nature of the the game is reflected in the information availability and the sequence of players' moves. The leader of the Stackelberg game chooses his/her action first, knowing the best response of the follower on the leader's action, and afterwards the follower acts depending on the action selected by the leader.

The Stackelberg game can be formalized as follows:

$$\begin{aligned} &\max_{a_L \in \Lambda_L} J_L(a_L, a_F^*) \\ &\quad \text{s.t. } a_F^* \in \arg\max_{a_F \in \Lambda_F} J_F(a_L, a_F). \end{aligned}$$

Stackelberg games are usually solved via backward induction. In the first step, one solves the follower's (inner) optimization problem given an arbitrary but fixed action of the leader. In the second step, one solves the leader's (outer) optimization problem knowing the set of optimal (also called "best" in academic literature) responses of the follower to each action of the leader.

A solution (a_L^*, a_F^*) to a Stackelberg game is called a Stackelberg equilibrium if it satisfies two conditions:

$$a_F^* \in \arg\max_{a_F \in \Lambda_F} J_F(a_L^*, a_F); \tag{SEC_1}$$

$$J_L(a_L, a_F) \leq J_L(a_L^*, a_F^*) \ \forall (a_L, a_F) \in \Lambda_L \times \Lambda_F \text{ s.t. } a_F \in \arg\max_{\tilde{a}_F \in \Lambda_F} J_F(a_L, \tilde{a}_F). \tag{SEC_2}$$

Condition (SEC_1) means that a_F^* is an optimal solution to the optimization problem of the follower. Condition (SEC_2) means that in case there are several optimal responses of the follower to an action of the leader, the follower selects among them the response that is most favorable for the leader, see, e.g., Bressan (2011)). Therefore, a Stackelberg game is a so-called optimistic bi-level optimization problem. For more details on optimistic and pessimistic bi-level optimization problems, see, e.g., Wiesemann et al. (2013), Zemkoho

(2016), Liu et al. (2018). For more theoretical insights into Stackelberg games, see, e.g., Fudenberg and Tirole (1991), Bressan (2011), and references therein.

Having provided the intuition about a Stackelberg game in general, we will now concertize it in the insureance-reinsurance context. We consider the same financial-market model as in Chapter 4 with a bank account, one risky asset that represents a non-reinsurable fund (fund in the individual investment strategy of the insurer) and one risky asset that represents a reinsurable fund. In the market, there are three parties: one representative client, one insurer and one reinsurer. The representative client would like to buy an equity-linked product from the insurer. Therefore, the client pays an initial contribution $v_I > 0$ to the insurer and expects to receive a capital guarantee $G_T > 0$ at product maturity. The insurer invests the initial capital v_I in assets S_0 and S_1. To increase the chances of ensuring a capital guarantee G_T to the client, the insurance company can buy at $t = 0$ reinsurance from the reinsurer. At the end of the investment period, the insurer receives the payment of the reinsurance contract. We model reinsurance as a put option with a benchmark portfolio as the underlying asset and G_T as the option's strike.

As in Chapter 4, we assume that the benchmark portfolio is a constant-mix (CM) portfolio with respect to S_0 and S_2. We denote the benchmark-portfolio value at time t by $V^{v_I,\pi_B}(t)$, $t \in [0,T]$. It satisfies SDE (4.2) with $V^{v_I,\pi_B}(0) = v_I$. This benchmark portfolio is in general not equal to the insurer's individual portfolio but should have high correlation so that the reinsurance indeed provides downside protection to the insurer. The choice of the constant-mix investment strategy for the benchmark portfolio makes the Stackelberg game analytically tractable, as for other benchmark strategies we may not be able to analytically derive the fair price of the option, which should be determined at the product inception time $t = 0$. Estimating the option price via Monte-Carlo simulations would be possible though.

According to Theorem 2.2.1, the fair price of the option Put at time t in the basic financial market is given by

$$Put(t) = \widetilde{Z}(t)^{-1}\mathbb{E}\left[\widetilde{Z}(T)(G_T - V^{v_I,\pi_B}(T))^+ \big| \mathcal{F}(t)\right]. \tag{5.1}$$

In contrast to Chapter 4, the reinsurance contract can be settled only at time $t = 0$ (when the put option is issued) and is fixed. We assume that the insurer can buy ξ_I reinsurance contracts (put options) and call ξ_I the reinsurance strategy of the insurer. Furthermore, we assume that the price of the reinsurance contract at time $t = 0$ equals $(1+\eta_R)Put(0)$, where $\eta_R \geq 0$ is the safety loading chosen by the reinsurer and $Put(0)$ is the fair price of the put option in the basic financial market. This resembles a so-called expected value principle in reinsurance pricing, for which the reinsurance price is given by $(1 + \eta)\mathbb{E}[X]$, where η is the safety loading and $\mathbb{E}[X]$ is the expected non-discounted loss under the real-world measure. In our case, the reinsurance premium is given by the expected discounted loss of the benchmark portfolio under the risk-neutral measure.

Note that the expected loss of the benchmark portfolio is not equal to the expected loss of the insurer.

We denote the insurer's relative portfolio process by $\pi_I(t) = (\pi_{I,1}(t), \pi_{I,2}(t))^\top$, $t \in [0,T]$. To model the insurer's preference to follow an individual investment strategy that is different from the strategy that can be reinsured, we impose an allocation constraint $\pi_I \in \mathbb{R} \times \{0\} =: C_{\pi_I}$. The wealth process of the insurer satisfies the following SDE:

$$\begin{aligned} dV_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t) =& (1-\pi_{I,1}(t)-\pi_{I,2}(t))V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_0(t)}{S_0(t)} \\ &+ \pi_{I,1}(t)V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_1(t)}{S_1(t)} + \pi_{I,2}(t)V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_2(t)}{S_2(t)}, \\ V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(0) =& v_I - \xi_I(1+\eta_R)Put(0) =: v_{I,0}(\xi_I,\eta_R). \end{aligned} \tag{5.2}$$

The insurer's total terminal wealth is given by the terminal value of the insurer's investment portfolio plus the payment from the reinsurance:

$$V^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T).$$

Similarly, we denote the reinsurer's relative portfolio process by $\pi_R = (\pi_{R,1}(t), \pi_{R,2}(t))^\top$, $t \in [0,T]$. The corresponding wealth process is given by

$$\begin{aligned} dV_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(t) =& (1-\pi_{R,1}(t)-\pi_{R,2}(t))V_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(t)\frac{dS_0(t)}{S_0(t)} \\ &+ \pi_{R,1}(t)V_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(t)\frac{dS_1(t)}{S_1(t)} + \pi_{R,2}(t)V_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(t)\frac{dS_2(t)}{S_2(t)}, \\ V_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(0) =& v_R + \xi_I(1+\eta_R)Put(0) =: v_{R,0}(\xi_I,\eta_R), \end{aligned} \tag{5.3}$$

where $v_R > 0$ is the initial wealth of the reinsurer before a reinsurance contract is issued. The reinsurer's total terminal wealth is given by terminal value of the reinsurer's investment portfolio less the payment of the reinsurance contract:

$$V^{v_{R,0}(\xi_I,\eta_R),\pi_R}(T) - \xi_I Put(T).$$

Further, we assume that the following condition holds:

$$\begin{aligned} &0 \le \eta_R \le \eta_R^{\max}; \\ &0 \le \xi_I \le \xi_I^{\max} := \min\left\{\frac{v_I}{(1+\eta_R)Put(0)}, \bar{\xi}\right\}, \end{aligned}$$

where $\eta_R^{\max} > 0$, $\bar{\xi} > 0$ is a constant independent of η_R. The condition on η_R ensures that the reinsurance company has an upper bound on the safety loading it can charge. The condition on ξ_I ensures that the insurance company has enough money for buying reinsurance at $t = 0$ and that it does not speculate by buying an excessive amount of reinsurance. For instance, we can choose $\bar{\xi}$ to be close to 1.

We denote by $U_R(\cdot)$ and $U_I(\cdot)$ the utility functions of the reinsurer and the insurer respectively as per Definition 2.3.1 with $\bar{v} = 0$.

The set of admissible strategies of the insurer is defined in the following way:

$$\begin{aligned}\Lambda_I := \{(\pi_I, \xi_I) |\ \pi_I \in \mathcal{A}_c^\pi(v_{I,0}(\xi_I, \eta_R), C_{\pi_I}),\ \xi_I \in [0, \xi_I^{\max}],\\ \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))^-] < +\infty]\},\end{aligned}$$

where $\mathcal{A}_c^\pi(v_{I,0}(\xi_I, \eta_R), C_{\pi_I}) := \{\pi_I | \pi_I \in \mathcal{A}_u^\pi(v_{I,0}(\xi_I, \eta_R)), \pi_I \in C_{\pi_I}\}$ with $\mathcal{A}_u^\pi(v_{I,0}(\xi_I, \eta_R))$ being the set of unconstrained portfolio processes according to Definition 2.1.4.

The set of admissible strategies of the reinsurer is defined by

$$\begin{aligned}\Lambda_R := \{(\pi_R, \eta_R) |\ \pi_R \in \mathcal{A}_u^\pi(v_{R,0}(\xi_I, \eta_R)),\ \eta_R \in [0, \eta_R^{\max}],\\ \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I,\eta_R),\pi_R}(T) - \xi_I Put(T))^-] < +\infty\}.\end{aligned}$$

Finally, we state the Stackelberg game between the reinsurer and the insurer:

$$\begin{aligned}&\sup_{(\pi_R,\eta_R)\in\Lambda_R} \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\pi_R}(T) - \xi_I^*(\eta_R) Put(T))]; \qquad \text{(SG)}\\ &\quad \text{s.t. } (\pi_I^*(\cdot|\eta_R), \xi_I^*(\eta_R)) \in \arg\max_{(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))].\end{aligned}$$

5.2 Solution to the Stackelberg game

In this section, we apply backward induction to solve the Stackelberg game (SG). First, we solve the optimization problem of the insurance company

$$\sup_{(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))] \qquad (P_I)$$

for each admissible safety loading $\eta_R \in [0, \eta_R^{\max}]$. We denote by $(\pi_I^*(\cdot|\eta_R), \xi_I^*(\eta_R)) \in \Lambda_I$ the solution to (P_I), which is the optimal response of the insurer to the reinsurer's action (π_R, η_R).

Second, we solve the optimization problem of the reinsurance company, which knows for each of its actions the best response(s) of the insurer:

$$\sup_{(\pi_R,\eta_R)\in\Lambda_R} \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\pi_R}(T) - \xi_I^*(\eta_R) Put(T))]. \qquad (P_R^{\pi_R,\eta_R})$$

We denote the solution to $(P_R^{\pi_R,\eta_R})$ by $(\pi_R^*, \eta_R^*) \in \Lambda_R$. Then the Stackelberg equilibrium in the Stackelberg game (SG) is given by $(\pi_R^*(\cdot), \eta_R^*, \pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$, as it satisfies Conditions (SEC_1) and (SEC_2).

5.2.1 Solution to the optimization problem of the insurer

In this subsection, we solve the insurer's optimization problem (P_I) given an arbitrary but fixed admissible safety loading $\eta_R \in [0, \eta_R^{\max}]$. Problem (P_I) has two challenges in comparison to standard portfolio optimization problems. First, there is a constraint on the relative portfolio process, since the insurer prefers not to invest in S_2 due to its individual investment strategy. Second, there is a fixed-term investment in the put option, which results in an additional control variable and influences the total terminal wealth of the insurer.

We solve (P_I) as follows. First, we consider a family of auxiliary financial markets as per Cvitanic and Karatzas (1992). For each auxiliary market, we derive the corresponding optimal allocation-unconstrained relative portfolio process and reinsurance amount via the generalized MA as per Desmettre and Seifried (2016). Second, we find the optimal auxiliary market and show that the optimal strategy in it coincides with the solution to the original problem (P_I).

The allocation constraint C_{π_I} is convex. Its support function is given by

$$\delta(x) := -\inf_{y \in C_{\pi_I}} (x^\top y) = -\inf_{y_1 \in \mathbb{R}} (x_1 y_1) = \begin{cases} 0, & \text{if } x_1 = 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

As in Section 4.2, we introduce the class $\mathcal{D}$ of dual processes ν:

$$\mathcal{D} := \Bigg\{ \nu = \big((\nu_1(t), \nu_2(t))^\top \big)_{t \in [0,T]} \Big| \nu \text{ progressively measurable}, \\ \mathbb{E}\left[\int_0^T \|\nu(t)\|^2 \, dt \right] < \infty, \, \mathbb{E}\left[\int_0^T \delta(\nu(t)) dt \right] < \infty \Bigg\}.$$

It holds for $\nu \in \mathcal{D}$ that $\nu_1(t) = 0$ $\mathbb{Q}$-a.s. for all $t \in [0, T]$. Each $\nu \in \mathcal{D}$ corresponds to an auxiliary market $\mathcal{M}_\nu$ as in (4.11) with the related auxiliary market price of risk $\gamma_\nu(t)$ defined in (4.12) and the pricing kernel $\widetilde{Z}_\nu(t)$ defined in (4.13).

In any $\mathcal{M}_\nu$, we have an allocation-unconstrained utility-maximization problem with two controls – the relative portfolio process and the amount of put options. To apply the generalized MA as per Desmettre and Seifried (2016), we only need the terminal payoff of the put option (fixed-term security in the terminology of Desmettre and Seifried (2016)) and its initial price, which can be any positive number, not necessarily the fair price as per Theorem 2.2.1. Thus, we are only interested in $Put(T)$ and $Put(0)$. $Put(T)$ is a random variable that is $\mathcal{F}(T)$-measurable. By (5.1), $Put(T) = (G_T - V^{v_I,\pi_B}(T))^+$ and $Put(0) = \mathbb{E}\left[\widetilde{Z}(T)(G_T - V^{v_I,\pi_B}(T))^+\right]$. Note that in general we have:

$$Put(0) \overset{(5.1)}{=} \mathbb{E}[\widetilde{Z}(T)Put(T)] \neq \mathbb{E}[\widetilde{Z}_\nu(T)Put(T)].$$

The insurer's wealth process $V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}$ in the auxiliary market $\mathcal{M}_\nu$ is given by

$$\begin{aligned} dV_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t) =& (1-\pi_{I,1}(t)-\pi_{I,2}(t))V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_0^\nu(t)}{S_0^\nu(t)} \\ &+ \pi_{I,1}(t)V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_1^\nu(t)}{S_1^\nu(t)} + \pi_{I,2}(t)V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_2^\nu(t)}{S_2^\nu(t)} \\ =& (1-\pi_{I,1}(t)-\pi_{I,2}(t))V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_0(t)}{S_0(t)} \\ &+ \pi_{I,1}(t)V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_1(t)}{S_1(t)} + \pi_{I,2}(t)V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\frac{dS_2(t)}{S_2(t)} \\ &+ V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t)\underbrace{\left(\pi_I(t)^\top\nu(t)\right)}_{\geq 0, \text{ if } \pi_I(t)\in C_{\pi_I}} dt, \\ V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(0) =& v_I - \xi_I(1+\eta_R)Put(0) = v_{I,0}(\xi_I,\eta_R). \end{aligned}$$

The unconstrained optimization problem of the insurer in $\mathcal{M}_\nu$ is given by

$$\sup_{(\pi_I,\xi_I)\in\Lambda_I^\nu} \mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))], \qquad (P_I^\nu)$$

where

$$\begin{aligned} \Lambda_I^\nu := \{(\pi_I,\xi_I) |\ & \pi_I \in \mathcal{A}_u^\pi(v_{I,0}(\xi_I,\eta_R)),\ \xi_I \in [0,\xi_I^{\max}], \\ & \mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))^-] < \infty\}. \end{aligned}$$

For the optimal ν^*, which is to be found, the solution to the unconstrained problem (P_I^ν) will satisfy the constraint $\pi_2(t) = 0$ $\mathbb{Q}$-a.s. $\forall t \in [0,T]$. We denote the solution to (P_I^ν) by $(\pi_\nu^*, \xi_\nu^*) \in \underset{(\pi_I,\xi_I)\in\Lambda_I^\nu}{\arg\sup}\ \mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))]$.

Next we define the random utility function by

$$\hat{U}_I(x) := U_I(x + \xi_I Put(T))$$

for $x \in [0,\infty)$, where $\xi_I \in [0,\xi^{\max}]$. The utility function $\hat{U}_I(\cdot)$ is random, since $Put(T)$ is a random variable. Hence, $\hat{U}_I : [0,+\infty) \to [U_I(\xi_I Put(T)), +\infty)$ and $\hat{U}_I(\cdot)$ is continuously differentiable, strictly increasing and strictly concave. Therefore, it holds

$\hat{U}_I' : [0, +\infty) \to (0, U_I'(\xi_I Put(T))]$ and

$$\hat{U}_I'(x) = U_I'(x + \xi_I Put(T)).$$

We denote the inverse of $\hat{U}_I'(\cdot)$ by $\hat{I}_I : (0, +\infty) \to [0, +\infty)$. For $\lambda \in (0, U_I'(\xi_I Put(T))]$, we have $\hat{I}_I(\lambda) = I_I(\lambda) - \xi_I Put(T)$, where $I_I(\cdot)$ is the inverse of $U_I'(\cdot)$. So $\hat{I}_I(\cdot)$ is bijective on $(0, U_I'(\xi_I Put(T))]$. For convenience, we set $\hat{I}_I(\lambda) := 0$ for $\lambda > U_I'(\xi_I Put(T))$.

In the next proposition we provide the solution to the insurer's optimization problem in each auxiliary market.

Proposition 5.2.1 (Optimal solution to (P_I^ν))**.** *Assume that for all* $\lambda \in (0, +\infty)$

$$\mathbb{E}[\widetilde{Z}_\nu(T) I_I(\lambda \widetilde{Z}_\nu(T))] < +\infty \text{ and } \mathbb{E}[U_I(I_I(\lambda \widetilde{Z}_\nu(T)))] < +\infty \tag{5.4}$$

holds. Then, there exists a solution (π_ν^*, ξ_ν^*) *to the unconstrained optimization problem of the insurer* (P_I^ν)*, where*

$$\xi_\nu^* \in \underset{\xi_I \in [0, \xi^{max}]}{arg\ max}\ h_I(\xi_I).$$

The function $h_I(\cdot)$ *is given by*

$$h_I(\xi_I) := \mathbb{E}[U_I(\max\{I_I(\lambda^*(\xi_I)\widetilde{Z}_\nu(T)), \xi_I Put(T)\})],$$

where the Lagrange multiplier $\lambda^* := \lambda^*(\xi_I)$ *is given by the budget constraint*

$$\mathbb{E}[\widetilde{Z}_\nu(T) \hat{I}_I(\lambda^* \widetilde{Z}_\nu(T))] = v_I - \xi_I(1 + \eta_R) Put(0).$$

The optimal terminal wealth $V_\nu^*(T) := V_\nu^{v_{I,0}(\xi_\nu^*, \eta_R), \pi_\nu^*}(T)$ *is given by*

$$V_\nu^*(T) = \hat{I}_I(\lambda^*(\xi_\nu^*)\widetilde{Z}_\nu(T)) = \max\{I_I(\lambda^*(\xi_\nu^*)\widetilde{Z}_\nu(T)) - \xi_\nu^* Put(T), 0\}.$$

and the optimal wealth process V_ν^* *is given by*

$$V_\nu^*(t) = \widetilde{Z}_\nu(t)^{-1} \mathbb{E}[\widetilde{Z}_\nu(T) V_\nu^*(T) | \mathcal{F}(t)]$$

for $t \in [0, T]$.

If $\hat{I}(\cdot)$ *and* $\frac{d\hat{I}_I}{d\lambda}(\cdot)$ *are polynomially bounded*[5] *at* 0 *and* $+\infty$*, the optimal relative portfolio*

[5] As per Desmettre and Seifried (2016), a function $f : (0, +\infty) \to \mathbb{R}$ is called polynomially bounded at 0 and $+\infty$, if there exist $c, k \in (0, +\infty)$ such that for all $\lambda \in (0, +\infty)$ the following holds:

$$|f(\lambda)| \leq c\left(\lambda + \frac{1}{\lambda}\right)^k$$

process π^*_ν is given by

$$\pi^*_\nu(t)V^*_\nu(t) = -(\sigma^\top)^{-1}\gamma_\nu \widetilde{Z}_\nu(t)^{-1}\mathbb{E}\left[\widetilde{Z}_\nu(T)\lambda^*(\xi^*_\nu)\widetilde{Z}_\nu(T)\frac{d\hat{I}_I}{d\lambda}\left(\lambda^*(\xi^*_I)\widetilde{Z}_\nu(T)\right)\middle|\mathcal{F}(t)\right] \quad (5.5)$$

$\mathbb{Q}$-a.s. for all $t \in [0,T]$.

Proof. See Appendix C.1. □

By Proposition 5.2.1 we know the solution (π^*_ν, ξ^*_ν) to the unconstrained optimization problem of the insurer (P^ν_I), if some technical conditions hold. In the next proposition, we show when the solution to the insurer's original optimization problem (P_I) and the insurer's unconstrained optimization problem (P^ν_I) coincide.

Proposition 5.2.2 (Optimal solution to (P_I))**.** *Suppose that there exists $\nu^* \in \mathcal{D}$ such that for the optimal solution $(\pi^*_{\nu^*}, \xi^*_{\nu^*})$ to $(P^{\nu^*}_I)$ we have $\pi^*_{\nu^*}(t) \in C_{\pi_I}$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. Then $(\pi^*_I, \xi^*_I) := (\pi^*_{\nu^*}, \xi^*_{\nu^*})$ is optimal for the constrained optimization problem of the insurer (P_I).*

Proof. See Appendix C.1. □

Remark to Proposition 5.2.2. For a deterministic utility function, ν^* can be found as in Example 15.1 of Cvitanic and Karatzas (1992). Since in our case $\hat{U}(\cdot)$ is a random utility function, the search for the optimal ν^* is a bit more involved. In Section 5.3, we equip the insurer with a power-utility function and explicitly derive ν^* that satisfies the conditions in Proposition 5.2.2.

5.2.2 Solution to the optimization problem of the reinsurer

In Subsection 5.2.1, we learned about the optimal responses (investment strategy and amount of reinsurance) of the insurer to an arbitrary admissible safety loading η_R. We write $(\pi^*_I(t|\eta_R, \xi^*_I(\eta_R)), \xi^*_I(\eta_R))$ to emphasize the dependence of the insurer's optimal response on $\eta_R \in [0, \eta^{\max}_R]$. In this subsection, we solve the optimization problem (P^{π_R,η_R}_R) of the reinsurer that knows how the insurer responds to different η_R.

Similarly to the insurer's problem (P_I), the reinsurer's problem (P^{π_R,η_R}_R) has two controls, namely the relative portfolio process π_R and the safety loading η_R. However, it does not have a constraint on the relative portfolio process, since we assume that the reinsurer has a larger investment universe in our model and can hedge the put option. We solve the problem via the replicating-strategies approach introduced in Korn and Trautmann (1999).

We solve $(P_R^{\pi_R,\eta_R})$ in the following way. First, we transform the optimization problem $(P_R^{\pi_R,\eta_R})$ into an optimization problem $(P_R^{\varphi_R,\eta_R})$ w.r.t. the trading strategy φ_R and η_R. Second, we use the idea of Desmettre and Seifried (2016) and solve the optimization problem $(P_R^{\varphi_R,\eta_R})$ given a fixed $\eta_R \in [0, \eta_R^{\max}]$. For this, we transform the optimization problem $(P_R^{\varphi_R,\eta_R})$ into the optimization problem $(P_R^{\varphi_R|\eta_R})$ w.r.t. the trading strategy φ_R given an arbitrarily fixed admissible η_R. Problem $(P_R^{\varphi_R|\eta_R})$ is in its turn transformed into the problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$ w.r.t. the trading strategy φ_R given fixed η_R and a fixed position $\xi(t) = -\xi_I^*(\eta_R)$ in Put. Then, we solve the optimization problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$ by applying the replicating-strategies approach as per Korn and Trautmann (1999) and use Relation 2.6 to derive the optimal relative portfolio process $\pi_R^*(\cdot|\eta_R)$ from the optimal trading strategy $\varphi_R^*(\cdot|\eta_R)$. The optimization problem $(P_R^{\varphi_R|\eta_R})$ is stated below in this subsection. However, we state problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$ in Appendix C.1 in the proof of Proposition 5.2.4 to keep this subsection concise. Lastly, we solve the optimization problem $(P_R^{\pi_R,\eta_R})$ with respect to η_R for the given optimal portfolio process $\pi_R^*(\cdot|\eta_R)$.

As mentioned above, the first step is the transformation of $(P_R^{\pi_R,\eta_R})$ into the equivalent optimization problem $(P_R^{\varphi_R,\eta_R})$ with respect to the trading strategy φ_R. Denoting by $V_R^{v_{I,0}(\xi_I,\eta_R),\varphi_R}$ the reinsurer's wealth process controlled by the trading strategy $\varphi_R := (\varphi_{R,0}(t), \varphi_{R,1}(t), \varphi_{R,2}(t))_{t\in[0,T]}^{\top}$, we state the corresponding transformed problem:

$$\sup_{(\varphi_R,\eta_R)\in\Lambda_R^{\varphi_R}} \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R}(T) - \xi_I^*(\eta_R)Put(T))], \qquad (P_R^{\varphi_R,\eta_R})$$

where $\Lambda_R^{\varphi_R}$ is the set of all admissible trading strategies and safety loadings of the reinsurer:

$$\Lambda_R^{\varphi_R} := \{(\varphi_R, \eta_R)|\ \varphi_R \in \mathcal{A}_u^{\varphi}(v_{R,0}(\xi_I^*(\eta_R)),\ \eta_R \in [0, \eta_R^{\max}],$$
$$\mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R}(T) - \xi_I^*(\eta_R)Put(T))^-] < +\infty\}.$$

We solve $(P_R^{\varphi_R,\eta_R})$ by first deriving the optimal trading strategy for any admissible η_R and then by optimizing the η_R. So we fix an arbitrary $\eta_R \in [0, \eta_R^{\max}]$ and consider the following optimization problem

$$\sup_{\varphi_R:(\varphi_R,\eta_R)\in\Lambda_R^{\varphi_R}} \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R}(T) - \xi_I^*(\eta_R)Put(T))], \qquad (P_R^{\varphi_R|\eta_R})$$

where η_R in $(P_R^{\varphi_R|\eta_R})$ emphasizes that η_R is not a control variable there. To derive the optimal trading strategy φ_R^* in $(P_R^{\varphi_R|\eta_R})$, we need the following auxiliary lemma about the put-replication strategy.

Lemma 5.2.3. *The replicating strategy $\psi(t)$, $t \in [0,T]$, of the put option Put is given* *by*

$$\psi(t) = \left(\frac{Put(t) - \pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_0(t)}, 0, \frac{\pi^{CM} V^{v_I,\pi_B}(t)(\Phi(d_+) - 1)}{S_2(t)} \right)^\top, \quad (5.6)$$

where

$$d_+ := d_1(t, V^{v_I,\pi_B}(t), G_T, r, \sigma_2 \pi^{CM})$$

and $d_1(\cdot)$ *is defined in* (2.17).

Proof. See Appendix C.1. □

In the next proposition, we provide the solution to $(P_R^{\varphi_R|\eta_R})$. To prove it, we modify Theorem 5.1 in Korn and Trautmann (1999) in accordance with Remark 5.2 in Korn and Trautmann (1999) and apply the modified theorem to the case of a fixed position in a put option.

Proposition 5.2.4 (Optimal solution to $(P_R^{\varphi_R|\eta_R})$)**.** *Assume that for all* $\lambda \in (0, +\infty)$ *it holds*

$$\mathbb{E}[\widetilde{Z}(T) I_R(\lambda \widetilde{Z}(T))] < +\infty,$$

where $I_R(\cdot)$ *is the inverse function of* $U_R'(\cdot)$*.*

(a) There exists an optimal trading strategy φ_R^* *in the optimization problem* $(P_R^{\varphi_R|\eta_R})$*. The optimal total terminal wealth in the optimization problem* $(P_R^{\varphi_R|\eta_R})$ *is given by*

$$V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(T) - \xi_I^*(\eta_R) Put(T) = I_R(\lambda_R^*(\eta_R) \widetilde{Z}(T)),$$

where $\lambda_R^* \equiv \lambda_R^*(\eta_R)$ *is the Lagrange multiplier determined by*

$$\mathbb{E}[\widetilde{Z}(T) I_R(\lambda_R^* \widetilde{Z}(T))] = v_R + \xi_I^*(\eta_R) \eta_R Put(0). \quad (5.7)$$

(b) Let ψ *be the replicating strategy given by* (5.6) *and* ζ_R^* *the optimal trading strategy of the stock optimization problem, where the reinsurer only invests in the assets* S_0*,* S_1 *and* S_2 *(i.e., no investment in options). Then, the optimal trading strategy*

φ_R^* *to the optimization problem* $(P_R^{\varphi_R|\eta_R})$ *is given by*

$$\varphi_{R,0}^*(t) = \frac{V^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(t) - \sum_{i=1}^2 \varphi_{R,i}^*(t)S_i(t)}{S_0(t)};$$
$$\varphi_{R,1}^*(t) = \zeta_{R,1}^*(t);$$
$$\varphi_{R,2}^*(t) = \zeta_{R,2}^*(t) + \psi_2(t)\xi_I^*(\eta_R).$$

Proof. See Appendix C.1. □

Remark to Proposition 5.2.4. The proof of the above proposition does not rely the generalized MA, which we used to solve the insurer's optimization problem. Instead, we use a simpler method that does not require the inversion of random utilities and uses the completeness of the financial market in which the reinsurer operates.

Proposition 5.2.4 yields the optimal terminal portfolio value $V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(T)$ and the optimal trading strategy φ_R^* given an arbitrary but fixed admissible η_R in $(P_R^{\varphi_R,\eta_R})$. The optimal η_R can be calculated using the following proposition.

Proposition 5.2.5 (Optimal safety loading)**.** *Let* $\varphi_R^*(\cdot|\eta_R)$ *be the optimal trading strategy in the optimization problem* $(P_R^{\varphi_R,\eta_R})$ *for* $\eta_R \in [0, \eta^{max}]$. *Then, the optimal safety loading* η_R^* *of the reinsurer is given by*

$$\eta_R^* = \underset{\eta_R \in [0,\eta_R^{\max}]}{arg\ max}\ \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(T) - \xi_I^*(\eta_R)Put(T))].$$

Proof. See Appendix C.1. □

5.2.3 Stackelberg equilibrium

Proposition 5.2.6 (Stackelberg equilibrium)**.** *The Stackelberg equilibrium of the Stackelberg game* (SG) *is given by* $(\pi_R^*(\cdot|\eta_R^*), \eta_R^*, \pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$, *where*

- $\pi_R^*(\cdot|\eta_R^*)$ *is given by*

$$\pi_{R,i}^*(t|\eta_R^*) = \frac{\varphi_{R,i}^*(t|\eta_R^*) \cdot S_i(t)}{V_R^{v_{R,0}(\xi_I^*(\eta_R^*),\eta_R^*),\varphi_R^*}(t)},$$

  where φ_R^* *is given by Proposition 5.2.4,*

- η_R^* *is given by Proposition 5.2.5, and*

- $(\pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$ *are given by Proposition 5.2.1 and Proposition 5.2.2, such that*

$$\begin{aligned}\xi_I^*(\eta_R^*) = \max\{\xi_I^* |\ \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I^*,\eta_R^*),\pi_I^*}(T) + \xi_I^* Put(T))] \\ = \sup_{\xi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R^*),\pi_I^*}(T) + \xi_I Put(T))]\}. \quad (5.8)\end{aligned}$$

Proof. See Appendix C.1. □

Remarks to Proposition 5.2.6.

First, Equation (5.8) ensures that Condition (SEC_2) is fulfilled, i.e., if there exist more than one best response of the insurer to the reinsurer's optimal strategy, then in the Stackelberg equilibrium the insurer's best response is the one that is best also from the reinsurer's perspective.

Second, for the optimal portfolio processes of the reinsurer π_R^* and the insurer π_I^*, we get analytical representations that depend on the optimal safety loading η_R^* and the amount of reinsurance $\xi_I^*(\eta_R^*)$. Depending on the concrete utility function, η_R^* and $\xi_I^*(\eta_R^*)$ can be analytically derived in a closed form, which we show below for a power-utility function, or numerically calculated.

5.3 Explicit solutions for power utility functions

In this section, we explicitly derive the Stackelberg equilibrium when the preferences of each party are described be a power-utility function as per (2.39), i.e., for $v \in (0, +\infty)$

$$U_R(v) := \frac{1}{p_R} v^{p_R} \text{ and } U_I(v) := \frac{1}{p_I} v^{p_I} \quad (5.9)$$

with $p_R, p_I \in (-\infty, 1)\backslash\{0\}$. In addition, we assume that the upper limit $\xi_I^{\max}$ is fixed and equals $\bar{\xi} > 0$ with $\bar{\xi} < v_I / \left((1+\eta_R^{\max})Put(0)\right)$.

First, we derive the best response for the insurer (follower) given an arbitrary but fixed admissible strategy of the leader. For that, we will use the optimal unconstrained relative portfolio process of the insurer without reinsurance in an auxiliary market $\mathcal{M}_\nu$. As in the previous chapter, we denote this process by $\pi_{u,\nu}^*$. Recall from Corollary 4.2.4 that it is given by:

$$\pi_{u,\nu}^*(t) := \pi_{u,\nu}^*(p_I) := \frac{1}{1-p_I}(\sigma\sigma^\top)^{-1}(\mu + \nu - r\mathbf{1}_2). \quad (5.10)$$

Corollary 5.3.1 (Best response of the insurer)**.** *Assume that the insurer has a power-utility function* $U_I(\cdot)$ *as in* (5.9)*. Let* $\eta_R \in [0, \eta_R^{\max}]$ *be an arbitrary but fixed safety* *loading chosen by the reinsurer. Then the optimal response of the insurer is given by*

$$\xi_I^*(\eta_R) = \begin{cases} \bar{\xi}, & \text{if } \eta_R < \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)}, \\ \text{any } \tilde{\xi} \in [0,\bar{\xi}], & \text{if } \eta_R = \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)}, \\ 0, & \text{if } \eta_R > \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)}, \end{cases} \tag{5.11}$$

and $\pi_I^* := (\pi_I^*(t))_{t\in[0,T]}$ *such that for* $t \in [0,T]$

$$\pi_I^*(t|\eta_R) = \pi_{u,\nu^*}^*(p_I) \frac{V_I^{v_{I,0}(\xi_I^*(\eta_R),\eta_R),\pi_I^*}(t) + \xi_I^*(\eta_R)\left(\widetilde{Z}_{\nu^*}(t)\right)^{-1}\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)|\mathcal{F}(t)]}{V_I^{v_{I,0}(\xi_I^*(\eta_R),\eta_R),\pi_I^*}(t)}, \tag{5.12}$$

where $\nu^* \in \mathcal{D}$ *equals*

$$\nu^* = \left(0, \frac{\sigma_2\rho}{\sigma_1}(\mu_1 - r) - \mu_2 + r\right)^\top$$

and $\pi_{u,\nu^*}^*(p_I)$ *is defined by* (5.10) *for* $\nu = \nu^*$.

Proof. See Appendix C.2. □

Remark to Corollary 5.3.1. We can write (5.10) and (5.12) for $\nu = \nu^*$ as follows:

$$\pi_{u,\nu^*}^*(p_I) = \pi_u^*(p_I) + \underbrace{\frac{1}{1-p_I}(\sigma\sigma^\top)^{-1}\nu^*}_{\text{constraint correction}}; \tag{5.13}$$

$$\pi_I^*(t|\eta_R) = \pi_u^*(p_I) + \underbrace{\frac{1}{1-p_I}(\sigma\sigma^\top)^{-1}\nu^*}_{\text{constraint correction}} + \underbrace{\pi_{u,\nu^*}^*(p_I)\frac{\xi_I^*(\eta_R)\widetilde{Z}_{\nu^*}(t)^{-1}\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)|\mathcal{F}(t)]}{V_I^{v_{I,0}(\xi_I^*(\eta_R),\eta_R),\pi_I^*}(t)}}_{\text{reinsurance contract correction}}, \tag{5.14}$$

where π_u^* is the optimal unconstrained portfolio process in the original market $\mathcal{M}$. If the insurer has no constraint on π_I, then $\nu^* = (0,0)^\top$ the last term in (5.13) vanishes and it holds that $\pi_{u,\nu^*}^*(p_I) = \pi_u^*(p_I)$. In addition, if $\nu^* = (0,0)^\top$, then for any $\eta_R > 0$ we get $\xi_I^*(\eta_R) = 0$ due to (5.11) and, thus, $\pi_I^*(t|\eta_R) = \pi_{u,\nu^*}^*(p_I) = \pi_u^*(p_I)$.

Otherwise, the insurer's optimal relative portfolio process π_I^* equals the optimal unconstrained relative portfolio process with two correction terms that account for the availability of the reinsurance contract and the difference between the insurer's individual portfolio and the reinsured portfolio.

Denoting the reinsurer's optimal unconstrained strategy without the put option by

$$\pi_u^*(p_R) = \frac{1}{1-p_R}(\sigma\sigma^\top)^{-1}(\mu - r\mathbf{1}_2),$$

we provide in the next proposition the Stackelberg equilibrium for (SG).

Corollary 5.3.2 (Stackelberg equilibrium)**.** *Assume that the insurer and the reinsurer have power-utility functions as in* (5.9)*. Then the Stackelberg equilibrium in* (SG) *is* $(\pi_R^*(\cdot|\eta_R^*), \eta_R^*, \pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$*, where*

$$\eta_R^* = \min\left\{\frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)}{Put(0)}, \eta_R^{\max}\right\}, \tag{5.15}$$

$$\pi_R^*(t|\eta_R^*) = \pi_u^*(p_R)\frac{V_R^{v_{R,0}(\xi_I^*(\eta_R^*),\eta_R^*),\varphi_R^*}(t) - \xi_I^*(\eta_R^*)Put(t)}{V_R^{v_{R,0}(\xi_I^*(\eta_R^*),\eta_R^*),\varphi_R^*}(t)} + \underbrace{\begin{pmatrix} 0 \\ \frac{\pi^{CM}V^{v_I,\pi_B}(t)(\Phi(d_+)-1)}{V_R^{v_{R,0}(\xi_I^*(\eta_R^*),\eta_R^*),\varphi_R^*}(t)}\xi_I^*(\eta_R^*)\end{pmatrix}}_{\text{guarantee correction term}}, \tag{5.16}$$

$\xi_I^*(\eta_R^*) = \bar{\xi}$ *is given by* (5.11), $\pi_I^*(\cdot|\eta_R^*)$ *is given by* (5.12)*.*

Proof. See Appendix C.2. □

Remark to Corollary 5.3.2. If the insurer has no allocation constraint (i.e., $\nu^* = (0,0)^\top$), then $\xi_I^*(\eta_R^*) = 0$ for any $\eta_R > 0$ (see Remark to Corollary 5.3.1) and $\pi_R^*(t|\eta_R^*) = \pi_u^*(p_R)$ for all $t \in [0,T]$ (see (5.16)).

If the insurer offers no guarantee to its client (i.e., $G_T = 0$), then $d_+ = +\infty$ and the last term in (5.16) is equal to zero. Moreover, $Put(t) = 0$, $t \in [0,T]$. Thus, $\pi_R^*(t|\eta_R^*) = \pi_u^*(p_R)$ for all $t \in [0,T]$ too.

5.4 Numerical studies

In this section, we describe the numerical studies for power-utility maximizing insurance and reinsurance companies. We explain how the model parameters are selected in Subsection 5.4.1. In Subsection 5.4.2, we calculate the Stackelberg equilibrium for the base-case values of parameters from Subsection 5.4.1 and analyze the sensitivity of the Stackelberg equilibrium w.r.t. the changes in the behavior of the parties and the changes in the put option price. In Subsection 5.4.3, we investigate how the expected utilities of the parties change when one of the players decides not to follow the Stackelberg-equilibrium strategy.

5.4.1 Parameter selection

Parameter	**Symbol**	**Values**
Interest rate	r	1.02%
Drift coefficient for S_1	μ_1	17.52%
Drift coefficient for S_2	μ_2	12.37%
Diffusion coefficient for S_1	σ_1	23.66%
Diffusion coefficient for S_2	σ_2	21.98%
Correlation coefficient	ρ	80.12%
Benchmark CM strategy	π_B	$(0\%, 29.48\%)^\top$
Initial value of S_1	s_1	1
Initial value of S_2	s_2	1
Guarantee	G_T	100
Initial wealth of insure	v_I	100
Initial wealth of reinsurer	v_R	100
Relative risk aversion of insurer (RRA_I)	$1 - p_I$	10
Relative risk aversion of reinsurer (RRA_R)	$1 - p_R$	10
Time horizon	T	10
Maximal safety loading of reinsurer	$\eta_R^{\max}$	50%
Maximal amount of reinsurance	$\xi_I^{\max} = \bar{\xi}$	1.5

Table 5.1: Parameters for the numerical analysis.

For the majority of model parameters, we choose the same values as in Chapter 4, compare with Table 4.1. We set the insurer's initial wealth v_I to 100. It is a reasonable assumption that the reinsurer is a larger company that has more initial capital. If reinsurance is offered on the company-aggregated level, then the reinsurer's initial wealth should be indeed larger than the insurer's initial wealth of the insurer, as, e.g., in Chen and Shen (2018). However, we model reinsurance within a single insurance product. Thus, we assume that the initial product-related capital of the reinsurer coincides with the initial wealth of the insurer, i.e., $v_R = v_I = 100$.

As for the risk-aversion coefficients, we set in the base case $p_I = p_R = -9$. In our sensitivity analysis, we explore the situations where the parties have different risk aversion. In Chen and Shen (2018), the researchers also assume that both parties have the same risk-aversion coefficient, whereas Bai et al. (2022) model the insurer as a more risk-averse party than the reinsurer.

We set the maximal admissible safety loading $\eta_R^{\max}$ to 50%. This choice is in line with Chen and Shen (2018) and Chen et al. (2019), where the safety loading has an upper bound of 45%. We do not allow that the insurer can speculate with the reinsurance by holding a short position in it or by buying an excessive amount of it. Since the underlying of the put option is not the insurer's individual portfolio but a correlated benchmark portfolio, we set $\xi_I^{\max} = \bar{\xi} = 1.5$. In practice, a regulator may have to impose this type of constraint.

5.4.2 Stackelberg equilibrium and its sensitivity

In the base case, the Stackelberg equilibrium is given by

$$\pi_R^*(0) = (32.85\%, -9.23\%)^\top, \quad \eta_R^* = 20.86\%;$$
$$\pi_I^*(0) = (31.69\%, 0\%)^\top, \quad \xi_I^*(\eta_R^*) = 1.5.$$

In the equilibrium, the insurer purchases the maximal amount of reinsurance (i.e., $\xi_I^*(\eta_R^*) = \xi_I^{\max}$) and the reinsurer selects the maximal safety loading such that the insurer may still buy reinsurance, in which case the price of reinsurance is about 20% higher than the fair (risk-neutral) price of the corresponding put option.

Next, we conduct the sensitivity analysis. First, we check how the equilibrium changes as we vary the RRA coefficients of the reinsurer and the insurer. Afterwards, we explore the impact of the interest rate r, the time horizon T, and the capital guarantee G_T on the Stackelberg equilibrium.

Figure 5.1 shows the Stackelberg equilibrium for varying RRA coefficients of each party. As expected from Corollary 5.3.2, a change of the companies' RRA coefficients does not influence the optimal reinsurance amount $\xi_I^*(\eta_R^*)$ and the optimal safety loading η_R^*. The more risk averse a company is, the less it invests in or speculates with the risky assets.

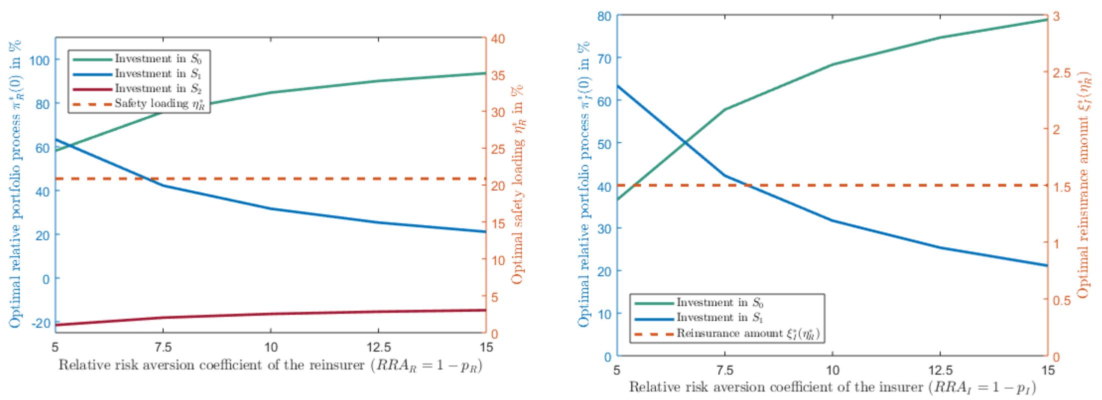


Figure 5.1: Sensitivity of the Stackelberg equilibrium w.r.t. RRA_R and RRA_I.

As it can be seen in Figure 5.2, the higher the interest rate $r \in \{-2\%, -1\%, 0\%, 1\%, 2\%\}$, the higher the optimal safety loading of the reinsurer. When r changes, the fair price of the put option $Put(0)$ in the basic market $\mathcal{M}$ decreases faster than the fair price of the put option $\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]$ in the auxiliary market $\mathcal{M}_{\nu^*}$ (see (5.15)). The interest rate has no influence on the reinsurance amount in the equilibrium, which is consistent with

Corollary 5.3.2. The higher the interest rate, the less the insurer invests in the risky asset. In contrast, due to the replication of the put-option position, the reinsurer invests more in risky assets when the interest rate grows.

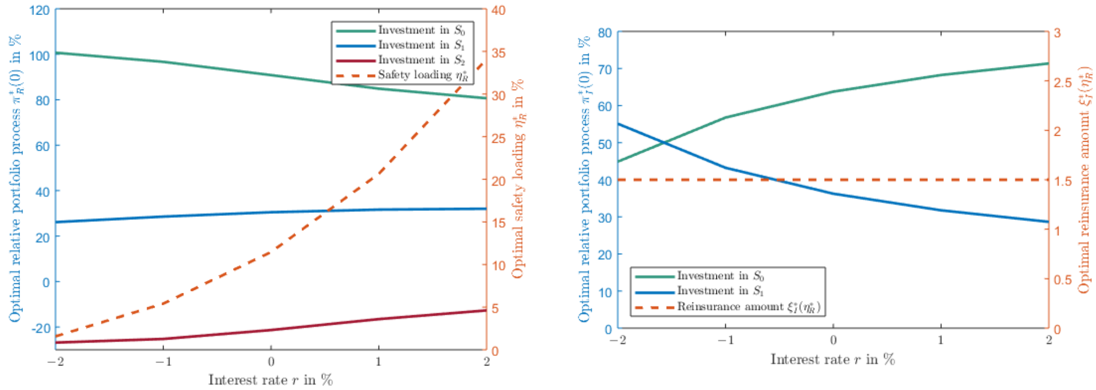


Figure 5.2: Sensitivity of the Stackelberg equilibrium w.r.t. r.

Figure 5.3 illustrates that the longer the investment period $T \in \{1, 5, 10, 15, 20\}$, the higher the optimal safety loading. However, with varying T, the optimal reinsurance strategy stays constant, whereas the optimal investment strategy of each party changes very little.

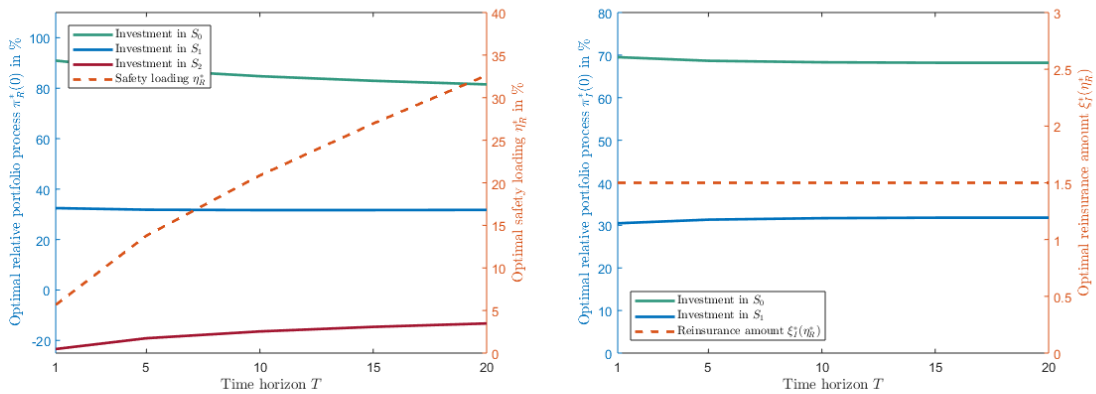


Figure 5.3: Sensitivity of the Stackelberg equilibrium w.r.t. T.

Finally, we look at the impact of the guarantee $G_T \in \{0.6 \cdot v_I, 0.7 \cdot v_I, 0.8 \cdot v_I, 0.9 \cdot v_I, 1 \cdot v_I, 1.1 \cdot v_I\}$ in the equilibrium investment-reinsurance strategies. Recall that G_T is the strike of the put option that models the reinsurance contract. According to Figure 5.4, the optimal safety loading is decreasing in G_T. This follows from the fact that the put option becomes more expensive as G_T increases, which is why the maximal safety loading (which increases the put option price) at which the insurer may still be willing to buy reinsurance decreases. On the contrary, the equilibrium reinsurance amount is constant despite the changes of G_T, namely $\xi_I = \xi_I^{\max} = 1.5$. This stems from the definition of the Stackelberg equilibrium, according to which, in case the follower has more than one best response to the leader's action, it is assumed that the follower selects the action that is also best from the leader's point of view. As for the optimal investment strategies, when G_T increases, the reinsurer invests the less in S_2 and more in the risk-free asset due to the hedge of its put-option position. With increasing G_T, the optimally behaving insurer invests more in the risky asset S_1.

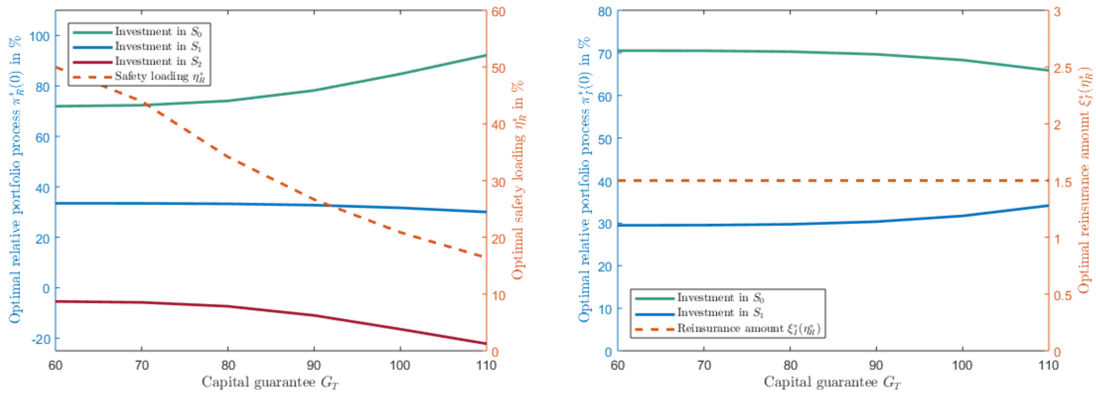


Figure 5.4: Sensitivity of the Stackelberg equilibrium w.r.t. G_T.

5.4.3 Impact of deviating from the Stackelberg equilibrium

According to (5.11) in Corollary 5.3.1, the insurer's best response to any $\eta_R \in [0, \eta_R^{\max}]$ is given by:

$$\xi_I^*(\eta_R) = \begin{cases} \bar{\xi}, & \text{if } \eta_R < \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)}{Put(0)}; \\ \text{any } \tilde{\xi} \in [0, \bar{\xi}], & \text{if } \eta_R = \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)}{Put(0)}; \\ 0, & \text{if } \eta_R > \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)}{Put(0)}. \end{cases}$$

So if the reinsurer selects the safety loading $\eta_R^* = \left(\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)\right)/Put(0)$, then the insurer becomes indifferent to the amount of reinsurance and can pick any $\xi_I \in [0, \bar{\xi}]$ without changing its expected utility. The Stackelberg equilibrium assumes that the insurer chooses ξ_I that is best from the perspective of the reinsurance company, i.e., $\xi_I^*(\eta_R^*) = \bar{\xi}$. However, in practice, the insurer could also decide to purchase a smaller amount of reinsurance. Therefore, the reinsurance company should consider charging a smaller safety loading than the equilibrium one η_R^* in order to ensure that the insurance company buys reinsurance. Two natural questions arise in this case. How the reinsurance company can reasonably choose a smaller safety loading? What is the benefit of each party in this case? These are the questions we answer in this subsection.

Denote a discounted safety loading by $\eta_R(l) = l \cdot \eta_R^*$ with $l \in (0,1)$. The reinsurance company is always better off when selling reinsurance with a discounted safety loading in comparison to not selling any reinsurance. This follows from the difference in the initial budgets of the reinsurance company and the reinsurer's ability to hedge its put-option position when reinsurance is sold. If the reinsurer sells reinsurance at a price $(1+\eta_R(l))Put(0)$, the company needs only $Put(0)$ capital to hedge its put-option position and can follow any investment strategy with the remaining budget $v_R + \eta_R(l)Put(0)$, which is larger than the initial budget v_R available to the reinsurer in case no reinsurance is sold.

Before providing several reasonable ways of how the reinsurer can choose the discounted safety loading $\eta_R(l)$, we introduce a tool for quantifying the monetary impact on each participant of the game if one combination of actions is followed instead of another one. "Combination of actions" means a specific choice of the safety loading as well as the investment strategy, which are the reinsurer's actions, and the amount of reinsurance as well as the investment strategy, which are the insurer's actions. For example, the Stackelberg equilibrium is compared with the case of no reinsurance along with the corresponding optimal investment strategies.

We generalize the concept of WEUL from Definition 2.3.5 to a **w**ealth-**e**quivalent **u**tility **c**hange (WEUC) by including static strategies in the comparison and by allowing the comparison of any two strategy choices, not only the optimal one with a suboptimal one. In the calculation of WEUC, the expected utility of a party in a so-called reference combination of actions (e.g., the Stackelberg equilibrium with a discounted safety loading) is compared to a so-called alternative combination of actions (e.g., no reinsurance and the corresponding optimal investment strategy of each party). WEUC equals the relative change of the party's initial wealth that is required to bring the party in the alternative action combination to the same expected utility as in the case of the reference combination of actions.

We denote a reference combination of actions by $(\tilde{\pi}_R(\cdot|\tilde{\eta}_R,\tilde{\xi}_I),\tilde{\eta}_R,\tilde{\pi}_I(\cdot|\tilde{\eta}_R,\tilde{\xi}_I),\tilde{\xi}_I)$ and an alternative combination of actions by $(\hat{\pi}_R(\cdot|\hat{\eta}_R,\hat{\xi}_I),\hat{\eta}_R,\hat{\pi}_I(\cdot|\hat{\eta}_R,\hat{\xi}_I),\hat{\xi}_I)$. Then the WEUC of the reinsurer is denoted by

$$WEUC_R(\tilde{\eta}_R,\tilde{\xi}_I,\hat{\eta}_R,\hat{\xi}_I) := WEUC_R((\tilde{\pi}_R(\cdot|\tilde{\eta}_R,\tilde{\xi}_I),\tilde{\eta}_R),(\hat{\pi}_R(\cdot|\hat{\eta}_R,\hat{\xi}_I),\hat{\eta}_R))$$

and satisfies the relation

$$\mathbb{E}\left[U_R\left(V_R^{v_{R,0}^{WEUC}(\hat{\xi}_I,\hat{\eta}_R),\hat{\pi}_R}(T) - \hat{\xi}_I Put(T)\right)\right] = \mathbb{E}\left[U_R\left(V_R^{v_{R,0}(\tilde{\xi}_I,\tilde{\eta}_R),\tilde{\pi}_R}(T) - \tilde{\xi}_I Put(T)\right)\right],$$

where

$$v_{R,0}^{WEUC}(\hat{\xi}_I,\hat{\eta}_R) = v_R \cdot (1 + WEUC_R(\tilde{\eta}_R,\tilde{\xi}_I,\hat{\eta}_R,\hat{\xi}_I)) + \hat{\xi}_I(1+\hat{\eta}_R)Put(0).$$

Analogously, the WEUC of the insurer is denoted by

$$WEUC_I(\tilde{\eta}_R, \tilde{\xi}_I, \hat{\eta}_R, \hat{\xi}_I) := WEUC_I((\tilde{\pi}_I(\cdot|\tilde{\eta}_R, \tilde{\xi}_I), \tilde{\xi}_I), (\hat{\pi}_I(\cdot|\hat{\eta}_R, \hat{\xi}_I), \hat{\xi}_I))$$

and satisfies the relation

$$\mathbb{E}\left[U_I\left(V_I^{v_{I,0}^{WEUC}(\hat{\xi}_I,\hat{\eta}_R),\hat{\pi}_I}(T) + \hat{\xi}_I Put(T)\right)\right] = \mathbb{E}\left[U_I\left(V_I^{v_{I,0}(\tilde{\xi}_I,\tilde{\eta}_R),\tilde{\pi}_I}(T) + \tilde{\xi}_I Put(T)\right)\right],$$

where

$$v_{I,0}^{WEUC}(\hat{\xi}_I, \hat{\eta}_R) = v_I \cdot (1 + WEUC_I(\tilde{\eta}_R, \tilde{\xi}_I, \hat{\eta}_R, \hat{\xi}_I)) - \hat{\xi}_I(1 + \hat{\eta}_R)Put(0).$$

The WEUC has an intuitive interpretation. If this quantity is positive, then the reference combination of actions is better for the party of interest than the alternative combination of actions. In this case, the WEUC indicates by which proportion the party has to increase its initial capital in the case of the alternative combination of actions so that it has the same expected utility as in the case of the reference combination of actions. If the WEUC is negative, then the reference combination of actions is worse for the considered party than the alternative combination of actions. In this case, the WEUC indicates by which proportion the party can decrease its initial capital in the case of the alternative combination of actions so that it has the same expected utility as in the case of the reference combination of actions. In the plots below, we indicate WEUC either in basis points or in percentage points.

Having introduced the concept of WEUC, we turn to the question how the reinsurer could choose the discount on the equilibrium safety loading to ensure that the insurer buys the maximal amount of reinsurance. One approach could be based on the reinsurer's WEUC. In this approach, the reinsurer calculates its WEUCs when the Stackelberg equilibrium (as a reference combination of actions) is compared with various alternative situations where all actions are the same as in the Stackelberg equilibrium but a safety loading is $\eta_R(l)$, $l \in (0,1)$. Then the reinsurer chooses the tolerance level of WEUC and derives the corresponding l. For example, consider the dark-blue line on Figure 5.5. It indicates that $WEUC_R > 0$ for any $l \in (0,1)$, which means that the reference combination of actions (Stackelberg equilibrium) is better than any alternative combination of actions with the discounted safety loading $\eta_R(l), l \in (0,1)$. Since the reinsurer cannot guarantee in practice that in the Stackelberg equilibrium the insurer buys the maximal amount of reinsurance, the reinsurer can consider the level $WEUC_R = 25$bp (see the turquoise dashed line) as the largest monetary loss it can tolerate when deviating from the Stackelberg equilibrium by providing a discount on the equilibrium safety loading. The corresponding parameter is $l = 79.27\%$ and defines the discounted safety loading $\eta_R(79.27\%) = 79.27\% \cdot 20.86\% \approx 16.54\%$ that the reinsurer is willing to offer.

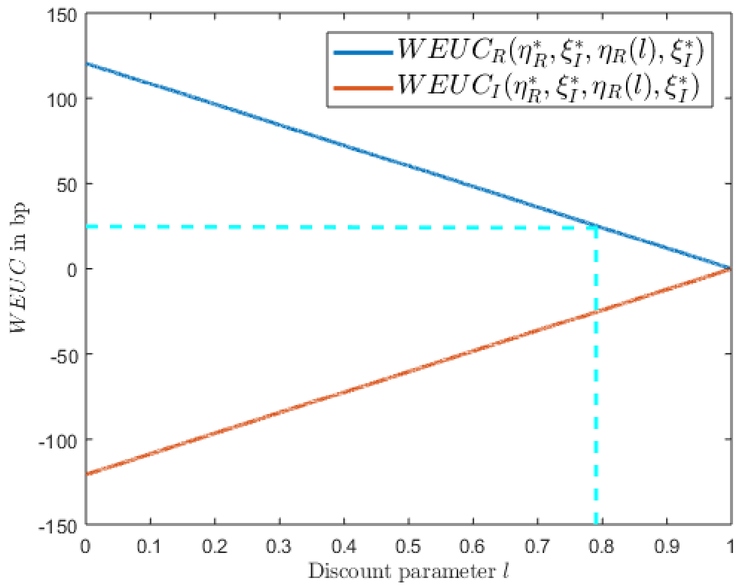


Figure 5.5: Comparison of WEUC: Stackelberg equilibrium (reference strategy) and the same action combination but with a discounted safety loading (alternative strategy).

Figure 5.5 also indicates that the gain of the insurer is of the same amount as the loss of the reinsurer when a discounted safety loading is offered instead of the equilibrium one. This property remains valid also when for other RRA coefficients of the companies. Hence, the insurer does not benefit more from deviating from the Stackelberg equilibrium as the reinsurer loses. For example, a discount of 5% on η_R^* would increase the WEUC of the reinsurer by around $6bp$ of the initial capital and decrease the insurer's WEUC by the same amount.

The other ways of choosing l can be related to reinsurance profitability from the perspective of the reinsurer. This can be measured by determining the probability that the reinsurer's total terminal wealth is lower than its initial wealth:

$$\mathbb{Q}(l) := \mathbb{Q}(V_R^{v_{R,0}(\eta_R(l),\xi_I^*),\pi_R^*(\cdot|\eta_R(l),\xi_I^*)}(T) - \xi_I^* Put(T) < v_R).$$

This measure shows the product profitability of the reinsurance for the reinsurer and is shown in Figure 5.6. The reinsurer can choose l in two different ways, which are based on:

- a tolerance level for the increase of the probability of loss. For example, the reinsurer is not willing to increase the probability of loss by more than 0.01% compared to the probability of loss $\mathbb{Q}(1) = 0.4413\%$ in the case of the Stackelberg equilibrium. In this example, the reinsurer chooses $l = 86.73\%$ (see green line in Figure 5.6).

- a maximal tolerable probability of loss. For example, if the reinsurer has a maximal admissible probability of loss 0.5%, then it chooses $l = 20.74\%$ (see purple line in Figure 5.6).

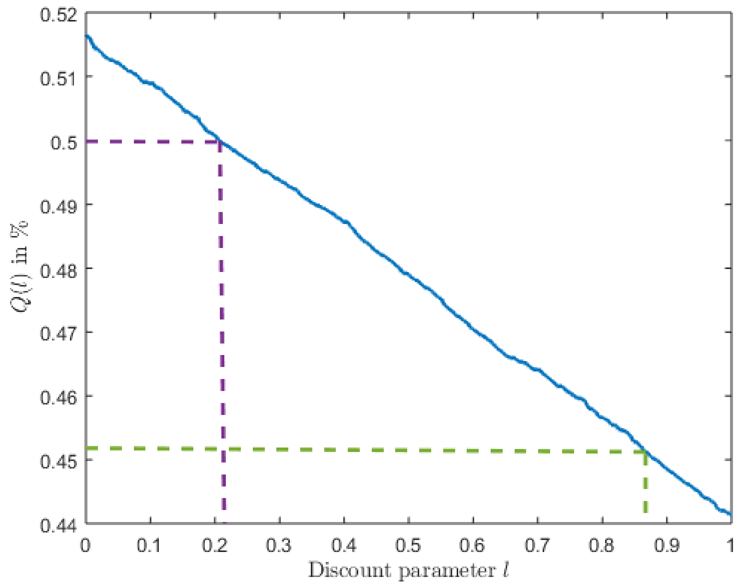


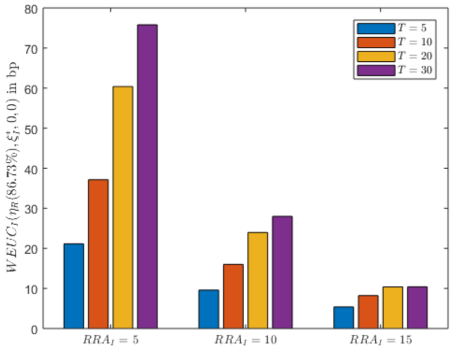
Figure 5.6: Probability of loss at time T for reinsurer.

The reinsurance company can also use other criteria for choosing $l \in (0, 1)$, e.g., other risk measures, as the standard deviation, or different performance measures, as the expected return or the adjusted Sharpe ratio.

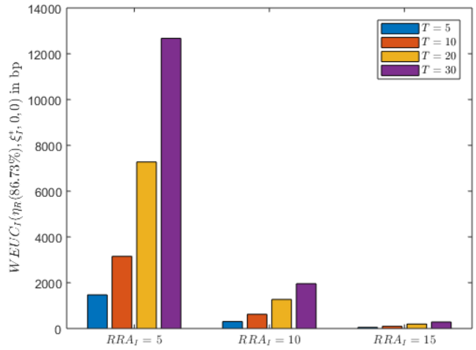
In the insurance industry, the analysis of the product profitability is a common procedure, which is also easier to communicate. Therefore, we select $l = 86.73\%$ for the remaining part of the numerical studies. In this case, the reinsurer limits the increase of the probability of loss (from the deal with the insurer) to 0.0001.

In the remaining analysis, we study how the insurer benefits from buying reinsurance with a safety loading of $\eta_R(86.73\%)$ (reference combination of actions) instead of buying no reinsurance (alternative combination of actions).

First, we determine the benefit of the insurance company if it follows the dynamic portfolio strategy and buys reinsurance $(\pi_I^*(\cdot|\eta_R(86.73\%), \xi_I^*), \xi_I^*)$ (reference) instead of the optimal investment strategy without reinsurance $(\pi_I^*(\cdot|0, 0), 0)$ (alternative). We compute $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$ for different RRA as well as T and plot the results in Subfigure 5.7a. Since $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0) > 0$, the insurance company needs less initial capital if it follows the dynamic portfolio strategy with reinsurance instead of the one without reinsurance. This means that the insurer can decrease the costs of its equity-linked product by buying reinsurance. The more risk averse the insurer, the



(a) WEUCs for the reference strategy $(\pi_I^*(\cdot|\eta_R(86.73\%), \xi_I^*), \xi_I^*)$ and the alternative strategy $(\pi_I^*(\cdot|0, 0), 0)$.



(b) WEUCs for the reference strategy $(\pi_I^*(\cdot|\eta_R(86.73\%), \xi_I^*), \xi_I^*)$ and the alternative strategy $((15\%, 0)^\top, 0)$.

Figure 5.7: Impact of relative risk aversion of insurer and investment horizon on $WEUC_I$.

lower $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$, and the longer the investment period, the higher $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$. If $RRA_I = 5$ and the product maturity is $T = 20$, then $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0) = 60bp$. If the insurer becomes more risk averse, then $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$ decreases strongly. For $RRA_I = 15$ and $T = 20$ we have $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0) = 10bp$.

Second, we consider the benefit of the insurance company if it follows the dynamic investment strategy with reinsurance $(\pi_I^*(\cdot|\eta_R(86.73\%), \xi_I^*), \xi_I^*)$ (reference) instead of the constant-mix (CM) strategy without reinsurance $((15\%, 0)^\top, 0)$ (alternative). Recall that we also considered the same CM strategy in Chapter 4, since it approximates the long-term investment strategy of an average life-insurance company. So, we calculate $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$ and analyze the impact of the relative risk aversion and investment horizon, see Figure 5.7b. Similarly to Subfigure 5.7a, the insurer needs less initial capital if it follows the dynamic investment strategy with reinsurance instead of the CM strategy without reinsurance. The more risk averse the insurance company, the lower $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$, and the longer the investment period, the higher $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$. If the insurer's RRA equals 5 and the company offers an equity-linked product with $T = 20$, then $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0) = 7275bp$. If the insurance company becomes more risk averse, then $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0)$ decreases substantially. For example, for $RRA_I = 15$ and $T = 20$ we obtain that $WEUC_I(\eta_R(86.73\%), \xi_I^*, 0, 0) = 287bp$.

6 Optimal investment under risk limitation and stochastic volatility

> Out of intense complexities, intense simplicities emerge.
>
> Sir Winston Churchill

So far we have considered the financial-market model described in Section 2.1. An important advantage of this market is its analytical tractability, which heavily relies on the assumption of constant market parameters and on the market completeness stated in Theorem 2.1.5. However, these properties are not supported by the empirical evidence of financial markets in real world, since interest rates, investment returns, volatility and other parameters describing the price dynamics of risky assets can vary over time. In this chapter, which is a reproduction of Escobar-Anel et al. (2022) with minor changes, we relax the assumption of constant volatility. In a stochastic-volatility setting, which implies the incompleteness of the financial market, we investigate the optimal investment strategies of a decision maker with risk limitation in the form of a Value-at-Risk (VaR) constraint. As described in detail later, the methodology we develop in this chapter opens the door to solving many previously unsolved portfolio-optimization problems with terminal-wealth constraints in incomplete markets.

Risk limitation using VaR plays a prominent role for both financial and insurance sectors of the world economy. Regulated by Basel III, banks have to comply with minimum capital reserve based on Expected Shortfall while the back-testing of bank-wide risk models is based on VaR, see, e.g., Basel Committee on Banking Supervision (2019). Regulated by Solvency II, insurance companies have to ensure that they hold enough capital to limit the risk of not satisfying liabilities. This capital requirement is calculated using a VaR measure at the confidence level of 99.5% on a 1-year period, see, e.g., Boonen (2017). Many insurance products contain in their design minimum guarantees, which implies effective constraints on the investment portfolios of insurance companies, see, e.g., Basak (1995), Boyle and Tian (2007).

There is ample evidence of time-dependent volatility in financial markets, see, e.g., Wiggins (1987) and Taylor (1994). One of the most popular stochastic-volatility models is the Heston model, introduced in the seminal paper Heston (1993). This model is highly regarded in the financial industry and is often used for option pricing. This raised interest in the optimal investment strategies in the setting of Heston's model. Kraft (2005) and Liu (2007) derived the optimal investment strategies for an investor in the setting of Heston's model without constraints on wealth or investment strategies.

Motivated by the above aspects and the absence of the results on the optimal VaR-constrained asset allocation under the Heston model, we consider a decision maker who maximizes his/her expected utility of terminal wealth while limiting the probability of loss (VaR constraint) in an incomplete financial market with one risk-free asset and one risky asset whose price-process dynamics is described by the Heston model. We answer the following questions:

1. How to derive the optimal VaR-constrained investment strategy and the optimal wealth process?
2. What is the impact of model parameters on the optimal VaR-constrained investment strategy?

As mentioned at the end of Section 2.1, the volatility of the risky asset is a mean-reverting stochastic process in Heston's model. We derive the optimal investment strategy by demonstrating that the optimal terminal wealth in the constrained optimization problem can be represented as a vega-neutral financial derivative on the optimal terminal wealth in the unconstrained optimization problem. To prove the result, we use a convenient financial derivative and match Hamilton-Jacobi-Bellman (HJB) equations under the real-world probability measure as well as an EMM. This generalizes the approach suggested by Kraft and Steffensen (2013) from the complete Black-Scholes financial market to the incomplete financial market driven by the Heston model.

Next we give an overview of relevant literature sources and structure it in two parts. First, we mention the sources on portfolio optimization with terminal-wealth constraints in complete markets. Second, we state papers that study optimal asset allocation with terminal-wealth constraints in incomplete markets.

In the case of market completeness, Basak and Shapiro (2001) and Kraft and Steffensen (2013) are the most influential relevant sources for this chapter. Basak and Shapiro (2001) solved a VaR-constrained portfolio optimization problem by embedding the constraint into utility maximization and relying on the martingale approach to derive the optimal investment strategies. Kraft and Steffensen (2013) showed that the same problem can be solved using dynamic programming. This is done by representing the optimal constrained terminal wealth as a conjectured financial derivative on the optimal unconstrained terminal wealth. Another relevant paper is Chen et al. (2018a), where the authors consider a wealth-constrained utility-maximization problem in a stochastic-volatility environment. The researchers complete the financial market with a traded financial option and derive the optimal investment strategies via the martingale approach.

In the area of constrained portfolio optimization in incomplete markets, closed-form solutions have remained elusive throughout the years, mostly due to the lack of techniques to tackle the problem. As mentioned in Chapters 4 and 5, Karatzas et al. (1991) consider an extension of the Black-Scholes market where the number of risk drivers is larger than the number of traded stocks, placing constraints on investment strategies, rather than wealth. Their idea is to complete the financial market by fictitious securities. This completion is based on a suitably parameterized family of fictitious securities, which correspond to exponential local martingales. The "right" completion should satisfy a certain minimality property. Gundel and Weber (2007) study this approach only for the optimal terminal wealth but do not derive the corresponding investment strategies. The explicit representation of the optimal terminal portfolio value is derived via certain worst-case measures, which can be characterized as minimizers of a dual problem. In parallel, He and Pearson (1991) apply a martingale approach to solve a consumption-portfolio problem in continuous time with incomplete markets and no-short-sale constraints on the investment strategy. They introduce a notion of minimax local martingale, transforming the dynamic problem into a static problem. Showing when the minimax local measure exists and how it is characterized, they derive conditions when the optimization has a solution, then linking the optimal strategies to the solution of a quasi-linear PDE.

Ntambara (2017) finds optimal trading strategies and portfolio values for the power-utility maximization problem under present expected shortfall constraints (same as the one considered in the context of a standard Black-Scholes market in Basak and Shapiro (2001)) in the incomplete market consisting of a geometric Brownian motion stock and a bank account where the interest rate follows a 1-factor Vasicek model. Assuming the existence of Lagrange multipliers, the author considers the Lagrange function and derives the optimal investment strategy as follows. First, for each fixed EMM $\widetilde{\mathbb{Q}}$, the optimal terminal portfolio value $V^*_{\widetilde{\mathbb{Q}}}(T)$ as a function of the pricing kernel is found from the primal problem using pointwise optimization. Second, $V^*_{\widetilde{\mathbb{Q}}}(T)$ is inserted into the dual problem and the optimal market prices of risk are found. Third, the replicating strategy is derived using Maliavin calculus. In order to find the optimal market prices of risk, which is the second step in the above-mentioned approach, one needs to know the distribution of the pricing kernel. When interest rates are modeled by a Cox-Ingersoll-Ross (CIR) process, the distribution of the pricing kernel is not known and one needs a numerical procedure for calculating it.

We contribute to the portfolio-optimization literature by generalizing the methodology of Kraft and Steffensen (2013) to the incomplete market due to stochastic volatility. To the best of our knowledge, we are first to derive the optimal investment strategies for a VaR-constrained decision maker in the incomplete market driven by the Heston model. Our methodology relies on the dynamic programming principle and, hence, can potentially circumvent the challenges that the martingale approach cannot handle. As a result, many previously unsolved portfolio-optimization problems may be finally solved.

The remainder of this chapter is organized as follows. In Section 6.1 we provide the problem setting and recapitulate the unconstrained utility-maximization problem under the Heston model, as per Kraft (2005). Section 6.1.1 presents the main theorem. Section 6.1.2 describes solutions to the power-utility maximization problem subject to VaR constraints in a Heston-model-based financial market. Section 6.2.2 reports details on numerical implementation and the results of numerical studies. Appendix D contains four sections. In Appendix D.1 we provide the results for the unconstrained optimization problem, which we need for solving the constrained one. In Appendix D.2, we prove the main results stated in Section 6.1.1. In Appendix D.3, we derive explicit formulas for computing the parameters of the synthetic financial derivative linking the solution to the constrained problem and the solution to the unconstrained one in the case of a power-utility function. In the last Appendix D.4, we provide an alternative way of deriving the optimal solution to the constrained portfolio optimization problem in the special case when the price process of the risky asset and the stochastic-volatility process are uncorrelated.

6.1 Constrained portfolio optimization problem and its solution

We consider the basic financial market from Chapter 2 for $n = 1$, but the risky asset S_1 has now one modification, namely its volatility is modeled by an additional stochastic mean-reverting process, as introduced in Heston (1993). The price dynamics of S_1 under the real-world measure $\mathbb{Q}$ is given by the following SDEs:

$$dS_1(t) = S_1(t)\left[\left(r + \gamma^{S_1} v(t)\right) dt + \sqrt{v(t)}\, dW_1^{\mathbb{Q}}(t)\right]; \tag{6.1}$$

$$dv(t) = \kappa\left(\theta - v(t)\right) dt + \sigma\rho\sqrt{v(t)}\, dW_1^{\mathbb{Q}}(t) \; + \sigma\sqrt{v(t)}\sqrt{1-\rho^2}\, dW_2^{\mathbb{Q}}(t) \tag{6.2}$$

with starting values $S_1(0) = s_1 > 0$ and $v(0) = v_0 > 0$, premium for volatility $\gamma^{S_1} > 0$, mean-reversion rate $\kappa > 0$, long-run mean $\theta > 0$, volatility of the variance $\sigma > 0$ and fulfilling Feller's condition $\kappa\theta > \frac{\sigma^2}{2}$. The portfolio value process under the real-world measure $\mathbb{Q}$ evolves according to:

$$dX^{x_0,\pi}(t) = X^{x_0,\pi}(t)\left[\left(r + \pi(t)\gamma^{S_1} v(t)\right) dt + \pi(t)\sqrt{v(t)}\, dW_1^{\mathbb{Q}}(t)\right], \qquad X^{x_0,\pi}(0) = x_0 > 0,$$

where $\pi(t)$ denotes the proportion of wealth invested in the risky asset at time $t \in [0, T]$, $1 - \pi(t)$ is the proportion of wealth invested in the cash account at time $t \in [0, T]$, and x_0 is the initial budget.

We consider the set of equivalent martingale measures that have the following Radon-

Nikodym derivatives w.r.t. $\mathbb{Q}$:

$$\frac{d\widetilde{\mathbb{Q}}(\gamma^{S_1}, \gamma^v(\cdot))}{d\mathbb{Q}} = \exp\Bigg(- \int_0^T \gamma^{S_1} \sqrt{v(s)}\, dW_1^{\mathbb{Q}}(s) - \int_0^T \gamma^v(s) \sqrt{v(s)}\, dW_2^{\mathbb{Q}}(s) - \frac{1}{2} \int_0^T \left(\left(\gamma^{S_1} \sqrt{v(s)}\right)^2 + \left(\gamma^v(s) \sqrt{v(s)}\right)^2 \right) ds \Bigg),$$

where $\widetilde{\mathbb{Q}}(\gamma^{S_1}, \gamma^v(\cdot))$ denotes a specific EMM, $\gamma^v(\cdot)$ is assumed to satisfy the Novikov's condition:

$$\mathbb{E}^{\mathbb{Q}} \left[\exp\left(\frac{1}{2} \int_0^T \left(\left(\gamma^{S_1} \sqrt{v(s)}\right)^2 + \left(\gamma^v(s) \sqrt{v(s)}\right)^2 \right) ds \right) \right] < +\infty.$$

To make notation concise, we will write only γ^v and $\widetilde{\mathbb{Q}}(\gamma^v)$, since only γ^v is a degree of freedom in the choice of the EMM. Moreover, we assume that γ^v is such that the Feller's condition is satisfied for the dynamics of $v(t)$ under $\widetilde{\mathbb{Q}}(\gamma^{S_1}, \gamma^v)$ (see (6.3) and (6.4) below). Heston (1993) assumes that the price of volatility risk is linear in the volatility process, i.e., in the square root of the variance process.

The Heston model under the EMM $\widetilde{\mathbb{Q}}(\gamma^v)$ is given by

$$dS_1(t) = S_1(t) \left[r dt + \sqrt{v(t)} dW_1^{\widetilde{\mathbb{Q}}}(t) \right]; \tag{6.3}$$

$$dv(t) = \tilde{\kappa} \left(\tilde{\theta} - v(t) \right) dt + \sigma\rho\sqrt{v(t)} dW_1^{\widetilde{\mathbb{Q}}}(t) + \sigma\sqrt{v(t)}\sqrt{1-\rho^2} dW_2^{\widetilde{\mathbb{Q}}}(t), \tag{6.4}$$

where $S_1(0) = s_1 > 0$, $v(0) = v_0 > 0$, $dW_1^{\mathbb{Q}}(t) = -\gamma^{S_1}\sqrt{v(t)}dt + dW_1^{\widetilde{\mathbb{Q}}}(t)$, $dW_2^{\mathbb{Q}}(t) = -\gamma^v\sqrt{v(t)}dt + dW_2^{\widetilde{\mathbb{Q}}}(t)$, $\tilde{\kappa} = \kappa + \sigma\gamma^{S_1}\rho + \sigma\gamma^v\sqrt{1-\rho^2}$, $\tilde{\theta} = \kappa\theta/\tilde{\kappa}$. Once again we emphasize that time could be impacting $\tilde{\kappa}$ and $\tilde{\theta}$ due to γ^v.

The wealth process under the EMM $\widetilde{\mathbb{Q}}(\gamma^v)$ evolves according to the SDE:

$$dX^{x_0,\pi}(t) = X^{x_0,\pi}(t) \left[r dt + \pi(t)\sqrt{v(t)} dW_1^{\widetilde{\mathbb{Q}}}(t) \right], \qquad X^{x_0,\pi}(0) = x_0 > 0.$$

Let us define $\mathcal{A}_u^{\pi}(x_0, v_0)$ to be the set of all admissible unconstrained relative portfolio processes:

$$\mathcal{A}_u^\pi(x_0, v_0) := \left\{ \pi = (\pi(t))_{t\in[0,T]} \;\middle|\; \pi \text{ is progressively measurable}, X^{x_0,\pi}(0) = x_0, \right.$$
$$\left. v(0) = v_0, \int_0^T \left(\pi(t) X^{x_0,\pi}(t)\right)^2 dt < \infty\, \mathbb{Q}\text{-a.s.} \right\}.$$

Let $\varepsilon \in [0, 1]$. Denoting by $C_X(\varepsilon) := \{X^{x_0,\pi}(T) \mid \mathbb{Q}\left(X^{x_0,\pi}(T) < K\right) = \varepsilon\}$, we define the set of all admissible constrained relative portfolio processes as follows:

$$\mathcal{A}_c^\pi\left(x_0, v_0, C_X(\varepsilon)\right) := \left\{\pi \in \mathcal{A}_u^\pi(x_0, v_0) \,|\, X^{x_0,\pi}(T) \in C_X(\varepsilon)\right\}.$$

Analogously, we denote by $\mathcal{A}_u^\pi(t, x, v)$ and $\mathcal{A}_c^\pi(t, x, v, C_X(\varepsilon))$ the corresponding sets of admissible relative portfolio processes π when the controlled process $X^{x,\pi}$ starts at time $t \in [0, T]$ with value $x > 0$, i.e., $X^{x,\pi}(t) = x$, and $v(t) = v$.

We assume that the decision maker maximizes the expected utility from terminal wealth with respect to a power-utility function $U(\cdot)$, defined in (2.39). The associated constrained value function is denoted by $\mathcal{V}^c\left(t, x, v\right)$. Hence, the main problem in this chapter is:

$$\begin{aligned} \mathcal{V}^c\left(t, x, v\right) &= \sup_{\pi\in\mathcal{A}_c^\pi(t,x,v,C_X(\varepsilon))} \mathbb{E}_{t,x,v}^{\mathbb{Q}}\left[U\left(X^{x,\pi}(T)\right)\right] \\ &= \sup_{\pi\in\mathcal{A}_c^\pi(t,x,v,C_X(\varepsilon))} \mathbb{E}_{t,x,v}^{\mathbb{Q}}\left[\frac{\left(X^{x,\pi}(T)\right)^p}{p}\right], \end{aligned} \tag{6.5}$$

where we write $\mathbb{E}_{t,x,v}^{\mathbb{M}}\left[\cdot\right] := \mathbb{E}^{\mathbb{M}}\left[\cdot | X^{x,\pi}(t) = x, v(t) = v\right]$ for $\mathbb{M} \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$. We denote the optimal investment strategy for (6.5) by $\pi_c^* = (\pi_c^*(s))_{s\in[t,T]}$ and the corresponding optimal welath process by $X^*(s) := X^{x,\pi_c^*}(s), s \in [t, T]$.

Due to the type of constraints, this problem can be rewritten with no constraints using a proper (optimal) Lagrange multiplier $\lambda_\varepsilon \in \mathbb{R}$:

$$\begin{aligned} \mathcal{V}^c\left(t, x, v\right) &= \sup_{\pi\in\mathcal{A}_c^\pi(t,x,v,C_X(\varepsilon))} \mathbb{E}_{t,x,v}^{\mathbb{Q}}\left[U\left(X^{x,\pi}(T)\right)\right] \\ &= \sup_{\pi\in\mathcal{A}_u^\pi(t,x,v)} \mathbb{E}_{t,x,v}^{\mathbb{Q}}\left[\overline{U}\left(X^{x,\pi}(T)\right)\right], \end{aligned} \tag{6.6}$$

where $\overline{U}\left(x\right)$ is a modified utility function for the problem without constraints on terminal wealth:

$$\overline{U}\left(x\right) = U\left(x\right) - \lambda_\varepsilon\left(1_{\{x<K\}} - \varepsilon\right).$$

We will solve (6.6) using the solution to the unconstrained optimization problem for a power-utility function. As the latter would lead to a different optimal wealth process and to gain in clarity, we denote the wealth process in the unconstrained problem by $Y^{y,\pi}(t), t \in [0, T]$, the optimal investment strategy for the unconstrained problem by

$\pi_u^* = (\pi_u^*(t))_{t\in[0,T]}$, and the optimal wealth process in the unconstrained problem by $Y^*(t) := Y^{y,\pi_u^*}(t), t \in [0,T]$. The value function corresponding to the unconstrained optimization problem is given by:

$$\mathcal{V}^u(t,y,v) = \max_{\pi\in\mathcal{A}_u^\pi(t,y,v)} \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[U(Y^\pi(T))\right] = \max_{\pi\in\mathcal{A}_u^\pi(t,y,v)} \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{(Y^{y,\pi}(T))^p}{p}\right]. \qquad (P_u)$$

The problem (P_u) is well-studied in the literature. In particular, for Heston's models whose parameters satisfy the following condition[1]:

$$\frac{p}{1-p}\gamma^{S_1}\left(\frac{\kappa\rho}{\sigma}+\frac{\gamma^{S_1}}{2}\right) < \frac{\kappa^2}{2\sigma^2}. \tag{6.7}$$

Kraft (2005) solve the unconstrained utility maximization problem. He uses the stochastic control approach, which we briefly described in Subsection 2.4.2, to derive a candidate solution and then provides a verification result. Kallsen and Muhle-Karbe (2010) combine the martingale approach (see Subsection 2.4.1), the concept of an opportunity process and the calculus of semimartingale characteristics to derive the optimal investment strategies for a decision maker who maximizes his/her expected power utility of terminal wealth in a market with one risky asset whose price dynamics follows a Heston model with parameters that may violate Condition (6.7). As many realistic parameterizations satisfy this condition, we focus on Heston's model parametrizations that satisfy Condition (6.7). For completeness of the exposition, we provide the optimal strategy, the optimal wealth and the value function in (P_u) in the following proposition.

Proposition 6.1.1. *Assume that* (6.7) *holds. Then the optimal investment strategy for* (P_u) *is given by:*

$$\pi_u^*(t) = -\frac{\gamma^{S_1}\mathcal{V}_y^u + \sigma\rho\mathcal{V}_{yv}^u}{x\mathcal{V}_{yy}^u} = \frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho}{1-p}b(t) \tag{6.8}$$

with $k_0 = \left(p\left(\gamma^{S_1}\right)^2\right)/(1-p)$, $k_1 = \kappa - \left(p\gamma^{S_1}\sigma\rho\right)/(1-p)$, $k_2 = \sigma^2 + (p\sigma^2\rho^2)/(1-p)$, $k_3 = \sqrt{k_1^2 - k_0k_2}$ *and*

$$b(t) = k_0\frac{\exp\left(k_3(T-t)\right)-1}{\exp\left(k_3(T-t)\right)(k_1+k_3) - k_1 + k_3}. \tag{6.9}$$

The value function is given by

$$\mathcal{V}^u(t,y,v) = \frac{y^p}{p}\exp(a(t)+b(t)v), \tag{6.10}$$

[1] Same as Condition (26) in Kraft (2005)

where $b(t)$ is defined by (6.9) *and*

$$a(t) = pr(T-t) + \frac{2\theta\kappa}{k_2}\ln\left(\frac{2k_3\exp\left(\frac{1}{2}(k_1+k_3)(T-t)\right)}{2k_3+(k_1+k_3)\left(\exp\left(k_3(T-t)\right)-1\right)}\right).$$

The optimal wealth Y^* has the following dynamics under $\mathbb{Q}$:

$$\begin{aligned} \mathrm{d}Y^*(t) = Y^*(t)\Bigg[&\left(r+\left(\frac{\gamma^{S_1}}{1-p}+\frac{\sigma\rho b(t)}{1-p}\right)\gamma^{S_1}v(t)\right)\mathrm{d}t \\ &+\left(\frac{\gamma^{S_1}}{1-p}+\frac{\sigma\rho b(t)}{1-p}\right)\sqrt{v(t)}\,\mathrm{d}W_1^{\mathbb{Q}}(t)\Bigg],\quad Y^*(0)=y>0. \end{aligned} \tag{6.11}$$

Proof. See Appendix D.1. □

In the next proposition we provide the characteristic functions of the logarithm of the optimal unconstrained wealth $Z^*(t) := \ln\left(Y^*(t)\right), t \in [0,T]$, under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}(\gamma^v)$.

Proposition 6.1.2. *The logarithm of the optimal unconstrained wealth has characteristic functions of the form:*

$$\phi^{Z^*(T),\mathbb{M}}(u;t,z,v) = \mathbb{E}^{\mathbb{M}}_{t,z,v}\left[\exp(iuZ^*(T))\right] = \exp\left(A^{\mathbb{M}}(T-t,u)+B^{\mathbb{M}}(T-t,u)v+iuz\right),$$

where $\mathbb{M}\in\{\mathbb{Q},\widetilde{\mathbb{Q}}\}$ and $A^{\mathbb{M}}$ and $B^{\mathbb{M}}$ satisfy ordinary differential equations (ODEs):

$$\begin{aligned} 0 &= -B^{\mathbb{Q}}_\tau(\tau,u)+\left(\pi^*(\tau)\sigma\rho iu-\kappa\right)B^{\mathbb{Q}}(\tau,u)+\frac{1}{2}\sigma^2\left(B^{\mathbb{Q}}(\tau,u)\right)^2 \\ &\quad -\frac{1}{2}\left(\pi^*(\tau)\right)^2\left(u^2+iu\right)+\pi^*(\tau)\gamma^{S_1}iu; \\ 0 &= -A^{\mathbb{Q}}_\tau(\tau,u)+riu+\kappa\theta B^{\mathbb{Q}}(\tau,u). \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} 0 &= -B^{\widetilde{\mathbb{Q}}}_\tau(\tau,u)+\left(\pi^*(\tau)\sigma\rho iu-\tilde{\kappa}\right)B^{\widetilde{\mathbb{Q}}}(\tau,u)+\frac{1}{2}\sigma^2\left(B^{\widetilde{\mathbb{Q}}}(\tau,u)\right)^2 \\ &\quad -\frac{1}{2}\left(\pi^*(\tau)\right)^2\left(u^2+iu\right); \\ 0 &= -A^{\widetilde{\mathbb{Q}}}_\tau(\tau,u)+riu+\tilde{\kappa}\tilde{\theta}B^{\widetilde{\mathbb{Q}}}(\tau,u). \end{aligned} \tag{6.13}$$

respectively, where $\tau := T-t$ and $\pi^* := \pi^*_u$ as per (6.8).

Proof. See Appendix D.1. □

Remarks to Proposition 6.1.2

1. The characteristic function of $\ln(Y^*(t))$ has the same structural form as the structural form of the characteristic function of $\ln(S_1(t))$. The latter function is known in closed form.
2. The ODEs for $B^{\mathbb{Q}}$ and $B^{\widetilde{\mathbb{Q}}}$ are of Riccati type, as in the case of the characteristic function of $\ln(S^*(T))$. However, here the coefficients of these Riccati ODEs are time-dependent. The analytical solutions of these ODEs are not known. Therefore, one has to compute them numerically.

The above proposition is needed for solving the constrained portfolio optimization problem in semi-closed form. In particular, we will use it for calculating expected values using Theorem 2.2.5, in particular:

$$\begin{aligned}\mathbb{E}^{\mathbb{M}}_{t,z,v}\left[g\left(Z^*(T)\right)\right] &= \int g(x) f^{\mathbb{M}}_{Z^*(T)}(x)\,dx \\ &\overset{\text{Th. 2.2.5}}{=} \int g\left(x\right)\left(\frac{1}{2\pi}\int \exp\left(-iux\right)\phi^{Z^*(T),\mathbb{M}}(u;t,z,v)du\right)dx \qquad (6.14)\\ &\overset{\text{Pr. 6.1.2}}{=} \frac{1}{2\pi}\int\int g\left(x\right)\exp\Big(-iu\left(x-z\right)+A^{\mathbb{M}}(T-t,u)+B^{\mathbb{M}}(T-t,u)v\Big)\,dudx,\end{aligned}$$

where $g(\cdot)$ is such that the expectation exists and is finite, and $f^{\mathbb{M}}_{Z^*(T)}(\cdot)$ is the conditional probability density function of $Z^*(T)$ given $Z^*(t)=z$.

6.1.1 Solution to the constrained problem

The optimal wealth X^* for the constrained problem (6.5) will be obtained using a financial derivative on the optimal wealth Y^* from the unconstrained problem (P_u). As per Proposition 6.1.1, we have the following SDEs of the optimal unconstrained wealth process and variance process under the EMM $\widetilde{\mathbb{Q}}(\gamma^v)$:

$$\begin{aligned}dY^*(t) &= Y^*(t)rdt + Y^*(t)\pi^*_u(t)\sqrt{v(t)}dW_1^{\widetilde{\mathbb{Q}}}(t); \qquad (6.15)\\ dv(t) &= \tilde{\kappa}\left(\tilde{\theta}-v(t)\right)dt+\sigma\sqrt{v(t)}\rho dW_1^{\widetilde{\mathbb{Q}}}(t)+\sigma\sqrt{v(t)}\sqrt{1-\rho^2}dW_2^{\widetilde{\mathbb{Q}}}(t).\end{aligned}$$

So we want to find a financial derivative (contingent claim) whose payoff is $D(\cdot)$ and the underlying portfolio is $Y^*(T)$. We denote the price of the contingent claim at time $t\in[0,T]$ by $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)$ such that $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(T,y,v)=D(y)$. This financial derivative should fulfill the budget constraint and the terminal-wealth constraint, i.e., $D(Y^*(T))\in\mathcal{A}^{\pi}_c(x_0,v_0)$ and $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(0,y_0,v_0)=x_0$.

By Theorem 2.2.2, the price of $D(Y^*(T))$ is given by

$$D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)=\mathbb{E}^{\widetilde{\mathbb{Q}}(\gamma^v)}_{t,y,v}\left[\exp\left(-r(T-t)\right)D(Y^*(T))\right]. \qquad (6.16)$$

Applying Theorem 2.2.8, we get the corresponding Feynman-Kac PDE for (6.16):

$$
\begin{aligned}
D_t^{\widetilde{\mathbb{Q}}(\gamma^v)} &= rD^{\widetilde{\mathbb{Q}}(\gamma^v)} - ryD_y^{\widetilde{\mathbb{Q}}(\gamma^v)} - \tilde{\kappa}\left(\tilde{\theta} - v\right) D_v^{\widetilde{\mathbb{Q}}(\gamma^v)} \\
&\quad - \frac{1}{2}v\left[y^2(\pi_u^*)^2 D_{yy}^{\widetilde{\mathbb{Q}}(\gamma^v)} + 2\sigma\rho y\pi_u^* D_{yv}^{\widetilde{\mathbb{Q}}(\gamma^v)} + \sigma^2 D_{vv}^{\widetilde{\mathbb{Q}}(\gamma^v)}\right]; \qquad (6.17) \\
D^{\widetilde{\mathbb{Q}}(\gamma^v)}(T, y, v) &= D(y).
\end{aligned}
$$

The expected utility of the financial derivative, based on the modified utility function $\overline{U}(\cdot)$, is:

$$
\overline{U}^{D,\mathbb{Q}}(t, y, v) = \mathbb{E}_{t,y,v}^{\mathbb{Q}}\left[\overline{U}(D(Y^*(T)))\right], \qquad (6.18)
$$

where the optimal wealth process under the real-world measure $\mathbb{Q}$ comes from Proposition 6.1.1. Again via Feynman-Kac Theorem 2.2.8, the investor's expected modified utility of the contingent claim ($\overline{U}^{D,\mathbb{Q}}(t, y, v)$) satisfies the following PDE:

$$
\begin{aligned}
0 &= \overline{U}_t^{D,\mathbb{Q}} + \left(r + \pi_u^*\gamma^{S_1}v\right) y\overline{U}_y^{D,\mathbb{Q}} + \kappa(\theta - v)\,\overline{U}_v^{D,\mathbb{Q}} \\
&\quad + \frac{1}{2}v\left[y^2(\pi_u^*)^2\overline{U}_{yy}^{D,\mathbb{Q}} + 2\sigma\rho y\pi_u^*\overline{U}_{yv}^{D,\mathbb{Q}} + \sigma^2\overline{U}_{vv}^{D,\mathbb{Q}}\right]; \qquad (6.19) \\
\overline{U}^{D,\mathbb{Q}}(T, y, v) &= \overline{U}(D(y)). \qquad (6.20)
\end{aligned}
$$

We show next that the wealth of the constrained problem (6.5) can be represented by the price $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v)$ of a financial derivative on Y^*, and the value function $\mathcal{V}^c(t, y, v)$ by the expected modified utility on the financial derivative $\overline{U}^{D,\mathbb{Q}}(t, y, v)$. The theorem next provides three conditions such that the PDEs and terminal conditions associated to $\mathcal{V}^c(t, x, v)$ and $\overline{U}^{D,\mathbb{Q}}(t, y, v)$ coincide, with $x = D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v)$.

Theorem 6.1.3 (Main theorem). *Assume that Condition* (6.7) *holds and the VaR constraint is feasible in* (6.5)*. Let* $D(\cdot)$*,* y*,* $\gamma^v(\cdot)$ *and* λ_ε *be such that* $\mathbb{Q}\left(D(Y^{y,\pi_u^*}(T)) < K\right) = \varepsilon$*,* $D(\cdot)$ *is non-decreasing on* $(0, +\infty)$ *and strictly increasing on a non-empty open subinterval of* $(0, +\infty)$ *and the following three conditions are satisfied at time* $t \in [0, T]$*:*

$$
\begin{aligned}
A &:= \frac{\overline{U}_{yy}^{D,\mathbb{Q}}(t,y,v)}{\overline{U}_y^{D,\mathbb{Q}}(t,y,v)} - \frac{D_{yy}^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)}{D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)} = -\frac{1-p}{y}; \qquad (6.21) \\
B &:= \frac{\overline{U}_{yv}^{D,\mathbb{Q}}(t,y,v)}{\overline{U}_y^{D,\mathbb{Q}}(t,y,v)} - \frac{D_{yv}^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)}{D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)} = b(t); \qquad (6.22) \\
& \qquad D_v^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v) = 0, \qquad (6.23)
\end{aligned}
$$

where $D^{\widetilde{\mathbb{Q}}(\gamma^v)}$ *is given by* (6.16), $\overline{U}^{D,\mathbb{Q}}$ *is defined in* (6.18), $Y^*(t) = y$, $v(t) = v$. *Then the candidate for the optimal terminal wealth is:*

$$\begin{aligned} X^{x,\pi_c^*}(T) &= D(Y^{y,\pi_u^*}(T)) \\ \text{with } x &= \mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}(\gamma^v)}\left[\exp\left(-r(T-t)\right) D(Y^{y,\pi_u^*}(T))\right] \left(=: D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)\right) \end{aligned} \tag{6.24}$$

and the candidates for the value function and the optimal relative portfolio process in (6.5) *at time* $t \in [0,T]$ *are:*

$$\left(\mathcal{V}^c\left(t,x,v\right) :=\right) \mathbb{E}_{t,x,v}^{\mathbb{Q}}\left[U\left(X^{x,\pi_c^*}(T)\right)\right] = \mathbb{E}_{t,y,v}^{\mathbb{Q}}\left[\overline{U}(D(Y^{y,\pi_u^*}(T)))\right] \left(=: \overline{U}^{D,\mathbb{Q}}(t,y,v)\right); \tag{6.25}$$

$$\pi_c^*(t) = \pi_u^*(t) \cdot y \cdot \frac{D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)}{D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)}. \tag{6.26}$$

If $\rho = 0$, *solely Conditions* (6.21) *and* (6.23) *are required.*

Proof. See Appendix D.2. □

Remarks to Theorem 6.1.3

1. We do not impose any condition on γ^v. Along with the parameters of the payoff function $D(\cdot)$, it is an important degree of freedom for ensuring Conditions (6.21) – (6.23), as we will see in following corollaries.
2. Condition (6.21) is the same as in Kraft and Steffensen (2013). Moreover, in the absence of stochastic volatility we recover their results for the complete Black-Scholes market.
3. Condition (6.23) means that the financial derivative with terminal payoff $D(\cdot)$ has to be vega-neutral at time t and the value v of the variance process[2]. The complexity lies in crafting this payoff function $D(\cdot)$.
4. If the optimal terminal wealth in the unconstrained problem with the initial capital x_0 satisfies the VaR constraint, then it is obviously the optimal wealth in the VaR-constrained optimization problem. In this case, $D(y) = y$, $\lambda_\varepsilon = 0$ and:
   - (6.21) $\iff$ $\overline{U}_y^{D,\mathbb{Q}}(t,y,v) = y^{p-1} G_A(t,v)$ for some function $G_A(t,v)$, which holds for $\mathcal{V}^u(t,y,v)$ from (6.10);
   - (6.22) $\iff$ $\overline{U}_y^{D,\mathbb{Q}}(t,y,v) = \exp\left(b(t)v\right) G_B(t,y)$ for some function $G_B(t,y)$, which holds for $\mathcal{V}^u(t,y,v)$ from (6.10);

[2] $\frac{\partial}{\partial v} D^{\widetilde{\mathbb{Q}}} = \frac{\partial D^{\widetilde{\mathbb{Q}}}}{\partial \sqrt{v}} \frac{\partial \sqrt{v}}{\partial v} = \frac{\partial D^{\widetilde{\mathbb{Q}}}}{\partial \sqrt{v}} \frac{1}{2\sqrt{v}}$

- (6.23) holds;
- (6.25) becomes $\mathcal{V}^c(t,y,v) = \overline{U}^{D,\mathbb{Q}}(t,y,v) \overset{\lambda_\varepsilon=0}{\equiv} \mathcal{V}^u(t,y,v)$;
- (6.26) $\pi_c^*(t) = y\left(\frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho}{1-p}b(t)\right)\frac{1}{y} = \pi_u^*(t)$.

Next we provide convenient sufficient conditions to facilitate the applications of our main theorem.

Lemma 6.1.4 (Sufficient condition for (6.21) and (6.22))**.** *Condition* (6.21) *is satisfied at time* $t \in [0,T]$ *given* $Y^*(t) = y$ *and* $v(t) = v$*, if there exists a function* $H(t,v)$ *such that the following sufficient condition holds:*

$$\overline{U}_y^{D,\mathbb{Q}}(t,y,v) = y^{p-1}H(t,v)D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v). \tag{SC$_0$}$$

Both Condition (6.21) *and Condition* (6.22) *are satisfied at time* $t \in [0,T]$ *given* $Y^*(t) = y$ *and* $v(t) = v$*, if* (SC$_0$) *holds with* $H(t,v) = h(t)\exp(b(t)v)$ *for some function* $h(t)$*, i.e.:*

$$\overline{U}_y^{D,\mathbb{Q}}(t,y,v) = y^{p-1}h(t)\exp(b(t)v)\,D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v). \tag{SC}$$

Proof. See Appendix D.2. □

Remarks to Lemma 6.1.4

1. In contrast to the sufficient condition in Kraft and Steffensen (2013), Condition (SC) has an additional term $\exp(b(t)v)$.
2. As we will see later, $h(t) = \exp(a(t))$ with $a(t)$ from Proposition 6.1.1.

6.1.2 Explicit formulas

Solving a VaR-constrained power-utility maximization problem in a complete Black-Scholes market, Kraft and Steffensen (2013) use a contingent claim $D^{BS}(\cdot)$ with the following payoff:

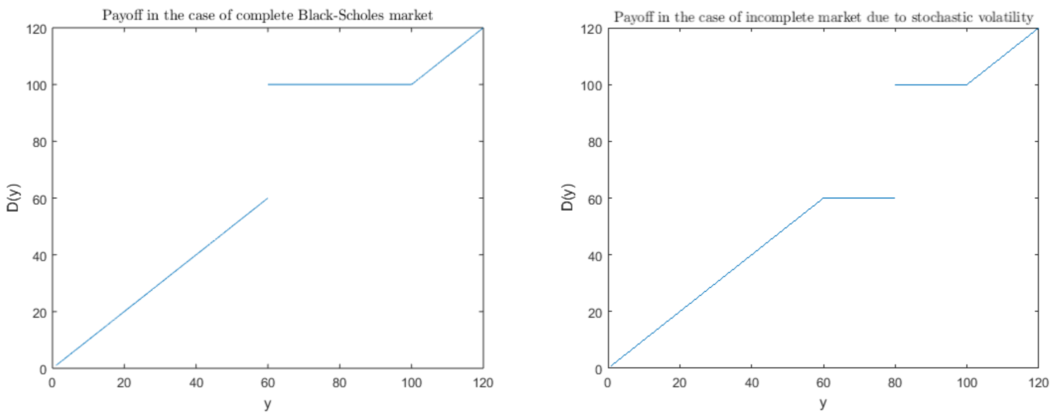
$$\begin{aligned} X^*(T) &= Y^*(T) + (K - Y^*(T))\mathbb{1}_{\{k_\varepsilon < Y^*(T) \le K\}} \\ &= Y^*(T) + (K - Y^*(T))\,\mathbb{1}_{\{Y^*(T) \le K\}} \\ &\quad - (k_\varepsilon - Y^*(T))\,\mathbb{1}_{\{Y^*(T) < k_\varepsilon\}} - (K - k_\varepsilon)\,\mathbb{1}_{\{Y^*(T) < k_\varepsilon\}} =: D^{BS}(Y^*(T)), \end{aligned} \tag{6.27}$$

where $0 \le k_\varepsilon \le K$.

The payoff (6.27) is illustrated in Figure 6.1a. It consists of a long position in the optimal unconstrained wealth, a long put option, a short put option with a lower strike and a binary put option. Our conjecture about the contingent claim $D(\cdot)$ can be seen as an extension of (6.27) with an additional degree of freedom that is needed to ensure the vega neutrality in (6.23):

$$\begin{aligned} X^*(T) &= Y^*(T) + (K - Y^*(T))\,\mathbb{1}_{\{k_\varepsilon \le Y^*(T) \le K\}} - (Y^*(T) - k_v)\,\mathbb{1}_{\{k_v \le Y^*(T) < k_\varepsilon\}} \\ &= Y^*(T) + (K - Y^*(T))\,\mathbb{1}_{\{Y^*(T) \le K\}} \\ &\quad - (k_v - Y^*(T))\,\mathbb{1}_{\{Y^*(T) < k_v\}} - (K - k_v)\,\mathbb{1}_{\{Y^*(T) < k_\varepsilon\}} =: D\left(Y^*(T)\right), \end{aligned} \tag{6.28}$$

with $0 \le k_v \le k_\varepsilon \le K$. The values of parameters k_v and k_ε are selected to ensure that Condition (6.23) and the VaR constraint are satisfied. Note that $\mathbb{Q}\left(X^*(T) < K\right) = \varepsilon \Leftrightarrow \mathbb{Q}\left(Y^*(T) < k_\varepsilon\right) = \varepsilon$ and $D(\cdot)$ has enough flexibility to ensure Condition (6.23), which means vega-neutrality of the financial derivative. This payoff is illustrated in Figure 6.1b.



(a) Example of Payoff (6.27), complete Black-Scholes market.

(b) Example of Payoff (6.28), incomplete stochastic volatility market.

Figure 6.1: Illustration of the payoffs $D(\cdot)$ for the Black-Scholes model and the Heston model.

The next corollary of our main theorem provides the solution to (6.5).

Corollary 6.1.5 (Solution to (6.5))**.** *Assume that Condition* (6.7) *holds and the VaR constraint is feasible in* (6.5)*. Set* $\gamma^v(t) = -\sigma\sqrt{1-\rho^2}b(t)$*, and let* $D(\cdot)$ *be given by* (6.28) *such that its degrees of freedom* $(y, k_v, k_\varepsilon)_t$ *satisfy the system of non-linear equations and inequalities (NLS):*

$$\begin{cases} h_B(y, k_v, k_\varepsilon) & := D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v) = x_t \\ h_{VN}(y, k_v, k_\varepsilon) & := D_v^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v) = 0 \\ h_{VaR}(y, k_v, k_\varepsilon) & := \mathbb{Q}\left(Y^*(T) < k_\varepsilon | Y^*(t) = y, v(t) = v\right) = \varepsilon \end{cases} \tag{NLS(y, k_v, k_ε)}$$

for the Lagrange multiplier

$$\lambda_\varepsilon = y^{p-1}\exp\left(a(t)+b(t)v\right)\left(K-k_v\right)\exp\left(-r(T-t)\right)\frac{f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(\ln k_\varepsilon)}{f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon)} - \frac{K^p-k_v^p}{p}, \quad (6.29)$$

where $f^{\mathbb{M}}_{Z^*(T)}(\cdot)$ *is the conditional density function of* $Z^*(T) := \ln(Y^*(T))$ *under the measure* $\mathbb{M} \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$ *given* $Y^*(t) = y$ *and* $v(t) = v$. *Then the candidate for the optimal terminal portfolio value is given by* (6.24)*, the candidate for the value function is given by* (6.25) *and the candidate for the solution to* (6.5) *is given by* (6.26)*.*

Proof. See Appendix D.2 □

Remarks to Corollary 6.1.5

1. The tuple $(y, k_v, k_\varepsilon)_t$ needs to be updated at every t in order to produce the right strategy.
2. The conditional density functions $f^{\mathbb{M}}_{Z^*(T)}(\cdot)$ can be calculated using the inversion of the characteristic functions of $Z^*(T)$ provided in Proposition 6.1.2.
3. The investor's value function $\mathcal{V}^c(t, x, v)$, the price of the derivative $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v)$ and its Greeks $D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}$ as well as $D_v^{\widetilde{\mathbb{Q}}(\gamma^v)}$ can be computed using the Carr-Madan approach to pricing options. We provide the corresponding formulas for $h_B(y, k_v, k_\varepsilon)$, $h_{VaR}(y, k_v, k_\varepsilon)$, $h_{VN}(y, k_v, k_\varepsilon)$ in Appendix D.3.

6.2 Numerical studies

In this section, first, we explain how we choose the model parameters and provide details on the solution procedure for the system of non-linear equations and inequalities (NLS(y, k_v, k_ε)). Second, we explore how the correlation coefficient and the volatility of the variance process influence the constrained optimal investment strategy. The latter can be considered as proxy for the magnitude of the market incompleteness.

6.2.1 Model parameterization and numerical procedure

We choose the Heston model parameters as in Table 4 in Escobar and Gschnaidtner (2016), the row corresponding to the average case of mentioned table. Note that the authors provide there parameterization under the EMM. In particular, we set: $\kappa = 3.6129$, $\theta = 0.0291$, $\sigma = 0.3$, $\rho = -0.4$, $v_0 = 0.03$. $\gamma^{S_1} = 1$ and $r = 3\%$. Under these parameters, $\gamma^v = 0.0238$, which leads to $\tilde{\kappa} = 3.5$, $\tilde{\theta} = 0.03$. We set $p = -2$, which corresponds to the relative risk aversion coefficient of 3, as also considered in Chen et al.

(2018a). We assume that the investor's time horizon is $T = 3$, his/her initial wealth is $x_0 = 100$, and the VaR constraint is specified by $K = 100$ and $\varepsilon = 1\%$ in the base case.

Solving a system of non-linear equations $\mathrm{NLS}(y, k_v, k_\varepsilon)$ requires numerical methods. First, we need to $A^{\mathbb{M}}$, $B^{\mathbb{M}}$, $\mathbb{M} \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$ appearing in the characteristic functions of $Z^*(T)$. As we mentioned in Remark 2 to Propostion 6.1.2, the ODEs for $B^{\mathbb{M}}$ have time-dependent complex-valued coefficients and are of Riccati type. To compute the solutions to those equations, we use a Matlab function `ode45` that is based on an explicit Runge-Kutta (4,5) formula. We choose a time grid of 10001 points, which corresponds to a time discretization step of $3 \cdot 10^{-4}$). Second, we compute the LHS of ($\mathrm{NLS}(y, k_v, k_\varepsilon)$) using the Carr-Madan approach, see Appendix D.3 for explicit formulas. As for dampening factors in this approach, we use $\alpha = 2$ for plain vanilla put options (the 2-nd and 3-rd terms in the financial derivative D) and $\alpha = 0.5$ for a digital put option (the 4-th term in D). Finally, the solution of $\mathrm{NLS}(y, k_v, k_\varepsilon)$ is computed by minimizing the sum of squared absolute errors, which is done with the help of a Matlab function `fmincon` with the sequential quadratic programming as the underlying non-linear optimization algorithm.

6.2.2 Numerical results

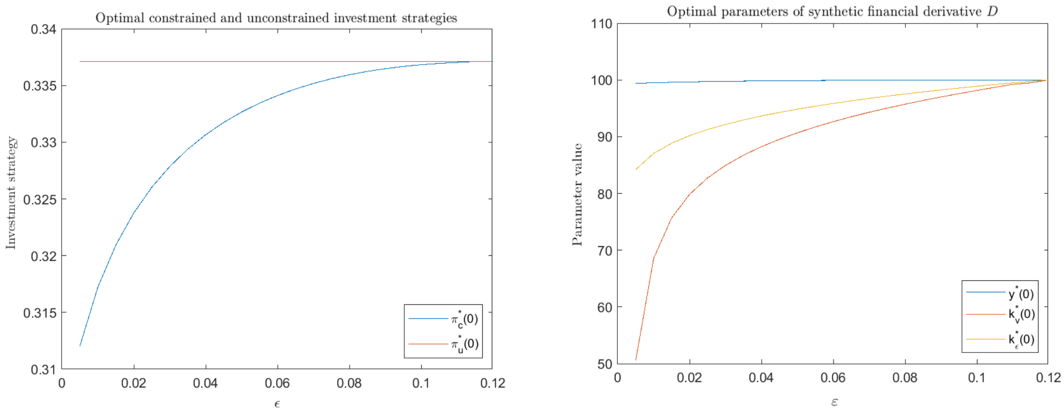
In this subsection, we first compute and interpret the optimal constrained investment strategy in the base case of $\varepsilon = 1\%$. Second, we conduct a sensitivity analysis of $\pi_c^*(0)$ and the optimal parameters of the synthetic derivative D with respect to ε. Third, we examine the impact of the RRA coefficient and the investment horizon on the optimal constrained investment strategy. Fourth, we examine the influence of the correlation coefficient ρ on $\pi_c^*(0)$. Finally, we examine the sensitivity of $\pi_c^*(0)$ with respect to the volatility σ of the variance process and the mean-reversion rate κ of the variance process.

In the base case, the optimal unconstrained investment strategy at time $t = 0$ is equal to 33.71%. The optimal constrained investment strategy at time $t = 0$ equals to 31.72%. The optimal terminal wealth in the constrained problem equals a financial derivative on the optimal unconstrained wealth with the following parameters: $y^*(0) = 99.5$, $k_v^*(0) = 68.55$, $k_\varepsilon^*(0) = 87.96$. It means that the optimal terminal wealth in the constrained optimization problem given the starting value $x_0 = 100$ is equal to a financial derivative that consists of:

1. a long position in the optimal unconstrained wealth $Y^*(T)$ with a starting value of $Y^*(0) = y^*(0) = 99.5$;
2. a long position in one put option on the optimal unconstrained wealth $Y^*(T)$ and with strike $K = 100$;

3. a short position in one put option on the optimal unconstrained wealth $Y^*(T)$ and with strike $k_v^*(0) = 68.55$;
4. a short position in $K - k_v^*(0) = 31.45$ digital-put options on the optimal unconstrained wealth $Y^*(T)$ and with strike $k_\varepsilon^*(0) = 87.96$.

Next, we analyze the impact of ε. Denote by $\varepsilon_u := \mathbb{Q}\left(Y^{x_0,\pi_u^*}(T) < K\right) \approx 12\%$ the probability that the optimal terminal unconstrained portfolio value is below K. Consider now Figure 6.2, which consists of two subfigures. In Subfigure 6.2a we see that for increasing ε the constrained optimal investment strategy becomes closer to the unconstrained one. This is intuitive, as the closer ε is to ε_u , the more the optimal constrained investment strategy should resemble the optimal unconstrained one. As Subfigure 6.2b indicates, the larger $\varepsilon < \varepsilon_u$, the larger the optimal initial capital of the underlying of the financial derivative D (the optimal unconstrained portfolio) and the higher are the thresholds k_ε and k_v. This is consistent with our previous finding that ε closer to ε_u leads to the optimal constrained investment strategy that is closer to the unconstrained one. The same holds for the optimal terminal wealth, as increasing k_ε and k_v mean that the optimal payoff of the derivative D is closer to the identity function (see Remark 4 to Theorem 6.1.3).



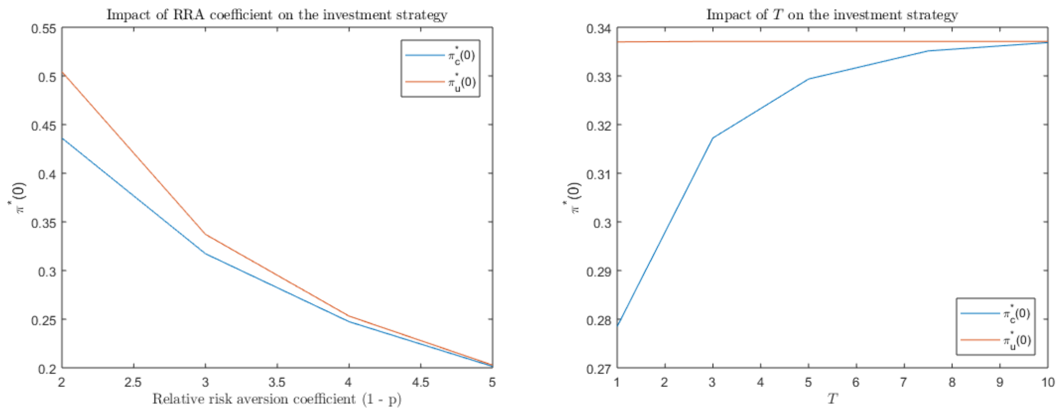
(a) $\pi_c^*(0)$ vs ε.

(b) $D^{\widetilde{\mathbb{Q}}}$ parameters vs ε.

Figure 6.2: The impact of VaR-probability threshold on the solution to Problem (6.5).

Next, we investigate the influence of the investor's risk aversion and time horizon on the optimal investment strategies. Consider Figure 6.3. On the left, Figure 6.3a, we see that both constrained and unconstrained investment strategies are decreasing in the RRA coefficient $1-p$. For example, a more risk averse investor with the RRA coefficient of 4, allocates only 25% of the capital into the risky asset. We also observe that the difference between those strategies shrinks as the investor becomes more risk averse. For the RRA coefficient of 5 ($p = -4$), the probability of the optimal terminal unconstrained wealth being smaller than $K = 100$ is slightly higher than 1%, which is why the optimal unconstrained strategy and the optimal constrained strategy for $\varepsilon = 1\%$ almost coincide.

On the right, Figure 6.3b, we notice than the optimal constrained strategy is increasing in the time horizon and approaches the unconstrained one. A decision maker with 1-year investment horizon will allocate 28% of his/her money into the risky asset. However, over a longer time period, e.g., 5 years, the investor put more money into the risky asset while still ensuring the desired VaR constraint, i.e., he/she will invest 33% of the money into S_1. For $T = 10$, the probability of the optimal terminal unconstrained wealth being smaller than $K = 100$ is around 1%, which is why the optimal unconstrained strategy and the optimal constrained strategy for $\varepsilon = 1\%$ almost coincide.



(a) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs $RRA = 1 - p$.

(b) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs T.

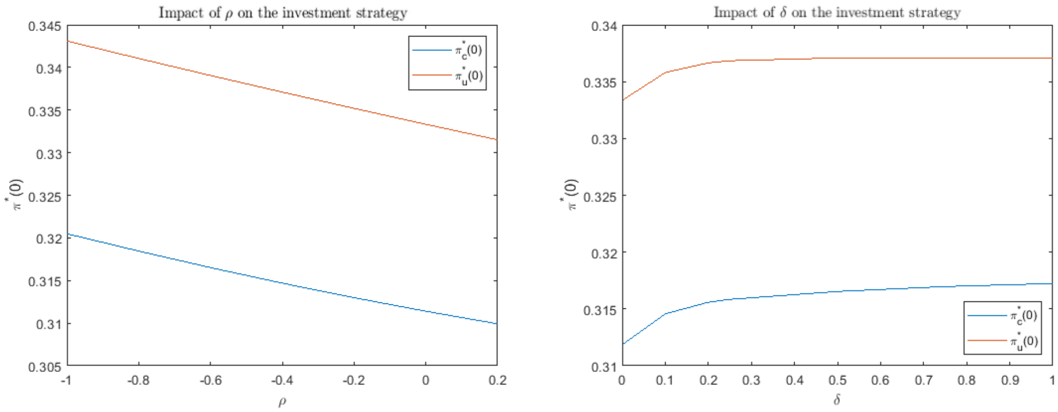
Figure 6.3: The impact of risk aversion and time horizon on the optimal investment strategies.

Now we check the impact of the correlation coefficient on the optimal constrained investment strategy. We plot $\pi_c^*(0)$ for $\rho \in \{-1, -0.8, -0.6, -0.4, -0.2, 0, 0.2\}$ in Figure 6.4a, since the literature on the calibration of the Heston model suggests that the correlation coefficient is mainly negative[3], see, e.g., Liu and Pan (2003), Escobar and Gschnaidtner (2016). We see that an increasing correlation coefficient decreases the optimal initial proportion of the money invested in the risky asset. This is consistent with the optimal unconstrained problem. Indeed, recall from (6.26) that $\pi_c^*(0) = \pi_u^*(0) \cdot y \cdot D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(0, y, v) / D^{\widetilde{\mathbb{Q}}(\gamma^v)}(0, y, v)$. For $p = -2 < 0$, $\pi_u^*(0) = \left(\frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho}{1-p} b(0)\right)$ is decreasing in ρ, as $b(0) < 0$ and $\sigma > 0$. Deriving analytically the impact of ρ on $y D_y^{\widetilde{\mathbb{Q}}(\gamma^v)}(0, y, v) / D^{\widetilde{\mathbb{Q}}(\gamma^v)}(0, y, v)$ is quite challenging, as the optimal parameters of D are determined numerically by solving $(\mathrm{NLS}(y, k_v, k_\varepsilon))$.

Finally, we investigate how the simultaneous decrease of parameters σ and κ influences the optimal constrained investment strategy. These parameters can be considered as a measure for the magnitude of the market incompleteness, since the lower σ, the less influence $W_2^{\mathbb{Q}}$ has on S_1. We fix $\varepsilon = 1\%$ and plot in Figure 6.4b $\pi_c^*(0)$ for decreasing

[3]This is the so-called "leverage effect", i.e., the empirical observation that rising asset prices are frequently accompanied by declining volatility and vice versa.

$\sigma_\delta := \sigma \cdot \delta$ and $\kappa_\delta := \kappa \cdot \delta$ with $\delta \in \{1, 0.75, 0.5, 0.25, 0.001\}$. We see that the smaller σ_δ and κ_δ, the larger the optimal initial constrained investment strategy. As we can see, the impact of ρ, σ, κ on $\pi_c^*(0)$ is quite small. For instance, a decrease in the correlation coefficient from -40% to -60% leads to an increase of the optimal initial constrained investment strategy only by 20 basis points, namely from 31.7% to 31.9%. Such a low sensitivity can be explained by the fact that the model parameterization, which we used so far, corresponds to a non-turbulent market and an investor with a moderate risk-aversion coefficient.

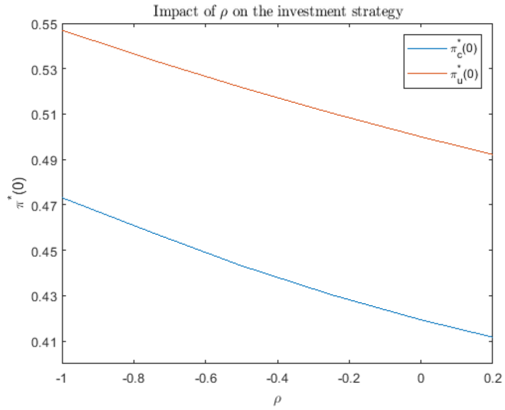


(a) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs ρ.

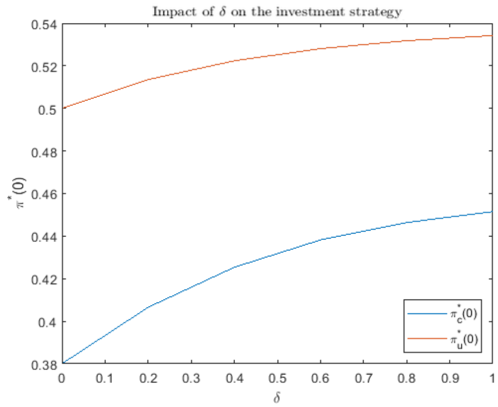
(b) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs δ.

Figure 6.4: The impact of ρ, σ, κ on the optimal investment strategies.

Consider now the same time horizon $T = 3$, but a decision maker with a smaller relative-risk aversion and who invests in a more turbulent market than we had before, i.e., higher initial value of the variance process, higher long-term average variance and lower mean reversion rate. In particular, we set $p = -1$ and use the values of the parameters of the Heston model such that they are consistent with Schoutens et al. (2004): $v_0 = 0.0654$, $\tilde{\theta} = 0.0707$, $\tilde{\kappa} = 0.6067$, $\sigma = 0.2928$, $\rho = -0.7571$. Next, we plot in Figure 6.5 the sensitivity of the optimal constrained investment strategy w.r.t. ρ, σ, and κ. In contrast to Figure 6.4, the sensitivity of the optimal constrained investment strategies w.r.t. the correlation coefficient, mean-reversion rate and the volatility of the variance process is higher in a more volatile market. For example, according to the Subfigure 6.5a, a decrease of the correlation coefficient from -40% to -60% leads to an increase of the initial optimal constrained investment strategy by more than 1%, namely from 42.7% to approximately 44%. Looking at $\delta = 1$ and $\delta = 0.75$ in Subfigure 6.5b, we see that a decrease of the volatility from 29.28% to 21.96% and the real-world-measure mean-reversion rate of the variance process from 0.8171 to 0.6128 would require a rational investor to decrease his/her initial constrained investment strategy by approximately 0.7%, namely from 45.2% to 44.5%. The behavior is similar to the one of the optimal unconstrained investment strategy. It can have the following economic interpretation. The infinitesimal Sharpe ratio of the risky asset is $\gamma^{S_1}\sqrt{v(t)}$. It is negatively correlated with the Wiener process $W_1^{\mathbb{Q}}(t)$ driving the stock returns. As a result, low return "today" tend to happen when $dW_1^{\mathbb{Q}}(t)$ is negative and $dW_2^{\mathbb{Q}}(t)$ is positive, which, in turn, pushes the "tomorrow's" Sharpe ratio higher and may give hope to the investor for good investment in the risky asset. Consequently, an investor increases his/her position in the risky asset in comparison to the Black-Scholes market. The "more" incompleteness an investor sees in the market, the more chances he/she sees for making profit with the risky asset investment and the corresponding correction term will be larger.



(a) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs ρ.



(b) $\pi_c^*(0)$ and $\pi_u^*(0)$ vs δ.

Figure 6.5: The impact of ρ, σ, κ on the optimal investment strategies in the setting of a more turbulent market and a less risk-averse investor.

7 Summary and conclusions

In this dissertation, we derived and analyzed the optimal investment and risk-sharing strategies for financial institutions that provide guarantees on their investment services, e.g., a guarantee to achieve a specific performance or a promise to cover a part of potential losses. Working in the context of the expected-utility framework and dynamic investment strategies, we contributed to the methodology of dynamic portfolio optimization, the design of equity-linked insurance products, and the discussion on the optimal risk-sharing in hedge funds. Below we review the structure of the thesis and summarize its main components. Moreover, we highlight the most important results obtained in this dissertation and provide an outlook for further research.

In Chapter 3, we analyzed the risk sharing between the manager of a hedge fund and a representative investor. Risk sharing followed via the first-loss fee schemes for which the investor's assets and the manager's deposit account are segregated. In Theorem 3.2.3, we derived the optimal fund's value by solving the optimization problem of the manager, whose utility function is non-concave due to his/her first-loss scheme. We tackled the non-concavity of the objective function using the concept of a concave envelope. Afterwards, we proved the existence of first-best Pareto optimal first-loss fee structures and computed the set of such Pareto optimal fee schemes. We proposed to maximize the fund's Sharpe ratio on this set to select the mutually preferred fee structure. The resulting single first-loss fee structure can be considered fair by both the hedge-fund manager and the investor.

In our numerical studies, we chose the model parameters that account for the historic popularity of the traditional 2%&20% arrangement in the hedge-fund sector and are consistent with the relevant literature as well as the current interest-rate environment. For the calibrated model, we found that the Pareto optimal fee structure that is closest to the 2%&20% scheme has a 0% management fee and a 20.3% performance fee. This is a possible explanation to the recently observed trend of decreasing management fees in hedge funds that still use the traditional scheme. According to our results, the traditional fee recommended in Escobar-Anel et al. (2018) (0% management fee and 30.7% performance fee) is not Pareto optimal in the presence of the first-loss coverage guarantee. The fee structure recommended in He and Kou (2018) (30% performance fee and 10% first-loss coverage guarantee) is not Pareto optimal either. The reason for that may be that we considered a different type of the first-loss scheme in comparison with He and Kou (2018). In our numerical studies, the preferred fee structures had the management fee usually close to 5%, the performance fee mainly in the range $30\% - 50\%$ and the first-loss coverage in the range 15% − 30%. We observed that if this fee structure was replaced by the traditional 2%&20% fee arrangement, the manager would be worse off in terms of his/her expected utility, whereas the investor would be much better off. Thus, we also considered only those Pareto optimal fee structures that yielded both parties at least the utility he/she had with a reasonable traditional fee structure. In that setting, the preferred management fee was 5%, performance fee was about 48%, and first-loss coverage guarantee was around 24%. All these findings shed light on the first-loss scheme that maturing hedge funds with first-loss compensation might be using in the future.

The sensitivity analysis showed that the more risk-averse the investor, the higher the preferred performance fee and the preferred first-loss coverage guarantee. For increasing risk aversion of the manager, the preferred performance fee and the first-loss coverage guarantee tend to decrease. This is different to the findings in He and Kou (2018). However, in contrast to our setting, the authors assume that the manager's preferences are modeled by a so-called S-shaped utility (see, e.g., Tversky and Kahneman (1992)) and consider hedge funds with commingled assets of investors and managers. We observed that the preferred first-loss coverage guarantee is decreasing in the market price of risk and increasing in the interest rates. We also found that the derived preferred fee structures substantially decrease the hedge fund's volatility in comparison to the traditional schemes as well as other reported first-loss fee structures. Possible future research could deal with investigating the preferred fee structures in a more realistic financial-market model with transaction costs and jumps in the risky-asset price process.

In Chapter 4, we shifted our focus from risk sharing in hedge funds to the optimal investment and risk-sharing strategies of insurance companies in the context of equity-linked insurance products with capital guarantees. Reinsurance in these products, if present, usually constrained the insurance companies in the investment strategies. Therefore, we considered an original modification of such products such that the insurer may follow an individual investment strategy and buy only a standardized reinsurance on some benchmark strategy. For an insurer offering such equity-linked products, we derived and analyzed its optimal investment and risk-sharing strategies in the presence of a Value-at-Risk constraint and a no-short-selling constraint on the risky fund as well as dynamically adjusted reinsurance. In particular, we showed in Propositions 4.2.1 and 4.2.2 how to transform the problem with a dynamically traded put option modeling reinsurance to an equivalent problem in terms of basic assets. Next in Lemma 4.2.3 we used the concept of auxiliary markets from Cvitanic and Karatzas (1992) and provided a condition under which the solution to an allocation-constrained and wealth-constrained portfolio optimization problem coincides with the allocation-unconstrained and wealth-constrained problem in the "right" auxiliary market. Using this lemma and the methodology of Basak and Shapiro (2001), we derived in Proposition 4.2.6 the solution to the equivalent problem of the insurer with a VaR constraint on terminal wealth and constraints on risky-assets allocations. The methodology we used for solving the insurer's problem allows for solving utility-maximization problems with other types of simultaneous terminal-wealth and allocation constraints in presence of continuously traded options, if the solution to the wealth-constrained and allocation-unconstrained problem has a multiplicative form of some positive (random) variable times the wealth-unconstrained and allocation-unconstrained solution. Finally, in Proposition 4.2.7, we derived the conditions under which it is optimal for the insurer to buy reinsurance in the management of equity-linked products.

In our numerical studies, we calibrated the model to the German market and found that fairly priced dynamically-adjusted reinsurance can allow insurers – without loss in their expected utility – decrease product costs and offer significantly higher capital guarantees to their clients. We reached the former conclusion by analyzing the wealth-equivalent utility loss. We inferred the latter benefit of reinsurance from the analysis of the guarantee-equivalent utility gain. In a numerical example, we showed that the inclusion of the optimally managed reinsurance to an equity-linked insurance product running for 10 years can allow the insurer to guarantee (at product maturity with 99.5% probability) that the client will receive at least 110% of the client's initial contribution without any loss of the insurer's expected utility. Moreover, such a product with optimal reinsurance can allow the insurer to guarantee (at product maturity with 99.5% probability) that the client will receive at least 128% of the client's initial endowment without any loss of the insurer's expected utility in comparison to the one obtained by a constant-mix strategy with 85% bonds and 15% stocks. We believe that our results may motivate a closer cooperation among insurers and reinsurers towards reversing the trend of decreasing capital guarantees embedded in equity-linked products, which is currently observed in insurance markets in many countries.

Our inference in Chapter 4 relies on the model assumptions. Therefore, possible future research could deal with the exploration of the optimality and economic implications of reinsurance in the design of equity-linked insurance products when more complex models or reinsurable portfolios are considered. For instance, reinsurance could be modeled as a passport option on the insurer's actual portfolio. A passport option would give the insurer the right to follow any admissible trading strategy, while the reinsurer would be obliged to cover any net losses on the strategy. Usually, such options are quite expensive due to the flexibility of the underlying portfolio.

In Chapter 5, we made the model from Chapter 4 more realistic with respect to the insurer-reinsurer interaction. In particular, we considered a Stackelberg game where the reinsurance company is the leader maximizing its expected utility by selecting its optimal investment strategy as well as a safety loading in the reinsurance contract. The insurance company is the follower maximizing its expected utility by selecting its investment strategy and the amount of reinsurance the company purchases at the price offered by the reinsurer. In contrast to Chapter 4, we assumed that the reinsurance is only purchased at the beginning of the investment horizon and is not continuously adjusted. We analytically derived the equilibrium of the Stackelberg game via backward induction. First, we derived in Propositions 5.2.1 and 5.2.2 the optimal action of the insurer by combining in a novel way the concept of auxiliary markets by Cvitanic and Karatzas (1992) and the generalized martingale approach by Desmettre and Seifried

(2016). Second, we derived in Propositions 5.2.4 and 5.2.5 the optimal action of the reinsurer via the replicating-strategies approach by Korn and Trautmann (1999). We found that in the Stackelberg equilibrium the reinsurer selects the largest safety loading of the reinsurance contract such that the insurer may still be willing to sign it. However, for this equilibrium value of the safety loading, the insurer becomes indifferent to the amount of reinsurance due to its high price. Thus, in practice, the reinsurer should consider offering a reinsurance contract with a safety loading that is lower than the equilibrium one. It is still beneficial for the reinsurer as long as the safety loading is positive.

In the corresponding numerical studies, we equipped each party with a power-utility function and calibrated the model similarly to Chapter 4. After computing the Stackelberg equilbrium, we presented several possible ways of rationally choosing the final discounted safety loading by the reinsurer. Moreover, if the reinsurance company offers a discount on the safety loading, the insurance company can substantially decrease product costs for the client when switching from an old investment strategy, e.g., a constant-mix strategy, without reinsurance to the new dynamic investment strategy with static reinsurance. In our numerical studies, the cost benefits for the insurer varied from a few basis points to 126.68%, depending on the investment horizon, the insurer's risk aversion and the old strategy used in comparison.

In practice, some equity-linked insurance products may have a death benefit or a surrender guarantee for the policyholder. Therefore, potential research could deal with including these actuarial risks into the model and exploring the analytical tractability of the corresponding Stackelberg game as well as the economic implications of its equilibrium. Including stochastic interest rates or no-short-selling and VaR constraints into the game and exploring the corresponding equilibrium could be another interesting line of research.

In Chapter 6, we abstracted away from specific institutional investors and studied the optimal investment problem under risk limitation via a VaR constraint in the incomplete Heston-model-based financial market. In our main Theorem 6.1.3 we solved this problem by generalizing the methodology of Kraft and Steffensen (2013) from the complete market to the incomplete market. In particular, we demonstrated that the value function in the constrained problem can be represented as an expected modified utility of a vega-neutral (synthetic) financial derivative on the optimal unconstrained wealth, and the optimal wealth and the optimal investment strategy in the constrained problem are similarly linked to those in the unconstrained problem. Using HJB PDEs, Feynman-Kac theorem, and Fourier transforms, we proved in Corollary 6.1.5 that the synthetic financial derivative consists of plain-vanilla put options and a digital-put option in the case of a VaR constraint.

Our numerical studies in Chapter 6 required solving PDEs of Riccatti type with complex-valued time-dependent coefficients, since we had to price the option on the optimal unconstrained wealth instead of the stock, in contrast to Heston (1993). Moreover, we derived and used option-pricing formulas in the spirit of Carr-Madan to make the calculation of the Greeks more stable for a digital-put option. In the sensitivity analysis of the optimal investment strategies, we found that the closer is the probability threshold in the VaR constraint to the probability of the optimal unconstrained portfolio to fall below the VaR capital threshold, the closer is the payoff of the synthetic financial derivative to the identity function, i.e., the payoff from holding just the optimal unconstrained portfolio as the underlying asset of the synthetic financial derivative. The risky investment of the optimal investment strategy is decreasing in the correlation coefficient. Furthermore, we varied the volatility as well as the mean-reversion rate of the variance process, which can be seen as a proxy for the magnitude of the market incompleteness. As these parameters increased, the risky investment of the optimal constrained investment strategy also increased. Finally, we observed that the optimal investment strategy is more sensitive to the changes of the above-mentioned parameters in case of less risk-averse investors acting in more turbulent markets, i.e., markets with large initial variance as well as long-term variance level and a low mean-reversion rate.

The methodology in Chapter 6 has a huge potential for future research and enlarging the class of analytically tractable portfolio optimization problems. For example, our results can be further extended to solving portfolio optimization problems in other incomplete markets, e.g., with stochastic market price of risk. Another topic for future research could be the investigation of other types of terminal-wealth or intermediate-wealth constraints, e.g., limiting the expected shortfall of terminal wealth or bounding from below the intermediate wealth of the investor. Last but not least, other types of utility functions are also worth exploring in the future.

To sum up, we contributed to the academic literature by solving novel optimal investment and risk-sharing problems, analyzing the optimal strategies from the economic perspective, and by pushing the boundaries of portfolio optimization methods that lead to closed-form solutions.

A Appendix to Chapter 3

In this appendix, we provide the proofs of theoretical results stated in Chapter 3. Section A.1 contains the proofs of main results. In Section A.2 we provide auxiliary theoretical results and their proofs.

A.1 Proofs of main results

Proof of Lemma 3.2.2. We show how to construct for $\widetilde{U}_M(\cdot)$ its concave envelope $\tilde{u}_M(\cdot)$, the uniqueness of $\tilde{\chi}_1$ will follow. For $\tilde{u}_M(\cdot)$, four cases are possible, which are illustrated in Figure A.1. Note that $\widetilde{U}_M(\cdot)$ has the following properties:

(i) $\widetilde{U}_{M,1} : (0, \widetilde{\mathcal{X}}_1) \to \mathbb{R}$ is constant[1];

(ii) $\widetilde{U}_{M,2} : (\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2) \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable;

(iii) $\widetilde{U}_{M,3} : (\widetilde{\mathcal{X}}_2, +\infty) \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable with $\lim_{v\uparrow+\infty} \widetilde{U}'_{M,3}(v) = 0$;

(iv) $\widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-) \geq \widetilde{U}'_{M,3}(\widetilde{\mathcal{X}}_2+)$;

(v) $U_{M,i}(\widetilde{\mathcal{X}}_i-) = U_{M,i+1}(\widetilde{\mathcal{X}}_i+)$ for $i \in \{1, 2\}$.

Denote $s(v) := (\widetilde{U}_M(v) - \widetilde{U}_M(0))/v$ the slope of a line that passes through the points $(0, \widetilde{U}_M(0))$ and $(v, \widetilde{U}_M(v))$. Since $\widetilde{U}_M$ is nondecreasing, $s(v) \geq 0\, \forall v > 0$.

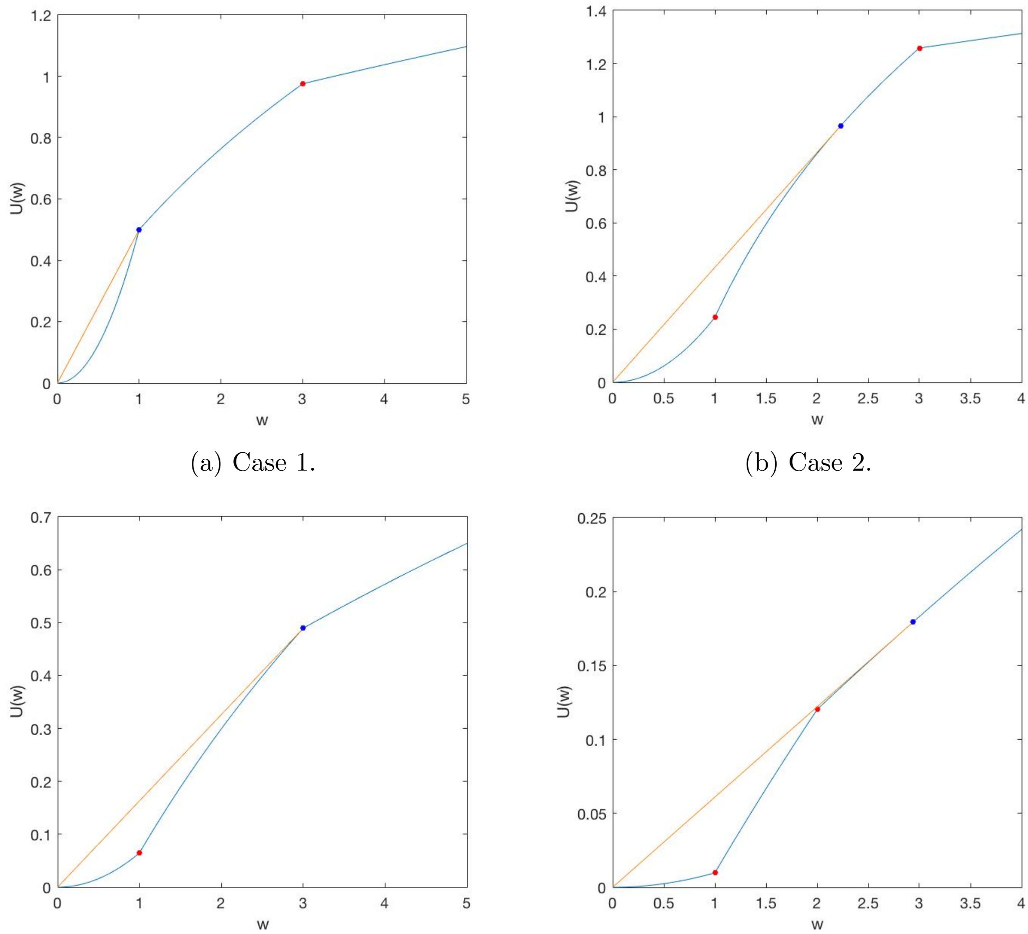
Due to properties (ii)-(v) $\widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-) \geq \widetilde{U}'_{M,3}(\widetilde{\mathcal{X}}_2+)$, $\widetilde{U}_{M,2}(\cdot)$ and $\widetilde{U}_{M,3}(\cdot)$ are strictly concave on $(\widetilde{\mathcal{X}}_1; \widetilde{\mathcal{X}}_2)$ and $(\widetilde{\mathcal{X}}_2, +\infty)$ respectively, $\widetilde{U}_{M,2}(\widetilde{\mathcal{X}}_2-) = \widetilde{U}_{M,3}(\widetilde{\mathcal{X}}_2+)$, we conclude that $\widetilde{U}_M(v)$ is strictly concave on $(\widetilde{\mathcal{X}}_1, +\infty)$.

Since $\widetilde{U}_{M,1}(v)$ is convex (see property (i)), we get:

$$\widetilde{U}_{M,1}(v) \leq \widetilde{U}_{M,1}(0) + s(\widetilde{\mathcal{X}}_1)v, \quad \forall v \in [0, \widetilde{\mathcal{X}}_1]. \tag{A.1}$$

Therefore, $\tilde{\chi}_1 \geq \widetilde{\mathcal{X}}_1$. Next we partition the set of functions $\widetilde{U}_M(\cdot)$, satisfying the assumptions of this lemma, into four subsets, depending on the values of $s(\widetilde{\mathcal{X}}_1) \geq 0, s(\widetilde{\mathcal{X}}_2) > 0$.

[1]The proof is also valid, if (i) is less restrictive, i.e., $\widetilde{U}_{M,1} : (0, \widetilde{\mathcal{X}}_1) \to \mathbb{R}$ is convex and non-decreasing



(a) Case 1. (b) Case 2.

(c) Case 3. (d) Case 4.

Figure A.1: Utility functions and their concave envelopes.

We show that in each case there is a unique $\tilde{\chi}_1$ – the change point between the linear part of the concave envelope and $\widetilde{U}_M(\cdot)$.

Case 1: $s(\widetilde{\mathcal{X}}_1) \in [\widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+), +\infty)$ and any $s(\widetilde{\mathcal{X}}_2)$.

Since the linear function $g(v) = \widetilde{U}_M(0) + s(\widetilde{\mathcal{X}}_1)v$ is obviously concave on $[0, \widetilde{\mathcal{X}}_1]$, $\widetilde{U}_M(\cdot)$ is strictly concave on $[\widetilde{\mathcal{X}}_1, +\infty)$ and $g(\widetilde{\mathcal{X}}_1-) = s(\widetilde{\mathcal{X}}_1) \geq \widetilde{U}_M(\widetilde{\mathcal{X}}_1+) = \widetilde{U}_{M,2}(\widetilde{\mathcal{X}}_1+)$, we conclude that the function

$$(\widetilde{U}_M(0) + s(\widetilde{\mathcal{X}}_1)v)\mathbb{1}_{[0,\widetilde{\mathcal{X}}_1)}(v) + \widetilde{U}_M(v)\mathbb{1}_{[\widetilde{\mathcal{X}}_1,+\infty)}(v)$$

is concave. The candidate for the concave envelope of $\widetilde{U}_M(\cdot)$ is

$$\tilde{u}_M(v) = \begin{cases} -\infty, & v < 0; \\ \widetilde{U}_M(0) + s(\widetilde{\mathcal{X}}_1)v, & v \in [0, \widetilde{\mathcal{X}}_1); \\ \widetilde{U}_{M,2}(v), & v \in [\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2); \\ \widetilde{U}_{M,3}(v), & v \in [\widetilde{\mathcal{X}}_2, +\infty). \end{cases}$$

Take any concave function $g(\cdot)$ such that $g(v) \geq \widetilde{U}_M(v)\ \forall v \in \mathbb{R}$. We get that $g(v) \geq \widetilde{U}_M(v) = \tilde{u}_M(v)$ for $v \in (-\infty, 0] \cup [\widetilde{\mathcal{X}}_1, +\infty)$. Since $\tilde{u}_M(\cdot)$ is linear on $(0, \widetilde{\mathcal{X}}_1)$ and $g(\cdot)$ is concave, we get for any $v = \lambda\widetilde{\mathcal{X}}_1$ with $\lambda \in (0, 1)$:

$$\begin{aligned} g(v) &= g(\lambda\widetilde{\mathcal{X}}_1 + (1-\lambda)0) \geq \lambda g(\widetilde{\mathcal{X}}_1) + (1-\lambda)g(0) \geq \lambda\widetilde{U}_M(\widetilde{\mathcal{X}}_1) + (1-\lambda)\widetilde{U}_M(0) \\ &= \lambda\tilde{u}_M(\widetilde{\mathcal{X}}_1) + (1-\lambda)\tilde{u}_M(0) = \tilde{u}_M(\lambda\widetilde{\mathcal{X}}_1 + (1-\lambda)0) = \tilde{u}_M(v). \end{aligned}$$

Then according to Definition 3.2.1, $\tilde{u}_M(\cdot)$ is the concave envelope of $\widetilde{U}_M(\cdot)$. Writing it down in the form (3.5) is straightforward setting $\tilde{\chi}_1 = \widetilde{\mathcal{X}}_1$ and $\tilde{\chi}_2 = \widetilde{\mathcal{X}}_2$. Figure A.1a provides graphical illustration to this case. Red markers stand for the change points of $\widetilde{U}_M(\cdot)$, i.e., $\widetilde{\mathcal{X}}_1$, $\widetilde{\mathcal{X}}_2$. The blue marker corresponds to the point $\tilde{\chi}_1$.

Case 2: $s(\widetilde{\mathcal{X}}_1) \in [0, \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+))$ and $s(\widetilde{\mathcal{X}}_2) \in (\widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-), +\infty)$.

Note that:

$$\begin{aligned} s(\widetilde{\mathcal{X}}_1) < \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+) &\Leftrightarrow \widetilde{U}_{M,2}(\widetilde{\mathcal{X}}_1) - \widetilde{U}_{M,1}(0) - \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+)\widetilde{\mathcal{X}}_1 < 0; \\ s(\widetilde{\mathcal{X}}_1) > \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-) &\Leftrightarrow \widetilde{U}_{M,2}(\widetilde{\mathcal{X}}_2) - \widetilde{U}_{M,1}(0) - \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-)\widetilde{\mathcal{X}}_2 > 0. \end{aligned} \tag{A.2}$$

Using (A.2) and property (ii), it is easy to show that there exists a unique $\tilde{\chi}_1 \in (\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$, such that $s(\tilde{\chi}_1) = \widetilde{U}'_{M,2}(\tilde{\chi}_1)$.

Now we show that $s(v)$ is strictly increasing on $(\widetilde{\mathcal{X}}_1, \tilde{\chi}_1)$. Consider the derivative of $s(v)$ on $(\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$:

$$s'(v) = \frac{\widetilde{U}'_{M,2}(v)v - \widetilde{U}_{M,2}(v) + \widetilde{U}_{M,1}(0)}{v^2} = \frac{-a_2(v)}{v^2}.$$

Since $a_2(v)$ is strictly increasing on $(\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$ and $a_2(\tilde{\chi}_1) = 0$, we conclude that $a_2(v) < 0$ for $v \in (\widetilde{\mathcal{X}}_1, \tilde{\chi}_1)$. Therefore, $s'(v) = \frac{-a_2(v)}{v^2} > 0$ for $v \in (\widetilde{\mathcal{X}}_1, \tilde{\chi}_1)$, whence $s(\widetilde{\mathcal{X}}_1) < s(\tilde{\chi}_1)$.

Using $s(\widetilde{\mathcal{X}}_1) < s(\tilde{\chi}_1)$ and strict concavity and differentiability of $\widetilde{U}_{M,2}(\cdot)$ on $(\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$ (see property (ii)), we obtain:

$$\begin{aligned}\widetilde{U}_{M,1}(v) &\leq \widetilde{U}_M(0) + s(\widetilde{\mathcal{X}}_1)v < \widetilde{U}_M(0) + s(\tilde{\chi}_1)v, \quad \forall v \in [0, \widetilde{\mathcal{X}}_1);\\ \widetilde{U}_{M,2}(v) &< \widetilde{U}_{M,2}(\tilde{\chi}_1) + \underbrace{\widetilde{U}'_{M,2}(\tilde{\chi}_1)}_{=s(\tilde{\chi}_1)}(v - \tilde{\chi}_1)\\ &= \widetilde{U}_{M,2}(\tilde{\chi}_1) + \frac{\widetilde{U}_{M,2}(\tilde{\chi}_1) - \widetilde{U}_{M,1}(0)}{\tilde{\chi}_1}(v - \tilde{\chi}_1)\\ &= \widetilde{U}_{M,2}(\tilde{\chi}_1) + s(\tilde{\chi}_1)v - (\widetilde{U}_{M,2}(\tilde{\chi}_1) - \widetilde{U}_M(0))\\ &= \widetilde{U}_M(0) + s(\tilde{\chi}_1)v, \ \forall v \in [\widetilde{\mathcal{X}}_1, \tilde{\chi}_1).\end{aligned}$$

Finally, we can check similarly to Case 1 that the function:

$$\tilde{u}_M(v) = \begin{cases} -\infty, & v < 0;\\ \widetilde{U}_M(0) + s(\tilde{\chi}_1)v, & v \in [0, \tilde{\chi}_1);\\ \widetilde{U}_{M,2}(v), & v \in [\tilde{\chi}_1, \widetilde{\mathcal{X}}_2);\\ \widetilde{U}_{M,3}(v), & v \in [\widetilde{\mathcal{X}}_2, +\infty)\end{cases}$$

is indeed the concave envelope of $\widetilde{U}_M(\cdot)$ in the sense of Definition 3.2.1. In terms of (3.5), we have a unique $\tilde{\chi}_1 \in (\widetilde{\mathcal{X}}_1, \widetilde{\mathcal{X}}_2)$ and can set $\tilde{\chi}_2 = \widetilde{\mathcal{X}}_2$. Figure A.1b corresponds to Case 2.

Case 3: $s(\widetilde{\mathcal{X}}_1) \in [0, \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+)]$, $s(\widetilde{\mathcal{X}}_2) \in [\widetilde{U}'_{M,3}(\widetilde{\mathcal{X}}_2+), \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_2-)]$. This case is similar to Case 1 and is illustrated in Figure A.1c. In terms of (3.5), we have $\tilde{\chi}_1 = \tilde{\chi}_2 = \widetilde{\mathcal{X}}_2$.

Case 4: $s(\widetilde{\mathcal{X}}_1) \in [0, \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+)]$, $s(\widetilde{\mathcal{X}}_2) \in (0, \widetilde{U}'_{M,3}(\widetilde{\mathcal{X}}_2+))$. This case is analogous to Case 2 and is illustrated in Figure A.1d. In terms of (3.5), we have a unique $\tilde{\chi}_1 \in (\widetilde{\mathcal{X}}_2, +\infty)$ and can set $\tilde{\chi}_2 = \tilde{\chi}_1$.

We have covered all possible cases, i.e., classified all possible values of $s(\widetilde{\mathcal{X}}_1) \geq 0, s(\widetilde{\mathcal{X}}_2) > 0$. Any $\widetilde{U}_M(\cdot)$ that has properties (i)-(v) is related to exactly one of the four described cases. □

Proof of Theorem 3.2.3.
1. The equation $h(\lambda_v) - v_0 = 0$ has a unique solution $\lambda_v^* \in (0, +\infty)$ due to the fact that under the integrability assumption (3.7) the function $h(\lambda_v) - v_0$ is continuous, strictly decreasing, $\lim_{\lambda_v \uparrow +\infty}(h(\lambda_v) - v_0) = -v_0 < 0$ as well as $\lim_{\lambda_v \downarrow 0}(h(\lambda_v) - v_0) = +\infty$.
2. Denote $V^*(T) := v^*(\lambda_v^*, \tilde{Z}(T))$. Take any admissible solution $\bar{V}(T)$ of problem

(P_M^{conc}). It must satisfy the budget constraint $\mathbb{E}[\tilde{Z}(T)\bar{V}(T)] \leq v_0$. Then:

$$\begin{aligned}
\mathbb{E}[\tilde{u}_M(V^*(T))] - \mathbb{E}[\tilde{u}_M(\bar{V}(T))] &= \mathbb{E}[\tilde{u}_M(V^*(T))] - \lambda_v^* v_0 - \mathbb{E}[\tilde{u}_M(\bar{V}(T)] + \lambda_v^* v_0 \\
&\geq \mathbb{E}[\tilde{u}_M(V^*(T))] - \lambda_v^* v_0 - \mathbb{E}[\tilde{u}_M(\bar{V}(T))] + \lambda_v^* \mathbb{E}[\tilde{Z}(T)\bar{V}(T)] \\
&= (\mathbb{E}[\tilde{u}_M(V^*(T))] - \lambda_v^* \mathbb{E}[\tilde{Z}(T)V^*(T)]) - (\mathbb{E}[\tilde{u}_M(\bar{V}(T))] \\
&\quad - \lambda_v^* \mathbb{E}[\tilde{Z}(T)\bar{V}(T)]) \\
&= \mathbb{E}[(\tilde{u}_M(V^*(T)) - \lambda_v^* \tilde{Z}(T)V^*(T)) - (\tilde{u}_M(\bar{V}(T)) \\
&\quad - \lambda_v^* \tilde{Z}(T)\bar{V}(T))] \overset{Lem.A.2.2}{\geq} 0.
\end{aligned}$$

Hence, $V^*(T)$ is optimal. It is $\mathbb{Q}$-a.s. unique, since $\mathbb{Q}(\widetilde{Z}(T) = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v^*) = 0$.
3. Observe that the initial utility function $\widetilde{U}_M(v)$ and its concave envelope $\tilde{u}_M(v)$ satisfy the following properties:

$$\begin{aligned}
&\{v \in \mathbb{R} : \widetilde{U}_M(v) < \tilde{u}_M(v)\} = (0, \tilde{\chi}_1); \\
&\{v \in \mathbb{R} : \widetilde{U}_M(v) = \tilde{u}_M(v)\} = \mathbb{R} \setminus (0, \tilde{\chi}_1).
\end{aligned} \tag{A.3}$$

Let $V^*(T)$ be the optimal solution of Problem (P_M^{conc}). Since the feasible regions in Problem (P_M) and Problem (P_M^{conc}) coincide, $V^*(T)$ is feasible in the former (original, non-concave) problem. Note that $\mathbb{Q}(V^*(T) \in \{0\} \cup [\tilde{\chi}_1, +\infty)) = 1$ due to the definition of $\tilde{\chi}_1$. Denote $V(T)$ any other feasible wealth in Problem (P_M) such that $\mathbb{Q}(V(T) = V^*(T)) \neq 1$. Then:

$$\begin{aligned}
\mathbb{E}[\widetilde{U}_M(V^*(T))] - \mathbb{E}[\widetilde{U}_M(V(T))] &\overset{(A.3)}{=} \underbrace{\mathbb{E}\left[(\tilde{u}_M(V^*(T)) - \tilde{u}_M(V(T))\, \mathbb{1}_{\{V^*(T) \notin (0,\tilde{\chi}_1)\}}\right]}_{\geq 0, \text{ due to optimality of } V^*(T) \text{ in Problem } (P_M^{conc})} \\
&\quad + \underbrace{\mathbb{E}\left[\left(\widetilde{U}_M(V^*(T)) - \widetilde{U}_M(V(T)\right) \mathbb{1}_{\{V^*(T) \in (0,\tilde{\chi}_1)\}}\right]}_{=0, \text{ since } \mathbb{Q}(V^*(T) \in (0,\tilde{\chi}_1))=0} \\
&\geq 0.
\end{aligned}$$

Hence, $V^*(T)$ is the $\mathbb{Q}$-a.s. unique optimal solution of Problem (P_M). □

Proof of Proposition 3.2.4. From Theorem 3.2.3, we know that $V^*(T) = v(\lambda_v^*, \widetilde{Z}(T))$ for λ_v^* such that $\mathbb{E}\left[V^*(T)\widetilde{Z}(T)\right] = v_0$ and $v(\lambda_v, \tilde{z})$ solving (3.6). The function $v(\lambda_v, \tilde{z})$, derived in the supplementary Lemma A.2.2, Appendix A.2, is continuous w.r.t. (m, α, c). Hence, the function $\widetilde{\mathcal{V}}_M(m, \alpha, c)$ is continuous w.r.t. $(m, \alpha, c) \in \mathcal{P}$ as a superposition of continuous functions.

Fix $\widetilde{\mathcal{V}}_M^{RUL} \in \left[\min_{(m,\alpha,c)\in\mathcal{P}} \widetilde{\mathcal{V}}_M(m, \alpha, c),\ \max_{(m,\alpha,c)\in\mathcal{P}} \widetilde{\mathcal{V}}_M(m, \alpha, c)\right]$. Then the set

$\mathcal{R} = \left\{(m,\alpha,c) : \widetilde{\mathcal{V}}_M(m,\alpha,c) \geq \widetilde{\mathcal{V}}_M^{RUL}\right\}$ is non-empty due to the choice of $\widetilde{\mathcal{V}}_M^{RUL}$ and closed due to the continuity of $\widetilde{\mathcal{V}}_M(m,\alpha,c)$. The set $\mathcal{P}$ is non-empty, closed and bounded. Hence, $\mathcal{R} \cap \mathcal{P}$ is a non-empty, closed and bounded set.

The function $\widetilde{\mathcal{V}}_I(m,\alpha,c)$ is also continuous for $(m,\alpha,c) \in \mathcal{P}$ as a superposition of continuous functions. Therefore, by Theorem 2.5.3, there exists $(m^*,\alpha^*,c^*) \in \mathcal{R} \cap \mathcal{P}$ such that $\widetilde{\mathcal{V}}_I(m^*,\alpha^*,c^*) \geq \widetilde{\mathcal{V}}_I(m,\alpha,c)\, \forall (m,\alpha,c) \in \mathcal{R} \cap \mathcal{P}$. □

Proof of Corollary 3.2.5. This corollary is a direct application of Lemma 3.2.2 (the construction of the concave envelope of the manager's utility function) and Theorem 3.2.3.

Recall that $\widetilde{\mathcal{X}}_1 = (1+m-c)v_0$, $\widetilde{\mathcal{X}}_2 = (1+m)v_0$, $s(v) = (\widetilde{U}_M(v) - \widetilde{U}_M(0))/v$. Then:

$$\begin{aligned}
\widetilde{U}'_M(\widetilde{\mathcal{X}}_1+) &= \lim_{v \downarrow \widetilde{\mathcal{X}}_1} \widetilde{U}'_{M,2}(v) = \lim_{v \downarrow \widetilde{\mathcal{X}}_1} (v - v_0 + a_M)^{p_M-1} = ((m-c)v_0 + a_M)^{p_M-1};\\
\widetilde{U}'_M(\widetilde{\mathcal{X}}_2-) &= \lim_{v \uparrow \widetilde{\mathcal{X}}_2} \widetilde{U}'_{M,2}(v) = \lim_{v \uparrow \widetilde{\mathcal{X}}_2} (v - v_0 + a_M)^{p_M-1} = (mv_0 + a_M)^{p_M-1};\\
\widetilde{U}'_M(\widetilde{\mathcal{X}}_2+) &= \lim_{v \downarrow \widetilde{\mathcal{X}}_2} \widetilde{U}'_{M,3}(v) = \lim_{v \downarrow \widetilde{\mathcal{X}}_2} (\alpha v + mv_0 - \alpha(1+m)v_0 + a_M)^{p_M-1}\alpha\\
&= \alpha(mv_0 + a_M)^{p_M-1}.
\end{aligned} \tag{A.4}$$

Since it holds that $s(\widetilde{\mathcal{X}}_1) = 0 < \widetilde{U}'_{M,2}(\widetilde{\mathcal{X}}_1+)$, we can easily verify that Case 2, Case 3 and Case 4 from Lemma 3.2.2 correspond to $\mathcal{P}_A$ (Case A), $\mathcal{P}_B$ (Case B) and $\mathcal{P}_C$ (Case C) respectively, where $\mathcal{P}_X$, $X \in \{A,B,C\}$, is defined in (3.10).

Consider Case A, $(m,\alpha,c) \in \mathcal{P}_A$. According to Lemma 3.2.2, there exists a unique $\tilde{\chi}_1 > \widetilde{\mathcal{X}}_2$ solving:

$$\begin{aligned}
s(v) = \widetilde{U}'_M(v) &\Leftrightarrow \frac{(\alpha v + (m - \alpha(1+m))v_0 + a_M)^{p_M} - (v_0(m-c) + a_M)^{p_M}}{p_M v} = \widetilde{U}'_{M,3}(v)\\
&\Leftrightarrow (\alpha v + (m - \alpha(1+m))v_0 + a_M)^{p_M-1}((1-p_M)\alpha v + (m - \alpha(1+m))v_0 + a_M)\\
&\qquad = (v_0(m-c) + a_M)^{p_M}.
\end{aligned}$$

As per Corollary 46 in Havrylenko (2018), the concave envelope of $\widetilde{U}_M(\cdot)$ in this case has the following form:

$$\tilde{u}_M(V(T)) = \begin{cases} -\infty, & V(T) < 0;\\ \dfrac{1}{p_M}(v_0(m-c) + a_M)^{p_M} + s(\tilde{\chi}_1)V(T), & V(T) \in [0, \tilde{\chi}_1);\\ \underbrace{\dfrac{1}{p_M}(\alpha V(T) + (m - \alpha(1+m))v_0 + a_M)^{p_M}}_{=\widetilde{U}_M(V(T)) = \widetilde{U}_{M,3}(V(T))\ \forall V(T) \in [\tilde{\chi}_1, +\infty)}, & V(T) \in [\tilde{\chi}_1, +\infty). \end{cases} \tag{A.5}$$

Denote by $I_2(\cdot), I_3(\cdot)$ the inverse functions of the marginal utilities $\widetilde{U}'_{M,2}(\cdot), \widetilde{U}'_{M,3}(\cdot)$:

$$I_2(v) = v^{-\frac{1}{1-p_M}} + v_0 - a_M,\ I_3(v) = \alpha^{\frac{1}{1-p_M}-1} v^{-\frac{1}{1-p_M}} + (1 + m - \alpha^{-1}m)v_0 - \alpha^{-1}a_M.$$

Then for $v \in [0, +\infty)$ the concave envelope $\tilde{u}_M(v)$ has the form of (3.5) for $\tilde{u}_{M,2}(\cdot) = \tilde{u}_{M,3}(\cdot)$ and $\tilde{\chi}_2 = \tilde{\chi}_1$. Using that $\tilde{u}_{M,2}(\cdot) = \tilde{u}_{M,3}(\cdot)$, $\tilde{u}'_{M,1}(\tilde{\chi}_1-) = \tilde{u}'_{M,2}(\tilde{\chi}_1+) = s(\tilde{\chi}_1) = \alpha(\alpha\tilde{\chi}_1 + (m - \alpha(1+m))v_0 + a_M)^{p_M-1}$ and setting $\tilde{v}_1 = \tilde{\chi}_1$, we obtain by applying Lemma A.2.2 that

$$v^*(\lambda_v, \tilde{z}) = \left(\alpha^{1/(1-p_M)-1}(\lambda_v \tilde{z})^{-1/(1-p_M)} + (1 + m - \alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \mathbb{1}_{(0,\, s(\tilde{\chi}_1)/\lambda_v]}(\tilde{z})$$

solves $\max_{v \geq 0}\{\tilde{u}_M(v) - \lambda_v \cdot \tilde{z} \cdot v\}$.

Using the supplementary Lemma A.2.1 in Appendix A.2, one can easily show that the integrability condition in Theorem 3.2.3 holds for $\forall \lambda_v > 0$. Therefore, we may apply Theorem 3.2.3 and conclude that:

$$\begin{aligned} V^*(T) = v^*(\lambda_v^*, \widetilde{Z}(T)) = &\Big(\alpha^{1/(1-p_M)-1}(\lambda_v^* \widetilde{Z}(T))^{-1/(1-p_M)} + (1 + m - \alpha^{-1}m)v_0 \\ &- \alpha^{-1}a_M\Big) \mathbb{1}_{\left\{\widetilde{Z}(T) \in \left(0,\, s(\tilde{\chi}_1^A)/\lambda_v^*\right]\right\}}, \end{aligned}$$

where $\lambda_v^* \in (0, +\infty)$ is the unique solution of the equation $\mathbb{E}\left[\widetilde{Z}(T) v^*(\lambda_v, \widetilde{Z}(T))\right] = v_0$ and $\tilde{\chi}_1^A := \tilde{\chi}_1$ to emphasize the correspondence of this concavification to Case A.

Case B and Case C are proven analogously to Case A. For details see Corollary 46 and Corollary 47 in Havrylenko (2018). □

Proof of Proposition 3.2.7. In the proof we mainly use Corollary 3.2.5.

Case A: $(m, \alpha, c) \in \mathcal{P}_A$. Denote $E_1 = \left(0, s(\tilde{\chi}_1^A)/\lambda_v^*\right]$. According to (3.11)

$$V^*(T) = \left(\alpha^{1/(1-p_M)-1}(\lambda_v^* \widetilde{Z}(T))^{-1/(1-p_M)} + (1 + m - \alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \mathbb{1}_{\left\{\widetilde{Z}(T) \in E_1\right\}}.$$

Using that $s(\tilde{\chi}_1^A) = \alpha(\alpha\tilde{\chi}_1^A + (m - \alpha(1 + m))v_0 + a_M)^{p_M-1}$ and the fact that $V^*(T) = v^*(\lambda_v^*, \widetilde{Z}(T))$ is a non-increasing function of $\widetilde{Z}(T)$, one can easily verify that:

$$\begin{aligned} &V_T^*(\omega) \geq \tilde{\chi}_1^A > (1 + m)v_0 \quad &&\forall \omega \in \left\{\omega \in \Omega : \widetilde{Z}_T(\omega) \in E_1\right\}; \\ &V_T^*(\omega) = 0 &&\forall \omega \in \left\{\omega \in \Omega : \widetilde{Z}_T(\omega) \notin E_1\right\}, \end{aligned} \tag{A.6}$$

where we write $V_T^*(\omega)$ for the value of the random variable $V^*(T)$ given $\omega \in \Omega$ and $\widetilde{Z}_T(\omega)$ for the value of $\widetilde{Z}(T)$ given $\omega \in \Omega$. Therefore, it holds:

$$\begin{aligned}\mathbb{Q}\Big(\omega \in \Omega : &V_T^*(\omega) \in \Big\{v \in \mathbb{R} : \tilde{u}_M(v) \neq \widetilde{U}_M(v)\Big\}\Big) \\ &= \mathbb{Q}\left(\omega \in \Omega : V_T^*(\omega) \in (0, \tilde{\chi}_1^A)\right) = 0.\end{aligned} \tag{A.7}$$

Then:

$$\begin{aligned}
\widetilde{\mathcal{V}}_M(m,\alpha,c) &= \mathbb{E}\left[\widetilde{U}_M(V^*(T))\right] \overset{(A.7)}{=} \mathbb{E}\left[\tilde{u}_M(V^*(T))\right] \\
&\underset{V^*(T)\geq 0}{\overset{(A.5)}{=}} \mathbb{E}\left[\left(\widetilde{U}_M(0) + s(\tilde{\chi}_1^A)V^*(T)\right)\mathbb{1}_{\left\{V^*(T)\in[0,\tilde{\chi}_1^A)\right\}}\right] \\
&\quad + \mathbb{E}\Big[p_M^{-1}(\alpha V^*(T) + (m - \alpha(1+m))v_0 + a_M)^{p_M} \\
&\quad \cdot \mathbb{1}_{\left\{V^*(T)\in[\tilde{\chi}_1^A,+\infty)\right\}}\Big] \\
&\underset{(3.11)}{\overset{(A.6)}{=}} \mathbb{E}\Big[(\widetilde{U}_M(0) + s(\tilde{\chi}_1^A)\cdot 0)\mathbb{1}_{\{Z_T\notin E_1\}}\Big] + p_M^{-1} \\
&\quad \cdot \mathbb{E}\Big[\Big(\alpha\left(\alpha^{1/(1-p_M)-1}(\lambda_v^*\widetilde{Z}(T))^{-1/(1-p_M)} + (1+m-\alpha^{-1}m)v_0\right. \\
&\quad \left.-\alpha^{-1}a_M\right) + (m-\alpha(1+m))v_0 + a_M\Big)^{p_M}\mathbb{1}_{\left\{\widetilde{Z}(T)\in E_1\right\}}\Big] \\
&= \widetilde{U}_M(0)\mathbb{E}\left[\mathbb{1}_{\left\{\widetilde{Z}(T)\notin E_1\right\}}\right] + p_M^{-1}\left(\lambda_v^*\right)^{1-1/(1-p_M)}\alpha^{1/(1-p_M)-1} \\
&\quad \cdot \mathbb{E}\left[\widetilde{Z}(T)^{1-1/(1-p_M)}\mathbb{1}_{\left\{\widetilde{Z}(T)\in E_1\right\}}\right] \\
&\underset{\widetilde{Z}(T)>0}{\overset{\text{Lem.A.2.1}}{=}} \widetilde{U}_M(0)(\Phi(d_2^A) - \Phi(-\infty)) + (\lambda_v^*)^{1-1/(1-p_M)}\alpha^{1/(1-p_M)-1} \\
&\quad \cdot \exp\left((1/(1-p_M)-1)\left(r+0.5\gamma^2\right)T + 0.5(1/(1-p_M)-1)^2\gamma^2 T\right) \\
&\quad \cdot \left(\Phi\left(d_1^A + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right) - \Phi\left(d_2^A + (1-(1-p_M)^{-1})\gamma\sqrt{T}\right)\right)\cdot p_M^{-1},
\end{aligned}$$

where

$$d_1^A = +\infty, \quad d_2^A = \frac{\log\left(\lambda_v^*/s(\tilde{\chi}_1^A)\right) - \left(r+0.5\gamma^2\right)T}{\gamma\sqrt{T}}.$$

Case B and Case C follow similarly. For details, see Proposition 52 in Havrylenko (2018). □

Proof of Proposition 3.2.8. In the proof of this proposition we mainly use Corollary 3.2.5.

Case A: $(m,\alpha,c)\in\mathcal{P}_A$. Denote $E_1=\big(0,s(\tilde\chi_1^A)/\lambda_v^*\big]$. We obtain:

$$
\begin{aligned}
\widetilde{\mathcal{V}}_I(m,\alpha,c) &= \mathbb{E}\left[p_I^{-1}(I(V^*(T))+a_I)^{p_I}\right]\\
&\overset{(3.1)}{=} p_I^{-1}\mathbb{E}\Bigg[\Big((V^*(T)+v_0(c-m))\mathbb{1}_{\{V^*(T)\in[0,(1+m-c)v_0)\}}\\
&\quad + v_0\mathbb{1}_{\{V^*(T)\in[(1+m-c)v_0,(1+m)v_0)\}}+\Big(V^*(T)-mv_0\\
&\quad -\alpha(V^*(T)-(1+m)v_0)\Big)\mathbb{1}_{\{V^*(T)\in[(1+m)v_0,+\infty)\}}+a_I\Big)^{p_I}\Bigg]\\
&\underset{(A.6)}{\overset{(3.11)}{=}} p_I^{-1}\mathbb{E}\Bigg[\Big((0+v_0(c-m))\mathbb{1}_{\{\widetilde{Z}(T)\notin E_1\}}+v_0\mathbb{1}_{\varnothing}+\Big((1-\alpha)\\
&\quad\cdot\Big(\alpha^{1/(1-p_M)-1}(\lambda_v^*\widetilde{Z}(T))^{-1/(1-p_M)}+(1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M\Big)\\
&\quad+(\alpha(1+m)-m)v_0\Big)\mathbb{1}_{\{\widetilde{Z}(T)\in E_1\}}+a_I\Big)^{p_I}\Bigg]\\
&= p_I^{-1}\mathbb{E}\left[(v_0(c-m)+a_I)^{p_I}\,\mathbb{1}_{\{\widetilde{Z}(T)\notin E_1\}}\right]+p_I^{-1}\\
&\quad\cdot\mathbb{E}\Bigg[\Big((1-\alpha)\alpha^{1/(1-p_M)-1}(\lambda_v^*)^{-1/(1-p_M)}\widetilde{Z}(T)^{-1/(1-p_M)}\\
&\quad+(1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M-(\alpha(1+m)-m)v_0\\
&\quad+a_M+(\alpha(1+m)-m)v_0+a_I\Big)^{p_I}\mathbb{1}_{\{\widetilde{Z}(T)\in E_1\}}\Bigg]\\
&= p_I^{-1}\,(v_0(c-m)+a_I)^{p_I}\,\mathbb{E}\left[\mathbb{1}_{\{\widetilde{Z}(T)\notin E_1\}}\right]+p_I^{-1}\\
&\quad\cdot\mathbb{E}\Bigg[\Big((1-\alpha)\alpha^{1/(1-p_M)-1}(\lambda_v^*)^{-1/(1-p_M)}\widetilde{Z}(T)^{-1/(1-p_M)}+(1+m-\alpha^{-1}m)v_0\\
&\quad+a_M(1-\alpha^{-1})+a_I\Big)^{p_I}\mathbb{1}_{\{\widetilde{Z}(T)\in E_1\}}\Bigg]\\
&\underset{\widetilde{Z}(T)>0}{\overset{\text{Lem.A.2.1}}{=}} p_I^{-1}\,(v_0(c-m)+a_I)^{p_I}\,\big(\Phi(d_2^A)-\Phi(-\infty)\big)\\
&\quad+p_I^{-1}\mathbb{E}\Bigg[\Big((1-\alpha)\alpha^{1/(1-p_M)-1}(\lambda_v^*)^{-1/(1-p_M)}\widetilde{Z}(T)^{-1/(1-p_M)}\\
&\quad+(1+m-\alpha^{-1}m)v_0+a_M(1-\alpha^{-1})+a_I\Big)^{p_I}\mathbb{1}_{\{\widetilde{Z}(T)\in E_1\}}\Bigg]\\
&= p_I^{-1}\,(v_0(c-m)+a_I)^{p_I}\,\Phi(d_2^A)\\
&\quad+p_I^{-1}\mathbb{E}\left[\Big(k\widetilde{Z}(T)^{-1/(1-p_M)}+l\Big)^{p_I}\,\mathbb{1}_{\{\widetilde{Z}(T)\in E_1\}}\right],
\end{aligned}
$$

where $k=(1-\alpha)\alpha^{1/(1-p_M)-1}(\lambda_v^*)^{-1/(1-p_M)}$ and $l=(1+m-\alpha^{-1}m)v_0+a_M(1-\alpha^{-1})+a_I$, and $d_2^A(\cdot)$ is defined in Proposition 3.2.7.

The derivation of $\widetilde{\mathcal{V}}_I(m,\alpha,c)$ for $(m,\alpha,c)\in\mathcal{P}_B$ and for $(m,\alpha,c)\in\mathcal{P}_C$ is done similarly to Case A. For details, see Proposition 53 in Havrylenko (2018). □

A.2 Auxiliary results with proofs

Lemma A.2.1. *Let $\widetilde{Z}(T)$ be the state price density process at time T, $a > 0, b > 0$ such that $a < b$. Then for any $k \in \mathbb{R}$ it holds:*

$$\mathbb{E}\left[\left(\widetilde{Z}(T)\right)^k \mathbb{1}_{\{a<\widetilde{Z}(T)<b\}}\middle|\mathcal{F}(t)\right] = \left(\widetilde{Z}(t)\right)^k \exp\left(-k\left(r+\frac{\gamma^2}{2}\right)(T-t)+\frac{k^2\gamma^2}{2}(T-t)\right) \cdot\left(\Phi(d_1+k\gamma\sqrt{T-t})-\Phi(d_2+k\gamma\sqrt{T-t})\right),$$

where

$$d_1 = \frac{\log\left(\frac{\widetilde{Z}(t)}{a}\right)-\left(r+\frac{\gamma^2}{2}\right)(T-t)}{\gamma\sqrt{T-t}},\ d_2 = \frac{\log\left(\frac{\widetilde{Z}(t)}{b}\right)-\left(r+\frac{\gamma^2}{2}\right)(T-t)}{\gamma\sqrt{T-t}}. \tag{A.8}$$

Proof. Let d_1 and d_2 be as defined above. Then:

$$\begin{aligned}
\mathbb{E}\left[\left(\widetilde{Z}(T)\right)^k \mathbb{1}_{\{a<\widetilde{Z}(T)<b\}}\middle|\mathcal{F}(t)\right] &= \left(\widetilde{Z}(t)\right)^k \mathbb{E}\left[\left(\frac{\widetilde{Z}(T)}{\widetilde{Z}(t)}\right)^k \mathbb{1}_{\{a<\widetilde{Z}(T)<b\}}\middle|\mathcal{F}(t)\right] \\
&= \left(\widetilde{Z}(t)\right)^k \mathbb{E}\left[\left(\frac{\widetilde{Z}(T)}{\widetilde{Z}(t)}\right)^k \mathbb{1}_{\left\{\frac{a}{\widetilde{Z}(t)}<\frac{\widetilde{Z}(T)}{\widetilde{Z}(t)}<\frac{b}{\widetilde{Z}(t)}\right\}}\middle|\mathcal{F}(t)\right] \\
&\overset{(2.4),(A.8)}{=} \left(\widetilde{Z}(t)\right)^k \mathbb{E}\left[\left(\frac{\widetilde{Z}(T)}{\widetilde{Z}(t)}\right)^k \mathbb{1}_{\left\{d_2<\frac{W(T)-W(t)}{\sqrt{T-t}}<d_1\right\}}\middle|\mathcal{F}(t)\right] \\
&= \left(\widetilde{Z}(t)\right)^k \int_{d_2}^{d_1} \exp\left(-k\gamma\sqrt{T-t}x-k\left(r+\frac{\gamma^2}{2}\right)(T-t)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \\
&= \left(\widetilde{Z}(t)\right)^k \exp\left(-k\left(r+\frac{\gamma^2}{2}\right)(T-t)\right) \\
&\quad \cdot\int_{d_2}^{d_1}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x+k\gamma\sqrt{T-t})^2+\frac{1}{2}k^2\gamma^2(T-t)\right)dx
\end{aligned}$$

$$
\begin{aligned}
&= \left(\widetilde{Z}(t)\right)^k \exp\left(-k\left(r+\frac{\gamma^2}{2}\right)(T-t)+\frac{k^2\gamma^2}{2}(T-t)\right) \\
&\quad \cdot \int_{d_2}^{d_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\overbrace{x+k\gamma\sqrt{T-t}}^{=z})^2\right)\, dx \\
&= \left(\widetilde{Z}(t)\right)^k \exp\left(-k\left(r+\frac{\gamma^2}{2}\right)(T-t)+\frac{k^2\gamma^2}{2}(T-t)\right) \\
&\quad \cdot \int_{d_2+k\gamma\sqrt{T-t}}^{d_1+k\gamma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)\, dz \\
&= \left(\widetilde{Z}(t)\right)^k \exp\left(-k\left(r+\frac{\gamma^2}{2}\right)(T-t)+\frac{k^2\gamma^2}{2}(T-t)\right) \\
&\quad \cdot \left(\Phi(d_1+k\gamma\sqrt{T-t})-\Phi(d_2+k\gamma\sqrt{T-t})\right).
\end{aligned}
$$

□

Lemma A.2.2 (Solution to pointwise optimization problem)**.** *Let* $\lambda_v \in (0,+\infty)$ *be any fixed number. Then the expression*

$$
v^* := v^*(\lambda_v,\tilde{z}) = \begin{cases}
I_3(\lambda_v\tilde{z}), & \text{if } \tilde{z} \in (\tilde{u}'_{M,3}(\tilde{\chi}_3-)/\lambda_v, \tilde{u}'_{M,3}(\tilde{\chi}_2+)/\lambda_v), \\
\tilde{\chi}_2, & \text{if } \tilde{z} \in [\tilde{u}'_{M,3}(\tilde{\chi}_2+)/\lambda_v, \tilde{u}'_{M,2}(\tilde{\chi}_2-)/\lambda_v], \\
I_2(\lambda_v\tilde{z}), & \text{if } \tilde{z} \in (\tilde{u}'_{M,2}(\tilde{\chi}_2-)/\lambda_v, \tilde{u}'_{M,2}(\tilde{\chi}_1+)/\lambda_v), \\
\tilde{\chi}_1, & \text{if } \tilde{z} \in [\tilde{u}'_{M,2}(\tilde{\chi}_1+)/\lambda_v, \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v), \\
\tilde{v}_1, & \text{if } \tilde{z} = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v, \\
\tilde{\chi}_0, & \text{if } \tilde{z} \in (\tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v, +\infty),
\end{cases} \tag{A.9}
$$

solves for all $\tilde{z} \in (0,+\infty)$ *the problem*

$$
\max_{v\geq 0}\{\tilde{u}_M(v) - \lambda_v \cdot \tilde{z} \cdot v\}, \tag{A.10}
$$

where $\tilde{u}_M$ *is defined in (3.5),* $I_i(\cdot) := (\tilde{u}'_{M,i})^{-1}(\cdot)$ *for* $i \in \{2,3\}$*,* $\tilde{v}_1$ *is any number from* $[\tilde{\chi}_0, \tilde{\chi}_1]$.

Proof of Lemma A.2.2. The piecewise structure of the objective function (A.10) motivates us to consider three subproblems, which are derived from the initial optimization problem by restricting the feasibility region to $[\tilde{\chi}_0, \tilde{\chi}_1]$, $[\tilde{\chi}_1, \tilde{\chi}_2]$ and $[\tilde{\chi}_2, \tilde{\chi}_3)$. Note that if we allow $\tilde{\chi}_0 < 0$, then the first subproblem will be restricted to $[0, \tilde{\chi}_1]$ due to the constraint $v \geq 0$ in (3.6). We derive optimal solutions of the subproblems depending on parameters $\lambda_v \in (0,+\infty)$ and $\tilde{z} \in (0,+\infty)$. Then for any fixed λ_v and $\tilde{z}$ we compare the solutions of the three subproblems to find the global optimizer of the initial problem (3.6). Finally, we write the global maximizer as a function of λ_v and $\tilde{z}$.

Consider the optimization problem:

$$\max_{v\in[\tilde{\chi}_0,\tilde{\chi}_1]}\{\tilde{u}_{M,1}(v)-\lambda_v\tilde{z}v\}. \tag{P_1}$$

According to (3.5), we $\tilde{u}_{M,1}(v)$ is linear and strictly increasing, whence we write $\tilde{u}_{M,1}(v) = a_1v+b_1$ with $a_1 > 0$. Hence, the first subproblem is $\max_{v\in[0,\tilde{\chi}_1]}\{(a_1v+b_1)-\lambda_v\tilde{z}v\}$. Due to linearity of the objective function, we easily get the optimal solution depending on parameter values:

$$v_1^* = \begin{cases} 0, & \text{if } \tilde{z} > a_1/\lambda_v; \\ \tilde{v}_1 & \text{if } \tilde{z} = a_1/\lambda_v; \\ \tilde{\chi}_1, & \text{if } \tilde{z} < a_1/\lambda_v, \end{cases}$$

where $\tilde{v}_1$ is any number from the interval $[0,\tilde{\chi}_1]$. So the optimum of this subproblem is unique for $\tilde{z} \neq a_1/\lambda_v = \tilde{u}'_{M,1}(0+)/\lambda_v = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v$.

Consider the second optimization subproblem:

$$\max_{v\in[\tilde{\chi}_1,\tilde{\chi}_2]}\{\tilde{u}_{M,2}(v)-\lambda_v\tilde{z}v\} \Leftrightarrow \min_{v\in[\tilde{\chi}_1,\tilde{\chi}_2]}\{-\tilde{u}_{M,2}(v)+\lambda_v\tilde{z}v\}. \tag{P_2}$$

The minimum exists because the objective function is continuous and the feasible set is compact. Constraints are linear and the objective function is strictly convex. The latter property follows from the fact that the function $-\tilde{u}_{M,2}(v)$ is strictly convex due to strict concavity of $\tilde{u}_{M,3}(v)$. The Slater condition from Definition 2.5.8 obviously holds. Therefore, the Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality by Theorem 2.5.9. Due to strict convexity, the minimum is unique.

Using KKT conditions, we obtain the following optimal solution of the problem (P_2) depending on the value of $\lambda_v\tilde{z}$:

$$v_2^*(\lambda_v,\tilde{z}) = \begin{cases} \tilde{\chi}_1, & \text{if } \lambda_v\tilde{z} \geq \tilde{u}'_{M,2}(\tilde{\chi}_1+); \\ I_2(\lambda_v\tilde{z}), & \text{if } \tilde{u}'_{M,2}(\tilde{\chi}_2-) < \lambda_v\tilde{z} < \tilde{u}'_{M,2}(\tilde{\chi}_1+); \\ \tilde{\chi}_2, & \text{if } \tilde{u}'_{M,2}(\tilde{\chi}_2-) \geq \lambda_v\tilde{z}. \end{cases} \tag{A.11}$$

Consider the optimization subproblem:

$$\max_{v\in[\tilde{\chi}_2,+\infty)}\{\tilde{u}_{M,3}(v)-\lambda_v\tilde{z}v\} \Leftrightarrow \min_{v\in[\tilde{\chi}_2,+\infty)}\{-\tilde{u}_{M,3}(v)+\lambda_v\tilde{z}v\}. \tag{P_3}$$

Analogously to the previous case, the optimal solution exists and is unique for any $\lambda_v\tilde{z} > 0$. Using the corresponding KKT conditions, we obtain the optimal solution of (P_3):

$$v_3^* := v_3^*(\lambda_v,\tilde{z}) = \begin{cases} \tilde{\chi}_2, & \text{if } \lambda_v\tilde{z} \geq \tilde{u}'_{M,3}(\tilde{\chi}_2+); \\ I_3(\lambda_v\tilde{z}), & \text{if } \tilde{u}'_{M,3}(\tilde{\chi}_2+) > \lambda_v\tilde{z}. \end{cases} \tag{A.12}$$

$\lambda_v \tilde{z} \in$	v_1^*	v_2^*	v_3^*
$(0, \tilde{u}'_{M,3}(\tilde{\chi}_2+))$	$\tilde{\chi}_1$	$\tilde{\chi}_2$	$I_3(\lambda_v \tilde{z})$
$[\tilde{u}'_{M,3}(\tilde{\chi}_2+), \tilde{u}'_{M,2}(\tilde{\chi}_2-)]$	$\tilde{\chi}_1$	$\tilde{\chi}_2$	$\tilde{\chi}_2$
$(\tilde{u}'_{M,2}(\tilde{\chi}_2-), \tilde{u}'_{M,2}(\tilde{\chi}_1+))$	$\tilde{\chi}_1$	$I_2(\lambda_v \tilde{z})$	$\tilde{\chi}_2$
$[\tilde{u}'_{M,2}(\tilde{\chi}_1+), \tilde{u}'_{M,1}(\tilde{\chi}_1-))$	$\tilde{\chi}_1$	$\tilde{\chi}_1$	$\tilde{\chi}_2$
$\{\tilde{u}'_{M,1}(\tilde{\chi}_1-)\}$	$\tilde{v}_1$	$\tilde{\chi}_1$	$\tilde{\chi}_2$
$(\tilde{u}'_{M,1}(\tilde{\chi}_1-), +\infty)$	0	$\tilde{\chi}_1$	$\tilde{\chi}_2$

Table A.1: Optimal solutions of (P_1), (P_2), (P_3) depending on the value of $\lambda_v \tilde{z}$.

Now we show that the optimal solution of the problem $\max_{v\in[0;+\infty)}\{\tilde{u}_M(v) - \lambda_v \tilde{z} v\}$ is given by (A.9).

Denote $f(v) = \tilde{u}_M(v) - \lambda_v \tilde{z} v$ and $f_i(v) = \tilde{u}_{M,i}(v) - \lambda_v \tilde{z} v$, $i \in \{1, 2, 3\}$. We distinguish between 6 cases, depending on the value of $\lambda_v \tilde{z}$ and prove only the first two of them, as the remaining cases are analogous.

1) If $\tilde{z} > \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v = a_1/\lambda_v$, then $v_1^* = 0$, $v_2^* = \tilde{\chi}_1$, $v_2^* = \tilde{\chi}_2$. We show now that $v_1^* = 0$ maximizes $f(v)$ on $v \geq 0$.

Take any $\bar{v} \in [0, \tilde{\chi}_1)$. Then:

$$f(v_1^*) - f(\bar{v}) = f_1(v_1^*) - f_1(\bar{v}) \geq f_1(v_1^*) - \max_{v\in[0,\tilde{\chi}_1]} f_1(v) = f_1(v_1^*) - f_1(v_1^*) = 0$$

Take any $\bar{v} \in [\tilde{\chi}_1, \tilde{\chi}_2)$. Then:

$$\begin{aligned} f(v_1^*) - f(\bar{v}) &= f_1(v_1^*) - f_2(\bar{v}) \geq f_1(v_1^*) - \max_{v\in[\tilde{\chi}_1,\tilde{\chi}_2]} f_2(v) = f_1(v_1^*) - f_2(v_2^*) \\ &= f_1(v_1^*) - f_2(\tilde{\chi}_1) = f_1(v_1^*) - f_1(\tilde{\chi}_1) \geq f_1(v_1^*) - \max_{v\in[0,\tilde{\chi}_1]} f_1(v) \\ &= f_1(v_1^*) - f_1(v_1^*) = 0. \end{aligned}$$

We used optimal solutions of problems (P_1) and (P_2) for the corresponding values of $\tilde{z}$ as well as the fact that $f_1(\tilde{\chi}_1) = f_2(\tilde{\chi}_1)$.

Take any $\bar{v} \in [\tilde{\chi}_2, +\infty)$ and obtain:

$$\begin{aligned} f(v_1^*) - f(\bar{v}) &= f_1(v_1^*) - f_3(\bar{v}) \geq f_1(v_1^*) - \max_{v\in[\tilde{\chi}_2,+\infty)} f_3(v) = f_1(v_1^*) - f_3(v_3^*) \\ &= f_1(v_1^*) - f_3(\tilde{\chi}_2) = f_1(v_1^*) - f_2(\tilde{\chi}_2) \geq f_1(v_1^*) - \max_{v\in[\tilde{\chi}_1,\tilde{\chi}_2]} f_2(v) \\ &= f_1(v_1^*) - f_1(\tilde{\chi}_1) \geq f_1(v_1^*) - \max_{v\in[0,\tilde{\chi}_1]} f_1(v) = f_1(v_1^*) - f_1(v_1^*) = 0. \end{aligned}$$

We used optimal solutions of problems (P_1), (P_2), (P_3) for the corresponding values of $\tilde{z}$ as well as the fact that $f_1(\tilde{\chi}_1) = f_2(\tilde{\chi}_1)$, $f_2(\tilde{\chi}_2) = f_3(\tilde{\chi}_2)$.

We conclude that for any $\lambda_v > 0$ and $\tilde{z} > \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v$ the optimal solution of the problem $\max_{v\geq 0}\{\tilde{u}_M(v) - \lambda_v \tilde{z} v\}$ is 0.

2) If $\tilde{z} = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v = a_1/\lambda_v$, then according to Table A.1 $v_1^* = \tilde{v}_1$, $v_2^* = \tilde{\chi}_1$, $v_2^* = \tilde{\chi}_2$, where $\tilde{v}_1 \in [0, \tilde{\chi}_1]$. In this case $v_1^* = \tilde{v}_1$ solves $\max_{v\geq 0} f(v)$. It can be concluded from the fact that $\tilde{z} = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v \geq \tilde{u}'_{M,2}(\tilde{\chi}_1+)/\lambda_v > \tilde{u}'_{M,2}(\tilde{\chi}_2-)/\lambda_v \geq \tilde{u}'_{M,3}(\tilde{\chi}_2+)/\lambda_v$ and the following relations

$$f(v_1^*) = f_1(v^*) = f_1(\tilde{\chi}_1) = f_2(\tilde{\chi}_1) = f_2(v_2^*) = \max_{v\in[\tilde{\chi}_1,\tilde{\chi}_2]} f_2(v) > f_2(\tilde{\chi}_2) = f_3(\tilde{\chi}_2) = f_3(v_3^*)$$
$$= \max_{v\in[\tilde{\chi}_2,+\infty)} f_3(v).$$

First, we used that $f_1(v) = (a_1 v + b_1) - \lambda_v(a_1/\lambda_v)v = b_1$ is constant. Then we used the continuity of $f(\cdot)$ and the optimal solutions v_2^* and v_3^* of the corresponding subproblems for $\tilde{z} = \tilde{u}'_{M,1}(\tilde{\chi}_1-)/\lambda_v$.

The remaining cases are proven analogously to the previous two. For details, see Lemma 37 in Havrylenko (2018). Taking all six cases into consideration, we conclude that $v^*(y, \tilde{z})$ given by (A.9) solves Problem (3.6). □

Proposition A.2.3 (Equations for computing λ_v^*)**.** *Let $\tilde{\chi}_1$ and $s(\tilde{\chi}_1)$ be as defined in Corollary 3.2.5 depending on cases. Let ξ_1, $d_1^A(\cdot)$, $d_2^A(\cdot)$, $d_1^B(\cdot)$, $d_2^B(\cdot)$, $d_3^B(\cdot)$, $d_1^C(\cdot)$, $d_2^C(\cdot)$, $d_3^C(\cdot)$, $d_4^C(\cdot)$ be as defined in Proposition 3.2.7. Denote:*

$$\xi_2 = \exp\left(-\left(r + 0.5\gamma^2\right) T + 0.5\gamma^2 T\right).$$

Then the explicit equation for computing the unique λ_v^* is given in
Case A:

$$\lambda_v^{-1/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1\left(\Phi\left(d_1^A(\lambda_v) - \frac{p_M}{1-p_M}\gamma\sqrt{T}\right) - \Phi\left(d_2^A(\lambda_v) - \frac{p_M}{1-p_M}\gamma\sqrt{T}\right)\right)$$
$$+\left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right)\xi_2\left(\Phi\left(d_1^A(\lambda_v) + \gamma\sqrt{T}\right) - \Phi\left(d_2^A(\lambda_v) + \gamma\sqrt{T}\right)\right) = v_0;$$

Case B:

$$\lambda_v^{-1/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1\left(\Phi\left(d_1^B(\lambda_v) - \frac{p_M}{1-p_M}\gamma\sqrt{T}\right) - \Phi\left(d_2^B(\lambda_v) - \frac{p_M}{1-p_M}\gamma\sqrt{T}\right)\right)$$
$$+\left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right)\xi_2\left(\Phi\left(d_1^B(\lambda_v) + \gamma\sqrt{T}\right) - \Phi\left(d_2^B(\lambda_v) + \gamma\sqrt{T}\right)\right)$$
$$+(1+m)v_0\xi_2\left(\Phi\left(d_2^B(\lambda_v) + \gamma\sqrt{T}\right) - \Phi\left(d_3^B(\lambda_v) + \gamma\sqrt{T}\right)\right) = v_0;$$

Case C:

$$
\begin{aligned}
&\lambda_v^{-1/(1-p_M)}\alpha^{p_M/(1-p_M)}\xi_1\left(\Phi\left(d_1^C(\lambda_v)-\frac{p_M}{1-p_M}\gamma\sqrt{T}\right)-\Phi\left(d_2^C(\lambda_v)-\frac{p_M}{1-p_M}\gamma\sqrt{T}\right)\right)\\
&+\left((1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M\right)\xi_2\left(\Phi\left(d_1^C(\lambda_v)+\gamma\sqrt{T}\right)-\Phi\left(d_2^C(\lambda_v)+\gamma\sqrt{T}\right)\right)\\
&+(1+m)v_0\xi_2\left(\Phi\left(d_2^C(\lambda_v)+\gamma\sqrt{T}\right)-\Phi\left(d_3^C(\lambda_v)+\gamma\sqrt{T}\right)\right)\\
&+\lambda_v^{-1/(1-p_M)}\xi_1\left(\Phi\left(d_3^C(\lambda_v)-\frac{p_M}{1-p_M}\gamma\sqrt{T}\right)-\Phi\left(d_4^C(\lambda_v)-\frac{p_M}{1-p_M}\gamma\sqrt{T}\right)\right)\\
&+(v_0-a_M)\xi_2\left(\Phi\left(d_3^C(\lambda_v)+\gamma\sqrt{T}\right)-\Phi\left(d_4^C(\lambda_v)+\gamma\sqrt{T}\right)\right)=v_0.
\end{aligned}
$$

Proof. By Proposition 3.2.3, there exists a unique λ_v^* solving $\mathbb{E}\left[\widetilde{Z}(T)V^*(T)\right]=v_0$. With the help of Corollary 3.2.5 and Lemma A.2.1, the explicit form of $\mathbb{E}\left[\widetilde{Z}(T)V^*(T)\right]$ can be obtained via straightforward but lengthy calculations. For details, see Proposition 48 in Havrylenko (2018). □

Proposition A.2.4 (First and second moment of the hedge-fund's optimal terminal value)**.** *Let the manager's preferences be determined by $\widetilde{U}_M$ as per* (3.8)*. Let λ_v^* be as defined in Corollary 3.2.5 and $d_1^A(\cdot)$, $d_2^A(\cdot)$, $d_1^B(\cdot)$, $d_2^B(\cdot)$, $d_3^B(\cdot)$, $d_1^C(\cdot)$, $d_2^C(\cdot)$, $d_3^C(\cdot)$, $d_4^C(\cdot)$ be as defined in Proposition 3.2.7. Denote:*

$$
\begin{aligned}
\xi_3&=\exp\left((1-p_M)^{-1}\left(r+0.5\gamma^2\right)T+0.5(1-p_M)^{-2}\gamma^2T\right);\\
\xi_4&=\exp\left(2(1-p_M)^{-1}\left(r+0.5\gamma^2\right)T+2(1-p_M)^{-2}\gamma^2T\right).
\end{aligned}
$$

The first two moments of the fund's optimal terminal value equal in
Case A:

$$
\begin{aligned}
\mathbb{E}\left[V^*(T)\right]=&\left(\lambda_v^*\right)^{-\frac{1}{1-p_M}}\alpha^{\frac{p_M}{1-p_M}}\xi_3\left(\Phi\left(d_1^A-\frac{\gamma\sqrt{T}}{1-p_M}\right)-\Phi\left(d_2^A-\frac{\gamma\sqrt{T}}{1-p_M}\right)\right)\\
&+\left((1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M\right)\left(\Phi\left(d_1^A\right)-\Phi\left(d_2^A\right)\right);
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[(V^*(T))^2\right]=&\,\alpha^{2\frac{p_M}{1-p_M}}\left(\lambda_v^*\right)^{-\frac{2}{1-p_M}}\xi_4\cdot\left(\Phi\left(d_1^A-\frac{2\gamma\sqrt{T}}{1-p_M}\right)-\Phi\left(d_2^A-\frac{2\gamma\sqrt{T}}{1-p_M}\right)\right)\\
&+2\alpha^{\frac{p_M}{1-p_M}}\left(\lambda_v^*\right)^{-\frac{1}{1-p_M}}\left((1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M\right)\xi_3\\
&\cdot\left(\Phi\left(d_1^A-\frac{1}{1-p_M}\gamma\sqrt{T}\right)-\Phi\left(d_2^A-\frac{1}{1-p_M}\gamma\sqrt{T}\right)\right)\\
&+\left((1+m-\alpha^{-1}m)v_0-\alpha^{-1}a_M\right)^2\left(\Phi\left(d_1^A\right)-\Phi\left(d_2^A\right)\right)\\
&+(1+m)^2v_0^2\left(\Phi\left(d_2^A\right)-\Phi\left(d_3^A\right)\right);
\end{aligned}
$$

Case B:

$$\mathbb{E}\left[V^*(T)\right] = (\lambda_v^*)^{-\frac{1}{1-p_M}} \alpha^{\frac{p_M}{1-p_M}} \xi_3 \left(\Phi\left(d_1^B(\lambda_v^*) - \frac{\gamma\sqrt{T}}{1-p_M} \right) - \Phi\left(d_2^B(\lambda_v^*) - \frac{\gamma\sqrt{T}}{1-p_M} \right) \right)$$
$$+ \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \left(\Phi\left(d_1^B(\lambda_v^*)\right) - \Phi\left(d_2^B(\lambda_v^*)\right)\right)$$
$$+ (1+m)v_0 \left(\Phi\left(d_2^B(\lambda_v^*)\right) - \Phi\left(d_3^B(\lambda_v^*)\right)\right);$$

$$\mathbb{E}\left[(V^*(T))^2\right] = \alpha^{\frac{2p_M}{1-p_M}} (\lambda_v^*)^{-\frac{2}{1-p_M}} \xi_4$$
$$\cdot \left(\Phi\left(d_1^B(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) - \Phi\left(d_2^B(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) \right)$$
$$+ 2\alpha^{\frac{p_M}{1-p_M}} (\lambda_v^*)^{-1/(1-p_M)} \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \xi_3$$
$$\cdot \left(\Phi\left(d_1^B(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) - \Phi\left(d_2^B(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) \right)$$
$$+ \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right)^2 \left(\Phi\left(d_1^B(\lambda_v^*)\right) - \Phi\left(d_2^B(\lambda_v^*)\right)\right)$$
$$+ (1+m)^2 v_0^2 \left(\Phi\left(d_2^B(\lambda_v^*)\right) - \Phi\left(d_3^B(\lambda_v^*)\right)\right);$$

Case C:

$$\mathbb{E}\left[V^*(T)\right] = (\lambda_v^*)^{-\frac{1}{1-p_M}} \alpha^{\frac{p_M}{1-p_M}} \xi_3 \left(\Phi\left(d_1^C(\lambda_v^*) - \frac{\gamma\sqrt{T}}{1-p_M} \right) - \Phi\left(d_2^C(\lambda_v^*) - \frac{\gamma\sqrt{T}}{1-p_M} \right) \right)$$
$$+ \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \left(\Phi\left(d_1^C(\lambda_v^*)\right) - \Phi\left(d_2^C(\lambda_v^*)\right)\right)$$
$$+ (1+m)v_0 \left(\Phi\left(d_2^C(\lambda_v^*)\right) - \Phi\left(d_3^C(\lambda_v^*)\right)\right)$$
$$+ (\lambda_v^*)^{-\frac{1}{1-p_M}} \xi_3 \left(\Phi\left(d_3^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) - \Phi\left(d_4^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) \right)$$
$$+ (v_0 - a_M) \left(\Phi\left(d_3^C(\lambda_v^*)\right) - \Phi\left(d_4^C(\lambda_v^*)\right)\right);$$

$$\mathbb{E}\left[(V^*(T))^2\right] = \alpha^{\frac{2p_M}{1-p_M}} (\lambda_v^*)^{-\frac{2}{1-p_M}} \xi_4$$
$$\cdot \left(\Phi\left(d_1^C(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) - \Phi\left(d_2^C(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) \right)$$
$$+ 2\alpha^{\frac{p_M}{1-p_M}} (\lambda_v^*)^{-\frac{1}{1-p_M}} \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right) \xi_3$$
$$\cdot \left(\Phi\left(d_1^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) - \Phi\left(d_2^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T} \right) \right)$$
$$+ \left((1+m-\alpha^{-1}m)v_0 - \alpha^{-1}a_M\right)^2 \cdot \left(\Phi\left(d_1^C(\lambda_v^*)\right) - \Phi\left(d_2^C(\lambda_v^*)\right)\right)$$
$$+ (1+m)^2 v_0^2 \left(\Phi\left(d_2^C(\lambda_v^*)\right) - \Phi\left(d_3^C(\lambda_v^*)\right)\right)$$
$$+ (\lambda_v^*)^{-\frac{2}{1-p_M}} \xi_4 \left(\Phi\left(d_3^C(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) - \Phi\left(d_4^C(\lambda_v^*) - \frac{2}{1-p_M}\gamma\sqrt{T} \right) \right)$$
$$+ 2(\lambda_v^*)^{-\frac{1}{1-p_M}} (v_0 - a_M)\xi_3$$

$$\cdot \left(\Phi\left(d_3^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T}\right) - \Phi\left(d_4^C(\lambda_v^*) - (1-p_M)^{-1}\gamma\sqrt{T}\right)\right)$$
$$+ (v_0 - a_M)^2 \left(\Phi\left(d_3^C(\lambda_v^*)\right) - \Phi\left(d_4^C(\lambda_v^*)\right)\right).$$

Proof. With the help of Corollary 3.2.5 and Lemma A.2.1, the explicit form of $\mathbb{E}[V^*(T)]$ and $\mathbb{E}\left[(V^*(T))^2\right]$ can be obtained via straightforward but lengthy calculations. For the complete derivation of $\mathbb{E}[V^*(T)]$ see Proposition 49 in Havrylenko (2018). The complete derivation of $\mathbb{E}\left[(V^*(T))^2\right]$ is provided in Proposition 50 in Havrylenko (2018). □

B Appendix to Chapter 4

Here we provide proofs to theoretical results from Chapter 4. Appendix B.1 contains the proofs of the main results. Appendix B.2 contains auxiliary results and their proofs.

B.1 Proofs of main results

Proof of Proposition 4.2.1. The dynamics of the insurer's portfolio with respect to S_0, S_1, Put is given by:

$$\begin{aligned}
d\bar{V}^{v_0,\bar{\pi}}(t) &= \bar{\varphi}_0(t)dS_0(t) + \bar{\varphi}_1(t)dS_1(t) + \bar{\varphi}_2(t)dPut(t) \\
&= \bar{\varphi}_0(t)S_0(t)rdt + \bar{\varphi}_1(t)S_1(t)(\mu_1 dt + \sigma_1 dW_1^{\mathbb{Q}}(t)) \\
&\quad + \bar{\varphi}_2(t)\big(((\Phi(d_+)-1)V^{v_0,\pi_B}(t)\pi_B^{CM}(\mu_2 - r) + rPut(t))dt \\
&\quad + (\Phi(d_+)-1)V^{v_0,\pi_B}(t)\pi_B^{CM}\sigma_2\left(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t)\right)\Big) \\
&= \Big(\bar{\varphi}_0(t)S_0(t)r + \bar{\varphi}_1(t)S_1(t)\mu_1 + \bar{\varphi}_2(t)\big((\Phi(d_+)-1)\,V^{v_0,\pi_B}(t)\pi_B^{CM}(\mu_2 - r) \\
&\quad + rPut(t)\big)\Big)dt + \big(\bar{\varphi}_1(t)S_1(t)\sigma_1 + \bar{\varphi}_2(t)(\Phi(d_+)-1)V^{v_0,\pi_B}(t)\pi_B^{CM}\sigma_2\rho\big)\,dW_1^{\mathbb{Q}}(t) \\
&\quad + \left(\bar{\varphi}_2(t)(\Phi(d_+)-1)V^{v_0,\pi_B}(t)\pi_B^{CM}\sigma_2\sqrt{1-\rho^2}\right)dW_2^{\mathbb{Q}}(t).
\end{aligned} \tag{B.1}$$

The dynamics of a portfolio with respect to S_0, S_1, S_2 is given by:

$$\begin{aligned}
dV^{v_0,\pi}(t) &= \varphi_0(t)dS_0(t) + \varphi_1(t)dS_1(t) + \varphi_2(t)dS_2(t) \\
&= \varphi_0(t)S_0(t)rdt + \varphi_1(t)S_1(t)(\mu_1 dt + \sigma_1 dW_1^{\mathbb{Q}}(t)) \\
&\quad + \varphi_2(t)S_2(t)(\mu_2 dt + \sigma_2(\rho dW_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t))) \\
&= (\varphi_0(t)S_0(t)r + \varphi_1(t)S_1(t)\mu_1 + \varphi_2(t)S_2(t)\mu_2)\,dt \\
&\quad + (\varphi_1(t)S_1(t)\sigma_1 + \varphi_2(t)S_2(t)\sigma_2\rho)\,dW_1^{\mathbb{Q}}(t) \\
&\quad + \left(\varphi_2(t)S_2(t)\sigma_2\sqrt{1-\rho^2}\right)dW_2^{\mathbb{Q}}(t).
\end{aligned} \tag{B.2}$$

Equating the coefficients next to the terms dt, $dW_1^{\mathbb{Q}}(t)$ and $dW_2^{\mathbb{Q}}(t)$ in (B.1) and (B.2), we get the following link:

$$\begin{cases} \bar{\varphi}_0(t) = \varphi_0(t) + \varphi_2(t)\dfrac{S_2(t)}{S_0(t)}\left(1 - \dfrac{Put(t)}{\pi_B^{CM} V^{v_0,\pi_B}(t)(\Phi(d_+) - 1)}\right); \\ \bar{\varphi}_1(t) = \varphi_1(t); \\ \bar{\varphi}_2(t) = \dfrac{\varphi_2(t) S_2(t)}{\pi_B^{CM} V^{v_0,\pi_B}(t)(\Phi(d_+) - 1)}. \end{cases} \tag{B.3}$$

Using (B.3) and the relation (4.5) between the investment strategies $\varphi(\bar{\varphi})$ and the relative portfolio processes $\pi(\bar{\pi})$, we conclude that the claim of the proposition follows. □

Proof of Proposition 4.2.2. We prove the claim of the proposition by contradiction.

Let π^* be the solution to (P_{ε,C_π}). Then $\bar{\pi}^* = (\Psi(t)\pi^*(t))_{t\in[0,T]} \in \bar{\mathcal{A}}_c^{\bar{\pi}}\left(v_0, \bar{C}_{\bar{V}}(\varepsilon), \bar{C}_{\bar{\pi}}\right)$ according to (4.9).

Assume that $\exists\ \bar{\pi}^{**} \in \bar{\mathcal{A}}_c^{\bar{\pi}}\left(v_0, \bar{C}_{\bar{V}}(\varepsilon), \bar{C}_{\bar{\pi}}\right)$ such that:

$$\mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\bar{\pi}^{**}}(T)\right)\right] > \mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\bar{\pi}^{*}}(T)\right)\right]. \tag{B.4}$$

Then $\pi^{**} := \left(\Psi^{-1}(t)\bar{\pi}^{**}(t)\right)_{t\in[0,T]} \in \mathcal{A}_c^{\pi}\left(v_0, C_V(\varepsilon), C_\pi\right)$ and:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\pi^{**}}(T)\right)\right] &\overset{\text{Prop.4.2.1}}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\bar{\pi}^{**}}(T)\right)\right] \overset{\text{(B.4)}}{>} \mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\bar{\pi}^{*}}(T)\right)\right] \\ &\overset{\text{Prop.4.2.1}}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\pi^{*}}(T)\right)\right], \end{aligned}$$

which contradicts the optimality of π^* for (P_{ε,C_π}). The claim follows. □

Proof of Lemma 4.2.3. Due to Condition (4.16), namely $\pi_\nu(t)^\top \nu(t) = 0$ $\mathbb{Q}$-a.s., $\forall t \in [0,T]$, the SDEs for the wealth processes $V_\nu^{v_0,\pi_\nu}$ and V^{v_0,π_ν} coincide. Since both processes start from the same initial wealth v_0, we conclude that

$$V_\nu^{v_0,\pi_\nu}(t) = V^{v_0,\pi_\nu}(t) \ \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T]. \tag{B.5}$$

Next we prove the admissibility of π_ν for the original problem (P_{ε,C_π}). Obviously, the budget constraint is satisfied. The allocation constraint is satisfied due to the second part of Condition (4.16). The VaR constraint is satisfied, since:

$$\varepsilon \overset{(i)}{\geq} \mathbb{Q}\left(V_\nu^{v_0,\pi_\nu}(t) < G_T\right) \overset{\text{(B.5)}}{=} \mathbb{Q}\left(V^{v_0,\pi_\nu}(t) < G_T\right),$$

where we used in (i) that $\pi_\nu \in \mathcal{A}_{c,\nu}^{\pi}\left(v_0, C_{V_\nu}(\varepsilon)\right)$. The remaining technical conditions for $\pi_\nu \in \mathcal{A}_c^{\pi}\left(v_0, C_V(\varepsilon)\right)$ immediately follow from the admissibility of $\pi_\nu \in \mathcal{A}_{c,\nu}^{\pi}\left(v_0, C_{V_\nu}(\varepsilon)\right)$.

Finally, we prove by contradiction the optimality of π_ν for the original problem (P_{ε,C_π}). Assume that there exists $\tilde{\pi} \in \mathcal{A}_c^\pi(v_0, C_V(\varepsilon))$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\pi_\nu}(T)\right)\right] < \mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\tilde{\pi}}(T)\right)\right]. \tag{B.6}$$

Then we obtain:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\pi_\nu}(T)\right)\right] &\overset{(B.5)}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(V_\nu^{v_0,\pi_\nu}(T)\right)\right] \\ &\overset{(B.6)}{<} \mathbb{E}^{\mathbb{Q}}\left[U\left(V^{v_0,\tilde{\pi}}(T)\right)\right] \overset{(i)}{\leq} \mathbb{E}^{\mathbb{Q}}\left[U\left(V_\nu^{v_0,\tilde{\pi}}(T)\right)\right],\end{aligned} \tag{B.7}$$

where we used in (i) the strict increasingness of $U(\cdot)$ and the fact that $V^{v_0,\tilde{\pi}}(T) \leq V_\nu^{v_0,\tilde{\pi}}(T)$, which follows from $\nu \in \mathcal{D}$ and SDE (4.15).

As (B.7) contradicts the optimality of π_ν for (P_ε^ν), the assumption regarding the existence of the above-mentioned $\tilde{\pi}$ is false. Therefore, π_ν is the optimal relative portfolio process for the original problem (P_{ε,C_π}). □

Proof of Corollary 4.2.4. Recall that we are considering a power-utility function $U(x) = \frac{1}{p}x^p$ with $p < 1,\ p \neq 0$ and the set of allocation constraints C_π is a convex cone.

For the case $0 < p < 1$, the statement of Corollary 4.2.4 is an immediate consequence of Theorem 10.1 and Theorem 15.3 from Cvitanic and Karatzas (1992).

For the case $p < 0$, we momentarily emphasize the explicit dependence of $\pi_{u,\nu}^*$ as in (4.18) on p as "$\pi_{u,\nu}^*(p)$". Let $p = p^- < 0$ and $p^+ \in (0,1)$ be arbitrary but fixed.

Then, $\pi_{u,\nu}^*(p^+)$ solves (P_{1,C_π}) with risk aversion parameter p^+, and $\pi_{u,\nu}^*(p^+)$ satisfies equations (4.16). Clearly, this implies

$$\pi_{u,\nu^*}^*(p^-) = \underbrace{\frac{1-p^+}{1-p^-}}_{\geq 0}\underbrace{\pi_{u,\nu^*}^*(p^+)}_{\in C_\pi} \in C_\pi,$$

because C_π is a convex cone, as well as

$$\pi_{u,\nu^*}^*(p^-)^\top \nu^* = \frac{1-p^+}{1-p^-}\pi_{u,\nu^*}^*(p^+)^\top \nu^* \overset{(4.16)}{=} 0.$$

Moreover, $\pi_{u,\nu^*}^*(p^-)$ is the optimal portfolio process for $(P_1^{\nu^*})$ with power utility with $p = p^-$. Hence, $\pi_{u,\nu^*}^*(p^-)$ satisfies (4.16) and is the optimal relative portfolio process for the primal problem (P_{1,C_π}). □

Proof of Corollary 4.2.5. The problem considered in Corollary 4.2.5 is precisely as in Section 2 of Basak and Shapiro (2001). Thus, we relate $(y, \widetilde{Z}_\nu(t))$ and $(v_{D_\nu}, V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(t))$

for arbitrary $v_{D_\nu}, y \geq 0$ in such a way that the statements of Corollary 4.2.5 follow directly from Basak and Shapiro (2001).

For any $v_{D_\nu} \geq 0$, the optimal unconstrained wealth process $V_\nu^{v_{D_\nu},\pi^*_{u,\nu}}(t)$, $t \in [0,T]$, is given as

$$\begin{aligned}
V_\nu^{v_{D_\nu},\pi^*_{u,\nu}}(t) &= v_{D_\nu} \exp\Big(\big(r + \underbrace{(\mu + \nu - r\mathbf{1}_2)^\top \pi^*_{u,\nu}}_{=\frac{1}{1-p}\|\gamma_\nu\|^2} - \underbrace{\frac{1}{2}\|\sigma^\top \pi^*_{u,\nu}\|^2}_{=\frac{1}{2(1-p)^2}\|\gamma_\nu\|^2}\big)t + \underbrace{\big(\pi^*_{u,\nu}\big)^\top \sigma}_{=\frac{1}{1-p}\gamma_\nu} W^{\mathbb{Q}}(t)\Big) \\
&= v_{D_\nu} \exp\Big(\big(r - \frac{1}{p-1}\|\gamma_\nu\|^2 - \frac{1}{2(p-1)^2}\|\gamma_\nu\|^2\big)t - \frac{1}{p-1}\gamma_\nu^\top W^{\mathbb{Q}}(t)\Big) \\
&= v_{D_\nu} \big(\widetilde{Z}_\nu(t)\big)^{\frac{1}{p-1}} \exp\Big(\big(r - \frac{1}{p-1}\|\gamma_\nu\|^2 - \frac{1}{2(p-1)^2}\|\gamma_\nu\|^2 + \frac{1}{p-1}(r + \frac{1}{2}\|\gamma_\nu\|^2)\big)t\Big) \\
&= v_{D_\nu} \big(\widetilde{Z}_\nu(t)\big)^{\frac{1}{p-1}} \exp\Big(\frac{p}{p-1}\big(r + \frac{1}{2}\|\gamma_\nu\|^2\big)t - \underbrace{\Big(\frac{1}{2} + \frac{1}{p-1} + \frac{1}{2(p-1)^2}\Big)}_{=\frac{p^2}{(p-1)^2}}\|\gamma_\nu\|^2 t\Big) \\
&= v_{D_\nu} \big(\widetilde{Z}_\nu(t)\big)^{\frac{1}{p-1}} \exp\Big(\underbrace{\frac{p}{p-1}\big(r + \frac{1}{2}\|\gamma_\nu\|^2\big)t - \Big(\frac{p}{p-1}\Big)^2 \frac{1}{2}\|\gamma_\nu\|^2 t}_{\overset{(4.21)}{=}\Gamma_\nu(t) - \Gamma_\nu(0)}\Big) \\
&= v_{D_\nu} \big(\widetilde{Z}_\nu(t)\big)^{\frac{1}{p-1}} \exp\Big(\Gamma_\nu(t) - \Gamma_\nu(0)\Big).
\end{aligned}$$

Since $\Gamma_\nu(T) \overset{(4.21)}{=} 0$, the optimal unconstrained terminal wealth that started from the initial capital v_{D_ν} is equal to

$$V_\nu^{v_{D_\nu},\pi^*_{u,\nu}}(T) = v_{D_\nu} \big(\widetilde{Z}_\nu(T)\big)^{\frac{1}{p-1}} \exp\Big(-\Gamma_\nu(0)\Big).$$

Comparing it with the optimal unconstrained wealth $\big(y\widetilde{Z}_\nu(T)\big)^{\frac{1}{p-1}}$ written in terms of a Lagrange multiplier $y > 0$, we identify the following link:

$$y := \big(v_{D_\nu} \exp\left(\Gamma_\nu(0)\right)\big)^{p-1}, \tag{B.8}$$

which is a continuous bijective function from $(0, +\infty)$ to $(0, +\infty)$.

According to Equation (9) in Basak and Shapiro (2001), the optimal VaR-constrained terminal wealth is given by:

$$
\begin{aligned}
V_\nu^{v_0,\pi_\nu^*}(T;y) &= I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{\widetilde{Z}_\nu(T)<U'(G_T)/y\right\}} + G_T\mathbb{1}_{\left\{U'(G_T)/y\le\widetilde{Z}_\nu(T)<\bar{z}_\nu^\varepsilon\right\}} \\
&\quad + I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{\bar{z}_\nu^\varepsilon\le\widetilde{Z}_\nu(T)\right\}} \\
&\overset{y\geq 0}{=} I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{y\widetilde{Z}_\nu(T)<U'(G_T)\right\}} + G_T\mathbb{1}_{\left\{U'(G_T)\le y\widetilde{Z}_\nu(T)<y\bar{z}_\nu^\varepsilon\right\}} \\
&\quad + I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{y\bar{z}_\nu^\varepsilon\le y\widetilde{Z}_\nu(T)\right\}} \\
&\overset{I(\cdot)\downarrow}{=} I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{I(U'(G_T))<I\left(y\widetilde{Z}_\nu(T)\right)\right\}} + G_T\mathbb{1}_{\left\{I(y\bar{z}_\nu^\varepsilon)<I\left(y\widetilde{Z}_\nu(T)\right)\le I(U'(G_T))\right\}} \\
&\quad + I(y\widetilde{Z}_\nu(T))\mathbb{1}_{\left\{I\left(y\widetilde{Z}_\nu(T)\right)\le I(y\bar{z}_\nu^\varepsilon)\right\}} \\
&\overset{\text{(B.8)}}{=} V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\mathbb{1}_{\left\{G_T<V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\right\}} + G_T\mathbb{1}_{\left\{I(y\bar{z}_\nu^\varepsilon)<V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\le G_T\right\}} \\
&\quad + V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\mathbb{1}_{\left\{V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\le I(y\bar{z}_\nu^\varepsilon)\right\}} \\
&\overset{k_\nu^\varepsilon:=I(y\bar{z}_\nu^\varepsilon)}{=} V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T) + \left(G_T - V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\right)\mathbb{1}_{(k_\nu^\varepsilon,G_T]}\left(V^{v_{D_\nu}(y),\pi_{u,\nu}^*}(T)\right),
\end{aligned}
$$

where $I(\cdot)$ is the inverse function of $U'(\cdot)$, $\bar{z}_\nu^\varepsilon$ is such that $\mathbb{Q}\left(\widetilde{Z}_\nu(T) > \bar{z}_\nu^\varepsilon\right) = \varepsilon$, and the Lagrange multiplier $y \geq 0$ solves the budget constraint $\mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_\nu(T)V_\nu^{v_0,\pi_\nu^*}(T;y)\right] = v_0$.

Using (B.8), we also get the relation

$$
V_\nu^{v_{D_\nu},\pi_{u,\nu}^*}(t) = v_{D_\nu}\left(\widetilde{Z}_\nu(t)\right)^{\frac{1}{p-1}}\exp\Big(\Gamma_\nu(t)-\Gamma_\nu(0)\Big) = \left(y\widetilde{Z}_\nu(t)\right)^{\frac{1}{p-1}}\exp\left(\Gamma_\nu(t)\right). \tag{B.9}
$$

By plugging these results into Proposition 1 and Proposition 3 from Basak and Shapiro (2001) and rewriting their definitions of d_1 and d_2 in terms of k_ν^ε and G_T, the statements of Corollary 4.2.5 follow immediately. □

Proof of Proposition 4.2.6. According to Corollary 4.2.5, $\pi_{\nu^*}^*$ is optimal for $\left(P_\varepsilon^{\nu^*}\right)$ and has multiplicative structure

$$
\pi^*(t) = \beta_{\nu^*}^D(t, V^{v_{D_{\nu^*}},\pi_{u,\nu^*}^*}(t))\cdot\pi_{u,\nu^*}^*,\ t\in[0,T]. \tag{B.10}
$$

We continue by verifying that this candidate portfolio $\pi^* := \pi_{\nu^*}$ satisfies (4.16) and is, therefore, optimal for (P_{ε,C_π}) by Lemma 4.2.3.

For this purpose, Corollary 4.2.4 provides us with useful information about π_{u,ν^*}^* as well as ν^*:

$$
\pi_{u,\nu^*}^* \in C_\pi \quad \text{and} \quad (\nu^*)^\top\pi_{u,\nu^*}^* = 0 \quad \mathbb{Q}\text{-a.s.}. \tag{B.11}
$$

The multiplicative structure[1] (B.10) of π^* and (B.11) imply that

$$(\nu^*)^\top \pi^*(t) = \beta^D_{\nu^*}(t, V^{v_{D_{\nu^*}}, \pi^*_{u,\nu^*}}(t)) \cdot (\nu^*)^\top \pi^*_{u,\nu^*} = 0 \quad \forall t \in [0,T],\ \mathbb{Q}\text{-a.s.},$$

which means the first part of Condition (4.16) is fulfilled.

Since $\pi^*_{u,\nu^*} \in C_\pi$, $C_\pi = [0,\infty) \times (-\infty, 0]$ is a convex cone and $\beta^D_{\nu^*}(t,V) \geq 0$ from (4.22), we obtain that $\pi^*(t) \in C_\pi$ $\mathbb{Q}$-a.s. $\forall t \in [0,T]$. So the second part of Condition (4.16) is satisfied.

In summary, we have shown that $\pi^* = (\pi^*(t))_{t\in[0,T]}$ is optimal for (P^ν_ε), $\pi^*(t) \in C_\pi$ $\forall t \in [0,T]$ $\mathbb{Q}$-a.s., and $(\nu^*)^\top \pi^*(t) = 0$ $\forall t \in [0,T]$ $\mathbb{Q}$-a.s.. Hence, π^* and ν^* satisfy (4.16) and π^* is optimal for the primal problem (P_{ε,C_π}) according to Lemma 4.2.3. □

Proof of Proposition 4.2.7. Observe that:

$$\begin{aligned}
\bar{\pi}^*_2(t) > 0 &\overset{\text{Prop.4.2.2}}{\underset{(4.8)}{\iff}} \underbrace{\frac{Put(t)}{\pi^{CM}_B V^{v_0,\pi_B}(t)(\Phi(d_+)-1)}}_{<0} \pi^*_2(t) > 0 \iff \pi^*_2(t) < 0 \\
&\overset{(4.24)}{\iff} \beta^{D_{\nu^*}}_{\nu^*}(t, V^{\nu_f,\pi^*_{u,\nu^*}}(t)) \cdot \pi^*_{u,\nu^*,2}(t) < 0 \overset{\text{Prop.4.2.6}}{\underset{\beta>0}{\iff}} \pi^*_{u,\nu^*,2}(t) < 0 \\
&\overset{(4.18)}{\iff} \left(\frac{1}{1-p}\Sigma^{-1}(\mu + \nu^* - r \cdot \mathbf{1}_2)\right)^\top \begin{pmatrix} 0 \\ 1 \end{pmatrix} < 0.
\end{aligned}$$

The inverse of the volatility matrix is given by:

$$\sigma^{-1} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\sigma_2\rho & \sigma_1 \end{pmatrix}$$

and thus:

$$\begin{aligned}
\Sigma^{-1}(\mu + \nu^* - r\mathbf{1}_2) &= \left(\sigma^{-1}\right)^\top \sigma^{-1}(\mu + \nu^* - r\mathbf{1}_2) \\
&= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2} & -\sigma_2\rho \\ 0 & \sigma_1 \end{pmatrix} \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \begin{pmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\sigma_2\rho & \sigma_1 \end{pmatrix} \\
&\quad \cdot (\mu + \nu^* - r\mathbf{1}_2) \\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2 & -\sigma_1\sigma_2\rho \\ -\sigma_1\sigma_2\rho & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \mu_1 + \nu_1^* - r \\ \mu_2 + \nu_2^* - r \end{pmatrix} \\
&= \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{pmatrix} \sigma_2^2(\mu_1 + \nu_1^* - r) - \sigma_1\sigma_2\rho(\mu_2 + \nu_2^* - r) \\ \sigma_1^2(\mu_2 + \nu_2^* - r) - \sigma_1\sigma_2\rho(\mu_1 + \nu_1^* - r) \end{pmatrix}
\end{aligned}$$

[1] In general, allocation constraints that lead to $\delta(x) \neq 0$ in (4.10) do not satisfy the multiplicative property of π^*, e.g., $\bar{C}_{\bar{\pi}} = [0,a] \times [0,a]$ with $a > 0$, and need special treatment.

Hence, we obtain:

$$\left(\frac{1}{1-p}\Sigma^{-1}(\mu+\nu^*-r\cdot\mathbf{1}_2)\right)^\top\begin{pmatrix}0\\1\end{pmatrix}<0$$

$$\overset{1-p>0}{\underset{\sigma_1^2\sigma_2^2(1-\rho^2)>0}{\Longleftrightarrow}}\ \sigma_1^2(\mu_2+\nu_2^*-r)-\sigma_1\sigma_2\rho(\mu_1+\nu_1^*-r)<0$$

$$\Longleftrightarrow\ \frac{\mu_2+\nu_2^*-r}{\sigma_2}<\rho\cdot\frac{\mu_1+\nu_1^*-r}{\sigma_1}\ \Longleftrightarrow\ SR_2^{\nu^*}<\rho\cdot SR_1^{\nu^*}.$$

□

B.2 Proofs of auxiliary results

This appendix contains the proofs of auxiliary lemmas and propositions needed for Sections 4.2 and 4.3.

Lemma B.2.1 (Delta-hedging in $\mathcal{M}_\nu$)**.** *Fix a dual vector $\nu\in C_\pi$ and consider in $\mathcal{M}_\nu$ a financial derivative with payoff $D(V_\nu^{v_D,\pi}(T))$, where $D:[0,\infty)\to[0,\infty)$, $v_D>0$ is the initial wealth of $V_\nu^{v_D,\pi}$ and $\pi\in\mathbb{R}^2$ is a constant-mix strategy. Denote by*

$$D_\nu(t,V):=\exp\left(-r(T-t)\right)\mathbb{E}^{\widetilde{\mathbb{Q}}_\nu}[D(V_\nu^{v_D,\pi}(T))|\ V_\nu^{v_D,\pi}(t)=V]$$

the time-t value of $D(V_\nu^{v_D,\pi}(T))$, provided that $V_\nu^{v_D,\pi}(t)=V$. Furthermore, assume that $D_\nu(t,V)$ is once continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. V. Then $D(V_\nu^{v_D,\pi}(T))$ can be attained by trading in $\mathcal{M}_\nu$ according to the relative portfolio process

$$\pi_\nu^*(t):=\pi_\nu^*(t,V_\nu^{v_D,\pi}(t))=\underbrace{\frac{\frac{d}{dV}D_\nu(t,V_\nu^{v_D,\pi}(t))\cdot V_\nu^{v_D,\pi}(t)}{D_\nu(t,V_\nu^{v_D,\pi}(t))}}_{=:\beta_\nu^D(t,V_\nu^{v_D,\pi}(t))}\cdot\pi=\beta_\nu^D(t,V_\nu^{v_D,\pi}(t))\cdot\pi$$

with the necessary initial wealth satisfying

$$v_0=D_\nu(0,v_D).$$

Proof. Since π is a constant-mix strategy, the corresponding wealth process $V_\nu^{v_D,\pi}$ in $\mathcal{M}_\nu$ has the dynamics of a geometric Brownian motion:

$$\begin{aligned}\frac{dV_\nu^{v_D,\pi}(t)}{V_\nu^{v_D,\pi}(t)} &= [r + (\mu + \nu - r \cdot \mathbf{1}_2)^\top \pi]dt + \pi^\top \sigma dW^{\mathbb{Q}}(t) \\ &= rdt + \pi^\top \sigma \underbrace{(dW^{\mathbb{Q}}(t) + [\gamma + \sigma^{-1}\nu]dt)}_{=:dW^{\widetilde{\mathbb{Q}}_\nu}(t)}, \qquad V_\nu^{v_D,\pi}(0) = \quad v_D,\end{aligned}$$

where $W^{\widetilde{\mathbb{Q}}_\nu}$ is a Wiener process with respect to $\widetilde{\mathbb{Q}}_\nu$.

Since $D_\nu \in C^{(1,2)}([0,T) \times [0,\infty))$ is once continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. V, we can use Itô's formula to determine the hedging portfolio for $D(V_\nu^{v_D,\pi}(T))$. Furthermore, we get by applying Feynman-Kac Theorem 2.2.8 that $D_\nu(t,V)$ satisfies the following PDE:

$$\begin{aligned}0 &= \frac{d}{dt}D_\nu + \frac{1}{2}\|\sigma^\top\pi\|^2 \cdot V^2 \cdot \frac{d^2}{d^2V}D_\nu + r \cdot V \cdot \frac{d}{dV}D_\nu - r \cdot D_\nu; \\ D(V) &= D_\nu(T,V).\end{aligned} \tag{B.12}$$

Let π_ν^* be the portfolio process that replicates the payoff $D(V_\nu^{v_D,\pi}(T))$ in $\mathcal{M}_\nu$. The existence of π_ν^* is guaranteed by the market completeness of $\mathcal{M}_\nu$. Then, for no-arbitrage reasons, $V_\nu^{v_0,\pi_\nu^*}(t) \overset{!}{=} D_\nu(t, V_\nu^{v_D,\pi}(t))$ $\mathcal{L}[0,T) \otimes \mathbb{Q} - a.e.$. In particular, v_0 is determined through

$$v_0 = D_\nu(0, V_\nu^{v_D,\pi}(0)) = D_\nu(0, v_D).$$

Further, $V_\nu^{v_0,\pi_\nu^*}(t)$ and $D_\nu(t, V_\nu^{v_D,\pi}(t))$ follow the dynamics

$$\begin{aligned}dV_\nu^{v_0,\pi_\nu^*}(t) &= V_\nu^{v_0,\pi_\nu^*}(t) \cdot [rdt + \pi_\nu^*(t)^\top \sigma dW^{\widetilde{\mathbb{Q}}_\nu}(t)] \\ &= D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot [rdt + \pi_\nu^*(t)^\top \sigma dW^{\widetilde{\mathbb{Q}}_\nu}(t)]\end{aligned}$$

and

$$\begin{aligned}dD_\nu(t, V_\nu^{v_D,\pi}(t)) &\overset{\text{Itô}}{=} \frac{d}{dt}D_\nu(t, V_\nu^{v_D,\pi}(t))dt + \frac{d}{dV}D_\nu(t, V_\nu^{v_D,\pi}(t))dV_\nu^{v_D,\pi}(t) \\ &\quad + \frac{1}{2}\frac{d^2}{d^2V}D_\nu(t, V_\nu^{v_D,\pi}(t))d\langle V_\nu^{v_D,\pi}, V_\nu^{v_D,\pi}\rangle(t) \\ &= \frac{d}{dt}D_\nu(t, V_\nu^{v_D,\pi}(t))dt + r \cdot V_\nu^{v_D,\pi}(t) \cdot \frac{d}{dV}D_\nu(t, V_\nu^{v_D,\pi}(t))dt \\ &\quad + \frac{d}{dV}D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot V_\nu^{v_D,\pi}(t) \cdot (\pi)^\top \sigma dW^{\widetilde{\mathbb{Q}}_\nu}(t) \\ &\quad + \frac{1}{2}\|\sigma^\top\pi\|^2 \cdot (V_\nu^{v_D,\pi}(t))^2 \cdot \frac{d^2}{d^2V}D_\nu(t, V_\nu^{v_D,\pi}(t))dt \\ &\overset{(B.12)}{=} r \cdot D_\nu(t, V_\nu^{v_D,\pi}(t))dt + \frac{d}{dV}D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot V_\nu^{v_D,\pi}(t) \cdot (\pi)^\top \sigma dW^{\widetilde{\mathbb{Q}}_\nu}(t).\end{aligned}$$

Matching the diffusion coefficients provides the condition

$$D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot \sigma^\top \pi_\nu^*(t) \overset{!}{=} \frac{d}{dV} D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot V_\nu^{v_D,\pi}(t) \cdot \sigma^\top \pi \quad \mathcal{L}[0,T] \otimes \mathbb{Q} - a.e.$$

$$\Leftrightarrow \pi_\nu^*(t) \overset{!}{=} \underbrace{\frac{\frac{d}{dV} D_\nu(t, V_\nu^{v_D,\pi}(t)) \cdot V_\nu^{v_D,\pi}(t)}{D_\nu(t, V_\nu^{v_D,\pi}(t))}}_{=:\ \beta_\nu^D(t, V_\nu^{v_D,\pi}(t))} \cdot \pi = \beta_\nu^D(t, V_\nu^{v_D,\pi}(t)) \cdot \pi \quad \mathcal{L}[0,T] \otimes \mathbb{Q} - a.e..$$

□

Lemma B.2.2. *Let $p \in \mathbb{R}$, $-\infty \leq l \leq u \leq +\infty$, $X \overset{d}{=} N(0,1)$. Then:*

$$\mathbb{E}^{\mathbb{Q}}\left[e^{pX} \mathbb{1}_{(l,u]}(X)\right] = \exp\left(\frac{p^2}{2}\right)(\Phi(u-p) - \Phi(l-p)).$$

In particular, the moment generating function of X is given by:

$$f_X(p) := \mathbb{E}^{\mathbb{Q}}\left[e^{pX}\right] = \exp\left(\frac{1}{2}p^2\right). \tag{B.13}$$

Proof.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[e^{pX} \mathbb{1}_{(l,u]}(X)\right] &= \int_l^u e^{px} \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}}\, dx \\
&= \int_l^u e^{px} \exp\left(-\frac{1}{2}(x^2 - 2px + p^2 - p^2)\right) \frac{1}{\sqrt{2\pi}}\, dx \\
&= \exp\left(\frac{p^2}{2}\right) \int_l^u \exp\left(-\frac{1}{2}(x-p)^2\right) \frac{1}{\sqrt{2\pi}}\, dx \\
&= \exp\left(\frac{p^2}{2}\right) \int_{l-p}^{u-p} \exp\left(-\frac{y^2}{2}\right) \frac{1}{\sqrt{2\pi}}\, dy \\
&= \exp\left(p^2/2\right)(\Phi(u-p) - \Phi(l-p)).
\end{aligned}$$

The claim about the moment generating function follows, since:

$$\lim_{l \downarrow -\infty} \Phi(l-p) = 0, \ \lim_{u \uparrow +\infty} \Phi(u-p) = 1.$$

□

Lemma B.2.3. *Let $0 < l \leq u < +\infty$. Then:*

$$\begin{aligned}
& l < \quad V^{v_{D_{\nu^*}}, \pi^*_{u,\nu^*}}(T) \leq \quad u \\
\Longleftrightarrow\ & -d_2^{\nu^*}(l, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T} < \quad \frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}} \leq \quad -d_2^{\nu^*}(u, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}.
\end{aligned}$$

Proof. Using (B.9), we get:

$$
\begin{aligned}
& l < v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right) \left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}} \leq u \\
\overset{(4.14)}{\Longleftrightarrow}\ & l < v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right) \left(\exp\left(-\left(r+0.5\|\gamma_{\nu^*}\|^2\right)T - \gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)\right)\right)^{\frac{1}{p-1}} \leq u \\
\overset{\frac{v_{D_{\nu^*}}}{\exp(\Gamma_{\nu^*}(0))}>0}{\Longleftrightarrow}\ & \frac{l\exp\left(\Gamma_{\nu^*}(0)\right)}{v_{D_{\nu^*}}} < \exp\left(\frac{1}{1-p}\left(r+0.5\|\gamma_{\nu^*}\|^2\right)T + \frac{1}{1-p}\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)\right) \\
& \leq \frac{u\exp\left(\Gamma_{\nu^*}(0)\right)}{v_{D_{\nu^*}}} \\
\overset{\ln(\cdot)\uparrow}{\Longleftrightarrow}\ & \ln\left(\frac{l\exp\left(\Gamma_{\nu^*}(0)\right)}{v_{D_{\nu^*}}}\right) - \frac{\left(r+0.5\|\gamma_{\nu^*}\|^2\right)T}{1-p} < \frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{1-p} \\
& \leq \ln\left(\frac{u\exp\left(\Gamma_{\nu^*}(0)\right)}{v_{D_{\nu^*}}}\right) - \frac{\left(r+0.5\|\gamma_{\nu^*}\|^2\right)T}{1-p} \\
\overset{\frac{\|\gamma_{\nu^*}\|\sqrt{T}}{1-p}>0}{\Longleftrightarrow}\ & \frac{(1-p)\ln\left(\frac{l\exp(\Gamma_{\nu^*}(0))}{v_{D_{\nu^*}}}\right) - \left(r+0.5\|\gamma_{\nu^*}\|^2\right)T}{\|\gamma_{\nu^*}\|\sqrt{T}} < \frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{\|\gamma_{\nu^*}\|\sqrt{T}} \\
& \leq \frac{(1-p)\ln\left(\frac{u\exp(\Gamma_{\nu^*}(0))}{v_{D_{\nu^*}}}\right) - \left(r+0.5\|\gamma_{\nu^*}\|^2\right)T}{\|\gamma_{\nu^*}\|\sqrt{T}} \\
\overset{(4.21)}{\Longleftrightarrow}\ & -d_2^{\nu^*}(l, v_{D_{\nu^*}}, 0) - \|\gamma_{\nu^*}\|\sqrt{T} < \frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{\|\gamma_{\nu^*}\|\sqrt{T}} \leq -d_2^{\nu^*}(u, v_{D_{\nu^*}}, 0) - \|\gamma_{\nu^*}\|\sqrt{T}.
\end{aligned}
$$

□

Proposition B.2.4 (Explicit form of equations for calculation of $v_{D_{\nu^*}}, k_{\nu^*}^{\varepsilon}$)**.**
The explicit form of the budget constraint in (4.23) *is given by:*

$$
\begin{aligned}
& v_{D_{\nu^*}} \cdot \left(1 + \Phi(d_1^{\nu^*}(G_T, v_{D_{\nu^*}}, 0)) - \Phi(d_1^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0))\right) \\
& + \exp\left(-rT\right) G_T \left(\Phi(d_2^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)) - \Phi(d_2^{\nu^*}(G_T, v_{D_{\nu^*}}, 0))\right) - v_0 = 0;
\end{aligned}
$$

The explicit form of the probability constraint in (4.23) *is given by:*

$$
\Phi(d_2^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)) + \varepsilon - 1 = 0.
$$

Proof. First, we simplify the budget constraint:

$$
\begin{aligned}
&\exp(-rT)\,\mathbb{E}^{\widetilde{\mathbb{Q}}_{\nu^*}}[f(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))] \overset{\partial\widetilde{\mathbb{Q}}_{\nu^*}/\partial\mathbb{Q}}{=} \mathbb{E}^{\mathbb{Q}}[\widetilde{Z}_{\nu^*}(T)f(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))]\\
&\overset{f\text{ def.}}{=} \mathbb{E}^{\mathbb{Q}}\Big[\widetilde{Z}_{\nu^*}(T)\Big(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T) + \big(G_T - V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\big)\\
&\qquad \cdot \mathbb{1}_{[k^\varepsilon_{\nu^*},G_T]}(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))\Big)\Big]\\
&\overset{(B.9)}{=} \mathbb{E}^{\mathbb{Q}}\Bigg[\widetilde{Z}_{\nu^*}(T)v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\mathbb{1}_{(0,k^\varepsilon_{\nu^*})}\Bigg(v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\\
&\qquad \cdot\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\Bigg)\Bigg] + \mathbb{E}^{\mathbb{Q}}\Bigg[\widetilde{Z}_{\nu^*}(T)G_T\mathbb{1}_{[k^\varepsilon_{\nu^*},G_T]}\Bigg(v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\\
&\qquad \cdot\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\Bigg)\Bigg] + \mathbb{E}^{\mathbb{Q}}\Bigg[\widetilde{Z}_{\nu^*}(T)v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\\
&\qquad \cdot\mathbb{1}_{(G_T,+\infty)}\Bigg(v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\Bigg)\Bigg] =: E_1 + E_2 + E_3.
\end{aligned}
$$

Take $0 < l < k^\varepsilon_{\nu^*}$ and calculate:

$$
\begin{aligned}
E_1(l) &= \mathbb{E}^{\mathbb{Q}}\Bigg[\widetilde{Z}_{\nu^*}(T)v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\\
&\qquad \cdot\mathbb{1}_{(l,k^\varepsilon_{\nu^*})}\Bigg(v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\Bigg)\Bigg]\\
&\overset{(4.14)}{=} v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\,\mathbb{E}^{\mathbb{Q}}\Bigg[\Big(\exp\Big(-\left(r+0.5||\gamma_{\nu^*}||^2\right)T - \gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)\Big)\Big)^{\frac{p}{p-1}}\\
&\qquad \cdot\mathbb{1}_{(l,k^\varepsilon_{\nu^*})}\Bigg(v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\Bigg)\Bigg]\\
&\overset{\text{Lem. }B.2.3}{=} v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right)\\
&\qquad \cdot\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\\
&\qquad \cdot\mathbb{1}_{(-d_2^{\nu^*}(l,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T},-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}]}\left(\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\Bigg]\\
&\overset{(i)}{=} v_{D_{\nu^*}}\exp(-\Gamma_{\nu^*}(0))\exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right)\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}X\right)\\
&\qquad \cdot\mathbb{1}_{(-d_2^{\nu^*}(l,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T},-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}]}(X)\Bigg]
\end{aligned}
$$

$$\overset{(ii)}{=} v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right) \exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right)$$
$$\cdot \exp\left(\frac{1}{2}\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)^2\right)\left(\Phi\left(-d_2^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}\right.\right.$$
$$\left.\left. - \frac{p}{p-1}||\gamma_{\nu^*}||\sqrt{T}\right) - \Phi\left(-d_2^{\nu^*}(l, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T} - \frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)\right)$$
$$\overset{(iii)}{=} v_{D_{\nu^*}} \cdot \left(1 - \Phi\left(d_2^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0) + \frac{1}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)\right.$$
$$\left. - \left(1 - \Phi\left(d_2^{\nu^*}(l, v_{D_{\nu^*}}, 0) + \frac{1}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)\right)\right)$$
$$\overset{(4.21)}{=} v_{D_{\nu^*}} \cdot \left(\Phi\left(d_1^{\nu^*}(l, v_{D_{\nu^*}}, 0)\right) - \Phi\left(d_1^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)\right)\right),$$

where we use in (i) $X \overset{d}{=} N(0,1) \overset{d}{=} \frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}$, in (ii) Lemma B.2.2, and in (iii) $\Phi(-x) = 1 - \Phi(x)$ as well as (4.21). Hence, we get:

$$E_1 = \lim_{l \downarrow 0} E_1(l) = \lim_{l \downarrow 0}\left(v_{D_{\nu^*}}\left(\Phi\left(d_1^{\nu^*}(l, v_{D_{\nu^*}}, 0)\right) - \Phi\left(d_1^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)\right)\right)\right)$$
$$\overset{\Phi \text{ is cts.}}{=} v_{D_{\nu^*}}\left(\Phi\left(\lim_{l \downarrow 0} d_1^{\nu^*}(l, v_{D_{\nu^*}}, 0)\right) - \Phi\left(d_1^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)\right)\right)$$
$$\underset{p<1}{\overset{(4.21)}{=}} v_{D_{\nu^*}}\left(1 - \Phi\left(d_1^{\nu^*}(k_{\nu^*}^{\varepsilon}, v_{D_{\nu^*}}, 0)\right)\right),$$

where we also used in the last equality $\lim_{l \downarrow 0} \ln(l) = -\infty$ and $\lim_{u \uparrow +\infty} \Phi(u) = 1$.

Replacing 0 by G_T and $k_{\nu^*}^{\varepsilon}$ by u and considering $\lim_{u \uparrow +\infty}$ we obtain:

$$E_3 = \lim_{u \uparrow +\infty} E_3(u) = v_{D_{\nu^*}} \cdot \left(\Phi\left(d_1^{\nu^*}(G_T, v_{D_{\nu^*}}, 0)\right) - \Phi\left(\lim_{u \uparrow +\infty} d_1^{\nu^*}(u, v_{D_{\nu^*}}, 0)\right)\right)$$
$$\underset{p<1}{\overset{(4.21)}{=}} v_{D_{\nu^*}} \cdot \Phi\left(d_1^{\nu^*}(G_T, v_{D_{\nu^*}}, 0)\right),$$

where we also used in the last equality $\lim_{u \uparrow +\infty} \ln(u) = +\infty$ and $\lim_{l \downarrow 0} \Phi(l) = 0$.

Next we calculate E_2:

$$E_2 = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_{\nu^*}(T) G_T \mathbb{1}_{[k_{\nu^*}^{\varepsilon}, G_T]}\left(v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)\right]$$
$$\overset{(4.14)}{=} G_T \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\left(r+0.5||\gamma_{\nu^*}||^2\right)T - \gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)\right)\right.$$
$$\left. \cdot \mathbb{1}_{[k_{\nu^*}^{\varepsilon}, G_T]}\left(v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)\right]$$

$$
\begin{aligned}
&\overset{\text{Lem. } B.2.3}{=} G_T \exp\left(-\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right)\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(-||\gamma_{\nu^*}||\sqrt{T}\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\\
&\qquad \cdot \mathbb{1}_{(-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T},-d_2^{\nu^*}(G_T,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}]}\left(\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\Bigg]\\
&\overset{\text{Lem. } B.2.2}{=} G_T \exp\left(-\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right)\exp\left(0.5||\gamma_{\nu^*}||^2T\right)\\
&\qquad \cdot\Bigg(\Phi\left(-d_2^{\nu^*}(G_T,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}-\left(-||\gamma_{\nu^*}||\sqrt{T}\right)\right)\\
&\qquad -\Phi\left(-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}-\left(-||\gamma_{\nu^*}||\sqrt{T}\right)\right)\Bigg)\\
&\overset{\Phi(-x)=1-\Phi(x)}{=} \exp(-rT)\,G_T\left(\Phi\left(d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)\right)-\Phi\left(d_2^{\nu^*}(G_T,v_{D_{\nu^*}},0)\right)\right),
\end{aligned}
$$

where we used in the fourth equality $X \overset{d}{=} N(0,1) \overset{d}{=} \frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}$.

Finally, we obtain the explicit form of the left-hand side of the budget constraint:

$$
\begin{aligned}
E_1+E_2+E_3 = v_{D_{\nu^*}}&\cdot\left(1-\Phi(d_1^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0))+\Phi(d_1^{\nu^*}(G_T,v_{D_{\nu^*}},0))\right)\\
&+\exp(-rT)\,G_T\cdot\left(\Phi\left(d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)\right)-\Phi\left(d_2^{\nu^*}(G_T,v_{D_{\nu^*}},0)\right)\right).
\end{aligned}
$$

Second, we simplify the left-hand side of the VaR-constraint:

$$
\begin{aligned}
&\mathbb{Q}\left(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))<G_T\right)\\
&\overset{f\,\text{def.}}{=}\mathbb{Q}\left(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)+\left(G_T-V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\mathbb{1}_{[k^\varepsilon_{\nu^*},G_T]}(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))<G_T\right)\\
&=\mathbb{Q}\left(V^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)<k^\varepsilon_{\nu^*}\right)\\
&\overset{(i)}{=}\mathbb{Q}\left(\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}<-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}\right)\\
&\overset{(ii)}{=}\Phi\left(-d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}\right)\\
&\overset{(iii)}{=}1-\Phi\left(d_2^{\nu^*}(k^\varepsilon_{\nu^*},v_{D_{\nu^*}},0)+||\gamma_{\nu^*}||\sqrt{T}\right),
\end{aligned}
$$

where we use in (i) Lemma B.2.3, in (ii) $\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}} \overset{d}{=} N(0,1)$, in (iii) $\Phi(-x)=1-\Phi(x)$.

The claim of the proposition follows. □

Remark to Proposition B.2.4: As argued in the proof of Proposition 4.2.6, the optimal portfolio for (P_{ε,C_π}) is also the optimal portfolio for $(P_\varepsilon^{\nu^*})$. Moreover, both corresponding wealth processes coincide according to (4.16) and, therefore, the present value of the optimal terminal payoff coincides in both $\mathcal{M}$ and $\mathcal{M}_{\nu^*}$. This means, in Equation (4.23) we can use either the budget equation in $\mathcal{M}$ or in $\mathcal{M}_{\nu^*}$ to determine the parameters $(v_{D_{\nu^*}}, k^{\varepsilon}_{\nu^*})$.

Proposition B.2.5. *The value function is given by:*

$$\mathbb{E}^{\mathbb{Q}}\big[U(\bar{V}^{v_0,\bar{\pi}^*}(T))\big] = \frac{1}{p}\left(v_{D_{\nu^*}}\right)^p \exp\left(\Gamma_{\nu^*}(0)(1-p)\right)$$
$$\cdot\Big(1-\Phi(d_1^{\nu^*}(k^{\varepsilon}, v_{D_{\nu^*}},0))+\Phi(d_1^{\nu^*}(G_T, v_{D_{\nu^*}},0))\Big)$$
$$+\frac{1}{p}\left(G_T\right)^p\left(\Phi\left(d_2^{\nu^*}(k^{\varepsilon}_{\nu^*}, v_{D_{\nu^*}},0)+\|\gamma_{\nu^*}\|\sqrt{T}\right)-\Phi\left(d_2^{\nu^*}(G_T, v_{D_{\nu^*}},0)+\|\gamma_{\nu^*}\|\sqrt{T}\right)\right).$$

Proof.

$$\mathbb{E}\big[U\left(\bar{V}^{v_0,\bar{\pi}^*}(T)\right)\big] \overset{\text{Pr.4.2.1}}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(\bar{V}^{v_0,\pi^*}(T)\right)\right] \overset{\text{Pr.4.2.6}}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(D_{\nu^*}(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\right]$$
$$\overset{f\text{ def.}}{=} \mathbb{E}^{\mathbb{Q}}\left[U\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)+\left(G_T-V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\mathbb{1}_{[k^{\varepsilon}_{\nu^*},G_T]}(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T))\right)\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[U\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\mathbb{1}_{(0,k^{\varepsilon}_{\nu^*})}\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\right]$$
$$+\mathbb{E}^{\mathbb{Q}}\left[U\left(G_T\right)\mathbb{1}_{[k^{\varepsilon}_{\nu^*},G_T]}\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\right]$$
$$+\mathbb{E}^{\mathbb{Q}}\left[U\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\mathbb{1}_{(G_T,+\infty)}\left(V_{\nu^*}^{v_{D_{\nu^*}},\pi^*_{u,\nu^*}}(T)\right)\right] =: E_1+E_2+E_3.$$

Take $0<l<k^{\varepsilon}$ and calculate:

$$E_1(l)=\mathbb{E}^{\mathbb{Q}}\Bigg[\frac{1}{p}\left(v_{D_{\nu^*}}\exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)^p$$
$$\cdot\mathbb{1}_{(l,k^{\varepsilon}_{\nu^*})}\left(v_{D_{\nu^*}}\exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)\Bigg]$$
$$\overset{(4.14)}{=}\frac{v^p_{D_{\nu^*}}}{p}\exp\left(-p\Gamma_{\nu^*}(0)\right)\mathbb{E}^{\mathbb{Q}}\Bigg[\left(\exp\left(-\left(r+0.5||\gamma_{\nu^*}||^2\right)T-\gamma_{\nu^*}^{\top}W^{\mathbb{Q}}(T)\right)\right)^{\frac{p}{p-1}}$$
$$\cdot\mathbb{1}_{(l,k^{\varepsilon}_{\nu^*})}\left(v_{D_{\nu^*}}\exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)\Bigg]$$

$$
\begin{aligned}
&\overset{\text{Lem. B.2.3}}{=} \frac{v_{D_{\nu^*}}^p}{p} \exp\left(-p\Gamma_{\nu^*}(0)\right) \exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right) \\
&\quad \cdot \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right) \\
&\quad \cdot \mathbb{1}_{(-d_2^{\nu^*}(l,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T},-d_2^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}]}\left(\frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\Bigg] \\
&\overset{(i)}{=} \frac{v_{D_{\nu^*}}^p}{p} \exp\left(-p\Gamma_{\nu^*}(0)\right) \exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right) \\
&\quad \cdot \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}X\right)\mathbb{1}_{(-d_2^{\nu^*}(l,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T},-d_2^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}]}(X)\right] \\
&\overset{\text{Lem. B.2.2}}{=} \frac{v_{D_{\nu^*}}^p}{p} \exp\left(-p\Gamma_{\nu^*}(0)\right) \exp\left(\frac{p}{1-p}\left(r+0.5||\gamma_{\nu^*}||^2\right)T\right) \\
&\quad \cdot \exp\left(\frac{1}{2}\left(\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)^2\right)\Bigg(\Phi\bigg(-d_2^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T} \\
&\quad -\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\bigg)-\Phi\left(-d_2^{\nu^*}(l,v_{D_{\nu^*}},0)-||\gamma_{\nu^*}||\sqrt{T}-\frac{p}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)\Bigg) \\
&\overset{(ii)}{=} \frac{v_{D_{\nu^*}}^p}{p} \exp\left((1-p)\Gamma_{\nu^*}(0)\right)\cdot\Bigg(1-\Phi\left(d_2^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)+\frac{1}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right) \\
&\qquad -\left(1-\Phi\left(d_2^{\nu^*}(l,v_{D_{\nu^*}},0)+\frac{1}{1-p}||\gamma_{\nu^*}||\sqrt{T}\right)\right)\Bigg) \\
&\overset{(4.21)}{=} \frac{v_{D_{\nu^*}}^p}{p} \exp\left((1-p)\Gamma_{\nu^*}(0)\right)\cdot\left(\Phi\left(d_1^{\nu^*}(l,v_{D_{\nu^*}},0)\right)-\Phi\left(d_1^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)\right)\right),
\end{aligned}
$$

where we use in (i) that $X \overset{d}{=} N(0,1) \overset{d}{=} \frac{\gamma_{\nu^*}^\top W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}$, in (ii) $\Phi(-x) = 1-\Phi(x)$ as well as (4.21). Hence, we get:

$$
\begin{aligned}
E_1 &= \lim_{l\downarrow 0} E_1(l) \\
&= \lim_{l\downarrow 0}\left(\frac{v_{D_{\nu^*}}^p}{p}\exp\left((1-p)\Gamma_{\nu^*}(0)\right)\left(\Phi\left(d_1^{\nu^*}(l,v_{D_{\nu^*}},0)\right)-\Phi\left(d_1^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)\right)\right)\right) \\
&\overset{\Phi\text{ is cts.}}{=} \frac{v_{D_{\nu^*}}^p}{p}\exp\left((1-p)\Gamma_{\nu^*}(0)\right)\left(\Phi\left(\lim_{l\downarrow 0} d_1^{\nu^*}(l,v_{D_{\nu^*}},0)\right)-\Phi\left(d_1^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)\right)\right) \\
&\underset{p<1}{\overset{(4.21)}{=}} \frac{v_{D_{\nu^*}}^p}{p}\exp\left((1-p)\Gamma_{\nu^*}(0)\right)\left(1-\Phi\left(d_1^{\nu^*}(k_{\nu^*}^\varepsilon,v_{D_{\nu^*}},0)\right)\right),
\end{aligned}
$$

where we also used in the last equality $\lim_{l\downarrow 0}\ln(l) = -\infty$ and $\lim_{u\uparrow+\infty}\Phi(u) = 1$.

Replacing 0 by G_T and $k^{\varepsilon}_{\nu^*}$ by u and considering $\lim_{u\uparrow+\infty}$ we obtain:

$$\begin{aligned}
E_3 &= \lim_{u\uparrow+\infty} E_3(u) = \frac{v^p_{D_{\nu^*}}}{p} \exp\left((1-p)\Gamma_{\nu^*}(0)\right) \\
&\quad \cdot \left(\Phi\left(d_1^{\nu^*}(G_T, v_{D_{\nu^*}}, 0)\right) - \Phi\left(\lim_{u\uparrow+\infty} d_1^{\nu^*}(u, v_{D_{\nu^*}}, 0)\right)\right) \\
&\overset{(4.21)}{\underset{p<1}{=}} \frac{v^p_{D_{\nu^*}}}{p} \exp\left((1-p)\Gamma_{\nu^*}(0)\right) \cdot \Phi\left(d_1^{\nu^*}(G_T, v_{D_{\nu^*}}, 0)\right),
\end{aligned}$$

where we also used in the last equality $\lim_{u\uparrow+\infty} \ln(u) = +\infty$ and $\lim_{l\downarrow 0} \Phi(l) = 0$.

Next we calculate E_2:

$$\begin{aligned}
E_2 &= \mathbb{E}^{\mathbb{Q}}\left[\frac{(G_T)^p}{p} \mathbb{1}_{[k^{\varepsilon}_{\nu^*}, G_T]}\left(v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right)\left(\widetilde{Z}_{\nu^*}(T)\right)^{\frac{1}{p-1}}\right)\right] \\
&\overset{(4.14)}{=} \frac{(G_T)^p}{p} \mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{1}_{[k^{\varepsilon}_{\nu^*}, G_T]}\Bigg(v_{D_{\nu^*}} \exp\left(-\Gamma_{\nu^*}(0)\right)\Bigg(\exp\Bigg(-\left(r + 0.5||\gamma_{\nu^*}||^2\right) T \\
&\qquad - \gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)\Bigg)\Bigg)^{\frac{1}{p-1}}\Bigg)\Bigg] \\
&\overset{(i)}{=} \frac{(G_T)^p}{p} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{(-d_2^{\nu^*}(k^{\varepsilon}_{\nu^*}, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}, -d_2^{\nu^*}(G_T, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}]}\left(\frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}}\right)\right] \\
&\overset{(ii)}{=} \frac{(G_T)^p}{p}\Bigg(\Phi\left(-d_2^{\nu^*}(G_T, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}\right) \\
&\qquad - \Phi\left(-d_2^{\nu^*}(k^{\varepsilon}_{\nu^*}, v_{D_{\nu^*}}, 0) - ||\gamma_{\nu^*}||\sqrt{T}\right)\Bigg) \\
&\overset{(iii)}{=} \frac{(G_T)^p}{p}\Bigg(\Phi\left(d_2^{\nu^*}(k^{\varepsilon}_{\nu^*}, v_{D_{\nu^*}}, 0) + \|\gamma_{\nu^*}\|\sqrt{T}\right) \\
&\qquad - \Phi\left(d_2^{\nu^*}(G_T, v_{D_{\nu^*}}, 0) + \|\gamma_{\nu^*}\|\sqrt{T}\right)\Bigg),
\end{aligned}$$

where we use in (i) Lemma B.2.3, in (ii) Lemma B.2.2 and $\frac{\gamma_{\nu^*}^{\top} W^{\mathbb{Q}}(T)}{||\gamma_{\nu^*}||\sqrt{T}} \sim N(0,1)$, in (iii) $\Phi(-x) = 1 - \Phi(x)$.

The claim of the proposition follows. □

C Appendix to Chapter 5

Here we provide proofs of theoretical results from Chapter 5. Appendix C.1 contains the proofs of theoretical results for general utility functions. In Appendix C.2 we provide the derivation of explicit formulas for power-utility functions.

C.1 Proofs for general utility functions

Proof of Proposition 5.2.1. The proof of this proposition is based on Corollary 3.4 and Theorem 4.1 in Desmettre and Seifried (2016). In $\mathcal{M}_\nu$, the reinsurance contract *Put* ("fixed-term investment" in Desmettre and Seifried (2016)) is spanned. Due to Assumption (5.4), we can apply Corollary 3.4 in Desmettre and Seifried (2016) and obtain ξ_ν^* ($\psi^\star$), $h_I(\cdot)$ ($\nu(\cdot)$), the optimal Lagrange multiplier λ^* ($\gamma^\star$), and the optimal terminal wealth $V_\nu^*(T)$ ($X_T^\star$), where we indicated in brackets the notation for the corresponding object in Desmettre and Seifried (2016).

Since also $\hat{I}_I(\cdot)$ and $d\hat{I}_I(\cdot)/d\lambda$ are polynomially bounded by the assumption of the proposition we prove, we can use Theorem 4.1 in Desmettre and Seifried (2016) and get the optimal relative portfolio process $\pi_\nu^*(t)$, denoted by $\pi_t^\star$, $t \in [0,T]$, in Desmettre and Seifried (2016). $\square$

Proof of Proposition 5.2.2. The proof of this proposition is based on the proof of Proposition 8.3 in Cvitanic and Karatzas (1992) and consists of two parts. First, we fix $\xi_I \in [0, \xi^{\max}]$ and prove that $\pi_{\nu^*}^* := \pi_{\nu^*}^*(\xi_I)$ is optimal for (P_I) given the fixed ξ_I. Second, we prove that $\xi_{\nu^*}^*$ from Proposition 5.2.1 is optimal for (P_I) given the optimal portfolio process $\pi_I^* \equiv \pi_{\nu^*}^*(\xi_I^*)$.

First part. Take an arbitrary but fixed $\xi_I \in [0, \xi^{\max}]$. For the initial wealth it holds that

$$V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(0) = V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(0). \tag{C.1}$$

Furthermore, let π_I be such that $(\pi_I, \xi_I) \in \Lambda_I$, i.e., it holds that $\pi_I(t) \in C_{\pi_I}$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. Then for all $\nu \in \mathcal{D}$ and $t \in [0,T]$ we have $\pi_I(t)^\top \nu(t) = 0$, whence

$$V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t) = V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(t) \geq 0 \quad \mathbb{Q} \text{ a.s.,} \tag{C.2}$$

where the equality follows from $\pi_I(t)^\top \nu(t) = 0$ and Equation (C.1). Using (C.2) for $t = T$, we obtain that

$$\begin{aligned}&\mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))^-]\\ &\qquad\qquad = \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))^-] < +\infty.\end{aligned}$$

Therefore, $(\pi_I, \xi_I) \in \Lambda_I^\nu$. It follows that $\Lambda_I \subset \Lambda_I^\nu$ and

$$\begin{aligned}&\sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))]\\ &\overset{(a)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))]\\ &\overset{(b)}{\leq} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^\nu} \mathbb{E}[U_I(V_\nu^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))], \qquad \text{(C.3)}\end{aligned}$$

where (a) follows from Equation (C.2), and (b) from $\Lambda_I \subset \Lambda_I^\nu$. Let $\nu^* \in \mathcal{D}$ and the optimal portfolio process $\pi_{\nu^*}^*$ of the unconstrained optimization problem of the insurer $(P_I^{\nu^*})$ given a fixed ξ_I be such that $(\pi_{\nu^*}^*, \xi_I) \in \Lambda_I^{\nu^*}$ and $\pi_{\nu^*}^*(t) \in C_{\pi_I}$ $\mathbb{Q}$-a.s. for all $t \in [0,T]$. Then for all $t \in [0,T]$ it holds

$$V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi_{\nu^*}^*}(t) \overset{\text{(C.2)}}{=} V_I^{v_{I,0}(\xi_I,\eta_R),\pi_{\nu^*}^*}(t). \qquad \text{(C.4)}$$

Hence, $(\pi_{\nu^*}^*, \xi_I) \in \Lambda_I$ and

$$\begin{aligned}&\sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^{\nu^*}} \mathbb{E}[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))]\\ &\overset{(a)}{=}\mathbb{E}[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi_{\nu^*}^*}(T) + \xi_I Put(T))] \qquad \text{(C.5)}\\ &\overset{(b)}{=}\mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_{\nu^*}^*}(T) + \xi_I Put(T))]\\ &\overset{(c)}{\leq} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))], \qquad \text{(C.6)}\end{aligned}$$

where (a) follows from the definition of $\pi_{\nu^*}^*$, (b) from Equation (C.4), and (c) from $(\pi_{\nu^*}^*, \xi_I) \in \Lambda_I$.
All in all, we have

$$\begin{aligned}&\mathbb{E}\Big[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi_{\nu^*}^*}(T) + \xi_I Put(T))\Big]\\ &\overset{(a)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I^{\nu^*}} \mathbb{E}\Big[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))\Big]\\ &\overset{(b)}{=} \sup_{\pi_I:(\pi_I,\xi_I)\in\Lambda_I} \mathbb{E}\Big[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi_I}(T) + \xi_I Put(T))\Big],\end{aligned}$$

where (a) follows from Equation (C.5) and (b) from Inequalities (C.3) and (C.6). Therefore, $\pi^*_{\nu^*}$ is optimal for the optimization problem of the insurer (P_I) given a fixed ξ_I.

Second part. Denote by $\xi^*_{\nu^*}$ the optimal reinsurance amount in the unconstrained optimization problem of the insurer $(P_I^{\nu^*})$ given $\pi^*_{\nu^*} = \pi^*_{\nu^*}(\xi^*_{\nu^*}) \in C_{\pi_I}$, i.e.:

$$\xi^*_{\nu^*} \in \underset{\xi_I \in [0,\xi^{\max}]}{\operatorname{argmax}} \, \mathbb{E}\Big[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi^*_{\nu^*}(\xi_I)}(T) + \xi_I Put(T))\Big]. \tag{C.7}$$

Observe that:

$$\{\xi_I |\ (\pi^*_{\nu^*}, \xi_I) \in \Lambda_I^{\nu^*}\} = [0, \xi^{\max}] = \{\xi_I |\ (\pi^*_{\nu^*}, \xi_I) \in \Lambda_I\}. \tag{C.8}$$

Then:

$$\begin{aligned}
\mathbb{E}\Big[U_I(V_I^{v_{I,0}(\xi^*_{\nu^*},\eta_R),\pi^*_{\nu^*}}(T) + \xi^*_{\nu^*} Put(T))\Big] &\overset{(a)}{=} \mathbb{E}\Big[U_I(V_{\nu^*}^{v_{I,0}(\xi^*_{\nu^*},\eta_R),\pi^*_{\nu^*}}(T) + \xi^*_{\nu^*} Put(T))\Big] \\
&\overset{(b)}{=} \sup_{\xi_I : (\pi^*_{\nu^*},\xi_I) \in \Lambda_I^{\nu^*}} \mathbb{E}\Big[U_I(V_{\nu^*}^{v_{I,0}(\xi_I,\eta_R),\pi^*_{\nu^*}}(T) + \xi_I Put(T))\Big] \\
&\overset{(c)}{=} \sup_{\xi_I : (\pi^*_{\nu^*},\xi_I) \in \Lambda_I} \mathbb{E}\Big[U_I(V_I^{v_{I,0}(\xi_I,\eta_R),\pi^*_{\nu^*}}(T) + \xi_I Put(T))\Big],
\end{aligned}$$

where we use in (a) Equation (C.4) for $t = T$, in (b) Equation (C.7), and in (c) Equations (C.2) and (C.8). Therefore, $\xi^*_{\nu^*}$ is optimal for the optimization problem of the insurer (P_I) given the portfolio process $\pi^*_{\nu^*}(\xi^*_{\nu^*}) \in C_{\pi_I}$.

Thus, we conclude that $(\pi^*_I, \xi^*_I) := (\pi^*_{\nu^*}, \xi^*_{\nu^*})$ is optimal for the optimization problem of the insurer (P_I). □

Proof of Lemma 5.2.3. Using Equation (2.14) and the fact that $V^{v_I,\pi_B}(t)$ is a geometric Brownian motion, we get the following price of the put option Put at time $t \in [0,T]$:

$$\begin{aligned}
Put(t) &= \widetilde{Z}(t)^{-1} \mathbb{E}[\widetilde{Z}(T) Put(T) | \mathcal{F}(t)] \\
&= \exp\left(-r(T-t)\right) G_T \Phi(-d_-) - V^{v_I,\pi_B}(t) \Phi(-d_+),
\end{aligned} \tag{C.9}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and

$$d_+ = d_1(t, V^{v_I,\pi_B}(t), G_T, r, \pi^{CM}\sigma_2), \quad d_- = d_2(t, V^{v_I,\pi_B}(t), G_T, r, \pi^{CM}\sigma_2),$$

where $d_1(\cdot)$ and $d_2(\cdot)$ are defined in (2.17) and (2.18) respectively.

Since the put-price function is continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. $V = V^{v_I,\pi_B}(t)$, we can apply delta-hedging Lemma B.2.1 for $\nu = (0,0)^\top$ and get that the relative portfolio process replicating the put option value is given by:

$$\pi^{Put}(t) = \frac{\frac{\partial Put(t,V^{v_I,\pi_B}(t))}{dV} V^{v_I,\pi_B}(t)}{Put(t,V^{v_I,\pi_B}(t))} \cdot \begin{pmatrix} 0 \\ \pi^{CM} \end{pmatrix}$$
$$\overset{(i)}{=} \begin{pmatrix} 0 \\ \frac{(\Phi(d_+)-1)\cdot V^{v_I,\pi_B}(t)\cdot \pi^{CM}}{Put(t,V^{v_I,\pi_B}(t))} \end{pmatrix}, t \in [0,T], \tag{C.10}$$

where we use in (i) the well-known formula of the delta of a put option $\partial Put(t,V)/\partial V = \Phi(d_+) - 1$.

The corresponding relative investment in the risk-free asset is given by:

$$\begin{aligned}\pi_0^{Put}(t) &= 1 - \pi^{Put}(t)^\top \mathbf{1}_2 \\ &= \frac{Put(t,V^{v_I,\pi_B}(t)) - (\Phi(d_+)-1)\cdot V^{v_I,\pi_B}(t)\cdot \pi^{CM}}{Put(t,V^{v_I,\pi_B}(t))}, t \in [0,T]. \end{aligned}\tag{C.11}$$

Applying Relation (2.6), we get:

$$\begin{aligned}\psi_0(t) &:= \varphi_0^{Put}(t) \overset{(2.6)}{=} \frac{\pi_0^{Put}(t)Put(t,V^{v_I,\pi_B}(t))}{S_0(t)} \\ &\overset{(C.11)}{=} \frac{Put(t,V^{v_I,\pi_B}(t)) - (\Phi(d_+)-1)\cdot V^{v_I,\pi_B}(t)\cdot \pi^{CM}}{S_0(t))}, \ t \in [0,T]; \\ \psi_1(t) &:= \varphi_1^{Put}(t) \overset{(2.6)}{=} \frac{\pi_1^{Put}(t)Put(t,V^{v_I,\pi_B}(t))}{S_1(t)} \overset{(C.10)}{=} 0, \ t \in [0,T]; \\ \psi_2(t) &:= \varphi_2^{Put}(t) \overset{(2.6)}{=} \frac{\pi_2^{Put}(t)Put(t,V^{v_I,\pi_B}(t))}{S_1(t)} \\ &\overset{(C.10)}{=} \frac{(\Phi(d_+)-1)\cdot V^{v_I,\pi_B}(t)\cdot \pi^{CM}}{Put(t,V^{v_I,\pi_B}(t))} \frac{Put(t,V^{v_I,\pi_B}(t))}{S_2(t)} \\ &= \frac{(\Phi(d_+)-1)\cdot V^{v_I,\pi_B}(t)\cdot \pi^{CM}}{S_2(t)}, \ t \in [0,T], \end{aligned}$$

which is exactly the replicating strategy in (5.6). □

Proof of Proposition 5.2.4. The proof is based on the proof of Theorem 4.1 in Korn and Trautmann (1999).

First, we define a new wealth process of the reinsurer with investment in the assets S_0, S_1 and S_2, and additionally in the put option Put. We denote by $\xi(t) \equiv -\xi_I^*(\eta_R)$ the trading strategy with respect to Put. The wealth process $V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}$ is given by

$$\begin{aligned} dV_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(t) =&\varphi_{R,0}(t)dS_0(t) + \varphi_{R,1}(t)dS_1(t) + \varphi_{R,2}(t)dS_2(t) \\ &+ \xi(t)dPut(t), \\ V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(0) =&v_R + \xi_I^*(\eta_R)\eta_R Put(0) =: \bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R). \end{aligned} \tag{C.12}$$

Note that $\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R)$ is not equal to $v_{R,0}(\xi_I^*(\eta_R),\eta_R)$:

$$\begin{aligned} \bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R) =&v_R + \xi_I^*(\eta_R)\eta_R Put(0) \\ =&v_R + \xi_I^*(\eta_R)(1+\eta_R)Put(0) - \xi_I^*(\eta_R)Put(0) \\ =&v_{R,0}(\xi_I^*(\eta_R),\eta_R) - \xi_I^*(\eta_R)Put(0). \end{aligned}$$

Since

$$\begin{aligned} V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(T) =&\varphi_{R,0}(T)S_0(T) + \varphi_{R,1}(T)S_1(T) + \varphi_{R,2}(T)S_2(T) \\ &+ \xi(T)Put(T) \end{aligned}$$

and the reinsurer has a short put position $-\xi_I^*(\eta_R)$, the optimization problem $(P_R^{\varphi_R|\eta_R})$ is equivalent to the optimization problem given by

$$\begin{aligned} \sup_{\varphi_R \in \Lambda_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)}} \quad & \mathbb{E}[U_R(V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(T))] \qquad (P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)}) \\ & \text{s.t. } \xi(t) \equiv -\xi_I^*(\eta_R) \ \forall t \in [0,T], \end{aligned}$$

where $\Lambda_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)}$ is the set of all admissible trading strategies φ_R for the optimization problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$:

$$\begin{aligned} \Lambda_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)} := \Bigg\{ & \varphi_R \in \mathbb{R}^3 \quad | \quad \varphi_R \text{ progressively measurable, self-financing,} \\ & \int_0^T |\varphi_0(t)|\, ds < +\infty, \int_0^T \varphi_i^2(t)\, ds < +\infty \ \mathbb{Q}\text{-a.s., } i \in \{1,2\}, \\ & V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(t) \geq 0 \ \mathbb{Q}\text{-a.s. } \forall t \in [0,T], \\ & \mathbb{E}[U_R(V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(T))^-] < +\infty \Bigg\}. \end{aligned}$$

Next we insert the replicating strategy (5.6) of Put to (C.12) and get:

$$\begin{aligned}V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(t) &= \varphi_{R,0}(t)S_0(t)+\varphi_{R,1}(t)S_1(t)+\varphi_{R,2}(t)S_2(t)+\xi(t)Put(t)\\ &\overset{(5.6)}{=} \varphi_{R,0}(t)S_0(t)+\varphi_{R,1}(t)S_1(t)+\varphi_{R,2}(t)S_2(t)+\xi(t)\psi_0(t)S_0(t)+\xi(t)\psi_2(t)S_2(t)\\ &= (\varphi_{R,0}(t)+\xi(t)\psi_0(t))S_0(t)+\varphi_{R,1}(t)S_1(t)+(\varphi_{R,2}(t)+\xi(t)\psi_2(t))S_2(t)\\ &=: \zeta_{R,0}(t)S_0(t)+\zeta_{R,1}(t)S_1(t)+\zeta_{R,2}(t)S_2(t),\end{aligned}$$

where

$$\begin{aligned}\zeta_R(t) &= (\zeta_{R,0}(t),\zeta_{R,1}(t),\zeta_{R,2}(t))^\top\\ &:= (\varphi_{R,0}(t)+\xi(t)\psi_0(t),\varphi_{R,1}(t),\varphi_{R,2}(t)+\xi(t)\psi_2(t))^\top\end{aligned} \tag{C.13}$$

is a self-financing trading strategy. Hence, the dynamics is given by

$$dV_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(t) = \zeta_{R,0}(t)dS_0(t)+\zeta_{R,1}(t)dS_1(t)+\zeta_{R,2}(t)dS_2(t).$$

The wealth process $V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}$ equals the wealth process of the reinsurer with respect to the trading strategy ζ_R (i.e., only an investment in the assets S_0, S_1 and S_2 without an investment in the put option Put). If the trading strategy φ_R is admissible for the optimization problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$, then the trading strategy ζ_R is admissible for the portfolio optimization problem $(P_R^{\zeta_R|\eta_R,\xi(t)=0})$:

$$V^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\zeta_R,0)}(t) = V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(t) \geq 0 \ \mathbb{Q}\text{-a.s. } \forall t\in[0,T]$$
$$\text{and}$$
$$\mathbb{E}[U_R(V^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\zeta_R,0)}(T))^-] = \mathbb{E}[U_R(V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R,\xi)}(T))^-] < +\infty.$$

Now we can use the standard martingale method, since the reinsurer has only an investment in the assets S_0, S_1 and S_2, and conclude that the optimal terminal wealth $V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\zeta_R^*,0)}$ for the problem $(P_R^{\zeta_R|\eta_R,\xi(t)=0})$ is given by

$$V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\zeta_R^*,0)}(T) = I_R(\lambda_R^*(\eta_R)\widetilde{Z}(T)),$$

where $\lambda_R^* \equiv \lambda_R^*(\eta_R)$ is determined by the budget constraint

$$\mathbb{E}[\widetilde{Z}(T)I_R(\lambda_R^*\widetilde{Z}(T))] = v_R + \xi_I^*(\eta_R)\eta_R Put(0), \tag{C.14}$$

and there exists the corresponding optimal trading strategy ζ_R^*. Therefore, there exists an optimal trading strategy φ_R^* for the optimization problem $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$ and the optimal wealth process $V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}$ is given by

$$V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}(T) = V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\zeta_R^*,0)}(T) = I_R(\lambda_R^*(\eta_R)\widetilde{Z}(T)).$$

From Relation (C.13), we get for the optimal trading strategy φ_R^* the following representation:

$$\varphi_{R,1}^*(t) = \zeta_{R,1}^*(t); \tag{C.15}$$
$$\varphi_{R,2}^*(t) = \zeta_{R,2}^*(t) - \xi(t)\psi_2(t) = \zeta_{R,2}^*(t) + \xi_I^*(\eta_R)\psi_2(t); \tag{C.16}$$
$$\begin{aligned}\varphi_{R,0}^*(t) =& \zeta_{0R}^*(t) - \xi(t)\psi_0(t)\\ =& \frac{V^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}(t) - \sum_{i=1}^2 \varphi_{R,i}^*(t)S_i(t) + \xi_I^*(\eta_R)Put(t)}{S_0(t)}.\end{aligned} \tag{C.17}$$

Since the optimization problems $(P_R^{\varphi_R|\eta_R})$ and $(P_R^{\varphi_R|\eta_R,\xi(t)=-\xi_I^*(\eta_R)})$ are equivalent, it holds that there exists an optimal trading strategy φ_R^* for the optimization problem $(P_R^{\varphi_R|\eta_R})$ given by (C.17), (C.15) and (C.16). The optimal terminal wealth of the reinsurer is given by

$$\begin{aligned}V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(T) &= V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}(T) + \xi_I^*(\eta_R)Put(T)\\ &= I_R(\lambda_R^*(\eta_R)\widetilde{Z}(T)) + \xi_I^*(\eta_R)Put(T)\end{aligned}$$

and the optimal wealth process by

$$\begin{aligned}V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(t) =& \widetilde{Z}(t)^{-1}\mathbb{E}\left[\widetilde{Z}(T)(V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}(T) + \xi_I^*(\eta_R)Put(T))|\mathcal{F}(t)\right]\\ =& V_R^{\bar{v}_{R,0}(\xi_I^*(\eta_R),\eta_R),(\varphi_R^*,\xi)}(t) + \xi_I^*(\eta_R)Put(t).\end{aligned}$$

Hence, it follows for the optimal trading strategy φ_R^* from (C.17), (C.15) and (C.16)

$$\begin{aligned}\varphi_{R,1}^*(t) =& \zeta_{R,1}^*(t);\\ \varphi_{R,2}^*(t) =& \zeta_{R,2}^*(t) + \xi_I^*(\eta_R)\psi_2(t);\\ \varphi_{R,0}^*(t) =& \frac{V^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(t) - \sum_{i=1}^2 \varphi_{R,i}^*(t)S_i(t)}{S_0(t)}.\end{aligned}$$

□

Proof of Proposition 5.2.5. From Proposition 5.2.4 we know the optimal strategy $\varphi_R^* = \varphi_R^*(\cdot|\eta_R)$ for Problem $(P_R^{\varphi_R|\eta_R})$ given an arbitrary but fixed $\eta_R \in [0, \eta^{\max}]$. Therefore, the original optimization problem reduces to the maximization of the help function $h_R(\cdot)$ defined by

$$\begin{aligned}h_R(\eta_R) :=& \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R),\eta_R),\varphi_R^*}(T) - \xi_I^*(\eta_R)Put(T))]\\ =& \mathbb{E}[U_R(I_R(\lambda_R^*(\xi_I^*(\eta_R),\eta_R)\widetilde{Z}(T)))],\end{aligned}$$

where $\lambda_R^* \equiv \lambda_R^*(\xi_I^*(\eta_R),\eta_R)$ is the Lagrange multiplier determined by Equation (C.14).

We show now that the maximum of this help function exists by showing the continuity of the map $\eta_R \mapsto h_R(\eta_R)$ and applying Theorem 2.5.3.

The first sub-step in proving the continuity is showing that $\xi_I^*(\eta_R)$ is continuous w.r.t. η_R. In the second sub-step, we address the continuity of $\lambda_R^*(\xi_I^*(\eta_R), \eta_R)$, and the continuity of $h_R(\eta_R)$ will follow.

According to Proposition 5.2.1, the map $\xi_I^*(\cdot)$ is given by

$$\xi_I^*(\eta_R) = \underset{\xi_I \in [0, \xi^{\max}(\eta_R)]}{\arg\max} \; h_I(\xi_I, \eta_R)$$

with $\xi^{\max}(\eta_R) = \min\{\bar{\xi}, \frac{v_I}{(1+\eta_R)Put(0)}\}$ and

$$h_I(\xi_I, \eta_R) := \mathbb{E}[U_I(\max\{I_I(\lambda_I^*(\xi_I, \eta_R)\widetilde{Z}_{\nu^*}(T)), \xi_I Put(T)\})],$$

where the Lagrange multiplier $\lambda_I^* \equiv \lambda_I^*(\xi_I, \eta_R)$ is given by the budget constraint of the insurer

$$\mathbb{E}[\widetilde{Z}_{\nu^*}(T)\hat{I}_I(\lambda_I^*\widetilde{Z}_{\nu^*}(T))] = v_I - \xi_I(1+\eta_R)Put(0).$$

The Lagrange multiplier $\lambda_I^*(\xi_I, \eta_R)$ is continuous with respect to ξ_I and η_R, since $\hat{I}_I(\cdot)$ is a continuous function and $v_I - \xi_I(1+\eta_R)Put(0)$ is continuous with respect to ξ_I and η_R. Since the functions $I_I(\cdot)$, max and $U_I(\cdot)$ are also continuous, $h_I(\cdot)$ is continuous with respect to ξ_I and η_R. Moreover, the strict concavity of $h_I(\cdot)$ with respect to ξ_I follows from the strict concavity of $U_I(\cdot)$ and the application of Lemma A.3 in Desmettre and Seifried (2016). Furthermore, the function $\eta_R \mapsto \xi_I^{\max}(\eta_R)$ is continuous. Therefore, the application of Theorem 2.5.4 yields that the function $\eta_R \mapsto \xi_I^*(\eta_R)$ is continuous too.

The Lagrange multiplier $\lambda_R^*(\xi_I^*(\eta_R), \eta_R)$ is continuous, since $I_R(\cdot)$ is a continuous function and $v_{R,0} + \xi_I^*(\eta_R)\eta_R Put(0)$ is continuous with respect to η_R. Furthermore, we have that the functions $I_R(\cdot)$ and $U_R(\cdot)$ are continuous. Therefore, the function $h_R(\cdot)$ is continuous with respect to η_R.

Having shown the continuity of $h_R(\cdot)$ and using the compactness of $[0, \eta_R^{\max}]$, we apply Theorem 2.5.3 and conclude that there exists η_R^* such that

$$\eta_R^* = \underset{\eta_R \in [0, \eta_R^{\max}]}{\arg\max} \; h_R(\eta_R).$$

□

Proof of Proposition 5.2.6. We show that Condition (SEC_1) and (SEC_2) are fulfilled:

(SEC_1) By Proposition 5.2.1 and Proposition 5.2.2, the optimal response $(\pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$ solves the optimization problem of the insurer. Thus, (SEC_1) holds by definition.

(SEC_2) By Proposition 5.2.4 and Proposition 5.2.5, the optimal strategy $(\pi_R^*(\cdot), \eta_R^*)$ solves the optimization problem of the reinsurer. If the insurer has more than one best response to the reinsurer's strategy (π_R^*, η_R^*), then we need to find $(\pi_I^*(\cdot|\eta_R^*), \xi_I^*(\eta_R^*))$ such that

$$\begin{aligned}\mathbb{E}\Big[U_R(V_R^{v_{R,0}(\xi_I,\eta_R^*),\pi_R^*}(T) - \xi_I Put(T))\Big]\\ \le \mathbb{E}[U_R(V_R^{v_{R,0}(\xi_I^*(\eta_R^*),\eta_R^*),\pi_R^*}(T) - \xi_I^*(\eta_R^*) Put(T))]\end{aligned} \tag{C.18}$$

for all ξ_I in the set of best responses of the insurer. Note that the reinsurer's value function does not depend on π_I. Thus, we focus only on ξ_I and show now that the reinsurer's value function is increasing in ξ_I.

For the total terminal wealth of the reinsurer it holds for $\xi_I \in [0, \xi^{\max}]$

$$V_R^{v_{R,0}(\xi_I,\eta_R^*),\pi_R^*}(T) - \xi_I Put(T) = I_R(\lambda_R^* \widetilde{Z}(T)), \tag{C.19}$$

where λ_R^* solves the budget constraint

$$\mathbb{E}[\widetilde{Z}(T) I_R(\lambda_R^* \widetilde{Z}(T))] = v_R + \xi_I \eta_R^* Put(0). \tag{C.20}$$

It holds

$$\begin{aligned}&\mathbb{E}\left[U_R(V_R^{v_{R,0}(\xi_I,\eta_R^*),\pi_R^*}(T) - \xi_I Put(T))\right] \text{ increasing w.r.t. } \xi_I\\ \overset{(a)}{\Leftrightarrow}\quad & V_R^{v_{R,0}(\xi_I,\eta_R^*),\pi_R^*}(T) - \xi_I Put(T) \text{ increasing w.r.t. } \xi_I\\ \overset{\text{(C.19)}}{\Leftrightarrow}\quad & I_R(\lambda_R^* \widetilde{Z}(T)) \text{ increasing w.r.t. } \xi_I\\ \overset{(b)}{\Leftrightarrow}\quad & \lambda_R^* \text{ decreasing w.r.t. } \xi_I,\end{aligned}$$

where we use in (a) that $U_R(\cdot)$ is an increasing function and in (b) that $I_R(\cdot)$ is decreasing due to the concavity of $U_R(\cdot)$ is a decreasing function and from Equation (C.19). By Equation (C.20), the Lagrange multiplier λ_R^* decreases if and only if ξ_I increases, since $I_R(\cdot)$ is a decreasing function. Hence, the value function of the reinsurer increases if ξ_I increases. Therefore, the Inequality (C.18) is fulfilled if the optimal reinsurance strategy of the insurer in the Stackelberg equilibrium is given by

$$\begin{aligned}\xi_I^*(\eta_R^*) = \max\{\xi_I^* |\ &\mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I^*,\eta_R^*),\pi_I^*}(T) + \xi_I^* Put(T))]\\ &= \sup_{\xi_I:(\xi_I,\pi_I)\in\Lambda_I} \mathbb{E}[U_I(V_I^{v_{I,0}(\xi_I,\eta_R^*),\pi_I^*}(T) + \xi_I Put(T))]\}.\end{aligned}$$

□

C.2 Explicit equilibrium for power-utility functions

Lemma C.2.1. *Let* $\xi^{\max} = \bar{\xi}$ *with* $\bar{\xi} < \frac{v_I}{(1+\eta_R)Put(0)}$ *for all* $\eta_R \in [0, \eta^{\max}]$*. Then the function* $h_I(\cdot)$ *from Proposition 5.2.1 is given by*

$$h_I(\xi) = \mathbb{E}[U_I(I_I(\lambda^*_\nu(\xi)\widetilde{Z}_{\nu^*}(T)))] \tag{C.21}$$

for $\xi \in [0, \xi^{\max}]$*, where* $I_I(\cdot)$ *is the inverse function of* $U_I'(\cdot)$ *and* $\lambda^*_\nu(\xi)$ *is the Lagrange multiplier given by*

$$\mathbb{E}[\widetilde{Z}_\nu(T)\hat{I}_I(\lambda^*_\nu(\xi)\widetilde{Z}_\nu(T))] = v_I - \xi(1+\eta_R)Put(0),$$

where $\hat{I}_I(\cdot)$ *is the inverse function of* $\hat{U}_I'(\cdot)$

Proof. Recall from Proposition 5.2.1:

$$h_I(\xi) := \mathbb{E}[U_I(\max\{I_I(\lambda^*_\nu(\xi)\widetilde{Z}_{\nu^*}(T)), \xi Put(T)\})],$$

where the Lagrange multiplier $\lambda^*_\nu(\xi)$ is given by

$$\mathbb{E}[\widetilde{Z}_\nu(T)\hat{I}_I(\lambda^*_\nu(\xi)\widetilde{Z}_\nu(T))] = v_I - \xi(1+\eta_R)Put(0).$$

$\hat{I}_I(\cdot)$ is the random inverse function of the insurer, which is bijective on $(0, U_I'(\xi Put(T))]$ and equals

$$\hat{I}_I(\lambda) = I_I(\lambda) - \xi Put(T)$$

for $\lambda \in (0, U_I'(\xi Put(T))]$. From this, it holds

$$\hat{I}_I(\lambda) > 0 \Leftrightarrow \lambda \in (0, U_I'(\xi Put(T))). \tag{C.22}$$

For the Lagrange multiplier $\lambda^*_\nu(\xi)$ it follows

$$\mathbb{E}[\widetilde{Z}_\nu(T)\hat{I}_I(\lambda^*_\nu(\xi)\widetilde{Z}_\nu(T))] = \underbrace{v_I - \xi(1+\eta_R)Put(0)}_{>0} \Leftrightarrow \hat{I}_I(\lambda^*_\nu(\xi)\widetilde{Z}_\nu(T)) > 0 \ \mathbb{Q}\text{-a.s.}$$
$$\Leftrightarrow \lambda^*_\nu(\xi)\widetilde{Z}_\nu(T) < U_I'(\xi Put(T)) \ \mathbb{Q}\text{-a.s.} \Leftrightarrow I_I(\lambda^*_\nu(\xi)\widetilde{Z}_\nu(T)) > \xi Put(T) \ \mathbb{Q}\text{-a.s.},$$

where the second equivalence holds from C.22 and the third from the fact that $I_I(\cdot)$ is strictly decreasing. Hence, we get:

$$h_I(\xi) = \mathbb{E}[U_I(\max\{I_I(\lambda^*_\nu(\xi)\widetilde{Z}_{\nu^*}(T)), \xi Put(T)\})] = \mathbb{E}[U_I(I_I(\lambda^*_\nu(\xi)\widetilde{Z}_{\nu^*}(T)))].$$

□

Proof of Corollary 5.3.1. For $\nu \in \mathcal{D}$, the optimal unconstrained relative portfolio process

$\pi^*_{u,\nu}$ in the auxiliary market $\mathcal{M}_\nu$ is given by

$$\pi^*_{u,\nu}(p_I) := \pi^*_{u,\nu}(t) = \frac{1}{1-p_I}(\sigma\sigma^\top)^{-1}(\mu + \nu(t) - r\mathbf{1}_2).$$

For the optimal portfolio process of the insurer π^*_ν in the auxiliary market $\mathcal{M}_\nu$ it holds by Proposition 5.2.1

$$\pi^*_\nu(t)V^*_\nu(t) = \pi^*_{u,\nu}(p_I)(V^*_\nu(t) + \xi^*_\nu \left(\widetilde{Z}_\nu(t)\right)^{-1} \mathbb{E}[\widetilde{Z}_\nu(T)Put(T)1_{\{V^*_\nu(T)>0\}}|\mathcal{F}(t)]),$$

where V^*_ν is the optimal wealth process of the insurer in the auxiliary market $\mathcal{M}_\nu$. Since $\xi^{\max} = \bar{\xi}$ with $\bar{\xi} < \frac{v_I}{(1+\eta_R)Put(0)}$ for all $\eta_R \in [0, \eta^{\max}]$ and $\xi^*_\nu \le \bar{\xi}$, it holds $v_{I,0}(\xi^*_\nu, \eta_R) > 0$ and, therefore, $V^*_\nu(t) > 0$ for all $t \in [0,T]$. It follows that

$$\pi^*_\nu(t) = \pi^*_{u,\nu}(p_I)\frac{V^*_\nu(t) + \xi^*_\nu \left(\widetilde{Z}_\nu(t)\right)^{-1} \mathbb{E}[\widetilde{Z}_\nu(T)Put(T)|\mathcal{F}(t)]}{V^*_\nu(t)}.$$

We find now $\nu^* \in \mathcal{D}$ such that $\pi^*_{\nu^*}(t) \in C_{\pi_I} = \mathbb{R} \times \{0\}$ for all $t \in [0,T]$.

Since π^*_ν is given by $\pi^*_{u,\nu}$ multiplied by a random variable bigger than zero, it is sufficient to find $\nu^* \in \mathcal{D}$ such that $\pi^*_{u,\nu^*} \in C_{\pi_I}$. Hence,

$$\pi^*_{u,\nu^*}(p_I) \in C_{\pi_I} \Leftrightarrow \frac{1}{1-p_I}(\sigma\sigma^\top)^{-1}\begin{pmatrix} \mu_1 - r \\ \mu_2 + \nu^*_2(t) - r \end{pmatrix} \in C_{\pi_I}.$$

Since $\nu^* \in \mathcal{D}$ has to hold we have $\nu^*_1(t) = 0$. It follows

$$\nu^*(t) \equiv \nu^* = \begin{pmatrix} 0 \\ \frac{\sigma_2\rho}{\sigma_1}(\mu_1 - r) - \mu_2 + r \end{pmatrix}.$$

From Lemma C.2.1, we get for the function $h_I(\cdot)$ in the special case of a power-utility function with $p_I \in (-\infty, 1)\backslash\{0\}$ that

$$h_I(\xi) = \mathbb{E}\left[\frac{1}{p_I}\left(\lambda^*_I(\xi)\widetilde{Z}_{\nu^*}(T)\right)^{\frac{p_I}{p_I-1}}\right],$$

where the Lagrange multiplier $\lambda^*_I(\xi)$ is determined by the budget constraint

$$\mathbb{E}\left[\widetilde{Z}_{\nu^*}(T)\left((\lambda^*_I(\xi)\widetilde{Z}_{\nu^*}(T))^{\frac{1}{p_I-1}} - \xi Put(T)\right)\right] = v_I - \xi(1+\eta_R)Put(0).$$

Hence, we have:

$$\lambda^*_I(\xi) = (v_I - \xi(1+\eta_R)Put(0) + \xi\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)])^{p_I-1}\mathbb{E}\left[\widetilde{Z}_{\nu^*}(T)^{\frac{p_I}{p_I-1}}\right]^{1-p_I}$$

and, therefore,

$$h_I(\xi) = \mathbb{E}\left[\frac{1}{p_I}(\lambda_I^*(\xi)\widetilde{Z}_{\nu^*}(T))^{\frac{p_I}{p_I-1}}\right]$$
$$= \frac{1}{p_I}(v_I - \xi(1+\eta_R)Put(0) + \xi\mathbb{E}\left[\widetilde{Z}_{\nu^*}(T)Put(T)\right])^{p_I}\mathbb{E}\left[\widetilde{Z}_{\nu^*}(T)^{\frac{p_I}{p_I-1}}\right]^{1-p_I}.$$

It follows for the optimal reinsurance strategy $\xi_I^* = \xi_I^*(\eta_R)$:

$$\xi_I^* = \underset{\xi_I\in[0,\bar{\xi}]}{\arg\max}\left(\frac{1}{p_I}(v_I - \xi_I(1+\eta_R)Put(0) + \xi_I\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)])^{p_I}\mathbb{E}\Big[\widetilde{Z}_{\nu^*}(T)^{\frac{p_I}{p_I-1}}\Big]^{1-p_I}\right)$$
$$= \begin{cases}\bar{\xi}, & \text{if } -(1+\eta_R)Put(0) + \mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] > 0;\\ \text{any } \tilde{\xi}\in[0,\bar{\xi}], & \text{if } -(1+\eta_R)Put(0) + \mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] = 0;\\ 0, & \text{if } -(1+\eta_R)Put(0) + \mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] < 0;\end{cases}$$
$$= \begin{cases}\bar{\xi}, & \text{if } \eta_R < \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)};\\ \text{any } \tilde{\xi}\in[0,\bar{\xi}], & \text{if } \eta_R = \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)};\\ 0, & \text{if } \eta_R > \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)}.\end{cases}$$

□

Proof of Corollary 5.3.2. For the optimal reinsurance strategy $\xi_I^*(\eta_R)$, it holds by Corollary 5.3.1

$$\xi_I^*(\eta_R) = \begin{cases}\bar{\xi}, & \text{if } \eta_R < \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)};\\ \text{any } \tilde{\xi}\in[0,\bar{\xi}], & \text{if } \eta_R = \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)};\\ 0, & \text{if } \eta_R > \frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)]-Put(0)}{Put(0)}.\end{cases}$$

It follows from Proposition 5.2.5 and (C.19) for the optimal safety loading η_R^*

$$\eta_R^* = \underset{\eta_R\in[0,\eta^{\max}]}{\arg\max}\ \mathbb{E}\left[\frac{1}{p_R}(\lambda_R^*(\eta_R)\widetilde{Z}(T))^{\frac{p_R}{p_R-1}}\right]$$
$$\overset{(5.7)}{=} \underset{\eta_R\in[0,\eta^{\max}]}{\arg\max}\ \frac{1}{p_R}(v_R + \xi_I^*(\eta_R)\eta_R Put(0))^{p_R}\mathbb{E}\left[\widetilde{Z}(T)^{\frac{p_R}{p_R-1}}\right]^{1-p_R}$$
$$= \underset{\eta_R\in[0,\eta^{\max}]}{\arg\max}\ \xi_I^*(\eta_R)\eta_R.$$

Hence, the reinsurer chooses the largest $\eta_R \in [0, \eta^{\max}]$ such that $\xi_I^*(\eta_R) = \bar{\xi}$, i.e.,

$$\eta_R^* = \min\left\{\frac{\mathbb{E}[\widetilde{Z}_{\nu^*}(T)Put(T)] - Put(0)}{Put(0)}, \eta^{\max}\right\}.$$

According to the definition of the Stackelberg equilibrium (see Conditions (SEC_1) and (SEC_2)), the optimal reinsurance strategy of the insurer is given by $\xi_I^*(\eta_R^*) = \bar{\xi}$ (i.e., the insurer chooses the response to the optimal safety loading, which is the best for the reinsurer). $\square$

D Appendix to Chapter 6

Here we provide the proofs of the results from Chapter 6. Appendix D.1 is related to the results for unconstrained optimization problem (P_u), namely Propostions 6.1.1 and 6.1.2. Appendix D.2 contains the proofs of the main results stated in Section 6.1.1. Appendix D.3 has explicit formulas for computing the LHS of $(\mathrm{NLS}(y, k_v, k_\varepsilon))$ needed for determining the parameters of the synthetic financial derivative linking the solution to the constrained problem and the solution to the unconstrained one. Finally, in Appendix D.4, we provide an alternative derivation of the optimal solution to the constrained problem in the special case of $\rho = 0$.

D.1 Solution to the unconstrained optimization problem

Proof of Proposition 6.1.1. We denote for readability of this proof $\mathcal{V} := \mathcal{V}^u$. We face a two-dimensional control problem (P_u) with state process (Y, v) and consider the HJB equation:

$$0 = \mathcal{V}_t + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + \kappa(\theta - v)\mathcal{V}_v + \sup_\pi \Big\{ \underbrace{y(r + \pi\gamma^{S_1} v)\mathcal{V}_y + \frac{1}{2}\pi^2 y^2 v \mathcal{V}_{yy} + \pi y \sigma v \rho \mathcal{V}_{yv}}_{g(\pi)} \Big\}$$

and boundary condition $\mathcal{V}(T, y, v) = \frac{y^p}{p}$. Eliminating the sup results in a first-order condition for π:

$$\pi_u^* = -\frac{y\gamma^{S_1} v \mathcal{V}_y + y\sigma v \rho \mathcal{V}_{yv}}{y^2 v \mathcal{V}_{yy}} = -\frac{\gamma^{S_1}\mathcal{V}_y + \sigma\rho\mathcal{V}_{yv}}{y\mathcal{V}_{yy}} = -\gamma^{S_1}\frac{\mathcal{V}_y}{y\mathcal{V}_{yy}} - \sigma\rho\frac{\mathcal{V}_{yv}}{y\mathcal{V}_{yy}} \tag{D.1}$$

under the assumption that $\mathcal{V}_{yy} < 0$. Substituting the expression for π_u^* back into the HJB equation leads to the following non-linear PDE for the value function.

$$\begin{aligned} 0 &= \mathcal{V}_t + \frac{1}{2}\sigma^2 v \mathcal{V}_{vv} + \kappa(\theta - v)\mathcal{V}_v + yr\mathcal{V}_y - y\frac{\gamma^{S_1}\mathcal{V}_y + \sigma\rho\mathcal{V}_{yv}}{y\mathcal{V}_{yy}}\gamma^{S_1} v \mathcal{V}_y \\ &\quad + \frac{1}{2}\frac{(\gamma^{S_1}\mathcal{V}_y + \sigma\rho\mathcal{V}_{yv})^2}{y^2\mathcal{V}_{yy}^2} y^2 v \mathcal{V}_{yy} - \frac{\gamma^{S_1}\mathcal{V}_y + \sigma\rho\mathcal{V}_{yv}}{y\mathcal{V}_{yy}} y\sigma v\rho\mathcal{V}_{yv} \\ &= \mathcal{V}_t + \kappa\theta\mathcal{V}_v + yr\mathcal{V}_y + v\left(\frac{1}{2}\sigma^2\mathcal{V}_{vv} - \kappa\mathcal{V}_v - \frac{1}{2}\frac{\left(\gamma^{S_1}\mathcal{V}_y + \sigma\rho\mathcal{V}_{yv}\right)^2}{\mathcal{V}_{yy}}\right). \end{aligned} \tag{D.2}$$

To find the solution, we use the separation ansatz

$$\mathcal{V}(t,y,v) = \frac{y^p}{p} h(t,v) \quad \text{with} \quad h(T,v) = 1.$$

In this case, $\pi_u^*(t) = \frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho}{1-p}\frac{h_v}{h}$. We substitute the ansatz into the HJB equation and conclude that:

$$0 = h_t + \kappa\theta h_v + prh + v\left(\frac{1}{2}\sigma^2 h_{vv} - \kappa h_v + \frac{1}{2}\frac{p(\gamma^{S_1}h + \sigma\rho h_v)^2}{(1-p)h}\right). \tag{D.3}$$

The structure implies that $h(t,v)$ is exponentially affine

$$h(t,v) = \exp(a(\tau(t)) + b(\tau(t))v) =: h,$$

with time horizon $\tau(t) = T - t$ and, therefore, using boundary condition $h(T,v) = 1 \quad \forall v \in \mathbb{R}$ we get that:

$$a(0) = a(\tau(T)) = 0, b(0) = b(\tau(T)) = 0.$$

Using this structure of $h(t,v)$ and rearranging to emphasize the linearity in v leads to

$$\begin{aligned} 0 = &- a'(\tau)h + b(\tau)\kappa\theta h + prh + v\Big[- b'(\tau)h + b^2(\tau)\left(\frac{1}{2}\sigma^2 h + \frac{p\sigma^2\rho^2 h}{2(1-p)}\right) \\ &+ b(\tau)\left(-\kappa h + \frac{p\gamma^{S_1}\sigma\rho h}{1-p}\right) + \frac{p\left(\gamma^{S_1}\right)^2 h}{2(1-p)}\Big]. \end{aligned}$$

Cancelling h out leads to Riccati equations for a and b:

$$\begin{aligned} a'(\tau) &= \kappa\theta b(\tau) + pr; & \text{(D.4)} \\ b'(\tau) &= \frac{1}{2}\underbrace{\left(\sigma^2 + \frac{p\sigma^2\rho^2}{1-p}\right)}_{k_2} b^2(\tau) - \underbrace{\left(\kappa - \frac{p\gamma^{S_1}\sigma\rho}{1-p}\right)}_{k_1} b(\tau) + \frac{1}{2}\underbrace{\frac{p\left(\gamma^{S_1}\right)^2}{1-p}}_{k_0} \\ &= \frac{1}{2}k_2 b(\tau)^2 - k_1 b(\tau) + \frac{1}{2}k_0 & \text{(D.5)} \end{aligned}$$

and boundary conditions $a(0) = 0, b(0) = 0$ with constants k_0, k_1, k_2 that have to satisfy $k_1^2 - k_0 k_2 > 0$. Then according to Kraft (2005) and Kallsen and Muhle-Karbe (2010) the solution is given by:

$$a(\tau) = pr\tau + \frac{2\theta\kappa}{k_2}\ln\left(\frac{2k_3\exp\left(\frac{1}{2}(k_1+k_3)\tau\right)}{2k_3 + (k_1+k_3)\left(\exp\left(k_3\tau\right) - 1\right)}\right); \tag{D.6}$$

$$b(\tau) = k_0 \frac{\exp\left(k_3\tau\right) - 1}{\exp\left(k_3\tau\right)\left(k_1+k_3\right) - k_1 + k_3} \tag{D.7}$$

with $k_3 = \sqrt{k_1^2 - k_0 k_2}$. For the system to be well-defined, we have to check whether our constants fulfill $k_1^2 - k_0 k_2 > 0$. Therefore, we formulate the following requirement on the parameters:

$$\begin{aligned} k_1^2 - k_0 k_2 &= \kappa^2 - 2\kappa \frac{p}{1-p}\gamma^{S_1}\sigma\rho + \frac{p^2}{(1-p)^2}\left(\gamma^{S_1}\right)^2\sigma^2\rho^2 - \frac{p}{1-p}\left(\gamma^{S_1}\right)^2\sigma^2 \\ &\quad - \frac{p^2}{(1-p)^2}\left(\gamma^{S_1}\right)^2\sigma^2\rho^2 = \kappa^2 - \frac{p}{1-p}\gamma^{S_1}\sigma(2\kappa\rho + \gamma^{S_1}\sigma) > 0 \\ \Leftrightarrow \frac{p}{1-p}\gamma^{S_1}\left(\frac{\kappa\rho}{\sigma} + \frac{\gamma^{S_1}}{2}\right) &< \frac{\kappa^2}{2\sigma^2}, \end{aligned}$$

which is exactly what Kraft (2005) requires in his Equation (26). Note that the ansatz satisfies the assumption $\mathcal{V}_{yy} < 0$, since for $p < 1$ we have:

$$\underbrace{(p-1)}_{<0}\underbrace{y^{p-2}h(t,v)}_{>0} < 0.$$

□

Proof of Proposition 6.1.2. Applying Itô's lemma to the wealth process Y^* and the logarithmic function, we obtain the dynamics of Z^* under the measure $\mathbb{Q}$:

$$\begin{aligned} \mathrm{d}Z^*(t) &= \left(r + \left(\pi_u^*(t)\gamma^{S_1} - \frac{1}{2}\left(\pi_u^*(t)\right)^2\right)v(t)\right)\mathrm{d}t + \pi_u^*(t)\sqrt{v(t)}\,\mathrm{d}W_1^{\mathbb{Q}}(t); \\ \mathrm{d}v(t) &= \kappa\left(\theta - v(t)\right)\mathrm{d}t + \sigma\rho\sqrt{v(t)}\,\mathrm{d}W_1^{\mathbb{Q}}(t) + \sigma\rho\sqrt{v(t)}\sqrt{1-\rho^2}\,\mathrm{d}W_2^{\mathbb{Q}}(t). \end{aligned}$$

To make the notation within the proof concise, we write $\pi^*(t)$ for $\pi_u^*(t)$. According to Feynman-Kac Theorem 2.2.8, the characteristic function

$$\phi^{Z^*(T),\mathbb{Q}}(u;t,z,v) = \mathbb{E}_{t,z,v}^{\mathbb{Q}}\left[\exp(iuZ^*(T))\right]$$

satisfies the following relations under $\mathbb{Q}$:

$$\begin{aligned} 0 &= \phi_t^{Z^*(T),\mathbb{Q}} + (r + \left(\pi^*(t)\gamma^{S_1} - \frac{1}{2}(\pi^*(t))^2\right)v)\phi_z^{Z^*(T),\mathbb{Q}} + \kappa(\theta - v)\phi_v^{Z^*(T),\mathbb{Q}} \\ &\quad + \frac{1}{2}(\pi^*(t))^2 v\phi_{zz}^{Z^*(T),\mathbb{Q}} + \pi^*(t)v\sigma\rho\phi_{zv}^{Z^*(T),\mathbb{Q}} + \frac{1}{2}\sigma^2 v\phi_{vv}^{Z^*(T),\mathbb{Q}} \\ \exp(iuz) &= \phi^{Z^*(T),\mathbb{Q}}(u;T,z,v). \end{aligned}$$

Using the ansatz for the characteristic function:

$$\phi^{Z^*(T),\mathbb{Q}}(u;t,z,v) = \exp\left(A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v + iuz\right),$$

changing the variable $\tau = T - t$, substituting and grouping under $\mathbb{Q}$,we receive

$$\begin{aligned} 0 &= -A^{\mathbb{Q}}_{\tau}(\tau,u) - B^{\mathbb{Q}}_{\tau}(\tau,u)v + \left(r + \left(\pi^*(\tau)\gamma^{S_1} - \frac{1}{2}\left(\pi^*(\tau)\right)^2\right)v\right)iu \\ &\quad + \kappa(\theta - v)B^{\mathbb{Q}}(\tau,u) - \frac{1}{2}\left(\pi^*(\tau)\right)^2 vu^2 + \pi^*(\tau)v\sigma\rho iuB^{\mathbb{Q}}(\tau,u) + \frac{1}{2}\sigma^2 v\left(B^{\mathbb{Q}}(\tau,u)\right)^2 \end{aligned}$$

and thus

$$\begin{aligned} 0 &= -B^{\mathbb{Q}}_{\tau}(\tau,u) + \left(\pi^*(\tau)\sigma\rho iu - \kappa\right)B^{\mathbb{Q}}(\tau,u) + \frac{1}{2}\sigma^2\left(B^{\mathbb{Q}}(\tau,u)\right)^2 - \frac{1}{2}\left(\pi^*(\tau)\right)^2\left(u^2 + iu\right) \\ &\quad + \pi^*(\tau)\gamma^{S_1}iu; \\ 0 &= -A^{\mathbb{Q}}_{\tau}(\tau,u) + riu + \kappa\theta B^{\mathbb{Q}}(\tau,u). \end{aligned}$$

Analogously, we obtain the dynamics of Z^* under the measure $\widetilde{\mathbb{Q}} := \widetilde{\mathbb{Q}}(\gamma^v)$ is given by

$$\begin{aligned} \mathrm{d}Z^*(t) &= \left(r - \frac{1}{2}\left(\pi^*_u(t)\right)^2 v(t)\right)\mathrm{d}t + \pi^*_u(t)\sqrt{v(t)}\,\mathrm{d}W^{\widetilde{\mathbb{Q}}}_1(t); \\ \mathrm{d}v(t) &= \tilde{\kappa}\left(\tilde{\theta} - v(t)\right)\mathrm{d}t + \sigma\rho\sqrt{v(t)}\,\mathrm{d}W^{\widetilde{\mathbb{Q}}}_1(t) + \sigma\sqrt{v(t)}\sqrt{1-\rho^2}\,\mathrm{d}W^{\widetilde{\mathbb{Q}}}_2(t) \end{aligned}$$

with $\tilde{\kappa} = \kappa + \sigma\gamma^{S_1}\rho + \sigma\gamma^v\sqrt{1-\rho^2}$ and $\tilde{\theta} = \kappa\theta/\tilde{\kappa}$. Recall these parameters may be time dependent due to γ^v.

Again using Theorem 2.2.8 and the ansatz

$$\phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u;t,z,v) = \exp\left(A^{\widetilde{\mathbb{Q}}}(T-t,u) + B^{\widetilde{\mathbb{Q}}}(T-t,u)v + iuz\right)$$

we obtain

$$\begin{aligned} 0 &= -A^{\widetilde{\mathbb{Q}}}_{\tau}(\tau,u) - B^{\widetilde{\mathbb{Q}}}_{\tau}(\tau,u)v + \left(r - \frac{1}{2}\left(\pi^*(\tau)\right)^2 v\right)iu + \tilde{\kappa}(\tilde{\theta} - v)B^{\widetilde{\mathbb{Q}}}(\tau,u) \\ &\quad - \frac{1}{2}\left(\pi^*(\tau)\right)^2 vu^2 + \pi^*(\tau)v\sigma\rho iuB^{\widetilde{\mathbb{Q}}}(\tau,u) + \frac{1}{2}\sigma^2 v\left(B^{\widetilde{\mathbb{Q}}}(\tau,u)\right)^2 \end{aligned}$$

Hence:

$$\begin{aligned} 0 &= -B^{\widetilde{\mathbb{Q}}}_{\tau}(\tau,u) + \left(\pi^*(\tau)\sigma\rho iu - \tilde{\kappa}\right)B^{\widetilde{\mathbb{Q}}}(\tau,u) + \frac{1}{2}\sigma^2\left(B^{\widetilde{\mathbb{Q}}}(\tau,u)\right)^2 \\ &\quad - \frac{1}{2}\left(\pi^*(\tau)\right)^2\left(u^2 + iu\right); \\ 0 &= -A^{\widetilde{\mathbb{Q}}}_{\tau}(\tau,u) + riu + \tilde{\kappa}\tilde{\theta}B^{\widetilde{\mathbb{Q}}}(\tau,u). \end{aligned}$$

□

D.2 Proofs of main results

Proof of Theorem 6.1.3. Our proof is based on the fact that two functions are equal if they satisfy the same PDEs with the same terminal conditions. In the following, we:

1. use the dynamic programming approach to derive the HJB PDE of $\mathcal{V}^c(t,x,c)$, simplify it under the assumption that $\mathcal{V}^c_{xx}(t,x,v) < 0$ and get the optimal investment strategy π^*_c in terms of the (to be found) function $\mathcal{V}^c(t,x,v)$;
2. consider the PDE of $\overline{U}^{D,\mathbb{Q}}(t,y,v)$ obtained via the Feynman-Kac (FK) formula and change of variables from (t,y,v) to (t,x,v) via $x = D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)$, i.e., $\hat{\mathcal{V}}^c(t,x,v) := \overline{U}^{D,\mathbb{Q}}(t, \left(D^{\widetilde{\mathbb{Q}}(\gamma^v)}\right)^{-1}(t,x,v),v)$ is our ansatz for the value function in the constrained optimization problem;
3. simplify the PDE from Step 2 using the assumption (6.23) that $D^{\widetilde{\mathbb{Q}}}_v(t,y,v) = 0$ and using the PDE of $D^{\widetilde{\mathbb{Q}}}(t,y,v)$ obtained via the FK formula;
4. show that the resulting PDE in Step 3 coincides with the PDE of $\mathcal{V}^c(t,x,c)$:
   a) for case $\rho = 0$ if Condition (6.21) holds;
   b) for case $\rho \neq 0$ if both Conditions (6.21), (6.22) hold;
5. show that the terminal conditions in the PDEs from Step 1 and Step 4 coincide and that $\hat{\mathcal{V}}^c_{xx}(t,x,v) < 0$, which implies that $\hat{\mathcal{V}}^c(t,x,v)$ solves the HJB PDE of $\mathcal{V}^c(t,x,c)$ and enables the calculation of π^*_c from Step 1.

To make derivations in this proof more readable, we omit the arguments of the functions $\mathcal{V}^c(t,x,v)$, $D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v)$, $\overline{U}^{D,\mathbb{Q}}(t,y,v)$. We also omit the parameter γ^v of the EMM $\widetilde{\mathbb{Q}}(\gamma^v)$.

Step 1. HJB PDE of $\mathcal{V}^c$. Similarly to the unconstrained Problem (P_u), we face a two-dimensional control problem with state process (X,v) and consider the HJB PDE:

$$0 = \mathcal{V}^c_t + \frac{1}{2}\sigma^2 v\mathcal{V}^c_{vv} + \kappa(\theta - v)\mathcal{V}^c_v + \sup_{\pi}\left\{x(r + \pi\gamma^{S_1}v)\mathcal{V}^c_x + \frac{1}{2}\pi^2x^2v\mathcal{V}^c_{xx} + \pi x\sigma v\rho\mathcal{V}^c_{xv}\right\} \tag{D.8}$$

and the boundary condition $\mathcal{V}^c(T,x,v) = \overline{U}(x)$. Eliminating the sup results in a first-order condition for π:

$$\pi^*_c = -\frac{x\gamma^{S_1}v\mathcal{V}^c_x + x\sigma v\rho\mathcal{V}^c_{xv}}{x^2v\mathcal{V}^c_{xx}} = -\frac{\gamma^{S_1}\mathcal{V}^c_x + \sigma\rho\mathcal{V}^c_{xv}}{x\mathcal{V}^c_{xx}} \tag{D.9}$$

under the assumption that $\mathcal{V}^c_{xx} < 0$. Analogous to (D.2), we substitute the expression for π^*_c back into the HJB PDE (D.8) and get the following PDE for the value function $\mathcal{V}^c$:

$$\mathcal{V}^c_t + xr\mathcal{V}^c_x + \kappa\theta\mathcal{V}^c_v + v\left(\frac{1}{2}\sigma^2\mathcal{V}^c_{vv} - \kappa\mathcal{V}^c_v - \frac{1}{2}\frac{(\gamma^{S_1}\mathcal{V}^c_x + \sigma\rho\mathcal{V}^c_{xv})^2}{\mathcal{V}^c_{xx}}\right) = 0; \tag{D.10}$$

$$\mathcal{V}^c(T,x,v) = \overline{U}\,(x)\,. \tag{D.11}$$

Steps 2-4. PDE of $\overline{U}^{D,\mathbb{Q}}$ and a change of variables. Recall from (6.19) and (6.20) that the Feynman-Kac representation of $\overline{U}^{D,\mathbb{Q}}$ is given by:

$$\begin{aligned} 0 &= \overline{U}^{D,\mathbb{Q}}_t + \left(r + \pi^*_u\gamma^{S_1}v\right) y\overline{U}^{D,\mathbb{Q}}_y + \kappa\,(\theta - v)\,\overline{U}^{D,\mathbb{Q}}_v \\ &\quad + \frac{1}{2}v\left[y^2(\pi^*_u)^2\overline{U}^{D,\mathbb{Q}}_{yy} + 2\sigma\rho y\pi^*_u\overline{U}^{D,\mathbb{Q}}_{yv} + \sigma^2\overline{U}^{D,\mathbb{Q}}_{vv}\right]; \\ \overline{U}^{D,\mathbb{Q}}(T,y,v) &= \overline{U}(D(y)). \end{aligned}$$

We change variables as follows:

$$t = t, \quad x = D^{\widetilde{\mathbb{Q}}}(t,y,v), \quad v = v. \tag{D.12}$$

This change of variables leads to an equivalent PDE $\forall\,(t,y,v) \in [0,T] \times (0,+\infty) \times (0,+\infty)$, since:

$$\begin{vmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial v} \\ \frac{\partial x}{\partial t} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial v} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ D^{\widetilde{\mathbb{Q}}}_t & D^{\widetilde{\mathbb{Q}}}_y & D^{\widetilde{\mathbb{Q}}}_v \\ 0 & 0 & 1 \end{vmatrix} = D^{\widetilde{\mathbb{Q}}}_y \neq 0 \quad \forall\,(t,y,v) \in [0,T] \times (0,+\infty) \times (0,+\infty)$$

under the assumption of $D(\cdot)$ being non-decreasing on $(0,+\infty)$ with a strictly increasing part.[1]

Using the ansatz

$$\overline{U}^{D,\mathbb{Q}}(t,y,v) = \hat{\mathcal{V}}^c(t, D^{\widetilde{\mathbb{Q}}}(t,y,v), v), \tag{D.13}$$

we compute the corresponding derivatives that appear in the PDE of $\overline{U}^{D,\mathbb{Q}}$:

[1] We can even show that $D^{\widetilde{\mathbb{Q}}}_y > 0$.

$$\begin{aligned}
\overline{U}_t^{D,\mathbb{Q}} &= \hat{\mathcal{V}}_t^c + \hat{\mathcal{V}}_x^c D_t^{\widetilde{\mathbb{Q}}};\\
\overline{U}_y^{D,\mathbb{Q}} &= \hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}};\\
\overline{U}_v^{D,\mathbb{Q}} &= \hat{\mathcal{V}}_x^c D_v^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_v^c \overset{(6.23)}{=} \hat{\mathcal{V}}_v^c;\\
\overline{U}_{yy}^{D,\mathbb{Q}} &= \hat{\mathcal{V}}_{xx}^c (D_y^{\widetilde{\mathbb{Q}}})^2 + \hat{\mathcal{V}}_x^c D_{yy}^{\widetilde{\mathbb{Q}}};\\
\overline{U}_{yv}^{D,\mathbb{Q}} &= \hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_x^c D_{yv}^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_{xx}^c D_y^{\widetilde{\mathbb{Q}}} D_v^{\widetilde{\mathbb{Q}}} \overset{(6.23)}{=} \hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_x^c D_{yv}^{\widetilde{\mathbb{Q}}};\\
\overline{U}_{vv}^{D,\mathbb{Q}} &= 2\hat{\mathcal{V}}_{xv}^c D_v^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_x^c D_{vv}^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_{vv}^c + \hat{\mathcal{V}}_{xx}^c \left(D_v^{\widetilde{\mathbb{Q}}}\right)^2 \overset{(6.23)}{=} \hat{\mathcal{V}}_x^c D_{vv}^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_{vv}^c.
\end{aligned} \tag{D.14}$$

Next we substitute these derivatives into the PDE of $\overline{U}^{D,\mathbb{Q}}$, also use the PDE for $D_t^{\widetilde{\mathbb{Q}}}$ to simplify the equation, and then we cancel out terms and insert the assumption $D^{\widetilde{\mathbb{Q}}}(t,y,v) = x$.

$$\begin{aligned}
0 &\overset{(6.19)}{=} \overline{U}_t^{D,\mathbb{Q}} + yr\overline{U}_y^{D,\mathbb{Q}} + \kappa\theta\overline{U}_v^{D,\mathbb{Q}} + v\Bigg(\frac{1}{2}\sigma^2\overline{U}_{vv}^{D,\mathbb{Q}} - \kappa\overline{U}_v^{D,\mathbb{Q}} + y\gamma^{S_1}\pi_u^*\overline{U}_y^{D,\mathbb{Q}}\\
&\quad + \frac{1}{2}y^2(\pi_u^*)^2\overline{U}_{yy}^{D,\mathbb{Q}} + \sigma\rho y\pi_u^*\overline{U}_{yv}^{D,\mathbb{Q}}\Bigg)\\
&\overset{(D.14)}{=} \hat{\mathcal{V}}_t^c + \hat{\mathcal{V}}_x^c D_t^{\widetilde{\mathbb{Q}}} + yr\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}} + \kappa\theta\hat{\mathcal{V}}_v^c + v\Bigg[\frac{1}{2}\sigma^2\left(\hat{\mathcal{V}}_x^c D_{vv}^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_{vv}^c\right) - \kappa\hat{\mathcal{V}}_v^c + y\gamma^{S_1}\pi_u^*\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}}\\
&\quad + \frac{1}{2}y^2(\pi_u^*)^2\left(\hat{\mathcal{V}}_{xx}^c(D_y^{\widetilde{\mathbb{Q}}})^2 + \hat{\mathcal{V}}_x^c D_{yy}^{\widetilde{\mathbb{Q}}}\right) + \sigma\rho y\pi_u^*\left(\hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \hat{\mathcal{V}}_x^c D_{yv}^{\widetilde{\mathbb{Q}}}\right)\Bigg]\\
&\underset{(6.17)}{\overset{(6.23)}{=}} \hat{\mathcal{V}}_t^c + \hat{\mathcal{V}}_x^c rx + \kappa\theta\hat{\mathcal{V}}_v^c + v\Bigg(\frac{1}{2}\sigma^2\hat{\mathcal{V}}_{vv}^c - \kappa\hat{\mathcal{V}}_v^c + y\gamma^{S_1}\pi_u^*\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}}\\
&\quad + \frac{1}{2}y^2(\pi_u^*)^2\left(\hat{\mathcal{V}}_{xx}^c(D_y^{\widetilde{\mathbb{Q}}})^2\right) + \sigma\rho y\pi_u^*\left(\hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}}\right)\Bigg)\\
&\overset{(i)}{=} \hat{\mathcal{V}}_t^c + xr\hat{\mathcal{V}}_x^c + \kappa\theta\hat{\mathcal{V}}_v^c + v\left[\frac{1}{2}\sigma^2\hat{\mathcal{V}}_{vv}^c - \kappa\hat{\mathcal{V}}_v^c - \frac{1}{2}\frac{(\gamma^{S_1}\hat{\mathcal{V}}_x^c + \sigma\rho\hat{\mathcal{V}}_{xv}^c)^2}{\hat{\mathcal{V}}_{xx}^c}\right]\\
&\quad + v\left(y\gamma^{S_1}\pi_u^*\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}y^2(\pi_u^*)^2\hat{\mathcal{V}}_{xx}^c\left(D_y^{\widetilde{\mathbb{Q}}}\right)^2 + \sigma\rho y\pi_u^*\hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}\frac{(\gamma^{S_1}\hat{\mathcal{V}}_x^c + \sigma\rho\hat{\mathcal{V}}_{xv}^c)^2}{\hat{\mathcal{V}}_{xx}^c}\right),
\end{aligned}$$

where in (i) we added and subtracted the term $-v\frac{1}{2}\frac{(\gamma^{S_1}\hat{\mathcal{V}}_x^c+\sigma\rho\hat{\mathcal{V}}_{xv}^c)^2}{\hat{\mathcal{V}}_{xx}^c}$.

We show now that under Conditions (6.21) and (6.22), the term

$$C := y\gamma^{S_1}\pi_u^*\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}y^2(\pi_u^*)^2\hat{\mathcal{V}}_{xx}^c\left(D_y^{\widetilde{\mathbb{Q}}}\right)^2 + \sigma\rho y\pi_u^*\hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}\frac{(\gamma^{S_1}\hat{\mathcal{V}}_x^c + \sigma\rho\hat{\mathcal{V}}_{xv}^c)^2}{\hat{\mathcal{V}}_{xx}^c}$$

is zero. Expanding the brackets in the last term of C we get:

$$C = y\gamma^{S_1}\pi_u^*\hat{\mathcal{V}}_x^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}y^2(\pi_u^*)^2\hat{\mathcal{V}}_{xx}^c\left(D_y^{\widetilde{\mathbb{Q}}}\right)^2 + \sigma\rho y\pi_u^*\hat{\mathcal{V}}_{xv}^c D_y^{\widetilde{\mathbb{Q}}} + \frac{1}{2}\left(\gamma^{S_1}\right)^2\frac{\left(\hat{\mathcal{V}}_x^c\right)^2}{\hat{\mathcal{V}}_{xx}^c} + \gamma^{S_1}\sigma\rho\frac{\hat{\mathcal{V}}_x^c\hat{\mathcal{V}}_{xv}^c}{\hat{\mathcal{V}}_{xx}^c} + \frac{1}{2}\sigma^2\rho^2\frac{\left(\hat{\mathcal{V}}_{xv}^c\right)^2}{\hat{\mathcal{V}}_{xx}^c}.$$

Using (D.14), we obtain:

$$\begin{aligned} \hat{\mathcal{V}}_x^c &= \frac{\overline{U}_y^{D,\mathbb{Q}}}{D_y^{\widetilde{\mathbb{Q}}}}, \hat{\mathcal{V}}_{xx}^c = \frac{\overline{U}_{yy}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yy}^{\widetilde{\mathbb{Q}}}}{\left(D_y^{\widetilde{\mathbb{Q}}}\right)^3}, \\ \hat{\mathcal{V}}_{xv}^c &= \frac{1}{D_y^{\widetilde{\mathbb{Q}}}}\left(\overline{U}_{yv}^{D,\mathbb{Q}} - \hat{\mathcal{V}}_x^c D_{yv}^{\widetilde{\mathbb{Q}}}\right) = \frac{\overline{U}_{yv}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yv}^{\widetilde{\mathbb{Q}}}}{\left(D_y^{\widetilde{\mathbb{Q}}}\right)^2}. \end{aligned} \tag{D.15}$$

Inserting these expressions in C, we get:

$$\begin{aligned} C &= y\gamma^{S_1}\pi_u^*\overline{U}_y^{D,\mathbb{Q}} + \frac{1}{2}y^2(\pi_u^*)^2\frac{\overline{U}_{yy}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yy}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}} + \sigma\rho y\pi_u^*\frac{\overline{U}_{yv}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yv}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}} \\ &+ \frac{1}{2}\left(\gamma^{S_1}\right)^2\frac{\left(\overline{U}_y^{D,\mathbb{Q}}\right)^2 D_y^{\widetilde{\mathbb{Q}}}}{\overline{U}_{yy}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yy}^{\widetilde{\mathbb{Q}}}} + \gamma^{S_1}\sigma\rho\overline{U}_y^{D,\mathbb{Q}}\frac{\overline{U}_{yv}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yv}^{\widetilde{\mathbb{Q}}}}{\overline{U}_{yy}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yy}^{\widetilde{\mathbb{Q}}}} \\ &+ \frac{1}{2}\sigma^2\rho^2\frac{\left(\overline{U}_{yv}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yv}^{\widetilde{\mathbb{Q}}}\right)^2}{D_y^{\widetilde{\mathbb{Q}}}\left(\overline{U}_{yy}^{D,\mathbb{Q}}D_y^{\widetilde{\mathbb{Q}}} - \overline{U}_y^{D,\mathbb{Q}}D_{yy}^{\widetilde{\mathbb{Q}}}\right)} \\ &= y\gamma^{S_1}\pi_u^*\overline{U}_y^{D,\mathbb{Q}} + \frac{1}{2}y^2(\pi_u^*)^2\overline{U}_y^{D,\mathbb{Q}}\left(\frac{\overline{U}_{yy}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yy}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right) + \sigma\rho y\pi_u^*\overline{U}_y^{D,\mathbb{Q}}\left(\frac{\overline{U}_{yv}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yv}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right) \\ &+ \frac{1}{2}\left(\gamma^{S_1}\right)^2\frac{\overline{U}_y^{D,\mathbb{Q}}}{\left(\frac{\overline{U}_{yy}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yy}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right)} + \gamma^{S_1}\sigma\rho\overline{U}_y^{D,\mathbb{Q}}\frac{\left(\frac{\overline{U}_{yv}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yv}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right)}{\left(\frac{\overline{U}_{yy}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yy}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right)} \\ &+ \frac{1}{2}\sigma^2\rho^2\overline{U}_y^{D,\mathbb{Q}}\frac{\left(\frac{\overline{U}_{yv}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yv}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right)^2}{\left(\frac{\overline{U}_{yy}^{D,\mathbb{Q}}}{\overline{U}_y^{D,\mathbb{Q}}} - \frac{D_{yy}^{\widetilde{\mathbb{Q}}}}{D_y^{\widetilde{\mathbb{Q}}}}\right)}. \end{aligned}$$

Denoting

$$A = \left(\frac{\overline{U}^{D,\mathbb{Q}}_{yy}}{\overline{U}^{D,\mathbb{Q}}_{y}} - \frac{D^{\widetilde{\mathbb{Q}}}_{yy}}{D^{\widetilde{\mathbb{Q}}}_{y}} \right) \text{ and } B = \left(\frac{\overline{U}^{D,\mathbb{Q}}_{yv}}{\overline{U}^{D,\mathbb{Q}}_{y}} - \frac{D^{\widetilde{\mathbb{Q}}}_{yv}}{D^{\widetilde{\mathbb{Q}}}_{y}} \right),$$

we get:

$$\begin{aligned} C = \overline{U}^{D,\mathbb{Q}}_{y} \Bigg(& y\gamma^{S_1}\pi^*_u + \frac{1}{2}y^2(\pi^*_u)^2 A + \sigma\rho y \pi^*_u B + \frac{1}{2}\left(\gamma^{S_1}\right)^2 \frac{1}{A} + \gamma^{S_1}\sigma\rho\frac{B}{A} \\ & + \frac{1}{2}\sigma^2\rho^2\frac{B^2}{A}\Bigg). \end{aligned} \tag{D.16}$$

If $\rho = 0$, the term B disappears (i.e., no condition on B is required) and (D.16) becomes:

$$\begin{aligned} C = \overline{U}^{D,\mathbb{Q}}_{y} \left(y\gamma^{S_1}\pi^*_u + \frac{1}{2}y^2(\pi^*_u)^2 A + \frac{1}{2}\left(\gamma^{S_1}\right)^2 \frac{1}{A} \right) \overset{!}{=} 0 & \overset{\overline{U}^{D,\mathbb{Q}}_{y}>0}{\Longleftrightarrow} \frac{1}{2A}\left(\gamma^{S_1} + y\pi^*_u A\right)^2 \overset{!}{=} 0 \\ & \overset{(6.8)}{\Longleftrightarrow} A \overset{!}{=} -\frac{1-p}{y}, \end{aligned}$$

i.e., Condition (6.21) of this theorem. Thus, $\hat{\mathcal{V}}^c$ satisfies the PDE (D.10).

If $\rho \neq 0$, we insert $A = -\frac{1-p}{y}$ into (D.16) and get:

$$\begin{aligned} C &= \overline{U}^{D,\mathbb{Q}}_{y} \Bigg(y\gamma^{S_1}\pi^*_u + \frac{1}{2}y^2(\pi^*_u)^2\left(-\frac{1-p}{y}\right) + \sigma\rho y\pi^*_u B + \frac{1}{2}\left(\gamma^{S_1}\right)^2\left(-\frac{y}{1-p}\right) \\ &\quad +\gamma^{S_1}\sigma\rho B\left(-\frac{y}{1-p}\right) + \frac{1}{2}\sigma^2\rho^2 B^2\left(-\frac{y}{1-p}\right)\Bigg) \\ &\overset{(6.8)}{=} \frac{\overline{U}^{D,\mathbb{Q}}_{y} y}{1-p} \Bigg(\gamma^{S_1}(\gamma^{S_1} + \sigma\rho b(t)) - \frac{1}{2}(\gamma^{S_1} + \sigma\rho b(t))^2 + \sigma\rho(\gamma^{S_1} + \sigma\rho b(t))B \\ &\quad -\frac{1}{2}\left(\gamma^{S_1}\right)^2 - \gamma^{S_1}\sigma\rho B - \frac{1}{2}\sigma^2\rho^2 B^2 \Bigg) \\ &= \frac{\overline{U}^{D,\mathbb{Q}}_{y} y}{1-p} \Bigg(\left(\gamma^{S_1}\right)^2 + \gamma^{S_1}\sigma\rho b(t) - \frac{1}{2}\left(\gamma^{S_1}\right)^2 - \gamma^{S_1}\sigma\rho b(t) - \frac{1}{2}(\sigma\rho b(t))^2 + \sigma\rho\gamma^{S_1} B \\ &\quad +(\sigma\rho)^2 b(t) B - \frac{1}{2}\left(\gamma^{S_1}\right)^2 - \gamma^{S_1}\sigma\rho B - \frac{1}{2}\sigma^2\rho^2 B^2 \Bigg) \\ &= \frac{\overline{U}^{D,\mathbb{Q}}_{y} y}{1-p} \left(-\frac{1}{2}(\sigma\rho b(t))^2 + (\sigma\rho)^2 b(t) B - \frac{1}{2}\sigma^2\rho^2 B^2 \right) \\ &= \frac{\overline{U}^{D,\mathbb{Q}}_{y} y}{1-p} \frac{\sigma^2\rho^2}{2}\left(b(t) - B\right)^2. \end{aligned}$$

Hence, if $\rho \neq 0$, $A = -\frac{1-p}{y}$ and $B = b(t)$, i.e. Conditions (6.21) and (6.22) hold, then $C = 0$. Thus, we conclude that $\hat{\mathcal{V}}^c$ satisfies PDE (D.10).

Step 5. Concluding the value function and the optimal investment strategy. Having shown that $\hat{\mathcal{V}}^c$ satisfies the HJB PDE of $\mathcal{V}^c$ for any $\rho \in [-1, 1]$, we show now that $\hat{\mathcal{V}}^c$ satisfies the terminal condition of the HJB PDE of $\mathcal{V}^c$:

$$\hat{\mathcal{V}}^c(T, D^{\widetilde{\mathbb{Q}}}(T, y, v), v) \overset{(D.13)}{=} \overline{U}^{D,\mathbb{Q}}(T, y, v) \overset{(6.20)}{=} \overline{U}(D(y)) \overset{(6.16)}{=} \overline{U}(D^{\widetilde{\mathbb{Q}}}(T, y, v)),$$

i.e., (D.11) holds with $x = D^{\widetilde{\mathbb{Q}}}(T, y, v)$.

Next we prove that $\hat{\mathcal{V}}^c$ satisfies the assumption of concavity in x. Observe that:

$$\begin{aligned} \hat{\mathcal{V}}^c_{xx} &\overset{(D.15)}{=} \frac{\overline{U}^{D,\mathbb{Q}}_{yy} D^{\widetilde{\mathbb{Q}}}_y - \overline{U}^{D,\mathbb{Q}}_y D^{\widetilde{\mathbb{Q}}}_{yy}}{\left(D^{\widetilde{\mathbb{Q}}}_y\right)^3} \overset{\text{Def. } A}{=} \left(D^{\widetilde{\mathbb{Q}}}_y\right)^{-3} A \overline{U}^{D,\mathbb{Q}}_y D^{\widetilde{\mathbb{Q}}}_y \\ &\overset{(6.21)}{=} \underbrace{\left(D^{\widetilde{\mathbb{Q}}}_y\right)^{-2}}_{>0} \underbrace{\left(-\frac{1-p}{y}\right)}_{<0} \overline{U}^{D,\mathbb{Q}}_y, \end{aligned} \tag{D.17}$$

since $y > 0, p < 1$. If $\overline{U}^{D,\mathbb{Q}}_y > 0$, then $\hat{\mathcal{V}}^c_{xx} < 0$.

Take any $y > 0$ and $\Delta y > 0$. Obviously, $Y^*(T)(\omega)|_{Y^*(t)=y+\Delta y} > Y^*(T)(\omega)|_{Y^*(t)=y}\ \forall \omega \in \Omega$. By assumption of the theorem, $D(\cdot)$ is non-decreasing on $(0, +\infty)$ with a strictly increasing part. Denote by $(\underline{d}, \overline{d}) \subset (0, +\infty)$ the subinterval where $D(\cdot)$ is strictly increasing. Denote $\mathcal{S}(y) = \{\omega \in \Omega : Y^*(T)(\omega) \in (\underline{d}, \overline{d}) | Y^*(t)(\omega) = y\}$. Then, according to (6.1) and (6.2), $\mathbb{Q}(\mathcal{S}(y)) > 0\ \forall y > 0$. Using that $\overline{U}(\cdot)$ is strictly increasing due to the strict increasingness of U, $\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\mathbb{1}_{\{D(Y^*(T))<K\}} - \varepsilon\right] = 0$ by the construction of D, we obtain that $\overline{U}^{D,\mathbb{Q}}$ is strictly increasing in y as follows:

$$\begin{aligned} \overline{U}^{D,\mathbb{Q}}(t, y+\Delta y, v) &= \mathbb{E}^{\mathbb{Q}}_{t,y+\Delta y,v}\left[\overline{U}(D(Y^*(T)))\right] \\ &= \mathbb{E}^{\mathbb{Q}}_{t,y+\Delta y,v}\left[\overline{U}(D(Y^*(T)))\mathbb{1}_{\{\mathcal{S}(y+\Delta y)\}}\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}_{t,y+\Delta y,v}\left[\overline{U}(D(Y^*(T)))\mathbb{1}_{\{\Omega\setminus\mathcal{S}(y+\Delta y)\}}\right] \\ &> \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\mathbb{1}_{\{\mathcal{S}(y+\Delta y)\}}\right] \\ &\quad + \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\mathbb{1}_{\{\Omega\setminus\mathcal{S}(y+\Delta y)\}}\right] \\ &= \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\right] = \overline{U}^{D,\mathbb{Q}}(t, y, v). \end{aligned}$$

So $\overline{U}^{D,\mathbb{Q}}$ is strictly increasing in y. Therefore, $\overline{U}^{D,\mathbb{Q}}_y > 0$, and via (D.17) we obtain that $\hat{\mathcal{V}}^c_{xx} < 0$.

Since $\hat{\mathcal{V}}^c$ satisfies the PDE of $\mathcal{V}^c$, the corresponding terminal condition, and $\hat{\mathcal{V}}^c_{xx} < 0$, we conclude that it is a candidate for the value function in the constrained optimization problem. Thus, we can now calculate the candidate for the optimal investment strategy. Plugging

$$\frac{\hat{\mathcal{V}}^c_{xv}}{\hat{\mathcal{V}}^c_{xx}} \overset{(D.15)}{=} \frac{\overline{U}^{D,\mathbb{Q}}_{yv} D^{\widetilde{\mathbb{Q}}}_y - \overline{U}^{D,\mathbb{Q}}_y D^{\widetilde{\mathbb{Q}}}_{yv}}{\left(D^{\widetilde{\mathbb{Q}}}_y\right)^2} \frac{\left(D^{\widetilde{\mathbb{Q}}}_y\right)^3}{\overline{U}^{D,\mathbb{Q}}_{yy} D^{\widetilde{\mathbb{Q}}}_y - \overline{U}^{D,\mathbb{Q}}_y D^{\widetilde{\mathbb{Q}}}_{yy}} = \frac{B}{A} D^{\widetilde{\mathbb{Q}}}_y = -\frac{yb(t)}{1-p} D^{\widetilde{\mathbb{Q}}}_y.$$

and $\hat{\mathcal{V}}^c_x$ as well as $\hat{\mathcal{V}}^c_{xx}$ from (D.15) into (D.9), we obtain the optimal control in the constrained portfolio optimization problem:

$$\pi^*_c(t) = -\frac{\gamma^{S_1}\hat{\mathcal{V}}^c_x}{x\hat{\mathcal{V}}^c_{xx}} - \frac{\sigma\rho\hat{\mathcal{V}}^c_{xv}}{x\hat{\mathcal{V}}^c_{xx}} = \frac{y\gamma^{S_1}}{1-p}\frac{D^{\widetilde{\mathbb{Q}}}_y}{D^{\widetilde{\mathbb{Q}}}} + \frac{y\sigma\rho}{1-p} b(t) \frac{D^{\widetilde{\mathbb{Q}}}_y}{D^{\widetilde{\mathbb{Q}}}} = \pi^*_u(t)\frac{yD^{\widetilde{\mathbb{Q}}}_y}{D^{\widetilde{\mathbb{Q}}}}.$$

□

Remark The above proof uses $D(\cdot)$ to ensure a matching of the terminal condition and the necessary Conditions (6.21)–(6.23). The choice of γ^v is crucial for ensuring the Conditions (6.21)–(6.23).

Proof of Lemma 6.1.4. If $\overline{U}^{D,\mathbb{Q}}_y = y^{p-1}H(t,v)D^{\widetilde{\mathbb{Q}}}_y$, then:

$$A = \frac{\overline{U}^{D,\mathbb{Q}}_{yy}}{\overline{U}^{D,\mathbb{Q}}_y} - \frac{D^{\widetilde{\mathbb{Q}}}_{yy}}{D^{\widetilde{\mathbb{Q}}}_y} = \frac{(p-1)\,H(t,v)D^{\widetilde{\mathbb{Q}}}_y y^{p-2} + H(t,v)D^{\widetilde{\mathbb{Q}}}_{yy}y^{p-1}}{H(t,v)D^{\widetilde{\mathbb{Q}}}_y y^{p-1}} - \frac{D^{\widetilde{\mathbb{Q}}}_{yy}}{D^{\widetilde{\mathbb{Q}}}_y} = -\frac{1-p}{y}, \tag{D.18}$$

i.e., Condition (6.21) holds.

If $H(t,v) = h(t)\exp(b(t)v)$, where $H(t,v)$ does not depend on y, then we have in addition:

$$B = \frac{\overline{U}^{D,\mathbb{Q}}_{yv}}{\overline{U}^{D,\mathbb{Q}}_y} - \frac{D^{\widetilde{\mathbb{Q}}}_{yv}}{D^{\widetilde{\mathbb{Q}}}_y} = \frac{b(t)D^{\widetilde{\mathbb{Q}}}_y y^{p-1}H(t,v) + D^{\widetilde{\mathbb{Q}}}_{yv}y^{p-1}H(t,v)}{D^{\widetilde{\mathbb{Q}}}_y y^{p-1}H(t,v)} - \frac{D^{\widetilde{\mathbb{Q}}}_{yv}}{D^{\widetilde{\mathbb{Q}}}_y} = b(t),$$

i.e., both Conditions (6.21) and (6.22) are satisfied. □

Proof of Corollary 6.1.5. Here we prove that for the Heston model and power-utility function conditions (6.21), (6.22) and (6.23) in Theorem 6.1.3 hold for a specific choice of $D(\cdot)$ and $\gamma^v(\cdot)$. Then we apply Theorem 6.1.3 to derive the optimal solution to (6.5) and provide more explicit formulas for computing the optimal solution and the value function.

Recall that $D(\cdot)$ has the form:

$$D\left(Y^*(T)\right) = Y^*(T) + \left(K - Y^*(T)\right) 1_{\{k_\varepsilon \le Y^*(T) \le K\}} - \left(Y^*(T) - k_v\right) 1_{\{k_\varepsilon \le Y^*(T) < k_\varepsilon\}}$$

with $k_v \le k_\varepsilon \le K$. Hence, we can rewrite $D(\cdot)$ and $\overline{U}^D(y) := \overline{U}(D(y))$ as follows:

$$\begin{aligned}
D(y) &= y + (K - y)\, 1_{\{y \le K\}} - (K - y)\, 1_{\{y < k_\varepsilon\}} + (k_v - y)\, 1_{\{y < k_\varepsilon\}} \\
&\quad - (k_v - y)\, 1_{\{y < k_v\}} \\
&= y + (K - y)\, 1_{\{y \le K\}} - (k_\varepsilon - y)\, 1_{\{y < k_v\}} - (K - k_\varepsilon)\, 1_{\{y < k_\varepsilon\}} \\
&=: D_1(y) + D_2(y) - D_3(y) - D_4(y) \qquad \text{(D.19)} \\
\overline{U}(D(y)) &= U(D(y)) - \lambda_\varepsilon \left(1_{\{D(y) < K\}} - \varepsilon\right) \\
&= \frac{1}{p}\left(y + (K - y)\, 1_{\{y < K\}} - (k_v - y)\, 1_{\{y < k_v\}} - (K - k_v)\, 1_{\{y < k_\varepsilon\}}\right)^p \\
&\quad -\lambda_\varepsilon 1_{\{y < k_\varepsilon\}} + \lambda_\varepsilon \varepsilon \\
&= \frac{y^p}{p} + \frac{1}{p}\left(K^p - y^p\right) 1_{\{y \le K\}} - \frac{1}{p}\left(k_v^p - y^p\right) 1_{\{y < k_v\}} \\
&\quad -\frac{1}{p}\left(\left(K^p - k_v^p\right) 1_{\{y < k_\varepsilon\}} + p\lambda_\varepsilon 1_{\{y < k_\varepsilon\}}\right) + \lambda_\varepsilon \varepsilon \\
&=: \overline{U}_1^D(y) + \overline{U}_2^D(y) - \overline{U}_3^D(y) - \overline{U}_4^D(y) + \lambda_\varepsilon \varepsilon \qquad \text{(D.20)}
\end{aligned}$$

Note:

$$\begin{aligned}
\{y \in \mathbb{R} : D(y) < K\} &= \left\{y \in \mathbb{R} : y + (K - y)\, 1_{\{k_\varepsilon \le y \le K\}} + (k_v - y)\, 1_{\{k_\varepsilon \le y < k_\varepsilon\}} < K\right\} \\
&= \{y \in \mathbb{R} : y < k_\varepsilon\}
\end{aligned}$$

The proof contains three parts.

Part 1. Show that Conditions (6.21) and (6.22) hold. By Lemma 6.1.4, it is sufficient to show that (SC) holds:

$$\overline{U}_y^{D,\mathbb{Q}} = y^{p-1} h(t) \exp\left(b(t)v\right) D_y^{\widetilde{\mathbb{Q}}}.$$

This involves checking three cases, as the second and third terms are structurally the same, whereas the fifth term is independent of y:

Term 1 D_1 and $\overline{U}_1^D$,

Terms 2 and 3 D_2 and $\overline{U}_2^D$, D_3 and $\overline{U}_3^D$. This involves writing the sufficient condition in terms of expectations leading to a new representation (ESC Put), then proving the equality via four steps:

Step 1 use FK theorem to derive the PDE of LHS of (ESC Put);

Step 2 use FK theorem to derive the PDE of expectation term in the RHS of (ESC Put);

Step 3 show that the terminal value of the LHS is equal to the value of the RHS, i.e., check that the terminal conditions of the corresponding PDEs are equal;

Step 4 show that RHS of (ESC Put) solves the PDE for LHS of (ESC Put).

Term 4 D_4 and $\overline{U}_4^D$.

Part 2. Addressing Condition (6.23).

Part 3. Application of Theorem 6.1.3.

We write for $i \in \{1, 2, 3, 4\}$:

$$\overline{U}^{(i)}(t, y, v) := \mathbb{E}_{t,y,v}^{\mathbb{Q}} \left[\overline{U}_i^D \left(Y^*(T) \right) \right];$$
$$D^{(i)}(t, y, v) := \mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}} \left[\exp\left(-r(T-t)\right) D_i \left(Y^*(T) \right) \right].$$

Part 1. Term 1.
For the first term of the modified utility function and the corresponding first piece of the financial derivative on the unconstrained optimal wealth, we can check the sufficient condition (SC) by explicitly calculating its LHS and RHS.

In LHS, $\overline{U}^{(1)}$ is the optimum of the objective function in (P_u), which is known due to Proposition 6.1.1:

$$\overline{U}^{(1)} = \frac{y^p}{p} \exp(a(t) + b(t)v) \quad \Rightarrow \quad \overline{U}_y^{(1)} = y^{p-1} \exp(a(t) + b(t)v).$$

As for RHS, $D_1(y) = y$ and $\exp(-rt)Y(t)$ is a martingale under any EMM $\widetilde{\mathbb{Q}}$. Thus, we have $D^{(1)} = y \Rightarrow D_y^{(1)} = 1$.

We conclude that for any $\rho \in [-1, 1]$ and any EMM $\widetilde{\mathbb{Q}}$ the following holds:

$$\overline{U}_y^{(1)} = y^{p-1} \exp(a(t) + b(t)v) \cdot 1 = y^{p-1} \underbrace{\exp\left(a(t)\right)}_{=h(t)} \exp\left(b(t)v\right) D_y^{(1)}.$$

Part 1. Terms 2 and 3.
We show now that the same relation holds for the second and third terms of the modified utility function, i.e., the utility of a put option on the unconstrained optimal wealth is

linked to a price under a convenient EMM $\widetilde{\mathbb{Q}}(\gamma^v)$ of a put option on the unconstrained optimal wealth. For simplicity of presentation we will write $\widetilde{\mathbb{Q}}$ instead of $\widetilde{\mathbb{Q}}(\gamma^v)$.

Recall from (6.14) that the expected values under $\mathbb{M} \in \{\mathbb{Q}, \widetilde{\mathbb{Q}}\}$ can be computed as

$$\begin{aligned}\mathbb{E}^{\mathbb{M}}_{t,z,v}\left[g\left(Z^*(T)\right)\right] &= \int g\left(x\right)\left(\frac{1}{2\pi}\int \exp\left(-iux\right)\phi^{Z^*(T),\mathbb{M}}(u;t,z,v)du\right)dx\\ &= \frac{1}{2\pi}\int\int g\left(x\right)\exp\left(-iu\left(x-z\right)+A^{\mathbb{M}}(T-t,u)+B^{\mathbb{M}}(T-t,u)v\right)dudx,\end{aligned}$$

where $\phi^{Z^*(T),\mathbb{M}}$ is the characteristic function of $Z^*(T)$ given in Proposition 6.1.2.

Changing variables, $Z^*(T) = \ln\left(Y^*(T)\right)$, $z = x - \ln y$, and using the inverse Fourier transform of $Z^*(T)$, we obtain $\forall\, i \in \{1,2,3,4\}$:

$$\begin{aligned}\overline{U}^{(i)} &= \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}^D_i(Y^*(T))\right]\\ &= \int \overline{U}^D_i(\exp\left(x\right))\left(\frac{1}{2\pi}\int \exp\left(-iux\right)\phi^{Z^*(T),\mathbb{Q}}(u;t,\ln y,v)du\right)dx\\ &= \frac{1}{2\pi}\int\int \overline{U}^D_i(\exp\left(x\right))\exp\Big(-iu\left(x-\ln y\right)+A^{\mathbb{Q}}(T-t,u)\\ &\quad + B^{\mathbb{Q}}(T-t,u)v\Big)du\,dx\\ &= \frac{1}{2\pi}\int\int \overline{U}^D_i(y\exp\left(z\right))\exp\Big(-iuz+A^{\mathbb{Q}}(T-t,u)+B^{\mathbb{Q}}(T-t,u)v\Big)du\,dz \quad \text{(D.21)}\\ D^{(i)} &= \exp\left(-r(T-t)\right)\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[D_i(Y^*(T))\right]\\ &= \int\int D_i(y\exp\left(z\right))\exp\left(-iuz+A^{\widetilde{\mathbb{Q}}}(T-t,u)+B^{\widetilde{\mathbb{Q}}}(T-t,u)v\right)du\,dz \quad \text{(D.22)}\\ &\quad\cdot\frac{\exp\left(-r(T-t)\right)}{2\pi}.\end{aligned}$$

For $\overline{U}^D_2(y) = \frac{1}{p}\left(K^p - y^p\right)1_{\{y<K\}}$ with any $K > 0$ fixed, we receive, using (D.21):

$$\begin{aligned}\overline{U}^{(2)} &= \frac{1}{2\pi}\frac{1}{p}\int\int\left(K^p - \exp\left(p(z+\ln y)\right)\right)1_{\{z<\ln K-\ln y\}}\\ &\quad\cdot\exp\left(-iuz+A^{\mathbb{Q}}(T-t,u)+B^{\mathbb{Q}}(T-t,u)v\right)dudz\\ &= \frac{1}{2\pi}\frac{K^p}{p}\int\int 1_{\{z<\ln K-\ln y\}}\exp\Big(-iuz+A^{\mathbb{Q}}(T-t,u)+B^{\mathbb{Q}}(T-t,u)v\Big)dudz\\ &\quad-\frac{1}{2\pi}\frac{1}{p}\int\int 1_{\{z<\ln K-\ln y\}}\exp\Big(p\ln y+pz-iuz+A^{\mathbb{Q}}(T-t,u)\\ &\quad+B^{\mathbb{Q}}(T-t,u)v\Big)dudz\end{aligned}$$

$$= \frac{1}{2\pi}\frac{K^p}{p}\int_{-\infty}^{\ln(K/y)}\int_{-\infty}^{+\infty}\exp\Big(-iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,dudz$$

$$-\frac{1}{2\pi}\frac{1}{p}\int_{-\infty}^{\ln(K/y)}\int_{-\infty}^{+\infty} y^p\exp\Big(pz - iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,dudz.$$

Taking the derivative of $\overline{U}^{(2)}$ yields:

$$\overline{U}^{(2)}_y = \frac{\partial}{\partial y}\left(\frac{1}{2\pi}\frac{K^p}{p}\int_{-\infty}^{\ln(K/y)}\underbrace{\int_{-\infty}^{+\infty}\exp\Big(-iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du\,dz}_{=:g_1(y,z)}\right)$$

$$-\frac{\partial}{\partial y}\left(\frac{1}{2\pi}\frac{1}{p}\int_{-\infty}^{\ln(K/y)}\underbrace{\int_{-\infty}^{+\infty} y^p\exp\Big(pz - iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du\,dz}_{=:g_2(y,z)}\right)$$

$$\overset{(a)}{=} \frac{1}{2\pi}\frac{K^p}{p}\left(g_1(y,\ln(K/y))\left(-\frac{1}{y}\right) - \lim_{c\downarrow-\infty}\Big(g_1(y,c)\underbrace{\frac{\partial c}{\partial y}}_{=0}\Big) + \int_{-\infty}^{\ln(K/y)}\underbrace{\frac{\partial}{\partial y}g_1(y,z)\,dz}_{=0}\right)$$

$$-\frac{1}{2\pi}\frac{1}{p}\left(g_2(y,\ln(K/y))\left(-\frac{1}{y}\right) - \lim_{c\downarrow-\infty}\Big(g_2(y,c)\underbrace{\frac{\partial c}{\partial y}}_{=0}\Big) + \int_{-\infty}^{\ln(K/y)}\frac{\partial}{\partial y}g_2(y,z)dz\right)$$

$$= -\frac{1}{2\pi}\frac{K^p}{p}\frac{1}{y}\int_{-\infty}^{+\infty}\exp\Big(-iu\ln(K/y) + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du$$

$$+\frac{1}{2\pi}\frac{1}{p}y^{p-1}\int_{-\infty}^{+\infty}\exp\Big(p\ln(K/y) - iu\ln(K/y) + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du$$

$$-\frac{1}{2\pi}\frac{1}{p}\int_{-\infty}^{\ln(K/y)}\int_{-\infty}^{+\infty} py^{p-1}\exp\Big(pz - iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du\,dz$$

$$
\begin{aligned}
&\overset{(b)}{=} \cancel{-\frac{1}{2\pi}\frac{K^p}{p}\frac{1}{y}\int_{-\infty}^{+\infty} \exp\Big(-iu\ln(K/y) + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du} \\
&\quad \cancel{+\frac{1}{2\pi}\frac{K^p}{p}\frac{1}{y}\int_{-\infty}^{+\infty} \exp\Big(-iu\ln(K/y) + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du} \\
&\quad -\frac{1}{2\pi}\frac{1}{\cancel{p}}\int_{-\infty}^{\ln(K/y)}\int_{-\infty}^{+\infty} \cancel{p}y^{p-1}\exp\Big(pz - iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du\,dz \\
&= -\frac{y^{p-1}}{2\pi}\int\int 1_{\{z<\ln K - \ln y\}}\exp\Big(pz - iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,dudz,
\end{aligned}
$$

where in (a) we use Theorem 2.5.1 and in (b) we use $\exp(p\ln(K/y)) = K^p/y^p$.

Next we reconstruct the stochastic representation of $\overline{U}_y^{(2)}$:

$$
\begin{aligned}
\overline{U}_y^{(2)} &\overset{x=z+\ln(y)}{=} -\frac{y^{p-1}}{2\pi}\int\int 1_{\{x<\ln K\}}\exp\Big(p(x-\ln(y)) - iu(x-\ln(y)) \\
&\quad + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)dudx \\
&\overset{-p\ln(y)=\ln(y^{-p})}{=} -\frac{y^{p-1}y^{-p}}{2\pi}\int\int 1_{\{x<\ln K\}}\exp(px - iux)\exp\Big(iu\ln(y) \\
&\quad + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)dudx \\
&= -y^{-1}\int 1_{\{x<\ln K\}}\exp(px)\underbrace{\left(\frac{1}{2\pi}\int\exp(-iux)\,\phi^{Z^*(T),\mathbb{Q}}(u;t,\ln(y),v)du\right)}_{\text{density of } Z^*(T) \text{ evaluated at } x \text{ given } Z^*(t)=\ln(y)}dx \\
&= -y^{-1}\mathbb{E}^{\mathbb{Q}}\left[\exp(pZ^*(T))\,1_{\{Z^*(T)<\ln K\}}|Z^*(t)=\ln(y), v(t)=v\right] \\
&\overset{Z^*(t):=\ln(Y^*(t))}{=} -y^{-1}\mathbb{E}^{\mathbb{Q}}\left[(Y^*(T))^p 1_{\{Y^*(T)<K\}}|Y^*(t)=y, v(t)=v\right].
\end{aligned}
$$

Applying the previous result for $p=1$ under the measure $\widetilde{\mathbb{Q}}$ instead of $\mathbb{Q}$, we receive the following expression for $D_2(x) = (K-y)\,1_{\{y<K\}}$ with $K>0$ a given parameter:

$$
\begin{aligned}
D^{(2)} &= \frac{1}{2\pi}\int\int D_2(y\exp(z))\exp\Big(-iuz + A^{\widetilde{\mathbb{Q}}}(T-t,u) + B^{\widetilde{\mathbb{Q}}}(T-t,u)v\Big)\,dudz \\
&\quad \cdot\exp(-r(T-t)) \\
&= -y^{-1}\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp(-r(T-t))\,Y^*(T)1_{\{Y^*(T)<K\}}|Y^*(t)=y, v(t)=v\right].
\end{aligned}
$$

Therefore, proving Condition (SC) for the second and the third terms of the auxiliary utility function is equivalent to proving the following condition:

$$-y^{-1}\mathbb{E}^{\mathbb{Q}}\left[(Y^*(T))^p\, 1_{\{Y^*(T)<K\}}|Y^*(t)=y, v(t)=v\right] \overset{!}{=} y^{p-1}\exp(a(t)+b(t)v)\left(-y^{-1}\right)$$
$$\cdot\, \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(-r(T-t)\right)Y^*(T)1_{\{Y^*(T)<K\}}|Y^*(t)=y, v(t)=v\right],$$

which, in turn, is equivalent to the following one:

$$\underbrace{\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[(Y^*(T))^p\, 1_{\{Y^*(T)<K\}}\right]}_{=:g^{\mathbb{Q}}(t,y,v)}$$
$$\overset{!}{=} y^{p-1}\exp(a(t)+b(t)v)\underbrace{\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)Y^*(T)1_{\{Y^*(T)<K\}}\right]}_{=:g^{\widetilde{\mathbb{Q}}}(t,y,v)}, \qquad \text{(ESC Put)}$$

where ESC stands for equivalent sufficient condition.

We prove now (ESC Put) via four steps.

Part 1. Terms 2 and 3. Step 1. FK PDE for LHS of (ESC Put) Recall that $\pi^*_u(t) = \frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho b(t)}{1-p}$ and under the measure $\mathbb{Q}$ we have:

$$dY^*(t) = Y^*(t)\left[\left(r+\pi^*_u(t)\gamma^{S_1}v(t)\right)dt + \pi^*_u(t)\sqrt{v(t)}dW_1^{\mathbb{Q}}(t)\right];$$
$$dv(t) = \kappa\left(\theta - v(t)\right)dt + \sigma\rho\sqrt{v(t)}\,dW_1^{\mathbb{Q}}(t) + \sigma\sqrt{v(t)}\sqrt{1-\rho^2}dW_2^{\mathbb{Q}}(t).$$

Then $\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[(Y^*(T))^p\, 1_{\{Y^*(T)<K\}}\right] = g^{\mathbb{Q}}(t,y,v)$ has the following FK representation:

$$0 = g^{\mathbb{Q}}_t + y(r+\pi^*_u(t)\gamma^{S_1}v)g^{\mathbb{Q}}_y + \kappa\left(\theta - v\right)g^{\mathbb{Q}}_v + \frac{1}{2}vy^2\left(\pi^*_u(t)\right)^2 g^{\mathbb{Q}}_{yy}$$
$$+\frac{1}{2}v\sigma^2 g^{\mathbb{Q}}_{vv} + \rho\sigma y v\pi^*_u(t)g^{\mathbb{Q}}_{yv};$$
$$y^p 1_{\{y<K\}} = g^{\mathbb{Q}}(T,y,v).$$

Part 1. Terms 2 and 3. Step 2. FK PDE for $\widetilde{\mathbb{Q}}$-expectation in RHS of (ESC Put)

Recall that under the measure $\widetilde{\mathbb{Q}}$ we have:

$$dY^*(t) = Y^*(t)rdt + Y^*(t)\pi^*_u(t)\sqrt{v(t)}dW_1^{\widetilde{\mathbb{Q}}}(t)$$
$$dv(t) = \tilde{\kappa}\left(\tilde{\theta} - v(t)\right)dt + \sigma\sqrt{v(t)}\rho dW_1^{\widetilde{\mathbb{Q}}}(t) + \sigma\sqrt{v(t)}\sqrt{1-\rho^2}dW_2^{\widetilde{\mathbb{Q}}}(t)$$

Then

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(-r(T-t)\right)Y^*(T)1_{\{Y^*(T)<K\}}|Y^*(t)=y, v(t)=v\right] = g^{\widetilde{\mathbb{Q}}}(T,y,v)$$

has the following FK representation:

$$
\begin{aligned}
0 &= g_t^{\widetilde{\mathbb{Q}}} - rg^{\widetilde{\mathbb{Q}}} + yrg_y^{\widetilde{\mathbb{Q}}} + \tilde{\kappa}\left(\tilde{\theta} - v\right) g_v^{\widetilde{\mathbb{Q}}} + \frac{1}{2}vy^2\left(\pi_u^*(t)\right)^2 g_{yy}^{\widetilde{\mathbb{Q}}} \\
&\quad + \frac{1}{2}v\sigma^2 g_{vv}^{\widetilde{\mathbb{Q}}} + \rho\sigma yv\pi_u^*(t) g_{yv}^{\widetilde{\mathbb{Q}}}; \\
y1_{\{y<K\}} &= g^{\widetilde{\mathbb{Q}}}(T, y, v).
\end{aligned}
$$

Part 1. Terms 2 and 3. Step 3. Equality of terminal conditions

Consider the ansatz $g^{\mathbb{Q}}(t, y, v) = y^{p-1}\exp(a(t) + b(t)v)g^{\widetilde{\mathbb{Q}}}(t, y, v)$ with $a(T) = b(T) = 0$. Then:

$$
\begin{aligned}
g^{\mathbb{Q}}(T, y, v) &= y^p 1_{\{y<K\}} = y^{p-1}y1_{\{y<K\}} = y^{p-1}y1_{\{y<K\}}\exp(a(T) + b(T)v) \\
&= y^{p-1}\exp(a(T) + b(T)v)g^{\widetilde{\mathbb{Q}}}(T, y, v),
\end{aligned}
$$

i.e., the LHS and RHS coincide at time $t = T$.

Part 1. Terms 2 and 3. Step 4. Verifying $g^{\mathbb{Q}}(t, y, v) = y^{p-1}\exp(a(t)+b(t)v)g^{\widetilde{\mathbb{Q}}}(t, y, v)$ via PDEs

Let us calculate the necessary partial derivatives of $g^{\mathbb{Q}}$, which appear in its FK PDE:

$$
\begin{aligned}
g_t^{\mathbb{Q}} &= \frac{\partial}{\partial t}\left(y^{p-1}\exp(a(t) + b(t)v)g^{\widetilde{\mathbb{Q}}}(t, y, v)\right) \\
&= y^{p-1}\exp(a(t) + b(t))\left(a'(t) + b'(t)v\right) g^{\widetilde{\mathbb{Q}}} + y^{p-1}\exp(a(t) + b(t))g_t^{\widetilde{\mathbb{Q}}} \\
&= y^{p-1}\exp(a(t) + b(t))\left(\left(a'(t) + b'(t)v\right) g^{\widetilde{\mathbb{Q}}} + g_t^{\widetilde{\mathbb{Q}}}\right); \\
g_y^{\mathbb{Q}} &= \exp(a(t) + b(t)v)\left((p-1)y^{p-2}g^{\widetilde{\mathbb{Q}}} + y^{p-1}g_y^{\widetilde{\mathbb{Q}}}\right) \\
&= y^{p-2}\exp(a(t) + b(t)v)\left((p-1)g^{\widetilde{\mathbb{Q}}} + yg_y^{\widetilde{\mathbb{Q}}}\right); \\
g_v^{\mathbb{Q}} &= y^{p-1}\left(\frac{\partial\exp(a(t) + b(t)v)}{\partial v}g^{\widetilde{\mathbb{Q}}} + \exp(a(t) + b(t)v)g_v^{\widetilde{\mathbb{Q}}}\right) \\
&= y^{p-1}\exp(a(t) + b(t)v)\left(b(t)g^{\widetilde{\mathbb{Q}}} + g_v^{\widetilde{\mathbb{Q}}}\right); \\
g_{yy}^{\mathbb{Q}} &= \frac{\partial}{\partial y}\left(g_y^{\mathbb{Q}}\right) = \exp(a(t) + b(t)v)\frac{\partial}{\partial y}\left((p-1)y^{p-2}g^{\widetilde{\mathbb{Q}}} + y^{p-1}g_y^{\widetilde{\mathbb{Q}}}\right) \\
&= \exp(a(t) + b(t)v)\Big((p-1)\left((p-2)y^{p-3}g^{\widetilde{\mathbb{Q}}} + y^{p-2}g_y^{\widetilde{\mathbb{Q}}}\right) \\
&\quad + \left((p-1)y^{p-2}g_y^{\widetilde{\mathbb{Q}}}\right) + y^{p-1}g_{yy}^{\widetilde{\mathbb{Q}}}\Big) \\
&= y^{p-3}\exp(a(t) + b(t)v)\left((p-1)(p-2)g^{\widetilde{\mathbb{Q}}} + 2(p-1)yg_y^{\widetilde{\mathbb{Q}}} + y^2 g_{yy}^{\widetilde{\mathbb{Q}}}\right);
\end{aligned}
$$

$$
\begin{aligned}
g^{\mathbb{Q}}_{vv} &= \frac{\partial}{\partial v}\left(g^{\mathbb{Q}}_{v}\right) = \frac{\partial}{\partial v}\left(y^{p-1}\exp(a(t)+b(t)v)\left(b(t)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{v}\right)\right)\\
&= y^{p-1}\left(\exp(a(t)+b(t)v)b(t)\left(b(t)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{v}\right) + \exp(a(t)+b(t)v)\left(b(t)g^{\widetilde{\mathbb{Q}}}_{v} + g^{\widetilde{\mathbb{Q}}}_{vv}\right)\right)\\
&= y^{p-1}\exp(a(t)+b(t)v)\left((b(t))^2 g^{\widetilde{\mathbb{Q}}} + 2b(t)g^{\widetilde{\mathbb{Q}}}_{v} + g^{\widetilde{\mathbb{Q}}}_{vv}\right);\\
g^{\mathbb{Q}}_{yv} &= \frac{\partial}{\partial y}\left(g^{\mathbb{Q}}_{v}\right) = \frac{\partial}{\partial y}\left(y^{p-1}\exp(a(t)+b(t)v)\left(b(t)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{v}\right)\right)\\
&= \exp(a(t)+b(t)v)\left((p-1)y^{p-2}\left(b(t)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{v}\right) + y^{p-1}\left(b(t)g^{\widetilde{\mathbb{Q}}}_{y} + g^{\widetilde{\mathbb{Q}}}_{yv}\right)\right)\\
&= y^{p-2}\exp(a(t)+b(t)v)\left((p-1)b(t)g^{\widetilde{\mathbb{Q}}} + (p-1)g^{\widetilde{\mathbb{Q}}}_{v} + yb(t)g^{\widetilde{\mathbb{Q}}}_{y} + yg^{\widetilde{\mathbb{Q}}}_{yv}\right).
\end{aligned}
$$

We plug those partial derivatives in the LHS PDE, i.e., FK PDE of $g^{\mathbb{Q}}$, and get:

$$
\begin{aligned}
0 &= y^{p-1}\exp(a(t)+b(t))\left(\left(a'(t)+b'(t)v\right)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{t}\right)\\
&\quad + y(r+\pi^*_u(t)\gamma^{S_1}v)y^{p-2}\exp(a(t)+b(t)v)\left((p-1)g^{\widetilde{\mathbb{Q}}} + yg^{\widetilde{\mathbb{Q}}}_{y}\right)\\
&\quad + \kappa\left(\theta - v\right)y^{p-1}\exp(a(t)+b(t)v)\left(b(t)g^{\widetilde{\mathbb{Q}}} + g^{\widetilde{\mathbb{Q}}}_{v}\right)\\
&\quad + \frac{1}{2}vy^2\left(\pi^*_u(t)\right)^2 y^{p-3}\exp(a(t)+b(t)v)\left((p-1)(p-2)g^{\widetilde{\mathbb{Q}}} + 2(p-1)yg^{\widetilde{\mathbb{Q}}}_{y} + y^2 g^{\widetilde{\mathbb{Q}}}_{yy}\right)\\
&\quad + \frac{1}{2}v\sigma^2 y^{p-1}\exp(a(t)+b(t)v)\left((b(t))^2 g^{\widetilde{\mathbb{Q}}} + 2b(t)g^{\widetilde{\mathbb{Q}}}_{v} + g^{\widetilde{\mathbb{Q}}}_{vv}\right)\\
&\quad + \rho\sigma yv\pi^*_u(t)y^{p-2}\exp(a(t)+b(t)v)\left((p-1)b(t)g^{\widetilde{\mathbb{Q}}} + (p-1)g^{\widetilde{\mathbb{Q}}}_{v} + yb(t)g^{\widetilde{\mathbb{Q}}}_{y} + yg^{\widetilde{\mathbb{Q}}}_{yv}\right).
\end{aligned}
$$

Since $\forall\, y > 0, v > 0$, we have $y^{p-1}\exp(a(t)+b(t)v) > 0$ and can divide by this term both sides of the PDE:

$$
\begin{aligned}
0 &= \left(a'(t)+b'(t)v\right)g^{\widetilde{\mathbb{Q}}} + \underline{g^{\widetilde{\mathbb{Q}}}_{t}} + (r+\pi^*_u(t)\gamma^{S_1}v)\left((p-\underline{1)g^{\widetilde{\mathbb{Q}}}} + \underline{yg^{\widetilde{\mathbb{Q}}}_{y}}\right)\\
&\quad + \kappa\left(\theta - v\right)\left(b(t)g^{\widetilde{\mathbb{Q}}} + \underline{g^{\widetilde{\mathbb{Q}}}_{v}}\right)\\
&\quad + \frac{1}{2}v\left(\pi^*_u(t)\right)^2\left((p-1)(p-2)g^{\widetilde{\mathbb{Q}}} + 2(p-1)yg^{\widetilde{\mathbb{Q}}}_{y} + \underline{y^2 g^{\widetilde{\mathbb{Q}}}_{yy}}\right)\\
&\quad + \frac{1}{2}v\sigma^2\left((b(t))^2 g^{\widetilde{\mathbb{Q}}} + 2b(t)g^{\widetilde{\mathbb{Q}}}_{v} + \underline{g^{\widetilde{\mathbb{Q}}}_{vv}}\right)\\
&\quad + \rho\sigma v\pi^*_u(t)\left((p-1)b(t)g^{\widetilde{\mathbb{Q}}} + (p-1)g^{\widetilde{\mathbb{Q}}}_{v} + yb(t)g^{\widetilde{\mathbb{Q}}}_{y} + \underline{yg^{\widetilde{\mathbb{Q}}}_{yv}}\right),
\end{aligned}
$$

where we underlined terms related to the $g^{\widetilde{\mathbb{Q}}}$ PDE.

Collecting the underlined terms in the previous equation, we get:

$$\begin{aligned}
0 = &\left(a'(t) + b'(t)v\right) g^{\widetilde{\mathbb{Q}}} + r\gamma g^{\widetilde{\mathbb{Q}}} + \pi_u^*(t)\gamma^{S_1} v \left((p-1)g^{\widetilde{\mathbb{Q}}} + y g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \kappa\left(\theta - v\right) b(t) g^{\widetilde{\mathbb{Q}}} \\
&+ \frac{1}{2} v \left(\pi_u^*(t)\right)^2 \left((p-1)(p-2) g^{\widetilde{\mathbb{Q}}} + 2(p-1) y g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \frac{1}{2} v\sigma^2 \left(\left(b(t)\right)^2 g^{\widetilde{\mathbb{Q}}} + 2b(t) g_v^{\widetilde{\mathbb{Q}}}\right) \\
&+ \rho\sigma v \pi_u^*(t) \left((p-1) b(t) g^{\widetilde{\mathbb{Q}}} + (p-1) g_v^{\widetilde{\mathbb{Q}}} + y b(t) g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \Bigg[\underline{g_t^{\widetilde{\mathbb{Q}}} - r g^{\widetilde{\mathbb{Q}}} + r y g_y^{\widetilde{\mathbb{Q}}} + \kappa\left(\theta - v\right) g_v^{\widetilde{\mathbb{Q}}} + \frac{1}{2} v \left(\pi_u^*(t)\right)^2 y^2 g_{yy}^{\widetilde{\mathbb{Q}}} + \frac{1}{2} v\sigma^2 g_{vv}^{\widetilde{\mathbb{Q}}}} \\
&\underline{+\rho\sigma v \pi_u^*(t) y g_{yv}^{\widetilde{\mathbb{Q}}}}\Bigg].
\end{aligned}$$

Next we use the link between the variance process parameters under the different measures according to (6.4):

$$\begin{aligned}
\kappa\left(\theta - v\right) &\overset{(i)}{=} \tilde{\kappa}\tilde{\theta} - \kappa v \overset{(ii)}{=} \tilde{\kappa}\tilde{\theta} - \left(\tilde{\kappa} - \sigma\gamma^{S_1}\rho - \sigma\gamma^{v}\sqrt{1-\rho^2}\right) v \\
&= \tilde{\kappa}\left(\tilde{\theta} - v\right) + \sigma\gamma^{S_1}\rho v + \sigma\gamma^{v}\sqrt{1-\rho^2} v,
\end{aligned}$$

where (i) refers to $\tilde{\theta} = \theta\kappa/\tilde{\kappa}$, (ii) refers to $\tilde{\kappa} = \kappa + \sigma\gamma^{S_1}\rho + \sigma\gamma^{v}\sqrt{1-\rho^2}$. Taking this as well as PDE of $g^{\widetilde{\mathbb{Q}}}$ into account, we get:

$$\begin{aligned}
0 = &\left(a'(t) + b'(t)v\right) g^{\widetilde{\mathbb{Q}}} + r\gamma g^{\widetilde{\mathbb{Q}}} + \pi_u^*(t)\gamma^{S_1} v \left((p-1)g^{\widetilde{\mathbb{Q}}} + y g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \kappa\left(\theta - v\right) b(t) g^{\widetilde{\mathbb{Q}}} \\
&+ \frac{1}{2} v \left(\pi_u^*(t)\right)^2 \left((p-1)(p-2) g^{\widetilde{\mathbb{Q}}} + 2(p-1) y g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \frac{1}{2} v\sigma^2 \left(\left(b(t)\right)^2 g^{\widetilde{\mathbb{Q}}} + 2b(t) g_v^{\widetilde{\mathbb{Q}}}\right) \\
&+ \rho\sigma v \pi_u^*(t) \left((p-1) b(t) g^{\widetilde{\mathbb{Q}}} + (p-1) g_v^{\widetilde{\mathbb{Q}}} + y b(t) g_y^{\widetilde{\mathbb{Q}}}\right) \\
&+ \sigma\gamma^{S_1}\rho v g_v^{\widetilde{\mathbb{Q}}} + \sigma\gamma^{v}\sqrt{1-\rho^2} v g_v^{\widetilde{\mathbb{Q}}}.
\end{aligned}$$

Using the ODEs for $a(\tau), b(\tau)$ from (D.4) (D.5) and the relation $\tau = T - t$, we conclude that:

$$
\begin{aligned}
a'(t) &= -\kappa\theta b(t) - pr;\\
b'(t) &= -\frac{1}{2}\underbrace{\left(\sigma^2 + \frac{p\sigma^2\rho^2}{1-p}\right)}_{k_2} b^2(t) + \underbrace{\left(\kappa - \frac{p\gamma^{S_1}\sigma\rho}{1-p}\right)}_{k_1} b(t) - \frac{1}{2}\underbrace{\frac{p\left(\gamma^{S_1}\right)^2}{1-p}}_{k_0}.
\end{aligned}
$$

Plugging the representation of $a'(t)$ and $b'(t)$ in the key relation we want to prove, we get:

$$
\begin{aligned}
0 = &\left(-\kappa\theta b(t) - pr + v\cdot\left(-\frac{1}{2}\left(\sigma^2 + \frac{p\sigma^2\rho^2}{1-p}\right)(b(t))^2 + \left(\kappa - \frac{p\gamma^{S_1}\sigma\rho}{1-p}\right)b(t)\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{p\left(\gamma^{S_1}\right)^2}{1-p}\right)\right)g^{\widetilde{\mathbb{Q}}} + r\gamma g^{\widetilde{\mathbb{Q}}} + \pi_u^*(t)\gamma^{S_1}v\left((p-1)g^{\widetilde{\mathbb{Q}}} + yg_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\kappa(\theta - v)\,b(t)g^{\widetilde{\mathbb{Q}}}\\
&+\frac{1}{2}v\left(\pi_u^*(t)\right)^2\left((p-1)(p-2)g^{\widetilde{\mathbb{Q}}} + 2(p-1)yg_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\frac{1}{2}v\sigma^2\left((b(t))^2 g^{\widetilde{\mathbb{Q}}} + 2b(t)g_v^{\widetilde{\mathbb{Q}}}\right)\\
&+\rho\sigma v\pi_u^*(t)\left((p-1)b(t)g^{\widetilde{\mathbb{Q}}} + (p-1)g_v^{\widetilde{\mathbb{Q}}} + yb(t)g_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\sigma\gamma^{S_1}\rho v g_v^{\widetilde{\mathbb{Q}}} + \sigma\gamma^v\sqrt{1-\rho^2}\,v g_v^{\widetilde{\mathbb{Q}}}.
\end{aligned}
$$

Next we indicate terms to be cancelled out directly and plug in the representation of $\pi_u^*(t) = \frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho b(t)}{1-p}$:

$$
\begin{aligned}
0 = &\left(-\cancel{\kappa\theta b(t)} - \cancel{pr} + v\cdot\left(-\frac{1}{2}\left(\cancel{\sigma^2} + \frac{p\sigma^2\rho^2}{1-p}\right)(b(t))^2 + \left(\cancel{\kappa} - \frac{p\gamma^{S_1}\sigma\rho}{1-p}\right)b(t)\right.\right.\\
&\left.\left.-\frac{1}{2}\frac{p\left(\gamma^{S_1}\right)^2}{1-p}\right)\right)g^{\widetilde{\mathbb{Q}}} + \cancel{r\gamma g^{\widetilde{\mathbb{Q}}}} + \left(\frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho b(t)}{1-p}\right)\gamma^{S_1}v\left((p-1)g^{\widetilde{\mathbb{Q}}} + yg_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\kappa(\cancel{\theta} - \cancel{v})\,b(t)g^{\widetilde{\mathbb{Q}}}\\
&+\frac{1}{2}v\left(\frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho b(t)}{1-p}\right)^2\left((p-1)(p-2)g^{\widetilde{\mathbb{Q}}} + 2(p-1)yg_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\frac{1}{2}v\sigma^2\left(\cancel{(b(t))^2 g^{\widetilde{\mathbb{Q}}}} + 2b(t)g_v^{\widetilde{\mathbb{Q}}}\right)\\
&+\rho\sigma v\left(\frac{\gamma^{S_1}}{1-p} + \frac{\sigma\rho b(t)}{1-p}\right)\left((p-1)b(t)g^{\widetilde{\mathbb{Q}}} + (p-1)g_v^{\widetilde{\mathbb{Q}}} + yb(t)g_y^{\widetilde{\mathbb{Q}}}\right)\\
&+\sigma\gamma^{S_1}\rho v g_v^{\widetilde{\mathbb{Q}}} + \sigma\gamma^v\sqrt{1-\rho^2}\,v g_v^{\widetilde{\mathbb{Q}}}.
\end{aligned}
$$

Next we use that $\frac{-p}{1-p} = 1 - \frac{1}{1-p}$, $\frac{p-1}{1-p} = -1$, $\frac{(p-1)(p-2)}{(1-p)(1-p)} = 1 + \frac{1}{1-p}$, expand several brackets with multiple summation terms and move y, v to the beginning of the corresponding product where they appear[2]:

$$\begin{aligned}
0 = v \cdot &\left(\frac{1}{2}\left(1 - \frac{1}{1-p}\right)\sigma^2\rho^2(b(t))^2 + \left(1 - \frac{1}{1-p}\right)\gamma^{S_1}\sigma\rho b(t) + \frac{1}{2}\left(1 - \frac{1}{1-p}\right)\left(\gamma^{S_1}\right)^2\right) \\
&\cdot g^{\widetilde{\mathbb{Q}}} - v\left(\gamma^{S_1} + \sigma\rho b(t)\right)\gamma^{S_1} g^{\widetilde{\mathbb{Q}}} + vy\gamma^{S_1}(1-p)^{-1}\left(\gamma^{S_1} + \sigma\rho b(t)\right) g_y^{\widetilde{\mathbb{Q}}} \\
&+ v\frac{1}{2}\left(1 + \frac{1}{1-p}\right)\left(\gamma^{S_1} + \sigma\rho b(t)\right)^2 g^{\widetilde{\mathbb{Q}}} + vy(p-1)^{-1}\left(\gamma^{S_1} + \sigma\rho b(t)\right)^2 g_y^{\widetilde{\mathbb{Q}}} \\
&+ v\sigma^2 b(t) g_v^{\widetilde{\mathbb{Q}}} - v\sigma\rho\left(\gamma^{S_1} + \sigma\rho b(t)\right) b(t) g^{\widetilde{\mathbb{Q}}} \\
&- v\sigma\rho\left(\gamma^{S_1} + \sigma\rho b(t)\right) g_v^{\widetilde{\mathbb{Q}}} + vy\sigma\rho(1-p)^{-1}\left(\gamma^{S_1} + \sigma\rho b(t)\right) b(t) g_y^{\widetilde{\mathbb{Q}}} + v\gamma^{S_1}\sigma\rho g_v^{\widetilde{\mathbb{Q}}} \\
&+ \sigma\gamma^v\sqrt{1-\rho^2}\, v g_v^{\widetilde{\mathbb{Q}}}.
\end{aligned}$$

The above equality is true for any $y > 0, v > 0$ if the the terms next to $vg^{\widetilde{\mathbb{Q}}}$, $vg_v^{\widetilde{\mathbb{Q}}}$, $vyg_y^{\widetilde{\mathbb{Q}}}$ are 0.

Coefficient next to $vg^{\widetilde{\mathbb{Q}}}$

Collecting all terms next to $vg^{\widetilde{\mathbb{Q}}}$ yields:

$$\begin{aligned}
0 &= \frac{1}{2}\left(1 - \frac{1}{1-p}\right)\sigma^2\rho^2(b(t))^2 + \left(1 - \frac{1}{1-p}\right)\gamma^{S_1}\sigma\rho b(t) + \frac{1}{2}\left(1 - \frac{1}{1-p}\right)\left(\gamma^{S_1}\right)^2 \\
&\quad - \left(\gamma^{S_1} + \sigma\rho b(t)\right)\gamma^{S_1} + \frac{1}{2}\left(1 + \frac{1}{1-p}\right)\left(\gamma^{S_1} + \sigma\rho b(t)\right)^2 - \sigma\rho\left(\gamma^{S_1} + \sigma\rho b(t)\right) b(t) \\
&= \frac{1}{2}\left(1 - \frac{1}{1-p}\right)\sigma^2\rho^2(b(t))^2 + \left(1 - \frac{1}{1-p}\right)\gamma^{S_1}\sigma\rho b(t) + \frac{1}{2}\left(1 - \frac{1}{1-p}\right)\left(\gamma^{S_1}\right)^2 \\
&\quad - \left(\gamma^{S_1}\right)^2 - \gamma^{S_1}\sigma\rho b(t) + \frac{1}{2}\left(1 + \frac{1}{1-p}\right)\left(\left(\gamma^{S_1}\right)^2 + 2\gamma^{S_1}\sigma\rho b(t) + (\sigma\rho b(t))^2\right) \\
&\quad - \gamma^{S_1}\sigma\rho b(t) - \sigma^2\rho^2(b(t))^2.
\end{aligned}$$

We show that the above equality is true by showing that the coefficients next to $(b(t))^2$, $b(t)^1$ and $b(t)^0$ are all equal to 0.

[2] For a better overview of the relation to be shown

For the coefficient next to $(b(t))^2$ we obtain:

$$\begin{aligned}&\frac{1}{2}\left(1-\frac{1}{1-p}\right)\sigma^2\rho^2+\frac{1}{2}\left(1+\frac{1}{1-p}\right)\sigma^2\rho^2-\sigma^2\rho^2\\&=\sigma^2\rho^2\left(\frac{1}{2}-\frac{1}{2(1-p)}+\frac{1}{2}+\frac{1}{2(1-p)}-1\right)=0.\end{aligned}$$

For the coefficient next to $(b(t))^1$ we obtain:

$$\begin{aligned}&\left(1-\frac{1}{1-p}\right)\gamma^{S_1}\sigma\rho-\gamma^{S_1}\sigma\rho+\frac{1}{2}\left(1+\frac{1}{1-p}\right)2\gamma^{S_1}\sigma\rho-\gamma^{S_1}\sigma\rho\\&=\gamma^{S_1}\sigma\rho\left(1-\frac{1}{1-p}-1+\left(1+\frac{1}{1-p}\right)-1\right)=0.\end{aligned}$$

For the coefficient next to $(b(t))^0$ we obtain:

$$\begin{aligned}&\frac{1}{2}\left(1-\frac{1}{1-p}\right)\left(\gamma^{S_1}\right)^2-\left(\gamma^{S_1}\right)^2+\frac{1}{2}\left(1+\frac{1}{1-p}\right)\left(\gamma^{S_1}\right)^2\\&=\left(\gamma^{S_1}\right)^2\left(\frac{1}{2}-\frac{1}{2}\frac{1}{1-p}-1+\frac{1}{2}+\frac{1}{2}\frac{1}{1-p}\right)=0.\end{aligned}$$

Hence, the coefficient next to $vg^{\widetilde{\mathbb{Q}}}$ is 0, i.e., $vg^{\widetilde{\mathbb{Q}}}$ vanishes in the relation we are proving.

Coefficient next to $vyg^{\widetilde{\mathbb{Q}}}_y$ The coefficient next to $vyg^{\widetilde{\mathbb{Q}}}_y$ is equal to:

$$\begin{aligned}&\gamma^{S_1}(1-p)^{-1}\left(\gamma^{S_1}+\sigma\rho b(t)\right)+(p-1)^{-1}\left(\gamma^{S_1}+\sigma\rho b(t)\right)^2+\sigma\rho(1-p)^{-1}\left(\gamma^{S_1}+\sigma\rho b(t)\right)\\&\cdot b(t)=(1-p)^{-1}\left(\gamma^{S_1}\left(\gamma^{S_1}+\sigma\rho b(t)\right)-\left(\gamma^{S_1}+\sigma\rho b(t)\right)^2+\sigma\rho b(t)\left(\gamma^{S_1}+\sigma\rho b(t)\right)\right)\\&=(1-p)^{-1}\left(\left(\gamma^{S_1}+\sigma\rho b(t)\right)^2-\left(\gamma^{S_1}+\sigma\rho b(t)\right)^2\right)=0.\end{aligned}$$

Hence, the coefficient next to $vyg^{\widetilde{\mathbb{Q}}}_y$ is 0, i.e., $vyg^{\widetilde{\mathbb{Q}}}_y$ vanishes in the relation we are proving.

Coefficient next to $vg^{\widetilde{\mathbb{Q}}}_v$ The coefficient next to $vg^{\widetilde{\mathbb{Q}}}_v$ is equal to:

$$\begin{aligned}&\sigma^2b(t)-\sigma\rho\left(\gamma^{S_1}+\sigma\rho b(t)\right)+\gamma^{S_1}\sigma\rho+\gamma^v\sigma\sqrt{1-\rho^2}\\&=\sigma^2b(t)-\sigma\rho\gamma^{S_1}-\sigma^2\rho^2b(t)+\gamma^{S_1}\sigma\rho+\gamma^v\sigma\sqrt{1-\rho^2}\\&=b(t)\sigma^2\left(1-\rho^2\right)+\gamma^v\sigma\sqrt{1-\rho^2}.\end{aligned}$$

The coefficient next to $vg^{\widetilde{\mathbb{Q}}}_v$ is equal to zero if $\gamma^v=-\sigma\sqrt{1-\rho^2}b(t)$. This is equivalent to picking a convenient change of measure on the variance process.

So for $\gamma^v = -\sigma\sqrt{1-\rho^2}b(t)$ (ESC Put) holds also for the second and third piece of the modified utility function:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{1}{p}\left(K^p - (Y^*(T))^p\right)1_{\{Y^*(T)<K\}}\right] = y^{p-1}\exp\left(a(t)+b(t)v\right) \cdot \frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp(-r(T-t))\left(K-Y^*(T)\right)1_{\{Y^*(T)<K\}}\right] \quad \forall K>0,\ p<1.$$

Part 1. Term 4. i.e. binary option

Now we show that for a specific choice of λ_ε, the last piece of the modified utility function also satisfies the same (SC), in particular $\overline{U}^{(4)}_y = y^{p-1}\exp\left(a(t)+b(t)v\right)D^{(4)}_y$.

For $\overline{U}_4(y) = \frac{1}{p}\left(K^p - k_\varepsilon^p + p\lambda_\varepsilon\right)1_{\{y<k_\varepsilon\}}$ in (D.21) we get:

$$\begin{aligned}
\overline{U}^{(4)} &= \frac{1}{2\pi}\frac{1}{p}\int\int\left(K^p - k_v^p + p\lambda_\varepsilon\right)1_{\{z<\ln k_\varepsilon - \ln y\}}\exp\Big(-iuz + A^{\mathbb{Q}}(T-t,u) \\
&\quad + B^{\mathbb{Q}}(T-t,u)v\Big)dudz \\
&= \frac{1}{2\pi}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p}\int_{-\infty}^{\ln(k_\varepsilon/y)}\underbrace{\int_{-\infty}^{+\infty}\exp\Big(-iuz + A^{\mathbb{Q}}(T-t,u) + B^{\mathbb{Q}}(T-t,u)v\Big)\,du}_{=:g(y,z)}\,dz.
\end{aligned}$$

Taking the partial derivative w.r.t. y, we obtain:

$$\begin{aligned}
\overline{U}^{(4)}_y &\overset{(i)}{=} \frac{1}{2\pi}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p}\left(g(y,\ln(k_\varepsilon/y))\left(-\frac{1}{y}\right) - \lim_{c\downarrow-\infty}\left(g(y,c)\underbrace{\frac{\partial c}{\partial y}}_{=0}\right)\right. \\
&\quad \left. + \int_{-\infty}^{\ln(k_\varepsilon/y)}\underbrace{\frac{\partial}{\partial y}g(y,z)}_{=0}\,dz\right) \\
&\overset{g}{=} -\frac{1}{y}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\Big(-iu\left(\ln k_\varepsilon - \ln y\right) + A^{\mathbb{Q}}(T-t,u) \\
&\quad + B^{\mathbb{Q}}(T-t,u)v\Big)du \\
&= -\frac{1}{y}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(-iu\ln k_\varepsilon\right)\exp\Big(iu\ln y + A^{\mathbb{Q}}(T-t,u) \\
&\quad + B^{\mathbb{Q}}(T-t,u)v\Big)du
\end{aligned}$$

$$= -\frac{1}{y}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(-iu\ln k_\varepsilon\right)\phi^{Z^*(T),\mathbb{Q}}(u;t,\ln(y),v)du}_{=:f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon)}$$

$$= -\frac{1}{y}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p} f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon),$$

where $f^{\mathbb{Q}}_{Z^*(T)}(\cdot)$ denotes the $\mathbb{Q}$-density of $Z^*(T)$ given $Z^*(t) = \ln(y)$ and we use in (i) Theorem 2.5.1.

Applying the previous result for $p = 1$, $\lambda_\varepsilon = 0$ and working under the measure $\widetilde{\mathbb{Q}}$ instead of $\mathbb{Q}$, we get for $D_4(y) = (K - k_v)\, 1_{\{y<k_\varepsilon\}}$ in (D.22) the following:

$$\begin{aligned} D^{(4)} &= \frac{1}{2\pi}\exp\left(-r(T-t)\right)\int\int (K-k_\varepsilon)\, 1_{\{z<\ln k_\varepsilon - \ln y\}} \exp\Big(-iuz + A^{\widetilde{\mathbb{Q}}}(T-t,u) \\ &\quad + B^{\widetilde{\mathbb{Q}}}(T-t,u)v\Big)dudz \\ &= -\frac{1}{y}\,(K-k_v)\exp\left(-r(T-t)\right) f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(\ln k_\varepsilon), \end{aligned}$$

where $f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}$ denotes the $\widetilde{\mathbb{Q}}$-density of $Z^*(T) = \ln(Y^*(T))$.

Hence, the condition equivalent to (SC) in the context of the fourth piece is given by:

$$\begin{aligned} &\cancel{\left(-\frac{1}{y}\right)}\frac{K^p - k_v^p + p\lambda_\varepsilon}{p} f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon) \overset{!}{=} y^{p-1}\exp\left(a(t)+b(t)v\right)\cancel{\left(-\frac{1}{y}\right)} \\ &\qquad \cdot (K-k_v)\exp\left(-r(T-t)\right) f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(\ln k_\varepsilon) \\ \Longleftrightarrow\ &\frac{K^p - k_v^p + p\lambda_\varepsilon}{p} f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon) \overset{!}{=} y^{p-1}\exp\left(a(t)+b(t)v\right)(K-k_v) \\ &\qquad \cdot \exp\left(-r(T-t)\right) f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(\ln k_\varepsilon). \end{aligned} \tag{ESC Binary}$$

Condition (ESC Binary) is satisfied if λ_ε is chosen as follows:

$$\lambda_\varepsilon = y^{p-1}\exp\left(a(t)+b(t)v\right)(K-k_v)\exp\left(-r(T-t)\right)\frac{f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(\ln k_\varepsilon)}{f^{\mathbb{Q}}_{Z^*(T)}(\ln k_\varepsilon)} - \frac{K^p - k_v^p}{p}.$$

We have shown that for specific γ^v, λ_ε, k_ε, k_v as above it holds that $\forall i \in \{1,2,3,4\}$ $\overline{U}_y^{(i)} = y^{p-1}\exp\left(a(t)+b(t)v\right)D_y^{(i)}$. Thus, $\overline{U}_y^{D,\mathbb{Q}} = y^{p-1}\exp\left(a(t)+b(t)v\right)D_y^{\widetilde{\mathbb{Q}}}$ and, by Lemma 6.1.4, both Conditions (6.21) and (6.22) in Theorem 6.1.3 are satisfied.

Part 2. Condition (6.23) is satisfied due to the assumption that (y, k_v, k_ε) solves

(NLS(y, k_v, k_ε)):

$$\begin{cases} D^{\widetilde{\mathbb{Q}}}(t, y, v) = x, & \mathcal{V}^c\left(t, D^{\widetilde{\mathbb{Q}}}(t, y, v), v\right) = \overline{U}^{D,\mathbb{Q}}(t, y, v); \\ \mathbb{Q}\left(Y^*(T) < k_\varepsilon | Y^*(t) = y, v(t) = v\right) = \varepsilon, & \text{VaR-constraint;} \\ D_v^{\widetilde{\mathbb{Q}}}(t, y, v) = 0, & \text{Condition (6.23),} \end{cases}$$

in particular the third equation. Note that there are three variables and three equations.

Part 3. In summary, Conditions (6.21) – (6.23) are satisfied as argued in Parts 1 and 2. Thus, we can apply Theorem 6.1.3 for $\gamma^v = -\sigma\sqrt{1-\rho^2}b(t)$ and conclude that

$$\begin{aligned} X^{x,\pi_c^*}(T) &= D(Y^{y,\pi_u^*}(T)) \quad \text{with} \quad x = \overbrace{\mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}(\gamma^v)}\left[\exp\left(-r(T-t)\right) D(Y^{y,\pi_u^*}(T))\right]}^{D^{\widetilde{\mathbb{Q}}}(t,y,v)=}; \\ \mathcal{V}^c(t, x, v) &= \overline{U}^{D,\mathbb{Q}}(t, y, v); \\ \pi_c^*(t) &= \pi_u^*(t) \cdot y \cdot \frac{D_y^{\widetilde{\mathbb{Q}}}(t, y, v)}{D^{\widetilde{\mathbb{Q}}}(t, y, v)}. \end{aligned}$$

□

D.3 Explicit formulas for the left-hand side of NLS(y, k_v, k_ε)

In this section of the appendix, we provide the representation of the equations in (NLS(y, k_v, k_ε)) in the spirit of Carr and Madan (1999).

Budget equation. First, we provide a formula for the price of a plain vanilla put option. Second, we derive the formula for the price of a digital put option. Afterwards, we will provide the formula for the LHS of the budget equation, which combines the obtained formulas in the previous two steps.

Put option. Take any $\alpha_P > 1$ and any strike $K > 0$. Denote $k = \ln(K)$. Analogously to Equation (3.50) in Fabrice (2013), pages 82-83, we can get:

$$\begin{aligned} Put(k) &:= Put(Y^*(T), K) = \mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}}\left[\exp\left(-r(T-t)\right)\left(K - Y^*(T)\right)^+\right] \\ &= \frac{\exp(\alpha k)}{\pi} \int_0^{+\infty} \text{Real}\Bigg(\frac{\exp\left(-r(T-t)\right)\exp\left(-iuk\right)}{\alpha_P^2 - \alpha_P - u^2 + iu(1-2\alpha_P)} \\ &\qquad \cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + (\alpha_P - 1)i; t, \ln y, v)\Bigg) du. \end{aligned} \tag{D.23}$$

Digital put option. Let $K > 0$ be an arbitrary but fixed strike of a digital put option with the nominal payment of 1 monetary unit. Denote $k = \ln(K)$. Then the price of such a digital put option is given by:

$$\begin{aligned}
DigPut(k) &:= DigPut(Y^*(T), K) = \mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)\mathbb{1}_{\{Y^*(T)<K\}}\right] \\
&\overset{Def}{=} \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\exp\left(-r(T-t)\right)\mathbb{1}_{\{Z^*(T)<k\}}|Z^*(t) = \ln(y), v(t) = v\right] \\
&= \exp\left(-r(T-t)\right)\widetilde{\mathbb{Q}}\left(Z^*(T) < k|Z^*(t) = \ln(y), v(t) = v\right) \\
&= \exp\left(-r(T-t)\right)\int_{-\infty}^{k} f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\,dz.
\end{aligned} \tag{D.24}$$

Take any $\alpha_{DP} > 0$ and consider the following dampened price of a digital put option:

$$DigPut^{(\alpha_{DP})}(k) = \exp\left(-\alpha_{DP}k\right)DigPut(k). \tag{D.25}$$

Then the Fourier transform of $DigPut^{(\alpha_{DP})}(k)$ is given by:

$$\begin{aligned}
\phi^{DigPut^{(\alpha_{DP})}}(k) &= \int_{-\infty}^{+\infty}\exp\left(iuk\right)DigPut^{(\alpha_{DP})}(k)\,dk \\
&\overset{(D.25)}{=} \int_{-\infty}^{+\infty}\exp\left(iuk\right)\exp\left(-\alpha_{DP}k\right)DigPut(k)\,dk \\
&\overset{(D.24)}{=} \int_{-\infty}^{+\infty}\exp\left(iuk\right)\exp\left(-\alpha_{DP}k\right)\exp\left(-r(T-t)\right)\int_{-\infty}^{k} f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\,dz\,dk \\
&\overset{(i)}{=} \int_{-\infty}^{+\infty}\int_{z}^{+\infty}\exp\left(iuk\right)\exp\left(-\alpha_{DP}k\right)\exp\left(-r(T-t)\right)f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\,dk\,dz \\
&= \int_{-\infty}^{+\infty}\exp\left(-r(T-t)\right)f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\left(\int_{z}^{+\infty}\exp\left(iuk\right)\exp\left(-\alpha_{DP}k\right)\,dk\right)dz \\
&\overset{\alpha_{DP}>0}{=} \int_{-\infty}^{+\infty}\exp\left(-r(T-t)\right)f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\frac{\exp\left(iuz-\alpha_{DP}z\right)}{\alpha_{DP}-iu}\,dz \\
&= \frac{\exp\left(-r(T-t)\right)}{\alpha_{DP}-iu}\int_{-\infty}^{+\infty}\exp\left(iz(u-\alpha_{DP}/i)\right)f^{\widetilde{\mathbb{Q}}}_{Z^*(T)}(z)\,dz
\end{aligned}$$

$$= \frac{\exp(-r(T-t))}{\alpha_{DP} - iu} \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u - \alpha_{DP}/i; t, \ln y, v)$$
$$= \frac{\exp(-r(T-t))}{\alpha_{DP} - iu} \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + \alpha_{DP} i; t, \ln y, v), \tag{D.26}$$

where in (i) we change the order of integration.

Therefore, the price of a digital put option is given by:

$$\begin{aligned}
DigPut(k) &= \exp(\alpha_{DP} k) \exp(-\alpha_{DP} k)\, DigPut(k) \\
&\overset{(D.25)}{=} \exp(\alpha_{DP} k)\, DigPut^{(\alpha_{DP})}(k) \\
&\overset{(2.21)}{=} \exp(\alpha_{DP} k) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Real}\Big(\exp(-iuk)\, \phi^{DigPut^{(\alpha_{DP})}}(u)\Big)\, du \\
&\overset{(D.26)}{=} \exp(\alpha_{DP} k) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Real}\Bigg(\exp(-iuk) \frac{\exp(-r(T-t))}{\alpha_{DP} - iu} \\
&\qquad \cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + \alpha_{DP} i; t, \ln y, v)\Bigg)\, du \\
&= \exp(\alpha_{DP} k) \frac{1}{\pi} \int_{0}^{+\infty} \text{Real}\Bigg(\exp(-iuk) \frac{\exp(-r(T-t))}{\alpha_{DP} - iu} \\
&\qquad \cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + \alpha_{DP} i; t, \ln y, v)\Bigg)\, du.
\end{aligned} \tag{D.27}$$

Therefore, the budget equation in (NLS(y, k_v, k_ε)) can be written as follows:

$$\begin{aligned}
D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t, y, v) &= y + \frac{\exp(\alpha_P \ln(K))}{\pi} \int_0^{+\infty} \text{Real}\Bigg(\frac{\exp(-r(T-t)) \exp(-iu \ln(K))}{\alpha_P^2 - \alpha_P - u^2 + iu(1 - 2\alpha_P)} \\
&\cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + (\alpha_P - 1)i; t, \ln y, v)\Bigg) du - \frac{\exp(\alpha_P \ln(k_v))}{\pi} \\
&\cdot \int_0^{+\infty} \text{Real}\Bigg(\frac{\exp(-r(T-t)) \exp(-iu \ln(k_v))}{\alpha_P^2 - \alpha_P - u^2 + iu(1 - 2\alpha_P)} \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + (\alpha_P - 1)i; t, \ln y, v)\Bigg) du \\
&- (K - k_v) \exp(\alpha_{DP} \ln(k_\varepsilon)) \frac{1}{\pi} \int_0^{+\infty} \text{Real}\Bigg(\exp(-iu \ln(k_\varepsilon)) \frac{\exp(-r(T-t))}{\alpha_{DP} - iu} \\
&\cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + \alpha_{DP} i; t, \ln y, v)\Bigg)\, du.
\end{aligned}$$

VaR equation. The LHS of the VaR equation can be obtained from Equation (D.27) by considering the measure $\mathbb{Q}$ instead of $\widetilde{\mathbb{Q}}$ and setting $r = 0$.

$$\mathbb{Q}\left(Y^*(T) < k_\varepsilon | Y^*(t) = y, v(t) = v\right) = \exp\left(\alpha_{DP} \ln\left(k_\varepsilon\right)\right) \cdot \frac{1}{\pi} \int_0^{+\infty} \text{Real}\left(\frac{\exp\left(-iu \ln\left(k_\varepsilon\right)\right)}{\alpha_{DP} - iu} \phi^{Z^*(T),\mathbb{Q}}(u + \alpha_{DP} i; t, \ln y, v)\right) du.$$

Vega equation. Differentiating the budget equation w.r.t v, we get:

$$\begin{aligned} D_v^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v) = {} & \frac{\exp\left(\alpha_P \ln(K)\right)}{\pi} \int_0^{+\infty} \text{Real}\Bigg(\phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + (\alpha_P - 1)i; t, \ln y, v) \\ & \cdot \frac{\exp\left(-r(T-t)\right) \exp\left(-iu \ln(K)\right) B^{\widetilde{\mathbb{Q}}}(T-t, u + (\alpha_P - 1)i)}{\alpha_P^2 - \alpha_P - u^2 + iu(1 - 2\alpha_P)} \Bigg) du - \frac{\exp\left(\alpha_P \ln(k_v)\right)}{\pi} \\ & \cdot \int_0^{+\infty} \text{Real}\Bigg(\frac{\exp\left(-r(T-t)\right) \exp\left(-iu \ln(k_v)\right) B^{\widetilde{\mathbb{Q}}}(T-t, u + (\alpha_P - 1)i)}{\alpha_P^2 - \alpha_P - u^2 + iu(1 - 2\alpha_P)} \\ & \cdot \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + (\alpha_P - 1)i; t, \ln y, v) \Bigg) du \\ & - (K - k_v) \exp\left(\alpha_{DP} \ln\left(k_\varepsilon\right)\right) \frac{1}{2\pi} \int_{-0}^{+\infty} \text{Real}\Bigg(\exp\left(-iu \ln\left(k_\varepsilon\right)\right) \\ & \cdot \frac{\exp\left(-r(T-t)\right) B^{\widetilde{\mathbb{Q}}}(T-t, u + \alpha_{DP} i)}{\alpha_{DP} - iu} \phi^{Z^*(T),\widetilde{\mathbb{Q}}}(u + \alpha_{DP} i; t, \ln y, v) \Bigg) du. \end{aligned}$$

D.4 Alternative proof for the case $\rho = 0$

In this section, we present a corollary to Theorem 6.1.3, which provides the solution to Problem (6.5) for the case $\rho = 0$. We consider this special case, as it is simpler due to fewer conditions from Theorem 6.1.3 to be verified. To prove the corollary, we will need the following lemma.

Lemma D.4.1. *Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{M})$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. Let $y^* \in \mathbb{R}$, $\delta > 0$, and $\mathcal{I}(y^*, \delta) := (y^* - \delta, y^* + \delta)$. Suppose that the following conditions hold:*

1. $\mathbb{E}^{\mathbb{M}}\left[f(\tilde{y}, X)\right] < +\infty \ \forall \, \tilde{y} \in \mathcal{I}(y^*, \delta)$;

2. $\left.\frac{\partial f(y,X)}{\partial y}\right|_{y=\tilde{y}} =: \frac{\partial}{\partial y} f(\tilde{y}, X)$ *exists and is continuous at each* $\tilde{y} \in \mathcal{I}(y^*, \delta)$*;*

3. *there exists a random variable* B *on* $(\Omega, \mathcal{F}, \mathbb{M})$ *such that* $\left|\frac{\partial}{\partial y} f(\tilde{y}, X)\right| \le B$ $\mathbb{M}$*-a.s.* $\forall\, \tilde{y} \in \mathcal{I}(y^*, \delta)$.

Then:

$$\frac{\partial}{\partial y}\left(\mathbb{E}^{\mathbb{M}}\left[f(y,X)\right]\right)\Big|_{y=y^*} = \mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial y} f(y^*, X)\right]. \tag{D.28}$$

Proof. First, we transform the LHS of (D.28) using Theorem 2.5.2. Second, we apply Theorem 2.5.5.

Take any sequence $\{h_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} h_n = 0$, e.g., $h_n = 1/n$ $\forall n \in \mathbb{N}$. Then the LHS of (D.28) can expressed as:

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\mathbb{E}^{\mathbb{M}}\left[f(y,X)\right]\right)\Big|_{y=y^*} &\overset{(i)}{=} \lim_{n\to+\infty} \frac{\mathbb{E}^{\mathbb{M}}\left[f(y^*+h_n, X)\right] - \mathbb{E}^{\mathbb{M}}\left[f(y^*, X)\right]}{y^* + h_n - y^*} \\
&\overset{(ii)}{=} \lim_{n\to+\infty} \mathbb{E}^{\mathbb{M}}\left[\frac{f(y^*+h_n, X) - f(y^*, X)}{h_n}\right] \\
&\overset{(iii)}{=} \lim_{n\to+\infty} \mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial y} f(y_n, X)\right],
\end{aligned}$$

where $y_n \in (y^*, y^*+h_n)$, we use in (i) the definition of a derivative and the differentiability of $f(y,X)$ w.r.t. y, in (ii) the linearity of expectation, in (iii) Theorem 2.5.2.

By the last assumption of the lemma we know that $\exists \delta > 0$ such that $\forall\, \tilde{y} \in \mathcal{I}(y^*, \delta)$

$$\left|\frac{\partial}{\partial y} f(\tilde{y}, X)\right| \le B$$

for some random variable B with $\mathbb{E}\left[B\right] < +\infty$.

Therefore, $\exists n^* \in \mathbb{N}$ such that $\forall n \ge n^*$:

$$\left|\frac{\partial}{\partial y} f(y_n, X)\right| \le B.$$

Thus, we obtain:

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial y} f(y_n, X)\right] &= \lim_{n^* \le n\to+\infty} \mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial y} f(y_n, X)\right] \\
&\overset{(i)}{=} \mathbb{E}^{\mathbb{M}}\left[\lim_{n^* \le n\to+\infty} \frac{\partial}{\partial y} f(y_n, X)\right] \overset{(ii)}{=} \mathbb{E}^{\mathbb{M}}\left[\frac{\partial}{\partial y} f(y^*, X)\right],
\end{aligned}$$

where we use in (i) Theorem 2.5.5 for the sequence $\left\{\frac{\partial}{\partial y} f(y_n, X)\right\}_{n\geq n^*}$ and the dominating variable B, in (ii) the definition of a derivative and $\lim_{n^*\leq n\to+\infty} y_n = y^*$. □

Corollary D.4.2 (Solution to (6.5) for $\rho = 0$). *Assume that $\rho = 0$, $\frac{p}{1-p}\left(\gamma^{S_1}\right)^2 < \frac{\kappa^2}{2\sigma^2}$ holds*[3] *and the VaR constraint is feasible in* (6.5). *If $\pi_u^* \in \mathcal{A}_c^\pi(x_0, v_0)$, then π_u^* solves* (6.5). *If $\pi_u^* \notin \mathcal{A}_c^\pi(x_0, v_0)$, set $\gamma^v = 0$ and let $D(\cdot)$ be given by* (6.28) *such that its degrees of freedom $(y, k_v, k_\varepsilon)_t$ satisfy the system of non-linear equations (SNLE):*

$$\begin{cases} h_B(y,k_v,k_\varepsilon) & := D^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v) = x_t; \\ h_{VN}(y,k_v,k_\varepsilon) & := D_v^{\widetilde{\mathbb{Q}}(\gamma^v)}(t,y,v) = 0; \\ h_{VaR}(y,k_v,k_\varepsilon) & := \mathbb{Q}\left(Y^*(T) < k_\varepsilon | Y^*(t) = y, v(t) = v\right) = \varepsilon; \end{cases}$$

for the Lagrange multiplier

$$\lambda_\varepsilon = (K - k_v)\, k_\varepsilon^{p-1} - \frac{K^p - k_v^p}{p}. \tag{D.29}$$

Then the candidate for the value function is given by (6.25) *and the candidate for the solution to* (6.5) *is given by* (6.26).

Proof of Corollary D.4.2. Before stating the proof outline and the corresponding details, we remind the reader the form of $D(\cdot)$ as per (D.19) and the form of $\overline{U}^D(\cdot)$ as per (D.20):

$$\begin{aligned} D(y) &= y + (K-y)\,1_{\{y\leq K\}} - (k_\varepsilon - y)\,1_{\{y<k_v\}} - (K - k_\varepsilon)\,1_{\{y<k_\varepsilon\}} \\ &= D_1(y) + D_2(y) - D_3(y) - D_4(y); \\ \overline{U}(D(y)) &= \frac{y^p}{p} + \frac{1}{p}\left(K^p - y^p\right)1_{\{y<K\}} - \frac{1}{p}\left(k_v^p - y^p\right)1_{\{y<k_v\}} \\ &\quad -\frac{1}{p}\left(\left(K^p - k_v^p\right)1_{\{y<k_\varepsilon\}} + p\lambda_\varepsilon 1_{\{y<k_\varepsilon\}}\right) + \lambda_\varepsilon\varepsilon \\ &= \overline{U}_1^D(y) + \overline{U}_2^D(y) - \overline{U}_3^D(y) - \overline{U}_4^D(y) + \lambda_\varepsilon\varepsilon. \end{aligned}$$

Next we provide the overview of the proof, which consists of three parts.

Part 1. Show analytically that Condition (6.21) holds for our choice of D by proving Sufficient Condition (SC_0). This involves three steps:

[3]This is a slightly stronger version of Condition (6.7)

Step (a) We first condition on the path of the variance process and prove:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right] = y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\Big[h(t,\{v(s)\})\exp(-r(T-t)) \cdot D(Y^*(T))|\mathcal{F}_v\Big],$$

where $\mathcal{F}_v$ is the σ-algebra generated by the process $(v(s))_{s\in[t,T]}$, $h(t,\{v(s)\}) := \exp\left(\int_t^T p\left(r + 0.5\left(\gamma^{S_1}\right)^2 v(s)/(1-p)\right) ds\right)$. This step contains three cases:

Term 1 D_1 and $\overline{U}^D_1$,

Terms 2 and 3 D_2 and $\overline{U}^D_2$, D_3 and $\overline{U}^D_3$,

Term 4 D_4 and $\overline{U}^D_4$,

Step (b) Via iterated expectations, we derive:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\right] = y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\Big[h(t,\{v(s)\})\exp(-r(T-t)) \cdot D(Y^*(T))\Big].$$

Step (c) The previous representation can be expressed as per (SC_0):
$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\right] = y^{p-1}H(t,v)\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp(-r(T-t))D(Y^*(T))\right].$

Part 2. Addressing Condition (6.23).

Part 3. Application of Theorem 6.1.3

Part 1.

The expectations of D and $\overline{U}^D$ are equal, when the expectations of each one of the four terms D_i and $\overline{U}^D_i$, $i = 1,..,4$, are equal. In the matching of the terms, we have three types of financial derivative components: the underlying portfolio (the optimal unconstrained portfolio, D_1), a put on the underlying portfolio (D_2 and D_3), and a binary option on the underlying portfolio (D_4).

Next we show that for the suggested D we can choose the right λ_ε and γ^v, such that each component of the financial derivative $D^{\widetilde{\mathbb{Q}}}$ satisfies (SC_0). To make the formulas more readable, we write $\widetilde{\mathbb{Q}}$ instead of $\widetilde{\mathbb{Q}}(\gamma^v)$. We write LHS for the "left-hand side", RHS for the "right-hand side".

Part 1. Step (a) We prove here

$$\underbrace{\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right]}_{=:\widetilde{U}_y(t,y,\{v_s\})} = y^{p-1}\underbrace{\exp\left(\int_t^T\left(pr+\frac{\left(\gamma^{S_1}\right)^2 v(s)p}{2(1-p)}\right)ds\right)}_{=:h(t,\{v(s)\})} \cdot \underbrace{\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)D(Y^*(T))|\mathcal{F}_v\right]}_{=:\widetilde{D}_y(t,y,\{v_s\})}, \tag{D.30}$$

where $\{v(s)\} := (v(s))_{s\in[t,T]}$.

To prove (D.30), it is sufficient to show that the equality holds term-wise:

$$\widetilde{U}^{(i)}_y(t,y,\{v_s\}) = y^{p-1}h(t,\{v(s)\})\widetilde{D}^{(i)}_y(t,y,\{v_s\}),\ i=1,\ldots,4. \tag{D.31}$$

Using (6.8) and the assumption in the corollary $\rho=0$, we get $\pi^*_u(t)=\gamma^{S_1}/(1-p)$. Hence, according to (6.11) and (6.15), the dynamics of the optimal unconstrained wealth under the real-world measure and any EMM:

$$dY^*(t) = Y^*(t)\left[\left(r+\frac{\left(\gamma^{S_1}\right)^2}{1-p}v(t)\right)dt+\frac{\gamma^{S_1}}{1-p}\sqrt{v(t)}dW_1^{\mathbb{Q}}(t)\right] \quad \text{under } \mathbb{Q};$$

$$dY^*(t) = Y^*(t)\left[rdt+\frac{\gamma^{S_1}}{1-p}\sqrt{v(t)}dW_1^{\widetilde{\mathbb{Q}}}(t)\right] \quad \text{under } \widetilde{\mathbb{Q}}.$$

Denote the volatility coefficient of $Y^*(t)$ by:

$$\hat{\sigma}(t) := \frac{\gamma^{S_1}}{1-p}\sqrt{v(t)}. \tag{D.32}$$

Applying Itô's lemma to the process Y^* (under $\mathbb{Q}$) and the function $f(x)=x^p$, we get:

$$\begin{aligned} d\left((Y^*(t))^p\right) &= p\left(Y^*(t)\right)^{p-1}d(Y^*(t)) + \frac{1}{2}p(p-1)\left(Y^*(t)\right)^{p-2}d\langle Y^*(\bullet),Y^*(\bullet)\rangle(t) \\ &= p\left(Y^*(t)\right)^{p-1}Y^*(t)\left(\left(r+\frac{\left(\gamma^{S_1}\right)^2}{1-p}v(t)\right)dt+\frac{\gamma^{S_1}}{1-p}\sqrt{v(t)}dW_1^{\mathbb{Q}}(t)\right) \\ &\quad + \frac{1}{2}p(p-1)\left(Y^*(t)\right)^{p-2}\left(Y^*(t)\right)^2\left(\frac{\gamma^{S_1}}{1-p}\sqrt{v(t)}\right)^2 dt \end{aligned}$$

$$= (Y^*(t))^p \left(p \left(r + \frac{\left(\gamma^{S_1}\right)^2}{1-p} v(t) + \frac{1}{2}(p-1) \frac{\left(\gamma^{S_1}\right)^2}{(1-p)^2} v(t) \right) dt \right.$$
$$\left. + p \frac{\gamma^{S_1}}{1-p} \sqrt{v(t)} dW_1^{\mathbb{Q}}(t) \right)$$
$$= (Y^*(t))^p \left(\left(pr + \underbrace{\frac{\left(\gamma^{S_1}\right)^2 p}{2(1-p)} v(t)}_{\overset{(D.32)}{=} \frac{1}{2} p(1-p)(\hat{\sigma}(t))^2} \right) dt + \underbrace{p \frac{\gamma^{S_1}}{1-p} \sqrt{v(t)}}_{\overset{(D.32)}{=} p\hat{\sigma}(t)} dW_1^{\mathbb{Q}}(t) \right). \tag{D.33}$$

Denote the drift coefficient of $(Y^*(t))^p$ by:

$$\hat{r}(t) := pr + \frac{p(1-p)}{2} (\hat{\sigma}(t))^2 \tag{D.34}$$

Define:

$$\bar{r} \;\; := \;\; \frac{1}{T-t} \int_t^T \hat{r}(s)\, ds \,,\, \bar{\sigma}^2 := \frac{1}{T-t} \int_t^T \left(\hat{\sigma}(s)\right)^2 \, ds. \tag{D.35}$$

Part 1. Step (a). Term 1 – the unconstrained optimal portfolio. Let us first calculate $\widetilde{U}_y^{(1)}(t, y, \{v(s)\}) = \frac{\partial}{\partial y} \mathbb{E}^{\mathbb{Q}}_{t,y,v} \left[\frac{(Y^*)^p}{p} | \mathcal{F}_v \right]$.

First, we show that $\exp\left(-\int\limits_0^t \hat{r}(s) ds\right) (Y^*(t))^p$ is a martingale under $\mathbb{Q}$. Indeed, it is $\mathcal{F}(t)$-measurable and satisfies an SDE:

$$d\left(\exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p\right) \overset{It\hat{o}}{=} \exp\left(-\int\limits_0^t \hat{r}(s)ds\right) d\left((Y^*(t))^p\right)$$
$$+ (Y^*(t))^p \, d\left(\exp\left(-\int\limits_0^t \hat{r}(s)ds\right)\right) + d\left\langle \exp\left(-\int\limits_0^\bullet \hat{r}(s)ds\right), (Y^*(\bullet))^p \right\rangle(t)$$
$$= \exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p \left(\hat{r}(t)dt + p\hat{\sigma}(t)dW_1^{\mathbb{Q}}(t)\right)$$
$$- \hat{r}(t) \exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p \, dt$$
$$= \exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p \, p\hat{\sigma}(t)dW_1^{\mathbb{Q}}(t). \tag{D.36}$$

The quadratic covariation term is equal to 0, since

$$d\left(\exp\left(-\int_0^t \hat{r}(s)ds\right)\right) = -\hat{r}(t)\exp\left(-\int_0^t \hat{r}(s)ds\right)dt$$

is without $dW_1^{\mathbb{Q}}(t)$ term. Denote by $\widetilde{Y}(t) := \exp\left(-\int_0^t \hat{r}(s)ds\right)(Y^*(t))^p, t \in [0,T]$. Using Itô's lemma for the process $\widetilde{Y}(t)$, $t \in [0,T]$, and the function $f(x) = \ln(x)$, we get:

$$\begin{aligned}
d\ln\left(\widetilde{Y}(t)\right) &\overset{Itô}{=} \frac{\partial}{dx}\ln\left(\widetilde{Y}(t)\right) d\widetilde{Y}(t) + \frac{1}{2}\frac{\partial^2}{dx^2}\ln\left(\widetilde{Y}(t)\right) d\langle\widetilde{Y}(\bullet),\widetilde{Y}(\bullet)\rangle(t) \\
&\overset{(D.36)}{=} \frac{1}{\widetilde{Y}(t)}\widetilde{Y}(t)p\hat{\sigma}(t)\,dW_1^{\mathbb{Q}}(t) - \frac{1}{2}\frac{1}{\left(\widetilde{Y}(t)\right)^2}\left(\widetilde{Y}(t)\right)^2 p^2\left(\hat{\sigma}(t)\right)^2 dt \\
&= p\hat{\sigma}(t)\,dW_1^{\mathbb{Q}}(t) - \frac{1}{2}p^2\left(\hat{\sigma}(t)\right)^2 dt.
\end{aligned}$$

Integrating and taking $\exp(\cdot)$ on both sides of the equality, we obtain:

$$\widetilde{Y}(t) = \widetilde{Y}(0)\exp\left(-\frac{1}{2}\int_0^t p^2\left(\hat{\sigma}(s)\right)^2 ds + \int_0^t p\hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s)\right).$$

Using the definition of $\widetilde{Y}$ and (D.32), we conclude that:

$$\begin{aligned}
\widetilde{Y}(t) &= \exp\left(-\int_0^t \hat{r}(s)ds\right)(Y^*(t))^p \\
&= (Y(0))^p\exp\left(-\frac{1}{2}\int_0^t \frac{p^2\left(\gamma^{S_1}\right)^2}{(1-p)^2}v(s)\,ds + \int_0^t \frac{p\gamma^{S_1}}{1-p}\sqrt{v(s)}\,dW_1^{\mathbb{Q}}(s)\right).
\end{aligned} \tag{D.37}$$

The stochastic exponential term in (D.37) is a martingale, which is proven identically to the proof of Theorem A.3[4] on page 248 in Kruse and Nögel (2005) with a minor adjustment due to the factor $\frac{p\gamma^{S_1}}{1-p}$. Thus, $\exp\left(-\int_0^t \hat{r}(s)ds\right)(Y^*(t))^p$ is a martingale. In

[4]The proof relies on Novikov's condition and the (known) conditional density function of $v(s)$ given $v(t)$, $0 \le t \le s \le T$, which is closely related to a non-central Chi-squared distribution.

particular, we have:

$$\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(-\int_0^T \hat{r}(s)ds\right)(Y^*(T))^p\right] = \exp\left(-\int_0^t \hat{r}(s)ds\right) y^p. \tag{D.38}$$

Therefore, we obtain:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{(Y^*(T))^p}{p}\bigg|\mathcal{F}_v\right] &= \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(\int_0^T \hat{r}(s)ds\right)\exp\left(\int_0^T -\hat{r}(s)ds\right)\frac{(Y^*(T))^p}{p}\bigg|\mathcal{F}_v\right] \\
&\overset{(i)}{=} \exp\left(\int_0^T \hat{r}(s)ds\right)\frac{1}{p}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(\int_0^T -\hat{r}(s)ds\right)(Y^*(T))^p\bigg|\mathcal{F}_v\right] \\
&\overset{(D.38)}{=} \exp\left(\int_0^T \hat{r}(s)ds\right)\exp\left(\int_0^t -\hat{r}(s)ds\right)\frac{y^p}{p} \\
&= \exp\left(\int_t^T \hat{r}(s)ds\right)\frac{y^p}{p} = \exp\left(\bar{r}(T-t)\right)\frac{y^p}{p},
\end{aligned} \tag{D.39}$$

where we use in (i) $\mathcal{F}_v$-measurability of $\exp\left(\int_0^T \hat{r}(s)ds\right)$. Hence, we get that:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{(Y^*(T))^p}{p}|\mathcal{F}_v\right] = y^{p-1}\exp\left(\bar{r}(T-t)\right). \tag{D.40}$$

Consider now the process $\exp(-rt)Y^*(t)$. Using $p = 1$ in the previous calculation, we can easily verify that this process is a martingale under the measure $\widetilde{\mathbb{Q}}$:

$$\begin{aligned}
\widetilde{D}^{(1)}(t, y, \{v(s)\}) &\overset{\text{Def.}\widetilde{D}^{(1)}}{=} \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp(-r(T-t))D_1(Y^*(T))|\,\mathcal{F}_v\right] \\
&\overset{(D.19)}{=} \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp(-r(T-t))Y^*(T)|\,\mathcal{F}_v\right] \\
&= \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[Y^*(T)|\,\mathcal{F}_v\right] \\
&\underset{(D.34)}{\overset{(D.39)}{=}} \exp(-r(T-t))\exp(r(T-t))y = y.
\end{aligned}$$

Therefore, $\widetilde{D}^{(1)}_y(t, y, \{v(s)\}) = 1$ and we conclude that

$$\widetilde{U}^{(1)}_y(t, y, \{v(s)\}) = y^{p-1}h(t, \{v(s)\})\widetilde{D}^{(1)}_y(t, y, \{v(s)\})$$

holds with $h(t, \{v(s)\}) = \exp(\bar{r}(T-t))$.

Part 1. Step (a). Terms 2 and 3 – put options on the unconstrained optimal portfolio. Here we show that the following holds:

$$\widetilde{U}_y^{(i)}(t, y, \{v(s)\}) = y^{p-1} h(t, \{v(s)\}) \widetilde{D}_y^{(i)}(t, y, \{v(s)\}) \quad \text{for } i \in \{2, 3\}.$$

We do that by proving the following relation $\forall k > 0$:

$$\begin{aligned} \frac{\partial}{\partial y} \mathbb{E}_{t,y,v}^{\mathbb{Q}} \left[\frac{1}{p} \left(k^p - (Y^*(T))^p \right) \mathbb{1}_{\{Y^*(T)<k\}} \middle| \mathcal{F}_v \right] &= y^{p-1} \exp(\bar{r}(T-t)) \\ \cdot \frac{\partial}{\partial y} \mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}} &\left[\exp(-\bar{r}(T-t)) (k - Y^*(T)) \mathbb{1}_{\{Y^*(T)<k\}} \middle| \mathcal{F}_v \right], \end{aligned} \tag{D.41}$$

which has as special cases $\widetilde{U}_y^{(i)}(t, y, \{v(s)\}) = y^{p-1} h(t, \{v(s)\}) \widetilde{D}_y^{(i)}(t, y, \{v(s)\})$ for $i = 2$ $(k = K)$ and $i = 3$ $(k = k_v)$.

To prove (D.41), we show for an arbitrary constant $k > 0$:

1. $\frac{\partial}{\partial y} \mathbb{E}_{t,y,v}^{\mathbb{Q}} \left[\exp(-r(T-t)) (k - Y^*(T)) \mathbb{1}_{\{Y^*(T)<k\}} \middle| \mathcal{F}_v \right] = \Phi(d_1(t, y, k, r, \bar{\sigma})) - 1$
2. $\mathbb{E}_{t,y,v}^{\mathbb{Q}} \left[\frac{1}{p} \left(k^p - (Y^*(T))^p \right) \mathbb{1}_{\{Y^*(T)<k\}} \middle| \mathcal{F}_v \right] = \exp(\bar{r}(T-t)) \frac{1}{p} Put(t, y^p, k^p, \bar{r}, p\bar{\sigma})$ for any $p \in (0, 1)$ and
$\mathbb{E}_{t,y,v}^{\mathbb{Q}} \left[\frac{1}{p} \left(k^p - (Y^*(T))^p \right) \mathbb{1}_{\{Y^*(T)<k\}} \middle| \mathcal{F}_v \right] = -\exp(\bar{r}(T-t)) \frac{1}{p} Call(t, y^p, k^p, \bar{r}, p\bar{\sigma})$ for any $p \in (-\infty, 0)$
3. $d_1(t, y^p, k^p, \bar{r}, p\bar{\sigma}) = d_1(t, y, k, r, \bar{\sigma})$
4. the results of Steps 1, 2, 3 yield (D.41),

where d_1, Put, $Call$ are defined in (2.17), (2.14), (2.13) respectively.

Part 1. Step (a) 1. We get:

$$\begin{aligned} \frac{\partial}{\partial y} \mathbb{E}_{t,y,v}^{\mathbb{Q}} \Big[\exp(-r(T-t)) (k - Y^*(T)) \mathbb{1}_{\{Y^*(T)<k\}} \Big| \mathcal{F}_v \Big] &\overset{(i)}{=} \frac{\partial}{\partial y} Put(t, y, k, r, \bar{\sigma}) \\ &\overset{(ii)}{=} \Phi(d_1(t, y, k, r, \bar{\sigma})) - 1, \end{aligned} \tag{D.42}$$

where we use in (i) the put-pricing formula due to the measurability of $\{v(s)\}$ w.r.t. $\mathcal{F}_v$, in (ii) the formula for the put-option's delta.

Part 1. Step (a) 2. For $p \in (0,1)$, we have:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\,\mathcal{F}_v\right]\\
&\quad = \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(\bar{r}(T-t)\right)\exp\left(-\bar{r}(T-t)\right)\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\,\mathcal{F}_v\right]\\
&\quad \overset{(i)}{=} \exp\left(\bar{r}(T-t)\right)\frac{1}{p}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(-\bar{r}(T-t)\right)\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{(Y^*(T))^p<k^p\}}\middle|\,\mathcal{F}_v\right]\\
&\quad \overset{(ii)}{=} \exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right), \qquad \text{(D.43)}
\end{aligned}$$

where in (i) we use that $\exp\left(\bar{r}(T-t)\right)$ is $\mathcal{F}_v$-measurable and the function x^p is increasing for $p \in (0,1)$, in (ii) we use that $\exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p$ is a $\mathbb{Q}$-martingale and given $\mathcal{F}_v$ we can apply Formula (2.14) for pricing put options in a Black-Scholes market.

For $p \in (-\infty,0)$, we have:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\,\mathcal{F}_v\right]\\
&\quad = \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(\bar{r}(T-t)\right)\exp\left(-\bar{r}(T-t)\right)\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\,\mathcal{F}_v\right]\\
&\quad \overset{(i)}{=} -\exp\left(\bar{r}(T-t)\right)\frac{1}{p}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(-\bar{r}(T-t)\right)\left((Y^*(T))^p - k^p\right)\mathbb{1}_{\{(Y^*(T))^p>k^p\}}\middle|\,\mathcal{F}_v\right]\\
&\quad \overset{(ii)}{=} -\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Call\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right), \qquad \text{(D.44)}
\end{aligned}$$

where in (i) we use that $\exp\left(\bar{r}(T-t)\right)$ is $\mathcal{F}_v$-measurable and the function x^p is decreasing for $p \in (-\infty,0)$, in (ii) we use that $\exp\left(-\int\limits_0^t \hat{r}(s)ds\right)(Y^*(t))^p$ is a $\mathbb{Q}$-martingale and given $\mathcal{F}_v$ we can apply Formula 2.13 for pricing call options in a Black-Scholes market.

Part 1. Step (a) 3. We obtain:

$$\begin{aligned}
d_1\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right) &\overset{(2.17)}{=} \frac{\ln(y^p/k^p) + \left(\bar{r} + \frac{1}{2}\left(p\bar{\sigma}\right)^2\right)(T-t)}{\sqrt{(p\bar{\sigma})^2(T-t)}}\\
&\overset{(D.34)}{=} \frac{\ln\left((y/k)^p\right) + \int_t^T \left(pr + \frac{p(1-p)}{2}\left(\hat{\sigma}(s)\right)^2\right)ds + \frac{1}{2}p^2\bar{\sigma}^2(T-t)}{\sqrt{(p\bar{\sigma})^2(T-t)}}\\
&\overset{(D.35)}{=} \frac{p\ln(y/k) + pr(T-t) + \frac{1}{2}\left(p - p^2 + p^2\right)\bar{\sigma}^2(T-t)}{|p|\bar{\sigma}\sqrt{T-t}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p\ln(y/k) + pr(T-t) + \frac{1}{2}p\bar{\sigma}^2(T-t)}{|p|\bar{\sigma}\sqrt{(T-t)}} \\
&= sign(p)\frac{\ln(y/k) + (r + \frac{1}{2}\bar{\sigma}^2)(T-t)}{\bar{\sigma}\sqrt{(T-t)}} \\
&= sign(p) d_1\left(t, y, k, r, \bar{\sigma}\right). \qquad \text{(D.45)}
\end{aligned}
$$

Step (a) 4. We obtain for $p \in (0,1)$:

$$
\begin{aligned}
&\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\mathcal{F}_v\right] \\
&\overset{\text{(D.43)}}{=} \frac{\partial}{\partial y}\left(\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right) \\
&\overset{(i)}{=} \exp\left(\bar{r}(T-t)\right)\frac{1}{p}\frac{\partial}{\partial\left(y^p\right)}\left(Put\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right)\frac{\partial}{\partial y}\left(y^p\right) \\
&\overset{(ii)}{=} \exp\left(\bar{r}(T-t)\right)\frac{1}{p}\left(\Phi\left(d_1\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right) - 1\right)py^{p-1} \\
&\overset{\text{(D.45)},p>0}{=} \exp\left(\bar{r}(T-t)\right)\left(\Phi\left(d_1\left(t, y, k, r, \bar{\sigma}\right)\right) - 1\right)y^{p-1} \\
&\overset{\text{(D.42)}}{=} \exp\left(\bar{r}(T-t)\right)y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)\left(k - Y^*(T)\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\mathcal{F}_v\right],
\end{aligned}
$$

where we use in (i) the Chain rule, in (ii) the delta of a put option.

We obtain for $p < 0$:

$$
\begin{aligned}
&\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{1}{p}\left(k^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\mathcal{F}_v\right] \\
&\overset{\text{(D.44)}}{=} \frac{\partial}{\partial y}\left(-\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Call\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right) \\
&\overset{(i)}{=} -\exp\left(\bar{r}(T-t)\right)\frac{1}{p}\frac{\partial}{\partial\left(y^p\right)}\left(Call\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right)\frac{\partial}{\partial y}\left(y^p\right) \\
&\overset{(ii)}{=} -\exp\left(\bar{r}(T-t)\right)\frac{1}{p}\Phi\left(d_1\left(t, y^p, k^p, \bar{r}, p\bar{\sigma}\right)\right)\frac{\partial}{\partial y}\left(y^p\right) \\
&\overset{\text{(D.45)},p<0}{=} -\exp\left(\bar{r}(T-t)\right)\Phi\left(-d_1\left(t, y, k, r, \bar{\sigma}\right)\right)y^{p-1} \\
&\overset{(iii)}{=} -\exp\left(\bar{r}(T-t)\right)\left(1 - \Phi\left(d_1\left(t, y, k, r, \bar{\sigma}\right)\right)\right)y^{p-1} \\
&= \exp\left(\bar{r}(T-t)\right)\left(\Phi\left(d_1\left(t, y, k, r, \bar{\sigma}\right)\right) - 1\right)y^{p-1} \\
&\overset{\text{(D.42)}}{=} \exp\left(\bar{r}(T-t)\right)y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)\left(k - Y^*(T)\right)\mathbb{1}_{\{Y^*(T)<k\}}\middle|\mathcal{F}_v\right],
\end{aligned}
$$

where we use in (i) the Chain rule, in (ii) the delta of a call option, in (iii) the property $\Phi(-x) = 1 - \Phi(x)$.

So for any $p \in (-\infty, 0) \cup (0, 1)$ we have shown that (D.31) holds for $i \in \{2, 3\}$.

Part 1. Step (a). Term 4 – binary option on the optimal unconstrained portfolio. Here we show that the following holds for the proper Lagrange multiplier:

$$
\begin{aligned}
&\widetilde{U}_y^{(4)}(t, y, \{v(s)\}) = y^{p-1} h(t, \{v(s)\}) \widetilde{D}_y^{(4)}(t, y, \{v(s)\}) \\
\overset{Def.}{\Longleftrightarrow}\ & \frac{\partial}{\partial y} \mathbb{E}^{\mathbb{Q}}_{t,y,v} \left[\frac{1}{p} \left(K^p - k_v^p + p\lambda_\varepsilon \right) \mathbb{1}_{\{Y^*(T) < k_\varepsilon\}} | \mathcal{F}_v \right] = y^{p-1} h(t, \{v(s)\}) \\
& \qquad \cdot \frac{\partial}{\partial y} \mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v} \left[\exp\left(-r(T-t)\right) (K - k_v) \mathbb{1}_{\{Y^*(T) < k_\varepsilon\}} | \mathcal{F}_v \right] \\
\Longleftrightarrow\ & \frac{1}{p} \left(K^p - k_v^p + p\lambda_\varepsilon \right) \frac{\partial}{\partial y} \mathbb{Q}_{t,y,v}(Y^*(T) < k_\varepsilon | \mathcal{F}_v) = y^{p-1} h(t, \{v(s)\})(K - k_v) \\
& \qquad \cdot \exp\left(-r(T-t)\right) \frac{\partial}{\partial y} \widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T) < k_\varepsilon | \mathcal{F}_v). \qquad \text{(D.46)}
\end{aligned}
$$

To prove (D.46), we use the relation[5]:

$$
\begin{aligned}
\frac{\partial}{\partial y} \mathbb{Q}_{t,y,v}(Y^*(T) < k | \mathcal{F}_v) &= y^{p-1} \underbrace{\exp\left(\bar{r}(T-t)\right)}_{=h(t,\{v(s)\})} \frac{1}{k^{p-1}} \\
& \quad \cdot \exp\left(-r(T-t)\right) \frac{\partial}{\partial y} \widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T) < k | \mathcal{F}_v),
\end{aligned} \qquad \text{(D.47)}
$$

which we show by calculating explicitly conditional probabilities on both sides of the relation.

First, we calculate RHS of (D.47). Recall that under the EMM $\widetilde{\mathbb{Q}}$ we have:

$$
dY^*(t) = Y^*(t) \left[r dt + \hat{\sigma}(t) dW_1^{\widetilde{\mathbb{Q}}}(t) \right].
$$

Since $\hat{\sigma}(t)$ is $\mathcal{F}_v$-measurable, as per (D.32), we get:

$$
Y^*(T) | Y^*(t) = y, \mathcal{F}_v \quad \overset{d.}{=} \quad y \exp\left(\int_t^T \left(r - \frac{1}{2} (\hat{\sigma}(s))^2 \right) ds + \int_t^T \hat{\sigma}(s)\, dW_1^{\widetilde{\mathbb{Q}}}(s) \right), \qquad \text{(D.48)}
$$

where $\overset{d.}{=}$ stands for "equal in distribution".

[5] This is an extension of Equation (34) in Kraft and Steffensen (2013) to a Black-Scholes market with time-dependent coefficients

Therefore, we get:

$$
\begin{aligned}
&\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T)<k|\mathcal{F}_v)\\
&\overset{(D.48)}{=}\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}\left(y\exp\left(\int_t^T\left(r-\frac{1}{2}\left(\hat{\sigma}(s)\right)^2\right)ds+\int_t^T\hat{\sigma}(s)\,dW_1^{\widetilde{\mathbb{Q}}}(s)\right)<k\middle|\mathcal{F}_v\right)\\
&\quad\cdot\exp\left(-r(T-t)\right)\\
&\overset{(D.35)}{=}\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}\left(\int_t^T\hat{\sigma}(s)\,dW_1^{\widetilde{\mathbb{Q}}}(s)<\ln\left(k/y\right)-\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)\middle|\mathcal{F}_v\right)\\
&=\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}\left(\frac{\int_t^T\hat{\sigma}(s)\,dW_1^{\widetilde{\mathbb{Q}}}(s)}{\bar{\sigma}\sqrt{T-t}}<\frac{\ln\left(k/y\right)-\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}}\middle|\mathcal{F}_v\right)\\
&\overset{(i)}{=}\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\Phi\left(\frac{\ln\left(k/y\right)-\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}}\right)\\
&=\exp\left(-r(T-t)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln\left(k/y\right)-\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\bar{\sigma}\sqrt{T-t}}\right)^2\right)\\
&\quad\cdot\frac{1}{\bar{\sigma}\sqrt{T-t}}\left(-\frac{1}{y}\right)\\
&=\exp\left(-r(T-t)\right)\exp\Bigg(-\frac{1}{2}\Bigg(\ln\left(\frac{k}{y}\right)^2-2\ln\left(\frac{k}{y}\right)\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)\\
&\quad+\left(\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)\right)^2\Bigg)/\left(\bar{\sigma}^2(T-t)\right)\Bigg)\cdot\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)\\
&=\exp\Bigg(-\frac{1}{2}\Bigg(\ln\left(\frac{k}{y}\right)^2-2\ln\left(\frac{k}{y}\right)\left(r+\frac{1}{2}\bar{\sigma}^2-\bar{\sigma}^2\right)(T-t)+2r\bar{\sigma}^2(T-t)^2\\
&\quad+\left(\left(r-\frac{1}{2}\bar{\sigma}^2\right)(T-t)\right)^2\Bigg)/\left(\bar{\sigma}^2(T-t)\right)\Bigg)\cdot\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)\\
&=\exp\Bigg(-\frac{1}{2}\Bigg(\ln\left(\frac{k}{y}\right)^2-2\ln\left(\frac{k}{y}\right)\left(r+\frac{1}{2}\bar{\sigma}^2\right)(T-t)+2\ln\left(\frac{k}{y}\right)\bar{\sigma}^2(T-t)\\
&\quad+\left(\left(r+\frac{1}{2}\bar{\sigma}^2\right)(T-t)\right)^2\Bigg)/\left(\bar{\sigma}^2(T-t)\right)\Bigg)\cdot\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)
\end{aligned}
$$

$$= \exp\left(-\frac{1}{2}\frac{\ln\left(\frac{k}{y}\right)^2 - 2\ln\left(\frac{k}{y}\right)\left(r+\frac{1}{2}\bar{\sigma}^2\right)(T-t) + \left(\left(r+\frac{1}{2}\bar{\sigma}^2\right)(T-t)\right)^2}{\bar{\sigma}^2(T-t)}\right)$$

$$\cdot \exp\left(-\ln\left(\frac{k}{y}\right)\right)\cdot\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)$$

$$= \exp\left(-\frac{1}{2}\left(\frac{\ln\left(\frac{k}{y}\right) - \left(r+\frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)^2\right)\frac{y}{k}\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right),$$

where we use in (i) $\left.\frac{\int_t^T \hat{\sigma}(s)\,dW_1^{\widetilde{\mathbb{Q}}}(s)}{\bar{\sigma}\sqrt{T-t}}\right|\mathcal{F}_v \overset{d.}{=} N(0,1)$.

Thus, we get:

$$\begin{aligned}&\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T)<k|\mathcal{F}_v)\\ &\qquad\overset{(2.17)}{=}\frac{y}{k}f_{N(0,1)}\left(d_1\left(t,y,k,r,\bar{\sigma}\right)\right)\frac{\partial}{\partial y}d_1\left(t,y,k,r,\bar{\sigma}\right),\end{aligned} \tag{D.49}$$

where $f_{N(0,1)}$ is the probability density function of a standard normal random variable. Next we calculate the LHS in (D.46). Recall that under the $\mathbb{Q}$-measure (see also (D.32)):

$$dY^*(t) = Y^*(t)\left[\left(r+(1-p)\left(\hat{\sigma}(t)\right)^2\right)dt + \hat{\sigma}(t)dW_1^{\mathbb{Q}}(t)\right].$$

Since $\hat{\sigma}(t)$ is $\mathcal{F}_v$-measurable, as per (D.32), we get under the $\mathbb{Q}$-measure:

$$\begin{aligned}Y^*(T)|_{Y^*(t)=y,\mathcal{F}_v} \overset{d.}{=} y\exp\Bigg(&\int_t^T\left(r+(1-p)\left(\hat{\sigma}(s)\right)^2-\frac{1}{2}\left(\hat{\sigma}(s)\right)^2\right)ds\\ &+\int_t^T\hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s)\Bigg).\end{aligned} \tag{D.50}$$

So:

$$\begin{aligned}\frac{\partial}{\partial y}\mathbb{Q}_{t,y,v}(Y^*(T)<k|\mathcal{F}_v) &\overset{(D.50)}{=} \frac{\partial}{\partial y}\mathbb{Q}\Bigg(y\exp\Bigg(\int_t^T\left(r+(1-p)\left(\hat{\sigma}(s)\right)^2-\frac{1}{2}\left(\hat{\sigma}(s)\right)^2\right)ds\\ &\quad+\int_t^T\hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s)\Bigg)<k\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\overset{(D.35)}{=} \frac{\partial}{\partial y}\mathbb{Q}\left(\int_t^T \hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s) < \ln(k/y) - \left(r + (1-p)\bar{\sigma}^2 - \frac{1}{2}\bar{\sigma}^2\right)(T-t)\middle|\,\mathcal{F}_v\right)\\
&= \frac{\partial}{\partial y}\mathbb{Q}\left(\frac{\int_t^T \hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s)}{\sqrt{\bar{\sigma}^2(T-t)}} < \frac{\ln(k/y) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\middle|\,\mathcal{F}_v\right)\\
&\overset{(i)}{=} \frac{\partial}{\partial y}\Phi\left(\frac{\ln(k/y) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right) \qquad \text{(D.51)}\\
&= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln(k/y) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)^2\right)\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\left(-\frac{1}{y}\right)\\
&= \exp\Bigg(-\frac{1}{2}\frac{1}{\bar{\sigma}^2(T-t)}\left(\ln\left(\frac{k}{y}\right)\right)^2 - 2\ln\left(\frac{k}{y}\right)\left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)\\
&\quad + \left(\left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)\right)^2\Bigg)\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)\\
&= \exp\Bigg(-\frac{1}{2}\frac{1}{\bar{\sigma}^2(T-t)}\Bigg(\left(\ln\left(\frac{k}{y}\right)\right)^2 - 2\ln\left(\frac{k}{y}\right)\left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)\\
&\quad + \left(\left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)\right)^2 + 2p\ln\left(\frac{k}{y}\right)\bar{\sigma}^2(T-t) - 2\left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)p\bar{\sigma}^2\\
&\quad \cdot (T-t) + \left(p\bar{\sigma}^2(T-t)\right)^2\Bigg)\Bigg)\left(-\frac{1}{\sqrt{2\pi}}\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)\\
&= -\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln\left(\frac{k}{y}\right) - \left(r + \frac{1}{2}\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)^2\right)\exp\left(-p\ln\left(\frac{k}{y}\right)\right)\\
&\quad \cdot \exp\left(\left(pr + \frac{1}{2}p\bar{\sigma}^2\right)(T-t) - \frac{1}{2}p^2\bar{\sigma}^2(T-t)\right)\left(\frac{1}{y\sqrt{\bar{\sigma}^2(T-t)}}\right)\\
&\underset{(D.34)}{\overset{(2.17)}{=}} f_{N(0,1)}\left(d_1(t,y,k,r,\bar{\sigma})\right)\left(\frac{y}{k}\right)^p \exp\left(\bar{r}(T-t)\right)\frac{\partial}{\partial y}d_1(t,y,k,r,\bar{\sigma}) \qquad \text{(D.52)}\\
&\overset{(D.49)}{=} y^{p-1}\exp\left(\bar{r}(T-t)\right)\frac{1}{k^{p-1}}\exp\left(-r(T-t)\right)\frac{\partial}{\partial y}\widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T) < k|\mathcal{F}_v),
\end{aligned}
$$

where we use in (i) that $\left.\frac{\int_t^T \hat{\sigma}(s)\,dW_1^{\mathbb{Q}}(s)}{\sqrt{\bar{\sigma}^2(T-t)}}\right|\mathcal{F}_v \overset{d.}{=} N(0,1)$. So (D.47) is proven.

Plugging (D.47) for $k = k_\varepsilon$ in the left-hand side of (D.46), we obtain:

$$\begin{aligned}&\frac{1}{p}\left(K^p - k_v^p + p\lambda_\varepsilon\right) y^{p-1} \exp\left(\bar{r}(T-t)\right) \frac{1}{k_\varepsilon^{p-1}} \exp\left(-r(T-t)\right) \frac{\partial}{\partial y} \widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T) < k_\varepsilon | \mathcal{F}_v)\\ &\quad = y^{p-1} h(t,\{v(s)\})(K - k_v) \exp\left(-r(T-t)\right) \frac{\partial}{\partial y} \widetilde{\mathbb{Q}}_{t,y,v}(Y^*(T) < k_\varepsilon | \mathcal{F}_v).\end{aligned}$$

Since $h(t,\{v(s)\}) = \exp\left(\bar{r}(T-t)\right)$, we conclude that (D.46) holds when λ_ε satisfies the following equation:

$$\frac{1}{p}\left(K^p - k_v^p + p\lambda_\varepsilon\right) \frac{1}{k_\varepsilon^{p-1}} \overset{!}{=} (K - k_v) \overset{p\neq 0}{\Leftrightarrow} \lambda_\varepsilon \overset{!}{=} (K - k_v)\, k_\varepsilon^{p-1} - \frac{K^p - k_v^p}{p}.$$

So for any $p \in (-\infty, 0) \cup (0,1)$, we can find λ_ε that ensures that (D.31) holds for $i = 4$.

Since we have shown that for D and λ_ε as above

$$\widetilde{U}_y^{(i)}(t,y,\{v(s)\}) = y^{p-1} h(t,\{v(s)\}) \widetilde{D}_y^{(i)}(t,y,\{v(s)\}) i \in \{1,2,3,4\},$$

we conclude that (D.30) holds, i.e.:

$$\begin{aligned}\widetilde{U}_y(t,y,\{v(s)\}) &= \frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right]\\ &= y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))|\mathcal{F}_v\right]\\ &= y^{p-1} h(t,\{v(s)\})\widetilde{D}_y(t,y,\{v(s)\}).\end{aligned}$$

Part 1. Step (b) We take on both sides of the above equality expectation w.r.t. $\mathbb{Q}$ and get:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right]\right]\\ &\qquad = \mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))|\mathcal{F}_v\right]\right].\end{aligned}$$

Since the variance process has the same dynamics under both $\mathbb{Q}$ and $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}(0)$ for $\rho = 0$ and $\gamma^v = 0$, we obtain that:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right]\right]\\ &\quad = \mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))|\mathcal{F}_v\right]\right].\end{aligned} \tag{D.53}$$

Next we show that the following holds by applying Lemma D.4.1:

$$\begin{aligned}&\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))|\mathcal{F}_v\right]\right]\\&\quad= y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))|\mathcal{F}_v\right]\right].\end{aligned} \tag{D.54}$$

We will use the following equality:

$$\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\exp\left(\bar{r}(T-t)\right)\right] = \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\int_t^T \hat{r}(s)\,ds\right)\right] = \exp(a(t)+b(t)v), \tag{D.55}$$

which follows from the definition of the value function, Equation (6.10) and Equation (D.39).

Consider the first term of the LHS in (D.53) with $\overline{U}(D(y))$ as in (D.20). We want to show that

$$\begin{aligned}&\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_1(Y^*(T))|\mathcal{F}_v\right]\right]\\&\quad= \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_1(Y^*(T))|\mathcal{F}_v\right]\right]\end{aligned} \tag{D.56}$$

holds for any $y^* \in (0,+\infty)$. Take an arbitrary but fixed $y^* \in (0,+\infty)$. The inner conditional expectation in (D.56) equals:

$$\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_1(Y^*(T))|\mathcal{F}_v\right] \overset{(i)}{=} \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{(Y^*(T))^p}{p}|\mathcal{F}_v\right] \overset{(D.39)}{=} \exp\left(\bar{r}(T-t)\right)\frac{y^p}{p}\bigg|_{y=y^*},$$

where we use in (i) the definition of $\overline{U}^D_1(\cdot)$ as per (D.20). Therefore, we can rewrite (D.56) as:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\bar{r}(T-t)\right)\frac{y^p}{p}\bigg|_{y=y^*}\right] = \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\frac{\partial}{\partial y}\exp\left(\bar{r}(T-t)\right)\frac{y^p}{p}\bigg|_{y=y^*}\right]. \tag{D.57}$$

We want to use now Lemma D.4.1. Take $\delta \in (0,y^*)$ and denote by $\mathcal{I}(y^*,\delta) := (y^*-\delta, y^*+\delta)$. Set $f(y,\bar{r}) := \exp\left(\bar{r}(T-t)\right)\cdot y^p/p$, i.e., $\bar{r}$ is a random variable that plays the role of X in the statement of Lemma D.4.1. Next we verify that all conditions of the lemma are satisfied. The first condition holds, since $\forall\,\tilde{y} \in \mathcal{I}(y^*,\delta)$:

$$\mathbb{E}^{\mathbb{Q}}_{t,v}\left[f(\tilde{y},\bar{r})\right] \overset{\text{Def.}}{=} \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\bar{r}(T-t)\right)\frac{\tilde{y}^p}{p}\right] \overset{(D.55)}{=} \frac{\tilde{y}^p}{p}\exp(a(t)+b(t)v) < +\infty.$$

The second condition, obviously, holds, i.e., $\partial f(\tilde{y},\bar{r})/\partial y$ exists and is continuous at any $\tilde{y} \in \mathcal{I}(y^*,\delta)$. As for the third condition, we can choose the dominating variable

$B := (y^* - \delta)^{p-1} \exp\left(\bar{r}(T-t)\right)$, since for any $\tilde{y} \in \mathcal{I}(y^*, \delta)$:

$$\left|\frac{\partial}{\partial y} f(\tilde{y}, \bar{r})\right| \overset{\text{Def.}}{=} \left|\frac{\partial}{\partial y} \exp\left(\bar{r}(T-t)\right) \frac{y^p}{p}\Big|_{y=\tilde{y}}\right| = \left|\tilde{y}^{p-1} \exp\left(\bar{r}(T-t)\right)\right|$$
$$\underset{y^*-\delta<\tilde{y}}{\overset{p<1}{<}} (y^* - \delta)^{p-1} \exp\left(\bar{r}(T-t)\right) = B$$

and $\mathbb{E}^{\mathbb{Q}}_{t,v}[B] < +\infty$ due to $y^* - \delta > 0$ and (D.55). Thus, we can apply Lemma D.4.1 and conclude that (D.56) holds for $y^* \in (0, +\infty)$ chosen at the beginning. Since it was arbitrarily chosen, (D.56) is fulfilled for all $y^* \in (0, +\infty)$.

Consider now the second term of the LHS in (D.53) and assume that $p \in (0, 1)$[6]. We aim at proving

$$\begin{aligned}\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}&\left[\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_2(Y^*(T))|\mathcal{F}_v\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_2(Y^*(T))|\mathcal{F}_v\right]\right]\end{aligned} \tag{D.58}$$

for all $y^* \in (0, +\infty)$. Take an arbitrary but fixed $y^* \in (0, +\infty)$. The inner conditional expectation in (D.58) equals:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_2(Y^*(T))|\mathcal{F}_v\right] &\overset{(i)}{=} \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{1}{p}\left(K^p - (Y^*(T))^p\right)\mathbb{1}_{\{Y^*(T)<K\}}\middle|\mathcal{F}_v\right] \\ &\overset{(ii)}{=} \exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, K^p, \bar{r}, p\bar{\sigma}\right)|_{y=y^*},\end{aligned}$$

where we use in (i) the definition of $\overline{U}^D_2(\cdot)$ as per (D.20), in (ii) (D.43) for $k = K$. Therefore, we can rewrite (D.58) as:

$$\begin{aligned}\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,v}&\left[\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, K^p, \bar{r}, p\bar{\sigma}\right)|_{y=y^*}\right] \\ &= \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\frac{\partial}{\partial y}\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, K^p, \bar{r}, p\bar{\sigma}\right)|_{y=y^*}\right].\end{aligned} \tag{D.59}$$

Again we plan to use Lemma D.4.1 to justify the interchangeability of the partial derivative and the expectation operators. Take $\delta \in (0, y^*)$ and let $\mathcal{I}(y^*, \delta) := (y^* - \delta, y^* + \delta)$. Since $\bar{r}$ can seen as a function of $\bar{\sigma}$ according to (D.35), we set

$$f(y, \bar{\sigma}) := \exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t, y^p, K^p, \bar{r}, p\bar{\sigma}\right)$$

[6]The case of $p \in (-\infty, 0)$ is analogous, call options are considered instead of the put options

and verify that all three conditions of the lemma hold. The first condition is satisfied:

$$\mathbb{E}^{\mathbb{Q}}_{t,v}\left[f(\tilde{y},\bar{\sigma})\right] \overset{\text{Def.}}{=} \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t,\tilde{y}^p,K^p,\bar{r},p\bar{\sigma}\right)\right] \overset{(i)}{\leq} \frac{1}{p}K^p\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\bar{r}(T-t)\right)\right]$$
$$\overset{(D.55)}{=} \frac{1}{p}K^p\exp(a(t)+b(t)v) < +\infty \qquad \forall\, \tilde{y} \in \mathcal{I}(y^*,\delta),$$

where we use in (i) that the price of a put option is not higher than its strike. The second condition, obviously, holds, i.e., $\partial f(\tilde{y},\bar{\sigma})/\partial y$ exists and is continuous at each $\tilde{y} \in \mathcal{I}(y^*,\delta)$. As for the third condition, we can choose the dominating random variable $B := (y^*-\delta)^{p-1}\exp\left(\bar{r}(T-t)\right)$, since for any $\tilde{y} \in \mathcal{I}(y^*,\delta)$:

$$\begin{aligned}\left|\frac{\partial}{\partial y}f(\tilde{y},\bar{\sigma})\right| &\overset{\text{Def.}}{=} \left|\frac{\partial}{\partial y}\left(\exp\left(\bar{r}(T-t)\right)\frac{1}{p}Put\left(t,y^p,K^p,\bar{r},p\bar{\sigma}\right)\right)\bigg|_{y=\tilde{y}}\right| \\ &\overset{(i)}{=} \left|\exp\left(\bar{r}(T-t)\right)\left(\Phi\left(d_1\left(t,\tilde{y},K,r,\bar{\sigma}\right)\right)-1\right)\tilde{y}^{p-1}\right| \\ &\overset{(ii)}{\leq} (y^*-\delta)^{p-1}\exp\left(\bar{r}(T-t)\right)\left|\Phi\left(d_1\left(t,\bar{y},K,r,\bar{\sigma}\right)\right)-1\right| \\ &\overset{(iii)}{\leq} (y^*-\delta)^{p-1}\exp\left(\bar{r}(T-t)\right) = B,\end{aligned}$$

where we use in (i) the result from Part 1, Step (a) 4 above, in (ii) $p<1$ and $\exp(\cdot)>0$, in (iii) $\Phi(\cdot) \in [0,1]$ as a distribution function. Since $\mathbb{E}^{\mathbb{Q}}_{t,v}[B] < +\infty$ due to $y^*-\delta > 0$ and (D.55), we can apply Lemma D.4.1 and conclude that (D.58) holds for chosen $y^* \in (0,+\infty)$. As y^* was arbitrarily fixed, we conclude that (D.58) is satisfied for any $y^* \in (0,+\infty)$.

The third term is of the same form as the second one but with a different strike. Thus, the interchangeability of the partial derivative and the expectation operators in this case follows analogously, i.e., (D.58) holds also for $\overline{U}^D_3(\cdot)$.

Finally, we consider the fourth term and prove that for any $y^* \in (0,+\infty)$ the following holds:

$$\begin{aligned}&\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_4(Y^*(T))|\mathcal{F}_v\right]\right] \\ &\qquad = \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_4(Y^*(T))|\mathcal{F}_v\right]\right].\end{aligned} \tag{D.60}$$

Take an arbitrary but fixed y^*. The inner conditional expectation in (D.60) equals:

$$\mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\overline{U}^D_4(Y^*(T))|\mathcal{F}_v\right] \overset{(i)}{=} \mathbb{E}^{\mathbb{Q}}_{t,y=y^*,v}\left[\frac{1}{p}\left(K^p - k^p_v + p\lambda_\varepsilon\right)\mathbb{1}_{\{Y^*(T)<k_\varepsilon\}}|\mathcal{F}_v\right]$$
$$= \frac{1}{p}\left(K^p - k^p_v + p\lambda_\varepsilon\right)\mathbb{Q}_{t,y=y^*,v}\left(Y^*(T) < k_\varepsilon\right)$$
$$\overset{(ii)}{=} \frac{1}{p}\left(K^p - k^p_v + p\lambda_\varepsilon\right)\Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\Bigg|_{y=y^*},$$

where we use in (i) the definition of $\overline{U}^D_4(\cdot)$ as per (D.20), in (ii) (D.51). Taking also into account that $\left(K^p - k^p_v + p\lambda_\varepsilon\right)/p$ is a constant independent of y, we conclude that (D.60) is equivalent to:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\Bigg|_{y=y^*}\right]$$
$$= \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\frac{\partial}{\partial y}\Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\Bigg|_{y=y^*}\right]. \tag{D.61}$$

As before, we want to use Lemma D.4.1 to justify the interchangeability of the partial derivative and the expectation operators. Take $\delta \in (0, y^*)$ and let $\mathcal{I}(y^*, \delta) := (y^* - \delta, y^* + \delta)$. Since $\bar{r}$ can seen as a function of $\bar{\sigma}$ according to (D.35), we set

$$f(y, \bar{\sigma}) := \Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)$$

and verify that all three conditions of the lemma hold. The first condition is satisfied:

$$\mathbb{E}^{\mathbb{Q}}_{t,v}\left[f(\tilde{y}, \bar{\sigma})\right] \overset{\text{Def.}}{=} \mathbb{E}^{\mathbb{Q}}_{t,v}\left[\Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\right] \overset{(i)}{\leq} 1 \quad \forall\, \tilde{y} \in \mathcal{I}(y^*, \delta),$$

where we use in (i) $\Phi(\cdot) \in [0, 1]$. The second condition, obviously, holds, i.e., $\partial f(\tilde{y}, \bar{\sigma})/\partial y$ exists and is continuous at each $\tilde{y} \in \mathcal{I}(y^*, \delta)$. As for the third condition, we can choose the dominating random variable

$$B := (y^* - \delta)^{p-1} k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right),$$

since for any $\tilde{y} \in \mathcal{I}(y^*, \delta)$:

$$\left|\frac{\partial}{\partial y}f(\tilde{y}, \bar{\sigma})\right| \overset{\text{Def.}}{=} \left|\frac{\partial}{\partial y}\Phi\left(\frac{\ln\left(k_\varepsilon/y\right) - \left(r + \frac{1}{2}\bar{\sigma}^2 - p\bar{\sigma}^2\right)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\Bigg|_{y=\tilde{y}}\right|$$

$$
\begin{aligned}
&\overset{(i)}{=} \left| f_{N(0,1)}\left(d_1\left(t,\tilde{y},k_\varepsilon,r,\bar{\sigma}\right)\right)\left(\frac{\tilde{y}}{k_\varepsilon}\right)^p \exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\tilde{y}\sqrt{\bar{\sigma}^2(T-t)}}\right)\right| \\
&\overset{(ii)}{=} \tilde{y}^{p-1}k_\varepsilon{}^{-p} f_{N(0,1)}\left(d_1\left(t,\tilde{y},k_\varepsilon,r,\bar{\sigma}\right)\right)\exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right) \\
&\overset{(iii)}{\leq} (y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right) = B,
\end{aligned}
$$

where we use in (i) Equations (D.52) and (2.17), in (ii) the fact that each multiplication term in the previous line is positive, in (iii) $p < 1$ and $f_{N(0,1)}(\cdot) \leq 1/\sqrt{2\pi}$.

The dominating variable B is integrable:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{t,v}\left[B\right] &\overset{\text{Def.}}{=} \mathbb{E}^{\mathbb{Q}}_{t,v}\left[(y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\right] \\
&= (y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\bar{r}(T-t)\right)\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right)\right] \\
&\overset{(i)}{\leq} (y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\sqrt{\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\left(\exp\left(\bar{r}(T-t)\right)\right)^2\right]\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\left(\frac{1}{\sqrt{\bar{\sigma}^2(T-t)}}\right)^2\right]} \\
&\underset{\text{(D.32)}}{\overset{\text{(D.34)}}{=}} (y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\left|\frac{1-p}{\gamma^{S_1}}\right|\left(\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\left(\int_t^T v(s)\,ds\right)^{-1}\right]\right)^{\frac{1}{2}} \\
&\quad\cdot\left(\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(2\int_t^T\left(pr+\frac{p(1-p)}{2}\left(\frac{\gamma^{S_1}}{1-p}\sqrt{v(s)}\right)^2\right)ds\right)\right]\right)^{\frac{1}{2}} \\
&= \underbrace{(y^*-\delta)^{p-1}k_\varepsilon{}^{-p}\left(\sqrt{2\pi}\right)^{-1}\left|\frac{1-p}{\gamma^{S_1}}\right|}_{\overset{(ii)}{<}+\infty}\Bigg(\underbrace{\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\left(\int_t^T v(s)\,ds\right)^{-1}\right]}_{\overset{(iii)}{<}+\infty}\Bigg)^{\frac{1}{2}} \\
&\quad\cdot\underbrace{\exp\left(2pr(T-t)\right)}_{<+\infty}\Bigg(\underbrace{\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(\frac{p\left(\gamma^{S_1}\right)^2}{1-p}\int_t^T v(s)\,ds\right)\right]}_{\overset{(iv)}{<}+\infty}\Bigg)^{\frac{1}{2}},
\end{aligned}
$$

where we use in (i) the Cauchy-Schwartz inequality, in (ii) the inequality $y^*-\delta > 0$ due to the choice of δ, in (iii) we use Theorem 4.1a[7] for $r = -1$ in Dufresne (2001), in (iv) we use the assumption of our corollary $\frac{p}{1-p}\left(\gamma^{S_1}\right)^2 \leq \frac{\kappa^2}{2\sigma^2}$ and Proposition 5.1[8] in Kraft (2005).

Since all three conditions of Lemma D.4.1 hold, we conclude that (D.60) holds at $y^* \in (0, +\infty)$. As y^* was arbitrarily fixed, we conclude that (D.60) is satisfied for any $y^* \in (0, +\infty)$.

So far we have shown that the LHS of (D.53) equals the LHS of (D.54) by showing the equality for each of the four pieces of the modified utility. Analogously, we can show via Lemma D.4.1 that the RHS of the (D.53) equals the RHS (D.54) holds, and, thus, (D.54) is equivalent to (D.53).

Applying to (D.54) the tower property of conditional expectation, we obtain:

$$\frac{\partial}{\partial y}\mathbb{E}^{\mathbb{Q}}_{t,y,v}\left[\overline{U}(D(Y^*(T)))\right] = y^{p-1}\frac{\partial}{\partial y}\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))\right].$$

Part 1. Step (c) we show that there exists a function $H(t, v)$ such that:

$$\underbrace{\mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[h(t,\{v(s)\})\exp(-r(T-t))D(Y^*(T))\right]}_{=:\hat{D}(t,y,v)} = H(t,v)D^{\widetilde{\mathbb{Q}}}(t,y,v).$$

Recall that the function (where $\widetilde{\mathbb{Q}}$ is $\widetilde{\mathbb{Q}}(\gamma^v = 0)$)

$$D^{\widetilde{\mathbb{Q}}}(t,y,v) = \mathbb{E}^{\widetilde{\mathbb{Q}}}_{t,y,v}\left[\exp\left(-r(T-t)\right)D(Y^*(T))\right]$$

has the following FK representation given $\rho = 0$ and $\gamma^v = 0$:

$$\begin{aligned}
&D^{\widetilde{\mathbb{Q}}}_t = rD^{\widetilde{\mathbb{Q}}} - ryD^{\widetilde{\mathbb{Q}}}_y - \kappa\left(\theta - v\right)D^{\widetilde{\mathbb{Q}}}_v - \frac{1}{2}v\left(y^2(\pi^*_u)^2 D^{\widetilde{\mathbb{Q}}}_{yy} + \sigma^2 D^{\widetilde{\mathbb{Q}}}_{vv}\right);\\
&D^{\widetilde{\mathbb{Q}}}(T,y,v) = D(y).
\end{aligned}$$

The function $\hat{D}(t,y,v)$ has the following FK representation:

$$\begin{aligned}
&\widehat{D}_t = \left(r - \hat{r}(t,v)\right)\widehat{D} - ry\widehat{D}_y - \kappa\left(\theta - v\right)\widehat{D}_v - \frac{1}{2}v\left(y^2(\pi^*_u)^2\widehat{D}_{yy} + \sigma^2\widehat{D}_{vv}\right);\\
&\widehat{D}(T,y,v) = D(y),
\end{aligned}$$

[7] It states that expectation of the integrated square-root process raised to any finite power is finite

[8] This proposition provides conditions under which the expression $\mathbb{E}^{\mathbb{Q}}_{t,v}\left[\exp\left(-av(T) - b\int_t^T v(s)\,ds\right)\right]$ is well-defined

where $\hat{r}(t,v) = p\Big(r + \frac{(\gamma^{S_1})^2 v}{2(1-p)}\Big)$.

We plug the ansatz $\widehat{D}(t,y,v) = D^{\widetilde{\mathbb{Q}}}(t,y,v)H(t,v)$ into the FK PDE for $\widehat{D}$:

$$D_t^{\widetilde{\mathbb{Q}}} H + D^{\widetilde{\mathbb{Q}}} H_t = (r - \hat{r}(t,v))\, D^{\widetilde{\mathbb{Q}}} H - ryD_y^{\widetilde{\mathbb{Q}}} H - \kappa(\theta - v)\left(D_v^{\widetilde{\mathbb{Q}}} H + D^{\widetilde{\mathbb{Q}}} H_v\right)$$
$$-\frac{1}{2} v y^2 (\pi_u^*(t))^2 D_{yy}^{\widetilde{\mathbb{Q}}} H - \frac{1}{2} v\sigma^2 \left(D^{\widetilde{\mathbb{Q}}} H_{vv} + D_{vv}^{\widetilde{\mathbb{Q}}} H + 2D_v^{\widetilde{\mathbb{Q}}} H_v\right).$$

Collecting the terms related to the PDE for $D^{\widetilde{\mathbb{Q}}}$, we get:

$$\underbrace{H\left(D_t^{\widetilde{\mathbb{Q}}} - rD^{\widetilde{\mathbb{Q}}} + ryD_y^{\widetilde{\mathbb{Q}}} + \kappa(\theta - v)\, D_v^{\widetilde{\mathbb{Q}}} + \frac{1}{2} v\left(y^2(\pi_u^*)^2 D_{yy}^{\widetilde{\mathbb{Q}}} - \sigma^2 D_{vv}^{\widetilde{\mathbb{Q}}}\right)\right)}_{=0}$$
$$= -D^{\widetilde{\mathbb{Q}}} H_t - \hat{r}(t,v) D^{\widetilde{\mathbb{Q}}} H - \kappa(\theta - v)\, D^{\widetilde{\mathbb{Q}}} H_v - \frac{1}{2} v\sigma^2 D^{\widetilde{\mathbb{Q}}} H_{vv} - v\sigma^2 \underbrace{D_v^{\widetilde{\mathbb{Q}}}}_{=0} H_v.$$

Using the PDE for $D^{\widetilde{\mathbb{Q}}}$, Condition (6.23) $D_v^{\widetilde{\mathbb{Q}}} = 0$ and $D^{\widetilde{\mathbb{Q}}} > 0$, we obtain a PDE for $H(t,v)$, which is independent of y:

$$\begin{aligned} 0 &= -H_t - \hat{r}(t,v)H - \kappa(\theta - v)\, H_v - \frac{1}{2} v\sigma^2 H_{vv}; \\ H(T,v) &= 1. \end{aligned}$$

Plugging the explicit form of $\hat{r}(t,v) = p(r + \frac{(\gamma^{S_1})^2 v}{2(1-p)})$ in the above PDE for $H(t,v)$ and collecting the terms multiplied by v, we obtain the same PDE as (D.3) for $\rho = 0$, namely:

$$0 = H_t + \kappa\theta H_v + prH + v\left(\frac{1}{2}\sigma^2 H_{vv} - \kappa H_v + \frac{1}{2}\frac{p\left(\gamma^{S_1}\right)^2}{1-p} H\right).$$

Thus, we conclude that $H(t,v) = \exp\left(a(\tau(t)) + b(\tau(t))v\right)$, where $a(\tau(t))$ and $b(\tau(t))$ are given by (D.6) and (D.7) respectively for $\rho = 0$.

Hence, $\hat{D}(t,y,v) = H(t,v)D^{\widetilde{\mathbb{Q}}}(t,y,v) = \exp\left(a(t) + b(t)v\right) D^{\widetilde{\mathbb{Q}}}(t,y,v)$, which completes the proof that (SC_0) holds for $\rho = 0$ and $\gamma^v = 0$. By Lemma 6.1.4, Condition (6.21) holds.

Part 2. Condition (6.23) is satisfied due to the assumption that (y, k_v, k_ε) solve SNLE

(NLS(y, k_v, k_ε)):

$$\begin{cases} D^{\widetilde{\mathbb{Q}}}(t,y,v) = x, & \mathcal{V}^c\left(t, D^{\widetilde{\mathbb{Q}}}(t,y,v), v\right) = \overline{U}^{D,\mathbb{Q}}(t,y,v); \\ \mathbb{Q}\left(Y^*(T) < k_\varepsilon | Y^*(t) = y, v(t) = v\right) = \varepsilon, & \text{VaR-constraint;} \\ D_v^{\widetilde{\mathbb{Q}}}(t,y,v) = 0, & \text{Condition (6.23),} \end{cases}$$

in particular the third equation. Note that there are three variables and three equations.

Part 3. Since $\rho = 0$ by assumption in this Corollary, Condition (6.21) holds as per Part 1, and Condition (6.23) is satisfied as per Part 2, we can apply Theorem 6.1.3 for $\gamma^v = 0$ and conclude that

$$\begin{aligned} X^{x,\pi_c^*}(T) &= D(Y^{y,\pi_u^*}(T)) \quad \text{with} \quad x = \overbrace{\mathbb{E}_{t,y,v}^{\widetilde{\mathbb{Q}}(\gamma^v)}\left[\exp\left(-r(T-t)\right) D(Y^{y,\pi_u^*}(T))\right]}^{D^{\widetilde{\mathbb{Q}}}(t,y,v)=}; \\ \mathcal{V}^c\left(t,x,v\right) &= \overline{U}^{D,\mathbb{Q}}(t,y,v); \\ \pi_c^*(t) &= \pi_u^*(t) \cdot y \cdot \frac{D_y^{\widetilde{\mathbb{Q}}}(t,y,v)}{D^{\widetilde{\mathbb{Q}}}(t,y,v)}. \end{aligned}$$

□

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