Soft-collinear Gravity and Soft Theorems

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Soft-kollineare Gravitation und Soft Theorems

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Abstract

In this thesis, we construct the soft-collinear Lagrangian for gravity systematically beyond leading power in the power-counting parameter and provide a set of minimal building blocks for the N-jet operators. We find that the effective theory is covariant with respect to an emergent soft background field that is obscured in the full theory. The emission of a soft gluon and graviton from a non-radiative process is investigated and an operatorial version of the soft theorem is obtained.

Zusammenfassung

In dieser Dissertation wird die soft-kollineare Lagrangedichte für Gravitation systematisch in höhere Ordnungen im Entwicklungsparameter konstruiert und eine minimale Menge von Bausteinen für die N-jet Operatoren präsentiert. Die effektive Theorie ist kovariant bezüglich eines emergenten soften Hintergrundfeldes, was nicht aus der vollen Theorie ersichtlich ist. Zusätzlich wird die Emission eines soften Gluon bzw. Graviton betrachtet und eine operatorielle Darstellung des Soft Theorems bestimmt.

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Chapter 1

Introduction

It would be difficult to pretend that the gravitational infrared divergence problem is very urgent. My reasons for now attacking this question are: (1) Because I can. [...] (2) Because something might go wrong and this would be interesting. Unfortunately, nothing does go wrong.

— Steven Weinberg [1]

Soft physics, and the related infrared divergences, are among the oldest conceptual problems in quantum field theory. Already in 1937, Bloch and Nordsieck [2] uncovered that in Quantum Mechanics and Quantum Electrodynamics (QED), soft photons break the perturbative expansion as the radiative corrections and real emissions of soft photons lead to divergences.

This can be seen directly at the amplitude level. The emission of a photon with momentum k from an external (massless) fermion leg carrying momentum p takes the form

$$e\overline{u}(p)\gamma^\mu \frac{\not{p}+\not{k}}{(p+k)^2+i0}\mathcal{A}\,,$$

where $\mathcal{A}$ denotes the rest of the amplitude. If the photon and fermion are on-shell, $p^2=k^2=0$, one finds that the denominator of the propagator takes the form

$$2p\cdot k = E_p\omega_k(1-\cos\theta)\,,$$

where $E_p=|\boldsymbol{p}|$ and $\omega_k=|\boldsymbol{k}|$. If either the fermion or the photon is soft, meaning $E_p\to 0$ or $\omega_k\to 0$, the denominator becomes small and the propagator diverges. The same is true for the collinear limit $\theta\to 0$, when the photon is emitted in the same direction as the fermion.[1] Physically, in these limits, the intermediate particle goes on-shell $(p+k)^2\to 0$. This not only holds for real-emission processes but also for virtual corrections, which cause further infrared divergences. At this point, one can start to wonder if these real and virtual divergences are related, as they have the same origin.

By investigating these divergences, Bloch and Nordsieck uncovered what is now known as the *eikonal approximation* and performed the first true all-order computation in perturbative QED [2], proving that real and virtual corrections cancel out by virtue of soft exponentiation, and thus providing a proof that infrared-safe observables exist in QED. Since then, much has been learnt about soft physics [3, 4], and, more importantly, the focus has switched to Quantum Chromodynamics (QCD) [5–7].

Some of the intuition and knowledge obtained from studying soft effects in QED did transfer over to its non-Abelian counterpart, even though the naive cancellation is violated and a more careful treatment is necessary, culminating in the Kinoshita-Lee-Nauenberg theorem [8–12]. However, in QED, no massless charged particles exist, and hence there are no true collinear divergences. This is in contrast to QCD, where the massless gluon carries a colour charge and collinear singularities pose a threat. The rigorous and systematic study of the singular structure has led to the important ideas of factorisation and universality, which have proven valuable in QCD. For an overview see [13, 14].

[1] If the fermion is massive, the collinear divergence is controlled by the fermion mass as $E_p(1-|\boldsymbol{v}|\cos\theta)$. In this case, there are no true divergences but there can be large enhancements.

Compared to gauge theory, which is strongly connected with elementary particle physics, gravity is often viewed as fundamentally different. It is usually formulated in terms of geometric objects that encode deep ideas and intuition about the classical properties of space-time, very distinct from the microscopic nature of quantum gauge theories. Despite this, the quantisation of gravity has a long history [15–24], although it is not a story of great success. When viewed as a quantum theory, the gravitational interactions are accompanied by a dimensionful coupling that causes the theory to be non-renormalisable and invalid at high energies, corresponding to short distances. This seems to rule out gravity as a fundamental quantum theory, unless it is modified at high energies.

However, as a low-energy effective theory, gravity can be viewed as a perfectly consistent quantum theory [25, 26]. For this low energy setting, corresponding to small curvatures, the Einstein-Hilbert action is simply the theory of a massless spin-2 field on flat Minkowski space,[2] very similar to a gauge theory. Its action can be constructed in an analogous way as the one for a massless spin-1 field in gauge theory, based on locality and gauge-invariance. In gravity, this gauge symmetry corresponds to invariance under local translations. From this point of view, gravity and gauge theory seem to be quite similar in nature.

One example, where these similarities can be made more precise, is the soft theorem [1,27–29]. For single gauge-boson emission, the soft theorem, in this context also called the Low-Burnett-Kroll (LBK) amplitude, states that the amplitude is universal up to next-to-leading power in the soft expansion. It can be derived only from gauge-invariance and the form of the eikonal factor, as we show in Section 1.1. The LBK amplitude takes the form

$$\mathcal{A}_{\text{rad}} = -g \sum_{i=1}^{n} t_i^a \, \overline{u}(p_i) \left(\frac{p_i \cdot \varepsilon^a(k)}{p_i \cdot k} + \frac{k_\nu \varepsilon^a_\mu(k) J_i^{\mu\nu}}{p_i \cdot k} \right) \mathcal{A}_{\text{nr}} + \mathcal{O}(k) \,,$$

and relates the radiative amplitude $\mathcal{A}_{\text{rad}}$ with an additional soft gauge boson emitted, to the non-radiative amplitude $\mathcal{A}_{\text{nr}}$. Here, t_i^a are the gauge generators, p_i is the momentum of the emitting particle, and $\overline{u}(p)$ is its polarisation vector. The soft gauge particle carries momentum k and has the polarisation vector $\varepsilon^a_\mu(k)$. In the subleading term, the angular momentum operator $J_i^{\mu\nu} = L_i^{\mu\nu} + \Sigma_i^{\mu\nu}$ appears.

In the following years, Weinberg [1] investigated soft and collinear divergences in gravity and generalised the soft theorem to include gravitons. It was subsequently further generalised in the late 1960s [30,31], where also a subleading term was identified. In the same article [1], Weinberg also found diagrammatic cancellations for the collinear singularities in gravity—an important result, crucial for the consistency of gravity. Unlike QCD, which is confining and hence there are no *asymptotic* massless colour-charged states in Nature, gravity features a number of massless charged particles, including photons and the graviton itself. The presence of collinear divergences would either be a catastrophe or of vital interest. (Un)Fortunately, there are no such divergences. With this clarification, the interest in soft and collinear gravity faded.

In 2014, however, the soft graviton theorem had a surprising spike in popularity. Motivated by asymptotic symmetries and using spinor-helicity methods, the soft graviton amplitude was reconsidered, and a *third* universal term was discovered [29]. The gravitational soft theorem is thus expressed as

$$\mathcal{A}_{\text{rad}} = \frac{\kappa}{2} \sum_i \overline{u}(p_i) \left(\frac{\varepsilon_{\mu\nu}(k) p_i^\mu p_i^\nu}{p_i \cdot k} + \frac{\varepsilon_{\mu\nu}(k) p_i^\mu k_\rho J_i^{\nu\rho}}{p_i \cdot k} + \frac{1}{2} \frac{\varepsilon_{\mu\nu}(k) k_\rho k_\sigma J_i^{\rho\mu} J_i^{\sigma\nu}}{p_i \cdot k} \right) \mathcal{A}_{\text{nr}} + \mathcal{O}(k^2) \,,$$

and takes a remarkably similar form to the gauge-theory result—except for the presence of a third term. This renewed interest in soft gravitons, soft theorems [32–37], and the gauge-theory result, but it was shown that there are no further universal terms.

[2] Or some other classical background.

This sparks the question: how can we understand why there are only two universal terms in gauge theory, but three terms in gravity? In what way is the theorem affected by the gauge symmetry, and how can we deduce its form from the underlying theory?

Unfortunately, neither the spinor-helicity formalism nor the conventional derivation provides deep insights into these questions. In the spinor-helicity derivation [29], the third universal term appears simply from the little-group scaling of the graviton amplitude. In the old-fashioned derivation [33], the gravitational gauge-invariance can be used "twice" to yield this term. While it seems clear that the underlying reason is the different structure of the gauge symmetry in gravity, it is not so obvious how this can be made more transparent.

The existence of soft theorems shows that infrared effects can be viewed as universal long-distance effects and are indeed independent of the underlying short-distance (hard) physics.

Another lesson learnt from the study of infrared divergences is that the conventional perturbative approach of quantum field theory is not suited for studying infrared effects. Often, all-order computations and summations have to be performed to resolve the infrared problems, which are cumbersome in standard perturbation theory.

These two points motivate an investigation of infrared effects using methods from effective field theories (EFTs). They offer a convenient and systematic separation of the long- and short-distance physics and provide the possibility to perform all-order computations using the perturbative expansion of the effective theory, based on power-counting and not the coupling expansion.

One of the tools that emerged to study these effects systematically is soft-collinear effective theory (SCET) [38–41]. SCET is one of the more complicated effective theories in high-energy particle physics. Here, the dynamic degrees of freedom are energetic ("collinear") and soft modes of the full-theory particles, while hard modes are integrated out. This yields an expansion around the classical light-cone of the energetic particle. In the Lagrangian construction, it was understood early on [41, 42] that gauge symmetry heavily constrains the form of the Lagrangian and is essential in the all-order construction.

However, while SCET QCD is quite advanced and developed, soft-collinear gravity has not received much attention, besides the leading-power construction [43] and a first construction of the subleading collinear sector [44].

Moreover, while the subleading LBK amplitude has been investigated using SCET [36, 45], the subleading terms of the soft graviton theorem have not yet been considered, and a direct comparison between gauge-theory and gravity was not possible before, since the framework of soft-collinear gravity did not exist.

Therefore, this thesis focuses on the investigation of the subleading soft-collinear interactions in gravity as well as the systematic construction of the effective Lagrangian. With this tool at hand, we examine the soft graviton emission using the SCET point of view, to see if the effective theory can provide additional insights for the number or form of the universal terms.

1.1 Low-Burnett-Kroll Amplitude and Soft Theorem

In the following section, we review the classical derivation of both the LBK amplitude [27, 28] as well as the soft theorem [33], pointing out how gauge symmetry constrains the subleading terms.

Soft Photon Emission

Consider a generic scattering amplitude $i \to f$ with N external legs, which we take to be scalar particles for simplicity. We are interested in the emission of an additional soft photon with momentum k from this process, $i \to f + \gamma$. Generically, there are two classes of diagrams contributing to this process, as depicted in Fig. 1.1. For the leading-order in the soft momentum, corresponding to $\mathcal{O}(1/k)$, only the emission from the external legs contributes as the intermediate

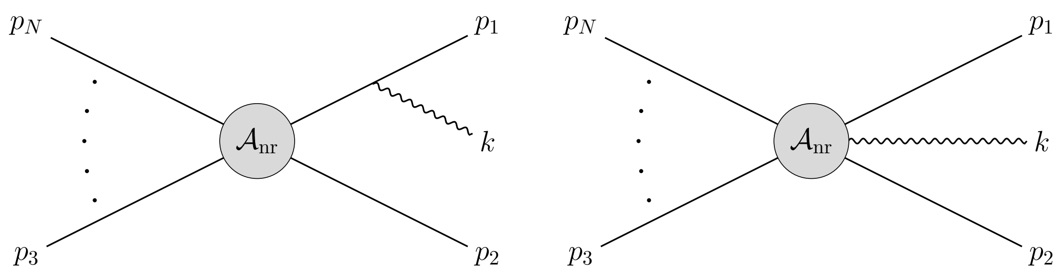


Figure 1.1: The two classes of diagrams contributing to the single soft-emission process. The first one corresponds to emissions off the external legs, which contribute already at leading power $\mathcal{O}(\frac{1}{k})$. The second class are emissions that originate from the hard scattering and start to contribute at next-to-leading power $\mathcal{O}(k^0)$.

propagator becomes singular. The second type of diagrams comes into effect at next-to-soft power $\mathcal{O}(k^0)$.

The emission from the external legs splits into the sum over the individual legs, where one immediately finds the well-known spin-independent eikonal emission[3] [1, 27, 28]

$$\mathcal{A}_{\text{rad}}(\{p_i\}; k) = \sum_{i=1}^{N} eQ_i \frac{p_i \cdot \varepsilon(k)}{p_i \cdot k} \mathcal{A}_{\text{nr}}(p_1, \ldots, p_i, \ldots, p_N) \,. \tag{1.1.1}$$

Using this result, one could now consider the emission of n soft photons and derive soft exponentiation [1]. However, we do not proceed in this direction and instead consider the subleading terms.

At the next-to-soft order, the emission from the hard vertex can contribute. We decompose the amplitude as

$$\mathcal{A}^{\mu}_{\text{rad}}(\{p_i\}; k) = \sum_{i=1}^{N} eQ_i \frac{p_i^{\mu}}{p_i \cdot k} \mathcal{A}_{\text{nr}}(p_1, \ldots, p_i + k, \ldots, p_N) + \mathcal{R}^{\mu}(\{p_i\}; k) \,, \tag{1.1.2}$$

where we stripped off the polarisation vector $\varepsilon^{\mu}(k)$ as $\mathcal{A}_{\text{rad}} = \varepsilon_{\mu}\mathcal{A}^{\mu}_{\text{rad}}$ and introduced the remainder term $\mathcal{R}^{\mu}$. Note that the first term reduces to the eikonal term (1.1.1) in the soft limit, but can generate higher-order contributions when Taylor-expanding the non-radiative amplitude in the small momentum k. The second term corresponds to the contributions from the second type of diagrams in Fig. 1.1.

We can now constrain the terms appearing inside $\mathcal{R}^{\mu}$ by performing a gauge transformation $\varepsilon^{\mu}(k) \to \varepsilon^{\mu}(k) + k^{\mu}\alpha$. Since the amplitude (1.1.2) must be gauge-invariant, this yields

$$0 = \sum_{i=1}^{N} eQ_i \mathcal{A}_{\text{nr}}(p_1, \ldots, p_i + k, \ldots, p_N) + k_{\mu}\mathcal{R}^{\mu}(\{p_i\}; k) \,. \tag{1.1.3}$$

Expanding this result for small k, we find at $\mathcal{O}(k^0)$

$$0 = \sum_{i=1}^{N} eQ_i \mathcal{A}_{\text{nr}}(p_1, \ldots, p_i, \ldots, p_N) \,, \tag{1.1.4}$$

which is satisfied if charge-conservation $\sum_i Q_i = 0$ is applied. At the next order in k, we can relate the derivative of $\mathcal{A}_{\text{nr}}$ and the remainder term $\mathcal{R}^{\mu}$ as

$$0 = \sum_{i=1}^{N} eQ_i k^{\mu} \frac{\partial}{\partial p_i^{\mu}} \mathcal{A}_{\text{nr}}(\{p_i\}) + k_{\mu}\mathcal{R}^{\mu}(\{p_i\}; 0) \,, \tag{1.1.5}$$

[3] We consider all particles to be outgoing, so there are no relative signs between the legs.

which fixes $\mathcal{R}^\mu$ to be

$$\mathcal{R}^\mu(\{p_i\};0) = -\sum_{i=1}^{N} eQ_i \frac{\partial}{\partial p_{i\mu}} \mathcal{A}_{\mathrm{nr}}(\{p_i\}) \,. \tag{1.1.6}$$

Inserting this result in (1.1.2), we see that the subleading term of the radiative amplitude $\mathcal{A}_{\mathrm{rad}}$ is completely determined in terms of the non-radiative amplitude $\mathcal{A}_{\mathrm{nr}}$ and its derivative as

$$\mathcal{A}^\mu_{\mathrm{rad}}(\{p_i\};k) = \sum_{i=1}^{N} eQ_i \left(\frac{p_i^\mu}{p_i\cdot k} k^\nu \frac{\partial}{\partial p_i^\nu} \mathcal{A}_{\mathrm{nr}}(\{p_i\}) - \frac{\partial}{\partial p_{i\mu}} \mathcal{A}_{\mathrm{nr}}(\{p_i\}) \right) . \tag{1.1.7}$$

We can now simplify the expression by introducing $\frac{p_i\cdot k}{p_i\cdot k}$ in the second term, which yields

$$\begin{aligned} \mathcal{A}_{\mathrm{rad}}(\{p_i\};k) &= \sum_{i=1}^{N} eQ_i \frac{k_\mu \varepsilon_\nu(k)}{p_i\cdot k} \left(p^\nu \frac{\partial}{\partial p_{i\mu}} - p_i^\mu \frac{\partial}{\partial p_{i\nu}} \right) \mathcal{A}_{\mathrm{nr}}(\{p_i\}) \\ &= \sum_{i=1}^{N} eQ_i \frac{k_\nu \varepsilon_\mu(k)}{p_i\cdot k} L_i^{\mu\nu} \mathcal{A}_{\mathrm{nr}}(\{p_i\}) \,, \end{aligned} \tag{1.1.8}$$

where we identified the orbital angular momentum operator $L_i^{\mu\nu} = p_i^\mu \frac{\partial}{\partial p_{i\nu}} - p_i^\nu \frac{\partial}{\partial p_{i\mu}}$ in the last line. For fields with non-zero spin, one finds in its place the full angular momentum operator $J_i^{\mu\nu} = L_i^{\mu\nu} + \Sigma_i^{\mu\nu}$, where $\Sigma_i^{\mu\nu}$ is the spin-operator in the given representation of the particle i. Note that this term is manifestly gauge-invariant due to the antisymmetry of the angular momentum $J_i^{\mu\nu}$. Therefore, one has no further constraints that can be employed to completely restrict the higher-order contributions.

In summary, the LBK theorem states that the amplitude for single soft photon emission takes the form

$$\mathcal{A}_{\mathrm{rad}}(\{p_i\};k) = \sum_{i=1}^{N} eQ_i \left(\frac{p_i\cdot\varepsilon(k)}{p_i\cdot k} + \frac{k_\nu \varepsilon_\mu(k)}{p_i\cdot k} J_i^{\mu\nu} \right) \mathcal{A}_{\mathrm{nr}}(\{p_i\}) + \mathcal{O}(k) \,, \tag{1.1.9}$$

where the higher-order terms are no longer universal. This statement can also be extended to QCD [28], where one finds the same expression (replacing the charge Q_i by the colour-generator t^a and the coupling e by g).

Soft Graviton Emission

Let us now investigate the same process in gravity, i.e. we consider the emission of a soft graviton from the non-radiative N-scalar process. This derivation follows [32]. The contributing diagrams take the same form as in Fig. 1.1. At the leading order, one finds that only the emissions from the external legs are relevant, which again results in the spin-independent eikonal term [1]

$$\mathcal{A}^{\mu\nu}_{\mathrm{rad}}(\{p_i\};k) = \frac{\kappa}{2} \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{p_i\cdot k} \mathcal{A}_{\mathrm{nr}}(\{p_i\}) \,, \tag{1.1.10}$$

where we employ the stripped amplitude $\mathcal{A}_{\mathrm{rad}} = \varepsilon_{\mu\nu}\mathcal{A}^{\mu\nu}_{\mathrm{rad}}$. At the subleading order, the emission from the hard vertex can start to contribute, and we can split the amplitude like in (1.1.2) as

$$\mathcal{A}^{\mu\nu}_{\mathrm{rad}}(\{p_i\};k) = \frac{\kappa}{2} \sum_{i=1}^{n} \frac{p_i^\mu p_i^\nu}{p_i\cdot k} \mathcal{A}_{\mathrm{nr}}(p_1,\ldots,p_i+k,\ldots,p_n) + \mathcal{R}^{\mu\nu}(\{p_i\};k) \,, \tag{1.1.11}$$

where we introduced the (symmetric) remainder term $\mathcal{R}^{\mu\nu}$. This amplitude must be gauge-invariant under the transformations of the graviton-polarisation

$$\varepsilon_{\mu\nu}(k) \to \varepsilon_{\mu\nu}(k) + k_\mu \alpha_\nu + k_\nu \alpha_\mu \,, \tag{1.1.12}$$

which gives the constraint

$$\begin{aligned} 0 &= k_\mu \mathcal{A}^{\mu\nu}_{\rm rad}(\{p_i\};k) \\ &= \frac{\kappa}{2}\sum_{i=1}^{n} p_i^\nu \mathcal{A}_{\rm nr}(p_1,\ldots,p_i+k,\ldots,p_n) + k_\mu \mathcal{R}^{\mu\nu}(\{p_i\};k)\,. \end{aligned} \tag{1.1.13}$$

We now expand this equation in small k to find relations between the remainder $\mathcal{R}^{\mu\nu}$ and the derivative of the non-radiative amplitude. At leading-power, (1.1.13) implies

$$0 = \frac{\kappa}{2}\left(\sum_{i=1}^{N} p_i^\nu\right)\mathcal{A}_{\rm nr}(\{p_i\})\,, \tag{1.1.14}$$

which is valid if momentum-conservation $\sum_i p_i^\nu = 0$ is imposed. At next-to-leading power, we obtain a relation between $\mathcal{R}^{\mu\nu}$ and the derivative of the non-radiative amplitude similar to (1.1.5), which reads[4]

$$\mathcal{R}^{\mu\nu}(\{p_i\};0) = -\frac{\kappa}{2}\sum_{i=1}^{N} p_i^\nu \frac{\partial}{\partial p_{i\mu}}\mathcal{A}_{\rm nr}(\{p_i\})\,. \tag{1.1.15}$$

Inserting this in the amplitude (1.1.11), we find that also in gravity, the next-to-leading order $\mathcal{O}(k^0)$ can be expressed in terms of the non-radiative amplitude as

$$\mathcal{A}^{(1)\,\mu\nu}_{\rm rad}(\{p_i\};k) = \frac{\kappa}{2}\sum_{i=1}^{N}\left(\frac{p_i^\mu p_i^\nu}{p_i\cdot k}k^\rho \frac{\partial}{\partial p_{i\rho}}\mathcal{A}_{\rm nr}(\{p_i\}) - p_i^\nu \frac{\partial}{\partial p_{i\mu}}\mathcal{A}_{\rm nr}(\{p_i\})\right)\,. \tag{1.1.16}$$

Similar to the corresponding equation in gauge theory (1.1.8), one can combine both terms into the orbital angular momentum as

$$\mathcal{A}^{(1)}_{\rm rad}(\{p_i\};k) = \frac{\kappa}{2}\sum_{i=1}^{N}\left(\frac{k_\rho \varepsilon_{\mu\nu}(k)}{p_i\cdot k}p_i^\mu L_i^{\nu\rho}\mathcal{A}_{\rm nr}(\{p_i\})\right)\,. \tag{1.1.17}$$

However, there is an important difference compared to the result (1.1.8) in gauge theory: the subleading term (1.1.17) is *not* manifestly gauge-invariant. Performing a gauge transformation, we find the condition

$$\sum_{i=1}^{N}\frac{p_i^\nu}{p_i\cdot k}\left(k_\mu\alpha_\nu + k_\nu\alpha_\mu\right)k_\rho L_i^{\mu\rho} = 0\,. \tag{1.1.18}$$

The first term vanishes identically due to the antisymmetry of $L_i^{\mu\nu}$. The second term, however, yields

$$\alpha_\mu k_\rho\left(\sum_{i=1}^{N} L_i^{\mu\rho}\right) = 0\,, \tag{1.1.19}$$

which holds true if *angular momentum conservation* is imposed. It seems that we have not yet completely exploited all constraints that gauge-invariance has to offer. This motivates us to go one order beyond, to sub-subleading order in the soft momentum. The second-order $\mathcal{O}(k^2)$ of the constraint (1.1.13) reads

$$0 = \frac{\kappa}{2}\sum_{i=1}^{N} p_i^\nu \frac{1}{2}k^\rho k^\sigma \frac{\partial}{\partial p_{i\rho}}\frac{\partial}{\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\}) + k_\mu k_\rho\left[\frac{\partial}{\partial k_\rho}\mathcal{R}^{\mu\nu}\right](\{p_i\},0)\,, \tag{1.1.20}$$

[4] This relation comes multiplied by k_μ, which we have dropped. Strictly speaking one has to be careful that one does not miss any terms when doing this. However, here and in the next order in k it works out since there are no gauge-invariant objects that could be missed [32].

which fixes the *symmetric* combination of the first-derivative of the remainder term as

$$\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}+\frac{\partial \mathcal{R}^{\rho\nu}}{\partial k_\mu}\right](\{p_i\};0)=-\frac{\kappa}{2}\sum_{i=1}^{N}p_i^\nu\frac{\partial^2}{\partial p_{i\mu}\partial p_{i\rho}}\mathcal{A}_{\rm nr}(\{p_i\})\,. \tag{1.1.21}$$

Using this in the soft-emission amplitude, we can rewrite the sub-subleading term as

$$\begin{aligned}\mathcal{A}^{(2)\,\mu\nu}_{\rm rad}(\{p_i\};k)&=\frac{\kappa}{2}\sum_{i=1}^{N}\frac{p_i^\mu p_i^\nu}{p_i\cdot k}\frac{1}{2}k^\rho k^\sigma\frac{\partial^2}{\partial p_{i\rho}\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\})+k_\rho\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}\right](\{p_i\};0)\\&=\frac{\kappa}{2}\sum_{i=1}^{N}\left(\frac{p_i^\mu p_i^\nu}{p_i\cdot k}\frac{1}{2}k^\rho k^\sigma\frac{\partial^2}{\partial p_{i\rho}\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\})-\frac{1}{2}k_\rho p_i^\nu\frac{\partial^2}{\partial p_{i\mu}\partial p_{i\rho}}\mathcal{A}_{\rm nr}(\{p_i\})\right)\\&\quad+\frac{1}{2}k_\rho\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}-\frac{\partial \mathcal{R}^{\rho\nu}}{\partial k_\mu}\right](\{p_i\};0)\\&=\frac{\kappa}{2}\sum_{i=1}^{N}\frac{p_i^\nu}{p_i\cdot k}k_\rho k_\sigma\frac{1}{2}L_i^{\mu\rho}\frac{\partial}{\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\})+\frac{1}{2}k_\rho\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}-\frac{\partial \mathcal{R}^{\rho\nu}}{\partial k_\mu}\right](\{p_i\};0)\,,\end{aligned} \tag{1.1.22}$$

where the angular momentum $L_i^{\mu\nu}$ can be identified in the second line. Half of the sub-subleading term has already been reduced to derivatives of the non-radiative amplitude. To constrain the remaining terms, we use gauge-invariance again, but now with the second index $k_\nu\mathcal{A}^{\mu\nu}_{\rm rad}(\{p_i\},k)=0$. Applying this to (1.1.22) yields the constraint

$$0=\frac{\kappa}{2}\sum_{i=1}^{N}k_\rho k_\sigma L_i^{\mu\rho}\frac{\partial}{\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\})+k_\rho k_\nu\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}-\frac{\partial \mathcal{R}^{\rho\nu}}{\partial k_\mu}\right](\{p_i\};0)\,, \tag{1.1.23}$$

which determines the antisymmetric combination in terms of the non-radiative amplitude

$$\left[\frac{\partial \mathcal{R}^{\mu\nu}}{\partial k_\rho}-\frac{\partial \mathcal{R}^{\rho\nu}}{\partial k_\mu}\right](\{p_i\};0)=-\frac{\kappa}{2}\sum_{i=1}^{N}L_i^{\mu\rho}\frac{\partial}{\partial p_{i\nu}}\mathcal{A}_{\rm nr}(\{p_i\})\,. \tag{1.1.24}$$

Using this relation in (1.1.22), we find

$$\begin{aligned}\mathcal{A}^{(2)\,\mu\nu}_{\rm rad}&=\frac{\kappa}{2}\sum_{i=1}^{N}\left(\frac{p_i^\nu}{p_i\cdot k}k_\rho k_\sigma\frac{1}{2}L_i^{\mu\rho}\frac{\partial}{\partial p_{i\sigma}}\mathcal{A}_{\rm nr}(\{p_i\})-\frac{1}{2}k_\rho L_i^{\mu\rho}\frac{\partial}{\partial p_{i\nu}}\mathcal{A}_{\rm nr}(\{p_i\})\right)\\&=-\frac{\kappa}{2}\sum_{i=1}^{N}\frac{1}{p_i\cdot k}k_\rho L_i^{\mu\rho}k_\sigma\left(p_i^\nu\frac{\partial}{\partial p_{i\sigma}}-p_i^\sigma\frac{\partial}{\partial p_{i\nu}}\right)\mathcal{A}_{\rm nr}(\{p_i\})\\&=-\frac{\kappa}{2}\sum_{i=1}^{N}\frac{1}{2}\frac{k_\rho k_\sigma}{p_i\cdot k}L_i^{\mu\rho}L_i^{\nu\sigma}\mathcal{A}_{\rm nr}(\{p_i\})\,.\end{aligned} \tag{1.1.25}$$

Note that the notation is slightly misleading since the angular momentum operators do not act on each other when computing the explicit amplitude. However, for an on-shell amplitude, it does not matter if the angular momenta are taken to act on each other or not, the difference between both versions vanishes using on-shell properties and equations of motion [32].

In summary, we find that for the soft graviton emission, one can determine *three* universal terms using the constraints from gauge symmetry, which take the form

$$\mathcal{A}^{\mu\nu}_{\rm rad}=\frac{\kappa}{2}\sum_{i=1}^{N}\left(\frac{p_i^\nu}{p_i\cdot k}p_i^\mu+\frac{p_i^\nu}{p_i\cdot k}k_\rho J_i^{\mu\rho}+\frac{1}{2}\frac{1}{p_i\cdot k}k_\rho k_\sigma J_i^{\mu\rho}J_i^{\nu\sigma}\right)\mathcal{A}_{\rm nr}(\{p_i\})+\mathcal{O}(k^2)\,. \tag{1.1.26}$$

In [32], it is explicitly verified that there are no further constraints due to the gauge symmetry, and thus the soft emission at the next order $\mathcal{O}(k^2)$ is not universal. While the actual derivation is surprisingly simple[5] it offers no real explanation as to why gravity has three universal terms compared to the two in gauge theory, it just happens this way. In addition, it remains unclear why the angular momentum appears, even twice in the sub-subleading term, and what the precise relation between the gauge-theory and gravitational result is. In Chapter 7, we reconsider these soft theorems both for gauge theory and gravity from the point of view of the soft-collinear effective theory. This will provide a new explanation as well as some intuition for both the number of universal terms as well as their form.

Outline. This thesis is organised as follows: we begin in Chapter 2 with the construction of the soft-collinear effective theory for a self-interacting scalar field, introducing all the necessary concepts and notation, as well as performing an example matching computation. Then, in Chapter 3, we extend the discussion to include gauge symmetries and point out the necessary steps required for deriving the effective action. In Chapter 4, we provide a quick review of perturbative gravity, matter fields coupled to gravity, and the vierbein formalism for including half-integer spin fields. Equipped with this knowledge, a theory describing purely-collinear gravitons and matter fields is constructed in Chapter 5, where the first analogies to the gauge-theory situation can be made. In Chapter 6, the full soft-collinear effective theory for gravitation is derived for a scalar matter field, and the extension to higher-spin fields is briefly discussed. The effective theory is then employed in Chapter 7 to derive the soft theorem for both gauge theory and gravity at the Lagrangian level. A discussion of loop corrections for the gravitational case is included. Finally, in Chapter 8, we change the focus and briefly examine an application of these EFT concepts to the dynamics of a light (or massless) scalar field in a deSitter space-time. In Chapter 9 we conclude.

This thesis is mainly based on the following publications and preprints

- M. Beneke, P. Hager and R. Szafron, *Soft-collinear gravity beyond the leading power, JHEP* **03** (2022) 080 [2112.04983]
- M. Beneke, P. Hager and R. Szafron, *Gravitational soft theorem from emergent soft gauge symmetries, JHEP* **03** (2022) 199 [2110.02969]
- M. Beneke, P. Hager and A. F. Sanfilippo, *Double copy for Lagrangians at trilinear order, JHEP* **02** (2022) 083 [2106.09054]
- M. Beneke, P. Hager and R. Szafron, *Soft-Collinear Gravity and Soft Theorems*, [2210.09336]
- M. Beneke, P. Hager and D. Schwienbacher, *Soft-collinear gravity with fermionic matter*, [2212.02525]

and work yet to appear

- M. Beneke, P. Hager and A. F. Sanfilippo, *in preparation*

[5] If one is aware of the tricks and knows that there is a third universal term. Recall that it took almost fifty years between [1] and [46] until the sub-subleading term was discovered.

Chapter 2

Scalar SCET

Soft-collinear effective theory is the theory describing the interactions of collinear and soft particles with themselves. It is one of the most advanced and successful effective theories in high-energy particle physics and has greatly contributed to pushing the accuracy of modern collider experiments. Originally constructed in the context of QCD [38–41], it has since been extended also to describe gravity [47, 48]. However, the theory contains many subtleties, especially when featuring a gauge symmetry as in QCD or gravity. Thus, in this section, we present the construction of SCET for one of the simplest possible situations: a self-interacting scalar theory. This allows us to introduce the necessary concepts and notation without the additional complexities that gauge symmetries require. Instead, one can start to build some intuition for the basic principles of the construction. Later, in Chapter 3, we explain how gauge symmetries are consistently implemented and in what shape they affect and constrain the form of the theory. In this section, we are not interested in detailed applications of the theory, but rather in the principles underlying the systematic all-order construction.

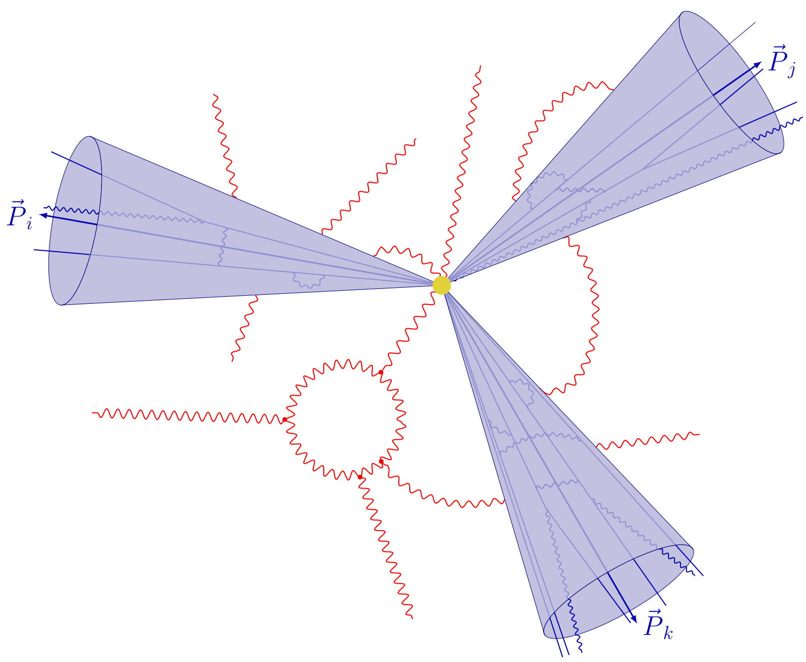


Figure 2.1: A prototypical process considered in SCET. The hard scattering at the origin creates a number of energetic particles, depicted in blue. These are assigned to different collinear sectors as indicated by the cones. Soft radiation, depicted in red, is emitted isotropically either directly from the hard scattering or from the classical trajectory of the energetic particles. It cannot resolve the internal structure of the cones.

2.1 Power-counting

In SCET, we consider processes where a hard scattering of order Q creates multiple energetic particles (jets). Each such jet is characterised by carrying a large momentum of order Q in the direction of a light-like reference vector n_{i-}^{μ}, i counting the number of distinct jets. In addition, we allow for (ultra-)soft isotropic radiation. The situation is depicted in Fig. 2.1. Within the sector corresponding to the direction n_{i-}^{μ}, one introduces a second light-like reference vector n_{i+}^{μ}, such that

$$n_{i+} \cdot n_{i-} = 2\,, \quad n_{i\pm}^2 = 0\,. \tag{2.1.1}$$

These two vectors, in combination with the remaining two transverse directions, then constitute a basis. A collinear momentum p can be decomposed in this basis as

$$p^{\mu} = n_{i+}p\,\frac{n_{i-}^{\mu}}{2} + n_{i-}p\,\frac{n_{i+}^{\mu}}{2} + p_{\perp}^{\mu}\,, \tag{2.1.2}$$

where the subscript $\perp$ is understood to be transverse to the $n_{i\pm}^{\mu}$ vectors. We expand the theory in the small parameter $\lambda \sim p_{\perp}/(n_{i+}p) \ll 1$, and the components of the collinear momentum-vector scale as

$$(n_{i+}p, p_{\perp}, n_{i-}p) \sim (1, \lambda, \lambda^2)Q\,. \tag{2.1.3}$$

In the following, we set the hard scale $Q = 1$ as is conventional. This implies that collinear momenta satisfy $p^2 \sim \lambda^2$. The (ultra-)soft particles carry momenta k satisfying $k^2 \sim \lambda^4$, and these momenta are taken to be isotropic, hence $k^{\mu} \sim \lambda^2$. We assume that no other modes, besides the hard modes that we integrate out, are relevant for the problem at hand.[1] This situation is usually denoted as SCET$_I$.

2.2 Field Content

The theory aims to reproduce the soft and collinear limits of the full-theory scattering amplitudes. The construction differs quite drastically from traditional effective theories, like the Fermi theory of weak decay or the Standard Model Effective Theory. In these conventional approaches, one is interested in the "light physics" up to some high energy scale Λ, and integrates out the heavy fields present in the full theory. The low-energy effective theory then only contains the light modes and higher-dimensional operators that reproduce the effects of the heavy modes.

In SCET, however, we wish to describe soft and collinear regions of all relevant particles in the full theory. Hence, instead of integrating out heavy modes, one integrates out fluctuations corresponding to certain regions of loop momenta, and the effective degrees of freedom are then the left-over modes. These modes must have a definite scaling in the power-counting parameter λ, e.g. $\phi \sim \lambda^{\alpha}$, which allows one to label them as hard, soft, and i-collinear. We call this a *homogeneous* scaling in λ. This means that the effective theory makes use of *multiple fields* describing the *same particle species*, but with the different fluctuations representing different kinematic regions.

The relevant modes for the effective theory are the *soft* and *collinear* modes. The *hard region*, given by momenta scaling as $p_{\mu} \sim 1$, is integrated out. Because of this, the theory is in general non-local along the respective n_{+}^{μ}-directions. This is the effect of integrating out the hard modes while having modes present in the effective theory that still depend on this hard scale. In SCET, the collinear fields still have a dependence on the large scale via the large momentum component $n_{+}p \sim 1$. The non-locality can be traced to operators featuring an arbitrary number of large derivatives $n_{+}\partial$, which all scale as $n_{+}\partial \sim 1$, and can be added in principle at any order in λ.

[1] This also means that we ignore the possibility of Glauber modes. For details, see [49].

However, this tower of derivatives may be traded for the non-locality in the n_+^μ direction, simply by rewriting the derivatives as a translation, using

$$\varphi(x + tn_+) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (n_+\partial)^k \varphi(x)\,. \tag{2.2.1}$$

Thus, instead of keeping infinite towers of derivatives $(n_+\partial)^m$, one allows for non-localities in the n_+^μ-direction of collinear objects. Often this non-locality also appears in the form of an inverse derivative operator [40]

$$\frac{1}{in_+\partial + i\varepsilon} f(x^\mu) = -i \int_{-\infty}^{0} ds\, f(x^\mu + sn_+^\mu)\,. \tag{2.2.2}$$

Let us now turn our attention to the scalar theory. To be concrete, consider the action

$$S[\varphi] = \int d^4x\, \frac{1}{2} \partial_\mu \varphi(x)\, \partial^\mu \varphi(x) - \frac{g_3}{3!} \varphi^3(x) - \frac{g_4}{4!} \varphi^4(x)\,. \tag{2.2.3}$$

We split the full-theory field φ into its collinear and soft modes, integrating out the hard region. In the EFT, this is manifested by introducing a collinear field φ_c (φ_{c_i} in case of multiple collinear directions) and a soft field φ_s for the original full-theory field φ.

Our approach is based on the position-space formulation of SCET developed in [40, 41]. In this formalism, we assign a *scaling* to the *position* argument of the fields, which is reciprocal to the momenta. In the Fourier transform, we impose $e^{ip\cdot x} \sim 1$ and obtain the scaling for collinear and purely-soft coordinates. Collinear fields depend on the coordinates

$$n_-x \sim 1\,, \quad x_\perp \sim \lambda^{-1}\,, \quad n_+x \sim \lambda^{-2}\,, \tag{2.2.4}$$

and the λ-scaling reflects the characteristic distance over which the fields exhibit substantial variations. For example, the soft field fluctuates only over large distances $x_s \sim \lambda^{-2}$. It cannot resolve the internal structures of a collinear jet, but only its large momentum and direction, as well as its charge (in the case of gauge theory). Consequently, this scaling of x also determines the scaling of derivatives when acting on collinear and soft fields. These scale precisely as collinear and soft momenta, namely

$$n_+\partial\varphi_c(x) \sim \varphi_c(x)\,, \quad \partial_\perp\varphi_c(x) \sim \lambda\varphi_c(x)\,, \quad n_-\partial\varphi_c(x) \sim \lambda^2\varphi_c(x)\,, \tag{2.2.5}$$

for collinear fields, and the isotropic

$$\partial_\mu\varphi_s(x) \sim \lambda^2\varphi_s(x) \tag{2.2.6}$$

for soft fields.

Unlike in conventional effective theories, the power-counting of the fields is not linked to their mass dimension, and the soft and collinear fields differ in their scaling. From their respective two-point functions, one can determine the power-counting of collinear and soft fields themselves [40]. For the scalar field, we have

$$\langle 0|\, T(\varphi(x)\varphi(y))\, |0\rangle = \int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\sim\lambda^4,\,\lambda^8}\, \underbrace{e^{-ip(x-y)}}_{\sim 1}\, \underbrace{\frac{i}{p^2 + i\varepsilon}}_{\sim\lambda^{-2},\,\lambda^{-4}}\,, \tag{2.2.7}$$

where we inserted the power-counting (2.1.3), namely $p^2 \sim \lambda^2$, $d^4p \sim \lambda^4$ for collinear, and $p^2 \sim \lambda^4$, $d^4p \sim \lambda^8$ for soft momenta, respectively. One obtains the scaling

$$\varphi_c \sim \lambda\,, \quad \varphi_s \sim \lambda^2\,, \tag{2.2.8}$$

for collinear fields φ_c and soft fields φ_s.

Finally, we need to assign the power counting to the d^4x measure in the effective action. If collinear fields are present in the integral, we take the counting of the collinear coordinates and assign $d^4x \sim \lambda^{-4}$. For purely soft fields, only the soft coordinate is relevant and we assign λ^{-8}. Notably, if both soft and collinear fields are present, we have collinear counting $d^4x \sim \lambda^{-4}$.

2.3 Light-front Multipole Expansion

Due to the power-counting of their momenta (and thus coordinate arguments), products containing both soft and collinear fields are *not homogeneous* in λ. In Fourier space, one finds, for example,

$$\varphi_c(x)\varphi_s(x) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k_s}{(2\pi)^4} e^{-i(p+k_s)\cdot x}\tilde{\varphi}_c(p)\tilde{\varphi}_s(k_s)\,. \tag{2.3.1}$$

Here, the product in the exponent is expanded as

$$(p+k_s)\cdot x = \frac{1}{2}\underbrace{(n_+p+n_+k_s)}_{1+\lambda^2}n_-x + \underbrace{(p_\perp + k_{s\perp})}_{\lambda+\lambda^2}\cdot x_\perp + \frac{1}{2}\underbrace{(n_-p+n_-k_s)}_{\lambda^2+\lambda^2}n_+x\,, \tag{2.3.2}$$

but unlike collinear momenta, we have $k_s^\mu \sim \lambda^2$ for all components. Thus, only the combination

$$n_-p + n_-k_s \sim \lambda^2 \tag{2.3.3}$$

scales homogeneously as $\mathcal{O}(\lambda^2)$, while the $k_{s\perp}$ and n_+k_s components are suppressed with respect to $p_\perp$ and n_+p. In an amplitude, this is taken care of by expanding away the suppressed soft components in the interaction vertices, and expanding propagators where necessary.

In the position-space formalism, this is accounted for by a *multipole expansion*. First, one expands the exponential in (2.3.1) as

$$e^{-i(p+k_s)\cdot x} = e^{-i(p+n_-k_s\frac{n_+}{2})\cdot x}\left(1 + ik_{s\perp}\cdot x_\perp + \frac{1}{2}in_+k_sn_-x + \dots\right)\,, \tag{2.3.4}$$

keeping only the homogeneous soft momentum n_-k_s in the exponent, and one rewrites this in an operatorial fashion as

$$e^{-i(p+k_s)\cdot x} = e^{-i(p+n_-k_s\frac{n_+}{2})\cdot x}\left(1 + x_\perp\cdot\partial_\perp + \frac{1}{2}n_-xn_+\partial + \dots\right)\,. \tag{2.3.5}$$

Thus, the original interaction (2.3.1) contributes as

$$\varphi_c(x)\varphi_s(x) = \varphi_c(x)\left(\varphi_s(x_-) + x_\perp\cdot[\partial\varphi_s](x_-) + \frac{1}{2}n_-x\,[n_+\partial\varphi_s](x_-) + \dots\right)\,, \tag{2.3.6}$$

where $x_-^\mu = n_+x\frac{n_-^\mu}{2}$, and the square brackets indicate that the field is evaluated at x_- *after* derivatives are taken. Thus, the expansion in small momenta at the amplitude level is equivalent to the so-called light-front multipole expansion [40, 41]

$$\varphi_s(x) = \varphi_s(x_-) + (x-x_-)^\alpha[\partial_\alpha\varphi_s](x_-) + \frac{1}{2}(x-x_-)^\alpha(x-x_-)^\beta[\partial_\alpha\partial_\beta\varphi_s](x_-) + \dots\,, \tag{2.3.7}$$

and we perform this for each soft field whenever it appears in a soft-collinear interaction. This guarantees that each term in the Lagrangian is manifestly homogeneous in λ, and in turn, this procedure generates an infinite tower of subleading soft-collinear interactions. In addition, one immediately notices two features: first, the soft fields are evaluated at x_-. This implies that in the momentum-conserving delta-functions that are present in the effective vertices, soft momenta only enter with the n_-k_s-component, and one sets the $k_{s\perp}$ and n_+k_s components to zero. This also leads to *eikonal* propagators for the collinear modes. Second, the coordinates $x_\perp$ and n_-x appear explicitly in the interactions, which modifies the Feynman rules of the effective theory. These coordinates lead to derivatives in momentum space that act on the momentum-conserving delta functions.

Note that the multipole expansion can be cast into an integral form, where $\varphi_s(x)$ is expressed in terms of $\varphi_s(x_-)$ and its kinetic 1-form $[\partial_\mu\varphi](x_-)$ as

$$\varphi_s(x) - \varphi_s(x_-) = \int_0^1 \mathrm{d}s\,(x-x_-)^\mu[\partial_\mu\varphi_s](y(s))\,, \tag{2.3.8}$$

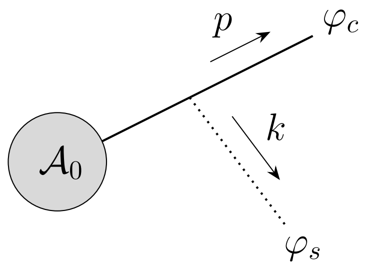


Figure 2.2: Soft scalar emission with momentum k, indicated by the dotted line, off an external collinear leg with momentum p connected to the non-radiative amplitude $\mathcal{A}_0$.

where $y(s) = x_- + s(x - x_-)$. Such identities also reappear later for the gauge and graviton fields and control the expansion to all orders in λ.

As an explicit example, we show how this multipole expansion precisely reproduces the expansion in small soft momenta. This serves as a small invitation, the systematic construction is explained in detail in the later Section 2.5. Consider a cubic interaction

$$\mathcal{L}_{\text{int}} = -\frac{g_3}{3!}\varphi^3(x)\,, \tag{2.3.9}$$

and the emission of a soft scalar with momentum k from an external leg with momentum p, as depicted in Fig. 2.2. For simplicity, we assume that the rest of the diagram, denoted by $\mathcal{A}_0$, does not depend on external momenta. Then the full-theory result is given by

$$\mathcal{A} = g_3 \frac{1}{(p+k)^2 + i0}\mathcal{A}_0\,. \tag{2.3.10}$$

We choose the external states to be on-shell, $p^2 = k^2 = 0$, and can use

$$(p+k)^2 = 2p\cdot k = \underbrace{n_+p\, n_-k}_{\sim\lambda^2} + 2\underbrace{p_\perp\cdot k_\perp}_{\sim\lambda^3} + \underbrace{n_-p\, n_+k}_{\sim\lambda^4}\,. \tag{2.3.11}$$

Thus, the propagator containing a soft and a collinear momentum is expanded as

$$\frac{i}{(p+k)^2} = \frac{i}{p_+k_-}\left(1 - \frac{2p_\perp\cdot k_\perp}{p_+k_-} - \frac{p_-k_+}{k_+p_-} + \frac{4(p_\perp\cdot k_\perp)^2}{(p_+k_-)^2}\right) + \mathcal{O}(\lambda^3)\,, \tag{2.3.12}$$

where we introduced the short-hand notation $n_+p \equiv p_+$, and the λ-expansion of the amplitude (2.3.10) is given by $\mathcal{A} = \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \mathcal{O}(\lambda^4)$,[2] where

$$\mathcal{A}^{(1)} = g_3\frac{1}{p_+k_-}\mathcal{A}_0\,, \tag{2.3.13}$$

$$\mathcal{A}^{(2)} = \left(-\frac{2p_\perp k_\perp}{p_+k_-}\right)\left(g_3\frac{1}{p_+k_-}\mathcal{A}_0\right)\,, \tag{2.3.14}$$

$$\mathcal{A}^{(3)} = \left(-\frac{p_-k_+}{p_+k_-} + \frac{4(p_\perp\cdot k_\perp)^2}{(p_+k_-)^2}\right)\left(g_3\frac{1}{p_+k_-}\mathcal{A}_0\right)\,. \tag{2.3.15}$$

In SCET, these results should be obtained directly from the Feynman rules of the homogeneous Lagrangian. Namely, for the considered three-point interaction (2.3.9), the effective Lagrangian contains an interaction

$$\mathcal{L}_{\text{int}} \supset -\frac{g_3}{2}\varphi_c^2(x)\varphi_s(x)\,, \tag{2.3.16}$$

[2]Here we assign $g_3 \sim \lambda$ for consistency. Therefore, the first term in the soft emission is already suppressed by $\mathcal{O}(\lambda)$, in contrast to the soft photon emission considered earlier.

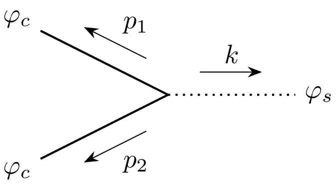
which yields after multipole expansion

$$\mathcal{L}_{\text{int}}^{(1)} = -\frac{g_3}{2}\varphi_c^2\varphi_s(x_-)\,, \tag{2.3.17}$$

$$\mathcal{L}_{\text{int}}^{(2)} = -\frac{g_3}{2}\varphi_c^2 x_\perp^\alpha\left[\partial_\alpha\varphi_s\right](x_-)\,, \tag{2.3.18}$$

$$\mathcal{L}_{\text{int}}^{(3)} = -\frac{g_3}{4}\varphi_c^2 n_- x\left[n_+\partial\varphi_s\right](x_-) - \frac{g_3}{4}\varphi_c^2 x_\perp^\alpha x_\perp^\beta\left[\partial_\alpha\partial_\beta\varphi_s\right](x_-)\,, \tag{2.3.19}$$

plus higher-order contributions. Recall that for soft fields, depending only on x_-, the momentum conserving delta-function only contains the n_-k component of the soft momentum k. Thus, whenever no explicit x^μ is present, we use this to impose momentum conservation. The Feynman rules are then given by



$$= -ig_3\begin{cases} 0\,, & \mathcal{O}(\lambda^0) \\ 1\,, & \mathcal{O}(\lambda^1) \\ -X_\perp^\mu k_\mu\,, & \mathcal{O}(\lambda^2) \\ \frac{1}{2}X_\perp^\alpha X_\perp^\beta k_\alpha k_\beta - \frac{1}{2}n_- X n_+ k\,, & \mathcal{O}(\lambda^3) \end{cases} \tag{2.3.20}$$

where X^μ is defined as [50]

$$X^\mu \equiv \partial^\mu\left[(2\pi)^4\delta^{(4)}\left(\sum p_{\text{in}} - \sum p_{\text{out}}\right)\right]\,, \tag{2.3.21}$$

and the derivative $\partial = \partial/\partial p_{\text{in}}$ or $\partial = -\partial/\partial p_{\text{out}}$ acts on incoming or outgoing momenta inside the delta function. If soft momenta k are present, one sets the $k_\perp$ and n_-k components to zero inside the momentum-conserving δ-function after the derivative is taken.[3]

At $\mathcal{O}(\lambda)$, one immediately realises that the SCET momentum conservation leads to the eikonal propagator of the internal line as

$$\int\frac{d^4\tilde{p}}{(2\pi)^4}\frac{i}{\tilde{p}^2}(2\pi)^4\delta^{(4)}\left(\tilde{p} - p - n_-k\frac{n_+}{2}\right) = \frac{i}{p_+k_-}\,, \tag{2.3.22}$$

and one recovers $\mathcal{A}^{(1)}$. For $\mathcal{A}^{(2)}$, one needs to evaluate $X_\perp^\mu$, the derivative acting on the delta-function, as defined in (2.3.21). The explicit contribution reads

$$\begin{aligned} \mathcal{A}^{(1)} &= \int\frac{d^4\tilde{p}}{(2\pi)^4}ig_3k_\mu\frac{i}{\tilde{p}^2}X_\perp^\mu\mathcal{A}_0 \\ &= -g_3k_\mu\int\frac{d^4\tilde{p}}{(2\pi)^4}\frac{1}{\tilde{p}^2}\frac{\partial}{\partial\tilde{p}_\mu}\delta^{(4)}(\tilde{p} - p - n_-k\frac{n_+}{2})\mathcal{A}_0 \\ &= -g_3k_\mu\int\frac{d^4\tilde{p}}{(2\pi)^4}\left(-\frac{\partial}{\partial\tilde{p}_\mu}\frac{1}{\tilde{p}^2}\mathcal{M}_0\right)\delta^{(4)}(\tilde{p} - p - n_-k\frac{n_+}{2}) \\ &= -g_3k_\mu\frac{2p_\perp^\mu}{(p_+k_-)^2}\mathcal{A}_0\,, \end{aligned} \tag{2.3.23}$$

and we reproduce (2.3.14). This way, one explicitly sees how the multipole expansion (2.3.7) reproduces the expansion in the soft momenta order by order in k. Thus, the only two ingredients required to construct the soft-collinear effective theory for interacting scalars are the mode decomposition and the multipole expansion. Already here we see a new feature of SCET compared to standard effective theories: the Lagrangian reproduces the soft emission from the external leg. However, the underlying scattering $\mathcal{A}_0$ is not described by the effective interaction. Instead, this scattering must be accounted for differently.

[3]In practice, one simply integrates by parts and sets the soft momenta to k_- directly inside the δ-function.

2.4 A Bird's Eye Perspective on the EFT

We are now ready to discuss the form and construction of the effective theory. First, we consider the effects of momentum conservation and homogeneity, which lead to a first manifestation of factorisation.

Recall that a collinear sector is characterised by its collinear momentum p_i, where the large component $n_{i+}p_i \sim 1$ is of order of the hard scale. Thus, to generate particles of different collinear sectors i, j with momenta p_i, p_j, one needs an underlying hard scattering to source these. A collinear particle of momentum p_i cannot source one with p_j without a hard momentum in the interaction vertex. This means that the physics within each collinear jet is described by a collinear Lagrangian $\mathcal{L}_i$, which only features modes of sector i, as well as interactions with the soft modes.

The effects of a hard scattering sourcing different collinear sectors and soft radiation are allocated in the so-called N-jet operators. These operators are the only place where Wilson coefficients appear, renormalisation takes place and a matching computation is necessary. Intuitively, one can think of the leading-order N-jet operators as the underlying non-radiative scattering amplitude, to which radiative corrections are considered.

One can understand their appearance in two ways. On the formal side, they are the result of integrating out hard modes. In conventional effective theories, integrating out heavy fields gives rise to higher-dimensional operators that account for the effects of these heavy fields. In complete analogy, one can think of these N-jet operators as the objects that mediate the effect of the hard regions that one integrated out. On the physical side, these objects must exist since one keeps all particles as valid external states in the effective theory, which can in general depend on hard momenta n_+P. Therefore, to describe scattering between such external particles, one requires additional ingredients on top of the effective Lagrangian that can account for hard momentum transfer. This is why these currents usually do *not* exist in a conventional EFT. But formally, they can be thought of as an analogue to the higher-dimensional operators in such EFTs.

The Lagrangians $\mathcal{L}_i$ and $\mathcal{L}_s$ correspond to the full theory in different kinematic limits expanded around certain backgrounds, and thus these are not renormalised by loop corrections. The purely-collinear theory corresponds to the full theory (in light-cone gauge if gauge symmetries are present), whereas the soft-collinear theory corresponds to the physics of a collinear fluctuation on top of an emergent soft background. The purely-soft Lagrangian corresponds again to the full theory, but with soft power counting. We make these statements more precise when we introduce gauge symmetries in the later sections. Crucially, these Lagrangians are not renormalised and exact to all orders in the couplings and in λ [40]. In Fig. 2.3, this factorisation into soft and collinear is depicted at the example of SCET QCD, considered in Chapter 3.

This split into soft and collinear modes can be viewed as a Lagrangian-level implementation of the method of regions [51]. Namely, the effective theory precisely reproduces the soft and collinear contributions of loop integrals. The hard contributions are absorbed into matching coefficients, which only appear in the N-jet operators that describe the (hard) interactions where multiple collinear sectors are present.

Note that this allocation of operators is quite distinct from the conventional EFT setup. Usually, one includes all higher-dimensional operators in the effective Lagrangian. Then, when computing a process, one decides which operators contribute and calculates their effect. In SCET, one can arrange the operators differently. In any given process, soft-collinear, purely-soft and purely-collinear interactions will always be relevant and must be included, since these can be added to the external legs. However, the kinematics of the process dictate what kind of underlying N-jet operator is relevant. Thus, the split into N-jet operators – the choice of which depends on the process at hand – and universal Lagrangian interactions – which always contribute – arises quite naturally.

The strategy of construction is quite simple. First, one constructs the effective soft-collinear

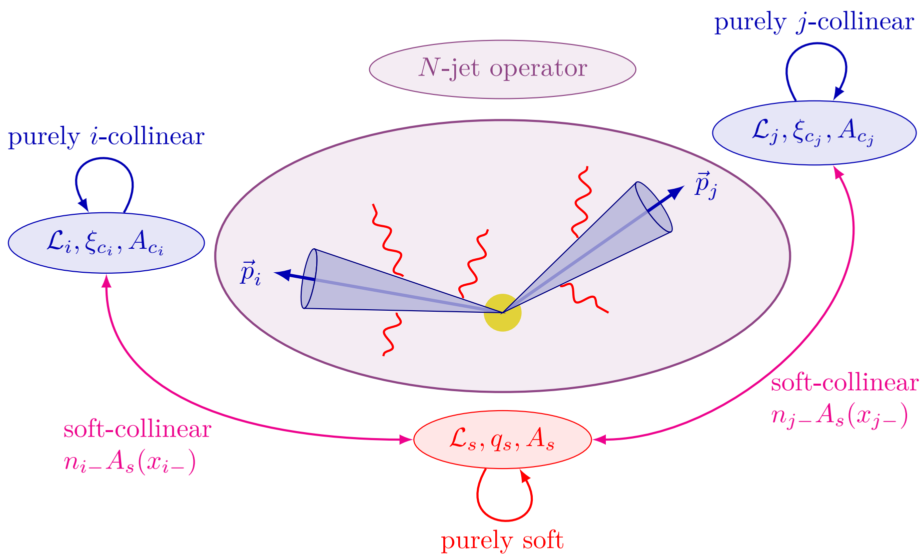


Figure 2.3: The factorisation present in the SCET QCD construction due to momentum conservation. Hard momenta are required to source modes of different collinear sectors. These objects are sorted into N-jet operators. The collinear sectors themselves are described by collinear Lagrangians $\mathcal{L}_i$, which feature purely-collinear and soft-collinear interactions. The soft field is described by a purely-soft Lagrangian $\mathcal{L}_s$. The appearance of $n_{i-}A_s(x_{i-})$ will be clear after Chapter 3.

and purely-soft Lagrangians. For the purely-scalar theory, this will turn out to be straightforward. Next, one investigates the possible building blocks that can appear in the N-jet operators, to find a minimal basis. Once these objects are identified, one can perform hard matching to the full theory and the construction is concluded. The result is the SCET Lagrangian and operator basis to the desired order in λ.

2.5 The Effective Lagrangian

We can now proceed with the construction of the effective collinear and soft Lagrangians. For the full theory, we consider the Lagrangian from (2.2.3), given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi(x)\,\partial^\mu\varphi(x) - \frac{g_3}{3!}\varphi^3(x) - \frac{g_4}{4!}\varphi^4(x)\,. \tag{2.5.1}$$

As explained above, the SCET Lagrangian takes the form

$$\mathcal{L}_{\text{SCET}} = \sum_i \mathcal{L}_i[\varphi_{c_i}, \varphi_s] + \mathcal{L}_{\text{soft}}[\varphi_s]\,, \tag{2.5.2}$$

where $\mathcal{L}_i$ is the soft-collinear Lagrangian containing only i-collinear and soft fields. There is no Lagrangian containing multiple collinear fields, as these only enter in the N-jet operators discussed further below. For the scalar theory, the full theory field is related to the EFT modes simply as

$$\varphi(x) = \varphi_c(x) + \varphi_s(x)\,. \tag{2.5.3}$$

We can interpret this split (2.5.3) as a collinear fluctuation on top of a soft background, and this interpretation is justified by the following construction. For the scalar field, this interpretation does not have any inherent advantage, but once we introduce gauge fields this intuition turns out to be useful. The effective theory describing the soft and collinear scalar modes can now be constructed in three steps:

(i) Introduce the decomposition (2.5.3) into the action (2.5.1). This yields a Lagrangian describing the collinear modes $\varphi_c(x)$ and the soft modes $\varphi_s(x)$, which is not yet homogeneous in λ. This is due to the presence of terms like $\int d^4x\, \varphi_c^2(x)\varphi_s(x)$, where the soft field is evaluated at the collinear argument x. However, as explained in Section 2.3, the soft field varies only over the large distance x_-, thus it should appear as $\varphi_s(x_-)$ in soft-collinear interactions.

(ii) To render each term manifestly homogeneous in λ, perform the multipole expansion (2.3.7) of soft fields in the soft-collinear interaction terms. The resulting theory takes the form of a collinear fluctuation $\varphi_c(x)$ in a soft background $\varphi_s(x_-)$, with an infinite tower of subleading terms controlled by the identity (2.3.8).

(iii) In the last step, expand these closed integrals and obtain a theory where each term has a definite and homogeneous power-counting in λ.

In step (i), we perform the expansion about the soft background $\varphi_s(x)$ by inserting (2.5.3) into the action (2.5.1). In the following, we adopt the convention that the argument of collinear modes is suppressed $\varphi_c(x) \equiv \varphi_c$, whereas for soft modes we keep it explicit. Inserting the decomposition yields the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\varphi_s(x)\partial^\mu\varphi_s(x) - \frac{g_3}{3!}\varphi_s^3(x) - \frac{g_4}{4!}\varphi_s^4(x) \\ &\quad + \frac{1}{2}\partial_\mu\varphi_c\partial^\mu\varphi_c - \frac{g_3}{3!}\varphi_c^3 - \frac{g_4}{4!}\varphi_c^4 - \frac{g_3}{2}\varphi_c^2\varphi_s(x) - \frac{g_4}{6}\varphi_c^3\varphi_s(x) - \frac{g_4}{4}\varphi_c^2\varphi_s^2(x)\,. \end{aligned}\tag{2.5.4}$$

Note that the Lagrangian does not contain terms linear in φ_c, like $\varphi_c\varphi_s^2(x)$. These terms violate momentum conservation since an energetic collinear particle cannot decay into purely soft ones. This is consistent with the expansion about a soft background $\varphi_s(x)$, as these linear terms are proportional to the soft equation of motion and can be dropped. This emphasises the point of view that already for a purely-scalar theory, the Lagrangian takes the form of a collinear fluctuation on a soft background.

The Lagrangian can be split into a (soft-)collinear $\mathcal{L}_c$ and a purely-soft part $\mathcal{L}_s$, where

$$\mathcal{L}_c = \frac{1}{2}\partial_\mu\varphi_c\partial^\mu\varphi_c - \frac{g_3}{3!}\varphi_c^3 - \frac{g_4}{4!}\varphi_c^4 - \frac{g_3}{2}\varphi_c^2\varphi_s(x) - \frac{g_4}{6}\varphi_c^3\varphi_s(x) - \frac{g_4}{4}\varphi_c^2\varphi_s^2(x)\,,\tag{2.5.5}$$

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu\varphi_s\partial^\mu\varphi_s - \frac{g_3}{3!}\varphi_s^3 - \frac{g_4}{4!}\varphi_s^4\,.\tag{2.5.6}$$

The purely soft theory is completely equivalent to the full theory of a (soft) scalar field φ_s. Since only soft fields are present, we count the integral $d^4x \sim \lambda^{-8}$, and the kinetic term is of order 1. Thus, in the purely-soft Lagrangian (2.5.6), we suppress the argument of the soft fields, as no multipole expansion is necessary.

The Lagrangian $\mathcal{L}_c$ contains a purely collinear part, which is simply the full-theory Lagrangian, as well as interactions between soft and collinear modes. However, the soft-collinear Lagrangian is not yet homogeneous in λ, since the soft fields must be multipole expanded about x_-. This is step (ii).

Performing this expansion and using the identities (2.3.8), one can express the all-order soft-collinear scalar Lagrangian as

$$\mathcal{L}_c^{(0)} = \frac{1}{2}\partial_\mu\varphi_c\partial^\mu\varphi_c - \frac{g_3}{3!}\varphi_c^3 - \frac{g_4}{4!}\varphi_c^4\,,\tag{2.5.7}$$

$$\begin{aligned}\mathcal{L}_{c,\,\mathrm{sub}} &= -\frac{g_3}{2}\varphi_c^2\varphi_s - \frac{g_4}{6}\varphi_c^3\varphi_s - \frac{g_4}{4}\varphi_c^2\varphi_s^2 \\ &\quad - \frac{g_3}{2}\varphi_c^2\int_0^1 \mathrm{d}s\,(x-x_-)^\mu\,[\partial_\mu\varphi_s]\,(y(s)) - \frac{g_4}{6}\varphi_c^3\int_0^1 \mathrm{d}s\,(x-x_-)^\mu\,[\partial_\mu\varphi_s]\,(y(s))\end{aligned}\tag{2.5.8}$$

$$- \frac{g_4}{2}\varphi_c^2\varphi_s \int_0^1 \mathrm{d}s\, (x-x_-)^\mu \left[\partial_\mu \varphi_s\right](y(s)) - \frac{g_4}{4}\varphi_c^2 \left(\int_0^1 \mathrm{d}s\, (x-x_-)^\mu \left[\partial_\mu \varphi_s\right](y(s)) \right)^2 .$$

Here, we dropped the argument for soft fields evaluated at x_-, $\varphi_s(x_-) \equiv \varphi_s$. Furthermore, we assigned a scaling $g_3 \sim \lambda$ to avoid a super-leading cubic interaction term.[4]

It is interesting to note the structure of the Lagrangian, as this is completely general and will manifest itself in the same form also for gauge theory and gravity. The purely-collinear interactions take the same form as the full theory. In fact, the purely collinear theory is completely equivalent to the full theory. The soft-collinear interactions give rise to a leading term, where $\varphi_s(x_-)$ appears, as well as an infinite tower of subleading terms. This theory corresponds to a collinear fluctuation on top of a new soft background, in this case $\varphi_s(x_-)$, and subleading terms expressed in terms of the kinetic 1-form $\partial_\mu \varphi_s$. In gauge theory, this structure will re-appear for the mode $n_- A_s(x_-)$, with the field-strength tensor $F_{s\mu\nu}$ in place of $\partial_\mu \varphi_s$. In gravity, the Riemann tensor appears instead.

The fact that both the purely-soft and purely-collinear Lagrangians are equivalent to the full theory (2.5.1) is easy to understand: without any external sources, there are no other scales present in either Lagrangian. Thus, without reference to any other scales, both theories are of course equivalent to the full theory, as one could simply boost the theory to a reference frame where all momenta scale homogeneously. It is the presence of other scales, either via soft-collinear interactions or via N-jet operators describing sources, that allows one to distinguish the soft and collinear modes, as one can now no longer boost to homogenise the momenta.

For practical computations, the structure presented in (2.5.8) is not useful, as it is not homogeneous in λ. If we expand the integrals in step (iii), the soft-collinear Lagrangian up to $\mathcal{O}(\lambda^2)$ is given by

$$\mathcal{L}_c = \mathcal{L}_c^{(0)} + \mathcal{L}_c^{(1)} + \mathcal{L}_c^{(2)} , \tag{2.5.9}$$

where

$$\mathcal{L}_c^{(0)} = \frac{1}{2}\partial_\mu \varphi_c \partial^\mu \varphi_c - \frac{g_3}{3!}\varphi_c^3 - \frac{g_4}{4!}\varphi_c^4 , \tag{2.5.10}$$

$$\mathcal{L}_c^{(1)} = -\frac{g_3}{2}\varphi_c^2\varphi_s - \frac{g_4}{6}\varphi_c^3\varphi_s , \tag{2.5.11}$$

$$\mathcal{L}_c^{(2)} = -\frac{g_4}{4}\varphi_c^2\varphi_s^2 - \frac{g_3}{2}\varphi_c^2 x_\perp^\mu \left[\partial_\mu \varphi_s\right] - \frac{g_4}{6}\varphi_c^3 x_\perp^\mu \left[\partial_\mu \varphi_s\right] , \tag{2.5.12}$$

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu \varphi_s \partial^\mu \varphi_s - \frac{g_3}{3!}\varphi_s^3 - \frac{g_4}{4!}\varphi_s^4 . \tag{2.5.13}$$

The explicitly appearing $x_\perp^\mu$, $n_- x$ due to the multipole expansion are a prominent feature of SCET and lead to rather complicated Feynman rules. These explicit coordinates correspond to the expansion of the suppressed components of soft momenta in soft-collinear vertices, as demonstrated in Section 2.3. The advantage of this Lagrangian is that any term has a definite power-counting in λ, and is thus homogeneous. This makes it straightforward to compute contributions at a desired order in λ, and no further expansions of the scattering amplitudes are necessary. In particular, this also holds for loop corrections. The systematic investigation of loop corrections is one of the main advantages of the Lagrangian approach compared to the explicit expansion of scattering amplitudes.

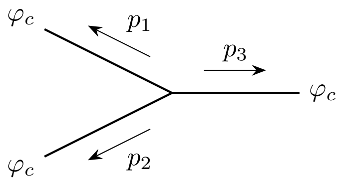
Note that this Lagrangian is constructed starting from a renormalised full-theory (2.2.3). Therefore, the effective Lagrangian is *not renormalised* to any order in the couplings g_3 and g_4 and in λ [40]. The only place where renormalisation takes place is in the N-jet operators.

The Feynman rules for the effective theory are readily derived. The propagator of the collinear and soft modes is the standard scalar propagator

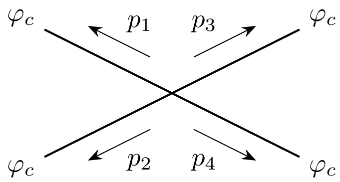
$$\frac{i}{p^2 + i\varepsilon} . \tag{2.5.14}$$

[4] Strictly speaking, with this choice the interaction remains super-leading for the purely-soft sector and should probably be integrated out. We will not worry about this in the following.

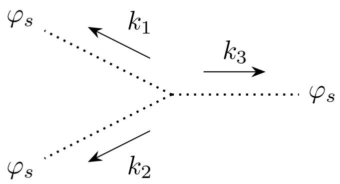
The purely-collinear and purely-soft vertices are only present at leading power,[5] and one finds the standard self-interaction



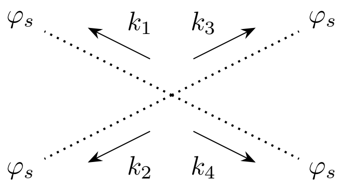
$$= -ig_3 \begin{cases} 1\,, & \mathcal{O}(\lambda^0) \\ 0\,, & \mathcal{O}(\lambda^1) \\ 0\,, & \mathcal{O}(\lambda^2) \end{cases}\,, \tag{2.5.15}$$



$$= -ig_4 \begin{cases} 1\,, & \mathcal{O}(\lambda^0) \\ 0\,, & \mathcal{O}(\lambda^1) \\ 0\,, & \mathcal{O}(\lambda^2) \end{cases}\,, \tag{2.5.16}$$

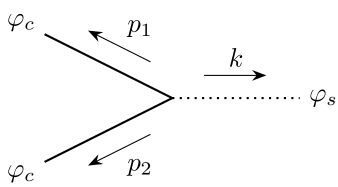


$$= -ig_3 \begin{cases} 1\,, & \mathcal{O}(\lambda^0) \\ 0\,, & \mathcal{O}(\lambda^1) \\ 0\,, & \mathcal{O}(\lambda^2) \end{cases}\,, \tag{2.5.17}$$

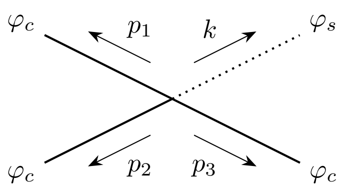


$$= -ig_4 \begin{cases} 1\,, & \mathcal{O}(\lambda^0) \\ 0\,, & \mathcal{O}(\lambda^1) \\ 0\,, & \mathcal{O}(\lambda^2) \end{cases} \tag{2.5.18}$$

In the soft-collinear sector, there are no leading-power interactions, and one obtains

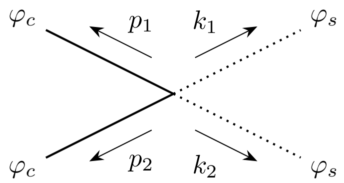


$$= -ig_3 \begin{cases} 0\,, & \mathcal{O}(\lambda^0) \\ 1\,, & \mathcal{O}(\lambda) \\ -X_\perp^\mu k_\mu\,, & \mathcal{O}(\lambda^2) \end{cases}\,, \tag{2.5.19}$$



$$= -ig_4 \begin{cases} 0\,, & \mathcal{O}(\lambda^0) \\ 1\,, & \mathcal{O}(\lambda) \\ -X_\perp^\mu k_\mu\,, & \mathcal{O}(\lambda^2) \end{cases}\,, \tag{2.5.20}$$

[5] With the given choice, either the purely-soft cubic interaction is super-leading, or the g_3 vertex should be suppressed by λ^2. Since we do not employ purely-soft cubic interactions in the following, we do not pursue this discussion further at this point. A similar subtlety arises when considering mass terms within SCET.



$$= -ig_4 \begin{cases} 0\,, & \mathcal{O}(\lambda^0) \\ 0\,, & \mathcal{O}(\lambda) \\ 1\,, & \mathcal{O}(\lambda^2) \end{cases} . \tag{2.5.21}$$

(2.5.22)

This is a crucial difference to both gauge theory and gravity. In the purely-scalar SCET, there are only leading-power purely-collinear interactions and no leading-power soft-collinear interactions. This is a first indication that there are no soft divergences in "pure matter" theories.

2.6 N-Jet Operators

As explained before, scattering processes which feature different collinear sectors are allocated to the N-jet operators. These objects are necessary, as the Lagrangian $\mathcal{L}_i$ only describes interactions within the same collinear sector, and with the soft background field. Notably, there are no interactions between different collinear sectors in the Lagrangian. These processes require a hard momentum to source the different collinear sectors and are not part of the Lagrangian interactions. Instead, the additional objects that describe such processes are the so-called N-jet operators. For the scalar theory, the minimal N-jet operator basis is trivial. Later, we will see how additional symmetries affect and constrain the operator basis.

A generic N-jet operator $\mathcal{J}$ in SCET takes the form of a light-ray operator [52]

$$\mathcal{J} = \int [dt]_N \, \widetilde{C}(t_{i_1}, t_{i_2}, \dots) J_s(0) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots)\,, \tag{2.6.1}$$

where $[dt]_N = \prod_{ik} dt_{i_k}$. Here, J_i denote the collinear and J_s the soft building blocks, and $\widetilde{C}(t_{i_1}, t_{i_2}, \dots)$ is the hard matching coefficient. The collinear operators are non-local along their respective collinear light-like direction n_{i+}^{μ}, indicated by the displacement t_{i_k}, due to their dependence on the large collinear momentum $n_{i+}p_i$, which is of the order of the hard scale. Thus, any field φ_{c_i} can in principle come with an arbitrary number of large derivatives $(n_{i+}\partial)^k \varphi_{c_i} \sim \varphi_{c_i}$. We trade this expansion in infinitely many large derivatives in favour of a non-locality along the light-cone, as explained in (2.2.1). This also eliminates any $n_{i+}\partial$ as possible building blocks in the operator basis. In the end, we evaluate the operator at $X = 0$, where the hard scattering takes.

At leading power, the elementary building blocks, denoted by J_i^{A0}, are simply the collinear scalar fields themselves,

$$J_i^{A0}(t_i) = \varphi_{c_i}(t_i) \sim \lambda\,. \tag{2.6.2}$$

The first important observation is that additional $\mathcal{O}(1)$ objects, namely the derivatives $n_{i+}\partial$, are already accounted for since the collinear operators are non-local along the n_{i+}^{μ}-direction. The derivative is then absorbed using integration by parts. Crucially, this shows that at a given order in λ, there are only a finite number of operators.

There are three possible ways to construct subleading operators starting from these J^{A0}:

(i) Adding derivatives $i\partial_\perp \sim \mathcal{O}(\lambda)$ or $in_{i-}\partial \sim \mathcal{O}(\lambda^2)$ to the $A0$ currents. For the simple scalar field, observe that the leading-power equation of motion reads

$$n_{i-}\partial\varphi_{c_i} = -\frac{\partial_\perp^2}{n_{i+}\partial}\varphi_{c_i} + \frac{g_3}{2}\frac{1}{n_{i+}\partial}\varphi_{c_i}^2 + \frac{g_4}{6}\frac{1}{n_{i+}\partial}\varphi_{c_i}^3 + \mathcal{O}(\lambda)\,. \tag{2.6.3}$$

Therefore, one can trade $n_{i-}\partial\varphi_{c_i}$ in favour of the leading-power collinear building blocks. At next-to-leading power, the equation of motion receives corrections from the soft-collinear interactions, and also soft building blocks will appear in the relation (2.6.3). In summary, $n_{i-}\partial$ can be eliminated at all orders in favour of soft and collinear building blocks, so derivative operators are characterised entirely by the number of $\partial_\perp$ derivatives added. We denote these types of operators by J_i^{An}, where n denotes the order in λ.

(ii) Adding more building blocks of the same collinear direction, as each block is itself of $\mathcal{O}(\lambda)$. An operator consisting of two (three) fields is denoted by J_i^{Bn} (J_i^{Cn}).

(iii) Adding $g_3 \sim \lambda$. This is specific to the scalar field with three-point interaction and will not re-appear in either the gauge-theory or gravitational setting.

Similarly, the leading-power soft building block corresponds to the soft scalar field

$$J_s(x) = \varphi_s(x) \sim \lambda^2 \,. \tag{2.6.4}$$

Subleading soft building blocks can be obtained by combining multiple soft scalars or adding derivatives. Since all soft derivatives have homogeneous counting in λ, there is no need to distinguish different directions.

In addition to these elementary building blocks, we incorporate the subleading Lagrangian interaction vertices by time-ordered product operators of the form

$$J_\varphi^{Tn}(t_{i_1}) = i\int d^4x T\left\{J_\varphi^{A0}(t_{i_1}), \mathcal{L}^{(n)}(x)\right\}\,, \tag{2.6.5}$$

and similar for subleading operators $J^{T(n+m)}$ formed with subleading currents J^{Am}. Of course, one can also form these products with B, C, ...-type currents.

2.7 An Example Matching

To familiarise ourselves with the formalism, we consider a simple matching computation for the theory described above. The process of interest is some hard scattering that creates four energetic scalar particles, well-separated in angle. This is what we later call the "non-radiative" hard matching, as we do not consider additional soft or collinear emissions. To perform the matching, we first compute the scattering amplitude in the full theory. This amplitude is then expanded in the power-counting parameter λ, and one decides based on the power-counting and fields present in this amplitude which operators contribute in SCET to deduce the matching coefficients from explicit matching. For the process at hand, we will find that the entire physics can be encapsulated by A-type operators. In addition, they can all be related to the leading $A0$ operator.

2.7.1 Full-theory Computation

We consider the scattering process $\varphi\varphi \to \varphi\varphi$ with momenta $p_1, \ldots, p_4$, all taken to be outgoing and on-shell, i.e. $p_i^2 = 0$. These momenta are collinear but well-separated in angle, meaning they belong to different collinear sectors with their own reference vectors $n_{1-}, \ldots, n_{4-}$. The contributing diagrams are given in Fig. 2.4, and are proportional to g_4 and g_3^2. The amplitude is computed in a straightforward fashion as

$$\mathcal{A} = -ig_4 + (-ig_3)^2\left(\frac{i}{(p_1+p_2)^2} + \frac{i}{(p_1+p_3)^2} + \frac{i}{(p_1+p_4)^2}\right)\,. \tag{2.7.1}$$

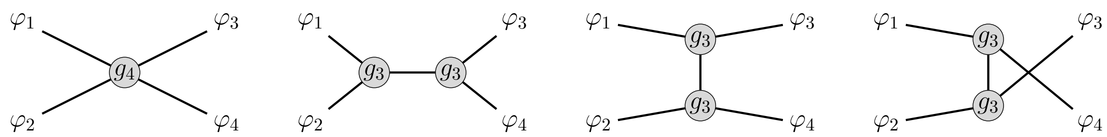


Figure 2.4: Diagrams contributing to $\varphi\varphi \to \varphi\varphi$. The first diagram is proportional to the four-point interaction g_4, while the remaining three stem from the three-point interaction and are proportional to g_3^2. The last diagram is the u-channel contribution where the lines are crossed.

Since the momenta belong to different collinear sectors, the sum $p_1 + p_2$ is not homogeneous in λ. We introduce their respective reference vectors and can decompose the Lorentz scalars in the denominators as

$$\begin{aligned} p_i \cdot p_j &= \left(n_{i+}p_i \frac{n_{i-}}{2} + p_{i\perp} + n_{i-}p_i \frac{n_{i+}}{2}\right) \cdot \left(n_{j+}p_j \frac{n_{j-}}{2} + p_{j\perp} + n_{j-}p_j \frac{n_{j+}}{2}\right) \\ &= \underbrace{\frac{n_{i-}n_{j-}}{4} n_{i+}p_i n_{j+}p_j}_{\sim 1} + \underbrace{n_{i-} \cdot p_{j\perp} n_{i+}p_i + n_{j-} \cdot p_{i\perp} n_{j+}p_j}_{\sim\lambda} \\ &\quad + \underbrace{\frac{n_{i-}n_{j+}}{4} n_{j-}p_j n_{i+}p_i + \frac{n_{j-}n_{i+}}{4} n_{i-}p_i n_{j+}p_j + p_{i\perp} \cdot p_{j\perp}}_{\sim\lambda^2} \\ &\quad + \underbrace{\frac{1}{2} n_{j+} \cdot p_{i\perp} n_{j-}p_j + \frac{1}{2} n_{i+} \cdot p_{j\perp} n_{i-}p_i}_{\sim\lambda^3} + \underbrace{\frac{n_{i+}n_{j+}}{4} n_{i-}p_i n_{j-}p_j}_{\sim\lambda^4} . \end{aligned} \tag{2.7.2}$$

Thus, we can expand the denominators appearing in the amplitude as

$$\begin{aligned} \frac{1}{2p_i \cdot p_j} = \frac{2}{n_{i-}n_{j-}} \frac{1}{p_{i+}p_{j+}} & \left(1 - \frac{2}{n_{i-}n_{j-}} \left(\frac{n_{i-}p_{j\perp}}{p_{i+}} + \frac{n_{j-}p_{i\perp}}{p_{j+}}\right)\right. \\ & + \left(\frac{2}{n_{i-}n_{j-}}\right)^2 \left(\frac{n_{i-}p_{j\perp}}{p_{i+}} + \frac{n_{j-}p_{i\perp}}{p_{j+}}\right)^2 \\ & \left. - \frac{2}{n_{i-}n_{j-}} \left(\frac{n_{i-}n_{j+}}{2} \frac{p_{j-}}{p_{j+}} + \frac{n_{j-}n_{i+}}{2} \frac{p_{i-}}{p_{i+}} + 2\frac{p_{i\perp} \cdot p_{j\perp}}{p_{i+}p_{j+}}\right)\right) + \mathcal{O}(\lambda^3) , \end{aligned} \tag{2.7.3}$$

where we introduced the short-hand notation $n_{i+}p_i \equiv p_{i+}$. The λ-expansion of the full amplitude is then given to $\mathcal{O}(\lambda)$ by

$$\mathcal{A}^{(0)} = -ig_4 - ig_3^2 \left(\frac{2}{n_{1-}n_{2-}} \frac{1}{p_{1+}p_{2+}} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)\right) , \tag{2.7.4}$$

$$\mathcal{A}^{(1)} = -ig_3^2 \left(\frac{2}{n_{1-}n_{2-}} \frac{1}{p_{1+}p_{2+}} \left(-\frac{2}{n_{1-}n_{2-}} \left(\frac{n_{1-} \cdot p_{2\perp}}{p_{1+}} + \frac{n_{2-} \cdot p_{1\perp}}{p_{2+}}\right)\right)\right) + (\text{perm}) . \tag{2.7.5}$$

Note that all subleading corrections (only relevant for the g_3-part) are related to the leading amplitude and are proportional to $p_{i\perp}$, via $n_{i-}p_i = -\frac{p_{i\perp}^2}{n_{i+}p_i}$. Thus, one could fix a reference frame where all momenta are aligned with their respective reference vectors, i.e. $p_{i\perp} = 0$. Then these subleading contributions identically vanish. One can exploit this property and deduce these subleading parts from a symmetry of the effective Lagrangian, called reparameterisation invariance (RPI) [53]. Then, one only has to perform the leading-order (LO) non-radiative matching, and all subleading parts are fixed by constraints from this symmetry. This is due to the fact that the full amplitude (2.7.1) depends on the Lorentz scalar $p_i \cdot p_j$. In the effective theory, this scalar must be expanded in λ, and we break the Lorentz symmetry. However, ultimately the effect of this UV Lorentz symmetry must still be present, and it re-appears in the form of RPI.

2.7.2 SCET Matching

Next, we perform the non-radiative matching at $\mathcal{O}(1)$ and $\mathcal{O}(\lambda)$ in SCET. The leading-order contribution in SCET must contain the four scalar fields. It is unique and given by the $A0$ operator

$$\mathcal{J}^{A0} = \int [dt]_4 \widetilde{C}(\{t_i\}) \varphi_1(t_1)\varphi_2(t_2)\varphi_3(t_3)\varphi_4(t_4)\,. \tag{2.7.6}$$

We can then match this to the LO-contribution of the full amplitude as

$$\begin{aligned}\mathcal{A}^{(0)} &= \langle p_1, p_2, p_3, p_4 | \, \mathcal{J}^{A0} \, |0\rangle \\ &= \int [dt]_4 e^{i\sum_j n_{j+}p_j t_j} \widetilde{C}^{A0}(t_1,\ldots,t_4) \equiv C^{A0}(n_{1+}p_1,\ldots,n_{4+}p_4)\,,\end{aligned} \tag{2.7.7}$$

and the leading-power momentum-space matching coefficient is determined to be

$$C^{A0}(n_{1+}p_1,\ldots n_{4+}p_4) = -ig_4 - ig_3^2\left(\frac{2}{n_{1-}n_{2-}}\frac{1}{p_{1+}p_{2+}} + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right)\,. \tag{2.7.8}$$

To obtain the position-space version, we simply perform a Fourier transformation. For the constant term, we find a δ-function, since

$$\int_{-\infty}^{\infty} dt_i\ (-ig_4)\, e^{in_{i+}p_i t_i}\delta(t_i) = -ig_4\,. \tag{2.7.9}$$

For the terms proportional to $(n_{i+}p)^{-1}$, we use

$$\int_{-\infty}^{\infty} dt_i\ \left(-ig_3^2\right) e^{in_{i+}p_i t_i}\theta(-t_i) = \left(-ig_3^2\right)\int_{-\infty}^{0} dt_i\ e^{in_{i+}p_i t_i} = \left(-ig_3^2\right)\frac{-i}{n_{i+}p_i}\,. \tag{2.7.10}$$

Therefore, the position-space matching coefficient reads

$$\begin{aligned}\widetilde{C}^{A0}(t_1,\ldots,t_4) &= -ig_4\delta(t_1)\delta(t_2)\delta(t_3)\delta(t_4) \\ &\quad - ig_3^2\left(\frac{2}{n_{1-}n_{2-}}\theta(-t_1)\theta(t_2) + (2\leftrightarrow 3) + (2\leftrightarrow 4)\right)\,.\end{aligned} \tag{2.7.11}$$

We see that (2.7.7) can be read as the simple statement that *the momentum-space matching coefficient corresponds to the non-radiative amplitude* order-by-order in λ.

For the non-radiative process we consider here, there is no collinear splitting, and the only way to construct a subleading operator is to attach a transverse derivative $\partial_\perp \sim \lambda$ to the fundamental building block φ_{c_i} i.e. to replace

$$\varphi_{c_i}(t_i n_{i+}) \to i\partial_\perp^\mu \varphi_{c_i}(t_i n_{i+}) \tag{2.7.12}$$

for one $J_i(t_i)$. The corresponding N-jet operator, now $\mathcal{O}(\lambda)$ suppressed compared to the leading-power (2.7.6), takes the form

$$\mathcal{J}^{A1} = \sum_j \int [dt]_4\, \widetilde{C}_j^{A1\,\mu}(t_1,\ldots,t_4)\, J^{A1}_{j\,\mu}(t_j)\left(\prod_{i\neq j} J_i^{A0}(t_i)\right)\,. \tag{2.7.13}$$

Again, the matrix element of this operator must reproduce the scattering amplitude at $\mathcal{O}(\lambda)$, which yields the condition

$$\mathcal{A}^{(1)} = \langle p_1,\ldots,p_4|\mathcal{J}^{A1}|0\rangle = -p^\mu_{j\perp}\int [dt]_4\, e^{i\sum_i n_{i+}p_i\, t_i}\widetilde{C}^{A1}_{j\,\mu}(t_1,\ldots t_4)$$

$$= -p^{\mu}_{j\perp} C^{A1}_{j\,\mu}(n_{1+}p_1, \ldots, n_{4+}p_4)\,, \tag{2.7.14}$$

and $C^{A1\mu}$ can be computed explicitly by comparing with (2.7.5). For example, the coefficient that comes with $i\partial^{\mu}_{\perp}\varphi_2$ is given by

$$C^{A1\mu}_{2} = -ig_3^2 \frac{2}{n_{1-}n_{2-}} \frac{1}{n_{1+}p_1 n_{2+}p_2} \left(\frac{2}{n_{1-}n_{2-}} \frac{n^{\mu}_{1-}}{n_{1+}p_1} \right)\,, \tag{2.7.15}$$

and it seems to contain the leading-power coefficient C^{A0}, which is sensible since this term stems from the Taylor expansion of the full-theory amplitude.

Indeed, instead of performing the matching order-by-order in λ, one can exploit the fact that for the non-radiative scattering captured by the A-type operators, the λ-expansion is entirely due to the expansion of the scalar products inside $\mathcal{A}$. Therefore, all subleading coefficients are in principle already determined from the leading-power result using RPI constraints. For the $A1$ current, this constraint reads [54]

$$C^{A1\,\mu}_{j}(n_{1+}p_1, \ldots, n_{N+}p_N) = -\sum_{k\neq j} \frac{2n^{\mu}_{k-}}{n_{k-}n_{j-}} \frac{\partial}{\partial n_{i+}p_i} C^{A0}(n_{1+}p_1, \ldots, n_{N+}p_N)\,. \tag{2.7.16}$$

Intuitively, if the Fourier transformation of the *leading-power matching coefficient* is simply the *non-radiative amplitude* depending only the leading-power momenta $n_{i+}p_i, n_{j+}p_j$, then this equation states that *the $A1$ matching coefficient* is simply the *next term in the Taylor expansion of the non-radiative amplitude*. Since the hard matching coefficient in SCET can only depend on the large components $n_{i+}p_i$, the equation (2.7.16) takes this somewhat cumbersome form.

In summary, the Fourier-transform of the hard matching coefficient C^{An} in the n-th power suppressed N-jet operator $\mathcal{J}^{(n)}$ corresponds, roughly speaking, to the n-th term in the Taylor expansion of the non-radiative amplitude.

These results are important in the derivation of the soft theorems from the effective theory, and we will come back to this later. The matching coefficients of the non-radiative amplitude can be inferred *to all orders* from the leading-power matching, which corresponds to the full theory in a special reference frame (where all p_i are aligned with their reference vectors).[6]

[6]This statement holds as long as the λ-expansion of the non-radiative theory is entirely due to the expansion of scalar products, and not due to additional higher-order contributions.

Chapter

QCD SCET 3

Now that we understand the basic principles underlying the SCET construction and formalism, we are ready to incorporate gauge symmetries. This complicates the discussion for two reasons, both of which are also present in gravity. First, the gauge symmetry itself must be modified. One must implement the split into soft and collinear modes, then determine the relevant gauge symmetry of each sector, and finally consistently modify these symmetries to respect the light-front multipole expansion. Second, the gauge fields contain modes that are not suppressed in the power counting. These components must be controlled to all orders in the power-counting parameter to obtain a finite operator basis. These two problems are readily dealt with by introducing a number of Wilson lines, which allow us to give closed expressions that hold to all orders in the power-counting parameter λ. These expressions can then be expanded in λ in a straightforward fashion and one obtains the effective Lagrangian where each term has a manifestly homogeneous power-counting, and the emergent gauge symmetry of the effective theory then also respects this counting. In the operator basis, the gauge symmetry allows us to identify a set of minimal soft and collinear building blocks.

We will see that it is the covariance with respect to this emergent gauge symmetry, and the constraints that follow from it, that ultimately give rise to the soft theorem from the EFT perspective.

The position-space derivation of the SCET Lagrangian was first presented in [40], and in full-generality featuring also non-Abelian symmetries in [41]. The following discussion follows closely the exposition in [47] by the author in collaboration with M. Beneke and R. Szafron, which expands on the previous derivations in a few key aspects related to the gauge symmetry and gauge condition, which prove valuable when considering the generalisation to gravity. Most of the results can already be found in [40, 41, 47] and [48].

3.1 Field Content

We begin our discussion with scalar QCD, the theory describing a scalar field ϕ^a in the fundamental representation of SU(N), and the corresponding gluon field A_μ.

The overall intuition from the previous purely-scalar consideration discussed in Section 2.4 still holds true, and most concepts can be carried over in a straightforward fashion.

Just as in the purely-scalar theory, we introduce collinear and soft modes for both fields, and use the two-point function (2.2.7) to determine their power counting. For the scalar, one obtains again the power counting (2.2.8)

$$\phi_c \sim \lambda, \quad \phi_s \sim \lambda^2 . \tag{3.1.1}$$

For the gluon, one finds in a general gauge [40]

$$\langle 0| T(A_\mu(x)A_\nu(y)) |0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\, \frac{i}{p^2+i\varepsilon} \left[-g_{\mu\nu} + (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right], \tag{3.1.2}$$

which determines

$$n_+A_c \sim 1\,, \quad A_{c\perp} \sim \lambda\,, \quad n_-A_c \sim \lambda^2\,, \quad A_{s\mu} \sim \lambda^2\,. \tag{3.1.3}$$

Note that the gluons, both soft and collinear, scale like the respective components of the momentum, i.e. $A_c \sim p_c$ and $A_s \sim k_s$. This means that the purely-collinear and purely-soft covariant derivatives, denoted by D_c and D_s, respectively, where $D_{c/s} = \partial - igA_{c/s}$, are homogeneous in the power-counting parameter λ, but the total soft-collinear covariant derivative, $D = \partial - igA_c - igA_s$, consisting of both soft and collinear fields, is not.

3.2 Gauge Symmetry

There is a major new feature that was not present in the previous discussion: QCD is invariant under a SU(N) gauge symmetry. This leads to additional complications when constructing the effective theory. Namely, the gauge symmetry is *extended* in a non-trivial fashion in the effective Lagrangian: the collinear fields take the role of *fluctuations* on top of the underlying *soft background*. This is a consistent way of implementing the desired soft and collinear physics. This structure is imposed by momentum conservation, as we argue in the following.

First, the full-theory gluon field A_μ transforms under a gauge transformation as

$$A_\mu \to U A_\mu U^\dagger + \frac{i}{g} U \left[\partial_\mu \, , U^\dagger\right] . \tag{3.2.1}$$

This field is decomposed into soft and collinear modes as

$$A_\mu(x) = A_{c\mu}(x) + A_{s\mu}(x) \, . \tag{3.2.2}$$

Both of these gluon fields should acquire a gauge transformation, and the sum of both fields must transform as the original full field. However, when performing a gauge transformation, the transformation matrices themselves are either soft or collinear objects, meaning the position-argument of the respective infinitesimal gauge parameter $\alpha_{c/s}(x)$ has either soft or collinear power-counting. One must ensure that the soft field does not transform under collinear gauge transformations, else the resulting transformed field is no longer soft. In that case, the notions of "soft" and "collinear" would not be gauge-invariant, which is problematic. On the other hand, the collinear field can transform under soft gauge without problem.

A consistent solution is to impose the transformations

$$\begin{aligned}
\text{collinear:} \quad & A_c \to U_c A_c U_c^\dagger + \frac{i}{g} U_c \left[D_s \, , U_c^\dagger\right] , && \phi_c \to U_c \phi_c \, , \\
& A_s \to A_s \, , && \phi_s \to \phi_s \, , \\
\text{soft:} \quad & A_c \to U_s A_c U_s^\dagger \, , && \phi_c \to U_s \phi_c \, , \\
& A_s \to U_s A_s U_s^\dagger + \frac{i}{g} U_s \left[\partial \, , U_s^\dagger\right] , && \phi_s \to U_s \phi_s \, .
\end{aligned} \tag{3.2.3}$$

Under collinear transformations, A_s does not transform. However, it appears in the form of the soft-covariant derivative $D_s^\mu = \partial^\mu - igA_s^\mu$ in the collinear transformation. Conversely, under a soft transformation, A_s has the standard transformation (3.2.1), and A_c has the covariant transformation of a (non-gauge) matter field in the adjoint representation.[1] Note that the x-argument of the local transformations U_c and U_s has the same scaling as the argument of collinear and soft fields, respectively. This ensures that the scaling of the gauge fields (3.1.3) remains unaltered by the gauge symmetry of the effective theory. One can verify that the transformation (3.2.3) has the property that the full field, defined as (3.2.2), indeed transforms as a standard gluon (3.2.1). Therefore we do not need to modify the naive decomposition (3.2.2) further.

In the purely-scalar theory, this line of reasoning led to the interpretation that the split $\varphi = \varphi_c + \varphi_s$ is a split into a collinear fluctuation on top of a soft background. Now, based on

[1] From the soft perspective, any collinear field, regardless if gauge or matter, has the same transformation, depending only on its representation.

the transformations (3.2.3), we again see that the decomposition (3.2.2) can be interpreted as the split into a collinear fluctuation on top of a soft background, and the gauge transformations immediately follow from this identification: the fluctuation (collinear mode) comes with its own gauge symmetry, covariant with respect to the non-trivial background configuration A_s (the soft mode). The soft background has the standard gauge transformation, and each fluctuation satisfies the homogeneous – non-gauge – matter transformation. Thus, momentum conservation imposes this extended symmetry, consisting of the two separate soft-background and collinear-fluctuation gauge symmetries, in the effective theory. In the case of multiple collinear modes, each mode would be interpreted as a fluctuation and would come with its own i-collinear gauge symmetry. However, all these collinear modes transform in the same way under the soft gauge transformation.

The soft matter fields are slightly more involved. Just as for gauge fields, the soft matter field cannot transform under collinear gauge transformations, as seen in (3.2.3). However, the naive decomposition

$$\phi(x) = \phi_c(x) + \phi_s(x) \tag{3.2.4}$$

is inconsistent with the proposed transformations (3.2.3), as the sum on the right-hand side does not transform like the full-theory scalar field on the left-hand side. Instead, one has to implement the transformation using a combination of Wilson lines. Namely, one decomposes the matter field as

$$\phi = \phi_c + W Z^\dagger \phi_s \,, \tag{3.2.5}$$

introducing the Wilson lines

$$W Z^\dagger = P \exp\left[ig \int_{-\infty}^{0} ds\, n_+ A(x + s n_+)\right] \bar{P} \exp\left[-ig \int_{-\infty}^{0} ds\, n_+ A_s(x + s n_+)\right] \,, \tag{3.2.6}$$

where P ($\bar{P}$) denotes (anti-)path-ordering. These Wilson lines transform as

$$\begin{aligned} &\text{collinear:} \quad && Z^\dagger \to Z^\dagger \,, && W \to U_c(x) W \,, \\ &\text{soft:} && Z^\dagger \to Z^\dagger U_s^\dagger(x) \,, \quad && W \to U_s(x) W \,, \end{aligned} \tag{3.2.7}$$

where we used $U(x - \infty n_+) = 1$, i.e. that the gauge transformation vanishes at infinity.

We can interpret the decomposition (3.2.5) as follows: The soft field $\phi_s(x)$ only transforms under soft gauge transformation $U_s(x)$. The semi-infinite Wilson line $Z^\dagger$, only containing the soft gluon field, moves this soft transformation from x to $-\infty n_+$, where we impose that the transformation vanishes. Thus, the combination $Z^\dagger \phi_s(x)$ is gauge-invariant under both soft and collinear transformations, since, by construction, a purely soft object does not have a collinear transformation. On the other hand, $W^\dagger$, the full-theory Wilson line,[2] can be used to render the full-theory scalar field gauge-invariant. Thus, when multiplying $Z^\dagger \phi_s(x)$ with W, one recovers the object $W Z^\dagger \phi_s(x)$ that transforms like an ordinary full-theory scalar field. Therefore, the right-hand side of the decomposition (3.2.5), combined with the proposed transformations (3.2.3), does indeed transform like the full-theory scalar field on the left-hand side.

One realises that for gauge theories, already at this first level of the mode decompositions, SCET requires non-local objects in order to implement consistent gauge transformations. These and other related Wilson lines play a central role in the later constructions. For more details on these, we refer to [40, 41].

3.3 Multipole Expansion and Redefinitions

Now that we understand the field content and the gauge symmetries, we can start the construction of the effective theory. Just like in the scalar case, as explained in Section 2.4, the effective

[2]We call it full-theory since only the full-theory gluon field $A = A_c + A_s$ appears.

theory separates into soft-collinear Lagrangians $\mathcal{L}_{c_i}$ and the N-jet operators. We first discuss how the effective Lagrangian is constructed, and later proceed with the analysis of a minimal operator basis for the N-jets. The basic principles are the same ones as outlined in Section 2.5: one inserts the field decompositions (3.2.2), (3.2.5) in the Lagrangian and then performs the multipole expansion as introduced in Section 2.3. However, the gauge symmetry leads to additional subtleties with regard to the multipole expansion, which we explain and address in the following sections.

3.3.1 Homogeneous Gauge Symmetry and Multipole Expansion

There is an immediate problem in the construction concerning the *soft* component of the gauge transformations. From (3.2.3), one sees that collinear matter fields transform under the soft gauge symmetry as

$$\phi_c(x) \to U_s(x)\phi_c(x)\,. \tag{3.3.1}$$

This poses a potential problem when performing the light-front multipole expansion about $x_- = n_+x\frac{n_-}{2}$ as explained in Section 2.3, since the soft gauge transformation $U_s(x)$ in (3.3.1) depends on the *collinear* coordinate x, which has the wrong scaling for a *soft* field. This means that $U_s(x)$ in (3.3.1) must also be multipole-expanded, and this in turn generates an infinite tower of subleading terms in the power-counting parameter λ, since

$$U_s(x)\phi_c(x) = U_s(x_-)\phi_c(x) + x_\perp^\alpha \left[\partial_\alpha U_s\right](x_-)\phi_c(x) + \mathcal{O}(\lambda^2\phi_c)\,. \tag{3.3.2}$$

In other words, after multipole expansion, the proper soft transformation should only depend on x_-, i.e. the collinear scalar field should transform as

$$\phi_c(x) \to U_s(x_-)\phi_c(x)\,, \tag{3.3.3}$$

without any subleading corrections. This way, the soft gauge transformation is *homogeneous* in λ and respects the soft multipole expansion.

The second problem, at first glance unrelated, concerns the transformation of the collinear gluon under the *collinear* gauge symmetry,

$$A_c(x) \to U_c(x)A_c(x)U_c^\dagger(x) + \frac{i}{g}U_c(x)\left[D_s(x)\,,U_c^\dagger(x)\right]\,. \tag{3.3.4}$$

Here, $D_s(x)$ appears, i.e. the full soft field $A_s(x)$. Even after consistently implementing the multipole expansion, the transformation

$$A_c(x) \to U_c(x)A_c(x)U_c^\dagger(x) + \frac{i}{g}U_c(x)\left[D_s(x_-),U_c^\dagger(x)\right] \tag{3.3.5}$$

is still inhomogeneous in λ, due to the power-counting of the gluon components (3.1.3). Namely, the gluon scales as

$$n_-A_c + n_-A_s \sim \lambda^2\,,\quad n_+A_c + n_+A_s \sim 1+\lambda^2\,,\quad A_{c\perp} + A_{s\perp} \sim \lambda + \lambda^2\,, \tag{3.3.6}$$

and only the n_-A_s component is the one that has the same scaling as the respective collinear component, while the remaining ones are subleading with respect to their collinear counterpart. Thus, the *homogeneous* collinear gauge transformation of the gluon should take the form

$$A_c(x) \to U_c(x)A_c(x)U_c^\dagger(x) + \frac{i}{g}U_c(x)\left[\hat{D}_s(x_-)\,,U_c^\dagger(x)\right]\,, \tag{3.3.7}$$

where

$$\hat{D}_s^\mu(x_-) = \partial^\mu - n_-A_s(x_-)\frac{n_+^\mu}{2}\,, \tag{3.3.8}$$

and only $n_- A_s(x_-)$ is present.

In summary, there are two issues one has to address: first, one needs to ensure that the *soft* gauge transformation of *collinear* fields respects the multipole expansion – this guarantees a soft gauge transformation *homogeneous* in the power-counting parameter λ. Second, one has to effectively *replace the soft background field* $A_s(x)$ by $n_- A_s(x_-)$. These two steps are necessary in order to construct manifestly homogeneous and gauge-covariant terms in λ, i.e. terms that have a definite power-counting and are not related to subleading terms via gauge symmetry.

3.3.2 The Static Multipole Expansion and Fixed-point Gauge

In order to explain how these two problems can be addressed, we first give a simpler example, to build some intuition. Namely, we consider the standard multipole expansion for an Abelian gauge field. To be precise, we consider a (hard) Dirac matter field $\psi(x)$,[3] which transforms under a U(1) gauge symmetry as

$$\psi(x) \to U(x)\psi(x) = e^{-ie\alpha(x)}\psi(x)\,, \tag{3.3.9}$$

coupled to a (soft) background vector potential $A_\mu(x)$ with the standard transformation

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{e}U(x)\left[\partial_\mu U^\dagger\right](x) = A_\mu(x) - \partial_\mu\alpha(x)\,, \tag{3.3.10}$$

for the Abelian field with gauge parameter $\alpha(x)$. The Lagrangian of these fields is then

$$\mathcal{L} = \bar\psi(x) i\gamma^\mu\partial_\mu\psi(x) + e\bar\psi(x)\gamma^\mu A_\mu(x)\psi(x)\,. \tag{3.3.11}$$

We are now interested in the limit where $A_\mu(x)$ is slowly varying compared to $\psi(x)$. That is, we assume a "soft" power-counting for $A_\mu(x) \sim \lambda^2 \equiv \varepsilon$, $\partial_\mu A(x) \sim \varepsilon^2$, and a hard power-counting for the field $\psi(x) \sim 1$ and the coordinate $x \sim 1$. Thus, one needs to perform the multipole expansion about $x = 0$ as

$$A_\mu(x) = A_\mu(0) + x^\alpha\,[\partial_\alpha A_\mu](0) + \frac{1}{2}x^\alpha x^\beta\,[\partial_\alpha\partial_\beta A_\mu]\,(0) + \mathcal{O}(\varepsilon^3 A_\mu)\,. \tag{3.3.12}$$

In this expansion, one can also use the order in x as power-counting, since each term adds a further power of ε, and we adopt this convention. Inserting the multipole-expansion (3.3.12) in the Lagrangian, one obtains to second order in x, or equivalently to $\mathcal{O}(\varepsilon^3)$,

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{O}(x^3)\,, \tag{3.3.13}$$

where the superscript indicates the order in ε, and the individual terms are given by

$$\mathcal{L}^{(0)} = \bar\psi i\gamma^\mu\partial_\mu\psi\,, \tag{3.3.14}$$

$$\mathcal{L}^{(1)} = e\bar\psi\gamma^\mu A_\mu\psi\,, \tag{3.3.15}$$

$$\mathcal{L}^{(2)} = e\bar\psi\gamma^\mu x^\alpha\,[\partial_\alpha A_\mu]\,\psi\,, \tag{3.3.16}$$

$$\mathcal{L}^{(3)} = e\bar\psi\gamma^\mu \frac{1}{2}x^\alpha x^\beta\,[\partial_\alpha\partial_\beta A_\mu]\,\psi\,. \tag{3.3.17}$$

Here, we used the short-hand notation $\psi \equiv \psi(x)$, $A_\mu \equiv A_\mu(0)$, and derivatives of A_μ are taken before setting $x = 0$, as indicated by the square brackets. The gauge transformation of ψ now mixes different orders of x, or ε, respectively, since the parameter of the transformation still depends on x, hence

$$\psi \to U(x)\psi = U(0)\psi + x^\alpha\,[\partial_\alpha U](0)\psi + \mathcal{O}(x^2)\,. \tag{3.3.18}$$

[3] We employ a Dirac field here simply because the Lagrangian is linear and thus less complicated.

In other words, the gauge transformation is not homogeneous in the power counting and does not respect the multipole expansion. The proper, homogeneous transformation of the matter field would correspond to a global transformation

$$\psi(x) \to U(0)\psi(x)\,. \tag{3.3.19}$$

The matter field should effectively no longer have a gauge transformation, and one would expect that the proper Lagrangian reflects this property, by consisting only of manifestly gauge-invariant terms, and without explicit $A_\mu(0)$ appearing. Consequently, there should be a way to simplify the Lagrangian.

One possibility is to employ the leading-power equation of motion

$$\partial\!\!\!/\psi = 0 + \mathcal{O}(\lambda)\,, \tag{3.3.20}$$

to simplify the subleading Lagrangian.[4] This is a standard operation in effective theories and often leads to simpler forms of the Lagrangian.[5] The term in $\mathcal{L}^{(1)}$ (3.3.15) can be rewritten as

$$\begin{aligned}\bar\psi A\!\!\!/(0)\psi &= \bar\psi\gamma^\mu\left[\partial_\mu x^\nu\right]A_\nu(0)\psi \\ &= -x^\nu A_\nu(0)\left((\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi\right) = 0\,.\end{aligned} \tag{3.3.21}$$

In $\mathcal{L}^{(2)}$ in (3.3.16), we have

$$\begin{aligned}\bar\psi\gamma^\mu x^\nu\left[\partial_\nu A_\mu\right](0)\psi &= \bar\psi\gamma^\mu x^\nu\left[\partial_\mu x^\alpha\right]\left[\partial_\nu A_\alpha\right](0)\psi \\ &= -\bar\psi\gamma^\nu x^\mu\left[\partial_\nu A_\mu\right](0)\psi + (\mathrm{eom})\,.\end{aligned} \tag{3.3.22}$$

There is no contribution from the subleading part of (3.3.20)

$$\partial\!\!\!/\psi = -eA\!\!\!/(0)\psi + \mathcal{O}(x)\,, \quad \bar\psi\overleftarrow{\partial\!\!\!/} = -e\bar\psi A\!\!\!/(0) + \mathcal{O}(x)\,, \tag{3.3.23}$$

where $\bar\psi\overleftarrow{\partial\!\!\!/} \equiv -(\partial_\mu\bar\psi)\gamma^\mu$, since the relative sign between ψ and $\bar\psi$ leads to cancellation. Thus, we can rewrite half of this term with the above identity (3.3.22) to obtain

$$\begin{aligned}\bar\psi\gamma^\mu x^\nu\left[\partial_\nu A_\mu\right](0)\psi &= \frac{1}{2}\bar\psi\gamma^\mu x^\nu\left[\partial_\nu A_\mu\right](0)\psi - \frac{1}{2}\bar\psi\gamma^\nu x^\mu\left[\partial_\nu A_\mu\right](0)\psi \\ &= \frac{1}{2}\bar\psi\gamma^\mu x^\nu F_{\nu\mu}\psi\,,\end{aligned} \tag{3.3.24}$$

where we introduced the field-strength tensor

$$F_{\mu\nu} \equiv \left[\partial_\mu A_\nu\right](0) - \left[\partial_\nu A_\mu\right](0)\,. \tag{3.3.25}$$

Next, the expression in $\mathcal{L}^{(3)}$ (3.3.17) can be rearranged in a similar fashion as

$$\begin{aligned}\bar\psi\gamma^\mu x^\alpha x^\beta\left[\partial_\alpha\partial_\beta A_\mu\right]\psi &= \bar\psi\gamma^\mu x^\alpha x^\beta\left[\partial_\mu x^\nu\right]\left[\partial_\alpha\partial_\beta A_\nu\right]\psi \\ &= -2\bar\psi\gamma^\mu x^\nu x^\alpha\left[\partial_\alpha\partial_\mu A_\nu\right]\psi\,.\end{aligned} \tag{3.3.26}$$

Accordingly, we can split the term in $\mathcal{L}^{(3)}$ as

$$\mathcal{L}^{(2)} = \frac{1}{2}e\bar\psi\gamma^\mu x^\alpha x^\beta\left[\partial_\alpha\partial_\beta A_\mu\right]\psi\left(\frac{2}{3}+\frac{1}{3}\right)$$

[4] Note that the relevant contribution of the subleading terms of this equation of motion up to $\mathcal{O}(\lambda^3)$ vanishes when inserted into the Lagrangian.

[5] Note that the application of equations of motion in the operator basis is not as straightforward in SCET. Here, equation-of-motion operators can mix into physical ones under renormalisation [54]. However, this is only relevant for the N-jet operators and not for this discussion.

$$
\begin{aligned}
&= \frac{1}{3} e\bar{\psi}\left(\gamma^\mu x^\alpha x^\beta \left[\partial_\alpha \partial_\beta A_\mu\right] - \gamma^\mu x^\nu x^\alpha \left[\partial_\alpha \partial_\mu A_\nu\right]\right)\psi \\
&= \frac{1}{3} e\bar{\psi}\gamma^\mu x^\nu x^\alpha \left[\partial_\alpha F_{\nu\mu}\right]\psi\,. \qquad (3.3.27)
\end{aligned}
$$

To summarise, we find the Lagrangian terms

$$\mathcal{L}^{(0)} = \bar{\psi} i\gamma^\mu \partial_\mu \psi\,, \tag{3.3.28}$$

$$\mathcal{L}^{(1)} = 0\,, \tag{3.3.29}$$

$$\mathcal{L}^{(2)} = \frac{1}{2} e F_{\mu\nu} \bar{\psi}\gamma^\mu x^\nu \psi\,, \tag{3.3.30}$$

$$\mathcal{L}^{(3)} = \frac{1}{3}\left[e\partial_\alpha F_{\nu\mu}\right]\bar{\psi}\gamma^\mu x^\nu x^\alpha \psi\,, \tag{3.3.31}$$

where the field-strength tensor and its derivative are evaluated at $x = 0$ *after* the derivatives are taken. Thus, the interactions with the background field can be completely expressed in terms of the field-strength tensor $F_{\mu\nu}$ and its derivatives. The couplings correspond to interactions with the dipole, quadrupole, and higher-pole moments of the matter field.

The application of equations of motion is equivalent to a field redefinition starting with

$$\psi(x) \to \hat{\psi}(x) = \psi(x) - iex^\alpha A_\alpha(0)\psi(x) + \mathcal{O}(x^2)\,. \tag{3.3.32}$$

This redefinition has an interesting effect on the gauge transformation. Namely, the redefined field $\hat{\psi}$ transforms as

$$
\begin{aligned}
\hat{\psi}(x) \to\ & \psi(x) - ie\alpha(0)\psi(x) - iex^\mu \left[\partial_\mu \alpha\right](0)\psi(x) \\
& - iex^\alpha A_\alpha(0)\psi(x) + iex^\mu\left[\partial_\mu \alpha\right](0)\psi(x) + \mathcal{O}(x^2, \alpha^2) \\
=\ & \hat{\psi}(x) - ie\alpha(0)\hat{\psi}(x) + \mathcal{O}(x^2,\alpha^2)\,. \qquad (3.3.33)
\end{aligned}
$$

Thus, the redefinition is actually responsible for *homogenising* the gauge transformation of the matter field. We can systematically extend this to all orders by employing a Wilson line. Namely, we desire an object that transports the gauge transformation of the matter field from point x to point $x = 0$ in a straight line. This is achieved by the Wilson line

$$V(x) = P\exp\left(ie\int_0^x dy^\mu\, A_\mu(y)\right) = P\exp\left(ie\int_0^1 ds\, x^\mu A_\mu(sx)\right)\,, \tag{3.3.34}$$

where P denotes path-ordering and is relevant if we generalise this to the non-Abelian situation. In the Abelian case, the Wilson line is just given by the exponential itself. This object transforms as

$$V(x) \to U(x)V(x)U^\dagger(0)\,. \tag{3.3.35}$$

Thus, we can redefine the matter field using this Wilson line as

$$\hat{\psi}(x) \equiv V^\dagger(x)\psi(x)\,, \tag{3.3.36}$$

which now has the desired transformation

$$\hat{\psi}(x) \to U(0)\hat{\psi}(x) \tag{3.3.37}$$

to all orders in x. When inserting this redefinition in the Lagrangian (3.3.11), one finds

$$\mathcal{L} = \bar{\hat{\psi}} i\gamma^\mu \partial_\mu \hat{\psi} + e\bar{\hat{\psi}}\left(V\gamma^\mu A_\mu(x)V^\dagger + \frac{i}{e}V\left[\partial_\mu V^\dagger\right]\right)\hat{\psi}\,. \tag{3.3.38}$$

The object in brackets in the second term is of special interest. This "dressed" gauge field

$$\tilde{A}_\mu(x) = V^\dagger A_\mu(x)V + \frac{i}{e}V^\dagger \partial_\mu V\,, \tag{3.3.39}$$

is a manifestly gauge-invariant object, that only transforms with a global transformation

$$\tilde{A}_\mu(x) \to U(0)\tilde{A}_\mu(x)U^\dagger(0)\,, \tag{3.3.40}$$

as can easily be seen by inserting (3.3.35) and using the standard photon transformation (3.3.10). In fact, this object corresponds to the gauge field in fixed-point, or Fock-Schwinger gauge, namely it satisfies

$$x^\mu \tilde{A}_\mu(x) = 0\,. \tag{3.3.41}$$

An interesting property of fixed-point gauge is that the gauge potential $\tilde{A}_\mu(x)$ can be expressed purely in terms of the field-strength tensor as

$$\tilde{A}_\nu(x) = \int_0^1 ds\; s x^\mu F_{\mu\nu}(sx)\,. \tag{3.3.42}$$

This identity holds even for non-Abelian theories, where the field-strength tensor has an additional commutator term. To prove this relation,[6] write

$$\begin{aligned}\tilde{A}_\mu(x) &= \int_0^1 ds\, \frac{d}{ds}\left(s\tilde{A}_\mu(sx)\right) = \int_0^1 ds\, \left(\tilde{A}_\mu(sx) + sx^\nu \frac{\partial}{\partial(sx)^\nu}\tilde{A}_\mu(sx)\right) \\ &= \int_0^1 ds\, \left(\tilde{A}_\mu(sx) - sx^\nu F_{\mu\nu}(sx) + sx^\nu \frac{\partial}{\partial(sx)^\mu}\tilde{A}_\nu(sx) + igsx^\nu\left[\tilde{A}_\mu\,,\tilde{A}_\nu\right](sx)\right)\,.\end{aligned} \tag{3.3.43}$$

In the non-Abelian case, the gauge condition (3.3.41) eliminates the additional commutator term, i.e. the last term in the second line of (3.3.43). Next, observe that after integration by parts,

$$sx^\nu \frac{\partial}{\partial(sx)^\mu}\tilde{A}_\nu(sx) = \frac{\partial}{\partial(sx)^\mu}(sx^\nu\tilde{A}_\nu(sx)) - \tilde{A}_\mu(sx) = -\tilde{A}_\mu(sx)\,, \tag{3.3.44}$$

which cancels with the first term in the second line of (3.3.43). Hence one is left with (3.3.42).

One immediately notices that in fixed-point gauge (3.3.41), the Wilson line V (3.3.34) simplifies to $V(x) = 1$. Thus, the Wilson line can be interpreted as a special gauge transformation that moves the generic field configuration $A_\mu(x)$ to a configuration $\tilde{A}_\mu(x)$ satisfying fixed-line gauge, according to

$$\tilde{A}_\mu(x) = V^\dagger A_\mu(x) V + \frac{i}{g}V^\dagger\partial_\mu V\,. \tag{3.3.45}$$

The field $\tilde{A}_\mu(x)$ then transforms only with the global transformation $U(0)$. By continuity, the gauge condition (3.3.41) implies $\tilde{A}_\mu(0) = 0$, so one finds $\tilde{A}_\mu(0) = 0$ in (3.3.42).

Let us now apply the identity (3.3.42) in the Lagrangian (3.3.38). we obtain

$$\mathcal{L} = \bar{\psi} i\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\nu\left(\int_0^1 ds\; sx^\mu F_{\mu\nu}(sx)\right)\psi\,, \tag{3.3.46}$$

and find that the interactions can be expressed using the field-strength tensor to all orders in x. The expansion of this integral in x yields

$$\int_0^1 ds\; sx^\mu F_{\mu\nu}(sx) = \frac{1}{2}x^\mu F_{\mu\nu} + \frac{1}{3}x^\mu x^\alpha\left[\partial_\alpha F_{\mu\nu}\right] + \mathcal{O}(x^3)\,, \tag{3.3.47}$$

where the field-strength tensors on the right-hand side are evaluated at $x = 0$. Consequently, we recover the Lagrangian (3.3.28) – (3.3.31), but to all orders in ε or x, in a closed form.

The circle now closes and we can start to appreciate the systematics behind the construction we just uncovered: In order to homogenise the gauge symmetry, we use the Wilson line V defined in (3.3.34) to transport the gauge transformation from point x to point $x = 0$. The matter

[6] We show this directly for the non-Abelian situation.

field $\psi(x)$ is redefined to $\hat{\psi}(x)$ according to (3.3.36) and now transforms with the homogeneous $U(0)$. Inserting this redefinition is then equivalent to "dressing" the gauge field, i.e. replacing $A_\mu(x) \to \tilde{A}_\mu(x)$. This dressed gauge field $\tilde{A}_\mu$ corresponds to the gauge field in fixed-point gauge (3.3.41). In this special gauge, we find that $\tilde{A}_\mu(0) = 0$, while the subleading terms in the multipole expansion can be expressed in terms of the field-strength tensor as (3.3.42). Thus, the Lagrangian expressed in terms of the redefined matter field $\hat{\psi}(x)$ *no longer has a residual gauge field*, and all interactions are due to couplings of the dipole, quadrupole, and higher-pole terms to the *field-strength tensor* and its derivatives. In other words, *one obtains a Lagrangian where each term is manifestly gauge-invariant by itself.* However, let us stress that *no gauge is fixed* in the Lagrangian itself. The gauge field still has the full transformation, but can only appear inside the gauge-invariant combination $F_{\mu\nu}(0)$.

We now wish to extend this construction to the more general situation, where we want to keep a residual dynamic background field. This situation arises naturally in the SCET context, as well as in non-relativistic QED and QCD. We consider the former case in detail, but the construction works completely analogously also for the latter case.

3.3.3 Light-front Multipole Expansion and Fixed-line Gauge

We now generalise the previous discussion to the situation encountered in SCET. The appearing fixed-line gauge was introduced for the first time in [41], and most of the results stated here can also be found in this reference. However, here we provide a more detailed exposition since understanding the systematics will be paramount to the more involved gravitational construction. The details can be found in [47].

In the following discussion, we focus purely on the *soft* gauge transformation and ignore the collinear gauge symmetry for the moment. As field content, we now consider the soft gluon field $A_{s\mu}(x)$ and some collinear matter field $\psi_c(x)$. The situation in SCET is in a sense more general than the standard multipole expansion discussed in the last section. Here, one has to perform a multipole expansion about the collinear light-cone coordinate $x_-^\mu = n_+x\,\frac{n_-^\mu}{2}$ instead of $x = 0$. Consequently, all soft fields have a left-over dependence on x_- and are dynamic. Therefore, the first step is to extend and adapt the previously introduced fixed-point gauge $x^\mu A_\mu(x) = 0$ to this new situation. The more general *fixed-line gauge* is defined by the condition [41]

$$(x - x_-)^\mu A_{s\mu}(x) = 0\,. \tag{3.3.48}$$

This gauge condition differs significantly from fixed-point gauge since it does not restrict the n_-A_s component at all. For the other components of the soft gluon, $A_{s\perp}$ and n_+A_s, however, the gauge condition (3.3.48) is equivalent to standard fixed-point gauge (3.3.41). For these components, one can immediately derive the identities corresponding to (3.3.42), and one finds

$$n_+A_s(x) = \int_0^1 ds\, s(x - x_-)^\mu n_+^\nu F_{s\mu\nu}(y(s))\,, \tag{3.3.49}$$

$$A_{s\nu_\perp}(x) = \int_0^1 ds\, s(x - x_-)^\mu F_{s\mu\nu_\perp}(y(s))\,, \tag{3.3.50}$$

where $y(s) = x_- + s(x - x_-)$. Similar to fixed-point gauge, $n_+A_s(x_-) = A_{s\perp}(x_-) = 0$. For n_-A_s, however, there is no constraint. Instead, one can treat this component in the same fashion as the soft scalar field φ_s in the purely-scalar theory, where we found (2.3.8)

$$\varphi_s(x) - \varphi_s(x_-) = \int_0^1 \mathrm{d}s\, (x - x_-)^\mu \left[\partial_\mu \varphi_s\right](y(s))\,, \tag{3.3.51}$$

simply by rewriting the multipole expansion. The corresponding result for n_-A_s then reads

$$n_-A_s(x) - n_-A_s(x_-) = \int ds\, (x - x_-)^\mu n_-^\nu \partial_\mu A_\nu(y(s))\,. \tag{3.3.52}$$

Using fixed-line gauge, one can write this in a manifestly gauge-covariant fashion by completing the right-hand side to the field-strength tensor

$$\begin{aligned}(x-x_-)^\mu n_-^\nu \partial_\mu A_\nu(y(s)) &= (x-x_-)^\mu n_-^\nu F_{\mu\nu}(y(s)) + (x-x_-)^\mu n_-^\nu \partial_\nu A_\mu(y(s)) \\ &\quad - ig(x-x_-)^\mu n_-^\nu \left[A_\mu\,, A_\nu\right](y(s)) \\ &= (x-x_-)^\mu n_-^\nu F_{\mu\nu}(y(s))\,. \end{aligned} \tag{3.3.53}$$

The additional second and third term in the first equality vanish in fixed-line gauge, by noticing

$$(x-x_-)^\mu A_\mu(y(s)) = \frac{1}{s}(y(s)-y_-(s))^\mu A_\mu(y(s)) = 0\,, \tag{3.3.54}$$

which eliminates the commutator term, while for the other term, one has

$$(x-x_-)^\mu n_-\partial A_\mu = n_-\partial\left[(x-x_-)^\mu A_\mu\right] = 0 \tag{3.3.55}$$

since $[n_-\partial(y-y_-)^\mu] = 0$. Thus, one obtains the identity

$$n_-A_s(x) - n_-A_s(x_-) = \int_0^1 ds\,(x-x_-)^\mu n_-^\nu F_{s\mu\nu}(y(s))\,, \tag{3.3.56}$$

where the kinetic 1-form $\partial_\mu\varphi$ of the scalar in (3.3.51) is replaced by the corresponding kinetic 2-form $F_{\mu\nu}$ of the vector.

These identities now allow us to again rewrite $A_{s\mu}(x)$ in terms of manifestly gauge-covariant field-strength tensor. However, from (3.3.56), notice that there is a *residual background field*, $n_-A_s(x_-)$, which is homogeneous in λ and dynamic in the effective theory. Accordingly, we rewrite $n_-A_s(x)$ in terms of the homogeneous soft background field $n_-A_s(x_-)$, as well as a tower of subleading terms proportional to the field-strength tensor and its derivatives. This emergent residual background field is necessary to obtain a theory that is covariant with respect to the *homogeneous* gauge symmetry consisting of transformations $U_s(x_-)$.

Let us go back to the example of a charged fermion with Lagrangian (3.3.11) and generalise the discussion from (3.3.12) – (3.3.46) to the expansion about x_-. Performing the multipole expansion about x_-^μ, we obtain the generalisation of (3.3.12),

$$A_{s\mu}(x) = A_{s\mu}(x_-) + x_\perp^\alpha\left[\partial_\alpha A_{s\mu}\right](x_-) + \mathcal{O}(\lambda^2 A_{s\mu})\,, \tag{3.3.57}$$

where now the counting in λ is relevant, and not the power in x or ε. Just as before, we also have to take into account the multipole expansion in the transformation of the matter fields

$$\psi_c(x) \to U_s(x)\psi_c(x) = U_s(x_-)\psi_c(x) + x_\perp^\alpha\left[\partial_\alpha U_s\right](x_-) + \mathcal{O}(\lambda^2 U_s)\,. \tag{3.3.58}$$

Again, the soft transformation mixes different orders in λ, as the homogeneous transformation would correspond to

$$\hat{\psi}_c(x) \to U_s(x_-)\hat{\psi}_c(x)\,. \tag{3.3.59}$$

Consequently, we can again construct a Wilson line, the analogue of V defined in (3.3.34), that moves a general field configuration into fixed-line gauge. The corresponding Wilson line, now denoted by $R(x)$, is given by [41]

$$R(x) = P\exp\left(ig\int_0^1 ds\,(x-x_-)^\mu A_{s\mu}\left(x+s(x-x_-)\right)\right)\,, \tag{3.3.60}$$

and is responsible for moving the gauge transformation from point x to x_- in a straight line along the directions orthogonal to x_-. Under soft gauge symmetry, it transforms as

$$R(x) \to U_s(x)R(x)U_s^\dagger(x_-)\,. \tag{3.3.61}$$

Using it, the collinear field can be redefined as

$$\hat{\psi}_c(x) = R^\dagger(x)\psi_c(x)\,, \tag{3.3.62}$$

and now transforms homogeneously with $U_s(x_-)$,

$$\hat{\psi}_c(x) \to U_s(x_-)\hat{\psi}_c(x)\,. \tag{3.3.63}$$

As an example, we insert the redefinition (3.3.62) in the leading term

$$\begin{aligned}\bar{\psi}_c\frac{\not{n}_+}{2}n_-D_s(x)\psi_c &= \bar{\hat{\psi}}_c\frac{\not{n}_+}{2}R^\dagger n_-D_s(x)R\hat{\psi}_c\\ &= \bar{\hat{\psi}}_c\frac{\not{n}_+}{2}\left(n_-\partial + g\left(R^\dagger n_-A_s(x)R + \frac{i}{g}R\left[n_-\partial R^\dagger\right]\right)\right)\hat{\psi}_c\,,\end{aligned} \tag{3.3.64}$$

where all fields are evaluated at x. To make use of the fixed-line gauge identities (3.3.49) – (3.3.56), add and subtract the homogeneous background field $n_-A_s(x_-)$, to find

$$\bar{\psi}_c\frac{\not{n}_+}{2}n_-D_s(x)\psi_c = \bar{\hat{\psi}}_c\frac{\not{n}_+}{2}\left(n_-D_s(x_-) + \left[R^\dagger in_-D_s(x)R - in_-D_s(x_-)\right]\right)\hat{\psi}_c\,. \tag{3.3.65}$$

The object in square brackets in the second term

$$\mathcal{A}_s(x) \equiv R^\dagger A_s(x)R + \frac{i}{g}R^\dagger\left[D_s\,, R\right]\,, \tag{3.3.66}$$

is the analogue of the "dressed" field $\tilde{A}(x)$ (3.3.39) of the previous section and can be seen to satisfy fixed-line gauge

$$(x-x_-)\cdot\mathcal{A}_s(x) = 0\,. \tag{3.3.67}$$

The identities (3.3.49) – (3.3.56) in fixed-line gauge can be promoted to the general case by undoing the transformation to fixed-line gauge with the gauge transformation $U = R^\dagger$, resulting in [41]

$$R^\dagger in_-D_s(x)R - in_-D_s(x_-) = \int_0^1 ds\,(x-x_-)^\mu n_-^\nu R^\dagger(y(s))gF_{s\mu\nu}(y(s))R(y(s))\,, \tag{3.3.68}$$

$$R^\dagger iD_{s\nu_\perp}(x)R - i\partial_{\nu_\perp} = \int_0^1 ds\,s(x-x_-)^\mu R^\dagger(y(s))gF_{s\mu\nu_\perp}(y(s))R(y(s))\,, \tag{3.3.69}$$

$$R^\dagger n_+D_s(x)R - in_+\partial = \int_0^1 ds\,s(x-x_-)^\mu n_+^\nu R^\dagger(y(s))gF_{s\mu\nu}(y(s))R(y(s))\,. \tag{3.3.70}$$

They give closed all-order expressions for the soft fields. Once expanded in λ, they generate an infinite tower of subleading terms. The Lagrangian expressed in terms of the hatted collinear fields contains the homogeneous soft background field $n_-A_s(x_-)$ in the covariant derivative $n_-D_s(x_-)$, as well as subleading interactions, expressed entirely in terms of $F_{s\mu\nu}$. Once the integrals are expanded in λ, these interactions will depend on $F_{s\mu\nu}$ and its (covariant) derivatives at x_-. Again, let us stress that the gauge-field in the Lagrangian has the *full* gauge transformation and is not gauge-fixed. It is due to the redefinitions that only $n_-A_s(x_-)$ appears inside the covariant derivative while all other components are allocated to the field-strength tensor.

We now understand that fixed-line gauge and the R Wilson line can be employed to render the matter transformation manifestly homogeneous in λ. But also the second problem explained in Section 3.3.1, namely the *collinear* gauge transformation of the collinear gluon, is addressed by this construction. To understand this, consider again the implication of the redefinitions like (3.3.62) using R: we effectively obtain a theory that is covariant not with respect to the full $A_{s\mu}(x)$, but rather only the residual component $n_-A_s(x_-)$. Therefore, the redefined collinear

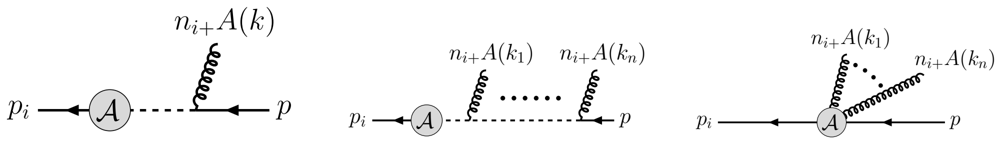


Figure 3.1: Diagrams depicting the emission of a single and multiple $n_{i+}A_c$ gluons from a leg of different collinearity in the full theory (first and second) and in the effective theory (third). The dashed line indicates propagators that must be integrated out. The third diagram is the corresponding one in the effective theory, where the gluons are emitted via an effective operator.

gluon $\hat{A}_c$ that should be used in the effective theory corresponds to a fluctuation on top of a background described by $n_-A_s(x_-)$ and transforms as

$$\begin{aligned} &\text{collinear:} \quad && \hat{A}_c \to U_c \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[D_s(x_-)\,, U_c^\dagger \right] , \\ &\text{soft:} && \hat{A}_c \to U_s(x_-) \hat{A}_c U_s^\dagger(x_-) \,. \end{aligned} \tag{3.3.71}$$

However, to relate this redefined $\hat{A}_c$ to the previous A_c, we require one additional ingredient.

3.3.4 Collinear Gluon emission: the Collinear Wilson Line

Besides its problematic collinear gauge transformation, which we now understand how to deal with, the collinear gluon faces a second difficulty: it contains a component n_+A_c that scales as $n_+A_c \sim 1$, as seen in (3.1.3). In principle, this implies that to any given process, one can attach an infinite number of emissions of n_+A_c, and they all contribute to the same order in λ. That is, it seems that there is no finite operator basis for the N-jet operators, and one would have to perform infinitely many matching computations at each order in λ, which would render the power-counting futile. However, this is not the case. Note that one can fix a gauge-condition independently in each collinear sector. Thus, it is a valid choice to impose $n_+A_c \equiv 0$ in each sector, i.e. to fix the respective collinear light-cone gauge. In this gauge, only the physical transverse components propagate, and so the first valid gluon building block is $A_{c\perp} \sim \lambda$. This is an indication that the n_+A_c component is a gauge artefact and can be controlled to all orders in the expansion. But there is a second problem: if such a gluon is emitted from a leg of different collinearity, the intermediate propagator is hard and must be integrated out. Both problems are connected and solved by the same tool – the collinear Wilson line W_c.

For the next part of this discussion, we neglect the soft background by setting $A_s \equiv 0$ and focus on the purely collinear theory, i.e. the interactions of collinear matter fields $\psi_c(x)$ with the collinear gluons $A_c(x)$.

We show how the emission of these collinear gluons can be absorbed in an effective operator that takes the form of a Wilson line, following [55]. The computation and results are very similar to the derivation of eikonal exponentiation [1]. Consider a scattering process featuring two energetic particles, i and j, where a gluon $n_{i+}A_c$ is emitted, as depicted in Fig. 3.1. If this gluon is emitted from the i-collinear particle, the intermediate propagator is also i-collinear, and the full process is captured by the SCET Lagrangian. However, emission of the gluon from the j-collinear particle results in an off-shell intermediate propagator that must be integrated out. Schematically, the amplitude takes the form

$$\overline{u}_i(p_i) \frac{i(\not{p} + \not{k})}{(p+k)^2 + i0} \mathcal{A}(\{p\}) i g t^a \gamma^\mu \left(n_{i+} A^a(k) \frac{n_{-\mu}}{2} \right) u_j(p) \,. \tag{3.3.72}$$

Note that since p and k are vectors of different collinear sectors, their product is indeed hard

$$(p+k)^2 = n_{i-} \cdot n_{j-}\, n_{j+}p\, n_{i+}k + \mathcal{O}(\lambda) \,. \tag{3.3.73}$$

To simplify the amplitude (3.3.72), we first expand the denominator, keeping only the leading term. Then, we move the $\not{n}_-$ all the way to the left, neglecting the subleading contributions and using the Dirac equation for $\overline{u}_j$, as well as the projection $\bar{u}_i \not{n}_{i-} = 0$.[7] The simplified amplitude is then schematically given by

$$\overline{u}_i(p_i)\left(-g\frac{n_{i+}A(k)}{n_{i+}k}\right)\mathcal{A}(\{p\})u_j(p)\,. \tag{3.3.74}$$

Extending this result to the emission of n gluons $n_{i+}A$, with momenta $k_1, k_2, \dots k_n$, one finds

$$\overline{u}_i(p_i)\sum_{\text{perm}}\frac{(-g)^n}{n!}\left(\frac{n_{i+}A(k_1)\dots n_{i+}A(k_n)}{[n_{i+}k_1][n_{i+}(k_1+k_2)]\dots[n_{i+}(k_1+\dots+k_n)]}\right)\mathcal{A}(\{p\})u_j(p)\,, \tag{3.3.75}$$

where all permutations of the gluon emissions must be taken into account. The last step is to perform the sum over the number of gluons. The result takes the form [55]

$$\overline{u}_i(p_i)W_{c_i}\mathcal{A}u_j(p)\,, \tag{3.3.76}$$

where W_{c_i} is given by

$$W_{c_i} = \sum_n\sum_{\text{perm}}\frac{(-g)^n}{n!}\left(\frac{n_{i+}A(k_1)\dots n_{i+}A(k_n)}{[n_{i+}k_1][n_{i+}(k_1+k_2)]\dots[n_{i+}(k_1+\dots+k_n)]}\right)\,. \tag{3.3.77}$$

This object is the momentum-space version of an i-collinear Wilson line W_{c_i}. In position space, suppressing the index i, this semi-infinite Wilson line reads

$$W_c(x) = P\exp\left(ig\int_{-\infty}^{0} ds\, n_+A_c(x+sn_+)\right)\,, \tag{3.3.78}$$

and is defined along a straight line in the n_-^μ direction going from point x to infinity, denoted by $x - \infty n_+$.

Under collinear gauge transformations, it behaves as

$$W_c(x) \to U_c(x)W_c(x)U_c^\dagger(x-\infty n_+)\,, \tag{3.3.79}$$

and we adopt the convention that the transformation falls off at infinity, resulting in

$$W_c(x) \to U_c(x)W_c(x)\,. \tag{3.3.80}$$

Moreover, this Wilson line has the defining property

$$in_+D_cW_c = W_c in_+\partial\,, \tag{3.3.81}$$

or, applied directly to the gluon field,

$$W_c^\dagger n_+A_cW_c + \frac{i}{g}W_c^\dagger\left[n_+\partial W_c\right] = 0\,. \tag{3.3.82}$$

In other words, the Wilson line W_c can be thought of as the transformation that takes a general gauge configuration $A_c(x)$ and moves it to light-cone gauge, where $n_+A_c(x) = 0$. This also explains why it appears in the matching computation (3.3.76): if light-cone gauge is fixed, $W_c = 1$ and no such emissions can take place. Accordingly, the Wilson line relates the amplitude in light-cone gauge to the one in a general gauge, thereby controlling the appearance of n_+A_c to all orders.

[7] This property is explained in the later Section 3.7 and follows from the expansion of the Dirac equation. For now, it is only important that this contribution vanishes and the result simplifies.

Applying the Wilson line to the other gluon components, one obtains for example

$$\mathcal{A}_{c\perp} = W_c^\dagger A_{c\perp} W_c + \frac{i}{g} W_c^\dagger \left[\partial_\perp W_c\right] , \tag{3.3.83}$$

which is manifestly gauge-invariant, i.e. $\mathcal{A}_{c\perp} \to \mathcal{A}_{c\perp}$. At the linear level,[8] this object takes the form

$$\mathcal{A}_{c\mu_\perp} = A_{c\mu_\perp} - \frac{\partial_{\mu_\perp}}{n_+\partial} A_{c+} + \mathcal{O}(gA_{c\mu_\perp}) . \tag{3.3.84}$$

One can also define manifestly gauge-invariant matter fields χ_c as

$$\chi_c = W_c^\dagger \psi_c . \tag{3.3.85}$$

This gives us the possibility to control the large n_+A_c component, by working in "covariant" light-cone gauge, i.e. by working with general gauge fields A_c, but always dressed with the collinear Wilson line W_c as in (3.3.83). Furthermore, since these objects $\mathcal{A}_c$ are manifestly collinear gauge-invariant, by employing these fields in the operator basis one is automatically collinear gauge-invariant in any direction, and there is no further need to check gauge-invariance explicitly. Therefore, the purely-collinear theory strongly urges one to use the Wilson line W_c and the collinear gauge-invariant objects whenever possible.

3.3.5 Field Redefinitions in the Soft-collinear Theory

Let us summarise the key insights of the past two sections. First, from the purely collinear theory, we learned that we should employ the collinear Wilson line W_c to construct manifestly collinear gauge-invariant building blocks. This controls the large components of the gluon n_+A_c. Second, we understood that the effective theory, once multipole expanded, should be formulated as a theory that is covariant with respect to the emergent soft background described only by $n_-A_s(x_-)$, the homogeneous component of the soft gluon that lives only on the classical trajectory of the energetic particles. In order to achieve this formulation, the Wilson line $R(x)$ (3.3.60) can be used to transport the gauge transformation.

However, recall that the collinear gluon has a non-trivial collinear gauge transformation once we include the soft background as seen in (3.2.3), and, in particular,

$$n_+A_c \to U_c n_+ A_c U_c^\dagger + \frac{i}{g} U_c \left[n_+D_s(x), U_c^\dagger\right] . \tag{3.3.86}$$

Therefore, in the soft-collinear setting, the Wilson line W_c as defined in (3.3.78), using the original gluon field A_c with transformation (3.2.3), does not have a good gauge transformation either under soft or collinear, as the covariantly-transforming gluon would be $n_+A_c + n_+A_s(x)$. The Wilson line should always be defined in terms of A_c and the relevant soft background field. However, from the previous discussion, we understand that the redefined matter fields are covariant with respect to the emerging homogeneous background $n_-A_s(x_-)$, so the redefined collinear gluon $\hat{A}_c$ should transform as (3.3.71), where in particular

$$n_+\hat{A}_c \to U_c n_+ \hat{A}_c U_c^\dagger + \frac{i}{g} U_c \left[n_+\partial U_c^\dagger\right] \tag{3.3.87}$$

has the standard transformation, without any reference to the background. Consequently, the Wilson line W_c defined in terms of $\hat{A}_c$ as

$$W_c(x) = P \exp\left(ig \int_{-\infty}^0 ds\, n_+\hat{A}_c(x+sn_+)\right) \tag{3.3.88}$$

[8] Which is exact in QED.

has well-defined collinear and soft gauge transformations, as

$$W_c(x) \xrightarrow{\text{coll.}} U_c(x)W_c(x)\,, \qquad W_c(x) \xrightarrow{\text{soft}} U_s(x_-)W_c(x)U_s^\dagger(x_-)\,. \tag{3.3.89}$$

We can now use this collinear Wilson line, in conjunction with the previously defined soft $R(x)$ (3.3.60) to redefine the collinear gluon and matter fields. To achieve the proper transformations (3.3.71), one first fixes collinear light-cone gauge $n_+A_c = 0$, eliminating the collinear gauge transformation. Then, the redefined and the original gluon field are related according to their soft gauge transformation as

$$A_{c\perp} = R(x)\hat{A}_{c\perp}R^\dagger(x)\,. \tag{3.3.90}$$

Similarly, the matter fields are related as

$$\phi_c(x) = R(x)\hat{\phi}_c(x)\,. \tag{3.3.91}$$

In collinear light-cone gauge, we have $\hat{A}_c = \hat{\mathcal{A}}_c$, and one can reinstate the gluon field in a generic gauge using the W_c Wilson line. Thus, the redefinition reads

$$A_{\perp c} = R\underbrace{\Big(W_c^\dagger\hat{A}_{c\perp}W_c + \frac{i}{g}W_c^\dagger\left[\partial_\perp\,, W_c\right]\Big)}_{=\hat{\mathcal{A}}_{\perp c}}R^\dagger\,, \tag{3.3.92}$$

$$n_-A_c = R\underbrace{\Big(W_c^\dagger n_-\hat{A}_c W + \frac{i}{g}W_c^\dagger\left[n_-D_s(x_-)\,, W_c\right]\Big)}_{=n_-\hat{\mathcal{A}}_c}R^\dagger\,, \tag{3.3.93}$$

where the object inside the brackets corresponds to the collinear-gauge-invariant building block $\hat{\mathcal{A}}_c$. For the matter field, one can reinstate the collinear gauge transformation in a similar fashion and one finds

$$\psi_c = RW_c^\dagger\hat{\psi}_c\,. \tag{3.3.94}$$

In summary, the effective theory should be expressed in terms of the homogeneously transforming fields $\hat{\phi}_c$, $\hat{A}_c$, which are covariant with respect to the soft background described by the component $n_-A_s(x_-)$. The other components of A_s, as well as the subleading terms from the multipole expansion, are expressed in terms of the field-strength tensor and its derivatives. These fields are related to the original fields ϕ_c, A_c via the soft Wilson line R and the collinear Wilson line W_c. Using the collinear Wilson lines W_c, one can further define manifestly collinear gauge-invariant building blocks.

With these definitions out of the way, the effective Lagrangian can now be constructed in a straightforward fashion, following closely the construction in the scalar case in Section 2.5.

3.4 Constructing the Effective Lagrangian

We are now in a position to construct the soft-collinear Lagrangian. The derivation is essentially the same as in the purely-scalar case presented in Section 2.5, with the additional complication of the gauge symmetry, as explained in the previous section. Consequently, there is an additional step in the construction where we make use of the R and W_c Wilson lines to relate the original fields A_c, ϕ_c, which are covariant with respect to the full $A_s(x)$, to the redefined fields $\hat{A}_c$, $\hat{\phi}_c$, which are now covariant with respect to $n_-A_s(x_-)$. This derivation was first presented in [41] and is also explained in detail in [47], which we follow closely, and consists of four steps:

(i) Like in the purely-scalar case, one introduces the decompositions given in (3.2.2), (3.2.5) into the Lagrangian. This yields the theory of a collinear fluctuation on top of a soft background $A_s(x)$, each sector with its own gauge symmetry. However, the Lagrangian is not yet homogeneous in λ for two reasons: First, in soft-collinear interactions, like $\int d^4x\, \psi_c(x)\psi_s(x)\ldots$, the x argument of the soft field has a different power-counting than the collinear measure d^4x, and it should be evaluated at x_-, which has the same scaling for both soft and collinear fields. This problem was also present in the purely-scalar theory. Second, there is a new problem. The soft gauge transformation is also evaluated at collinear x, and it must be homogenised in λ.

(ii) Next, to render soft-collinear interactions homogeneous, we perform the multipole expansion of the soft fields $A_s(x) = A_s(x_-) + \mathcal{O}(\lambda)$. This way, each term is now, in principle, manifestly homogeneous in λ. However, the soft gauge symmetry does not yet respect the multipole expansion, namely, collinear fields still transform with the full transformation $U_s(x)$ (3.2.3), but should only transform with the homogeneous $U_s(x_-)$. Thus, the gauge transformations still mix different orders in λ.

(iii) To remedy this, we redefine the collinear fields $\phi_c \to \hat{\phi}_c$ using the R Wilson line (3.3.60), so that the soft gauge transformation U_s of these fields respects the multipole expansion and depends only on x_-^μ. At the same time, this also makes the transformation homogeneous in λ. Expressed in these new fields, one finds a theory that is covariant with respect to the homogeneous background field $n_- A_s(x_-)$, which appears inside a new covariant derivative. All other terms due to the multipole expansion and due to the other gluon components are expressed in a manifestly gauge-covariant fashion in terms of the field-strength tensors, using identities like (3.3.49) and (3.3.56). This yields a closed all-order expression of the subleading Lagrangian.

(iv) As the final step, one can perform the λ expansion of this closed result to obtain the Lagrangian order-by-order in λ. Each term is now homogeneous in λ, that is, it has a definite scaling in λ and is covariant with respect to the homogeneous soft gauge symmetry due to $n_- A_s(x_-)$.

We first show how this construction works for the scalar QCD scenario, as presented in [47]. This procedure can be extended in a straightforward fashion to both fermionic [40, 41] and vectorial matter [48], since the construction only depends on the representation under soft gauge transformations.

3.4.1 Background-field Lagrangian

To begin, in step (i), we insert the decompositions (3.2.2), (3.2.5) into the scalar QCD Lagrangian to obtain

$$\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}\left[n_+ D\phi_c\right]^\dagger n_- D\phi_c + \frac{1}{2}\left[n_- D\phi_c\right]^\dagger n_+ D\phi_c \\
& + \left[D_{\mu_\perp}\phi_c\right]^\dagger D^{\mu_\perp}\phi_c + \left[D_{s\mu}\phi_s\right]^\dagger D_s^\mu \phi_s \\
& + \frac{1}{2}\left[n_+ D\phi_c\right]^\dagger n_- D W Z^\dagger \phi_s + \frac{1}{2}\left[n_+ D W Z^\dagger \phi_s\right]^\dagger n_- D\phi_c \\
& + \frac{1}{2}\left[n_- D\phi_c\right]^\dagger n_+ D W Z^\dagger \phi_s + \frac{1}{2}\left[n_- D W Z^\dagger \phi_s\right]^\dagger n_+ D\phi_c \\
& + \left[D_{\mu_\perp}\phi_c\right]^\dagger D^{\mu_\perp} W Z^\dagger \phi_s + \left[D_{\mu_\perp} W Z^\dagger \phi_s\right]^\dagger D^{\mu_\perp}\phi_c\,,
\end{aligned} \tag{3.4.1}$$

where $D_\mu = \partial_\mu - igA_{c\mu} - igA_{s\mu}(x)$. This Lagrangian is invariant under the full background-field gauge symmetry (3.2.3) and describes a collinear fluctuation on top of a soft background $A_s(x)$. To render the terms in this Lagrangian homogeneous, we perform the multipole expansion of all soft fields about x_-^μ, as in (2.3.7). At the same time, we can already employ the results from Section 3.3.5 and introduce the redefined fields $\hat{A}_c$, $\hat{\phi}$ (3.3.92), (3.3.94), which are covariant with respect to the homogeneous $n_-A_s(x_-)$, that is, they transform with a gauge symmetry that only depends on the parameter $\varepsilon_s(x_-)$ at x_-. This means that in practice, steps (ii) and (iii) are performed simultaneously.

Since the fields are covariant only to $n_-A_s(x_-)$, it is useful to introduce the corresponding soft-covariant derivative

$$D_{s\mu} = \partial_\mu - ign_-A_s(x_-)\,\frac{n_{+\mu}}{2}\,, \tag{3.4.2}$$

as well as the gauge-covariant "dressed" combination (3.3.66)

$$\mathcal{A}_s(x) \equiv R^\dagger A_s(x) R + \frac{i}{g} R^\dagger \left[D_s\,, R\right]\,, \tag{3.4.3}$$

where $D_{s\mu}$ is the one in (3.4.2). This combination $\mathcal{A}_s^\mu(x)$ corresponds to the manifestly gauge-covariant part of the original field, which can be expressed in terms of the field-strength tensor and its covariant derivatives using the identities (3.3.49) – (3.3.56). The full-theory gauge potential $A_s(x)$ splits into the homogeneous background field $n_-A_s(x_-)$, which only appears inside $n_-D_s(x_-)$, and the gauge-covariant object $\mathcal{A}_s(x)$, which is expressed in terms of the field-strength tensor $F_{s\mu\nu}(x_-)$ and its derivatives, as explained in detail following (3.3.64).

3.4.2 All-order Soft-collinear Lagrangian

We now discuss the technical implementation of steps (ii) and (iii) in detail. Since the computation of all terms in the Lagrangian is completely analogous, we explain in detail the derivation for the case $\phi_s = 0$ and give the final result containing all terms. We therefore focus only on the first three terms of the Lagrangian (3.4.1).

We want to make use of the field redefinitions (3.3.92) – (3.3.94), this requires us to fix collinear light-cone gauge for the unhatted fields. The Lagrangian (3.4.1), where we set $\phi_s = 0$, then reads

$$\mathcal{L} = \frac{1}{2}\left[n_+D_s(x)\phi_c\right]^\dagger n_-D\phi_c + \frac{1}{2}\left[n_-D\phi_c\right]^\dagger n_+D_s(x)\phi_c + (D_{\mu_\perp}\phi_c)^\dagger D^{\mu_\perp}\phi_c\,, \tag{3.4.4}$$

where $D_\mu = \partial_\mu - igA_{c\mu} - igA_{s\mu}(x)$, and here $D_{s\mu}(x) = \partial_\mu - igA_{s\mu}(x)$ still contains the full soft gauge field. We now insert the redefinitions (3.3.92) – (3.3.94), and express the Lagrangian in terms of the manifestly gauge-invariant $\hat{\chi}_c$ and $\hat{\mathcal{A}}_c$, which are defined in the very same equations. We obtain

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left[(n_+\partial - ign_+\mathcal{A}_s(x))\,\hat{\chi}_c\right]^\dagger \Big(n_-D_s - ign_-\hat{\mathcal{A}}_c - ign_-\mathcal{A}_s(x)\Big)\,\hat{\chi}_c + \text{h.c.} \\ &\quad + \Big[\Big(\partial_{\mu_\perp} - ig\hat{\mathcal{A}}_{c\mu_\perp} - ig\mathcal{A}_{s\mu_\perp}(x)\Big)\,\hat{\chi}_c\Big]^\dagger \Big(\partial^{\mu_\perp} - ig\hat{\mathcal{A}}_c^{\mu_\perp} - ig\mathcal{A}_s^{\mu_\perp}(x)\Big)\,\hat{\chi}_c\,,\end{aligned} \tag{3.4.5}$$

where now D_s is defined with respect to the homogeneous background field $n_-A_s(x_-)$ given in (3.4.2). Next, use the fixed-line gauge identities (3.3.49) – (3.3.56) to eliminate $A_{s\perp}(x)$, $n_+A_s(x)$ and the subleading terms from the multipole expansion of $n_-A_s(x)$, which are all collected inside $\mathcal{A}_s(x)$, in favour of the field-strength tensor. In addition, to simplify the notation, we introduce the Noether currents j_μ^a

$$\begin{aligned} n_+j^a &= i\hat{\chi}_c^\dagger t^a n_+\overleftrightarrow{\partial}\hat{\chi}_c\,, \\ j_{\mu_\perp}^a &= i\hat{\chi}_c^\dagger t^a \overleftrightarrow{\mathcal{D}}_{c\mu_\perp}\hat{\chi}_c\,, \\ n_-j^a &= i\hat{\chi}_c^\dagger t^a n_-\overleftrightarrow{\mathcal{D}}\hat{\chi}_c\,, \end{aligned} \tag{3.4.6}$$

and the covariant derivatives

$$\begin{aligned} \mathcal{D}_{c\mu_\perp} &= \partial_{\mu_\perp} - ig\hat{\mathcal{A}}_{c\mu_\perp}\,, \\ n_-\mathcal{D} &= n_-D_s - ign_-\hat{\mathcal{A}}_c\,. \end{aligned} \tag{3.4.7}$$

The left-right arrow indicates

$$i\hat{\chi}_c^\dagger t^a \overleftrightarrow{\mathcal{D}}_{c\mu_\perp}\hat{\chi}_c = i\left(\hat{\chi}_c^\dagger t^a \mathcal{D}_{c\mu_\perp}\hat{\chi}_c - \left[\mathcal{D}_{c\mu_\perp}\hat{\chi}_c^\dagger\right] t^a\hat{\chi}_c\right), \tag{3.4.8}$$

with t^a the colour generators in the representation of the scalar field. This yields the Lagrangian

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}_{\text{sub}}\,, \tag{3.4.9}$$

where the leading-power terms are

$$\begin{aligned}\mathcal{L}^{(0)} &= \frac{1}{2}\left[n_+\partial\hat{\chi}_c\right]^\dagger n_- D_s\hat{\chi}_c + \frac{1}{2}\left[n_- D_s\hat{\chi}_c\right]^\dagger n_+\partial\hat{\chi}_c + \partial_{\mu_\perp}\hat{\chi}_c^\dagger\partial^{\mu_\perp}\hat{\chi}_c\\ &\quad + \frac{1}{2}gn_-\hat{\mathcal{A}}_c^a n_+ j^a + g\hat{\mathcal{A}}^a_{c\mu_\perp} j^{a\mu_\perp} + g^2\hat{\mathcal{A}}^a_{c\mu_\perp}\hat{\mathcal{A}}_c^{b\mu_\perp}\hat{\chi}_c^\dagger t^a t^b\hat{\chi}_c\\ &= \frac{1}{2}\left[n_+\partial\hat{\chi}_c\right]^\dagger n_-\mathcal{D}\hat{\chi}_c + \frac{1}{2}\left[n_-\mathcal{D}\hat{\chi}_c\right]^\dagger n_+\partial\hat{\chi}_c + \left[\mathcal{D}_{c\mu_\perp}\hat{\chi}_c\right]^\dagger\mathcal{D}_c^{\mu_\perp}\hat{\chi}_c\,.\end{aligned} \tag{3.4.10}$$

The advantage of this construction is that the subleading terms are expressed to all orders in λ in a closed form via the integrals. One finds

$$\begin{aligned}\mathcal{L}_{\text{sub}} &= \frac{1}{2}gn_+ j^b\int_0^1 ds\,(x-x_-)^\mu n_-^\nu R^{ab}(y(s))F^a_{s\mu\nu}(y(s))\\ &\quad + gj^{b\nu_\perp}\int_0^1 ds\, s(x-x_-)^\mu R^{ab}(y(s))F^a_{s\mu\nu_\perp}(y(s))\\ &\quad + \frac{1}{2}gn_- j^b\int_0^1 ds\, s(x-x_-)^\mu n_+^\nu R^{ab}(y(s))F^a_{s\mu\nu}(y(s))\\ &\quad + \frac{1}{2}g^2\hat{\chi}_c^\dagger\left\{t^a, t^b\right\}\hat{\chi}_c\int_0^1 ds\,(x-x_-)^\mu R^{da}(y(s))n_-^\nu F^d_{s\mu\nu}(y(s))\\ &\quad\times\int_0^1 ds'\, s'(x-x_-)^\alpha n_+^\beta R^{eb}(y(s'))F^e_{s\alpha\beta}(y(s'))\\ &\quad + \frac{1}{2}g^2\hat{\chi}_c^\dagger\left\{t^a, t^b\right\}\hat{\chi}_c\int_0^1 ds\, s(x-x_-)^\mu R^{da}(y(s))F^d_{s\mu\nu}(y(s))\\ &\quad\times\eta_\perp^{\nu\beta}\int_0^1 ds'\, s'(x-x_-)^\alpha R^{eb}(y(s'))F^e_{s\alpha\beta}(y(s'))\,,\end{aligned} \tag{3.4.11}$$

where we introduced the adjoint R-Wilson line

$$R^{ab}(x)t^b = R^\dagger(x)t^a R(x)\,. \tag{3.4.12}$$

A few comments are in order. First, $\mathcal{L}^{(0)}$ contains all collinear interactions, as well as the soft-collinear interactions mediated by the covariant derivative $n_-D_s(x_-)$. It counts homogeneously as $\mathcal{O}(\lambda^0)$ and is the leading-power Lagrangian, as we show below. These leading soft-collinear interactions are precisely the eikonal terms that one finds in the diagrammatic approach.

The subleading terms in $\mathcal{L}_{\text{sub}}$ are not homogeneous in λ. All these interactions stem from the multipole expansion of the soft field. Using fixed-line gauge, we managed to re-express all these terms, including the relatively-suppressed components $A_{s\perp}(x)$ and $n_+A_s(x)$ (compared to the respective collinear modes), in a closed form as an integral over the field-strength tensor. Expanding these integrals yields an infinite tower of subleading soft-collinear interactions. Therefore, we see that this Lagrangian (in conjunction with the N-jet operators) describes the full soft-collinear physics to all orders in λ, even in a closed form. No additional conceptual ingredients are necessary to describe these limits beyond subleading power, and the form of the theory is completely determined by covariance with respect to the homogeneous soft background gauge symmetry mediated by $n_-A_s(x_-)$.

However, for practical computations, one should work with objects that have a homogeneous scaling in λ. Hence, we expand the integrals in λ and determine the Lagrangian to $\mathcal{O}(\lambda^2)$.

3.4.3 Expansion in λ

In this last step (iv), we only need to expand the integrals appearing in (3.4.11). The leading-power term (3.4.10), does not need any expansions and is already homogeneous in λ. Consequently, the leading-power Lagrangian is given by

$$\mathcal{L}^{(0)} = \frac{1}{2}\left[n_+ D_c \hat{\phi}_c\right]^\dagger n_- D\hat{\phi}_c + \frac{1}{2}\left[n_- D\hat{\phi}_c\right]^\dagger n_+ D_c\hat{\phi}_c + \left[D_{c\mu_\perp}\hat{\phi}_c\right]^\dagger D_c^{\mu_\perp}\hat{\phi}_c\,. \tag{3.4.13}$$

For the integrals in (3.4.11), use the expansions [41]

$$\int_0^1 ds\,(x-x_-)^\mu n_-^\nu R^\dagger(y(s)) gF_{s\mu\nu}(y(s))R(y(s)) = x_\perp^\mu n_-^\nu gF_{s\mu\nu} + \frac{1}{2} n_- x n_+^\mu n_-^\nu gF_{s\mu\nu} + \frac{1}{2} x_\perp^\mu x_{\perp\rho} n_-^\nu \left[D_s^\rho\,, gF_{s\mu\nu}\right] + \mathcal{O}(\lambda^5)\,, \tag{3.4.14}$$

$$\int_0^1 ds\,s(x-x_-)^\mu R^\dagger(y(s)) gF_{s\mu\nu_\perp}(y(s))R(y(s)) = \frac{1}{2}x_\perp^\mu gF_{s\mu\nu_\perp} + \mathcal{O}(\lambda^4)\,, \tag{3.4.15}$$

$$\int_0^1 ds\,s(x-x_-)^\mu n_+^\nu R^\dagger(y(s)) gF_{s\mu\nu}(y(s))R(y(s)) = \mathcal{O}(\lambda^3)\,. \tag{3.4.16}$$

Here, we do not need to expand all terms to a fixed order (e.g. $\mathcal{O}(\lambda^4)$), since the second and third integral appear power-suppressed in the Lagrangian, starting at $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$, respectively.

The subleading terms (3.4.11) then yield up to $\mathcal{O}(\lambda^2)$ the contributions

$$\mathcal{L}_\chi^{(1)} = \frac{1}{2} x_\perp^\mu n_-^\nu gF^a_{s\mu\nu} n_+ j^a\,, \tag{3.4.17}$$

$$\mathcal{L}_\chi^{(2)} = \frac{1}{4} n_- x n_+^\mu n_-^\nu gF^a_{s\mu\nu} n_+ j^a + \frac{1}{4} x_\perp^\mu x_{\perp\rho} n_-^\nu \mathrm{tr}\left(\left[D_s^\rho\,, gF_{s\mu\nu}\right] t^a\right) n_+ j^a + \frac{1}{2} x_\perp^\mu gF^a_{s\mu\nu_\perp} j^{a\nu_\perp}\,. \tag{3.4.18}$$

For any practical applications, the Noether current should also be expressed in terms of the non-invariant fields $\hat{\phi}_c$ and $\hat{A}_c$. The complete Lagrangian then reads

$$\mathcal{L}^{(0)} = \frac{1}{2}\left[n_+ D_c \hat{\phi}_c\right]^\dagger n_- D\hat{\phi}_c + \frac{1}{2}\left[n_- D\hat{\phi}_c\right]^\dagger n_+ D_c\hat{\phi}_c + \left[D_{c\mu_\perp}\hat{\phi}_c\right]^\dagger D_c^{\mu_\perp}\hat{\phi}_c\,, \tag{3.4.19}$$

$$\mathcal{L}_{\phi_c}^{(1)} = \frac{1}{2}\hat{\phi}_c^\dagger \left(x_\perp^\mu n_-^\nu W_c gF_{s\mu\nu} W_c^\dagger\right) i n_+ D_c \hat{\phi}_c + \text{h.c.}\,, \tag{3.4.20}$$

$$\mathcal{L}_{\phi_c}^{(2)} = \frac{1}{4}\hat{\phi}_c^\dagger \left(n_- x n_+^\mu n_-^\nu W_c gF_{s\mu\nu} W_c^\dagger\right) i n_+ D_c \hat{\phi}_c + \frac{1}{4}\hat{\phi}_c^\dagger \left(x_\perp^\mu n_-^\nu x_{\perp\rho} W_c \left[D_s^\rho,\, gF_{s\mu\nu}\right] W_c^\dagger\right) i n_+ D_c \hat{\phi}_c + \frac{1}{2}\hat{\phi}_c^\dagger \left(x_\perp^\mu W_c gF_{s\mu\nu} W_c^\dagger\right) i D^\nu_{c\perp}\hat{\phi}_c + \text{h.c.} \tag{3.4.21}$$

The construction of the terms containing soft matter fields ϕ_s proceeds in complete analogy to the previous discussion. The soft matter field gives rise to new contributions starting at $\mathcal{O}(\lambda)$, which take the form

$$\mathcal{L}^{(1)}_{\phi_c\phi_s} = \frac{1}{2}\left[n_+ D_c\hat{\phi}_c\right]^\dagger n_- \overleftarrow{D} W_c\phi_s + \left[D_c^{\mu_\perp}\hat{\phi}_c\right]^\dagger \overleftarrow{D}_{c\mu_\perp} W_c\phi_s + \frac{1}{2}\left[n_- D\hat{\phi}_c\right]^\dagger n_+ \overleftarrow{D}_c W_c\phi_s + \text{ h.c.}\,, \tag{3.4.22}$$

$$\mathcal{L}^{(2)}_{\phi_c\phi_s} = \left(\frac{1}{2}\left[i n_+ D_c\hat{\phi}_c\right]^\dagger n_- \overleftarrow{D} + \left[i D_c^{\mu_\perp}\hat{\phi}_c\right]^\dagger \overleftarrow{D}_{c\mu_\perp} + \frac{1}{2}\left[i n_- D\hat{\phi}_c\right]^\dagger n_+ \overleftarrow{D}_c\right) W_c\, x_\perp^\rho \left[D_{s\rho_\perp}\phi_s\right] + \left[D_{c\mu_\perp}\hat{\phi}_c\right]^\dagger W_c D_s^{\mu_\perp}\phi_s + \frac{1}{2}\phi_s^\dagger W_c^\dagger \left(n_+ D_c n_- D + D_{c\mu_\perp} D_c^{\mu_\perp}\right) W_c\phi_s + \text{h.c.} \tag{3.4.23}$$

These Lagrangians completely describe the soft-collinear physics of scalar matter particles coupled to Yang-Mills theory. Let us stress again that this Lagrangian is not renormalised and all coefficients are exact to all orders in the coupling α_s and the power-counting parameter λ [40] since the starting point is already a renormalised theory.

To describe processes that feature more than one collinear direction, one must include the N-jet operators that describe hard sources emitting multiple collinear particles. These operators are explained in the next section.

3.5 N-jet Operator Basis

The N-jet operators are introduced for the purely-scalar theory in Section 2.6. Most statements directly generalise to the gauge-theoretic situation. However, we now need to account for the gauge symmetry, which further constrains the form of the building blocks. Recall that these operators are required to mediate interactions between different collinear sectors, which depend on the underlying hard process, and are thus not part of the Lagrangian interactions. The operator basis of these objects in the position-space formalism has been worked out in [50, 52] and we state the most important results here.

A generic N-jet operator takes the same form as in the scalar case (2.6.1)

$$\mathcal{J} = \int [dt]_N \, \widetilde{C}(t_{i_1}, t_{i_2}, \dots) J_s(0) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots) \,, \tag{3.5.1}$$

where $[dt]_N = \prod_{ik} dt_{i_k}$. Here, J_i denote the collinear and J_s the soft building blocks, and $\widetilde{C}(t_{i_1}, t_{i_2}, \dots)$ is the hard matching coefficient.

In gauge theory, a colour-neutral on-shell amplitude is gauge-invariant under the full gauge symmetry (3.2.3). Therefore, when the matching is performed, the N-jet operator should be gauge-invariant as well. The gauge symmetry of the full theory factorises into a product of N collinear gauge symmetries, which only act on the respective collinear sector, and one collective soft gauge background symmetry, under which all sectors transform in a similar fashion. To ensure full-theory gauge-invariance, one must obtain an operator that is invariant under both the collinear as well as the soft gauge transformations, possibly after taking into account on-shell conditions and colour-neutrality.

For the collinear gauge symmetry, it is easiest to work in terms of manifestly gauge-invariant operators, while for the soft we employ covariant objects. Consequently, one works with operators that transform as

$$J_i(x) \xrightarrow{\text{col.}} J_i(x) \,, \qquad J_i(x) \xrightarrow{\text{soft}} U_s(x_{i-}) J_i(x) \,, \tag{3.5.2}$$

where the soft transformation depends on the representation of the building block, but, crucially, for J_i at x is always evaluated along the collinear light-cone coordinate x_{i-}. This can be achieved by working in terms of the redefined scalar fields $\hat{\phi}_{c_i}$ and $\hat{A}_{c_i}$, and employing the gauge-invariant building blocks constructed from them, i.e. the gauge-invariant scalar field $\hat{\chi}_{c_i}$ (3.3.85) and the gauge field $\hat{\mathcal{A}}_{c_i}$ (3.3.83). This restricts the collinear building blocks to combinations of the manifestly gauge-invariant fields and their derivatives. At $\mathcal{O}(\lambda)$, one has the building blocks

$$\hat{\chi}_{c_i} = W_{c_i}^\dagger \hat{\phi}_{c_i} \,, \qquad g\hat{\mathcal{A}}^\mu_{c_i \perp_i} = W_{c_i}^\dagger \left[iD^\mu_{c_i \perp_i} W_{c_i} \right] \,. \tag{3.5.3}$$

As explained in Section 2.6, the possible $\mathcal{O}(1)$ building blocks that can be constructed by adding $n_{i+}\partial$ are already accounted for by the non-locality. In gauge-theory, there is the additional $\mathcal{O}(1)$ object $n_{i+}\hat{A}_{c_i}$. However, since we work with the manifestly gauge-invariant building blocks, which are constructed using W_{c_i}, these building blocks are effectively evaluated in collinear light-cone gauge and satisfy $n_{i+}\hat{\mathcal{A}}_{c_i} = 0$. Therefore, *there are no* $\mathcal{O}(1)$ *building blocks* in the operator basis, and at any given order in λ, there are only finitely many possible operators. In summary, the elementary building blocks, denoted by J_i^{A0}, are given by

$$J_i^{A0}(t_i) \in \left\{ \hat{\chi}_{c_i}(t_i n_{i+}) \,, \hat{\chi}^\dagger_{c_i}(t_i n_{i+}) \,, \hat{\mathcal{A}}_{c_i \perp_i}(t_i n_{i+}) \right\} \,. \tag{3.5.4}$$

As explained in Section 2.6, there are only two possible ways to construct subleading operators from the building blocks J_i^{A0}, either by adding power-suppressed derivatives or by adding more collinear building blocks. In the scalar case, we managed to eliminate $n_{i-}\partial$ in favour of $\partial_\perp^2$ and other soft and collinear building blocks. In gauge-theory, one can similarly use the equations of motion to eliminate the soft-covariant $n_{i-}D$ as well as the power-suppressed $n_{i-}\hat{\mathcal{A}}_{c_i}$ in favour of the $A0$-building blocks (3.5.4) and transverse derivatives thereof, plus additional purely-soft building blocks. This is performed explicitly in the appendix of [52].

The soft building blocks are similarly constrained. Under gauge transformations, these operators must transform as

$$J_s(x) \xrightarrow{\text{col.}} J_s(x)\,, \quad J_s(x) \xrightarrow{\text{soft}} U_s(x) J_s(x)\,, \tag{3.5.5}$$

where the soft transformation depends on the representation, as above. The collinear transformation does not give any constraints, since soft fields are invariant by definition. Hence the only requirement is that J_s is a soft gauge-covariant object, for example

$$\phi_s(x) \sim \lambda^2\,, \quad F_{s\mu\nu} \sim \lambda^4\,, \quad iD_s^\mu \phi_s(x) \sim \lambda^4\,, \tag{3.5.6}$$

where $D_s^\mu = i\partial^\mu - igA_s^\mu(x)$ in the purely soft case. Let us stress again that the covariant derivative $n_- D_s$ is *not* a valid building block, since it can be eliminated by collinear equations of motion. Therefore, the first possible soft gluon building block is the field-strength tensor $F_{s\mu\nu} \sim \lambda^4$, which appears at next-to-next-to-soft order.

For $\mathcal{J}$ to be gauge-invariant, one has to impose colour-neutrality. This means that the soft transformations $U_s(0)$ of all collinear and soft building blocks together must combine to unity, $\prod_{\text{building blocks}} U_s(0) = 1$.

3.6 Soft Emission and the LBK Theorem

As an example application, we use the previously constructed effective Lagrangian and the operator basis to derive and prove the LBK theorem from the EFT perspective. The derivation amounts to a straightforward tree-level computation of a soft-emission process. However, it is instructive to perform this computation in detail to familiarise oneself with the non-standard SCET Feynman rules.

To set the stage, recall a main insight of the previous N-jet operator discussion: *there are no possible soft gluon building blocks* until $\mathcal{O}(\lambda^4)$, where the soft field-strength tensor $F_{s\mu\nu}$ enters. However, there *are* soft-collinear Lagrangian interactions starting already at leading power. This means that any contribution to the emission of a soft gluon, starting from leading power until next-to-next-to-leading power, *must stem from the universal*, that is process-independent, Lagrangian interactions.[9] Only at the sub-subleading level in the soft expansion, corresponding to $\mathcal{O}(\lambda^4)$, a soft building block can be added to the N-jet operator, and thus process-dependence can enter via a new matching coefficient.

Let us emphasise that this statement already implies that soft emission is universal not just at leading but also at next-to-soft order in gauge theory. Only at the third order in the soft momentum, there exists a possibility to add process-dependent contributions, and consequently, the soft theorem only contains *two universal terms*. Remarkably, this statement is manifest from the Lagrangian and the possible building blocks, and no computation is necessary to prove the *universality* of the soft theorem. The computation is only required to derive the precise form of the soft theorem, not the fact that it holds true.

In other words, just from these considerations we already know that the soft-emission amplitude $\mathcal{A}_{\text{rad}}$ is related to the non-radiative amplitude $\mathcal{A}$ as

$$\mathcal{A}_{\text{rad}} = S^{(0)}\,[\mathcal{A}] + S^{(2)}\,[\mathcal{A}] + \mathcal{O}(k_s)\,, \tag{3.6.1}$$

where $S^{(0)}$ and $S^{(2)}$ denote universal operators acting on the non-radiative amplitude.

[9] We will show below that there are no physical contributions at next-to-leading power. This can be understood from counting soft momenta $k_s \sim \lambda^2$: Next-to-soft corresponds to next-to-next-to-leading power in our counting.

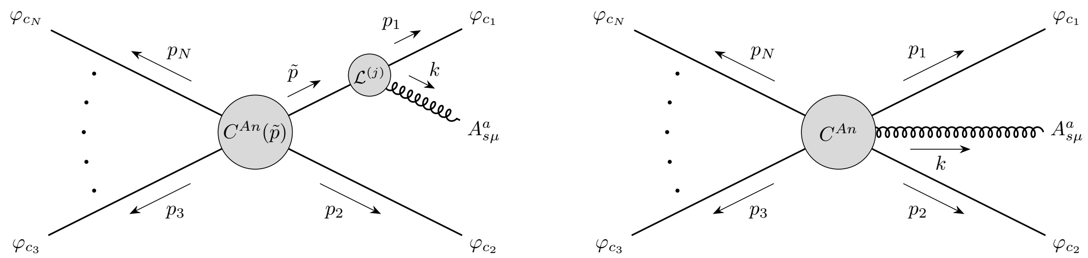


Figure 3.2: Two diagram classes contributing to the soft emission process in SCET. The first class represents Lagrangian insertions in the external legs via time-ordered product operators. These contributions are independent of the underlying hard amplitude. The second class stems from explicit soft building blocks J_s added to the N-jet operator. There are no soft gluon building blocks until $\mathcal{O}(\lambda^4)$, and these diagrams do not exist at $\mathcal{O}(\lambda^0)$ or $\mathcal{O}(\lambda^2)$.

Moreover, the Lagrangian interactions and the soft building blocks added directly to the N-jet operators correspond to different classes of Feynman diagrams, as depicted in Fig. 3.2. The Lagrangian interactions simply correspond to emission from the external legs, and thus only depend on the properties of these legs, i.e. on the precise form of the soft-collinear interactions. These interactions depend only on fundamental properties of the elementary particles, like spin, charge (and representation), or the kinematics of the scattering. The second type, adding a soft building block to the N-jet, corresponds to emissions directly from the hard vertex. Recall from the conventional derivation in Section 1.1, that these diagrams contributed at next-to-soft order, and one had to employ gauge-invariance to address these terms. Here, in the effective theory, these diagrams are manifestly absent.

Furthermore, since the Lagrangian $\mathcal{L}_i$ describes only a single collinear sector i, one can immediately conclude that the soft theorem reduces to a sum over the different legs, where each summand must be of the same form since it stems from the Lagrangian insertions. Therefore, one can specify (3.6.1) as

$$\mathcal{A}_{\mathrm{rad}} = \sum_{i=1}^{N} S_i^{(0)}\left[\mathcal{A}\right] + S_i^{(2)}\left[\mathcal{A}\right] + \mathcal{O}(k_s)\,, \tag{3.6.2}$$

where all S_i are of the same form. It is enough to compute the soft emission from a single leg to determine S_i, and we obtain the full soft emission amplitude by summing this contribution.

3.6.1 Assumptions and Relevant Operators

To derive the precise form of the soft theorem, we consider a process where a single soft gluon is emitted from N energetic scalar particles. Therefore, we are only interested in contributions to single-soft emission. To simplify the discussion, we will further assume that these N energetic particles are well-separated in angle, i.e. that they can be assigned to *different* collinear sectors, and no collinear splitting takes place.[10] In addition, we use a special reference frame for the radiative process: we choose a frame where each collinear reference vector is aligned with its momentum, i.e. $p_i^\mu = n_{i+}p_i\frac{n_{i-}^\mu}{2}$ and $p_{i\perp} = n_{i-}p_i = 0$. This choice is always possible, as long as each energetic particle is in its own collinear sector.

Note that one cannot choose this reference frame simultaneously for both the radiative and non-radiative computation, since the soft momentum k^μ enters in the hard vertex, and thus

[10] This assumption is not necessary to derive the soft theorem, but it simplifies the computation drastically.

we can either choose to align $p_i + k$ or p_i with n_{i+}, but not both. Therefore, we compute the non-radiative matching in a *general reference frame* with $p_{i\perp} \neq 0$.

Next, we discuss the possible building blocks that contribute. At leading power, there is only the operator

$$\mathcal{J}^{(0)} = \int [dt]_N\, \widetilde{C}^{A0}(t_1, \ldots, t_N) \prod_i J_i^{A0}(t_i)\,, \tag{3.6.3}$$

where the form of J_i^{A0} depends on the specific process at hand. Since we assume that we only have a single particle in each collinear direction, the subleading operators are completely determined by inserting the subleading building blocks

$$\begin{aligned} \mathcal{O}(\lambda^1): \quad & J^{A1\mu}_{\partial\chi_i^\dagger}(t_i) = i\partial^\mu_{i\perp}\chi_i^\dagger(t_i n_{i+})\,, \\ \mathcal{O}(\lambda^2): \quad & J^{A2\mu\nu}_{\partial^2\chi_i^\dagger}(t_i) = i\partial^\mu_{i\perp} i\partial^\nu_{i\perp}\chi_i^\dagger(t_i n_{i+})\,. \end{aligned} \tag{3.6.4}$$

Throughout the computation, we employ the all-outgoing convention for our process. Accordingly, the relevant building blocks are $\chi_{c_i}^\dagger$.

Note that the operator (3.6.3) contains a matching coefficient $\widetilde{C}^{A0}$, which is determined by "non-radiative" matching, i.e. matching to the underlying N-jet process *without* soft radiation. This non-radiative matching is precisely the one already performed in Section 2.7, now with slightly different building blocks and in the context of QCD. It still holds that, intuitively, the Fourier-transformed matching coefficient C^{An} is precisely the non-radiative amplitude order-by-order in λ.

Since there are no available soft building blocks, all contributions to the emission must stem from Lagrangian interactions, which we incorporate using time-ordered products that scale as

$$i \int d^4x\, T\{J^{Ak}(t_i), \mathcal{L}_i^{(n)}(x)\} \sim \mathcal{O}(\lambda^{k+n}) \tag{3.6.5}$$

compared to the leading-power current J^{A0}. Thanks to our choice of reference frame, there will only be three non-vanishing contributions to $\mathcal{O}(\lambda^2)$, stemming from

$$\int d^4x\, T\left\{\mathcal{J}^{(0)}, \mathcal{L}_\chi^{(0)}\right\}\,, \tag{3.6.6}$$

$$\int d^4x\, T\left\{\mathcal{J}^{(0)}, \mathcal{L}_\chi^{(2)}\right\}\,, \tag{3.6.7}$$

$$\int d^4x\, T\left\{\mathcal{J}^{(1)}, \mathcal{L}_\chi^{(1)}\right\}\,, \tag{3.6.8}$$

where $\mathcal{J}^{(1)}$ contains one subleading building block from (3.6.4). All other relevant terms vanish in our reference frame. Since we can relate $\mathcal{J}$ to the non-radiative amplitude, we immediately see the structure of the soft theorem appearing, namely as some soft-collinear operator acting on the non-radiative amplitude $\mathcal{J}$.

3.6.2 Non-radiative Matching

We begin by performing the non-radiative matching. This section is in some sense a more abstract version of the explicit matching performed in Section 2.7, and we keep the discussion rather brief.

In the full theory, we denote the non-radiative amplitude for the process by $\mathcal{A}$, and we need to relate this to the hard matching coefficients $\widetilde{C}$ of our operators by matching. In Section 2.7, we already found that the N-jet operator can be thought of as the non-radiative amplitude, and its subleading corrections then correspond to the Taylor-expansion in the subleading momenta.

This result makes the following discussion look overly technical and somewhat inconvenient at first glance, very reminiscent of the traditional derivation [27, 28], where one has to impose momentum conservation to relate the non-radiative and radiative amplitude before performing the expansion in the soft momenta. However, from the effective theory point of view, this is just a standard matching computation, which then yields a rather trivial result. Since the soft theorem is a tree-level statement, we will restrict the discussion to tree-level matching.

The full-theory amplitude $\mathcal{A}$, which depends on the scalar products of momenta, must be expanded in λ as

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{O}(\lambda^2)\,, \tag{3.6.9}$$

with

$$\mathcal{A}^{(0)} = \mathcal{A}\Big|_{p_i^\mu = n_{i+}p_i\, n_{i-}^\mu/2}\,, \tag{3.6.10}$$

$$\mathcal{A}^{(1)} = p_{i\perp}^\mu \left(\frac{\partial}{\partial p_{i\perp}^\mu}\mathcal{A}\right)\Bigg|_{p_i^\mu = n_{i+}p_i\, n_{i-}^\mu/2}\,. \tag{3.6.11}$$

At leading-power, the amplitude $\mathcal{A}^{(0)}$ must be reproduced by the matrix element of the leading-power N-jet operator (3.6.3), which yields the condition

$$\begin{aligned}\mathcal{A}^{(0)} &= \langle p_1, \ldots, p_N|\, \mathcal{J}^{(0)}\, |0\rangle \\ &= \int [dt]_N\, e^{i\sum_i n_{i+}p_i\, t_i}\widetilde{C}^{A0}(t_1, \ldots t_N) \equiv C^{A0}(n_{1+}p_1, \ldots, n_{N+}p_N)\,,\end{aligned} \tag{3.6.12}$$

where $C^{A0}(\{n_{i+}p_i\})$ is the Fourier-transformed matching coefficient.

At next-to-leading power, one can continue matching explicitly, or use the RPI constraints [53, 56] to determine the subleading matching coefficients. For C^{A1}, it reads (2.7.16)

$$C_j^{A1\,\mu}(n_{1+}p_1, \ldots, n_{N+}p_N) = -\sum_{k\neq j}\frac{2n_{k-}^\mu}{n_{k-}n_{j-}}\frac{\partial}{\partial n_{i+}p_i}C^{A0}(n_{1+}p_1, \ldots, n_{N+}p_N)\,. \tag{3.6.13}$$

3.6.3 Soft Theorem Computation

Let us now calculate the soft factors explicitly. The relevant interaction terms contributing to single soft emission are

$$\mathcal{L}^{(0)} \supset \frac{g}{2}n_- A_s^a(x_-)\left(\phi_c^\dagger t^a i n_+\partial\phi_c - i n_+\partial\phi_c^\dagger t^a\phi_c\right)\,, \tag{3.6.14}$$

$$\mathcal{L}_\chi^{(1)} \supset \frac{1}{2}x_\perp^\mu n_-^\nu gF_{s\mu\nu}^a n_+ j^a\,, \tag{3.6.15}$$

$$\mathcal{L}_\chi^{(2)} \supset \frac{1}{4}n_- x n_+^\mu n_-^\nu gF_{s\mu\nu}^a n_+ j^a + \frac{1}{4}x_\perp^\mu x_{\perp\rho} n_-^\nu \mathrm{tr}([\partial^\rho gF_{s\mu\nu}]\, t^a) n_+ j^a + \frac{1}{2}x_\perp^\mu gF_{s\mu\nu_\perp}^a j^{a\nu_\perp}\,, \tag{3.6.16}$$

where the Noether current reads

$$\begin{aligned} n_+ j^a &= i\hat{\phi}_c^\dagger t^a n_+\overleftrightarrow{\partial}\hat{\phi}_c\,, \\ j_{\mu_\perp}^a &= i\hat{\phi}_c^\dagger t^a\overleftrightarrow{\partial}_{\mu_\perp}\hat{\phi}_c\,.\end{aligned} \tag{3.6.17}$$

Here and in the following, we can replace the matter field ϕ_c with its gauge-invariant building block χ_c, since they differ only by additional collinear gluon fields, which are irrelevant to the process at hand. Consequently, we do not need to worry about ϕ_c appearing in the Lagrangian but χ_c appearing in the operator basis. For our considerations, they are equivalent. The

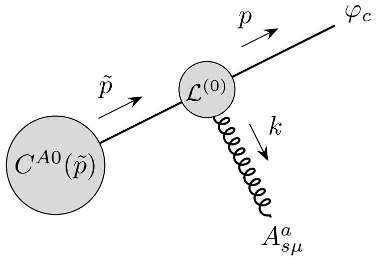
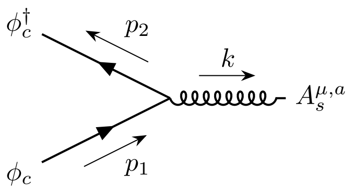


Figure 3.3: Leading-power contribution to the emission of a soft gluon. C^{A0} denotes the (non-radiative) leading-power matching coefficient and $\mathcal{L}^{(0)}$ denotes the insertion of the leading-power Lagrangian interaction.

corresponding Feynman rule for soft emission is given by



$$= igt^a \begin{cases} \frac{1}{2}(n_+p_1 + n_+p_2)n_-^\mu\,, & \mathcal{O}(\lambda^0) \\ -\frac{1}{2}X_\perp^\alpha n_-^\beta(n_+p_1 + n_+p_2)(k_\alpha\delta^\mu_\beta - k_\beta\delta^\mu_\alpha)\,, & \mathcal{O}(\lambda^1) \\ S^{\alpha\beta}(p_1,p_2)\frac{1}{2}(k_\alpha\delta^\mu_\beta - k_\beta\delta^\mu_\alpha)\,, & \mathcal{O}(\lambda^2) \end{cases} \tag{3.6.18}$$

where we defined

$$\begin{aligned} S^{\alpha\beta}(p_1,p_2) \equiv & -\frac{1}{2}n_+^\alpha n_-^\beta(n_+p_1 + n_+p_2)n_-X - (p_{1\perp}^\beta + p_{2\perp}^\beta)X_\perp^\alpha \\ & + \frac{1}{2}n_-^\beta k^\rho(n_+p_1 + n_+p_2)X_\perp^\alpha X_{\perp\rho}\,, \end{aligned} \tag{3.6.19}$$

and X is given in (2.3.21)

$$X^\mu \equiv \partial^\mu\left[(2\pi)^4\delta^{(4)}\left(\sum p_{\text{in}} - \sum p_{\text{out}}\right)\right]\,, \tag{3.6.20}$$

where the derivative $\partial = \partial/\partial p_{\text{in}}$ or $\partial = -\partial/\partial p_{\text{out}}$ acts on incoming or outgoing momenta inside the delta function. If soft momenta k are present, one sets the $k_\perp$ and n_-k components to zero inside the δ-function after moving the derivative via integration by parts. Notably, we keep the momentum-conserving δ-function of each vertex explicit as part of the Feynman rule. To derive these Feynman rules (3.6.18), one simply replaces $\partial_\mu = -ip_\mu$ for ingoing and $\partial_\mu = ip_\mu$ for outgoing momenta while the explicit x^μ in the Lagrangian correspond to iX^μ in the Feynman rule.

Leading Power

At leading-power, we consider the diagram in Fig. 3.3. Explicitly, the amplitude reads[11]

$$\mathcal{M}_i^{(0)} = \int \frac{d^4\tilde{p}}{(2\pi)^4}\, igt^a\frac{1}{2}(n_{i+}p_i + n_{i+}\tilde{p})n_{i-}\varepsilon(k)\frac{i}{\tilde{p}^2 + i0}C^{A0}(\tilde{p})(2\pi)^4\delta^{(4)}\left(\tilde{p} - p_i - n_{i-}k\frac{n_{i+}}{2}\right)\,, \tag{3.6.21}$$

[11]In the following, we denote the amplitude computed using SCET by $\mathcal{M}^{(n)}$, while we reserve $\mathcal{A}^{(n)}$ for the full-theory radiative result, to stress that this is really an independent calculation in the EFT and not simply matching. In the end, we will find that both are indeed the same.

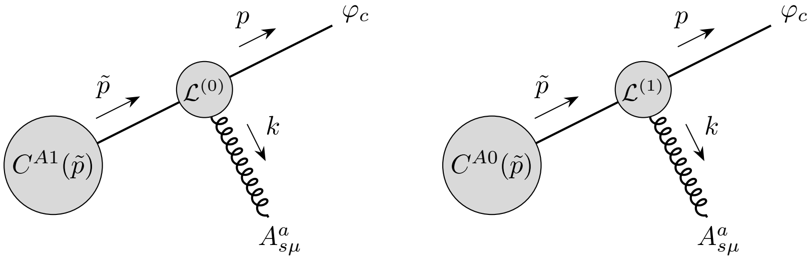


Figure 3.4: The two diagrams contributing to soft-gluon emission at $\mathcal{O}(\lambda)$. The first one corresponds to the leading-power emission from the subleading non-radiative amplitude, and the second one to the subleading-power emission from the leading non-radiative amplitude. In the first case, the subleading non-radiative amplitude is proportional to an explicit $p_\perp = 0$, so the contribution trivially vanishes. In the second case, one sees after evaluating the explicit $X_\perp$ that the result is likewise proportional to $p_\perp$ and vanishes in our reference frame.

where we abbreviated $C^{A0}(n_{1+}p_1, \ldots, n_{N+}p_N)$ to emphasise the relevant dependence on the internal $\tilde{p}$. Since there are no explicit X^μ, one can directly perform the integral over $\tilde{p}$. The momentum-conserving δ-function, in combination with our special reference frame, then yields the replacement $\tilde{p}^\mu = n_{i+}p_i\frac{n^\mu_{i-}}{2} + n_{i-}k\frac{n^\mu_{i+}}{2}$, which implies in particular

$$\frac{1}{\tilde{p}^2}\bigg|_{\tilde{p}=n_{i+}p_i\frac{n_{i-}}{2}+n_{i-}k\frac{n_{i+}}{2}} = \frac{1}{n_{i+}p_i n_{i-}k}\,. \tag{3.6.22}$$

Thus one obtains

$$\begin{aligned}\mathcal{M}^{(0)}_i &= -gt^a n_{i+}p_i n_{i-}\varepsilon(k)\frac{1}{n_{i+}p_i n_{i-}k}C^{A0}(\{n_{i+}p_i\}) \\ &= -gt^a n_{i-}\frac{n_{i-}\varepsilon(k)}{n_{i-}k}C^{A0}(\{n_{i+}p_i\})\,.\end{aligned} \tag{3.6.23}$$

This is already the well-known eikonal term of soft emission, and it immediately follows from the leading-power soft-collinear interactions.

Next-to-leading Power

By counting soft momenta, we expect the next independent contribution to appear at $\mathcal{O}(\lambda^2) \sim \mathcal{O}(k)$ compared to the leading-power amplitude. However, in principle, there could be a contribution that is related to the leading-power result by reparameterisation invariance. Recasting the leading result as $\mathcal{M} = S^{(0)}_i\mathcal{A}^{(0)}$, a possible NLP contribution is of the form $\mathcal{M} = S^{(0)}_i\mathcal{A}^{(1)}$. These types of terms are proportional to explicit $p_\perp$, and thus are absent in our choice of reference frame. However, we can verify this explicitly by inserting the $\mathcal{O}(\lambda)$ Feynman rule. There are two possible contributions, as depicted in Fig. 3.4. For the first diagram, note that C^{A1}_μ always appears in combination with $\tilde{p}^\mu_\perp$. The leading-power interaction does not feature any explicit $X_\perp$, therefore this term immediately vanishes after imposing momentum conservation, yielding $\tilde{p}_\perp = p_\perp = 0$. For the second diagram, we find

$$\mathcal{M}^{(1)} = -\int\frac{d^4\tilde{p}}{(2\pi)^4}\,igt^a\frac{1}{2}(n_{i+}p_i + n_{i+}\tilde{p})X^\alpha_\perp n^\beta_-(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)\frac{i}{\tilde{p}^2+i0}C^{A0}(\tilde{p})\,. \tag{3.6.24}$$

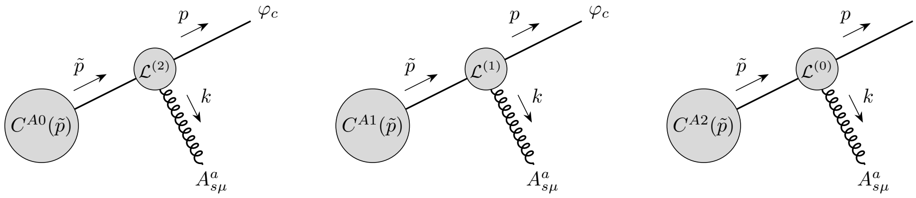


Figure 3.5: The diagrams contributing to soft-gluon emission at $\mathcal{O}(\lambda^2)$. The first one corresponds to the leading-power non-radiative amplitude with sub-subleading Lagrangian interaction, the second one to the subleading amplitude with subleading interaction, while the third one is the sub-subleading amplitude with leading-power interaction. The third diagram again trivially vanishes, since it is proportional to $p_\perp^\mu p_\perp^\nu$. The other two diagrams have non-zero contributions. For the second one, this is due to explicit dependence on $X_\perp$ which can eliminate the $p_\perp$ from the subleading amplitude.

The coefficient C^{A0} only depends on $n_{i+}\tilde{p}$, so the transverse derivative inside $X_\perp$, after integration by parts, can only act on the propagator. Here, one sees that

$$\frac{\partial}{\partial \tilde{p}_{\perp\mu}} \frac{1}{\tilde{p}^2} = -\frac{2\tilde{p}_\perp^\mu}{\tilde{p}^4}\,, \tag{3.6.25}$$

and after imposing momentum conservation, this term is proportional to $p_\perp = 0$. Therefore, there is indeed no contribution at $\mathcal{O}(\lambda)$ in our reference frame, as anticipated.

Next-to-next-to-leading Power (Next-to-soft)

At $\mathcal{O}(\lambda^2)$, corresponding to $\mathcal{O}(k)$ suppression, there are three diagrams, given in Fig. 3.5. The third diagram, consisting of the leading-power emission from the sub-subleading non-radiative amplitude $\mathcal{A}^{(2)}$, corresponding to C^{A2}, trivially vanishes in our reference frame, since the coefficient $C^{A2}_{\mu\nu}$ comes with $\tilde{p}_{\perp\mu}\tilde{p}_{\perp\nu}$, and after momentum conservation the external $p_\perp$ again vanish. We cannot use this argument for the second diagram, where $C^{A1}_\mu \tilde{p}_\perp^\mu$ appears since the subleading Feynman rule contains $X_\perp$, which can act on this transverse momentum and the diagram can thus yield a non-vanishing contribution. We begin by evaluating the first diagram. Here, we find

$$\mathcal{M}^{(2)}_{A0} = \int \frac{d^4\tilde{p}}{(2\pi)^4}\, igt^a S^{\alpha\beta}(p,\tilde{p})\frac{1}{2}(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)\frac{i}{\tilde{p}^2}C^{A0}(\tilde{p})\,, \tag{3.6.26}$$

where $S^{\alpha\beta}$ is given in (3.6.19) and reads

$$\begin{aligned} S^{\alpha\beta}(p,\tilde{p}) = &-\frac{1}{2}n_{i+}^\alpha n_{i-}^\beta(n_{i+}p + n_{i+}\tilde{p})n_{i-}X - (p_\perp^\beta + \tilde{p}_\perp^\beta)X_\perp^\alpha \\ &+\frac{1}{2}n_{i-}^\beta k^\rho(n_{i+}p + n_{i+}\tilde{p})X_\perp^\alpha X_{\perp\rho}\,. \end{aligned} \tag{3.6.27}$$

We consider the three summands of (3.6.27) individually. The first contribution is

$$\mathcal{M}^{(2)}_{A0,1} = -\frac{1}{4}gt^a\int \frac{d^4\tilde{p}}{(2\pi)^4}\,(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)n_{i+}^\alpha n_{i-}^\beta(n_{i+}p + n_{i+}\tilde{p})n_{i-}X\frac{1}{\tilde{p}^2}C^{A0}(\tilde{p})\,. \tag{3.6.28}$$

In our kinematic situation, $n_{i-}X$ is defined as (see (2.3.21))

$$n_{i-}X = (2\pi)^4 n_{i-}^\mu \frac{\partial}{\partial\tilde{p}^\mu}\delta^{(4)}(\tilde{p} - p - n_{i-}k\frac{n_{i+}}{2})\,. \tag{3.6.29}$$

We move the derivative by partial integration and then perform the $\tilde{p}$ integral over the δ-function. This yields

$$\mathcal{M}^{(2)}_{A0,1} = -\frac{1}{4} g t^a \int \frac{d^4\tilde{p}}{(2\pi)^4} \, (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) n^\alpha_{i+} n^\beta_{i-} n^\rho_{i-} \frac{\partial}{\partial \tilde{p}^\rho} \left[(n_{i+}p + n_{i+}\tilde{p}) \frac{1}{\tilde{p}^2} C^{A0}(\tilde{p}) \right] \\ \times (2\pi)^4 \delta^{(4)}(\tilde{p} - p - n_{i-}k \frac{n_{i+}}{2}) \,. \tag{3.6.30}$$

To evaluate the derivative, first observe that

$$n^\rho_{i-} \frac{\partial}{\partial \tilde{p}^\rho} (n_{i+}\tilde{p} \frac{1}{\tilde{p}^2}) = n_{i-} \cdot n_{i+} \frac{1}{\tilde{p}^2} + n_{i+}p(-2) \frac{n_{i-}\tilde{p}}{\tilde{p}^4} \\ = \frac{2}{n_{i+}p n_{i-}k} - \frac{2 n_{i+}p n_{i-}k}{(n_{i+}p n_{i-}k)^2} = 0 \,, \tag{3.6.31}$$

where we used $\tilde{p} = n_{i+}p\frac{n_{i-}}{2} + n_{i-}k\frac{n_{i+}}{2}$ and $n_{i+} \cdot n_{i-} = 2$ in the last line. Therefore, $n_{i-}X$ (and also $X_\perp$) do not act on the combination $n_{i+}\tilde{p}\frac{1}{\tilde{p}^2}$.[12] Thus we obtain two terms, one where the derivative acts on C^{A0}, and one where it acts on the propagator in the term proportional to $n_{i+}p\frac{1}{\tilde{p}^2}$. This yields

$$\mathcal{M}^{(2)}_{A0,1} = -\frac{1}{2} g t^a n^\alpha_{i+} n^\beta_{i-} (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) n_{i+}p \frac{1}{n_{i+}p \, n_{i-}k} \left[n_{i-} \cdot \frac{\partial}{\partial \tilde{p}} C^{A0}(\tilde{p}) \right]_{\tilde{p} = n_{i+}p\frac{n_{i-}}{2} + n_{i-}k\frac{n_{i+}}{2}} \\ + \frac{1}{2} g t^a n^\alpha_{i+} n^\beta_{i-} (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) \frac{1}{n_{i+}p \, n_{i-}k} C^{A0}(n_{i+}p) \,. \tag{3.6.32}$$

The second term in (3.6.27) yields a vanishing contribution, since

$$\mathcal{M}^{(2)}_{A0,2} = \frac{1}{2} g t^a \int \frac{d^4\tilde{p}}{(2\pi)^4} \, (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha)(p^\beta_\perp + \tilde{p}^\beta_\perp) X^\alpha_\perp \frac{1}{\tilde{p}^2} C^{A0}(\tilde{p}) \\ = -\frac{1}{2} g t^a (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) \eta^{\alpha\beta}_\perp \frac{1}{n_{i+}p \, n_{i-}k} C^{A0}(n_{i+}p) + (\text{terms} \sim p_\perp) \,, \tag{3.6.33}$$

which vanishes by antisymmetry and by setting $p_\perp = 0$. Lastly, the third term in (3.6.27) yields a non-vanishing contribution. First note that $X_\perp$ does not act on $n_{i+}\tilde{p}$, so we can already insert $n_{i+}\tilde{p} = n_{i+}p$.Then, we compute

$$\mathcal{M}^{(2)}_{A0,3} = -\frac{1}{2} g t^a \int \frac{d^4\tilde{p}}{(2\pi)^4} \, (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) n^\beta_{i-} k^\rho n_{i+}p X^\alpha_\perp X_{\perp\rho} \frac{1}{\tilde{p}^2} C^{A0}(\tilde{p}) \\ = (k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) n^\beta_{i-} k^\alpha_\perp \frac{n_{i+}p}{(n_{i+}p \, n_{i-}k)^2} C^{A0}(n_{i+}p) \,, \tag{3.6.34}$$

where we used

$$\frac{\partial}{\partial \tilde{p}_{\perp\mu}} \frac{\partial}{\partial \tilde{p}_{\perp\nu}} \frac{1}{\tilde{p}^2} = -\frac{2\eta^{\mu\nu}_\perp}{\tilde{p}^4} + (\text{terms} \sim p_\perp) \,, \tag{3.6.35}$$

as well as $p_\perp = 0$. Next, we use transversality and the on-shell condition

$$0 = k^\alpha \varepsilon_\alpha = \frac{1}{2} n_{i-}k n_{i+}\varepsilon + \frac{1}{2} n_{i+}k n_{i-}\varepsilon + k^\alpha_\perp \varepsilon_\alpha \,, \tag{3.6.36}$$

$$0 = k^2 = n_{i-}k n_{i+}k + k^\alpha_\perp k_{\perp\alpha} \,, \tag{3.6.37}$$

to rewrite the combination in (3.6.34) as

$$(k_\alpha \varepsilon_\beta - k_\beta \varepsilon_\alpha) n^\beta_{i-} k^\alpha_\perp = (k^\alpha_\perp k_{\perp\alpha} n_{i-}\varepsilon - n_{i-}k k^\alpha_\perp \varepsilon_\alpha$$

[12] This combination is precisely the usual eikonal propagator, which only depends on $n_{i-}\tilde{p}$. We will formalise and exploit this observation in Chapter 7

$$
\begin{aligned}
&= (-n_{i-}kn_{i+}kn_{i-}\varepsilon + n_{i-}k(\frac{1}{2}n_{i-}kn_{i+}\varepsilon + \frac{1}{2}n_{i-}kn_{i+}\varepsilon) \\
&= -\frac{1}{2}n_{i-}k(n_{i+}kn_{i-}\varepsilon - n_{i-}kn_{i+}\varepsilon)\,. \qquad (3.6.38)
\end{aligned}
$$

Using this result, we find

$$
\mathcal{M}^{(2)}_{A0,3} = -\frac{1}{2}gt^a n^\alpha_{i+}n^\beta_{i-}(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)\frac{1}{n_{i+}p\,n_{i-}k}C^{A0}(n_{i+}p)\,, \qquad (3.6.39)
$$

which precisely cancels the last term in (3.6.32). In conclusion, we obtain for the contribution from the first graph in Fig. 3.5 the amplitude

$$
\mathcal{M}^{(2)}_{A0} = -\frac{1}{2}gt^a n^\alpha_{i+}n^\beta_{i-}(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)\frac{n_{i+}p}{n_{i+}p\,n_{i-}k}\left[n_{i-}\cdot\frac{\partial}{\partial\tilde{p}}C^{A0}(\tilde{p})\right]_{\tilde{p}=n_{i+}p\frac{n_{i-}}{2}+n_{i-}k\frac{n_{i+}}{2}}\,. \qquad (3.6.40)
$$

We can now recast this into a more familiar form by introducing the orbital angular momentum in the given reference frame as

$$
\begin{aligned}
L^{\mu\nu}_i &= \frac{1}{4}n^{[\mu}_{i+}n^{\nu]}_{i-}n_{i+}p_i n^\alpha_{i-}\frac{\partial}{\partial p^\alpha_i} + \frac{1}{2}n_{i+}p_i n^{[\nu}_{i-}\frac{\partial}{\partial p_{i\perp\mu]}} \\
&= \frac{1}{2}n^{[\mu}_{i+}n^{\nu]}_{i-}n_{i+}p_i\frac{\partial}{\partial n_{i+}p_i} + \frac{1}{2}n_{i+}p_i n^{[\nu}_{i-}\frac{\partial}{\partial p_{i\perp\mu]}}\,, \qquad (3.6.41)
\end{aligned}
$$

where we used $n^\mu_{i-}\frac{\partial}{\partial p^\mu_i} = 2\frac{\partial}{\partial n_{i+}p_i}$. We see that (3.6.40) contains the first term of the angular momentum (3.6.41), namely

$$
\begin{aligned}
\mathcal{M}^{(2)}_{A0} &= -\frac{1}{2}gt^a n^\alpha_{i+}n^\beta_{i-}(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)\frac{n_{i+}p}{n_{i+}p\,n_{i-}k}\left[n_{i-}\cdot\frac{\partial}{\partial\tilde{p}}C^{A0}(\tilde{p})\right]_{\tilde{p}=p} \\
&= -gt^a\frac{1}{2}\frac{k_\alpha\varepsilon_\beta}{n_{i+}p\,n_{i-}k}n^{[\alpha}_{i+}n^{\beta]}_{i-}\left[L_{+-}C^{A0}\right] \\
&= -gt^a\frac{k_\alpha\varepsilon_\beta}{p\cdot k}\frac{n^{[\alpha}_{i+}n^{\beta]}_{i-}}{4}L_{+-}[\mathcal{A}]\,. \qquad (3.6.42)
\end{aligned}
$$

Next, we consider the second diagram in Fig. 3.5, where the subleading interaction of $\mathcal{O}(\lambda)$ is inserted in external legs of the subleading amplitude $\mathcal{A}^{(1)}$. Since $\mathcal{A}^{(1)}$ enters the diagram via $C^{A1\mu}\tilde{p}_{\perp\mu}$, and the $\mathcal{O}(\lambda)$ Feynman rule contains an $X_\perp$, there can be a non-zero contribution despite $p_\perp = 0$. Thus we compute

$$
\mathcal{M}^{(2)}_{A1} = -igt^a\int\frac{d^4\tilde{p}}{(2\pi)^4}\,(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)X^\alpha_\perp n^\beta_{i-}n_{i+}p\frac{i}{\tilde{p}^2}(-\tilde{p}_{\perp\mu}C^{A1\mu}(\tilde{p}))\,. \qquad (3.6.43)
$$

Since the external momenta satisfy $p_\perp = 0$, the only non-vanishing contribution arises when $X_\perp$ acts on the explicit $\tilde{p}_{\perp\mu}$. The relevant term reads

$$
\mathcal{M}^{(2)}_{A1} = gt^a(k_\alpha\varepsilon_\beta - k_\beta\varepsilon_\alpha)n^\beta_{i-}n_{i+}p\frac{1}{n_{i+}p\,n_{i-}k}C^{A1\alpha}(p)\,. \qquad (3.6.44)
$$

Identifying $C^{A1\alpha}(p)$ with the derivative of the non-radiative amplitude,

$$
C^{A1\alpha}(p) = \left(-\frac{\partial}{\partial p_{\perp\alpha}}\mathcal{A}^{(1)}\right)_{p=n_{i+}p\frac{n_{i-}}{2}}\,, \qquad (3.6.45)
$$

we see that this term can equally be rewritten using the angular momentum (3.6.41) as

$$
\mathcal{M}^{(2)}_{A1} = -gt^a\frac{k_\alpha\varepsilon_\beta}{n_{i+}p\,n_{i-}k}n^{[\beta}_{i-}n_{i+}p\left(\frac{\partial}{\partial p_{\perp\alpha]}}\mathcal{A}^{(1)}\right)_{p=n_{i+}p\frac{n_{i-}}{2}}
$$

$$= -gt^a \frac{k^\alpha \varepsilon_\beta}{2p \cdot k} \frac{1}{2} n_{i-}^{[\beta} L_{+\alpha_\perp]}[\mathcal{A}] \tag{3.6.46}$$

In summary, from (3.6.42) and (3.6.46), and summing over all legs, we find for the next-to-soft term

$$\mathcal{M}^{(2)} = -g \sum_{i=1}^{N} t_i^a \frac{k_\alpha \varepsilon_\beta(k) L_i^{\beta\alpha}}{p_i \cdot k} \mathcal{A}\,, \tag{3.6.47}$$

and thus we reproduce the well-known Low-Burnett-Kroll amplitude

$$\mathcal{A}_{\text{rad}} = -g \sum_{i=1}^{N} t_i^a \left(\frac{p_i \cdot \varepsilon(k)}{p_i \cdot k} + \frac{k_\nu \varepsilon_\mu^a(k) L_i^{\mu\nu}}{p_i \cdot k} \right) \mathcal{A}\,. \tag{3.6.48}$$

In conclusion, we see that the soft theorem follows from a straightforward computation of the tree-level emission process in the effective theory. The universality of the leading and next-to-soft term is an immediate consequence of the soft gauge symmetry, and the explicit form is readily computed from the effective Feynman rules. This derivation can now be extended to also include other matter fields besides scalar particles.

However, the derivation was slightly non-trivial and a number of accidental cancellations occurred. This poses a simple question: Can the soft-collinear Lagrangian be recast in a form that immediately yields the soft theorem as an emission vertex? The answer to this question is yes, and it will be explained in detail in the later Chapter 7.

3.7 Extension to Dirac Fermions

In this section, we generalise the notions of the previous sections to also describe fermionic matter. In the context of SCET, the effective theory was originally constructed with quarks in mind. Therefore, the fermionic construction preceded the scalar one [40, 41]. Most concepts of the previous discussion transfer over directly to the spinor fields.

The first difference compared to the scalar field is that the Dirac spinor describes four degrees of freedom compared to one in the scalar case. The spinor field $\psi(x)$ is a solution of the Dirac equation

$$(i\not{\partial} - m)\psi(x) = 0\,. \tag{3.7.1}$$

Setting $m = 0$ and going to momentum space, we can insert the power-counting of the collinear momenta to find

$$0 = \not{p} u(p) = \frac{1}{2} n_+ p \not{n}_- \psi(p) + \mathcal{O}(\lambda)\,, \tag{3.7.2}$$

i.e. the leading-power equation of motion gives the condition

$$\not{n}_- u(p) = 0\,, \tag{3.7.3}$$

so some components of the spinor are projected out. One can make this observation manifest by introducing the projection operators $P_\pm$ via

$$1 = P_+ + P_- = \frac{\not{n}_- \not{n}_+}{4} + \frac{\not{n}_+ \not{n}_-}{4}\,. \tag{3.7.4}$$

One can immediately verify that these objects are indeed projections and satisfy $P_\pm^2 = P_\pm$. The full spinor field can be decomposed into two components with definite projection properties as

$$\xi = P_+ \psi_c = \frac{\not{n}_- \not{n}_+}{4} \psi_c\,, \quad \eta = P_- \psi_c = \frac{\not{n}_+ \not{n}_-}{4} \psi_c\,, \tag{3.7.5}$$

and the power-counting follows from the two-point function [40]. For the first component $\xi(x)$, one finds

$$\begin{aligned} \langle 0|\, T(\xi(x)\xi(0))\,|0\rangle &= \frac{\not{n}_-\not{n}_+}{4}\,\langle 0|\, T(\psi_c(x)\psi_c(0))\,|0\rangle\,\frac{\not{n}_+\not{n}_-}{4} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\frac{i}{p^2+i\varepsilon}\,\frac{\not{n}_+\not{n}_-}{4}\not{p}\frac{\not{n}_+\not{n}_-}{4} \sim \lambda^4\frac{1}{\lambda^2} = \lambda^2\,. \end{aligned} \tag{3.7.6}$$

For the second component, the analogous computation yields

$$\begin{aligned} \langle 0|\, T(\eta(x)\eta(0))\,|0\rangle &= \frac{\not{n}_+\not{n}_-}{4}\,\langle 0|\, T(\psi_c(x)\psi_c(0))\,|0\rangle\,\frac{\not{n}_-\not{n}_+}{4} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\frac{i}{p^2+i\varepsilon}\,\frac{\not{n}_-\not{n}_+}{4}\not{p}\frac{\not{n}_-\not{n}_+}{4} \sim \lambda^4\lambda^2\frac{1}{\lambda^2} = \lambda^4\,. \end{aligned} \tag{3.7.7}$$

Therefore, the second component η is subleading compared to the leading spinor ξ. The soft quark scales as $q_s \sim \lambda^3$. Inserting the spinor decomposition (3.7.5) into the Lagrangian and using the projection properties, one finds

$$\begin{aligned} \mathcal{L}_c = \bar\psi_c i\not{D}\psi_c &= (\bar\xi+\bar\eta)\left[\frac{\not{n}_-}{2}in_+D + \frac{\not{n}_+}{2}in_-D + i\not{D}_\perp\right](\xi+\eta) \\ &= \bar\xi\frac{\not{n}_+}{2}in_-D\xi + \bar\xi i\not{D}_\perp\eta + \bar\eta i\not{D}_\perp\xi + \bar\eta\frac{\not{n}_-}{2}in_+D\eta\,. \end{aligned} \tag{3.7.8}$$

It is now convention to integrate out the subleading component η using its equation of motion

$$\eta = -\frac{1}{n_+D}\frac{\not{n}_+}{2}\not{D}_\perp\xi\,. \tag{3.7.9}$$

This yields a Lagrangian of the form

$$\mathcal{L}_c = \bar\xi\frac{\not{n}_+}{2}in_-D\xi + \bar\xi i\not{D}_\perp\frac{1}{in_+D}i\not{D}\frac{\not{n}_+}{2}\xi\,, \tag{3.7.10}$$

which forms the starting point of the EFT derivation. This is the main difference between the scalar field and the Dirac spinor. Since both fields behave the same way under gauge transformations, the redefinitions of the collinear fields can be directly transferred, and the construction of the effective Lagrangian proceeds in the same way. The fermionic Lagrangian is then given by [41]

$$\mathcal{L}^{(0)}_\xi = \bar\xi\left(in_-D + i\not{D}_\perp\frac{1}{in_+D}i\not{D}_\perp\right)\frac{\not{n}_+}{2}\xi + \bar q i\not{D}_s q + \mathcal{L}^{(1)}_\xi + \mathcal{L}^{(2)}_\xi + \mathcal{L}^{(1)}_{\xi q} + \mathcal{L}^{(2)}_{\xi q}\,, \tag{3.7.11}$$

where the interaction terms read

$$\mathcal{L}^{(1)}_\xi = \bar\xi_c\left(x^\mu_\perp n^\nu_-\, W_c\, gF^s_{\mu\nu}W_c^\dagger\right)\frac{\not{n}_+}{2}\xi_c\,, \tag{3.7.12}$$

$$\begin{aligned} \mathcal{L}^{(2)}_\xi &= \frac{1}{2}\bar\xi_c\left(n_-x\, n^\mu_+ n^\nu_-\, W_c\, gF^s_{\mu\nu}W_c^\dagger + x^\mu_\perp x_{\perp\rho}n^\nu_- W_c[D^\rho_s, gF^s_{\mu\nu}]W_c^\dagger\right)\frac{\not{n}_+}{2}\xi_c \\ &+ \frac{1}{2}\bar\xi_c\left(i\not{D}_\perp\frac{1}{in_+D}x^\mu_\perp\gamma^\nu_\perp\, W_c\, gF^s_{\mu\nu}W_c^\dagger + x^\mu_\perp\gamma^\nu_\perp\, W_c\, gF^s_{\mu\nu}W_c^\dagger\,\frac{1}{in_+D}\,i\not{D}_\perp\right)\frac{\not{n}_+}{2}\xi_c\,, \end{aligned} \tag{3.7.13}$$

$$\mathcal{L}^{(1)}_{\xi q} = \bar q W_c^\dagger i\not{D}_\perp\xi - \bar\xi i\overleftarrow{\not{D}}_\perp W_c q\,, \tag{3.7.14}$$

$$\mathcal{L}^{(2)}_{\xi q} = \bar q W_c^\dagger\left(in_-D + i\not{D}_\perp(in_+D)^{-1}i\not{D}_\perp\right)\frac{\not{n}_+}{2}\xi + \bar q\overleftarrow{D}^\mu_s x_{\perp\mu}W_c^\dagger i\not{D}_\perp\xi$$

$$-\bar{\xi}\frac{\not{n}_+}{2}\left(in_-\overleftarrow{D}+i\overleftarrow{\not{D}}_\perp\left(in_+\overleftarrow{D}\right)^{-1}i\overleftarrow{\not{D}}_\perp\right)W_c q-\bar{\xi}i\overleftarrow{\not{D}}_\perp W_c x_{\perp\mu}D_s^\mu q\,. \tag{3.7.15}$$

In the operator basis, the collinear gauge-invariant spinor field is defined just like the scalar field as (3.3.85)

$$\chi_c=W_c^{-1}\xi\,, \tag{3.7.16}$$

where only the leading component ξ enters. However, Lorentz invariance (or RPI in the case of SCET) imposes that the full spinor $\psi=\xi+\eta$ should appear in the amplitude and consequently in the operator basis. This manifests itself in a different relation of the $A1$ current (2.7.16), which now contains the effect of the subleading component

$$\begin{aligned} C_i^{A1\,\mu}(n_{1+}p_1,\ldots,n_{N+}p_N) &= \left[-\frac{\gamma_{i\perp}^\mu}{n_{i+}p_i}\frac{\not{n}_{i+}}{2}-\sum_{j\neq i}\frac{2n_{j-}^\mu}{n_{i-}\cdot n_{j-}}\frac{\partial}{\partial n_{i+}p_i}\right]C^{A0}(n_{1+}p_1,\ldots,n_{N+}p_N) \\ &\equiv C_{i,\,\mathrm{spin}}^{A1\mu}(n_{1+}p_1,\ldots,n_{N+}p_N)+C_{i,\,\mathrm{orbit}}^{A1\mu}(n_{1+}p_1,\ldots,n_{N+}p_N)\,. \end{aligned} \tag{3.7.17}$$

This additional, spin-dependent term will be crucial in the derivation of the soft theorem for fermionic fields in Section 7.3.

Chapter

4 Perturbative Gravity

In this section, we introduce the underlying "full theory" of SCET gravity, perturbative gravity. As a starting point, we discuss the Einstein-Hilbert action. The quantisation of this action has a long history [15–24], and it turns out that this action is not renormalisable in the strict sense. However, the action can be interpreted as the first term in an effective theory, where higher-order terms, which correspond to higher-derivative terms, are necessary to render the gravitational loops finite. In this way, one obtains a fully consistent quantum theory describing gravity at energies way below the Planck scale. This construction of a low-energy effective theory of gravity was pioneered by Donoghue in [25]. We introduce this approach and the necessary concepts, and refer for further details to the excellent reviews [26, 57].

Starting from the Einstein-Hilbert action, we investigate the gauge symmetries of gravitational theories, treating them in the same fashion as in ordinary gauge theory. Then, we perform the weak-field expansion, with emphasis on its effect on the full-theory gauge-invariance. At this point, it is useful to compare this expanded theory, which forms the basis of the SCET construction, to the situation encountered in gauge theory. Already at this level, one can identify substantial differences between both theories, and anticipate what effect this may have on the form of SCET gravity.

4.1 Gravitational Action

We consider a curved space-time described by the metric tensor $g_{\mu\nu}(x)$. Following general relativity, the metric tensor $g_{\mu\nu}$ obeys the Einstein-Hilbert action[1]

$$S_{\rm EH} = -\frac{2}{\kappa^2}\int \mathrm{d}^4x\, \sqrt{-g}R\,, \tag{4.1.1}$$

where $\kappa^2 = 32\pi G_N$, g is the metric determinant, and R is the Ricci scalar. It is obtained from the Riemann tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha\Gamma^{\mu}{}_{\beta\nu} - \partial_\beta\Gamma^{\mu}{}_{\alpha\nu} + \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\lambda}{}_{\beta\nu} - \Gamma^{\mu}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\alpha\nu}\,, \tag{4.1.2}$$

by contracting the indices as $R = g^{\alpha\beta}R^{\mu}{}_{\alpha\mu\beta}$. The Christoffel symbols $\Gamma^{\mu}{}_{\alpha\beta}$ are defined as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}\left(-\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha}\right)\,. \tag{4.1.3}$$

This action is invariant under a large class of transformations, the diffeomorphism group. These consist of coordinate transformations

$$x^\mu \to y^\mu(x)\,, \tag{4.1.4}$$

where $y^\mu(x)$ is a general invertible function of x. From this transformation, one can define the Jacobi matrices

$$U^{\mu}{}_{\alpha}(x) \equiv \frac{\partial y^\mu}{\partial x^\alpha}(x)\,, \quad U_{\nu}{}^{\beta}(x) \equiv \frac{\partial x^\beta}{\partial y^\nu}(x)\,, \tag{4.1.5}$$

[1] We employ metric signature $(+,-,-,-)$.

which are inverse with respect to each other, i.e. they satisfy $U^{\mu}{}_{\alpha}(x)U_{\nu}{}^{\alpha}(x) = \delta^{\mu}_{\nu}$.

Under such coordinate change, the metric tensor transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(y) = U_{\mu}{}^{\alpha}(x)U_{\nu}{}^{\beta}(x)g_{\alpha\beta}(x)\,, \tag{4.1.6}$$

and one can explicitly verify that the Riemann tensor is indeed a tensor and that the Ricci scalar is invariant $R(x) \to R'(y) = R(x)$. In addition, the measure d^4x transforms, and one can identify the invariant measure $d^4x\sqrt{-g}$. Thus, the action (4.1.1) is invariant under diffeomorphisms.

To quantise this action, one would perform a weak-field expansion

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \kappa h_{\mu\nu}(x) \tag{4.1.7}$$

around some background $\bar{g}_{\mu\nu}(x)$, identifying the fluctuation $h_{\mu\nu}$ as the graviton field. We explain this in detail down below. In this expanded theory, the quantisation proceeds along the same lines as in Yang-Mills, e.g. by performing gauge-fixing using a ghost Lagrangian. Then, from the quadratic action, one obtains the free propagator, and higher-order terms are treated as interactions. However, the coupling constant κ is dimensionful, as $\kappa \sim \sqrt{G_N} \sim \frac{1}{M_{\rm Pl}}$. This means that the theory of a spin-2 field is not renormalisable in the strict sense. Indeed, the local divergences encountered at the one-loop level take the schematic form[2] [24, 58]

$$\delta\mathcal{L} = \frac{1}{16\pi^2}\frac{2}{4-d}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right)\,, \tag{4.1.8}$$

and one immediately sees that they cannot be absorbed by renormalising the fields and couplings in (4.1.1).

Instead, one can now view the Einstein-Hilbert action (4.1.1) as the first term of an effective theory. At one-loop level (and beyond), this action is modified, and one would use

$$S_{\rm grav,EFT} = -\int \mathrm{d}^4x\,\sqrt{-g}\left(\Lambda + \frac{2}{\kappa^2}R - c_1R^2 - c_2R_{\mu\nu}R^{\mu\nu} + \dots\right)\,. \tag{4.1.9}$$

Here, we see that the additional terms R^2 and $R_{\mu\nu}R^{\mu\nu}$ can be used to absorb the divergences (4.1.8). We also introduced the cosmological constant Λ, which can be accommodated in the standard action (4.1.1). Note that the Ricci scalar contains two derivatives, $R \sim \partial^2$. Therefore, the new terms are higher-derivative terms $R^2 \sim \partial^4$, and are consequently suppressed at low energies. Equivalently, in situations of small curvature (characterised by small $h_{\mu\nu}$), these higher-order terms are suppressed compared to the leading R term. One can see that the effective theory quite naturally takes the structure of a derivative expansion, and at higher loop levels, one has to take into account more and more of these higher-order terms in the curvature.

From this perspective, the effective action (4.1.9) is a perfectly well-defined and predictive effective quantum theory, with the expansion parameter being either a low-energy parameter or, equivalently, small curvature. Crucially, for our purposes, the action has the same set of gauge symmetries as the original Einstein-Hilbert action, the diffeomorphisms. Since the SCET construction only depends on the underlying gauge symmetry, one can consider as "full theory" the action (4.1.1), or its higher-order modification (4.1.9) up to any desired order. The precise form of the SCET Lagrangian will change at higher orders, but the underlying all-order construction will be the exact same.

4.2 Diffeomorphism Invariance

The diffeomorphism group in gravity can be thought of as the analogue of $SU(N)$-invariance in Yang-Mills theory and is of special interest for constructing the soft-collinear effective theory.

[2] We regularise in $d = 4 - 2\varepsilon$ dimensions.

However, in its conventional form as coordinate transformations, as introduced in (4.1.4), the analogy is not very clear. This form can be viewed as passive transformations, where the coordinates are transformed, and tensor fields change accordingly. For our purposes, it is much more convenient to use an *active* point of view. That is, one reformulates the effect of a diffeomorphism as a *purely internal* transformation of the fields, and never transforms the coordinates directly.

To be precise, consider a local translation of the form

$$x^\mu \to y^\mu(x) = x^\mu + \varepsilon^\mu(x)\,, \tag{4.2.1}$$

with some (not necessarily infinitesimal) vector field $\varepsilon(x)$. A scalar field $\varphi(x)$ transforms as

$$\varphi(x) \to \varphi'(x') = \varphi(x)\,. \tag{4.2.2}$$

One can write the transformed scalar field $\varphi'(x')$ as

$$\varphi'(x') = \varphi'(x + \varepsilon(x)) = \left[T_\varepsilon \varphi'(x)\right]\,, \tag{4.2.3}$$

where the translation operator T_ε acting on some function $f(x)$ is defined simply by the Taylor expansion

$$T_\varepsilon f(x) = f(x) + \varepsilon^\alpha(x)\partial_\alpha f(x) + \frac{1}{2}\varepsilon^\alpha(x)\varepsilon^\beta(x)\partial_\alpha\partial_\beta f(x) + \mathcal{O}(\varepsilon^3)\,. \tag{4.2.4}$$

One can now evaluate the transformed scalar field φ' at the old coordinate x, defining the active transformation

$$\varphi(x) \to \varphi'(x) = \left[U(x)\varphi(x)\right]\,, \tag{4.2.5}$$

where $U(x)$ is then simply the inverse of the translation operator (4.2.4). Expanding in ε, this corresponds to

$$\begin{aligned}\left[U(x)\varphi(x)\right] &= \varphi(x) - \varepsilon^\alpha(x)\partial_\alpha\varphi(x) \\ &\quad + \frac{1}{2}\varepsilon^\alpha(x)\varepsilon^\beta(x)\partial_\alpha\partial_\beta\varphi(x) + \varepsilon^\alpha(x)\partial_\alpha\varepsilon^\beta(x)\partial_\beta\varphi(x) + \mathcal{O}(\varepsilon^3)\,.\end{aligned} \tag{4.2.6}$$

Note that we labelled this operation by $U(x)$ to stress the formal similarity with the $SU(N)$ gauge transformation. Indeed, by treating diffeomorphisms as active transformations, we are able to treat these transformations as *purely internal* ones, and we can use most of the intuition from the gauge theory side.

One can immediately notice two further points: first, the leading-order term of the transformation (4.2.6) is just the Lie-derivative of the scalar field

$$\pounds_\varepsilon\varphi = -\varepsilon^\alpha\partial_\alpha\varphi\,. \tag{4.2.7}$$

We will check below that this also holds for generic tensors, and one can easily verify that this even holds for the expansion about non-trivial backgrounds. Second, and more importantly, one sees that the inverse is not simply given by replacing $\varepsilon \to -\varepsilon$ in the definition of T_ε. This is due to our choice of translation (4.2.1) and accordingly taking T_ε to be (4.2.4). If instead, we would consider an infinitesimal transformation like

$$x^\mu \to x^\mu + \varepsilon^\mu(x) + \frac{1}{2}\varepsilon^\alpha\partial_\alpha\varepsilon^\mu(x) + \ldots\,, \tag{4.2.8}$$

corresponding to

$$x^\mu \to \exp(\varepsilon^\alpha\partial_\alpha)x^\mu\,, \tag{4.2.9}$$

this would be different, and the inverse would be defined as a translation with parameter $-\varepsilon(x)$. This is simply a choice one has to make at the beginning of the construction, and we opt for the simpler form of the local translation (4.2.1).

The metric tensor $g_{\mu\nu}(x)$ has the active transformation

$$\begin{aligned} g_{\mu\nu} \to & \Big[U(x){U_\mu}^\alpha(x){U_\nu}^\beta(x)g_{\alpha\beta}(x)\Big] \\ &= g_{\mu\nu}(x) - \nabla_\mu\varepsilon_\nu(x) - \nabla_\nu\varepsilon_\mu(x) + \mathcal{O}(\varepsilon^2)\,, \end{aligned} \tag{4.2.10}$$

where ∇_μ denotes the covariant derivative with respect to $g_{\mu\nu}$. For a generic tensor field ${T_\mu}^\nu(x)$, one finds analogously

$$\begin{aligned} {T_\mu}^\nu(x) \to & \Big[U(x){U_\mu}^\rho(x){U^\nu}_\sigma(x){T_\rho}^\sigma(x)\Big] \\ &= {T_\mu}^\nu(x) - \partial_\mu\varepsilon^\rho(x){T_\rho}^\nu(x) + \partial_\sigma\varepsilon^\nu(x){T_\mu}^\sigma(x) - \varepsilon^\alpha(x)\partial_\alpha{T_\mu}^\nu(x) + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \tag{4.2.11}$$

Again, comparing this to the Lie derivative

$$\mathcal{L}_\varepsilon{T_\mu}^\nu = {T_\mu}^\nu - \varepsilon^\alpha\partial_\alpha{T_\mu}^\nu - \partial_\mu\varepsilon^\rho{T_\rho}^\nu + \partial_\sigma\varepsilon^\nu{T_\mu}^\sigma\,, \tag{4.2.12}$$

one notices that these are precisely the linear terms in the transformation (4.2.11).

In summary, when adopting the active point of view, one is able to treat diffeomorphisms as purely internal transformations, that only act on the fields and not the coordinates, $\varphi(x) \to \varphi'(x)$. This, in turn, looks formally very similar to gauge theory. Notably, at leading order in the gauge-parameter ε, these active coordinate transformations correspond to the Lie derivative. At higher orders, they can be systematically constructed from the Taylor expansion.

Gauge-invariant expressions are of course invariant, regardless if one considers active or passive transformations. Therefore, one can derive a set of useful properties of these translation operators. These are given in Appendix A.

4.3 Weak-field Expansion

Next, we discuss the weak-field expansion and its effect on the gauge symmetry. For now, we take as action the first term, (4.1.1), and for the matter part, we consider the minimally-coupled scalar field[3]

$$S_\varphi = \int d^4x\,\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\,. \tag{4.3.1}$$

We perform the weak-field expansion (4.1.7) around flat-space, corresponding to $\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu}$. The expansion of the metric tensor

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{4.3.2}$$

is taken to be *exact*, that is, we do not consider any higher-order modifications in κ to (4.3.2). All other functions of the metric tensor, however, are in general given as infinite series in $h_{\mu\nu}$, or κ, respectively. For example, the inverse metric tensor $g^{\mu\nu}(x)$ can be determined from $g_{\mu\alpha}(x)g^{\alpha\nu}(x) = \delta_\mu^\nu$ to be

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h_\alpha^\nu - h^{\mu\alpha}h_{\alpha\beta}h^{\beta\nu} + \mathcal{O}(h^4)\,, \tag{4.3.3}$$

while the metric determinant is given by

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}) + \mathcal{O}(h^3)\,. \tag{4.3.4}$$

The action then turns into an infinite series in $h_{\mu\nu}$, resp. κ:

$$S = \sum_{k=0}^{\infty}\kappa^k S^{(k)}\,, \tag{4.3.5}$$

where the precise form of $S^{(k)}$ at higher orders depends on which terms one considers to be part of the "full theory", i.e. if one only considers Einstein-Hilbert (4.1.1), or also takes into account higher-order Riemann terms as in (4.1.9).

[3] One could add to this action a self-interaction, e.g. $-\frac{\lambda}{4!}\varphi^4$. However, this does not affect the SCET construction in any form, so we omit this term for now. We will add it back once we give the final Lagrangian.

4.3.1 Einstein-Hilbert Action

Using these expansions in the Einstein-Hilbert action (4.1.1), we obtain to leading order in h the quadratic Lagrangian

$$\mathcal{L}^{(0)}_{\rm EH} = \frac{1}{2}\left(\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} - \partial_\alpha h \partial^\alpha h - 2\partial_\mu h^{\mu\nu}(\partial_\alpha h^\alpha_\nu - \partial_\nu h)\right) . \tag{4.3.6}$$

To quantise this theory, we must now add a gauge-fixing term. We consider a generalised de Donder gauge $\partial^\alpha h_{\mu\alpha} = \partial_\mu h$ with parameter b by adding the gauge-fixing term

$$S_{\rm gf} = b\int {\rm d}^4x\,(\partial_\alpha h^\alpha_\mu - \frac{1}{2}\partial_\mu h)(\partial_\beta h^{\beta\mu} - \frac{1}{2}\partial^\mu h) . \tag{4.3.7}$$

Following the standard Faddeev-Popov procedure, the gauge-fixing term also introduces a ghost Lagrangian. However, like in the gauge theory situation, the ghost Lagrangian is not of interest for the construction of SCET, and we omit its discussion here as well. However, in off-shell computations, or scattering into unphysical polarisations, one has to keep in mind that ghosts are present in gravity as well.

One can then compute the graviton propagator in standard fashion by inverting the bilinear part, which yields

$$D_{\mu\nu,\alpha\beta} = \langle 0|T(h_{\mu\nu}(x)h_{\alpha\beta}(y))|0\rangle = i\int\frac{{\rm d}^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2+i0}(P_{\mu\nu,\alpha\beta} + \frac{1-b}{b}S_{\mu\nu,\alpha\beta}) , \tag{4.3.8}$$

where

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right) , \tag{4.3.9}$$

$$S_{\mu\nu,\alpha\beta} = \frac{1}{2p^2}\left(\eta_{\mu\alpha}p_\nu p_\beta + \eta_{\mu\beta}p_\nu p_\alpha + p_\mu p_\alpha \eta_{\nu\beta} + p_\mu p_\beta \eta_{\nu\alpha}\right) . \tag{4.3.10}$$

The self-interactions begin at the trilinear level, corresponding to $\mathcal{O}(\kappa)$ suppression, with the Lagrangian $\kappa\mathcal{L}^{(1)} \equiv \mathcal{L}_{h^3}$, which reads

$$\begin{aligned}\mathcal{L}^{(1)} = -\frac{1}{2}h^{\alpha\beta}\Big(& h\partial_\alpha\partial_\beta h + 2\partial_\mu h^{\mu\nu}\partial_\nu h_{\alpha\beta} + \partial_\alpha h_{\mu\nu}\partial_\beta h^{\mu\nu} + h_{\alpha\beta}\Box h + 2\partial^\mu h_{\mu\alpha}\partial_\beta h \\ &+ \partial_\alpha h_{\beta\mu}\partial^\mu h - h\partial^\mu\partial_\alpha h_{\beta\mu} - h^\mu_\alpha \Box h_{\mu\beta} + 2\partial_\mu h_{\beta\nu}\partial^\nu h^\mu_\alpha + 4\partial^\mu\partial^\nu h_{\alpha\nu}h_{\beta\nu}\Big) \\ & - \frac{1}{4}h\partial_\mu h\partial^\mu h + \frac{1}{4}h\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} .\end{aligned} \tag{4.3.11}$$

4.3.2 Scalar Action

Using the same expansions in the scalar action (4.3.1), one obtains the leading Lagrangian

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi , \tag{4.3.12}$$

which is just a free scalar theory. Correspondingly, we find the standard propagator (2.2.7)

$$\langle 0|\,T(\varphi(x)\varphi(y))\,|0\rangle = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\,\frac{i}{p^2+i\varepsilon} . \tag{4.3.13}$$

At higher-order in $h_{\mu\nu}$, we find the scalar-graviton interaction terms

$$\mathcal{L}^{(1)} = -\frac{1}{2}h_{\mu\nu}\left(\partial^\mu\varphi\,\partial^\nu\varphi - \eta^{\mu\nu}\frac{1}{2}\partial_\alpha\varphi\,\partial^\alpha\varphi\right) , \tag{4.3.14}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\left(h^{\mu\alpha}h^\nu_\alpha - \frac{1}{2}hh^{\mu\nu} + \frac{1}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta})\eta^{\mu\nu}\right)\partial_\mu\varphi\,\partial_\nu\varphi . \tag{4.3.15}$$

4.3.3 Weak-field Gauge-invariance

In the weak-field expansion, we slightly change conventions and redefine the parameter of local translations as

$$x^\mu \to x^\mu + \kappa\varepsilon^\mu(x)\,. \tag{4.3.16}$$

We use this choice so that the infinitesimal transformation of $h_{\mu\nu}$, determined from (4.2.10), is the simple shift

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - \partial_\mu\varepsilon_\nu(x) - \partial_\nu\varepsilon_\mu(x) + \mathcal{O}(\varepsilon^2)\,. \tag{4.3.17}$$

Note that this already implies that the gauge symmetry mixes different orders in κ. For the graviton field $h_{\mu\nu}$, which enters at $\mathcal{O}(\kappa)$ itself, the gauge symmetry corresponds to a linear shift, plus non-linear higher order terms in κ. For matter fields, however, this rescaling results in

$$\varphi \to \varphi - \kappa\varepsilon^\alpha\partial\alpha\varphi + \mathcal{O}(\kappa^2)\,. \tag{4.3.18}$$

This is a somewhat subtle feature of the weak-field expansion: once one performs the weak-field expansion, one has to truncate the expanded action at some order in $h_{\mu\nu}$. In turn, also the diffeomorphisms must be expanded and truncated. This means that the expanded action is only gauge-invariant order-by-order in h, and subleading gauge transformations are cancelled by the leading gauge-transformations of the subleading Lagrangians.

For example, the leading scalar action (4.3.12) is gauge-invariant by itself, since the scalar field does not have a leading-power gauge transformation. At $\mathcal{O}(\kappa)$, one has

$$\delta_\varepsilon\varphi = -\kappa\varepsilon^\alpha\partial_\alpha\varphi\,, \tag{4.3.19}$$

and in turn, the Lagrangian generates a subleading

$$\delta_\varepsilon^{(1)}\mathcal{L}^{(0)} = -\kappa\left(\varepsilon^\alpha\partial_\alpha\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\right) + \partial_\mu\varepsilon^\alpha\partial_\alpha\varphi\partial^\mu\varphi\right)\,. \tag{4.3.20}$$

For the subleading Lagrangian (4.3.14), one only has to consider the linear shift of $h_{\mu\nu}$, and one finds

$$\delta_\varepsilon^{(0)}\mathcal{L}^{(1)} = \kappa\partial_\mu\varepsilon_\nu\left(\partial^\mu\varphi\partial^\nu\varphi - \eta^{\mu\nu}\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi\right)\,. \tag{4.3.21}$$

Combining both terms, one obtains

$$\delta_\varepsilon^{(1)}\mathcal{L}^{(0)} + \delta_\varepsilon^{(0)}\mathcal{L}^{(1)} = -\kappa\partial_\alpha\left(\varepsilon^\alpha\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\right)\,, \tag{4.3.22}$$

which is a total derivative and thus the Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ is gauge-invariant up to $\mathcal{O}(\kappa)$. This generalises also to $\mathcal{L}^{(2)}$, where the sub-subleading gauge transformation of $\mathcal{L}^{(0)}$ is relevant.

An immediate consequence of these truncated diffeomorphisms is that we cannot have objects that are homogeneous in h – that is, are monomials in h – and gauge-invariant at the same time. Gauge-covariant objects, such as the Riemann tensor, are represented as a series order-by-order in h or κ. When working with a theory expanded in h, one finds that subleading terms, that is, higher-order terms in h, appear in precise combinations to yield a gauge-invariant theory. This is a generic feature of non-linearly realised symmetries.

4.4 The Vierbein Formalism

4.4.1 Spinors in Curved Space-time

A problem arises if one wants to consider half-integer representations of the Lorentz group in curved space-times, like spinors. These correspond to representations of the covering group $SL(2,\mathbb{C})$, and have no direct corresponding representation in the diffeomorphism group $GL(4)$

[59]. Therefore, to properly implement such half-integer fields, one makes use of the *vierbein* formalism. Intuitively, this is a literal implementation of the equivalence principle. Namely, one introduces a local inertial frame ξ^a_X, $a = 0, 1, 2, 3$ at every point X. These coordinates have the property that the metric at point X corresponds to the flat-space Minkowski metric. To relate these to the generic reference frame x^μ, one defines the corresponding vierbein $e^a{}_\mu(X)$ as

$$e^a{}_\mu(X) \equiv \left[\frac{\partial \xi^a_X(x)}{\partial x^\mu}\right]_{x=X} . \tag{4.4.1}$$

The generic metric tensor $g_{\mu\nu}(x)$ can then be expressed as

$$g_{\mu\nu}(x) = \eta_{ab} e^a{}_\mu(x) e^b{}_\nu(x) . \tag{4.4.2}$$

Such a local inertial frame is not unique. At each point X, one can perform a Lorentz transformation $\Lambda^a{}_b(X)$ that leaves the decomposition (4.4.2) invariant since the Minkowski metric is unchanged by Lorentz transformations. Therefore, the vierbein comes with its own *local* Lorentz transformation (LLT), where it transforms as

$$e^a{}_\mu(x) \to \Lambda^a{}_b(x) e^b{}_\mu(x) . \tag{4.4.3}$$

In addition, the vierbein transforms under diffeomorphisms, in this context also called "General Coordinate Transformations" (GCT), like a standard covariant tensor

$$e^a{}_\mu(x) \to e'^a{}_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} e^a{}_\nu(x) . \tag{4.4.4}$$

In the following, we adopt the convention that Greek indices from the middle of the alphabet, $\mu, \nu, \ldots$ are reserved for GCT tensors, while LLT indices are denoted by Latin ones $a, b, m, n, \ldots$ In addition, we use Greek indices from the beginning of the alphabet, $\alpha, \beta, \ldots$ to denote spinor-indices. Note that while GCT indices are still raised and lowered with the full metric $g_{\mu\nu}$, the LLT indices are raised and lowered with the flat Minkowski metric η_{ab}. It is useful to define the inverse vierbein $E^\mu{}_a(x)$ which satisfies

$$\begin{aligned} E^\mu{}_a e^a{}_\nu &= \delta^\mu_\nu , \\ E^\mu{}_a e^b{}_\mu &= \delta^b_a . \end{aligned} \tag{4.4.5}$$

Tensors can be defined either as standard GCT tensors $T^\mu{}_\alpha$, which transform under diffeomorphisms according to their representation

$$T^\mu{}_\alpha(x) \to \frac{\partial x'^\mu}{\partial x^\nu}(x) \frac{\partial x^\beta}{x'^\alpha}(x) T^\nu{}_\beta(x) , \tag{4.4.6}$$

or as LLT tensors $T^m{}_a$, which transform under GCT as a *scalar*, and under LLT according to their representation

$$T^m{}_a(x) \to \Lambda^m{}_n(x) \Lambda_a{}^b(x) T^n{}_b(x) . \tag{4.4.7}$$

These two versions of the tensor T are related by the vierbein as

$$T^m{}_a(x) = e^m{}_\mu(x) E^\alpha{}_a(x) T^\mu{}_\alpha(x) . \tag{4.4.8}$$

One can transform any GCT tensor field into a set of GCT scalars. This has the advantage that one can now implement also half-integer representations of the Lorentz group, simply by defining them in the local inertial frame. To promote these objects to a generic curved space-time, one then ensures that the Lagrangian describing these objects is *local* Lorentz invariant and forms a scalar. Once one arrives at a scalar Lagrangian, the invariant action is defined in the usual way by performing the integration $d^4x\sqrt{-g}$. We explain this in detail at the example of the Dirac fermion.

The Dirac spinor lives in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group, where it has a Lorentz transformation

$$\psi_\alpha(x)\to D_{\alpha\beta}(\Lambda)\psi_\beta(x)\,. \tag{4.4.9}$$

Here, Λ is the Lorentz transformation and $D_{\alpha\beta}(\Lambda)$ is the associated spinor transformation defined by

$$D_{\alpha\beta}(\Lambda)=e^{-\frac{i}{2}\lambda_{ab}J^{ab}}\,, \tag{4.4.10}$$

where λ_{ab} is the parameter of the infinitesimal Lorentz transformation and $J^{ab}=\frac{i}{4}\left[\gamma^a,\gamma^b\right]$ are the generators of the Lorentz group for the Dirac representation. In curved space-times, this transformation is now promoted to a *local* one, and the spinor field now transforms as

$$\psi_\alpha(x)\to D_{\alpha\beta}(\Lambda(x))\psi_\beta(x)\,. \tag{4.4.11}$$

The only difference is that now the Lorentz transformation corresponds to a local $\Lambda(x)$. As in gauge theory, a local Lorentz transformation poses a problem, since derivatives now also act on the local parameter and are no longer covariant. Therefore, one defines a Lorentz-covariant derivative

$$D_\mu\psi_\alpha\equiv\partial_\mu\psi_\alpha-i\left[\Omega_\mu\right]_{\alpha\beta}\psi_\beta\,. \tag{4.4.12}$$

Here, we introduce the object $\left[\Omega_\mu\right]_{\alpha\beta}$ as the gauge-field corresponding to local Lorentz transformations. It has the standard transformation of a gauge-connection

$$\Omega_\mu(x)\to D(\Lambda(x))\Omega_\mu D^{-1}(\Lambda(x))-i\left[\partial_\mu D(\Lambda(x))\right]D^{-1}(\Lambda(x))\,, \tag{4.4.13}$$

and can be defined in terms of the Lorentz generator J^{ab} as

$$\left[\Omega_\mu\right]_{\alpha\beta}=\frac{1}{2}\left[J^{ab}\right]_{\alpha\beta}\left[\omega_\mu\right]_{ab}(x)\,. \tag{4.4.14}$$

The coefficient $\left[\omega_\mu\right]_{ab}$ is called the *spin-connection*. Using the spin-connection, one can construct the gauge-field Ω_μ for any Lorentz representation by contracting it with the appropriate representation of the generator J^{ab}. The spin-connection can be computed from the vierbein as [59]

$$\left[\omega_\mu\right]_{ab}=g_{\lambda\nu}E_a{}^\lambda\left(\partial_\mu E_b{}^\nu+\Gamma^\nu{}_{\mu\rho}E_b{}^\rho\right)\,. \tag{4.4.15}$$

With the help of the Lorentz-covariant derivative (4.4.12), one can now define the Dirac action in curved space-times.

One starts from the flat-space Dirac Lagrangian

$$\mathcal{L}_\psi=\bar{\psi}i\gamma^\mu\partial_\mu\psi\,, \tag{4.4.16}$$

where $\bar{\psi}=\psi^\dagger\gamma^0$ as usual. One promotes the spinors to *local* Lorentz spinors, and changes the derivative to the Lorentz-covariant derivative accordingly. Thus, one obtains

$$\mathcal{L}'_\psi=\bar{\psi}i\gamma^\mu\left(\partial_\mu-i\Omega_\mu\right)\psi\,. \tag{4.4.17}$$

Using the transformations (4.4.11) and (4.4.13), one finds for the transformation of the Lagrangian

$$\begin{aligned}\bar{\psi}\gamma^a E^\mu{}_a D_\mu\psi&\to\bar{\psi}D^{-1}(\Lambda)\gamma^a\Lambda_a{}^bE^\mu{}_bD(\Lambda)D_\mu\psi\\&=\bar{\psi}\left[D^{-1}(\Lambda)\gamma^aD(\Lambda)\right]\Lambda_a{}^bE^\mu{}_bD_\mu\psi\\&=\bar{\psi}\gamma^aE^\mu{}_aD_\mu\psi\,,\end{aligned} \tag{4.4.18}$$

where we used in the last line that

$$D(\Lambda)\gamma^a D^{-1}(\Lambda) = \Lambda^a{}_b \gamma^b \,. \tag{4.4.19}$$

Therefore, the Lagrangian is indeed a local Lorentz scalar. Under GCT transformations, the spinor transforms like a scalar field[4]

$$\psi(x) \to \psi'(x) = U(x)\psi(x)\,, \tag{4.4.20}$$

while the spin-connection and vierbein transform like tensors according to their index position

$$\begin{aligned} \Omega_\mu &\to U(x) U_\mu{}^\nu(x) \Omega_\nu(x)\,, \\ E^\mu{}_a(x) &\to U(x) U^\mu{}_\nu(x) E^\nu{}_a(x)\,. \end{aligned} \tag{4.4.21}$$

Since all GCT indices are contracted, the Lagrangian itself transforms as a scalar field, and invariance follows from integration over $d^4x\sqrt{-g}$. Consequently, one defines the Dirac action in curved space-times as

$$S_\psi = \int d^4x \sqrt{-g}\, \bar{\psi} i \gamma^\mu \left(\partial_\mu - i\Omega_\mu\right)\psi \,. \tag{4.4.22}$$

This action is manifestly LLT and GCT invariant and describes a spin-$\frac{1}{2}$ field in curved space-time. Note that in this approach, the gamma matrices γ^a fulfil the standard Clifford algebra in the local inertial frame,

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}\,. \tag{4.4.23}$$

4.4.2 Weak-field Expansion

Next, we investigate how the vierbein and spin-connections are determined in the weak-field expansion. We consider the expansion around flat space, where the metric tensor is given by

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{4.4.24}$$

To determine the vierbein, we make the ansatz

$$e_\mu{}^a = \delta^\mu_a + \varphi_\mu{}^a\,, \tag{4.4.25}$$

where $\varphi_\mu{}^a$ denotes the vierbein fluctuation. From the metric condition (4.4.2)

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{4.4.26}$$

one can determine that the fluctuation φ satisfies to first order in h

$$\varphi_{\mu\nu} + \varphi_{\nu\mu} = h_{\mu\nu} + \mathcal{O}(h^2)\,, \tag{4.4.27}$$

where $\varphi_{\mu\nu} = \eta_{\mu\alpha}\delta^\alpha_a \varphi^a{}_\nu$. Here, one notices a subtlety: in the weak field expansion, we seem to mix the notions of LLT indices and GCT indices. Furthermore, the condition (4.4.27) only constrains the symmetric part of φ, which corresponds to 10 degrees of freedom. The remaining 6 antisymmetric components are not constrained. These components are of course linked to the local Lorentz symmetry. To see this directly, consider a parameterisation of the vierbein as

$$e_\mu{}^a(x) = \hat{\Lambda}^a{}_b(x)\hat{e}_\mu{}^b(x)\,, \tag{4.4.28}$$

where the hatted vierbein is gauge-fixed such that it is symmetric, i.e. one defines the 6 parameters of the local Lorentz transformation so that they precisely cancel out any antisymmetric

[4] We adopt the active point of view for the GCT transformations.

component of the vierbein. Then, from the metric condition (4.4.27), one immediately determines the symmetric vierbein in terms of the metric fluctuation $h_{\mu\nu}$ as

$$\hat{e}_\mu{}^a = \delta^a_\mu + \frac{1}{2}h^a_\mu - \frac{1}{8}h_{\mu\nu}h^{\nu a} + \mathcal{O}(h^3)\,. \tag{4.4.29}$$

It is important to stress that the result (4.4.29) only holds in "symmetric" gauge. If one fixes a different gauge, one can determine the non-vanishing antisymmetric components in terms of $h_{\mu\nu}$, and the weak-field expansion will be different. It is relevant to discuss this detail for one subtlety: unlike in gauge theory, where fixing different gauges has no effect for e.g. matching computations, the LLT gauge *has* an effect, since the asymptotic spinor states are also modified by the gauge choice.

To see this, insert the relation (4.4.28) into the action (4.4.22) to obtain

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\gamma^a E^\mu{}_a D_\mu\psi \\ &= \bar{\psi}\gamma^a \hat{\Lambda}_a{}^b \hat{E}^\mu{}_b D_\mu\psi \\ &= \bar{\psi}D^{-1}(\hat{\Lambda})\gamma^b \hat{E}^\mu{}_b D(\hat{\Lambda}) D_\mu\psi \\ &= \overline{\left(D(\hat{\Lambda})\psi\right)}\gamma^a \hat{E}^\mu{}_a D_\mu \left(D(\hat{\Lambda})\psi\right)\,.\end{aligned} \tag{4.4.30}$$

One notices that the relevant spinor in this gauge is the gauge-fixed spinor $D(\hat{\Lambda})\psi$. This is the object that one should use as asymptotic states, i.e. we define

$$\psi(x) = D^{-1}(\hat{\Lambda})\hat{\psi}(x)\,, \tag{4.4.31}$$

and the spinor $\hat{\psi}(x)$ has the standard mode decomposition

$$\hat{\psi}(x) = \sum_s \int d^3p\, \left(\hat{u}_s(p)e^{ipx}a(p)_s + v_s(p)e^{-ipx}a^\dagger(p)_s\right)\,, \tag{4.4.32}$$

with polarisations

$$\hat{u}_{\frac{1}{2}}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\,. \tag{4.4.33}$$

The important point is that these external spinors are defined only in the reference frame where the vierbein takes the form $\hat{e}_\mu{}^a$. If one chooses a different vierbein, it is related to the hatted one by a LLT $\Lambda(x)$. This means that we change the local reference frame. Thus, following a reasoning similar to (4.4.30), the spinors in this new frame are related to the hatted ones as

$$\psi(x) = D(\Lambda)\hat{\psi}\,, \tag{4.4.34}$$

which also affects the external polarisations $u_s(p)$. Therefore, if one performs matching computations for spinors in curved space-times, one has to ensure that one either fixes the same LLT gauge in both the full theory and the EFT or that one accounts for the different spinors. From now on, we will always fix the symmetric gauge unless otherwise specified, as this is the simplest and most natural gauge choice.

To summarise, we give the weak-field expansion of the Dirac action and all relevant objects in symmetric gauge. The vierbein and its inverse are given by

$$e^a{}_\mu = \delta^a_\mu + \frac{1}{2}h^a_\mu - \frac{1}{8}h^{a\beta}h_{\beta\mu} + \frac{1}{16}h^{a\beta}h_{\beta\nu}h^\nu_\mu + \mathcal{O}(h^4)\,, \tag{4.4.35}$$

$$E^\mu{}_a = \delta^\mu_a - \frac{1}{2}h^\mu_a + \frac{3}{8}h^{\mu\alpha}h_{\alpha a} - \frac{5}{16}h_{a\beta}h^{\beta\nu}h^\mu_\nu + \mathcal{O}(h^4)\,. \tag{4.4.36}$$

The spin-connection is determined from (4.4.15) to be

$$[\omega_\mu]_{ab} = -\,\partial_{[a}h_{b]\mu} - \frac{1}{2}\Big(h^\nu_{[a}\partial_{b]}h_{\mu\nu} - h^\nu_{[a}\partial_\nu h_{b]\mu} + \frac{1}{2}h^\nu_{[a}\partial_\mu h_{b]\nu}\Big) \tag{4.4.37}$$

$$+\,\frac{1}{4}\Big(h_{\nu[a}h^{\nu\lambda}\partial_\mu h_{b]\lambda} + \frac{3}{2}h_{\nu[a}h^{\nu\lambda}\partial_{b]}h_{\mu\lambda} - \frac{3}{2}h_{\nu[a}h^{\nu\lambda}\partial_\lambda h_{b]\mu} + h_{\nu[a}h_{\lambda b]}\partial^\lambda h^\nu_\mu\Big) + \mathcal{O}(h^4)\,, \tag{4.4.38}$$

where the antisymmetrisation only acts on *Latin* indices. Since we fix symmetric gauge, we can freely change between Latin and Greek indices, and keep the Latin ones only for clarity. The weak-field Dirac Lagrangian is then computed as

$$\mathcal{L} = \mathcal{L}^{(0)} + \kappa\mathcal{L}^{(1)} + \mathcal{O}(\kappa^2), \tag{4.4.39}$$

where the individual terms are given by

$$\mathcal{L}^{(0)} = i\bar{\psi}\gamma^\mu\partial_\mu\psi\,, \tag{4.4.40}$$

$$\mathcal{L}^{(1)} = \frac{i}{4}\bar{\psi}\gamma^\mu\left\{h^\alpha{}_\alpha, \partial_\mu\right\}\psi - \frac{i}{4}\bar{\psi}\gamma^a\left\{h^\mu{}_a, \partial_\mu\right\}\psi + \frac{1}{2}\bar{\psi}\left\{\gamma^\mu, \Omega_\mu\right\}\psi\,. \tag{4.4.41}$$

Here, $\{\cdot,\cdot\}$ denotes the anticommutator.

4.5 QCD and Gravity: a Comparison

At this point, it is instructive to compare gauge theory and gravity, to draw some conclusions on how the EFT construction might differ in the gravitational case.

At first glance, there are four main differences:

(i) The full theory considered in gravity, defined by the action (4.1.9), is itself an effective low-energy theory. Contrast this with Yang-Mills, where the full theory is a UV-complete, renormalisable theory.

(ii) The graviton $h_{\mu\nu}$ is defined as the fluctuation around a non-vanishing background configuration $\bar{g}_{\mu\nu}$. Therefore, one has to perform a weak-field expansion of the original action (4.1.9), and the full theory is defined as an infinite series in κ. This also affects the gauge transformations, which must be truncated. At this stage, the weak-field expansion is not motivated by the SCET power-counting, but by a small curvature expansion. In gauge theory, Yang-Mills does not need to be expanded around a background configuration.[5]

(iii) Yang-Mills features purely internal gauge transformations of some non-Abelian group. Therefore, these gauge transformations are not affected by the SCET kinematics, and generators have no scaling in λ, $T^a \sim \mathcal{O}(\lambda^0)$. In gravity, however, the gauge symmetry consists of local translations. These translations are generated by momenta P^μ, which have non-trivial scaling in λ. Moreover, soft and collinear momenta have different scaling. Therefore, soft and collinear gauge transformations *differ* in their power-counting. In gravity, it is therefore inevitable that gauge transformations mix different orders in λ.

(iv) In QCD, we encountered a large component $n_+A_c \sim \lambda^0$. If uncontrolled, this would render the power-counting meaningless, since there would not be a finite operator basis. The introduction of a collinear Wilson and gauge-invariant building blocks alleviates this problem, and only objects corresponding to the physical transverse gluons appear in the operator basis. In gravity, the situation is even worse: looking at the two-point function (4.3.8), one can anticipate $h_{++} \sim \lambda^{-1}$, one component is power-enhanced! This component must be controlled, otherwise one can enhance subleading contributions by adding arbitrarily many h_{++} to it.

[5]Unless one wants to consider more exotic situations, like instanton background.

Let us expand on these points in more detail. The first point, the full theory being an effective theory, is relevant for the high-energy behaviour of quantum gravity, and one would expect that this should not have a great impact on the soft and collinear limits. Indeed, this problem is more of a conceptual one: if one considers only the leading term R, i.e. pure Einstein-Hilbert, one can perform the weak-field expansion to arbitrary order in κ, and construct SCET to any order in λ from this. In principle, the SCET construction is even more general, since the SCET Lagrangian is not renormalised. However, if one wants to compute some scattering process involving quantum corrections, that is, loops, one needs to modify the full theory to account for these loops. In the matching computation, this will also require a modification of the SCET Lagrangian, since these higher-order terms in R are now part of the full theory.

It is important to stress that this does not affect the underlying construction in any form. The soft-collinear effective theory is derived purely from gauge symmetry considerations, and this gauge symmetry is not affected by the introduction of higher-order terms. Therefore, one only needs to specify the full theory, i.e. the action (4.1.9) up to some loop order, and can then use the construction presented below to construct the SCET up to this loop order. For the sake of simplicity, we will only consider the easiest scenario, Einstein-Hilbert (4.1.1) below. But the construction is completely general.

The second point is also not an actual problem. While the weak-field expansion is motivated by a small curvature expansion at the moment, we will see in the following section that the SCET power-counting also imposes a weak-field expansion. Intuitively this is clear since a purely-soft or purely-collinear theory must be equivalent to the full theory, which is a weak-field expansion. Therefore, our construction is in principle valid for any backgrounds and curvature regimes, as long as the kinematic situation is valid. In particular, this implies that the SCET Lagrangian is valid for trans-Planckian energies, as long as the momenta satisfy the SCET kinematics

Based on this explanation, we can also elaborate on point (iii). The λ-expansion is analogous to the weak-field expansion and leads to a non-linear realisation of the original diffeomorphism symmetry. Already in the full theory, we observed that an object cannot be diffeomorphism invariant and homogeneous in κ at the same time. That is, objects are only gauge-invariant order-by-order in the truncated diffeomorphisms.

Now, in SCET, we see that the gauge transformations are generated by an object that has non-trivial λ-counting. We also know that the expansion in λ must share similarities to a weak-field expansion and that the purely-soft and purely-collinear theories must be equivalent to the full theory. The only sensible conclusion is that again, the gauge transformations are realised non-linearly. Therefore, the gauge symmetry links different orders in λ. Again, any object cannot be homogeneous in λ, i.e. have a definite scaling λ^α, and gauge-invariant at the same time. But this feature is not surprising, it is directly inherited from the full theory, where it is present in a slightly different fashion.

However, the soft-collinear interactions are different from the full theory. Here, the guiding principle in QCD is to impose a homogeneous gauge transformation that respects the multipole expansion. This led to the discovery of the homogeneous background field $n_- A_s(x_-)$, and we could construct the soft-collinear interactions by formulating the theory with respect to this background field. In gravity, gauge transformations are inherently inhomogeneous in λ. Therefore, simply imposing homogeneous transformations does not lead to a sensible result, since we expect the "proper" EFT gauge symmetry to *not be homogeneous*, precisely because the generators are not. Therefore, we have to carefully understand what it means to "respect the multipole expansion" in the gravitational context.

Finally, point (iv), which seems to be the most distressing one, is surprisingly the point that is closest to the gauge theory situation. One can define analogues of the Wilson lines W_c used in QCD to control $n_+ A_c$. In gravity, these Wilson lines control the components $h_{+\mu}$. Recall that collinear momenta scale as $(n_+ P, P_\perp, n_- P) \sim (1, \lambda, \lambda^2)$. It turns out that the enhanced component h_{++} always comes with the suppressed momentum $n_- P \sim \lambda^2$ to yield a contribution

$h_{++}n_-P \sim \lambda$. This can be made manifest by constructing the collinear Wilson line, and one can verify explicitly that it is not possible to power-enhance operators. This Wilson line is then an object of the form $W_c = 1 + \mathcal{O}(\lambda)$. One can understand this fact intuitively by recalling that gravity does not have any collinear singularities. Indeed, this Wilson line shows that one cannot construct gauge-invariant leading-power interactions between collinear matter (or gravitons) with collinear gravitons. Surprisingly, the one aspect that seems the most different from gauge theory is actually treated formally in the exact same fashion. It is the EFT power-counting of the gauge charges that leads to a very different physical situation, but not the underlying formalism.

Besides these differences, we will see that the construction of the soft-collinear interactions proceeds along remarkably similar lines as in the gauge theory situation. Indeed, we will find that at the formal level, the construction is identical, once the correct Wilson lines and redefinitions are identified. Furthermore, most of the intuition developed for gauge theory still carries through in the gravitational situation.

Chapter 5

Collinear Gravity

As a first step, we introduce the purely-collinear theory. This means that we only consider the collinear modes, expanded around a trivial background, and do not introduce soft modes. This theory considered in isolation is of course quite trivial and should be equivalent to the full theory. Regardless, already in this simple theory, we can make contact with some subtleties of SCET gravity. First, we will see that the graviton field has a large component and one that is even power-enhanced. To control these, the analogues of gravitational Wilson lines are motivated, constructed, and introduced in the simpler setting of the purely-collinear theory. Then, we make contact with the first iteration of gauge-invariant building blocks. In the end, we give the purely-collinear Lagrangians for the matter theory and Einstein-Hilbert and verify that they are indeed equivalent to the full theory.

5.1 Power-counting and Field Content

We consider the minimally-coupled scalar field (4.3.1) and the Einstein-Hilbert action (4.1.1), where we use only a single collinear mode. For the scalar field, this is trivial, $\varphi_{\text{full}} = \varphi_c$, while for the graviton this implies[1]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{5.1.1}$$

For each collinear direction, we define the light-like reference vectors $n_{\pm}^{\mu}$ as usual and decompose the collinear momentum in this basis with the scaling (2.1.3), which we restate here $(n_+p\,, p_\perp\,, n_-p) \sim (1\,, \lambda\,, \lambda^2)\,.$

Inserting the decomposition (5.1.1) in the Einstein-Hilbert action (4.1.1) leads to an infinite series in $h_{\mu\nu}$. Recall that we also introduce a gauge-fixing term as explained in (4.3.7). At the moment, we do not assume any weak-field expansion. From the bilinear terms of this theory, one determines the two-point function (4.3.8), given by

$$\langle 0|T(h_{\mu\nu}(x)h_{\alpha\beta}(y))|0\rangle = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2+i0}\left(P_{\mu\nu,\alpha\beta} + \frac{1-b}{b}S_{\mu\nu,\alpha\beta}\right), \tag{5.1.2}$$

where

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right), \tag{5.1.3}$$

$$S_{\mu\nu,\alpha\beta} = \frac{1}{2p^2}\left(\eta_{\mu\alpha}p_\nu p_\beta + \eta_{\mu\beta}p_\nu p_\alpha + p_\mu p_\alpha\eta_{\nu\beta} + p_\mu p_\beta\eta_{\nu\alpha}\right). \tag{5.1.4}$$

Inserting the collinear scaling, notice that $P_{\mu\nu,\alpha\beta} \sim 1$ always, since it does not depend on momenta. The other combination, $S_{\mu\nu,\alpha\beta}$ has non-trivial λ-scaling. For example, for the $++$ and $\perp\perp$ modes, one obtains [43]

$$\begin{aligned} S_{++,++} &\sim \frac{1}{\lambda^2}\,, \quad P_{++,++} = 0\,, \\ S_{\perp\perp,\perp\perp} &\sim 1\,, \quad P_{\perp\perp,\perp\perp} \sim 1\,. \end{aligned} \tag{5.1.5}$$

[1] We now work in units where $\kappa = 1$, and $h_{\mu\nu}$ denotes the collinear fluctuation from here.

Then, the scaling of these modes is easily determined to be $h_{++} \sim \lambda^{-1}$, $h_{\perp\perp} \sim \lambda$. In general, one finds the scaling

$$h_{\mu\nu} \sim \frac{p^\mu p^\nu}{\lambda}\,, \tag{5.1.6}$$

which implies $h \sim \lambda$ for the trace. The scalar field scales as $\varphi_c \sim \lambda$ like before.

Notice that $h_{++} \sim \lambda^{-1}$ is power-counting enhanced, in addition to the large $h_{+\perp} \sim \lambda^0$. This enhanced component is a new feature that was not present in gauge theory. However, there are two things to observe. First, the full theory does not feature any collinear divergences. Therefore, when viewed in isolation, the collinear Lagrangian cannot have leading-power interactions between collinear gravitons and matter fields. Despite the super-leading nature of h_{++}, any possible contraction in the Lagrangian must be suppressed in λ. Indeed, any contraction of $h_{\mu\nu}$ with a collinear vector from *the same sector* yields power-suppression, since we have

$$h^{\mu\nu}V_\nu = \frac{1}{2}(h_+^\mu V_- + h_-^\mu V_+) + h_\perp^\mu V_\perp \sim \lambda V^\mu\,, \tag{5.1.7}$$

and thus the expansion in λ agrees with the weak-field expansion in the purely-collinear theory.

However, this argument does not work for N-jet operators, which feature multiple collinear directions. Here, any contraction between an i-collinear graviton and a j-collinear vector has the property

$$h_i^{\mu\nu}V_{j\nu} = \frac{n_{i-}^\mu}{2}\frac{n_{i-}n_{j-}}{4}h_{+_i+_i}V_{+_j} + \mathcal{O}(\lambda^0) \sim \mathcal{O}(\lambda^{-1})\,, \tag{5.1.8}$$

which counts as $\mathcal{O}(\lambda^{-1})$. Therefore, if one were to add h_{++} to a suppressed operator, its power-counting would *reduce* and it would contribute at an earlier order than anticipated. This is a serious problem and cannot be a feature of a sensible EFT. Moreover, on the physical side, such behaviour would cause collinear divergences, which are known to be absent in gravity [1,60].

Notice the analogy to QCD: the collinear gluon scales like a collinear momentum, with large component $n_+A_c \sim 1$. In the Lagrangian, however, derivatives only enter in a combination like $\partial^2 \sim \lambda^2$, so each large n_+A_c always comes with a small $n_-\partial$ or n_-A_c, so that the combination scales as λ^2. In the gravitational Lagrangian, we observe the same feature. The power-counting enhanced component always comes with a suppressed one.

In the N-jet, however, one could in principle add arbitrarily many n_+A_c to any given operator. However, we noticed that $n_+A_c = 0$ in light-cone gauge, and introduced collinear semi-infinite Wilson lines W_c that moved the fields to this light-cone gauge. This definition gave rise to gauge-invariant building blocks, which correspond to the physical polarisations of the gluon. In turn, the couplings of n_+A_c are controlled by this Wilson line.

In gravity, we use the same intuition: first, notice that gravitational light-cone gauge can be expressed as the condition $h_{+\mu} = 0$, thereby removing both the power-enhanced as well as the $\mathcal{O}(1)$ component. The idea is now straightforward: Using the gravitational gauge symmetry, we construct the analogues of the collinear Wilson lines, defining gauge-invariant building blocks. Then, when employing only these building blocks in the N-jet operator basis, one can never add explicit h_{++} or $h_{+\perp}$ to the N-jet, and the power-counting is saved.

5.2 Gauge Symmetry

In this section, we focus on the gauge symmetry of the collinear sector. They are inherited from the full theory, which is itself a weak-field expansion. Here, the diffeomorphisms are truncated at some order in κ. We already know that in the purely-collinear theory, the expansion in the parameter λ is equivalent to the weak-field expansion in κ, since any contraction of $h_{\mu\nu}$ with a collinear vector (of the same sector) is power-suppressed. Therefore, power-counting alone will allow us to truncate the diffeomorphisms at some order in λ.

To define the λ-scaling of the gauge-parameter ε, we impose a homogeneous linear transformation for the collinear graviton

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu \sim \mathcal{O}(h_{\mu\nu}) \,. \tag{5.2.1}$$

This leads to the scaling

$$n_+\varepsilon \sim \frac{1}{\lambda} \,, \quad n_-\varepsilon \sim \lambda \,, \quad \varepsilon^{\mu_\perp} \sim 1 \,. \tag{5.2.2}$$

When comparing this to the scaling of collinear coordinates

$$n_+x \sim \frac{1}{\lambda^2} \,, \quad n_-x \sim 1 \,, \quad x_\perp^\alpha \sim \frac{1}{\lambda} \,, \tag{5.2.3}$$

notice that ε^μ scales as λx^μ, consistent with the notion of a "small" translation, i.e. power-suppressed compared to x^μ. However, this λ-scaling also means that the gauge transformation mixes different orders in λ, and must be truncated if we consider the theory only to some finite order. For example, the scalar field transforms as (4.2.6)

$$\varphi_c \to \varphi_c - \underbrace{\varepsilon^\alpha \partial_\alpha \varphi_c}_{\sim\lambda\varphi_c} + \frac{1}{2} \underbrace{\varepsilon^\alpha \varepsilon^\beta \partial_\alpha \partial_\beta \varphi_c}_{\sim\lambda^2\varphi_c} + \mathcal{O}(\lambda^3 \varphi_c) \,, \tag{5.2.4}$$

and we see that the λ-scaling indeed agrees with the κ-scaling encountered previously. The collinear graviton inherits the non-linear transformation

$$h_{\mu\nu} \to \left[U_c \left(U_{c\mu}{}^\alpha U_{c\nu}{}^\beta (\eta_{\alpha\beta} + h_{\alpha\beta}) \right) \right] - \eta_{\mu\nu} \,. \tag{5.2.5}$$

Expanded to second order in λ, it reads

$$\begin{aligned} h'_{\mu\nu} = h_{\mu\nu} &- \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu - \partial_\mu \varepsilon^\alpha h_{\alpha\nu} - \partial_\nu \varepsilon^\alpha h_{\alpha\mu} - \varepsilon^\alpha \partial_\alpha h_{\mu\nu} \\ &+ \partial_\mu \varepsilon^\alpha \partial_\alpha \varepsilon_\nu + \partial_\nu \varepsilon^\alpha \partial_\alpha \varepsilon_\mu + \partial_\mu \varepsilon_\alpha \partial_\nu \varepsilon^\alpha + \varepsilon^\alpha \partial_\alpha (\partial_\mu \varepsilon_\nu + \partial_\mu \varepsilon_\nu) + \mathcal{O}(\lambda^3) \,, \end{aligned} \tag{5.2.6}$$

which is also completely equivalent to the transformation in the weak-field expansion.

Again, notice that since the gauge transformation mixes different orders in λ, one cannot construct an object that is simultaneously gauge-invariant and homogeneous in λ. We have to make a choice on which feature to prioritise.

5.3 Collinear Wilson Line

In QCD, the collinear Wilson line W_c was constructed to control n_+A_c to all orders in the operator basis. In gravity, we wish to find a similar object. Unfortunately, we do not have a notion of parallel transport for the coordinate transformations and the Lie derivative. Therefore, there is no standard concept of a Wilson line we can employ.

Instead, we take the explicit route: we compute an inverse gauge transformation that renders the fields gauge-invariant. The guiding principle is that this gauge transformation should move an arbitrary field configuration to light-cone gauge.

Since we denote the gauge transformation (which is an *inverse* translation) by $U(x)$, we define for a local translation

$$y = x + \theta_c[h] \tag{5.3.1}$$

the inverse gauge transformation as the translation

$$W_c^{-1} = T_{\theta_c[h]} \,. \tag{5.3.2}$$

In addition, we denote the Jacobi matrices as

$$W^\mu{}_\alpha = \frac{\partial y^\mu}{\partial x^\alpha} \,, \quad W_\mu{}^\alpha = \frac{\partial x^\alpha}{\partial y^\mu} \,. \tag{5.3.3}$$

Here, $\theta_c[h]$ is a parameter that is to be determined.

Intuitively, the new coordinate (5.3.1) corresponds to the choice of a new reference frame where we impose light-cone gauge $h_{\mu+}=0$. The metric tensor in this new gauge is given by[2]

$$\eta_{\mu\nu}+\mathfrak{h}_{\mu\nu}(x)=W^{\alpha}{}_{\mu}W^{\beta}{}_{\nu}\,[W_c^{-1}g_{\alpha\beta}(x)]\,, \tag{5.3.4}$$

where $g_{\alpha\beta}(x)=\eta_{\alpha\beta}+h_{\alpha\beta}(x)$. Gauge transformations are inhomogeneous in λ, therefore we anticipate that the parameter θ_c can be determined order by order in λ, and we introduce

$$\theta_c^{\mu}=\theta_c^{\mu(0)}+\theta_c^{\mu(1)}+\dots\,, \tag{5.3.5}$$

where the superscript denotes the λ-counting relative to the leading term, which is denoted as $\theta^{(0}$ for simplicity. Expanding the translation operator

$$W_c^{-1}=T_{\theta_c}=1+\theta_c^{\alpha}\partial_\alpha+\frac{1}{2}\theta_c^{\alpha}\theta_c^{\beta}\partial_\alpha\partial_\beta+\mathcal{O}(\theta_c^3)\,, \tag{5.3.6}$$

and inserting this in (5.3.4), one obtains the non-linear equation

$$\mathfrak{h}_{\mu\nu}=h_{\mu\nu}+\partial_\mu\theta_{c\nu}+\partial_\nu\theta_{c\mu}+\partial_\mu\theta_c^{\alpha}h_{\alpha\nu}+\partial_\nu\theta_c^{\alpha}h_{\alpha\mu}+\theta_c^{\alpha}\partial_\alpha h_{\mu\nu}+\partial_\mu\theta_c^{\alpha}\partial_\nu\theta_{c\alpha}+\mathcal{O}(\theta_c^3,\theta_c^2h)\,. \tag{5.3.7}$$

To determine θ_c, impose $\mathfrak{h}_{\mu+}=0$ and work order-by-order in λ. At leading-power, since the gauge transformation of $h_{\mu\nu}$ is defined to be homogeneous, one only needs to consider

$$\mathfrak{h}_{\mu\nu}=h_{\mu\nu}+\partial_\mu\theta_{c\nu}^{(0}+\partial_\nu\theta_{c\mu}^{(0}+\mathcal{O}(\lambda^2)\,. \tag{5.3.8}$$

For $\mathfrak{h}_{++}=0$, one finds

$$0=h_{++}+2\partial_+\theta_{c+}^{(0)}\,, \tag{5.3.9}$$

which yields[3]

$$\theta_{c+}^{(0)}=-\frac{1}{2}\frac{1}{n_+\partial}h_{++}\,. \tag{5.3.11}$$

Inserting this in the condition for $h_{+\mu}$, one finds

$$0=h_{+\mu}+\partial_+\theta_{c\mu}^{(0)}+\partial_\mu\theta_{c+}^{(0)}\,, \tag{5.3.12}$$

which is solved by

$$\theta_{c\mu}^{(0)}=-\frac{1}{n_+\partial}\left(h_{\mu+}-\frac{1}{2}\frac{\partial_\mu}{n_+\partial}h_{++}\right). \tag{5.3.13}$$

One order higher, the non-linear terms become relevant and the computation becomes slightly more involved. Regardless, one can still first compute θ_{c+} from $\mathfrak{h}_{++}=0$ and then insert this in $\mathfrak{h}_{+\mu}=0$. This results in

$$\theta_{c+}^{(1)}=-\frac{1}{2}\frac{1}{n_+\partial}\left(2n_+\partial\theta_c^{(0)\alpha}h_{\alpha+}+\theta_c^{(0)\alpha}\partial_\alpha h_{++}+n_+\partial\theta_c^{(0)\alpha}n_+\partial\theta_{c\alpha}^{(0)}\right), \tag{5.3.14}$$

$$\begin{aligned}\theta_{c\mu}^{(1)}=-\frac{1}{n_+\partial}\bigg(&\partial_\mu\theta_{c+}^{(1)}+\partial_\mu\theta_c^{(0)\alpha}h_{\alpha+}+n_+\partial\theta_c^{(0)\alpha}h_{\alpha\mu}+\theta_c^{(0)\alpha}\partial_\alpha h_{\mu+}\\&+\partial_\mu\theta_c^{(0)\alpha}n_+\partial\theta_{c\alpha}^{(0)}\bigg)\,.\end{aligned} \tag{5.3.15}$$

[2] Note the interchanged order compared to (4.2.10), since we consider an inverse gauge transformation.

[3] Recall that the inverse of $in_+\partial$ acting on a function $f(x^\mu)$ is defined as

$$\frac{1}{in_+\partial+i\epsilon}f(x^\mu)=-i\int_{-\infty}^{0}ds\,f(x^\mu+sn_+^\mu)\,. \tag{5.3.10}$$

At this point, it is instructive to explicitly compute $\mathfrak{h}_{\mu\nu}$ to the first order. Here, we find

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \frac{\partial_\mu}{n_+\partial}\left(h_{\nu+} - \frac{1}{2}\frac{\partial_\nu}{n_+\partial}h_{++}\right) - \frac{\partial_\nu}{n_+\partial}\left(h_{\mu+} - \frac{1}{2}\frac{\partial_\mu}{n_+\partial}h_{++}\right) + \mathcal{O}(\lambda h_{\mu\nu})\,. \tag{5.3.16}$$

Comparing this to QCD, we find that this looks remarkably similar to the gauge-invariant collinear gluon field (3.3.84),

$$\mathcal{A}_{c\mu_\perp} = \frac{1}{g}W_c^\dagger\left[iD_{c\mu_\perp}W_c\right] = A_{c\mu_\perp} - \frac{\partial_{\mu_\perp}}{n_+\partial}A_{c+} + \mathcal{O}(gA_{c\mu_\perp})\,, \tag{5.3.17}$$

which indicates that the W_c^{-1} we construced is indeed the correct object.

From the explicit result (5.3.13) – (5.3.15), we can determine its gauge transformation to be

$$\theta^\mu \to \theta^{(0)\mu} + \theta^{(1)\mu} + \varepsilon^\mu + \theta^{(0)\alpha}\left[\partial_\alpha\varepsilon^\mu\right] + \mathcal{O}(\lambda^2)\,. \tag{5.3.18}$$

Inserting this in the translation operator W_c^{-1}, we find the transformation

$$\begin{aligned}
W_c^{-1} &= 1 + \theta^\alpha\partial_\alpha + \frac{1}{2}\theta^\alpha\theta^\beta\partial_\alpha\partial_\beta + \ldots \\
&\to 1 + \theta^\alpha\partial_\alpha + \varepsilon^\alpha\partial_\alpha + \theta^{(0)\beta}\partial_\beta\varepsilon^\alpha\partial_\alpha + \frac{1}{2}\theta^\alpha\theta^\beta\partial_\alpha\partial_\beta + \varepsilon^\alpha\theta^\beta\partial_\alpha\partial_\beta + \ldots \\
&= \left(1 + \theta^\alpha\partial_\alpha + \frac{1}{2}\theta^\alpha\theta^\beta\partial_\alpha\partial_\beta + \ldots\right)\left(1 + \varepsilon^\alpha\partial_\alpha + \ldots\right) \\
&= W_c^{-1}U^{-1}(x)\,.
\end{aligned} \tag{5.3.19}$$

Here, we implicitly assumed that gauge transformations vanish at spatial $-\infty$, which is used when explicitly evaluating the inverse derivatives in the definition of θ.

Therefore, we find that the gravitational Wilson line transforms order-by-order as

$$W_c^{-1} \to W_c^{-1}U^{-1}(x)\,, \tag{5.3.20}$$

just like its gauge-theory counterpart. We can now employ this object in the same form as in QCD to construct gauge-invariant building blocks. Recall the scalar field transformation (4.2.5)

$$\varphi(x) \to \varphi'(x) = U(x)\varphi(x)\,, \tag{5.3.21}$$

where $U(x)$ is the inverse translation operator. Therefore, to render a field gauge-invariant, we apply W_c^{-1} like a translation to define

$$\chi_c = \left[W_c^{-1}\varphi\right]\,, \tag{5.3.22}$$

which is manifestly gauge-invariant. One can also explicitly verify that the metric tensor $\mathfrak{h}_{\mu\nu}$ defined in (5.3.4) is manifestly gauge-invariant. This also gives us a recipe to render any tensor field gauge-invariant: First, one applies W_c^{-1} to remove the scalar-like transformation. Then, for a tensor $T^\mu{}_\alpha$, one applies the Jacobi matrices as

$$\mathfrak{T}^\mu{}_\alpha = W_\nu{}^\mu W^\beta{}_\alpha\left[W_c^{-1}T^\nu{}_\beta(x)\right]\,. \tag{5.3.23}$$

The conceptual similarity to QCD is now immediate. In gauge theory, we employ $W_c^\dagger$ to fix light-cone gauge. This object depends on the dynamical field A_c via n_+A_c, and each field in the full theory is redefined according to its gauge transformation. In gravity, the corresponding object is given by the translation W_c^{-1}, which similarly depends on $h_{\mu\nu}$ via $h_{+\mu}$. To define gauge-invariant tensor fields, one applies the Wilson line according to the tensor representation.

There is an alternative way to derive this Wilson line, based on the geodesic equation. This approach was first presented in [61] and also used in [62]. It is presented in detail in the appendix of [47], including the proof that both parameters agree. In addition, similar to QCD in Section 3.3.4, it is possible to find this object by considering the emission of $h_{\mu+}$-polarisations. This is presented in the subsequent Section 5.5.

5.4 Manifestly Gauge-invariant Lagrangian

The purely-collinear Lagrangian is completely equivalent to the standard weak-field expansion since the λ-expansion immediately leads to the κ-expansion. However, we can express the Lagrangian in terms of the gauge-invariant building blocks. This yields

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_\mu \chi_c \partial^\mu \chi_c - \frac{\lambda_\varphi}{4!}\chi_c^4\,, \tag{5.4.1}$$

$$\mathcal{L}^{(1)} = -\frac{1}{2}\mathfrak{h}_{\mu\nu}(\partial^\mu\chi_c\partial^\nu\chi_c - \eta^{\mu\nu}\frac{1}{2}\partial_\alpha\chi_c\partial^\alpha\chi_c) - \frac{1}{2}\mathfrak{h}\frac{\lambda_\varphi}{4!}\chi_c^4\,, \tag{5.4.2}$$

$$\begin{aligned}\mathcal{L}^{(2)} = &\frac{1}{2}\Big(\mathfrak{h}^{\mu\alpha}\mathfrak{h}_\alpha^\nu - \frac{1}{2}\mathfrak{h}\mathfrak{h}^{\mu\nu} + \frac{1}{8}(\mathfrak{h}^2 - 2\mathfrak{h}^{\alpha\beta}\mathfrak{h}_{\alpha\beta})\eta^{\mu\nu}\Big)\partial_\mu\chi_c\partial_\nu\chi_c \\ &- \frac{1}{8}(\mathfrak{h}^2 - 2\mathfrak{h}^{\alpha\beta}\mathfrak{h}_{\alpha\beta})\frac{\lambda_\varphi}{4!}\chi_c^4\end{aligned} \tag{5.4.3}$$

up to $\mathcal{O}(\lambda^2)$. For brevity, have not yet expanded out the contractions. Note however, that e.g. for the first term in $\mathcal{L}^{(1)}$, one finds

$$\mathfrak{h}_{\mu\nu}\partial^\mu\chi_c\partial^\nu\chi_c = \mathfrak{h}_{\mu_\perp\nu_\perp}\partial_\perp^\mu\chi_c\partial_\perp^\nu\chi_c + \mathfrak{h}_{\mu_\perp -}\partial_\perp^\mu\chi_c n_+\partial\chi_c + \frac{1}{4}\mathfrak{h}_{--}n_+\partial\chi_c n_+\partial\chi_c\,, \tag{5.4.4}$$

since $\mathfrak{h}_{+\mu} = 0$. One can simplify this Lagrangian further by noting that the trace $\mathfrak{h}$ and $\mathfrak{h}_{-\mu}$ are subleading and can be eliminated by their equations of motion, which read up to $\mathcal{O}(\lambda)$)

$$\mathfrak{h} = \frac{1}{2}\Big(\mathfrak{h}_{\alpha_\perp\beta_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp} - \frac{1}{\partial_+^2}(\partial_+\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial_+\mathfrak{h}^{\alpha_\perp\beta_\perp})\Big)\,, \tag{5.4.5}$$

$$\begin{aligned}\mathfrak{h}_{-\mu_\perp} = &-2\frac{\partial^{\alpha_\perp}}{\partial_+}\mathfrak{h}_{\mu_\perp\alpha_\perp} + \Big(-\frac{\partial_{\mu_\perp}}{\partial_+^3}(\partial_+\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial_+\mathfrak{h}^{\alpha_\perp\beta_\perp}) \\ &+ \frac{1}{\partial_+^2}\Big(-2\partial_+^2\mathfrak{h}_{\mu_\perp\alpha_\perp}\frac{\partial_{\beta_\perp}}{\partial_+}\mathfrak{h}^{\alpha_\perp\beta_\perp} + 2\mathfrak{h}^{\alpha_\perp\beta_\perp}\partial_+\partial_{\alpha_\perp}\mathfrak{h}_{\mu_\perp\beta_\perp} + \partial_+\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial_{\mu_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp}\Big)\Big)\,.\end{aligned} \tag{5.4.6}$$

We do not perform this explicitly for the scalar Lagrangian, but only for the gravitational part. The Einstein-Hilbert Lagrangian is given up to $\mathcal{O}(\lambda)$ by

$$\mathcal{L}_{\rm EH}^{(0)} = \frac{1}{2}\partial_\mu\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial^\mu\mathfrak{h}^{\alpha_\perp\beta_\perp}\,, \tag{5.4.7}$$

$$\begin{aligned}\mathcal{L}_{\rm EH}^{(1)} = &\frac{1}{2}\Big(\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial_+^2\mathfrak{h}^{\alpha_\perp\beta_\perp}\frac{\partial_{\rho_\perp}\partial_{\sigma_\perp}}{\partial_+^2}\mathfrak{h}^{\rho_\perp\sigma_\perp} - 2\mathfrak{h}_{\alpha_\perp\beta_\perp}\partial_+\partial^{\rho_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp}\frac{\partial^{\sigma_\perp}}{\partial_+}\mathfrak{h}_{\rho_\perp\sigma_\perp} \\ &+ \mathfrak{h}_{\alpha_\perp\beta_\perp}\mathfrak{h}_{\rho_\perp\sigma_\perp}\partial^{\rho_\perp}\partial^{\sigma_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp} - 2\mathfrak{h}_{\alpha_\perp\sigma_\perp}\mathfrak{h}_{\beta_\perp\rho_\perp}\partial^{\rho_\perp}\partial^{\sigma_\perp}\mathfrak{h}^{\alpha_\perp\beta_\perp} \\ &- 4\mathfrak{h}_{\alpha_\perp\sigma_\perp}\partial_+\mathfrak{h}^{\alpha_\perp\beta_\perp}\frac{\partial^{\rho_\perp}\partial^{\sigma_\perp}}{\partial_+}\mathfrak{h}_{\rho_\perp\beta_\perp}\Big)\,.\end{aligned} \tag{5.4.8}$$

Note that $\mathfrak{h}_{\mu\nu}$ is not homogeneous in λ, but is instead defined order-by-order as $\mathfrak{h}_{\mu\nu} = \mathfrak{h}_{\mu\nu}^{(1)} + \mathfrak{h}_{\mu\nu}^{(2)} + \ldots$, where

$$\mathfrak{h}_{\mu\nu}^{(1)} = h_{\mu\nu} + \partial_\mu\theta_{c\nu}^{(0)} + \partial_\nu\theta_{c\mu}^{(0)}\,, \tag{5.4.9}$$

$$\mathfrak{h}_{\mu\nu}^{(2)} = \partial_\mu\theta_{c\nu}^{(1)} + \partial_\nu\theta_{c\mu}^{(1)} + \partial_\mu\theta_c^{(0)\alpha}h_{\alpha\nu} + \partial_\nu\theta_c^{(0)\alpha}h_{\alpha\mu} + \theta_c^{(0)\alpha}\partial_\alpha h_{\mu\nu} + \partial_\mu\theta_{c\alpha}^{(0)}\partial_\nu\theta_c^{(0)\alpha}\,, \tag{5.4.10}$$

and θ_c is defined according to (5.3.13) – (5.3.15). Each order is gauge-invariant under the truncated diffeomorphisms at that order, but only the full infinite series $\mathfrak{h}_{\mu\nu}$ is gauge-invariant under the full diffeomorphisms. Regardless, one can still consider the Lagrangian to be "homogeneous" in λ, in the sense that each term as a well-defined leading piece that determines its scaling in

λ. Furthermore, if one were to fix light-cone gauge, all subleading pieces would vanish, and the Lagrangian would be manifestly homogeneous. For practical computations, it is convenient to fix a standard covariant gauge, like de Donder gauge, and keep Wilson lines explicit if this is necessary.

Furthermore, these Lagrangians are expressed manifestly in terms of the physical graviton polarisations $\mathfrak{h}_{\perp\perp}$. They show that there are no leading-power collinear graviton interactions, and thus from Lagrangian interactions alone, it is not possible to obtain collinear divergences. In the N-jet operators, one also employs the collinear Wilson line to render the fields manifestly gauge-invariant. The leading and superleading graviton fields are contained in this collinear Wilson line, which takes the form $W_c \sim 1 + \mathcal{O}(\lambda)$, and there is no coupling to these components at leading power. Therefore, additional emission of these leading-power collinear gravitons is also suppressed at least by $\mathcal{O}(\lambda)$, and it is not possible to obtain collinear divergences.

Contrast this with QCD: here, the purely-collinear Lagrangian features leading-power interactions. In addition, the collinear Wilson lines W_c used in the N-jet operators, multiplying every field to yield a gauge-invariant building block, describe leading-power emissions of the large gluon components n_+A_c. Thus collinear emissions are not suppressed, and divergences are possible.

In summary, we found that the purely-collinear theory is completely equivalent to the standard weak-field expansion since the λ-expansion agrees with the κ-expansion. By imposing light-cone gauge, we constructed a collinear "Wilson line" W_c^{-1}, which is the precise analogue of the collinear Wilson line in gauge theory and has the same intuition as an inverse gauge transformation that moves a field configuration to light-cone gauge. Using this Wilson line, or light-cone gauge in general, we managed to write the Lagrangian in a form where only the physical polarisation $\mathfrak{h}_{\perp\perp}$ is present. The dangerous components h_{++} and $h_{+\perp}$ are controlled by the Wilson line. However, this Wilson line takes the form $W_c^{-1} \sim 1 + \mathcal{O}(\lambda)$, and one can conclude from this property that there are no collinear divergences in gravity.

5.5 Collinear Emission

In this section, we compute the process $\varphi_1\varphi_2 \to \varphi_3\varphi_4 h_1$, i.e. the emission of a 1-collinear graviton off a four-scalar scattering, and check that it is indeed reproduced by the Lagrangian. In addition, this serves as an explicit check that the collinear Wilson line indeed captures the emission of the large components $h_{\mu+}$, similar to the matching computed in Section 3.3.4.

For simplicity, we only consider the five diagrams corresponding to the scalar four-point interaction with graviton emission and not the ones where we have a graviton mediating the interaction.

5.5.1 Full-theory Computation

Non-radiative Amplitude

The non-radiative amplitude is simply given by the four-point vertex

$$M = -i\lambda_s \,. \tag{5.5.1}$$

The radiative amplitude consists of three different contributions: the emission directly from the scattering via the five-point $\varphi^4 h$-vertex, the emission from the leg of the same collinearity, and the emission from all other legs. We discuss these contributions in this order.

5-point Vertex

The 5-point vertex simply yields

$$M_5 = -\frac{i\kappa\lambda_s}{2} h \,, \tag{5.5.2}$$

where $h = h^{\alpha}{}_{\alpha}$ denotes the trace of the graviton polarisation tensor.

Emission off Leg 1

Here, we find the amplitude

$$
\begin{aligned}
M_1 &= -i\lambda_s \frac{-i}{(p_1+k)^2}\frac{i\kappa}{2}\Big(p_1^{\mu}(p_1+k)^{\nu} + p_1^{\nu}(p_1+k)^{\mu} - \eta^{\mu\nu}p_1\cdot(p_1+k)h_{\mu\nu} \\
&= -\frac{i\kappa\lambda_s}{2}\frac{1}{p_1\cdot k}(p^{\mu}p^{\nu} + p^{\mu}k^{\nu})h_{\mu\nu} + \frac{i\kappa\lambda_s}{4}h\,,
\end{aligned}
\tag{5.5.3}
$$

which is already homogeneous in λ and needs no further expansion.

Emission off Leg j

Next, consider the emission of the 1-collinear graviton with momentum k off a leg $j \neq 1$ with momentum p. In this case, the amplitude must be expanded in λ, as the intermediate propagator becomes off-shell and must be integrated out in the effective theory. We employ the relations

$$
\begin{aligned}
\frac{1}{p\cdot k} = \frac{4}{n_{1-}n_{j-}p_{j+}k_{1+}}\Big(1 &- \frac{2}{n_{1-}n_{j-}p_{j+}k_{1+}}(n_{1-}p_{j\perp}k_{1+} + n_{j-}k_{1\perp}p_{j+}) \\
&- \frac{1}{n_{1-}n_{j-}p_{j+}k_{1+}}(n_{1-}n_{j+}p_{j-}k_{1+} + n_{1+}n_{j-}p_{j+}k_{1-} + 4k_{1\perp}p_{j\perp}) \\
&+ (\frac{2}{n_{1-}n_{j-}p_{j+}k_{1+}})^2((n_{1-}p_{j\perp})^2 + 2n_{1-}p_{j\perp}n_{j-}k_{1\perp} + (n_{j-}k_{1\perp})^2)\Big) + \mathcal{O}(\lambda^3)\,,
\end{aligned}
\tag{5.5.4}
$$

as well as[4]

$$
\begin{aligned}
h_{\mu\nu}p^{\mu}p^{\nu} = (\frac{n_{1-}n_{j-}}{4})^2 h_{++}p_{j+}^2 &+ \frac{n_{1-}n_{j-}}{4}h_{++}p_{j+}n_{i-}p_{j\perp} + \frac{n_{1-}n_{j-}}{4}\frac{n_{1-}n_{j+}}{2}h_{++}p_{j+}p_{j-} \\
&+ (\frac{n_{1-}p_{j\perp}}{2})^2 h_{++} + \frac{n_{1-}n_{j-}}{2}h_{+a}p_{j\perp}^a p_{j+} + \frac{n_{1-}n_{j-}}{4}h_{+a}n_{j-}^a p_{j+}^2 \\
&+ \frac{n_{j-}^a}{2}h_{+a}p_{j+}n_{1-}p_{j\perp} + \frac{n_{1-}n_{j-}}{4}\frac{n_{1+}n_{j-}}{2}h_{+-}p_{j+}^2 + h_{ab}\frac{n_{j-}^a n_{j-}^b}{4}p_{j+}^2 + \mathcal{O}(\lambda^2)\,,
\end{aligned}
\tag{5.5.5}
$$

and

$$
h_{\mu\nu}p^{\mu}k^{\nu} = \frac{n_{1-}n_{j-}}{4}h_{+\nu}k^{\nu}p_{j+}\,. \tag{5.5.6}
$$

Inserting these expansions and keeping terms to $\mathcal{O}(\lambda)$, we find the amplitude

$$
\begin{aligned}
M_j = \frac{i\kappa\lambda_s}{4}h - \frac{i\kappa\lambda_s}{2}\Bigg(&k^{\nu}\frac{h_{+\nu}}{k_+} \\
&+ 2\Big(\frac{h_{++}}{k_+}\frac{n_{1-}^{\mu}}{2}\Big(\frac{n_{j-\mu}}{2}p_{j+} + p_{j\perp\mu} + \frac{n_{j+\mu}}{2}p_{j-}\Big) + \frac{h_{+a}}{k_+}\Big(\frac{n_{j-}^a}{2}p_{j+} + p_{j\perp}^a\Big) + \frac{h_{+-}}{k_+}\frac{n_{1+}n_{j-}}{4}p_{j+}\Big) \\
&- \frac{h_{++}}{(k_+)^2}\Big(k_+\frac{n_{1-}^{\mu}}{2}\Big(\frac{n_{j-\mu}}{2}p_{j+} + p_{j\perp\mu} + \frac{n_{j+\mu}}{2}p_{j-}\Big) + k_{\perp}^{\mu}\Big(\frac{n_{j-\mu}}{2}p_{j+} + p_{j\perp\mu}\Big) + k_-\frac{n_{1+}n_{j-}}{4}p_{j+}\Big) \\
&+ \frac{1}{n_{1-}n_{j-}p_{j+}k_+}n_{j-}^a n_{j-}^b p_{j+}^2\Big(h_{ab} - \Big(k_a\frac{h_{+b}}{k_+} - \frac{1}{2}k_ak_b\frac{h_{++}}{(k_+)^2}\Big) - \Big(k_b\frac{h_{+a}}{k_+} - \frac{1}{2}k_bk_a\frac{h_{++}}{(k_+)^2}\Big)\Big)\Bigg)\,.
\end{aligned}
\tag{5.5.7}
$$

When summing the diagrams for $j = 2, 3, 4$, we use momentum conservation

$$
p_1 + p_2 + p_3 + p_4 + k = 0\,, \tag{5.5.8}
$$

[4] Latin indices denote transverse components.

and find for the sum of the $j \neq 1$ diagrams the result

$$\sum_{j\neq 1} M_j = -\frac{i\kappa\lambda_s}{2}\left(-2\frac{h_{+\mu}}{k_+}p_1^\mu + \frac{h_{+\mu}}{k_+}k^\mu + \frac{h_{++}}{(k_+)^2}k\cdot p_1\right) + \frac{3}{4}i\kappa\lambda_s h$$
$$+ \frac{i\kappa\lambda_s}{2}\sum_j\left(\frac{n^a_{j-}n^b_{j-}p^2_{j+}}{n_{1-}n_{j-}p_{j+}k_+}\mathfrak{h}_{ab}\right) \tag{5.5.9}$$

The full-theory amplitude, expanded in λ is then given by

$$\begin{aligned} M &= -\frac{i\kappa\lambda_s}{2}\left(\frac{1}{p_1\cdot k}(p_1^\mu p_\nu^1 + p_1^\mu k^\nu)h_{\mu\nu} - h - 2\frac{h_{+\mu}}{k_+}p_1^\mu + k^\mu\frac{h_{+\mu}}{k_+} + k\cdot p\frac{h_{++}}{(k_+)^2}\right. \\ &\quad \left. + \sum_{j\neq 1}\left(\frac{n^a_{j-}n^b_{j-}p^2_{j+}}{n_{1-}n_{j-}p_{j+}k_+}\mathfrak{h}_{ab}\right)\right) \\ &= \frac{-i\kappa\lambda_s}{2}\left(\frac{1}{p_1\cdot k}\left((p_1^\mu p_1^\nu + p_1^\mu k^\nu)h_{\mu\nu} - p_1\cdot k p_1^\mu\frac{h_{+\mu}}{k_+} - p_1\cdot k p_1^\nu\frac{h_{+\nu}}{k_+} - p_1\cdot k k^\mu\frac{h_{+\mu}}{k_+}\right.\right. \\ &\quad \left.\left. + (p_1\cdot k)^2\frac{h_{++}}{(k_+)^2} - h + 2p_1\cdot k k^\mu\frac{h_{+\mu}}{k_+}\right) + \sum_{j\neq 1}\left(\frac{n^a_{j-}n^b_{j-}p^2_{j+}}{n_{1-}n_{j-}p_{j+}k_+}\mathfrak{h}_{ab}\right)\right) \\ &= -\frac{i\kappa\lambda_s}{2}\left(\frac{1}{p_1\cdot k}(p_1^\mu p_1^\nu + p_1^\mu k^\nu)\mathfrak{h}_{\mu\nu} - \mathfrak{h} + \sum_{j\neq 1}\left(\frac{n^a_{j-}n^b_{j-}p^2_{j+}}{n_{1-}n_{j-}p_{j+}k_+}\mathfrak{h}_{ab}\right)\right). \end{aligned} \tag{5.5.10}$$

As we can see, there is no direct appearance of $h_{+\mu}$ once we rewrite it in terms of the gauge-invariant combination $\mathfrak{h}$. This can be viewed as yet another explicit derivation of the collinear Wilson line.

Inspecting the result closer, notice that the first two terms come with the collinear propagator $\frac{1}{p_1\cdot k}$. This indicates that these terms are captured directly by the Lagrangian (5.4.1) – (5.4.3). The third and fourth term combine momenta of different collinear sectors and require an additional operator insertion of the gauge-invariant collinear graviton field $\mathfrak{h}$. Note that only the transverse components of $\mathfrak{h}$ appear in these terms.

At $\mathcal{O}(\lambda^2)$, the full amplitude receives the additional corrections

$$\begin{aligned} iM^{(2)} &= -\frac{i\kappa\lambda_s}{2}\sum_{j\neq 1}\left(\frac{n^a_{j-}p^b_{j\perp}}{n_{i-}n_{j-}}\frac{\mathfrak{h}_{ab}}{k_+} + \frac{n^a_{j-}n^\nu_{j-}}{n_{i-}n_{j-}}n_{i+\nu}\frac{\mathfrak{h}_{a-}}{k_+}p_{j+}\right. \\ &\quad \left. + \frac{2}{n_{i-}n_{j-}}\frac{n^a_{j-}k^\nu}{k_+}\mathfrak{h}_{a\nu} - \frac{2}{n_{i-}n_{j-}}\left(n_{i-}p_{j\perp} + \frac{n_{j-}k_\perp p_{j+}}{k_+}\right)\frac{n^a_{j-}n^b_{j-}}{n_{i-}n_{j-}}\frac{\mathfrak{h}_{ab}}{k_+}\right), \end{aligned} \tag{5.5.11}$$

which all take the form of a N-jet operator insertion. Note that here, $\mathfrak{h}$ is the linear combination, which reads

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - k_\mu\frac{h_{+\nu}}{k_+} - k_\nu\frac{h_{+\mu}}{k_+} + k_\mu k_\nu\frac{h_{++}}{k_+^2}\,. \tag{5.5.12}$$

This is to be expected since the full-theory amplitude is linear in h.

5.5.2 Matching

The non-radiative amplitude is matched to the 4-jet operator

$$\begin{aligned} \mathcal{J}_{\mathrm{nr}} &= \int \mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\widetilde{C}_{\mathrm{nr}}(t_1,t_2,t_3,t_4)\chi_1(t_1n_{1+})\chi_2(t_2n_{2+})\chi_3(t_3n_{3+})\chi_4(t_4n_{4+}) \\ &= \int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\, C_{\mathrm{nr}}(p_1,p_2,p_3,p_4)\tilde{\chi}_1(p_1)\tilde{\chi}_2(p_2)\tilde{\chi}_3(p_3)\tilde{\chi}_4(p_4)\,, \end{aligned} \tag{5.5.13}$$

where χ_i is the gauge-invariant building block (5.3.22) and the coefficient $C_{\rm nr}$ is determined by evaluating the matrix element

$$\begin{aligned}&\langle\varphi(q_1)\varphi(q_2)\varphi(q_3)\varphi(q_4)|\mathcal{J}|0\rangle = \\ &\qquad\int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4 C(p_1,p_2,p_3,p_4)\delta(q_1-p_1)\delta(q_2-p_2)\delta(q_3-p_3)\delta(q_4-p_4)\,.\end{aligned} \tag{5.5.14}$$

Here, we employ the Fourier-transformation of the operators $\tilde{\chi}(P_i)$, defined as

$$\tilde{J}^{A0}(P_i) = \int \mathrm{d}t\, e^{-itP_i} J^{A0}(t)\,, \tag{5.5.15}$$

which depend only on the large collinear momentum $n_{i+}P_i$. For brevity, we write these momenta simply as p_i. We find $C_{\rm nr} = -i\lambda$, i.e. the non-radiative matching coefficient is simply the stripped amplitude. The corresponding position-space coefficient is given by

$$\widetilde{C}(t_1,t_2,t_3,t_4) = \int \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\, e^{-it_1p_1-it_2p_2-it_3p_3-it_4p_4} C(p_1,p_2,p_3,p_4)\,, \tag{5.5.16}$$

which is computed to be

$$\widetilde{C}_{\rm nr}(t_1,t_2,t_3,t_4) = -i\lambda\delta(t_1)\delta(t_2)\delta(t_3)\delta(t_4)\,. \tag{5.5.17}$$

Next, we consider the time-ordered products of the subleading Lagrangian. Note that we have to re-express the building block $\mathfrak{h}_{\mu\nu}$ in terms of the original graviton field $h_{\mu\nu}$.

The contribution of the insertion of $\mathcal{L}^{(1)}$ in (5.4.2) yields

$$\begin{aligned}\langle f|J^{A0}|0\rangle_{\mathcal{L}^{(1)}\text{ insertion}} &= -i\lambda_s\int_{\tilde{p}}\frac{-i}{2\tilde{p}}(\frac{i\kappa}{2})(p_1^\mu\tilde{p}^\nu+\tilde{p}^\mu p_1^\nu-\eta^{\mu\nu}p_1\cdot\tilde{p})\mathfrak{h}^{(0)}_{\mu\nu}(k)\delta(\tilde{p}-(p_1+k))\\ &= -\frac{i\kappa\lambda_s}{2}\frac{1}{p_1\cdot k}\mathfrak{h}^{(0)}_{\mu\nu}(k)(p_1^\mu p_1^\nu+p_1^\mu k^\nu)+\frac{i\kappa\lambda_s}{4}\mathfrak{h}\,,\end{aligned} \tag{5.5.18}$$

where

$$\mathfrak{h}^{(0)}_{\mu\nu}(k) = h_{\mu\nu}(k) - k_\mu\frac{h_{+\nu}}{k_+} - k_\nu\frac{h_{+\mu}}{k_+} - k_\mu k_\nu\frac{h_{++}}{(k_+)^2} \tag{5.5.19}$$

is the linear gauge-invariant polarisation tensor. Note that since $\mathcal{L}^{(2)}$ only contains terms $\sim h^2\varphi^2$, there is *no* contribution of $T\{J^{A0},\mathcal{L}^{(2)}\}$ to this amplitude.

Subtracting (5.5.18) from the full amplitude, we are left with the remaining terms

$$\begin{aligned}M_{\rm rest} &= \frac{i\kappa\lambda_s}{4}\mathfrak{h} - \frac{i\kappa\lambda_s}{2}\sum_{j\neq 1}\left(\frac{n^a_{j-}n^b_{j-}}{n_{1-}n_{j-}}\frac{p_{j+}}{k_+}\mathfrak{h}_{ab}\right)\\ &\quad-\frac{i\kappa\lambda_s}{2}\sum_{j\neq 1}\left(\frac{n^a_{j-}p^b_{j\perp}}{n_{i-}n_{j-}}\frac{\mathfrak{h}_{ab}}{k_+}+\frac{n^a_{j-}n^\nu_{j-}}{n_{i-}n_{j-}}n_{i+\nu}\frac{\mathfrak{h}_{a-}}{k_+}p_{j+}\right.\\ &\qquad\left.+\frac{2}{n_{i-}n_{j-}}\frac{n^a_{j-}k^\nu}{k_+}\mathfrak{h}_{a\nu}-\frac{2}{n_{i-}n_{j-}}\left(n_{i-}p_{j\perp}+\frac{n_{j-}k_\perp p_{j+}}{k_+}\right)\frac{n^a_{j-}n^b_{j-}}{n_{i-}n_{j-}}\frac{\mathfrak{h}_{ab}}{k_+}\right).\end{aligned} \tag{5.5.20}$$

These terms must now be matched to the possible $A1$ and B-type currents. The $A1$-currents are obtained by adding a $\partial_\perp$ to the current, while the B-type current carries an additional $\mathfrak{h}^{(0)}_{\mu\nu}$ building block. For the B-type currents, we make use of momentum fractions x in the Fourier transformation. In position space, the B-type current is then defined as

$$\begin{aligned}\mathcal{J}^{B1}_{\mathfrak{h}\chi} = &\int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4\\ &\quad\widetilde{C}_{ab}(t_{1_1},t_{1_2},t_2,t_3,t_4)\mathfrak{h}^{ab}(t_{1_1}n_{1+})\chi_1(t_{1_2}n_{1+})\chi_2(t_2n_{2+})\chi_3(t_3n_{3+})\chi_4(t_4n_{4+})\,,\end{aligned} \tag{5.5.21}$$

where latin indices a, b indicate that $\mathfrak{h}_{ab}$ is purely transverse. As there are two fields in the 1-direction, we define the Fourier-transform as

$$J^{B1}(P_1, x) = P_1^2 \int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\, e^{-i(t_{1_1}xP_1 + t_{1_2}\bar{x}P_2)} J^{B1}(t_{1_1}, t_{1_2})\,, \tag{5.5.22}$$

where $\bar{x} = 1 - x$ and x is the momentum fraction.

Computing the overlap of the B1-current with the state $|h(k)\varphi(q)\rangle$, we find

$$\langle \mathfrak{h}_{\mu\nu}(k_+)\varphi(q_+)|J^{B1}_{h\varphi}(P, x)|0\rangle = P^2 \mathfrak{h}_{\mu\nu}(xP)\delta(xP - k)\delta(\bar{x}P - q)\,, \tag{5.5.23}$$

and we can now use

$$\delta(xP - k_+)\delta(xP - q_+) = \frac{1}{P}\delta(P - (k_+ + q_+))\delta(x - \frac{n_+k}{P})\,, \tag{5.5.24}$$

to find, omitting the overall momentum conservation,

$$\langle \mathfrak{h}_{\mu\nu}(k_+)\varphi(q_+)|J^{B1}_{h\varphi}(P, x)|0\rangle = P\mathfrak{h}_{\mu\nu}(xP)\delta(x - \frac{n_+k}{P})\,. \tag{5.5.25}$$

To reproduce the remaining terms in the amplitude (5.5.20), we require the following types of B1 and B2 currents:

$$\begin{aligned}
\mathcal{J}^{B1}_{\mathfrak{h}\chi} &= \int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 && (5.5.26)\\
&\quad \widetilde{C}_{ab}(t_{1_1}, t_{1_2}, t_2, t_3, t_4)\mathfrak{h}^{ab}(t_{1_1}n_{1+})\chi_1(t_{1_2}n_{1+})\chi_2(t_2n_{2+})\chi_3(t_3n_{3+})\chi_4(t_4n_{4+})\,,\\
\mathcal{J}^{B2}_{(\partial\mathfrak{h})\chi} &= \int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 && (5.5.27)\\
&\quad \widetilde{C}_{abc}(t_{1_1}, t_{1_2}, t_2, t_3, t_4)(i\partial^c_\perp\mathfrak{h}^{ab}(t_{1_1}n_{1+}))\chi_1(t_{1_2}n_{1+})\chi_2(t_2n_{2+})\chi_3(t_3n_{3+})\chi_4(t_4n_{4+})\,,\\
\mathcal{J}^{B2}_{\mathfrak{h}_-\chi} &= \int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 && (5.5.28)\\
&\quad \widetilde{C}_{a}(t_{1_1}, t_{1_2}, t_2, t_3, t_4)\mathfrak{h}^{a-}(t_{1_1}n_{1+})\chi_1(t_{1_2}n_{1+})\chi_2(t_2n_{2+})\chi_3(t_3n_{3+})\chi_4(t_4n_{4+})\,,\\
\mathcal{J}^{B1,A1}_{\mathfrak{h}\chi,\chi_2} &= \int \mathrm{d}t_{1_1}\mathrm{d}t_{1_2}\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4 && (5.5.29)\\
&\quad \widetilde{C}_{ab}(t_{1_1}, t_{1_2}, t_2, t_3, t_4)\mathfrak{h}^{ab}(t_{1_1}n_{1+})\chi_1(t_{1_2}n_{1+})(i\partial_{\perp b}\chi_2(t_2n_{2+}))\chi_3(t_3n_{3+})\chi_4(t_4n_{4+})\,,
\end{aligned}$$

where the 2 in the last current indicates the position of the derivative $\partial_\perp$. Note that using the equations of motion for $\mathfrak{h}_{\mu\nu}$, we could eliminate $\mathfrak{h}_{\mu-}$ and $\mathfrak{h}$ in favour of $\mathfrak{h}_{ab}$. However, we choose to keep these operators explicitly in the basis for now. We define the momentum fractions of the 1-direction so that φ carries $\bar{x}P$. The momentum-space matching coefficients are then computed as

$$C^{ab}_{\mathcal{J}^{B1}_{\mathfrak{h}\chi}}(x) = -\frac{i\kappa\lambda_s}{2}\sum_{j=2}^{4}\frac{n^a_{j-}n^b_{j-}}{n_{1-}n_{j-}}\frac{1}{xP_1}P_j + \frac{i\kappa\lambda_s}{4}\eta^{ab}_\perp\,, \tag{5.5.30}$$

$$\begin{aligned}
C^{abc}_{\mathcal{J}^{B2}_{(\partial\mathfrak{h})\chi}}(x) &= \frac{i\kappa\lambda_s}{2}\sum_{j=2}^{4}\frac{2}{n_{1-}n_{j-}}\left(n^c_{1-}\frac{n^a_{j-}n^b_{j-}}{n_{1-}n_{j-}}\frac{1}{xP_1}P_j - \frac{n^a_{j-}\eta^{bc}_\perp}{xP_1}\right)\\
&\quad + \frac{i\kappa\lambda_s}{2}\sum_{j=2}^{4}\frac{2}{n_{1-}n_{j-}}n^c_{j-}\frac{P_j}{xP_1}\frac{n^a_{j-}n^b_{j-}}{n_{1-}n_{j-}}\frac{1}{xP_1}P_2\,,
\end{aligned} \tag{5.5.31}$$

$$C^{a}_{\mathcal{J}^{B2}_{\mathfrak{h}_-\chi}}(x) = -\frac{i\kappa\lambda_s}{2}\sum_{j=2}^{4}\frac{n^a_{j-}n^\nu_{j-}}{n_{1-}n_{j-}}n_{1+\nu}\frac{1}{xP_1}P_j + \frac{1}{n_{i-}n_{j-}}n^a_{j-}\,, \tag{5.5.32}$$

$$C^{a}_{\mathcal{J}^{B1,A1}_{\mathfrak{h}\chi,\chi_2}}(x) = -\frac{i\kappa\lambda_s}{2}\frac{n^a_{j-}}{n_{1-}n_{j-}}\frac{1}{xP_1}\,, \tag{5.5.33}$$

and the full amplitude for collinear graviton emission is indeed reproduced by the effective theory. This explicit computation serves as a check that the collinear Wilson line W_c indeed also arises from an explicit matching computation, as we also verified for QCD in Section 3.3.4. It can be viewed as a third, independent derivation of the Wilson line, or as an explicit double-check of our construction.

Chapter 6

Soft-collinear Gravity

In this section, we present the full soft-collinear theory for gravity, the derivation of the effective Lagrangian as well as a discussion of the N-jet operator basis. Conceptually, the construction is very similar to the gauge-theory one presented in Chapter 3. However, on a technical level, the various definitions and expressions used in the derivation differ quite drastically. The main complication, similar to gauge-theory, arises after the multipole expansion of the soft fields and consists of the identification of the "homogeneous" soft background field. Once the proper background is identified, one can employ various Wilson lines to define redefined "homogeneously transforming" fields. Then, inserting these fields and using the same manipulations as in the gauge-theory case presented in Section 3.4 yields the full soft-collinear Lagrangian.

The derivation and construction follow closely [47] by the author in collaboration with M. Beneke and R. Szafron, where it was performed for the first time.

6.1 Power-counting, Field Content and Gauge Symmetry

The first step is to define the field content. As usual, we include only the soft and collinear modes of the full-theory fields in the effective theory. For the gravitational part, this means we consider collinear gravitons $h_{i\mu\nu}$, one for each collinear direction, and one soft graviton field $s_{\mu\nu}$. The split in the gravitational sector is implemented similarly to gauge theory (3.2.2) as a sum

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + s_{\mu\nu} \equiv g_{s\mu\nu} + h_{\mu\nu}\,. \tag{6.1.1}$$

In the following construction, it is convenient to employ the soft metric tensor $g_{s\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}$, since the effective theory turns out to be covariant with respect to this soft background $g_{s\mu\nu}$. Therefore, one can employ geometric notions and intuition throughout the construction. In the previous section, the scaling of the collinear graviton was determined as (5.1.6). From the same two-point function (4.3.8), the scaling of the soft $s_{\mu\nu}$ is found to be homogeneous,

$$s_{\mu\nu} \sim \lambda^2\,. \tag{6.1.2}$$

The gauge symmetry is affected in a non-trivial fashion by the split (6.1.1). Namely, the original full-theory diffeomorphism symmetry is extended to a semi-direct product of a soft and a collinear gauge symmetry. Since all fields have to be homogeneous in the power counting, one has to ensure that a soft field never transforms under collinear gauge transformations. Else, the soft field would turn collinear. Therefore, the original symmetry is now realised as two different symmetries.

Under full diffeomorphisms, the metric tensor transforms as (4.2.11)

$$\begin{aligned} g_{\mu\nu} &\to U\left[U_\mu{}^\alpha U_\nu{}^\beta g_{\alpha\beta}\right] \\ &= g_{\mu\nu} - \nabla_\mu\varepsilon_\nu - \nabla_\nu\varepsilon_\mu + \mathcal{O}(\varepsilon^2)\,, \end{aligned} \tag{6.1.3}$$

where ∇ is the standard covariant derivative with respect to the full metric tensor $g_{\mu\nu}$.

Now we consider the split (6.1.1). The left-hand side only transforms under full diffeomorphisms. Therefore, if the right-hand side transforms under soft or collinear diffeomorphisms, the sum of the transformations must correspond to a full-theory one. One consistent way of implementing the two separate symmetries – under the constraint that a soft field never transforms under the collinear diffeomorphism – is to treat the soft field as a background, and the collinear field as a fluctuation on top of this soft background.

Consider first the soft gauge transformations. Here, the soft background field should transform like an ordinary metric tensor. For the fluctuation $s_{\mu\nu}$, this implies that it should have the standard inhomogeneous gauge transformation (5.2.6). Inserting this transformation in the split (6.1.1) then implies the transformation for $h_{\mu\nu}$. Combined, they read

$$\begin{aligned} h_{\mu\nu} &\to \left[U_s \left(U_{s\mu}{}^{\alpha} U_{s\nu}{}^{\beta} h_{\alpha\beta}\right)\right] , \\ s_{\mu\nu} &\to \left[U_s U_{s\mu}{}^{\alpha} U_{s\nu}{}^{\beta} (\eta_{\alpha\beta} + s_{\alpha\beta})\right] - \eta_{\mu\nu} . \end{aligned} \tag{6.1.4}$$

Note that the collinear graviton $h_{\mu\nu}$ transforms like an ordinary rank-2 tensor field, and *not* like a graviton, whereas the soft graviton $s_{\mu\nu}$ has indeed the standard inhomogeneous gauge transformation. The soft transformation implies the standard transformation for the background metric $g_{s\mu\nu}$

$$g_{s\mu\nu} \to \left[U_s \left(U_{s\mu}{}^{\alpha} U_{s\nu}{}^{\beta} g_{s\alpha\beta}\right)\right] . \tag{6.1.5}$$

In addition, there is also the collinear symmetry related to $h_{\mu\nu}$. The soft graviton $s_{\mu\nu}$ must not transform under this symmetry, otherwise, the homogeneous scaling would be violated – the soft mode would acquire collinear fluctuations. Therefore, imposing $s_{\mu\nu} \to s_{\mu\nu}$ yields the transformations

$$\begin{aligned} h_{\mu\nu} &\to \left[U_c \left(U_{c\mu}{}^{\alpha} U_{c\nu}{}^{\beta} (g_{s\alpha\beta} + h_{\alpha\beta})\right)\right] - g_{s\mu\nu} , \\ s_{\mu\nu} &\to s_{\mu\nu} , \\ g_{s\mu\nu} &\to g_{s\mu\nu} , \end{aligned} \tag{6.1.6}$$

where we have also added the transformation of the soft background $g_{s\mu\nu}$. Comparing this to the standard graviton transformation in the weak-field expansion (5.2.6), one immediately sees that the transformation (6.1.6) of $h_{\mu\nu}$ is covariant with respect to the background $g_{s\mu\nu}$, which appears instead of $\eta_{\mu\nu}$. Indeed, the transformations (6.1.4) and (6.1.6) simply implement a small fluctuation $h_{\mu\nu}$ on top of a background $g_{s\mu\nu}$. The small fluctuation comes with its own gauge transformations, which we call collinear. Since the fluctuation is expanded around a non-trivial background, the gauge transformations are covariant with respect to this background. These are, however, on such small scales that the slowly-varying background is blind to them. Therefore, it does not transform under this symmetry. That is the essence of (6.1.6). On the other hand, the background has its own diffeomorphism symmetry, where it transforms in the standard fashion. From the point of view of the background, the fluctuation then transforms as an ordinary tensor field, the fluctuation is not special compared to any other matter field present in the theory. This is the statement of (6.1.4).

One can compare the transformations to the ones encountered in the gauge-theory situation, presented in (3.2.3). One immediately notices that the intuition is the same. Also in QCD, the soft gluon acts as a background field to the collinear one. This is evident from the soft gauge transformation (3.2.3), where the soft gluon has the standard (inhomogeneous) gauge transformation, but the collinear one transforms like a matter field in the adjoint representation. This is the same situation as depicted in (6.1.4) in gravity. For the collinear transformations in QCD, the collinear gluon transformation is covariant with respect to the soft background A_s, since the covariant derivative D_s appears in the inhomogeneous term instead of ∂. The soft background is invariant. In gravity, we find conceptually the same in (6.1.6). The inhomogeneous term in the graviton transformation (5.2.6) is characterised by $\eta_{\mu\nu}$, which changes to the soft background $g_{s\mu\nu}$ in (6.1.6). In both gravity and QCD, the gauge symmetry is implemented as a number of collinear fluctuations (with their own symmetry) on top of a soft background.

Therefore, the intuition and concepts derived for the gauge symmetries in QCD should directly transfer to the gravitational context.

As matter content, we consider the collinear and soft scalar fields $\varphi_c \sim \lambda$ and $\varphi_s \sim \lambda^2$. Under diffeomorphisms, a scalar transforms as

$$\begin{aligned}\varphi &\to [U\varphi] \\ &= \varphi - \varepsilon^\alpha \nabla_\alpha \varphi + \mathcal{O}(\varepsilon^2)\,.\end{aligned} \tag{6.1.7}$$

Under a soft transformation, both fields should transform as ordinary matter fields, which implies

$$\begin{aligned}\varphi_c &\to [U_s \varphi_c]\ , \\ \varphi_s &\to [U_s \varphi_s]\ .\end{aligned} \tag{6.1.8}$$

Under a collinear one, however, the soft field cannot transform, and thus we have to impose

$$\begin{aligned}\varphi_c &\to [U_c \varphi_c]\ , \\ \varphi_s &\to \varphi_s\ .\end{aligned} \tag{6.1.9}$$

In QCD, the situation is the exact formal analogy. Here, we employed the Wilson line $WZ^\dagger$ to relate full-theory and EFT fields (3.2.5). In gravity, the analogue is to the "Wilson line" WZ^{-1}

$$\varphi = \varphi_c + \left[WZ^{-1}\varphi_s\right]\ . \tag{6.1.10}$$

Here, W^{-1} and Z^{-1} are the standard gravitational "Wilson lines" encountered in the previous section (5.3.6), but expressed in terms of the full graviton and the soft graviton field, respectively, that is

$$W \equiv T^{-1}_{\theta[g_s+h]}\ , \qquad Z^{-1} \equiv T_{\theta[g_s]}\ , \tag{6.1.11}$$

where $\theta[g_{\mu\nu}]$ denotes the light-cone gauge parameter θ for a theory with metric tensor $g_{\mu\nu}$. In other words, the Wilson line W fixes light-cone gauge in the full theory, where the fluctuation is defined as $h_{\text{full},\,\mu\nu} = h_{\mu\nu} + s_{\mu\nu}$, while Z fixes light-cone gauge for the purely-soft theory $s_{\mu+} \equiv 0$. This is the exact formal counterpart to the gauge-theory situation (3.2.6).

In summary, the formal setup looks very similar to the QCD scenario. Two main issues arise at this point, both of which are also encountered in gauge theory:

- The collinear graviton features large components $h_{++} \sim \lambda^{-1}$, $h_{+\perp} \sim 1$, similar to the collinear gluon $n_+A_c \sim 1$. To control these components, we make use of the collinear Wilson line W_c^{-1} of the previous section (5.3.6). This Wilson line needs to be modified to account for the now non-trivial soft background.
- After performing the light-front multipole expansion, the soft gauge transformations (6.1.4) and (6.1.8) of collinear graviton and matter field will mix different orders of the multipole expansion. We need to identify the "homogeneous" background field that has a gauge symmetry which respects the multipole expansion.

6.2 Inhomogeneities in λ

This second issue, the identification of the correct background field, is the main complication in the gravitational setting compared to QCD. In gauge theory, it was straightforward to deduce the form of the homogeneous background field $n_-A_s(x_-)$. QCD SCET only has one source of inhomogeneities in λ, namely the multipole expansion. The background field $n_-A_s(x_-)$ is the only homogeneous component of the full covariant derivative, and covariance with respect to this field immediately leads to transformations that are homogeneous in λ.

In gravity, this situation is more complicated. Here, we have not one but two sources of inhomogeneities in λ. In addition to the light-front multipole expansion, the gravitational gauge charges, namely the momenta P^μ – most notably the collinear momentum $(n_+P, P_\perp, n_-P) \sim (1, \lambda, \lambda^2)$ – lead to gauge transformations that are inherently inhomogeneous in λ.

This second inhomogeneity already appears in both the full theory as well as the purely-collinear one. It is related to performing the weak-field expansion, which imposes a truncation of the gauge symmetry at some order in κ, correspondingly λ in the EFT. As a consequence, one has to decide if one expresses objects in a manifestly gauge-invariant fashion, which is inherently inhomogeneous in λ, or via manifestly homogeneous fields, which cannot be gauge-invariant. This seems to be in stark contrast to QCD, where gauge-invariant objects are also homogeneous in λ.

However, at the formal level, the situation is not as different as it seems. This is best explained using the collinear Wilson line. In QCD, it is defined as (3.3.78), and one immediately sees that it is inhomogeneous in the strong coupling g, but counts as $\mathcal{O}(\lambda^0)$. In gravity, however, the corresponding Wilson line takes the form (5.3.6). This object looks very similar to the gauge-theory one and is also inhomogeneous in the coupling κ. However, now the gauge charges have a scaling in λ, therefore the Wilson line is also inhomogeneous in λ. Hence this type of inhomogeneity seems unavoidable in gravity, and it turns out that it poses no problem for the EFT construction. If one forgets about the power-counting for a moment, the formal definitions in terms of the couplings are very similar, and one can borrow much intuition.

In gravity, this simply implies that there are intricate relations between subleading terms in the Lagrangian, which combine into geometric (gauge-covariant) objects, similar to RPI constraints in standard SCET.

The first type of inhomogeneity, on the other hand, is the one that is also present in QCD. In soft-collinear interactions, the soft fields $g_{s\mu\nu}(x)$, $\varphi_s(x)$ can only depend on the large coordinate x_-^μ. Therefore, the light-front multipole expansion (2.3.7) must be performed, e.g. for the metric tensor

$$\begin{aligned} g_{s\mu\nu}(x) &= g_{s\mu\nu}(x_-) + x_\perp^\alpha \left[\partial_\alpha g_{s\mu\nu}\right](x_-) \\ &\quad + \frac{1}{2} n_-x \left[n_+\partial g_{s\mu\nu}\right](x_-) + \frac{1}{2} x_\perp^\alpha x_\perp^\beta \left[\partial_\alpha \partial_\beta g_{s\mu\nu}\right](x_-) + \mathcal{O}(\lambda^3 g_{s\mu\nu})\,. \end{aligned} \tag{6.2.1}$$

This means that any soft field generates an infinite tower of subleading interactions, which are all of the *same* order in the coupling κ, and are not related to the non-linearities of gravity. This also applies to the gauge transformations, and one has to ensure that the gauge symmetry of the effective theory respects this multipole expansion.

However, due to the aforementioned inhomogeneous gauge symmetries, the tricky part is to identify this new background field. Whereas in QCD one could simply impose the homogeneous transformations $U_s(x_-)$, in gravity it is a priori not clear what the precise form of these transformations should be. For example, the naive diffeomorphism $\varphi_c \to U_s(x_-)\varphi_c$ is not the right transformation, as this does not lead to homogeneously transforming tensor fields. For example, the transverse derivative of a scalar field should intuitively transform like

$$\partial_\perp \varphi_c \to U_s(x_-) {U_s}_\perp{}^\alpha(x_-) \partial_\alpha \varphi_c\,. \tag{6.2.2}$$

However, with the above transformation, $\frac{\partial \varepsilon(x_-)}{\partial x_\perp} = 0$, so the Jacobian would not be generated, and the derivative would not transform like a tensor $T_\perp \to U_s(x_-) U_\perp{}^\alpha(x_-) T_\alpha$. Therefore, the identification of the homogeneous gauge symmetry is a non-trivial task in gravity.

6.3 Multipole Expansion and Normal Coordinates

In QCD, the guiding principle to modify the soft gauge symmetry is to find the transformation that is homogeneous in λ, which corresponds to $\phi_c \to U_s(x_-)\phi_c$. This transformation no longer mixes different orders in λ, and one can construct a Lagrangian where each term is manifestly gauge-covariant and homogeneous. In gravity, however, the gauge transformations are inherently inhomogeneous. Consequently, this is not a sensible approach, and we require a different guiding principle.

The end result in QCD is a soft sector that is expressed in terms of a number of building blocks that respect the multipole expansion, see (3.4.17) – (3.4.18). It features the soft-covariant derivative n_-D_s, which depends only on the homogeneous $n_-A_s(x_-)$, and the field-strength tensor $F_{s\mu\nu}$, as well as its (covariant) derivatives. Therefore, the aim is to find the analogous split into a "homogeneous" background field and gauge-covariant objects in gravity. In QCD, these fields can be identified by employing fixed-line gauge in the soft sector. In section Sections 3.3.2 and 3.3.3, we have seen explicitly how fixed-line gauge, and the simpler fixed-point version, follow quite naturally in the context of a multipole expansion. Accordingly, we will first consider the static multipole expansion in gravity and determine the analogue of fixed-point gauge. Only then we generalise this result to the light-front situation and determine the "homogeneous" background field. This circumvents the necessity for a homogeneous gauge transformation, as we work directly with the multipole expansion, the gauge-fixing, and its residual symmetries.

6.3.1 Fixed-point Gauge and Riemann Normal Coordinates

Consider a theory consisting of a scalar matter field $\varphi(x)$ and a metric field $g_{\mu\nu}(x)$. The metric field is multipole expanded about $x = 0$ as

$$g_{\mu\nu}(x) = g_{\mu\nu}(0) + x^\alpha \left[\partial_\alpha g_{\mu\nu}\right](0) + \frac{1}{2} x^\alpha x^\beta \left[\partial_\alpha \partial_\beta g_{\mu\nu}\right](0) + \mathcal{O}(x^3) \,. \tag{6.3.1}$$

In gauge theory, fixed-line gauge can be employed to render the subleading terms of this multipole expansion manifestly gauge-invariant, by expressing the derivatives of the gauge field in terms of the field-strength tensor. In gravity, a similar, well-known gauge condition exists, namely Riemann normal coordinates (RNC). In these coordinates, the metric tensor is expressed as

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(0) - \frac{1}{6} x^\alpha x^\beta (R^\rho{}_{\alpha\nu\beta}(0) g_{\rho\mu}(0) + R^\rho{}_{\alpha\mu\beta}(0) g_{\rho\nu}(0)) + \mathcal{O}(x^3) \,, \tag{6.3.2}$$

and the higher-order terms correspond to derivatives of the Riemann tensor. In addition, one can always diagonalise the metric at the origin to obtain the standard result

$$\check{g}_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} x^\alpha x^\beta R_{\alpha\mu\beta\nu}(0) + \mathcal{O}(x^3) \,, \tag{6.3.3}$$

where the metric tensor is expressed purely in terms of the flat-space Minkowski metric, the Riemann tensor, and its derivatives.

It is instructive to compare this result to QCD. Note that in fixed-point gauge, the leading term $A_\mu(0) = 0$, and the first derivative term is rewritten via the field-strength tensor. In contrast, gravity features a non-vanishing leading term $g_{\mu\nu}(0)$ or $\eta_{\mu\nu}$, depending on the reference frame, then the first derivative is vanishing $x^\alpha \left[\partial_\alpha g_{\mu\nu}\right](0) = 0$ and the second derivative terms are expressed via the Riemann tensor. Intuitively, one can understand this by comparing the gluon field A_μ not with the metric tensor, but with the Christoffel symbol $\Gamma^\mu{}_{\alpha\beta}$, which is the analogue of the gauge field from a geometric perspective, as both fields are the respective gauge connections. The RNC gauge condition applied to the Christoffel symbol reads

$$x^\alpha x^\beta \Gamma^\mu{}_{\alpha\beta}(x) = 0 \,, \tag{6.3.4}$$

which is similar to the fixed-point condition $x^\mu A_\mu(x) = 0$. In addition, the vanishing of the first-derivative term of the metric tensor corresponds simply to $\Gamma^\mu{}_{\alpha\beta}(0) = 0$, similar to $A_\mu(0) = 0$. Therefore, Riemann normal coordinates are the direct analogue of fixed-point gauge.

Since these coordinates must be extended to the more general light-front situation, it is instructive to derive the standard RNC and the metric tensor in the same formalism that is used in gauge theory in Section 3.3.3. In the following, we adopt the convention that a field which has no explicit argument is evaluated at $x = 0$, e.g $\Gamma^{\mu}{}_{\alpha\beta} \equiv \Gamma^{\mu}{}_{\alpha\beta}(0)$.

Riemann normal coordinates are defined such that in a small neighbourhood around the origin, a geodesic $y^{\mu}(s)$ satisfying $y^{\mu}(0) = 0$ and $y^{\mu}(1) = x^{\mu}$, i.e. passing through the origin at $s = 0$ and through point x at $s = 1$, is parametrised as a straight line

$$y^{\mu}(s) = sx^{\mu}\,. \tag{6.3.5}$$

To derive the coordinate transformation, one now considers this geodesic $y^{\mu}(s)$ in a generic reference frame, where it is not straight, and employs the geodesic equation.

In this generic reference frame, parametrise the geodesic as

$$y^{\mu}(s) = sx^{\mu} + v^{\mu}(s)\,, \tag{6.3.6}$$

where $v^{\mu}(s)$ satisfies $v^{\mu}(0) = 0$, since both frames are taken to have coinciding origins. Here, $v^{\mu}(s)$ can be thought of as the displacement of both frames. Note that $v^{\mu}(s)$ is in general also x-dependent, but for convenience, we suppress this dependence. The geodesic satisfies the geodesic equation

$$\frac{d^2y^{\mu}(s)}{ds^2} + \Gamma^{\mu}{}_{\alpha\beta}(y(s))\frac{dy^{\alpha}(s)}{ds}\frac{dy^{\beta}(s)}{ds} = 0\,. \tag{6.3.7}$$

One now inserts (6.3.6), expands the equation in $v^{\mu}(s)$, and solves it iteratively. Using

$$\Gamma^{\mu}{}_{\alpha\beta}(sx + v(s)) = \Gamma^{\mu}{}_{\alpha\beta}(sx) + v^{\rho}(s)\left[\partial_{\rho}\Gamma^{\mu}{}_{\alpha\beta}\right](sx) + \mathcal{O}(s^2)\,, \tag{6.3.8}$$

one finds at leading order

$$\frac{dv^{(0)\mu}(s)}{ds^2} = -\Gamma^{\mu}{}_{\alpha\beta}(sx)x^{\alpha}x^{\beta}\,, \tag{6.3.9}$$

and its solution

$$v^{(0)\mu}(s) = -\int_0^s ds' \int_0^{s'} ds''\, x^{\alpha}x^{\beta}\,\Gamma^{\mu}{}_{\alpha\beta}(xs'')\,. \tag{6.3.10}$$

To further evaluate this integral, one has to expand $\Gamma^{\mu}{}_{\alpha\beta}(xs'')$ around $x = 0$ and integrate term by term. Immediately, one sees that while this can be done to any desired order, there is in general no closed expression for $v^{\mu}(s)$. This general feature of gravity is in contrast to the QCD situation, where closed formulas for fixed-point gauge could be obtained (3.3.42).

To relate the generic reference frame, denoted by x, to the RNC frame $\tilde{x}$, use (6.3.6) for $s = 1$. The left-hand side corresponds to the generic reference frame, while x^{μ} on the right-hand side is the RNC coordinate. Both frames are displaced by v^{μ}. One finds

$$x^{\mu} = \tilde{x}^{\mu} - \frac{1}{2}\tilde{x}^{\alpha}\tilde{x}^{\beta}\Gamma^{\mu}{}_{\alpha\beta} + \frac{1}{6}\tilde{x}^{\alpha}\tilde{x}^{\beta}\tilde{x}^{\nu}\left(2\Gamma^{\mu}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\nu} - \left[\partial_{\nu}\Gamma^{\mu}{}_{\alpha\beta}\right]\right) + \mathcal{O}(\tilde{x}^4)\,. \tag{6.3.11}$$

The inverse can be computed order by order in x and is given by

$$\tilde{x}^{\mu} = x^{\mu} + \frac{1}{2}x^{\alpha}x^{\beta}\Gamma^{\mu}{}_{\alpha\beta} + \frac{1}{6}x^{\alpha}x^{\beta}x^{\nu}\left(\Gamma^{\mu}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\nu} + \left[\partial_{\nu}\Gamma^{\mu}{}_{\alpha\beta}\right]\right) + \mathcal{O}(x^4)\,, \tag{6.3.12}$$

which is the standard form of RNC.

To explicitly verify the form of the metric tensor (6.3.2), one computes

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}}(x)\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}}(x)g_{\alpha\beta}(x)\,. \tag{6.3.13}$$

Here, $g_{\mu\nu}(x)$ on the right-hand side must be multipole expanded

$$g_{\mu\nu}(x) = g_{\mu\nu} + x^{\alpha}\left[\partial_{\alpha}g_{\mu\nu}\right] + \frac{1}{2}x^{\alpha}x^{\beta}\left[\partial_{\alpha}\partial_{\beta}g_{\mu\nu}\right] + \mathcal{O}(x^3)\,. \tag{6.3.14}$$

Using metric compatibility $\nabla_\alpha g_{\mu\nu} = 0$, one can derive the relations

$$\partial_\alpha g_{\mu\nu} = \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} + \Gamma^\lambda{}_{\alpha\nu} g_{\lambda\mu} \,, \tag{6.3.15}$$

$$\partial_\beta\partial_\alpha g_{\mu\nu} = \left[\partial_\beta \Gamma^\lambda{}_{\alpha\mu}\right] g_{\lambda\nu} + \Gamma^\lambda{}_{\alpha\mu}\Gamma^\rho{}_{\beta\lambda} g_{\rho\nu} + \Gamma^\lambda{}_{\alpha\mu}\Gamma^\rho{}_{\beta\nu} g_{\rho\lambda} + (\mu \leftrightarrow \nu) \,. \tag{6.3.16}$$

The explicit evaluation of (6.3.13) is now straightforward and directly leads to (6.3.2).

In the following construction, we want to employ the same formalism that is also used in gauge theory. This means that an expression like (6.3.2) should not be computed explicitly from a gauge transformation (6.3.13). Instead, it should be defined using the analogue of the R Wilson line that was introduced in Section 3.3.3. Since we have access to the explicit coordinate transformation, it is straightforward to derive the R Wilson line – it is simply the translation to these coordinates.

This “Wilson line” can be thought of as the transformation that moves a generic field configuration to Riemann normal coordinates. More importantly, it must also transport the gauge transformation from a generic point x to the point $x = 0$, where only global transformations remain. Therefore, once we construct the R Wilson line, we can redefine the matter fields accordingly and find the “homogeneous” gauge transformations.

In the static example, similar to fixed-point gauge, we anticipate that the homogeneous gauge transformations will be global transformations.

6.3.2 The R Wilson Line for Riemann Normal Coordinates

With the explicit coordinate transformations (6.3.11) and (6.3.12), we can define the analogue of the R Wilson line. Recall from the collinear discussion in Section 5.3 that the “Wilson lines” in gravity take the form of translation operators (5.3.6)

$$R^{-1}_{\rm RNC}(x) \equiv T_{\theta_{\rm RNC}(x)} = 1 + \theta^\alpha_{\rm RNC}(x)\partial_\alpha + \frac{1}{2}\theta^\alpha_{\rm RNC}(x)\,\theta^\beta_{\rm RNC}(x)\partial_\alpha\partial_\beta + \mathcal{O}(\theta^3_{\rm RNC}) \,. \tag{6.3.17}$$

To construct such a Wilson line, one simply needs to determine the correct parameter, which we denote by $\theta_{\rm RNC}$. This parameter is obtained from the explicit coordinate transformation (6.3.11), interpreted as

$$x^\mu = [T_{\theta_{\rm RNC}} \tilde{x}^\mu] \,. \tag{6.3.18}$$

Expanding the translation operator and comparing it with the explicit form in (6.3.11), one obtains

$$\theta^\mu_{\rm RNC}(x) \equiv -\frac{1}{2} x^\alpha x^\beta \Gamma^\mu{}_{\alpha\beta} + \frac{1}{6} x^\alpha x^\beta x^\nu \left(2\Gamma^\mu{}_{\alpha\tau}\Gamma^\tau{}_{\beta\nu} - \left[\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right]\right) + \mathcal{O}(x^4) \,. \tag{6.3.19}$$

In addition, one requires the Jacobi-matrices

$$R_\mu{}^\alpha(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}(x) \,, \quad R^\mu{}_\alpha(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}(x) \,. \tag{6.3.20}$$

The metric tensor in Riemann normal coordinates is then defined by transforming it with the R Wilson line and Jacobians according to its representation, namely

$$\tilde{g}_{\mu\nu}(x) \equiv R^\alpha{}_\mu(x) R^\beta{}_\nu(x) \left[R^{-1}_{\rm RNC}(x) g_{\alpha\beta}(x)\right] \,. \tag{6.3.21}$$

Note that R is the inverse of a gauge transformation, hence the “dressing” is inverse to a gauge transformation (4.2.10). Replacing R with W_c, one sees that this is the same prescription that yields the gauge-invariant building block in collinear gravity (5.3.4).

The interpretation of this object is the same as in gauge theory. Starting from a generic metric field $g_{\mu\nu}$, one constructs the Wilson line $R[g_{\mu\nu}]$, which moves this field into a gauge corresponding to Riemann normal coordinates. This gauge-transformed field is defined as (6.3.21) and satisfies the properties of a metric in RNC. If one were to start from RNC directly, note that $R(x) = 1$ and there is no redefinition.

To explicitly evaluate (6.3.21), one first determines

$$\begin{aligned} R^{\alpha}{}_{\mu}(x) &= \delta^{\alpha}_{\mu} - x^{\rho}\Gamma^{\alpha}{}_{\rho\mu} + x^{\rho}x^{\sigma}\left(\frac{1}{3}\Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\rho\sigma} + \frac{2}{3}\Gamma^{\alpha}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\mu} - \frac{1}{6}\left[\partial_{\mu}\Gamma^{\alpha}{}_{\rho\sigma}\right] - \frac{1}{3}\left[\partial_{\rho}\Gamma^{\alpha}{}_{\sigma\mu}\right]\right) \\ &\quad + \mathcal{O}(x^3)\,, \end{aligned} \tag{6.3.22}$$

and using (6.3.15), one can check that (6.3.21) indeed agrees with (6.3.2).

The object $\tilde{g}_{\mu\nu}(x)$ as defined in (6.3.21) or explicitly (6.3.2) is expressed only in terms of the metric at the origin, $g_{\mu\nu}(0)$ and the manifestly gauge-invariant Riemann tensor. Note that there is still a residual gauge symmetry that respects this multipole expansion, namely the symmetry transformations of the residual background metric $g_{\mu\nu}(0)$ These transformations correspond to *global* transformations $A_{\mu}{}^{\alpha} \in \mathrm{GL}(1,3)$ and *global* translations ε^{μ}[1], i.e.

$$x^{\mu} \to A_{\alpha}{}^{\mu}x^{\alpha} + \varepsilon^{\mu}\,. \tag{6.3.23}$$

One immediately verifies that also the Riemann tensor $R^{\mu}{}_{\nu\alpha\beta}(0)$ transforms covariantly under these transformations, and the object (6.3.21) indeed covariantly transforms under (6.3.23).

If one applies $R(x)$ to a matter field, the effect is to change the gauge transformation from the generic diffeomorphism evaluated at x to the new set of transformations (6.3.23) evaluated at $x = 0$, similar to QCD where we move the gauge transformation from x to $x = 0$. Note, however, that the transformations are not simply the ones where one takes the parameter $\varepsilon(x) \to \varepsilon(0)$. Instead, they consist of two terms, one translation and one global linear transformation, as we already anticipated. These transformations lead to homogeneous and non-trivial tensor transformations since the global linear transformation is required for this.

Therefore, in gravity, the "homogeneous" residual gauge symmetry that respects the static multipole expansion does not just consist of global translations, but also of global linear transformations.

But one can achieve more than just this result. Note that the actual coordinate transformation (6.3.11) is non-trivial starting only at the second order in x. Consequently, there is still the possibility to modify the linear level of these transformations. One can use such a linear coordinate transformation to transform $g_{\mu\nu}(0) \to \eta_{\mu\nu}$ at the origin, thereby simplifying the form of the final metric tensor.

To compute this transformation, one can make use of the vierbein (4.4.2) in a global version. Namely, one can write the metric at the origin via

$$g_{\mu\nu}(0) = e_{\mu}{}^{\alpha}e_{\nu}{}^{\beta}\eta_{\alpha\beta}\,, \tag{6.3.24}$$

introducing the global "vierbein" $e_{\mu}{}^{\alpha}$. Since the metric tensor is symmetric, $g_{\mu\nu}(0)$ is a symmetric matrix and it can always be diagonalised as (6.3.24). This guarantees the existence of these vierbeins. These objects have the same properties and expansions as the usual vierbein, and one could think of this as introducing a "local inertial frame" at $x = 0$. However, this merely amounts to rotating the coordinate system such that the metric takes the standard Minkowski form. In the weak-field expansion, the vierbein is given by (4.4.29)

$$e_{\mu}{}^{\alpha} = \delta^{\alpha}_{\mu} + \frac{1}{2}h^{\alpha}_{\mu} - \frac{1}{8}h_{\mu\beta}h^{\beta\alpha} + \mathcal{O}(h^3)\,. \tag{6.3.25}$$

And one can modify the coordinate transformation (6.3.12) as $\check{x}^{\mu} = e_{\rho}{}^{\mu}\tilde{x}^{\rho}$, and find explicitly

$$\check{x}^{\mu} = e_{\rho}{}^{\mu}\left(x^{\rho} + \frac{1}{2}x^{\alpha}x^{\beta}\Gamma^{\rho}{}_{\alpha\beta} + \frac{1}{6}x^{\alpha}x^{\beta}x^{\nu}\left(\Gamma^{\rho}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\nu} + \left[\partial_{\nu}\Gamma^{\rho}{}_{\alpha\beta}\right]\right)\right) + \mathcal{O}(x^4)\,. \tag{6.3.26}$$

[1] Note that the Riemann normal coordinates are defined with respect to the origin at $x = 0$. Since we employ the active point of view, where coordinates do not transform, we can still perform a translation of the field without affecting the RNC. In the passive point of view, this simply amounts to also shifting the RNC to the new origin.

To obtain the new parameter $\check{\theta}^{\mu}_{\rm RNC}(x)$, which now diagonalises the metric, one again inverts (6.3.26) and reads it off. Here, one requires the inverse matrix $E^{\mu}{}_{\alpha}$, defined via

$$E^{\mu}{}_{\beta} e_{\alpha}{}^{\beta} = \delta^{\mu}_{\alpha}\,. \tag{6.3.27}$$

The parameter $\check{\theta}^{\mu}_{\rm RNC}(x)$ then reads

$$\begin{aligned}\check{\theta}^{\mu}_{\rm RNC}(x) &= (E^{\mu}{}_{\rho} - \delta^{\mu}_{\rho})x^{\rho} - \frac{1}{2}x^{\rho}x^{\sigma}E^{\alpha}{}_{\rho}E^{\beta}{}_{\sigma}\Gamma^{\mu}{}_{\alpha\beta} \\ &\quad + \frac{1}{6}x^{\rho}x^{\sigma}x^{\lambda}E^{\alpha}{}_{\rho}E^{\beta}{}_{\sigma}E^{\nu}{}_{\lambda}(2\Gamma^{\mu}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\nu} - \left[\partial_{\nu}\Gamma^{\mu}{}_{\alpha\beta}\right]) + \mathcal{O}(x^4)\,.\end{aligned} \tag{6.3.28}$$

Using the same definition (6.3.26) but now with the new parameter $\check{\theta}^{\mu}_{\rm RNC}(x)$ yields a metric tensor that satisfies the diagonalised (6.3.3) instead of (6.3.2).

It is now interesting to see how the residual transformation is affected. Before, we found that using the old parameter (6.3.19) leads to a residual symmetry consisting of global linear transformations and global translations, the symmetries of $g_{\mu\nu}(0)$. Now, however, the Minkowski metric appears in its place. Therefore, the residual symmetries are changed to be the symmetries of the Minkowski metric, that is, global Poincaré transformations. one can again immediately verify that (6.3.3) is indeed covariant under global Lorentz transformations and global translations.

In summary, Riemann normal coordinates are the direct analogue of fixed-point gauge in QCD found in Section 3.3.2. In these coordinates, the metric tensor can be expressed in terms of its value at the origin and manifestly gauge-invariant Riemann tensor terms. The gauge is completely fixed, and the residual transformations correspond to global transformations, which are the symmetries of the metric $g_{\mu\nu}(0)$ at the origin. One can modify the RNC at the linear level (6.3.24) to change the value of $g_{\mu\nu}(0)$. A convenient choice is to fix this to be the Minkowski metric $\eta_{\mu\nu}$. Then, the left-over global transformations correspond to its symmetries, namely global Poincaré transformations.

Applied to matter fields, the R Wilson line (6.3.17) with the corresponding parameter (6.3.28) moves the gauge transformations from generic diffeomorphisms at point x to global Poincaré transformations at point $x = 0$. This is precisely the intuition one has from gauge theory following Section 3.3.2, where the V Wilson line moves the transformation from point x to point $x = 0$. This residual transformation is the analogue of the homogeneous gauge transformation. Crucially, already in this toy model, the residual gauge transformation of gravity is more complicated than the gauge-theory analogue and does not just consist of global translations as one might naively expect, replacing $\varepsilon(x) \to \varepsilon(0)$ as in QCD. However, starting from the normal coordinates provides the gauge symmetry in a straightforward fashion, as they directly yield the residual background field.

6.3.3 Fixed-line Normal Coordinates

The next step is to extend the previous discussion to the scenario relevant in SCET, the light-front multipole expansion about $x^{\mu}_{-} = n_{+}x\frac{n^{\mu}_{-}}{2}$. For the metric tensor, this means that instead of (6.3.14) one uses the expansion

$$\begin{aligned}g_{s\mu\nu}(x) &= g_{s\mu\nu}(x_-) + x^{\alpha}_{\perp}\left[\partial_{\alpha}g_{s\mu\nu}\right](x_-) + \frac{1}{2}n_{-}x\left[n_{+}\partial g_{s\mu\nu}\right](x_-) \\ &\quad + \frac{1}{2}x^{\alpha}_{\perp}x^{\beta}_{\perp}\left[\partial_{\alpha}\partial_{\beta}g_{s\mu\nu}\right](x_-) + \mathcal{O}(\lambda^3 g_{s\mu\nu})\,.\end{aligned} \tag{6.3.29}$$

The main complication in this setting, similar to gauge theory, compare Section 3.3.3 is the x_- dependence of the fields in the expansion (6.3.29). The individual terms are now no longer constants but are dynamical. In gauge theory, the generalisation of fixed-point gauge is fixed-line gauge with condition $(x - x_-)^{\mu}A_{\mu}(x) = 0$, where the fixed-point condition is only applied in the directions transverse to x_-. Along the direction of x_-, there is an unconstrained background field $n_- A_s(x_-)$ with residual gauge symmetry. We can anticipate that there will be a similar result in gravity.

Therefore, the plan is to construct the RNC also only in the direction transverse to x_-. The immediate generalisation of the RNC gauge condition (6.3.4) is to adapt the fixed-line prescription as

$$(x - x_-)^\alpha (x - x_-)^\beta \Gamma^\mu{}_{\alpha\beta}(x) = 0 \,. \tag{6.3.30}$$

Note that this is not a complete gauge-fixing, since the components $\Gamma^\mu{}_{--}$ are not constrained. We denote these coordinates as fixed-line normal coordinates (FLNC). In the following, we derive the relevant translation parameter θ_{FLNC} using the geodesic equation similar to the previous section. To simplify the notation, we assume in the following that soft fields without argument are evaluated at x_-, i.e. $g_{s\mu\nu}(x_-) \equiv g_{s\mu\nu}$.

Instead of the parameterisation (6.3.5), a more suitable choice for the light-front setting is

$$y^\mu(s) = x_-^\mu + s(x - x_-)^\mu + v^\mu(s) \,, \tag{6.3.31}$$

where the geodesic is only straight in the directions transverse to x_-^μ, since these are the directions where we want to construct the RNC.

Following the exact same steps and computations as in the previous sections, i.e. using the geodesic equation (6.3.7) and solving for the displacement $v^\mu(s)$ iteratively, one determines the fixed-line analogue of (6.3.12) to be (6.3.12) is given by

$$\begin{aligned} \tilde{x}^\mu = x^\mu &+ \frac{1}{2}(x - x_-)^\alpha (x - x_-)^\beta \Gamma^\mu{}_{\alpha\beta} \\ &+ \frac{1}{6}(x - x_-)^\alpha (x - x_-)^\beta (x - x_-)^\nu \left(\Gamma^\mu{}_{\alpha\tau}\Gamma^\tau{}_{\beta\nu} + \left[\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right]\right) + \mathcal{O}((x - x_-)^4) \,. \end{aligned} \tag{6.3.32}$$

Note the similarity to the RNC result (6.3.12), in particular if one restricts to the transverse coordinates. This result is the equivalent of (6.3.12), and we have not yet performed a linear transformation to simplify the leading-order term in (6.3.29). This coordinate transformation will not change $g_{s\mu\nu}(x_-)$, and it will only partially eliminate the first-derivative terms since $\Gamma^\mu{}_{--}$ is not constrained by the gauge. Thus, there is a residual gauge symmetry, which is related to the symmetries of $g_{s\mu\nu}(x_-)$ as well as the Christoffel symbol. In order to cast this symmetry in a more useful form, we will again add a linear transformation in the transverse direction (that can now depend on x_-) to simplify the residual leading term as much as possible.

In the same spirit as before, we introduce the “vierbein” $e_\mu{}^\alpha(x_-)$ via

$$g_{s\mu\nu}(x_-) \equiv e_\mu{}^\alpha(x_-) e_\nu{}^\beta(x_-) \eta_{\alpha\beta} \,, \tag{6.3.33}$$

to diagonalise the metric tensor. This is the direct generalisation of the constant matrix $e_\mu{}^\alpha$ used in (6.3.24), and is formally also equivalent to a vierbein. However, let us emphasize that we do not introduce a local inertial frame, rather we perform a rotation of the coordinate system. The weak-field expansion is the standard (4.4.29)

$$e_\mu{}^\alpha = \delta_\mu^\alpha + \frac{1}{2}s_\mu^\alpha - \frac{1}{8}s_{\mu\beta}s^{\beta\alpha} + \mathcal{O}(s^3) \,. \tag{6.3.34}$$

Just as in (6.3.26), we now use the vierbein to rotate the transverse components of the FLNC coordinates $\tilde{x}^\mu$. The resulting new coordinate system $\check{x}$ is given by

$$\check{x}_- = \tilde{x}_- \,, \quad \check{x}_\perp^\mu = e_\alpha{}^{\mu_\perp}\tilde{x}^\alpha \,, \quad n_-\check{x} = n_{-\rho} e_\alpha{}^\rho \tilde{x}^\alpha \,, \tag{6.3.35}$$

where the $\tilde{x}_-$-coordinate is unchanged since we only rotate the transverse directions.

To determine the parameter $\theta_{\rm FLNC}$, use (6.3.32) to evaluate (6.3.35) to express $\check{x}$ in terms of the original coordinate x. The relation between both coordinate systems then reads

$$\begin{aligned} x^\mu = \check{x}^\mu &+ (E^\mu{}_\alpha - \delta^\mu_\alpha)(\check{x}-\check{x}_-)^\alpha - \frac{1}{2}(\check{x}-\check{x}_-)^\rho(\check{x}-\check{x}_-)^\sigma E^\alpha{}_\rho E^\beta{}_\sigma \Gamma^\mu{}_{\alpha\beta} \\ &+ \frac{1}{6}(\check{x}-\check{x}_-)^\rho(\check{x}-\check{x}_-)^\sigma(\check{x}-\check{x}_-)^\kappa E^\alpha{}_\rho E^\beta{}_\sigma E^\nu{}_\kappa (2\Gamma^\mu{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\nu} - \left[\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right]) + \mathcal{O}(\check{x}^3)\,, \end{aligned} \tag{6.3.36}$$

where the inverse "vierbein" $E^\mu{}_\alpha(x_-)$ is defined in the standard fashion (4.4.5)

$$E^\mu{}_\alpha(x_-) e_\nu{}^\alpha(x_-) = \delta^\mu_\nu\,, \tag{6.3.37}$$

and its weak-field expansion is similarly given by

$$E^\mu{}_\alpha = \delta^\mu_\alpha - \frac{1}{2}s^\mu_\alpha + \frac{3}{8}s^{\mu\beta}s_{\beta\alpha} + \mathcal{O}(s^3)\,. \tag{6.3.38}$$

From (6.3.36), the parameter $\theta_{\rm FLNC}$ can be read off as

$$\begin{aligned} \theta^\mu_{\rm FLNC}(x) \;\; = \;\; &(E^\mu{}_\rho - \delta^\mu_\rho)(x-x_-)^\rho - \frac{1}{2}(x-x_-)^\rho(x-x_-)^\sigma E^\alpha{}_\rho E^\beta{}_\sigma \Gamma^\mu{}_{\alpha\beta} \\ &+ \frac{1}{6}(x-x_-)^\rho(x-x_-)^\sigma(x-x_-)^\lambda E^\alpha{}_\rho E^\beta{}_\sigma E^\nu{}_\lambda (2\Gamma^\mu{}_{\alpha\tau}\Gamma^\tau{}_{\beta\nu} - \left[\partial_\nu \Gamma^\mu{}_{\alpha\beta}\right]) + \mathcal{O}(x^4)\,. \end{aligned} \tag{6.3.39}$$

Comparing this to (6.3.28), we again see a strong formal similarity. Basically, these coordinates are the standard (rotated) RNC in the transverse directions, only the x_- part differs. However, note that each vierbein and field appearing in (6.3.39) depends on x_- and thus lives on the classical trajectory of the energetic particles, and is not constant like in (6.3.28). Furthermore, these coordinates will not completely fix the gauge of the metric tensor, as was already anticipated from the gauge-fixing condition (6.3.30). Therefore, when evaluating the metric tensor in these special coordinates, we will find a non-trivial residual background field.

The next step is to construct the R "Wilson line" just as before (6.3.17). The parameter is given in (6.3.39), and the R Wilson line (6.3.17) is then defined as

$$R^{-1}_{\rm FLNC}(x) = T_{\theta_{\rm FLNC}(x)}\,. \tag{6.3.40}$$

The metric tensor in fixed-line gauge, denoted by $\check{g}_{s\mu\nu}(x)$, is then defined as in (6.3.21) as

$$\check{g}_{s\mu\nu}(x) \equiv R^\alpha{}_\mu(x) R^\beta{}_\nu(x) \left[R^{-1}_{\rm FLNC}(x) g_{s\alpha\beta}(x)\right]\,, \tag{6.3.41}$$

where the Jacobian $R^\alpha{}_\mu$ is defined as in (6.3.20), using $R^{-1}_{\rm FLNC}(x)$.

We now compute (6.3.41) explicitly to second order.

First, for the transverse components $\check{g}_{s\mu_\perp\nu_\perp}(x)$, determine the Jacobian $R^\alpha{}_{\mu_\perp}(x)$ (6.3.20) to be

$$\begin{aligned} R^\alpha{}_{\mu_\perp}(x) = E^\alpha{}_{\mu_\perp} &- (x-x_-)^\rho E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho \Gamma^\alpha{}_{\kappa\lambda} \\ &+ \frac{1}{6}(x-x_-)^\rho(x-x_-)^\sigma \Big(2E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho E^\nu{}_\sigma (2\Gamma^\alpha{}_{\nu\tau}\Gamma^\tau{}_{\kappa\lambda} - [\partial_\nu \Gamma^\alpha{}_{\kappa\lambda}]) \\ &+ E^\kappa{}_\rho E^\lambda{}_\sigma E^\nu{}_{\mu\perp} (2\Gamma^\alpha{}_{\nu\tau}\Gamma^\tau{}_{\kappa\lambda} - [\partial_\nu \Gamma^\alpha{}_{\kappa\lambda}]) \Big) + \mathcal{O}(x^3)\,. \end{aligned} \tag{6.3.42}$$

Evaluating (6.3.41), we find

$$\check{g}_{s\mu_\perp\nu_\perp}(x) = \eta_{\mu_\perp\nu_\perp} - \frac{1}{3}(x-x_-)^\rho(x-x_-)^\sigma E^\alpha{}_\rho E^\beta{}_\sigma E^\kappa{}_{\mu_\perp} E^\lambda{}_{\nu_\perp} R_{\alpha\kappa\beta\lambda} + \mathcal{O}(x^3)\,. \tag{6.3.43}$$

Note here that the transverse components in FLNC basically correspond to the standard RNC result (6.3.3). The main difference is that all objects still have residual dependence on x_-, but the overall form (including the appearance of the constant Minkowski metric) is very similar. The same also holds for the components $\check{g}_{\mu_\perp +}$ and $\check{g}_{++}$. Note that the same situation occurs in gauge theory with the components $n_+ A_s$ and $A_{s\perp}$, which basically satisfy the fixed-point identities.

If one index is contracted with n_-^μ, the results differ from the RNC result. For example, consider

$$\check{g}_{s\mu_\perp -}(x) = R^\alpha{}_{\mu_\perp}(x) R^\beta{}_-(x) \left[R^{-1}_{\mathrm{FLNC}}(x) g_{s\alpha\beta}(x) \right] . \tag{6.3.44}$$

Here, the combination $R^\alpha{}_{\mu_\perp}(x) R^\beta{}_-(x)$ is found to be

$$\begin{aligned} R^\alpha{}_{\mu_\perp}(x) R^\beta{}_-(x) &= E^\alpha{}_{\mu_\perp} n_-^\beta - y^\rho \left(E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho \Gamma^\alpha{}_{\kappa\lambda} n_-^\beta - E^\alpha{}_{\mu_\perp} \partial_- E^\beta{}_\rho \right) \\ &\quad + y^\rho y^\sigma \left(\frac{1}{6} n_-^\beta \left(2 E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho E^\nu{}_\sigma + E^\kappa{}_\rho E^\lambda{}_\sigma E^\nu{}_{\mu_\perp} \right) \left(2\Gamma^\alpha{}_{\nu\tau} \Gamma^\tau{}_{\kappa\lambda} - [\partial_\nu \Gamma^\alpha{}_{\kappa\lambda}] \right) \right. \\ &\quad \left. - \partial_- E^\beta{}_\rho E^\kappa{}_{\mu_\perp} E^\lambda{}_\sigma \Gamma^\alpha{}_{\kappa\lambda} - \frac{1}{2} E^\alpha{}_{\mu_\perp} \partial_- (E^\kappa{}_\rho E^\lambda{}_\sigma \Gamma^\beta{}_{\kappa\lambda}) \right) + \mathcal{O}(x^3) \,, \end{aligned} \tag{6.3.45}$$

where we defined $y^\rho \equiv (x - x_-)^\rho$. The remaining factor $\left[R^{-1}_{\mathrm{FLNC}}(x) g_{s\alpha\beta}(x) \right]$ is computed as

$$\begin{aligned} \left[R^{-1}_{\mathrm{FLNC}}(x) g_{s\alpha\beta}(x) \right] &= g_{s\alpha\beta} + y^\rho E^\kappa{}_\rho [\partial_\kappa g_{s\alpha\beta}] + \frac{1}{2} y^\rho y^\sigma E^\kappa{}_\rho E^\lambda{}_\sigma [\partial_\kappa \partial_\lambda g_{s\alpha\beta}] \\ &\quad - \frac{1}{2} y^\rho y^\sigma E^\kappa{}_\rho E^\lambda{}_\sigma \Gamma^\tau{}_{\kappa\lambda} [\partial_\tau g_{s\alpha\beta}] + \dots \,. \end{aligned} \tag{6.3.46}$$

Then one can simply multiply (6.3.45) and (6.3.46) and compute the result order by order in x. For the leading term at $\mathcal{O}(x^0)$, one finds

$$\check{g}^{(0)}_{s\mu_\perp -}(x) = E^\alpha{}_{\mu_\perp} g_{s\alpha -} = E^\alpha{}_{\mu_\perp} e_\alpha{}^\rho e_-{}^\sigma \eta_{\rho\sigma} = e_{-\mu_\perp} \tag{6.3.47}$$

which is simply the "vierbein" $e_\mu{}^\alpha(x_-)$ that diagonalises the metric tensor, as given in (6.3.33).

Next, at $\mathcal{O}(x)$, one can again use the identities (6.3.15) to manipulate the first-derivative terms, like in the RNC scenario. Then, one obtains

$$\begin{aligned} \check{g}^{(1)}_{s\mu_\perp -}(x) &= y^\rho \left(E^\alpha{}_{\mu_\perp} E^\kappa{}_\rho [\partial_\kappa g_{s\alpha -}] - E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho \Gamma^\alpha{}_{\kappa\lambda} g_{s\alpha -} + E^\alpha{}_{\mu_\perp} \left[\partial_- E^\beta{}_\rho \right] g_{s\alpha\beta} \right) \\ &= y^\rho \left(E^\alpha{}_{\mu_\perp} E^\kappa{}_\rho (\Gamma^\tau{}_{\kappa\alpha} g_{s\tau -} + \Gamma^\tau{}_{\kappa -} g_{s\tau\alpha}) - E^\kappa{}_{\mu_\perp} E^\lambda{}_\rho \Gamma^\alpha{}_{\kappa\lambda} g_{s\alpha -} + E^\alpha{}_{\mu_\perp} \left[\partial_- E^\beta{}_\rho \right] g_{s\alpha\beta} \right) \\ &= y^\rho \left(E^\alpha{}_{\mu_\perp} \left[\partial_- E^\beta{}_\rho \right] g_{s\alpha\beta} + E^\alpha{}_{\mu_\perp} E^\kappa{}_\rho \Gamma^\beta{}_{\kappa -} g_{s\beta\alpha} \right) \\ &= -y^\alpha [\Omega_-]_{\alpha\mu_\perp} \,. \end{aligned} \tag{6.3.48}$$

Here, we introduced a new object, the "spin-connection" $[\Omega_\mu]_{\alpha\beta}$. This object is defined like the standard spin-connection (4.4.37), but constructed from the "vierbein" $e_\mu{}^\alpha$ (6.3.33) as

$$[\Omega_\mu]^{\alpha\beta} = e_\nu{}^\alpha \left[\partial_\mu E^{\nu\beta} \right] + e_\nu{}^\alpha \Gamma^\nu{}_{\sigma\mu} E^{\sigma\beta} \,. \tag{6.3.49}$$

At the second order, one finds the result

$$\check{g}^{(2)}_{s\mu_\perp -}(x) = -\frac{2}{3} y^\alpha y^\beta E^\kappa{}_\alpha E^\lambda{}_\beta E^\rho{}_{\mu_\perp} n_-^\nu R_{\rho\kappa\nu\lambda} \,. \tag{6.3.50}$$

In summary, from (6.3.47), (6.3.48) and (6.3.50), one finds for the transverse-minus component of the metric tensor in FLNC

$$\check{g}_{s\mu_\perp -}(x) = e_{-\mu_\perp} - y^\alpha [\Omega_-]_{\alpha\mu_\perp} - \frac{2}{3} y^\alpha y^\beta E^\kappa{}_\alpha E^\lambda{}_\beta E^\rho{}_{\mu_\perp} n_-^\nu R_{\rho\kappa\nu\lambda} + \mathcal{O}(x^3) \,. \tag{6.3.51}$$

Performing the same computation for $\check{g}_{s+-}$, one obtains

$$\check{g}_{s+-}(x) = e_{-+} - y^\alpha \left[\Omega_-\right]_{\alpha+} - \frac{2}{3} y^\alpha y^\beta E^\kappa{}_\alpha E^\lambda{}_\beta E^\rho{}_+ n_-^\nu R_{\rho\kappa\nu\lambda} + \mathcal{O}(x^3)\,. \tag{6.3.52}$$

Finally, for $\check{g}_{s--}(x)$ the same computation yields

$$\begin{aligned}\check{g}_{s--}(x) = g_{s--} &- 2y^\rho \left[\Omega_-\right]_{\rho\alpha} e_-{}^\alpha - y^\rho y^\sigma E^\kappa{}_\rho E^\lambda{}_\sigma n_-^\mu n_-^\nu R_{\mu\kappa\nu\lambda} \\ &+ y^\rho y^\sigma \big(\left[\partial_- E^\mu{}_\rho\right] \left[\partial_- E^\nu{}_\sigma\right] g_{s\mu\nu} + 2 \left[\partial_- E^\mu{}_\rho\right] E^\kappa{}_\sigma \Gamma^\lambda{}_{\kappa -} g_{s\lambda\mu} \\ &+ E^\kappa{}_\rho E^\lambda{}_\sigma \Gamma^\alpha{}_{\lambda -} \Gamma^\beta{}_{\kappa -} g_{s\alpha\beta}\big) + \mathcal{O}(x^3)\,,\end{aligned} \tag{6.3.53}$$

which one can rewrite as

$$\begin{aligned}\check{g}_{s--}(x) \;=\; &(e_-{}^\alpha - y^\rho \left[\Omega_-\right]_\rho{}^\alpha)(e_-{}^\beta - y^\sigma \left[\Omega_-\right]_\sigma{}^\beta)\eta_{\alpha\beta} - y^\alpha y^\beta E^\kappa{}_\alpha E^\lambda{}_\beta n_-^\mu n_-^\nu R_{\mu\kappa\nu\lambda} \\ &+ \mathcal{O}(x^3)\,.\end{aligned} \tag{6.3.54}$$

In this form, the metric tensor g_{--} is decomposed as $g_{s--} = e_-{}^\alpha e_-{}^\beta \eta_{\alpha\beta}$, and one can read off the "residual vierbein".

At this point one can compare the results (6.3.43), (6.3.51), (6.3.54) in the light-front setting to the corresponding RNC result (6.3.3). As we noted before, the transverse components formally satisfy the standard RNC identities, but now with x_--dependent functions instead of constants. One can even express the leading term $g_{\perp\perp}(x_-) = \eta_{\mu\nu}$ using the additonal linear transformation. For the components where one index is contracted with n_-^μ, however, we see a non-trivial leading-order and first-derivative term, while the second-order (and higher-order) terms are expressed via the Riemann tensor. Basically, one can identify a non-vanishing "residual vierbein" that contains the vierbein at x_- as well as the spin-connection. It is useful to split the metric field into this residual background field, which is denoted by $\hat{g}_{s\mu\nu}$, and a gauge-covariant part, that contains the Riemann-tensor terms. This term is denoted by $\mathfrak{g}_{s\mu\nu}$.[2] Thus, we split

$$\check{g}_{s\mu\nu}(x) \equiv \hat{g}_{s\mu\nu}(x) + \mathfrak{g}_{s\mu\nu}(x)\,, \tag{6.3.55}$$

where the residual background field can be determined from (6.3.43), (6.3.51) (6.3.52), and (6.3.54) to be

$$\hat{g}_{s+-}(x) = e_{-+} - (x - x_-)^\alpha \left[\Omega_-\right]_{\alpha+}\,, \tag{6.3.56}$$

$$\hat{g}_{s\mu_\perp -}(x) = e_{-\mu_\perp} - (x - x_-)^\alpha \left[\Omega_-\right]_{\alpha\mu_\perp}\,, \tag{6.3.57}$$

$$\hat{g}_{s--}(x) = \left(e_-{}^\alpha - (x - x_-)^\rho \left[\Omega_-\right]_\rho{}^\alpha\right)\left(e_-{}^\beta - (x - x_-)^\sigma \left[\Omega_-\right]_\sigma{}^\beta\right)\eta_{\alpha\beta}\,, \tag{6.3.58}$$

$$\hat{g}_{s\mu_\perp\nu_\perp}(x) = \eta_{\mu_\perp\nu_\perp}\,. \tag{6.3.59}$$

We see that the residual background field has two independent contributions, $e_-{}^\alpha(x_-)$ and $\left[\Omega_-\right]_{\alpha\beta}(x_-)$. At first glance, one might conclude that the spin-connection $\left[\Omega_-\right]_{\alpha\beta}(x_-)$ is constructed from the vierbein $e_-{}^\alpha(x_-)$, and these two objects are not independent. Note, however, that in the effective theory, all soft objects must depend only on the light-cone variable x_-. Therefore, one has in particular $\partial_\perp e_-{}^\alpha(x_-) = 0$. Hence, the spin-connection, containing also transverse derivatives of the vierbein, is an object that cannot be constructed from $e_-{}^\alpha(x_-)$, but only from the vierbein evaluated at x. From the point of view of the effective theory, these two objects should be treated as truly independent fields, which also come with their own separate gauge symmetries, inherited from the full theory. In the soft weak-field expansion, these objects are given by

$$e_-{}^\alpha = \delta_-^\alpha + \frac{1}{2} s_-^\alpha - \frac{1}{8} s_{-\beta} s^{\beta\alpha} + \mathcal{O}(s^3)\,, \tag{6.3.60}$$

[2] This is the same split we performed in gauge theory, where the background corresponds to $n_- A_s(x_-)$, and the gauge-covariant part corresponds to $\mathcal{A}_s$ in (3.3.66).

$$[\Omega_-]_{\alpha\beta} = -\frac{1}{2}\left([\partial_\alpha s_{\beta -}] - [\partial_\beta s_{\alpha -}]\right) + \mathcal{O}(s^2)\,. \tag{6.3.61}$$

In the gauge-covariant part $\mathfrak{g}_{s\mu\nu}$, only the Riemann tensor and its derivatives appear. This result is very similar to the gauge-theory situation discussed in Section 3.4.3, where one obtains the soft background field $n_- A_s(x_-)$ as well as a tower of subleading terms that are expressed in terms of the soft field-strength tensor $F^s_{\mu\nu}$. These terms are collected in the covariant expression $\mathcal{A}_s$ (3.3.66), the analogue of $\mathfrak{g}_{s\mu\nu}$.

This two-fold homogeneous background field is one of the most important results for the construction of SCET gravity. Using background field methods similar to the gauge-theory case, one can construct the effective Lagrangian to be covariant with respect to this background field $\hat{g}_{s\mu\nu}$. Then, the soft gauge symmetry of this background automatically respects the multipole expansion and is the analogue of the homogeneous gauge transformation in the gauge-theory case. Crucially, with this background field, the effective theory is cast into a form very similar to an ordinary gauge theory. In the following, we show how the residual background metric $\hat{g}_{s\mu\nu}$ naturally gives rise to a soft-covariant derivative which contains the two gauge-fields. On the formal level, the soft-collinear sector of gravity is then analogous to gauge theory, where a gauge-covariant derivative mediates the leading-power interactions, while the subleading terms are expressed in terms of gauge-covariant building blocks.

6.3.4 Soft-covariant Derivative

In this section, we show how the residual background field $\hat{g}_{s\mu\nu}$ in fixed-line normal coordinates can be arranged into a soft-covariant derivative and determine its residual gauge symmetry.

The starting point is the split (6.3.55) of the metric tensor in fixed-line coordinates into the residual background metric $\hat{g}_{s\mu\nu}$ and the gauge-covariant Riemann-tensor terms $\mathfrak{g}_{s\mu\nu}$. The crucial observation is that the transverse components of the background metric are trivial, and only $\hat{g}_s^{\mu-}$ is non-trivial (6.3.56) – (6.3.59). This means that one index of the background metric must be contracted with n_-^μ to generate soft interactions. This, in turn, allows one to arrange this metric tensor as a covariant derivative. For example, consider the simple scalar expression

$$\hat{g}_s^{\mu\nu}\partial_\mu\varphi_c\,\partial_\nu\varphi_c\,. \tag{6.3.62}$$

Splitting the metric tensor into its light-cone components yields

$$\hat{g}_s^{\mu\nu}\partial_\mu\varphi_c\,\partial_\nu\varphi_c = \hat{g}_s^{\mu-}\partial_\mu\varphi_c\, n_+\partial\varphi_c + \eta_\perp^{\mu\nu}\partial_{\mu_\perp}\varphi_c\,\partial_{\nu_\perp}\varphi_c\,. \tag{6.3.63}$$

Note that $\hat{g}_s^{\mu-}$ always appears in combination with n_+^ν. Therefore, one can define a soft-covariant derivative $n_- D_s$ as

$$\begin{aligned} n_- D_s &\equiv \hat{g}_s^{\mu-}\partial_\mu \\ &= \partial_- - \frac{1}{2}s_{-\mu}\partial^\mu + \frac{1}{8}s_{+-}s_{--}\partial_+ + \frac{1}{16}s_{-\alpha_\perp}s_-^{\alpha_\perp}\partial_+ + \frac{1}{2}\left[\Omega_-\right]_{\mu\nu}J^{\mu\nu} + \mathcal{O}(\lambda^3)\,, \end{aligned} \tag{6.3.64}$$

where $J^{\mu\nu} = (x-x_-)^\mu\partial^\nu - (x-x_-)^\nu\partial^\mu \equiv (x-x_-)^{[\mu}\partial^{\nu]}$ is the orbital Lorentz generator or angular momentum.[3] Note that this covariant derivative is substantially different from the one encountered in gauge theory. First, it is a non-linear expression in the soft graviton $s_{\mu\nu}$, due to the weak-field expansion performed in gravity. Second, it depends on two *independent* objects, one stemming from $e_-{}^\alpha(x_-)$, and the spin-connection term $[\Omega_-]_{\alpha\beta}(x_-)$ proportional to $(x-x_-)$ from the multipole expansion. These fields begin at $\mathcal{O}(\lambda^2)$ and $\mathcal{O}(\lambda^4)$, respectively. The first field, the vierbein, couples to the linear momentum of the scalar field, and we see that the leading-order interaction is simply

$$-\frac{1}{2}s_{-\mu}\partial^\mu\,, \tag{6.3.65}$$

[3]"Orbital" here means that there is no additional spin-dependent term.

the minimal coupling of the soft graviton $s_{-\mu}(x_-)$ living on the collinear trajectory to the collinear momentum $\partial^\mu = -iP^\mu$ of the scalar particle. This term then receives higher-order non-linear corrections as is usual in gravity. The second term, the spin-connection $[\Omega_-]_{\alpha\beta}(x_-)$, can be seen to couple to the orbital angular momentum $J^{\mu\nu}$ along the light-cone of the scalar particle. Note that this is not the full angular momentum, since only the transverse coordinates $(x-x_-)^\mu$ appear in its definition. This angular momentum arises from combining the $(x-x_-)^\mu$ from the multipole expansion with the ∂^ν from the derivative of the scalar. This interaction term

$$\frac{1}{2}\left[\Omega_-\right]_{\mu\nu} J^{\mu\nu} \equiv \frac{1}{2}\Omega_- \tag{6.3.66}$$

quite naturally fits the standard form of the spin-connection term, where one can absorb the Lorentz generator in the definition of Ω_μ.

Let us stress the importance of this result. Simply from identifying the residual background metric in fixed-line gauge, one sees that the soft-collinear interactions can be arranged in a covariant derivative consisting of *two* independent gauge-fields. This is very unusual, as a scalar field does not feature any covariant derivatives in gravity. This covariant derivative controls the leading interactions and the gauge fields couple to the *linear momentum* as well as the *angular momentum* of the scalar field. Therefore, the relevant gauge charges for soft physics are not just the linear momentum, but also the angular momentum. This is a feature we encountered very early in the discussion already when discussing the soft theorem in Section 1.1. Here, one can realise that the first two terms of the soft theorem actually correspond to this two-fold gauge symmetry. This point will be expanded on in detail in the later Section 7.5.

For now, let us investigate the actual form of the residual symmetry of these fields. Since the effective theory will be constructed to be covariant with respect to the residual metric tensor $\hat{g}_{s\mu\nu}$, that is, this metric will be used to raise and lower indices, any generic tensors must be defined to transform according to its residual symmetry.

To find these transformations, first weak-field expand the metric as

$$\hat{g}_{s\mu\nu}(x) = \eta_{\mu\nu} + \hat{s}_{\mu\nu}(x)\,, \tag{6.3.67}$$

where $\hat{s}_{\mu\nu}$ can be determined in terms of $s_{\mu\nu}$ by comparison with (6.3.56) – (6.3.59) in weak-field expansion. The original soft graviton $s_{\mu\nu}(x)$ has the standard gauge transformation

$$s_{\mu\nu}(x) \to s_{\mu\nu}(x) - \partial_\mu\varepsilon_\nu(x) - \partial_\nu\varepsilon_\mu(x) + \mathcal{O}(s^2)\,. \tag{6.3.68}$$

Therefore, one can simply insert the transformation (6.3.68) in (6.3.67) to find the transformations of the residual fluctuation $\hat{s}_{\mu\nu}$. As a concrete example, consider the linear terms of $\hat{s}_{\mu_\perp -}(x)$. For this component, one finds from (6.3.59), (6.3.60), (6.3.61)

$$\hat{s}_{\mu_\perp -}(x) = \frac{1}{2}s_{\mu_\perp -} + \frac{1}{2}x_\perp^\alpha\left[\partial_{[\alpha}s_{\mu_\perp]-}\right] + \frac{1}{4}n_-x\left[\partial_{[+}s_{\mu_\perp]-}\right]\,. \tag{6.3.69}$$

Its transformation follows from inserting (6.3.68) and is given by

$$\begin{aligned}
\frac{1}{2}s_{\mu_\perp -} \to{}& \frac{1}{2}s_{\mu_\perp -} - \frac{1}{2}\partial_{\mu_\perp}\varepsilon_- - \frac{1}{2}\partial_-\varepsilon_{\mu_\perp}\\
={}& \frac{1}{2}s_{\mu_\perp -} - \partial_-\varepsilon_{\mu_\perp} + \frac{1}{2}\left(\partial_-\varepsilon_{\mu_\perp} - \partial_{\mu_\perp}\varepsilon_-\right)\\
={}& \frac{1}{2}s_{\mu_\perp -} - \partial_-\varepsilon_{\mu_\perp} + \omega_{-\mu_\perp}\,,
\end{aligned} \tag{6.3.70}$$

where we introduced the antisymmetric parameter

$$\omega_{\mu\nu} \equiv \frac{1}{2}\left(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu\right)\,. \tag{6.3.71}$$

By performing the same steps also for the other components, including non-linear terms and the spin-connection, one determines the infinitesimal gauge transformation of the residual metric field $\hat{s}_{\mu-}$ as

$$\hat{s}_{\mu-}(x) \to \hat{s}_{\mu-}(x) - \partial_- \varepsilon_\mu + \omega_{-\mu_\perp} - \partial_- \left((x-x_-)^\alpha \omega_{\mu\alpha}\right) + \dots \,. \tag{6.3.72}$$

This transformation corresponds to a coordinate transformation

$$x^\mu \to x^\mu + \varepsilon^\mu(x_-) + \omega^\mu{}_\nu(x_-)(x-x_-)^\nu \,, \tag{6.3.73}$$

and is generated by the *independent* parameters $\varepsilon_\mu(x_-)$ and the antisymmetric $\omega_{\mu\nu}(x_-)$. The first one corresponds to infinitesimal local translations, which are restricted to the classical trajectory x_-. The second parameter is also restricted to the collinear light-cone, and its corresponding generator is the residual angular momentum $J^{\mu\nu} = (x-x_-)^{[\mu}\partial^{\nu]}$. This corresponds to infinitesimal Lorentz transformations about $(x-x_-)$. In summary, the transformation (6.3.73) describes an infinitesimal *local Poincaré transformation* that is restricted to the classical trajectory of the energetic particles, i.e. all parameters can only depend on the light-cone component $x_-^\mu = n_+x\frac{n_-^\mu}{2}$. This is the analogue of the homogeneous gauge theory $U_s(x_-)$ in QCD that respects the multipole expansion. Note, however, that due to the scaling of the gauge charges, i.e. the momentum and angular momentum, this "homogeneous" gauge symmetry in gravity is *not* homogeneous in λ. It mixes different orders. Therefore, also the corresponding covariant derivative cannot be homogeneous and is instead defined as a non-linear object order-by-order in λ.

To extend this beyond the infinitesimal level, one can employ the closed forms (6.3.56) – (6.3.59) and perform the same calculations as above to find the (non-linear) transformation to any desired order in λ. Here, also the non-linear terms in the weak-field expansions of the vierbein $e_-{}^\alpha(x_-)$ (6.3.60) and spin-connection $[\Omega_-]_{\alpha\beta}(x_-)$ (6.3.61) are relevant and must be included. In addition, one has to use the full non-linear transformation of the fluctuation $s_{\mu\nu}(x)$ from (6.1.4) to the desired order.

On a related note, let us also check the transformation of derivatives under these Poincaré transformations. Since we only identified n_-D_s, this implies that the other derivatives should already transform covariantly, consistent with the idea that only the $n_-\partial$ derivative is sensitive to the local nature of the gauge transformations. We again consider the infinitesimal transformation (6.3.73),

$$\varphi_c'(x) = \varphi(x) - \varepsilon^\alpha \partial_\alpha \varphi(x) - \omega_{\alpha\beta}\,(x-x_-)^\beta \partial^\alpha \varphi(x) + \mathcal{O}(\varepsilon^2) \,. \tag{6.3.74}$$

For the transverse components $\partial_\perp$ and $n_+\partial$, one simply finds

$$\partial_{\mu_\perp}\varphi(x) \to T^{-1}_{\varepsilon+\omega}\left[\partial_{\mu_\perp}\varphi(x)\right] - \omega_{\mu_\perp\alpha}\partial^\alpha\varphi(x) + \mathcal{O}(\varepsilon^2) \,, \tag{6.3.75}$$

$$\partial_+\varphi(x) \to T^{-1}_{\varepsilon+\omega}\left[\partial_+\varphi(x)\right] - \omega_{+\alpha}\partial^\alpha\varphi(x) + \mathcal{O}(\varepsilon^2) \,, \tag{6.3.76}$$

where $T^{-1}_{\varepsilon+\omega}$ is defined as the translation

$$T^{-1}_{\varepsilon+\omega} = 1 - \varepsilon^\alpha\partial_\alpha - \omega_{\alpha\beta}\,(x-x_-)^\beta\partial^\alpha + \mathcal{O}(\varepsilon^2) \,. \tag{6.3.77}$$

These correspond to the standard transformations of a derivative under infinitesimal Poincaré transformations. The coordinate transformation is pulled in front of the derivative, and a rotation of the derivative is generated. For $n_-\partial$, however, the transformation is not correct. This derivative is sensitive to the dependence of the gauge parameters on x_- and acts on these as well. One obtains

$$n_-\partial\varphi(x) \to T^{-1}_{\varepsilon+\omega}\left[n_-\partial\varphi(x)\right] - n_-\partial\varepsilon^\alpha\partial_\alpha\varphi(x) - n_-\partial\omega_{\alpha\beta}\,(x-x_-)^\beta\partial^\alpha\varphi(x) + \mathcal{O}(\varepsilon^2) \,. \tag{6.3.78}$$

We already anticipated this result, since the effective theory contains the soft-covariant derivative n_-D_s (6.3.64). At the linear order, it is given by

$$n_-D_s = \partial_- - \frac{1}{2}s_{-\mu}\partial^\mu - \frac{1}{4}\left[\Omega_-\right]_{\mu\nu} J^{\mu\nu} + \dots\,, \tag{6.3.79}$$

and its infinitesimal transformation is indeed

$$n_-D_s\varphi(x) \to T^{-1}_{\varepsilon+\omega}\left[n_-D_s\varphi(x)\right] - \omega_{-\alpha}D_s^\alpha\varphi(x)\,, \tag{6.3.80}$$

where $D_s^\alpha = \frac{n_-^\alpha}{2}n_+\partial + \partial_\perp^\alpha + \frac{n_+^\alpha}{2}n_-D_s$. To check this, simply insert the transformation of $\hat{s}_{\mu\nu}$ from (6.3.72), which yields

$$\frac{1}{2}s_{-\mu} \to \frac{1}{2}s_{-\mu} - \partial_-\varepsilon_\mu + \omega_{-\mu} + \dots\,, \tag{6.3.81}$$

$$\left[\Omega_-\right]_{\alpha\beta} \to \left[\Omega_-\right]_{\alpha\beta} - \partial_-\omega_{\alpha\beta} + \dots \tag{6.3.82}$$

Therefore, we see that the set of derivatives $(n_-D_s, \partial_\perp, n_+\partial)$ indeed transform covariantly under the local Poincaré transformations. This explicitly verifies that the effective theory only features a non-trivial n_-D_s, analogous to gauge theory and that the previous geometric intuition in the construction of n_-D_s is indeed correct.

There is one additional important remark to make regarding the gauge-invariance of the action. The covariant derivative defined here is covariant in the sense that it transforms under these *local* Poincaré transformations just like an *ordinary* derivative would transform under *global* ones. This implies that a scalar object, e.g.

$$n_-D_s\varphi_c n_+\partial\varphi_c + \partial_{\mu_\perp}\varphi_c\partial^{\mu_\perp}\varphi_c \tag{6.3.83}$$

indeed transforms like a scalar field, namely as

$$n_-D_s\varphi_c n_+\partial\varphi_c + \partial_{\mu_\perp}\varphi_c\partial^{\mu_\perp}\varphi_c \to T^{-1}_{\varepsilon+\omega}\left[n_-D_s\varphi_c n_+\partial\varphi_c + \partial_{\mu_\perp}\varphi_c\partial^{\mu_\perp}\varphi_c\right]\,, \tag{6.3.84}$$

with the translation as defined in (6.3.77). Therefore, to render such a term manifestly gauge-invariant, one requires in addition the presence of the invariant measure $d^4x\sqrt{-\hat{g}_s}$. This is a standard feature of gravity. The Lagrangian always transforms like a scalar field,[4] and it is only the action, which comes with the invariant measure, that is manifestly invariant under diffeomorphisms.

In summary, when the theory is constructed to be covariant with respect to the residual metric field $\hat{g}_{s\mu\nu}$, the soft graviton can only appear inside the Lagrangian via the covariant derivative n_-D_s or inside the Riemann tensor terms, and in the definition of gauge-invariant collinear building blocks as we explain below (6.4.6), e.g. by raising or lowering indices with $\hat{g}_{s\mu\nu}$. In addition, it appears via the metric determinant in front of the Lagrangian, but only once to render it a scalar density. This already implies the soft-collinear factorisation of the effective theory.

Let us summarise the key insights of this section:

- In gravity, the analogue of fixed-point gauge are Riemann normal coordinates. These normal coordinates must be generalised to the *fixed-line* normal coordinates (6.3.39).

- Fixed-line normal coordinates are not a complete gauge-fixing. Instead, there is a residual background field $\hat{g}_{s\mu\nu}$ (6.3.56) – (6.3.59). This field contains two *independent* components, the leading $e_{-\mu}(x_-)$ and the subleading $\left[\Omega_-\right]_{\alpha\beta}(x_-)$, as given in (6.3.60) and (6.3.61). These two gauge fields are the counterpart of the homogeneous background field n_-A_s in gauge theory.

[4] Or, if one uses $\sqrt{g}\mathcal{L}$ as Lagrangian, like a scalar density.

- Any further sub-subleading and higher-order terms are expressed via the manifestly gauge-invariant Riemann tensor and can be constructed systematically to any desired order in λ. These terms correspond to the field-strength tensor terms in gauge theory (3.3.49) – (3.3.56), but *cannot* be expressed in a closed form.
- The residual symmetries of the "homogeneous" background field correspond to the effective gauge symmetry of SCET gravity. These symmetries correspond to local Poincaré transformations that are restricted to the classical trajectory x_-^μ of the energetic particles. The effective theory is constructed to be covariant with respect to this soft background symmetry.
- The two gauge fields can be allocated into a soft-covariant derivative n_-D_s (6.3.64).
- This restricts the appearance of the soft graviton: it can only appear inside the metric determinant g_s, inside the covariant derivative n_-D_s, inside collinear gauge-invariant building blocks to raise indices, and inside the Riemann tensor.

6.4 Redefinitions and Collinear Gauge-invariant Building Blocks

In the previous section, we discussed in detail the fixed-line normal coordinates. Here, we encountered the analogue of the R Wilson line that moves a field in FLNC and the corresponding residual metric field, which serves as background in the effective theory. In addition, we found that this metric field can be arranged as a covariant derivative. We undertook this detour in order to identify the set of homogeneous gauge transformations, i.e. the ones that respect the multipole expansion. This is important since the final Lagrangian should be expressed in terms of fields whose gauge transformation does not mix different orders in $(x - x_-)$. In gauge theory, we redefine the collinear fields and express them in terms of $\hat{\phi}_c$ (3.3.94) and $\hat{A}_c$ (3.3.92), which have homogeneous gauge transformations. These fields correspond to fields that transform covariantly with respect to the residual soft background $n_-A_s(x_-)$. Therefore, in gravity, we want to introduce analogues of the hatted fields, which transform covariantly with respect to the emergent background metric $\hat{g}_{s\mu\nu}$.

In gauge theory, these redefinitions use both the collinear Wilson line W_c, as well as the fixed-line Wilson line R, and are explained in detail in Section 3.3.5. The collinear Wilson line W_c is used to fix collinear light-cone gauge, and the R Wilson line then parallel transports the gauge symmetry to $x = x_-$.

In gravity, we already identified the analogue of W_c in the purely-collinear theory in Section 5.3. Here, the main motivation was to control the large components $h_{\mu+}$ of the collinear graviton field. Now in the soft-collinear setting, there is a small subtlety compared to the purely-collinear situation. Since our residual background field $\hat{g}_{s\mu\nu}$ has a non-vanishing component $\hat{g}_{s+-}$ (6.3.56), it will appear in the collinear gauge transformation of $\hat{h}_{+-}$. This can already be seen from the linear transformation

$$\hat{h}_{+-} \to \hat{h}_{+-} - \hat{\nabla}_+\varepsilon_- - \hat{\nabla}_-\varepsilon_+ \,, \tag{6.4.1}$$

and noting that

$$\hat{\nabla}_-\varepsilon_+ = \partial_-\varepsilon_+ - \hat{\Gamma}^\alpha{}_{+-}\varepsilon_\alpha \tag{6.4.2}$$

is not a purely-collinear object. Here, the Christoffel symbol $\hat{\Gamma}^\mu{}_{\alpha\beta}$ is the one constructed from the metric tensor $\hat{g}_{s\mu\nu} + \hat{h}_{\mu\nu}$, i.e. it is covariant with respect to the homogeneous soft background $\hat{g}_{s\mu\nu}$.

Therefore, the soft background will appear in the collinear Wilson line W in gravity, unlike in QCD, where only collinear fields n_+A_c are present.[5] This, however, only affects the relevant

[5] Note, however, that this is consistent with the notion of Wilson lines. They always have to be defined with respect to the relevant background symmetry in order to transform covariantly.

parameter θ_{LC}, while the formal definition of the Wilson line is the same (5.3.6), and it is again constructed by explicitly fixing light-cone gauge – now with respect to the soft background. The collinear Wilson line is thus given by

$$W_c^{-1} = T_{\theta_{\mathrm{LC}}} = 1 + \theta^{\alpha}_{\mathrm{LC}}\partial_\alpha + \mathcal{O}(\lambda^2)\,, \tag{6.4.3}$$

and the parameter θ_{LC} is determined to be

$$\theta^{\mu}_{\mathrm{LC}} = -\frac{1}{(n_+\partial)^2}\hat{\Gamma}^{\mu}_{++} + \frac{1}{(n_+\partial)^2}\left(2\hat{\Gamma}^{\mu}_{\tau+}\frac{1}{n_+\partial}\hat{\Gamma}^{\tau}_{++} + \partial_\nu\hat{\Gamma}^{\mu}_{++}\frac{1}{(n_+\partial)^2}\hat{\Gamma}^{\nu}_{++}\right) + \mathcal{O}(\lambda^2)\,. \tag{6.4.4}$$

If one sets $\hat{g}_{s\mu\nu} = \eta_{\mu\nu}$, one immediately recovers the purely-collinear result (5.3.13) – (5.3.15).

This Wilson line can now be employed to fix collinear light-cone gauge and define manifestly gauge-invariant collinear fields. For our situation, we require the gauge-invariant scalar $\hat{\chi}_c$ and graviton $\hat{\mathfrak{h}}_{\mu\nu}$. The scalar field is simply defined as

$$\hat{\chi}_c = \left[W_c^{-1}\hat{\varphi}_c\right]\,, \tag{6.4.5}$$

and the gauge-invariant graviton as

$$\hat{\mathfrak{h}}_{\mu\nu}(x) = W^{\rho}{}_{\mu}W^{\sigma}{}_{\nu}\left[W_c^{-1}(\hat{g}_{s\rho\sigma}(x) + \hat{h}_{\rho\sigma}(x))\right] - \hat{g}_{s\mu\nu}(x)\,. \tag{6.4.6}$$

Note the formal equivalence of these objects to the purely-collinear ones (5.3.4). Only the parameter inside W_c is constructed differently, now it is covariant with respect to $\hat{g}_{s\mu\nu}$ instead of $\eta_{\mu\nu}$.

Next, we employ the R Wilson line to relate the hatted fields to the original ones. This Wilson line is also defined as a translation operator (6.3.40)

$$R^{-1} \equiv T_{\theta_{\mathrm{FLNC}}} = 1 + \theta^{\alpha}_{\mathrm{FLNC}}\partial_\alpha + \mathcal{O}(\lambda^2)\,, \tag{6.4.7}$$

but with parameter θ_{FLNC} (6.3.39), which reads

$$\theta^{\mu}_{\mathrm{FLNC}} = (E^{\mu}{}_{\rho} - \delta^{\mu}_{\rho})(x-x_-)^{\rho} - \frac{1}{2}(x-x_-)^{\rho}(x-x_-)^{\sigma}E^{\alpha}{}_{\rho}E^{\beta}{}_{\sigma}\Gamma^{\mu}{}_{\alpha\beta} + \dots\,. \tag{6.4.8}$$

To redefine the collinear fields, one dresses them with R and its Jacobians according to their soft gauge transformation. For the scalar field, this reads

$$\varphi_c = \left[RW_c^{-1}\hat{\varphi}_c\right]\,, \tag{6.4.9}$$

while the graviton is redefined as

$$h_{\mu\nu} = \left[RR_{\mu}{}^{\alpha}R_{\nu}{}^{\beta}\left(W^{\rho}{}_{\alpha}W^{\sigma}{}_{\beta}\left[W_c^{-1}(\hat{h}_{\rho\sigma} + \hat{g}_{s\rho\sigma})\right] - \hat{g}_{s\alpha\beta}\right)\right]\,, \tag{6.4.10}$$

In these equations, we take the original fields on the left-hand side to be gauge-fixed to light-cone gauge. The hatted fields on the right-hand side are not gauge-fixed and have their "homogeneous" gauge transformation. Note that the objects on the right-hand side correspond to the gauge-invariant building blocks, dressed by their appropriate R Wilson lines.

Now we have all the necessary ingredients and can construct the Lagrangian as well as the N-jet operator basis of the effective theory.

6.5 Effective Theory Construction

In the previous sections, we determined the field content and power-counting of the EFT fields. In addition, we investigated fixed-line normal coordinates, the analogue of fixed-line gauge, in order to identify the homogeneous gauge transformations in gravity, which respect the multipole expansion and do not mix different orders in x. We found that these correspond to local Poincaré transformations that are restricted to the classical trajectory of the energetic particles. We used the residual background field $\hat{g}_{s\mu\nu}$ to identify these transformations, and defined new collinear fields $\hat{\varphi}_c$ and $\hat{h}_{\mu\nu}$ that transform covariantly with respect to this background. Then, we constructed the Wilson lines W and R to relate these fields to the original ones. With these ingredients at hand, it is now straightforward to construct the SCET Lagrangian.

As explained in the purely-scalar case Section 2.4, the effective theory again separates into soft-collinear Lagrangians $\mathcal{L}_{c_i}$ and the N-jet operators. We first construct the effective action, which follows closely the gauge-theory case.

The construction proceeds in the same four steps as in gauge-theory, Section 3.4:

(i) Introduce the split into soft and collinear modes in the full theory. We implement this by performing a weak-field expansion

$$g_{\mu\nu}(x) = g_{s\mu\nu}(x) + h_{\mu\nu}(x)\,, \tag{6.5.1}$$

where $h_{\mu\nu}$ is the collinear graviton and $g_{s\mu\nu}$ is a dynamical soft background. This split also duplicates the gauge symmetry. The background $g_{s\mu\nu}$ comes with the "soft" gauge symmetry (6.1.4), and the fluctuation $h_{\mu\nu}$ has its own "collinear" transformations (6.1.6).

(ii) Perform the multipole expansion of soft fields in soft-collinear interactions to render these homogeneous. However, collinear fields still transform under soft gauge symmetry with the full $U_s(x)$, and therefore the soft gauge symmetry mixes different orders in $(x - x_-)$.

(iii) Redefine the collinear fields as $\varphi_c \to \hat{\varphi}_c$ using the R and W Wilson lines. The new fields are defined to be covariant with respect to the emergent "homogeneous" background $\hat{g}_{s\mu\nu}(x)$ (6.3.56) – (6.3.59), and their soft gauge transformation respects the multipole expansion. This gives rise to a soft-covariant derivative n_-D_s (6.3.64), while all other subleading soft-collinear interactions are expressed in a manifestly gauge-covariant fashion via the Riemann tensor and its derivatives. Each term is manifestly gauge-covariant, but not homogeneous in λ, due to the inhomogeneous nature of gravitational gauge symmetry.

(iv) Perform the λ-expansion. This yields the fully expanded Lagrangian that can be used to perform practical computations

Note that this procedure is exactly the same as in gauge theory, including the necessary concepts like the Wilson lines. Only their actual technical implementation differs in gravity. However, unlike in gauge theory, one cannot give closed all-order results in step (iii) for the subleading Lagrangians. Instead, the gravitational Lagrangian is only defined order-by-order in λ.

The following derivation was presented for the first time in [47] where it was performed by the author in collaboration with M. Beneke and R. Szafron, and since this is now a straightforward computation, we follow this exposition closely.

6.5.1 Background-field Lagrangian

In step (i), we insert the decomposition (6.1.1) of the full metric tensor into a collinear fluctuation $h_{\mu\nu}$ on top of a soft background $g_{s\mu\nu}$ in the full theory. For the scalar field, we consider the action

$$S = \int \mathrm{d}^4x\, \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda_\varphi}{4!}\varphi^4\right)\,. \tag{6.5.2}$$

The Lagrangian is then given by a series in $h_{\mu\nu}$, which we denote by

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_{\varphi h} + \mathcal{L}_{\varphi hh} + \mathcal{O}(h^3)\,, \tag{6.5.3}$$

where the subscripts denote the order of the respective term in h. The individual terms are determined to be

$$\mathcal{L}_\varphi = \frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \sqrt{-g_s}\frac{\lambda_\varphi}{4!}\varphi^4\,, \tag{6.5.4}$$

$$\mathcal{L}_{\varphi h} = -\frac{1}{2}\sqrt{-g_s}\left(g_s^{\mu\alpha}g_s^{\nu\beta}h_{\alpha\beta} - \frac{1}{2}g_s^{\alpha\beta}h_{\alpha\beta}g_s^{\mu\nu}\right)\partial_\mu\varphi\partial_\nu\varphi - \sqrt{g_s}\left(\frac{1}{2}g_s^{\alpha\beta}h_{\alpha\beta}\right)\frac{\lambda_\varphi}{4!}\varphi^4\,, \tag{6.5.5}$$

$$\begin{aligned}\mathcal{L}_{\varphi hh} = \frac{1}{2}\sqrt{-g_s}\bigg(&g_s^{\mu\alpha}g_s^{\nu\beta}g_s^{\rho\sigma}h_{\alpha\rho}h_{\beta\sigma} - \frac{1}{2}g_s^{\alpha\beta}h_{\alpha\beta}g_s^{\mu\rho}g_s^{\nu\sigma}h_{\rho\sigma}\\ &+\frac{1}{8}g_s^{\mu\nu}(g_s^{\alpha\beta}h_{\alpha\beta})^2 - \frac{1}{4}g_s^{\mu\nu}g_s^{\rho\alpha}g_s^{\sigma\beta}h_{\rho\sigma}h_{\alpha\beta}\bigg)\partial_\mu\varphi\partial_\nu\varphi\\ - \sqrt{-g_s}&\left(\frac{1}{8}(g_s^{\alpha\beta}h_{\alpha\beta})^2 - \frac{1}{4}g_s^{\rho\alpha}g_s^{\sigma\beta}h_{\rho\sigma}h_{\alpha\beta}\right)\frac{\lambda_\varphi}{4!}\varphi^4\,.\end{aligned} \tag{6.5.6}$$

Here, the scalar field is given by $\varphi = \varphi_c + WZ^{-1}\varphi_s$ Since we have not yet expanded the soft background, soft gauge-invariance is manifest, since all fields transform covariantly with respect to the background $g_{s\mu\nu}$, which also contracts all appearing indices. To render the Lagrangian a scalar density, it is multiplied by the soft metric determinant. The theory is only invariant under collinear transformations order-by-order in $h_{\mu\nu}$, similar to standard weak-field expansion as explained in Section 4.3.3.

Next, in step (ii), we perform the light-front multipole expansion and redefine the collinear fields according to (6.4.9) and (6.4.10) in step (iii). Since the following equations can get quite lengthy, we focus on the details of individual terms to discuss the systematics of the construction.

6.5.2 Inserting the Redefinitions

Terms without Collinear Gravitons

To elucidate how the technical details of the construction work, we first consider the leading term $\mathcal{L}_\varphi$ from (6.5.4),

$$\frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}\partial_\mu\varphi_c\partial_\nu\varphi_c\,, \tag{6.5.7}$$

setting the soft scalar $\varphi_s = 0$ for simplicity. In this leading term, there are no explicit collinear gravitons present, up to the possibility of explicit appearances of the Wilson line W_c. To simplify the notation, $g_{s\mu\nu}(x)$, $R(x)$ and collinear objects like $\varphi_c(x)$ and $W_c(x)$ are always understood to be evaluated at x and we drop the argument. Soft fields that appear after multipole expansion are evaluated on the light-cone x_-^μ if no argument is given. Throughout this derivation, we employ the useful identities given in Appendix A.

The first step is to insert the redefinitions (6.4.9) and (6.4.10) in (6.5.7). This yields

$$\begin{aligned}&\frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}\left[\partial_\mu RW_c^{-1}\hat{\varphi}_c\right]\left[\partial_\nu RW_c^{-1}\hat{\varphi}_c\right]\\ &\quad= \frac{1}{2}\sqrt{-g_s}g_s^{\mu\nu}RR_\mu{}^\alpha R_\nu{}^\beta\left[\partial_\alpha W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\beta W_c^{-1}\hat{\varphi}_c\right]\\ &\quad= \frac{1}{2}\det(\underline{R})\left[R^{-1}\sqrt{-g_s}\right]\left[R^{-1}g_s^{\mu\nu}\right]R_\mu{}^\alpha R_\nu{}^\beta\left[\partial_\alpha W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\beta W_c^{-1}\hat{\varphi}_c\right]\,.\end{aligned} \tag{6.5.8}$$

In the last line, we used the product rule and determinant identities (A.0.10), (A.0.14), integrated by parts and dropped boundary terms. We use the same notation as in Appendix A and indicate the determinant of $[\underline{R}]^\mu{}_\alpha \equiv R^\mu{}_\alpha$ by the notation $\det(\underline{R})$. As before, derivatives act on all terms inside the square brackets.

To split off the gauge-covariant terms, introduce the residual background field $\hat{g}_{s\mu\nu}$, given in (6.3.56) – (6.3.59), by adding and subtracting

$$\frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\nu}\left[\partial_\mu W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\nu W_c^{-1}\hat{\varphi}_c\right]. \tag{6.5.9}$$

This yields

$$\begin{aligned}&\frac{1}{2}\left[\partial_\alpha W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\beta W_c^{-1}\hat{\varphi}_c\right]\Bigg(\sqrt{-\hat{g}_s}\hat{g}_s^{\alpha\beta}\\ &\quad+\det(\underline{R})\left[R^{-1}\sqrt{-g_s}\right]R_\mu{}^\alpha R_\nu{}^\beta\left[R^{-1}g_s^{\mu\nu}(x)\right]-\sqrt{-\hat{g}_s}\hat{g}_s^{\alpha\beta}\Bigg).\end{aligned} \tag{6.5.10}$$

This expression can be further simplified. First, write

$$R_\mu{}^\alpha R_\nu{}^\beta\left[R^{-1}g_s^{\mu\nu}(x)\right]=\eta^{\alpha\beta}-\eta^{\alpha\rho}\eta^{\beta\sigma}\left(R^\mu{}_\rho R^\nu{}_\sigma\left[R^{-1}s_{\mu\nu}(x)+\eta_{\mu\nu}\right]-\eta_{\rho\sigma}\right)+\dots \tag{6.5.11}$$

The expression in round brackets on the right-hand side is the metric tensor in fixed-line gauge $\check{g}_{s\mu\nu}$ (6.3.56) – (6.3.59). Furthermore, the right-hand side takes the form of a weak-field expansion, in the sense

$$\eta^{\alpha\beta}-\check{s}^{\alpha\beta}+\dots \tag{6.5.12}$$

This identity thus simply expresses the weak-field expansion of the inverse metric in terms of the metric $\check{g}_{s\mu\nu}$ with lowered indices, and the left-hand side is equivalent to this inverse metric $\check{g}_s^{\mu\nu}(x)$. Next, one can introduce the split into the residual background metric and the gauge-covariant field (6.3.55) to rewrite (6.5.11) as

$$R^\mu{}_\rho R^\nu{}_\sigma\left[R^{-1}s_{\mu\nu}(x)+\eta_{\mu\nu}\right]=\hat{g}_{s\rho\sigma}+\mathfrak{g}_{s\rho\sigma}, \tag{6.5.13}$$

The manifestly gauge-covariant part, which contains the Riemann tensor terms, is given by

$$\begin{aligned}\mathfrak{g}_{s\mu\nu}(x)=&-\frac{n_{+\mu}n_{+\nu}}{4}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}-\frac{n_{+\mu}}{2}\frac{2}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\nu_\perp\beta-}-\frac{n_{+\nu}}{2}\frac{2}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\mu_\perp\beta-}\\ &-\left(\frac{n_{+\mu}n_{-\nu}}{4}+\frac{n_{+\nu}n_{-\mu}}{4}\right)\frac{2}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha+\beta-}-\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\mu_\perp\beta\nu_\perp}\\ &-\frac{n_{-\mu}}{2}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\nu_\perp\beta+}-\frac{n_{-\nu}}{2}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha\mu_\perp\beta+}-\frac{n_{-\mu}n_{-\nu}}{4}\frac{1}{3}x_\perp^\alpha x_\perp^\beta R_{\alpha+\beta+}+\mathcal{O}(\lambda^3).\end{aligned} \tag{6.5.14}$$

For the effective Lagrangian, only the component $\mathfrak{g}_{--}$ will contribute to $\mathcal{O}(\lambda^2)$, since the other components are accompanied by (suppressed) collinear derivatives $\partial_\perp$ and $n_-\partial$ and enter only in higher order in λ.

By the same reasoning, the dressed object

$$\det(\underline{R})\left[R^{-1}\sqrt{-g_s}\right] \tag{6.5.15}$$

in (6.5.10) is simply the metric determinant in fixed-line normal coordinates $\sqrt{-\check{g}_{s\mu\nu}}$. We again expand this around the residual metric determinant $\sqrt{\hat{g}_s}$ as

$$\det(\underline{R})\,R^{-1}\sqrt{-g_s}=\sqrt{-\hat{g}_s}\,(1-\frac{1}{6}x_\perp^\alpha x_\perp^\beta R^\mu{}_{\alpha\mu\beta}+\dots). \tag{6.5.16}$$

However, the Riemann-tensor terms from this expansion are not relevant for the following construction and contribute only beyond $\mathcal{O}(\lambda^2)$.

In summary, up to $\mathcal{O}(\lambda^2)$, the leading term (6.5.7) of the full theory yields the effective Lagrangian

$$\mathcal{L}_\varphi=\frac{1}{2}\left[\partial_\alpha W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\beta W_c^{-1}\hat{\varphi}_c\right]\left(\sqrt{-\hat{g}_s}\hat{g}_s^{\alpha\beta}-\frac{1}{4}\sqrt{-\hat{g}_s}n_+^\alpha n_+^\beta x_\perp^\mu x_\perp^\nu R_{\mu-\nu-}\right). \tag{6.5.17}$$

This term can be simplified further, since the collinear Wilson lines cancel out in terms that only contain the residual metric $\hat{g}_{s\mu\nu}$ and collinear fields, like the first term. They do not cancel out if soft-covariant building blocks, like the Riemann tensor, are present. This is explained in detail further down below. On the other hand, one can simply introduce the gauge-invariant building blocks $\hat{\chi}_c = W_c^{-1}\hat{\varphi}_c$ to write this term as

$$\mathcal{L}_\varphi = \frac{1}{2}\left[\partial_\alpha\hat{\chi}_c\right]\left[\partial_\beta\hat{\chi}_c\right]\left(\sqrt{-\hat{g}_s}\hat{g}_s^{\alpha\beta} - \frac{1}{4}\sqrt{-\hat{g}_s}n_+^\alpha n_+^\beta x_\perp^\mu x_\perp^\nu R_{\mu-\nu-}\right). \tag{6.5.18}$$

Including Collinear Gravitons

Now that the construction is understood for the simplest case, we consider the terms that contain additional explicit collinear gravitons in (6.5.5), (6.5.6), namely

$$\mathcal{L}_{\varphi h} = \frac{1}{2}\sqrt{-g_s}\left(-g_s^{\mu\alpha}g_s^{\nu\beta}h_{\alpha\beta} + \frac{1}{2}g_s^{\alpha\beta}h_{\alpha\beta}g_s^{\mu\nu}\right)\partial_\mu\varphi_c\partial_\nu\varphi_c\,, \tag{6.5.19}$$

$$\begin{aligned}\mathcal{L}_{\varphi hh} = \frac{1}{2}\sqrt{-g_s}\Big(&g_s^{\mu\alpha}g_s^{\nu\beta}g_s^{\rho\sigma}h_{\alpha\rho}h_{\beta\sigma} - \frac{1}{2}g_s^{\alpha\beta}h_{\alpha\beta}g_s^{\mu\rho}g_s^{\nu\sigma}h_{\rho\sigma} + \frac{1}{8}g_s^{\mu\nu}(g_s^{\alpha\beta}h_{\alpha\beta})^2\\ &-\frac{1}{4}g_s^{\mu\nu}g_s^{\rho\alpha}g_s^{\sigma\beta}h_{\rho\sigma}h_{\alpha\beta}\Big)\partial_\mu\varphi_c\partial_\nu\varphi_c\,.\end{aligned} \tag{6.5.20}$$

For brevity, we again leave out the scalar self-interaction, since these terms are only multiplied by a metric determinant.

First, consider the $\mathcal{O}(h)$ term. We split this into two terms

$$\mathcal{L}_{\varphi h} = \mathcal{L}_{\varphi h,\,1} + \mathcal{L}_{\varphi h,\,2}\,, \tag{6.5.21}$$

where

$$\begin{aligned}\mathcal{L}_{\varphi h,\,1} &= -\frac{1}{2}\sqrt{-g_s}g_s^{\mu\alpha}g_s^{\nu\beta}h_{\alpha\beta}\partial_\mu\varphi_c\partial_\nu\\ \mathcal{L}_{\varphi h,\,2} &= \frac{1}{4}\sqrt{-g_s}g_s^{\alpha\beta}h_{\alpha\beta}g_s^{\mu\nu}\partial_\mu\varphi_c\partial_\nu\,,\end{aligned} \tag{6.5.22}$$

which we discuss separately for transparency. First, one introduces the redefined fields $\hat{\varphi}_c$ (6.4.9) and $\hat{h}_{\mu\nu}$ (6.4.10). For the first term in (6.5.19), this yields

$$\begin{aligned}\mathcal{L}_{\varphi h,\,1} = -\frac{1}{2}\sqrt{-g_s}g_s^{\mu\alpha}g_s^{\nu\beta}&\left[RR_\alpha{}^\kappa R_\beta{}^\lambda\left(W^\rho{}_\kappa W^\sigma{}_\lambda\left[W_c^{-1}(\hat{h}_{\rho\sigma}+\hat{g}_{s\rho\sigma})\right] - \hat{g}_{s\kappa\lambda}\right)\right]\\ &\times\left[\partial_\mu(RW_c^{-1}\hat{\varphi}_c)\right]\left[\partial_\nu(RW_c^{-1}\hat{\varphi}_c)\right]\,,\end{aligned} \tag{6.5.23}$$

and using the same manipulations as before (integration by parts and the useful identities in Appendix A), one obtains

$$\mathcal{L}_{\varphi h,\,1} = -\frac{1}{2}\det(\underline{R})\left[R^{-1}\sqrt{-g_s}\right]R_\alpha{}^\rho R_\mu{}^\kappa\left[R^{-1}g_s^{\mu\alpha}\right]R_\beta{}^\sigma R_\nu{}^\lambda\left[R^{-1}g_s^{\nu\beta}\right]\mathcal{M}_{\rho\sigma\kappa\lambda}\,, \tag{6.5.24}$$

where we defined

$$\mathcal{M}_{\rho\sigma\kappa\lambda} \equiv \left(W^\alpha{}_\rho W^\beta{}_\sigma\left[W_c^{-1}(\hat{h}_{\alpha\beta}+\hat{g}_{s\alpha\beta})\right] - \hat{g}_{s\rho\sigma}\right)\left[\partial_\kappa W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\lambda W_c^{-1}\hat{\varphi}_c\right]. \tag{6.5.25}$$

Just as before, one adds and subtracts the residual background metric

$$-\frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\mathcal{M}_{\alpha\beta\mu\nu}\,, \tag{6.5.26}$$

and performs the split into background field and manifestly gauge-covariant terms (6.5.13). This results in

$$\mathcal{L}_{\varphi h,\,1} = -\frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\mathcal{M}_{\alpha\beta\mu\nu}$$

$$
\begin{aligned}
&- \frac{1}{2}\Big(\det(\underline{R}) \left[R^{-1}\sqrt{-g_s} \right] R_\alpha{}^\rho R_\mu{}^\kappa \left[R^{-1} g_s^{\mu\alpha} \right] R_\beta{}^\sigma R_\nu{}^\lambda \left[R^{-1} g_s^{\nu\beta} \right] \\
&\quad - \sqrt{-\hat{g}_s}\hat{g}_s^{\rho\kappa}\hat{g}_s^{\sigma\lambda} \Big) \mathcal{M}_{\rho\sigma\kappa\lambda} \,. \qquad (6.5.27)
\end{aligned}
$$

Here, one can neglect the second and third line in their entirety, since these terms contribute only beyond $\mathcal{O}(\lambda^2)$.

Performing the same steps for the second term in $\mathcal{L}_{\varphi h}$ yields

$$
\begin{aligned}
\mathcal{L}_{\varphi h,\,2} &= \frac{1}{4}\sqrt{g}_s g_s^{\alpha\beta} g_s^{\mu\nu} \left[R R_\alpha{}^\kappa R_\beta{}^\lambda \left(W^\rho{}_\kappa W^\sigma{}_\lambda \left[W_c^{-1}(\hat{h}_{\rho\sigma} + \hat{g}_{s\rho\sigma}) \right] - \hat{g}_{s\kappa\lambda} \right) \right] \\
&\quad \times \left[\partial_\mu R W_c^{-1} \hat{\varphi}_c \right] \left[\partial_\nu R W_c^{-1} \hat{\varphi}_c \right] \qquad (6.5.28) \\
&= \frac{1}{4} \det(\underline{R}) \left[R^{-1}\sqrt{-g_s} \right] R_\alpha{}^\rho R_\beta{}^\sigma \left[R^{-1} g_s^{\alpha\beta} \right] R_\mu{}^\kappa R_\nu{}^\lambda \left[R^{-1} g_s^{\mu\nu} \right] \mathcal{M}_{\rho\sigma\kappa\lambda} \,,
\end{aligned}
$$

where $\mathcal{M}$ is the same expression (6.5.25). Again, introduce the residual background field via (6.5.26) to obtain

$$
\begin{aligned}
\mathcal{L}_{\varphi h,\,2} &= \frac{1}{4}\sqrt{-\hat{g}}_s \hat{g}_s^{\alpha\beta} \hat{g}_s^{\mu\nu} \mathcal{M}_{\alpha\beta\mu\nu} \\
&\quad + \frac{1}{4}\Big(\det(\underline{R}) \left[R^{-1}\sqrt{-g_s} \right] R_\alpha{}^\rho R_\beta{}^\sigma \left[R^{-1} g_s^{\alpha\beta} \right] R_\mu{}^\kappa R_\nu{}^\lambda \left[R^{-1} g_s^{\mu\nu} \right] \\
&\quad - \sqrt{-\hat{g}_s}\, \hat{g}_s^{\rho\sigma} \hat{g}_s^{\kappa\lambda} \Big) \mathcal{M}_{\rho\sigma\kappa\lambda} \,. \qquad (6.5.29)
\end{aligned}
$$

In this expression, only the first line contributes to $\mathcal{O}(\lambda^2)$, while the remaining terms are too suppressed. Combining both results (6.5.27) and (6.5.29), expanding $\mathcal{M}$ and introducing the gauge-invariant building blocks yields

$$
\mathcal{L}_{\varphi h} = \frac{1}{2}\sqrt{-\hat{g}_s} \left(-\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{\mathfrak{h}}_{\alpha\beta} + \frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta}\hat{g}_s^{\mu\nu} \right) \partial_\mu \hat{\chi}_c \partial_\nu \hat{\chi}_c \,. \qquad (6.5.30)
$$

For $\mathcal{L}^{(2)}$ (6.5.20), the exact same computation gives

$$
\begin{aligned}
\mathcal{L}_{\varphi hh} &= \frac{1}{2}\sqrt{-\hat{g}_s} \Big(\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{g}_s^{\rho\sigma}\hat{\mathfrak{h}}_{\alpha\rho}\hat{\mathfrak{h}}_{\beta\sigma} - \frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta}\hat{g}_s^{\mu\rho}\hat{g}_s^{\nu\sigma} h_{\rho\sigma} + \frac{1}{8}\hat{g}_s^{\mu\nu}(\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta})^2 \\
&\quad - \frac{1}{4}\hat{g}_s^{\mu\nu}\hat{g}_s^{\rho\alpha} g_s^{\sigma\beta}\hat{\mathfrak{h}}_{\rho\sigma}\hat{\mathfrak{h}}_{\alpha\beta} \Big) \partial_\mu \hat{\chi}_c \partial_\nu \hat{\chi}_c \,. \qquad (6.5.31)
\end{aligned}
$$

Removing Collinear Wilson Lines

In the QCD Lagrangian (3.4.19) – (3.4.23), note that the collinear Wilson line W_c only appears in the terms that contain the field-strength tensor, and not in the ones that contain the soft-covariant derivative. This can be easily understood since the Wilson lines are constructed to be covariant with respect to this residual background. They simply amount to a collinear gauge transformation, and these terms are gauge-invariant. Therefore, the Wilson lines must cancel out in the end. The same should be true in gravity, and we now explain in detail how this works. Here, they should cancel out in all terms that are covariant with respect to $\hat{g}_{s\mu\nu}$, and stay in terms that contain the $\mathfrak{g}_{s\mu\nu}$ field.

The terms that contain $\hat{g}_{s\mu\nu}$ are precisely the respective leading terms in the multipole expansion, which enter at $\mathcal{O}((x - x_-)^0)$, that appear in every order in λ due to the weak-field expansion in $\hat{h}_{\mu\nu}$. The first three terms are

$$
\mathcal{L}_\varphi\big|_{\mathfrak{g}_{s\mu\nu}=0} = \frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\nu} \left[\partial_\mu W_c^{-1} \hat{\varphi}_c \right] \left[\partial_\nu W_c^{-1} \hat{\varphi}_c \right] \,, \qquad (6.5.32)
$$

$$\mathcal{L}_{\varphi h}\big|_{\mathfrak{g}_{s\mu\nu}=0} = \left(-\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{\mathfrak{h}}_{\alpha\beta} + \frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta}\hat{g}_s^{\mu\nu}\right)\left[\partial_\mu W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\nu W_c^{-1}\hat{\varphi}_c\right], \tag{6.5.33}$$

$$\begin{aligned}\mathcal{L}_{\varphi hh}\big|_{\mathfrak{g}_{s\mu\nu}=0} = \frac{1}{2}\sqrt{-\hat{g}_s}\Big(&\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{g}_s^{\rho\sigma}\hat{\mathfrak{h}}_{\alpha\rho}\hat{\mathfrak{h}}_{\beta\sigma} - \frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta}\hat{g}_s^{\mu\rho}\hat{g}_s^{\nu\sigma}h_{\rho\sigma} + \frac{1}{8}\hat{g}_s^{\mu\nu}(\hat{g}_s^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta})^2 \\ &-\frac{1}{4}\hat{g}_s^{\mu\nu}\hat{g}_s^{\rho\alpha}\hat{g}_s^{\sigma\beta}\hat{\mathfrak{h}}_{\rho\sigma}\hat{\mathfrak{h}}_{\alpha\beta}\Big)\left[\partial_\mu W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\nu W_c^{-1}\hat{\varphi}_c\right]. \end{aligned}\tag{6.5.34}$$

If summed to all orders in $\hat{h}_{\mu\nu}$, i.e. undoing the weak-field expansion, these terms $\mathcal{L}_{\varphi h^i}\big|_{\mathfrak{g}_{s\mu\nu}=0}$ formally sum to the closed form

$$\sum_{i=0}^{\infty}\mathcal{L}_{\varphi h^i}\Big|_{\mathfrak{g}_{s\mu\nu}=0} = \sqrt{-\bar{g}}\bar{g}^{\mu\nu}\left[\partial_\mu\hat{\chi}_c\right]\left[\partial_\nu\hat{\chi}_c\right], \tag{6.5.35}$$

where we introduced the metric tensor

$$\bar{g}_{\mu\nu} = \hat{g}_{s\mu\nu} + \hat{\mathfrak{h}}_{\mu\nu}\,. \tag{6.5.36}$$

We can interpret this split as a fluctuation $\hat{\mathfrak{h}}_{\mu\nu}$ on top of the soft background $\hat{g}_{s\mu\nu}$. The Lagrangian density (6.5.35) takes the standard form of a scalar field in a curved space-time described by $\bar{g}_{\mu\nu}$, and therefore the theory is invariant under diffeomorphism. In particular, it is invariant under the transformations of the fluctuations $\hat{\chi}_c$ and $\hat{\mathfrak{h}}_{\mu\nu}$. The important observation is now that the dressings

$$\hat{\chi}_c = \left[W_c^{-1}\hat{\varphi}_c\right], \tag{6.5.37}$$

$$\hat{\mathfrak{h}}_{\mu\nu} = W^\alpha{}_\mu W^\beta{}_\nu\left[W_c^{-1}\left(\hat{h}_{\alpha\beta} + \hat{g}_{s\alpha\beta}\right)\right] - \hat{g}_{s\mu\nu}\,, \tag{6.5.38}$$

take the form of an inverse gauge transformation of the fields $\hat{\varphi}_c$ and $\hat{h}_{\mu\nu}$ by construction. Therefore, in the terms considered here, the collinear Wilson lines simply amount to a gauge transformation and cancel out precisely. Explicitly, one has

$$\bar{g}_{\mu\nu} = \hat{g}_{s\mu\nu} + \hat{\mathfrak{h}}_{\mu\nu} = W^\alpha{}_\mu W^\beta{}_\nu\left[W_c^{-1}\tilde{g}_{\alpha\beta}\right], \tag{6.5.39}$$

where $\tilde{g}_{\alpha\beta} = \hat{g}_{s\alpha\beta} + \hat{h}_{\alpha\beta}$ is the metric in an arbitrary (unfixed) gauge, with inverse

$$\bar{g}^{\mu\nu} = W_\alpha{}^\mu W_\beta{}^\nu\left[W_c^{-1}\tilde{g}^{\alpha\beta}\right]. \tag{6.5.40}$$

Insert this in the Lagrangian (6.5.35), to obtain

$$\begin{aligned}\mathcal{L}_{\bar{g}} &= \det(\underline{W})\left[W_c^{-1}\sqrt{-\tilde{g}}\right]W_\alpha{}^\mu W_\beta{}^\nu\left[W_c^{-1}\tilde{g}^{\alpha\beta}\right]\left[\partial_\mu W_c^{-1}\hat{\varphi}_c\right]\left[\partial_\nu W_c^{-1}\hat{\varphi}_c\right] \\ &= \det(\underline{W})\left[W_c^{-1}\sqrt{-\tilde{g}}\right]W_\alpha{}^\mu W_\beta{}^\nu\left[W_c^{-1}\tilde{g}^{\alpha\beta}\right]W^\rho{}_\mu\left[W_c^{-1}\partial_\rho\hat{\varphi}_c\right]W^\sigma{}_\nu\left[W_c^{-1}\partial_\sigma\hat{\varphi}_c\right] \\ &= \det(\underline{W})\left[W_c^{-1}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\left[\partial_\mu\hat{\varphi}_c\right]\left[\partial_\nu\hat{\varphi}_c\right]\right] \\ &= \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\left[\partial_\mu\hat{\varphi}_c\right]\left[\partial_\nu\hat{\varphi}_c\right] + \text{t.d.}\,,\end{aligned}\tag{6.5.41}$$

where we dropped a total derivative using the Wilson line identity (A.0.13).

6.5.3 The Soft-collinear Lagrangian

Simplifying the Lagrangian as much as possible, we find

$$\mathcal{L}_\varphi = \frac{1}{2}\sqrt{-\hat{g}_s}\hat{g}_s^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c - \frac{1}{8}\sqrt{-\hat{g}_s}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}(\partial_+W_c^{-1}\hat{\varphi}_c)^2 - \sqrt{-\hat{g}_s}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \tag{6.5.42}$$

$$\begin{aligned}
\mathcal{L}_{\varphi h} &= \frac{1}{2}\sqrt{-\hat{g}_s}\left(-\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{h}_{\alpha\beta}+\frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{h}_{\alpha\beta}\hat{g}_s^{\mu\nu}\right)\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c \\
&\quad -\sqrt{-\hat{g}_s}\left(\frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{h}_{\alpha\beta}\right)\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \qquad (6.5.43)\\
\mathcal{L}_{\varphi hh} &= \frac{1}{2}\sqrt{-\hat{g}_s}\left(\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{g}_s^{\rho\sigma}\hat{h}_{\alpha\rho}\hat{h}_{\beta\sigma}-\frac{1}{2}\hat{g}_s^{\alpha\beta}\hat{h}_{\alpha\beta}\hat{g}_s^{\mu\rho}\hat{g}_s^{\nu\sigma}\hat{h}_{\rho\sigma}+\frac{1}{8}\hat{g}_s^{\mu\nu}(\hat{g}_s^{\alpha\beta}\hat{h}_{\alpha\beta})^2\right.\\
&\qquad\left.-\frac{1}{4}\hat{g}_s^{\mu\nu}\hat{g}_s^{\rho\alpha}\hat{g}_s^{\sigma\beta}\hat{h}_{\rho\sigma}\hat{h}_{\alpha\beta}\right)\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c\\
&\quad -\sqrt{-\hat{g}_s}\left(\frac{1}{8}(\hat{g}_s^{\alpha\beta}\hat{h}_{\alpha\beta})^2-\frac{1}{4}\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{h}_{\mu\nu}\hat{h}_{\alpha\beta}\right)\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \qquad (6.5.44)
\end{aligned}$$

where we have now added the scalar self-interaction. To simplify the structure even further, introduce the soft-covariant derivative (6.3.64) and use the metric tensor $\hat{g}_{s\mu\nu}$ to raise and lower indices. The Lagrangians (6.5.42) – (6.5.44) are then given by

$$\mathcal{L} = \sqrt{-\hat{g}_s}\left(\mathcal{L}^{(0)}_{D_s}+\mathcal{L}^{(1)}_{D_s}+\mathcal{L}^{(2)}_{D_s}\right)\,, \qquad (6.5.45)$$

where the superscript indicates the leading λ-counting of each term. Up to $\mathcal{O}(\lambda^2)$, the individual terms are given by

$$\begin{aligned}
\mathcal{L}^{(0)}_{D_s} &= \frac{1}{2}\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c+\frac{1}{2}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c-\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \qquad (6.5.46)\\
\mathcal{L}^{(1)}_{D_s} &= -\frac{1}{2}\hat{h}^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c+\frac{1}{4}\hat{h}^{\beta_\perp}{}_{\beta_\perp}\left(\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c+\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c\right)-\frac{1}{2}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \qquad (6.5.47)\\
\mathcal{L}^{(2)}_{D_s} &= -\frac{1}{8}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}(\partial_+W_c^{-1}\hat{\varphi}_c)^2+\frac{1}{2}\hat{h}^{\mu\alpha}\hat{h}_\alpha{}^\nu\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c-\frac{1}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\hat{h}^{\mu\nu}\partial_\mu\hat{\varphi}_c\partial_\nu\hat{\varphi}_c\\
&\quad+\frac{1}{16}\left((\hat{h}^{\alpha_\perp}{}_{\alpha_\perp})^2-2\hat{h}^{\alpha\beta}\hat{h}_{\alpha\beta}\right)\left(\partial_+\hat{\varphi}_c D_-\hat{\varphi}_c+\partial_{\mu_\perp}\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c\right)\\
&\quad-\left(\frac{1}{8}(\hat{h}^{\alpha_\perp}{}_{\alpha_\perp})^2-\frac{1}{4}\hat{h}^{\mu\nu}\hat{h}_{\mu\nu}\right)\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,. \qquad (6.5.48)
\end{aligned}$$

Here, we defined the graviton with raised indices as

$$\hat{h}^{\mu\nu}\equiv\hat{g}_s^{\mu\alpha}\hat{g}_s^{\nu\beta}\hat{h}_{\alpha\beta}\,. \qquad (6.5.49)$$

Note that the background metric $\hat{g}_s^{\mu\nu}$ is not homogeneous and also has a λ-expansion.

This is the most concise and conceptually clear form the soft-collinear Lagrangian for a collinear scalar field interacting with soft and collinear gravitons. It includes explicitly all terms up to sub-subleading or next-to-soft order $\mathcal{O}(\lambda^2)$, and the inhomogeneous objects, like the Riemann tensor, the soft-covariant derivative or the collinear Wilson line, even implicitly contain an infinite tower of further subleading terms.

The Lagrangian (6.5.46) – (6.5.48) has a strong formal similarity to the gauge-theory Lagrangian (3.4.19) – (3.4.21). In the soft sector, there is a soft-covariant derivative n_-D_s which contains not one but *two* independent soft gauge fields, one related to translations and one related to Lorentz transformations, with corresponding charges momentum and angular momentum. This is the direct analogue of the soft-covariant derivative n_-D_s in QCD, which contains $n_-A_s(x_-)$ coupling to the (colour) charge of the energetic fields.

Besides this derivative, the theory features an infinite tower of subleading terms where the Riemann tensor couples to the quadrupole and higher-pole terms of the energetic particle. The same structure also arises in gauge theory, where the field-strength tensor couples to the dipole and higher-pole terms.

In the collinear sector, the two theories differ. Whereas QCD has a collinear-covariant derivative that gives rise to leading-power interactions, such an object is absent in gravity. Instead, the collinear interactions start at subleading order $\mathcal{O}(\lambda)$. This feature, in combination with the absence of leading-power collinear graviton building blocks in the N-jet operator basis, immediately implies the absence of collinear divergences in gravity as opposed to gauge theory. This absence is crucial, as collinear divergences would cause inconsistencies if massless charged particles are present. In gravity, such particles *exist* in Nature, like the photon or the graviton. In QCD, on the other hand, there are no massless charged particles in the asymptotic states, since QCD is confining.

Soft divergences, on the other hand, are present in both gauge theory and gravity. These can be controlled using the soft decoupling transformation [43], which introduces a purely-soft Wilson line in a form similar to W_c. It converts $n_- D_s \to n_- \partial$ and moves the leading-power (and subleading-power) soft effects into the sources directly. This is a manifestation of *soft exponentiation* [1].

If one is interested in a technical application of the effective theory, for example, if one wants to compute a soft emission process or some loop corrections explicitly, the form (6.5.46) – (6.5.48) is not suitable, since it must still be expanded. Therefore, we also provide the fully expanded Lagrangian density, including self-interactions, the gravitational coupling κ, and the soft scalar field down below. For brevity, it is expressed in collinear light-cone gauge $h_{+\mu} = 0$.

$$\mathcal{L}^{(0)} = \frac{1}{2}\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c + \frac{1}{2}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c - \frac{\kappa}{8}s_{--}(\partial_+\hat{\varphi}_c)^2 - \frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,, \tag{6.5.50}$$

$$\begin{aligned}
\mathcal{L}^{(1)} = &-\frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha-}\right]x_\perp^\alpha(\partial_+\hat{\varphi}_c)^2 - \frac{\kappa}{4}s_{\mu_\perp -}\partial^{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c \\
&- \frac{\kappa}{2}\left(\hat{h}^{\mu_\perp\nu_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c + \frac{1}{4}\hat{h}^{--}(\partial_+\hat{\varphi}_c)^2\right) \\
&+ \frac{\kappa}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\left(\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c - \frac{\kappa}{4}s_{--}(\partial_+\hat{\varphi}_c)^2 + \partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c\right) - \frac{\kappa}{2}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4\,,
\end{aligned} \tag{6.5.51}$$

$$\begin{aligned}
\mathcal{L}^{(2)} = &-\frac{\kappa}{16}\left[\partial_{[+}s_{-]-}\right]n_- x(\partial_+\hat{\varphi}_c)^2 - \frac{\kappa}{4}\left[\partial_{[\alpha_\perp}s_{\mu_\perp]-}\right]x_\perp^\alpha\partial^{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c \\
&+ \frac{\kappa^2}{32}s_{--}s_{+-}(\partial_+\hat{\varphi}_c)^2 + \frac{\kappa^2}{32}s_{-\alpha_\perp}s_-^{\alpha_\perp}(\partial_+\hat{\varphi}_c)^2 - \frac{1}{8}x_\perp^\alpha x_\perp^\beta R_{\alpha-\beta-}(\partial_+\hat{\varphi}_c)^2 \\
&+ \frac{\kappa}{8}s_{+-}\partial_{\alpha_\perp}\hat{\varphi}_c\partial^{\alpha_\perp}\hat{\varphi}_c - \frac{\kappa}{4}s_{+-}\frac{\lambda_\varphi}{4!}\hat{\varphi}_c^4 \\
&+ \frac{\kappa^2}{2}\left(\hat{h}^{\mu_\perp\alpha_\perp}\hat{h}^{\nu_\perp}_{\alpha_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp\alpha_\perp}\hat{h}_{\alpha_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c + \frac{1}{4}\hat{h}^{-\alpha_\perp}\hat{h}_{\alpha_\perp -}(\partial_+\hat{\varphi}_c)^2\right) \\
&- \frac{\kappa^2}{4}\hat{h}^{\alpha_\perp}{}_{\alpha_\perp}\left(\hat{h}^{\mu_\perp\nu_\perp}\partial_{\mu_\perp}\hat{\varphi}_c\partial_{\nu_\perp}\hat{\varphi}_c + \hat{h}^{\mu_\perp -}\partial_{\mu_\perp}\hat{\varphi}_c\partial_+\hat{\varphi}_c + \frac{1}{4}\hat{h}_{--}\partial_+\hat{\varphi}_c\partial_+\hat{\varphi}_c\right) \\
&+ \frac{\kappa^2}{16}\left((\hat{h}^{\alpha_\perp}_{\alpha_\perp})^2 - 2\hat{h}^{\alpha_\perp\beta_\perp}\hat{h}_{\alpha_\perp\beta_\perp}\right)\left(\partial_+\hat{\varphi}_c\partial_-\hat{\varphi}_c - \frac{\kappa}{4}s_{--}(\partial_+\hat{\varphi}_c)^2 + \partial_{\mu_\perp}\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c\right) \\
&+ \frac{\kappa^2}{4}\hat{h}^{\mu_\perp\alpha_\perp}s_{\alpha_\perp -}\partial_+\hat{\varphi}_c\partial_{\mu_\perp}\hat{\varphi}_c + \frac{\kappa^2}{8}\hat{h}^{-\alpha_\perp}s_{\alpha_\perp -}(\partial_+\hat{\varphi}_c)^2 \\
&- \frac{\kappa^2}{8}\hat{h}^{\alpha_\perp}_{\alpha_\perp}s_{\mu_\perp -}\partial_+\hat{\varphi}_c\partial^{\mu_\perp}\hat{\varphi}_c - \frac{\kappa^2}{16}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\left[\partial_{[\mu_\perp}s_{-]-}\right]x_\perp^\mu(\partial_+\hat{\varphi}_c)^2 \\
&- \kappa^2\frac{\lambda_\varphi}{4!}\hat{\varphi}_c{}^4\left((\hat{h}^{\alpha_\perp}_{\alpha_\perp})^2 - \frac{1}{4}\hat{h}^{\alpha_\perp\beta_\perp}\hat{h}_{\alpha_\perp\beta_\perp}\right)\,,
\end{aligned} \tag{6.5.52}$$

$$\mathcal{L}^{(1)}_{\varphi_s} = -\frac{\lambda_\varphi}{3!}\hat{\varphi}_c^3\varphi_s\,, \tag{6.5.53}$$

$$\mathcal{L}^{(2)}_{\varphi_s} = \frac{\kappa}{4}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\partial_+\hat{\varphi}_c\partial_-\varphi_s - \frac{\lambda_\varphi}{4}\hat{\varphi}_c{}^2\varphi_s^2 - \frac{\lambda_\varphi}{3!}\frac{\kappa}{2}\hat{h}^{\alpha_\perp}_{\alpha_\perp}\hat{\varphi}_c^3\varphi_s\,. \tag{6.5.54}$$

Note that in this soft-collinear Lagrangian, the order in κ no longer agrees with the order in λ like in the purely-collinear (and purely-soft) theories. Instead, each order in κ generates an infinite tower of subleading terms in λ already due to the multipole expansion.

This result can be further simplified by employing the equations of motion for both the matter and graviton fields, as performed in [63] below (112). This allows one to push many terms to

$\mathcal{O}(\lambda^3)$ and one obtains the simpler expressions

$$\mathcal{L}^{(1)} = -\frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha -}\right] x_\perp^\alpha (\partial_+ \hat{\varphi}_c)^2 - \frac{\kappa}{4} s_{\mu_\perp -} \partial^{\mu_\perp} \hat{\varphi}_c \partial_+ \hat{\varphi}_c \\ - \frac{\kappa}{2}\left(\hat{h}^{\mu_\perp \nu_\perp} \partial_{\mu_\perp} \hat{\varphi}_c \partial_{\nu_\perp} \hat{\varphi}_c + \hat{h}^{\mu_\perp -} \partial_{\mu_\perp} \hat{\varphi}_c \partial_+ \hat{\varphi}_c + \frac{1}{4} \hat{h}^{--} (\partial_+ \hat{\varphi}_c)^2\right) \tag{6.5.55}$$

$$\mathcal{L}^{(2)} = -\frac{\kappa}{16}\left[\partial_{[+} s_{-]-}\right] n_- x (\partial_+ \hat{\varphi}_c)^2 - \frac{\kappa}{4}\left[\partial_{[\alpha_\perp} s_{\mu_\perp]-}\right] x_\perp^\alpha \partial^{\mu_\perp} \hat{\varphi}_c \partial_+ \hat{\varphi}_c \\ + \frac{\kappa^2}{32} s_{--} s_{+-} (\partial_+ \hat{\varphi}_c)^2 + \frac{\kappa^2}{32} s_{-\alpha_\perp} s_-^{\alpha_\perp} (\partial_+ \hat{\varphi}_c)^2 - \frac{1}{8} x_\perp^\alpha x_\perp^\beta R_{\alpha - \beta -} (\partial_+ \hat{\varphi}_c)^2 \\ + \frac{\kappa}{8} s_{+-} \partial_{\alpha_\perp} \hat{\varphi}_c \partial^{\alpha_\perp} \hat{\varphi}_c - \frac{\kappa}{4} s_{+-} \frac{\lambda_\varphi}{4!} \hat{\varphi}_c^4 \\ + \frac{\kappa^2}{2}\left(\hat{h}^{\mu_\perp \alpha_\perp} \hat{h}_{\alpha_\perp}^{\nu_\perp} \partial_{\mu_\perp} \hat{\varphi}_c \partial_{\nu_\perp} \hat{\varphi}_c + \hat{h}^{\mu_\perp \alpha_\perp} \hat{h}_{\alpha_\perp -} \partial_{\mu_\perp} \hat{\varphi}_c \partial_+ \hat{\varphi}_c + \frac{1}{4} \hat{h}^{-\alpha_\perp} \hat{h}_{\alpha_\perp -} (\partial_+ \hat{\varphi}_c)^2\right) \\ + \frac{\kappa^2}{4} \hat{h}^{\mu_\perp \alpha_\perp} s_{\alpha_\perp -} \partial_+ \hat{\varphi}_c \partial_{\mu_\perp} \hat{\varphi}_c + \frac{\kappa^2}{8} \hat{h}^{-\alpha_\perp} s_{\alpha_\perp -} (\partial_+ \hat{\varphi}_c)^2 \,, \tag{6.5.56}$$

$$\mathcal{L}_{\varphi_s}^{(1)} = -\frac{\lambda_\varphi}{3!} \hat{\varphi}_c^3 \varphi_s \,, \tag{6.5.57}$$

$$\mathcal{L}_{\varphi_s}^{(2)} = -\frac{\lambda_\varphi}{4} \hat{\varphi}_c{}^2 \varphi_s^2 \,. \tag{6.5.58}$$

6.5.4 Graviton Lagrangian

The soft-collinear graviton Lagrangian can be derived in the same fashion, albeit the individual steps become more cumbersome and technical. The starting point is the Einstein-Hilbert action

$$S_{\text{EH}} = -2 \int d^4x \, \sqrt{-g} R \,, \tag{6.5.59}$$

where R denotes the Ricci scalar. The first step consists of inserting the decomposition (6.1.1) of the metric tensor $g_{\mu\nu}$ into a soft background $g_{s\mu\nu}$ and a collinear fluctuation $h_{\mu\nu}$. This yields

$$\mathcal{L}_{\text{EH}} = \mathcal{L}_s + \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{O}(h^4) \,, \tag{6.5.60}$$

where the term linear in the fluctuation, $\mathcal{L}_h$, is absent as usual due to equations of motion. The first term, $\mathcal{L}_s$, contains only soft gravitons and is the purely-soft Lagrangian. Here, all fields depend only on the soft coordinate x_s and no multipole expansion is necessary.

The purely-collinear theory begins in $\mathcal{L}_{hh}$, where the collinear kinetic term as well as the leading-power soft-collinear interactions appear. This Lagrangian is determined to be

$$\mathcal{L}_{hh} = \sqrt{-g_s}\left(\frac{1}{2}\nabla_\mu h_{\alpha\beta} \nabla^\mu h^{\alpha\beta} - \frac{1}{2}\nabla_\mu h \nabla^\mu h + \nabla_\alpha h^{\alpha\beta} \nabla_\beta h - \nabla_\alpha h^{\alpha\beta} \nabla_\mu h_\beta^\mu \right. \\ \left. - 4R_{\alpha\beta} h^{\alpha\mu} h_\mu^\beta + 2R_{\alpha\beta\mu\nu} h^{\alpha\mu} h^{\beta\nu} + R_{\alpha\beta} h h^{\alpha\beta} - \frac{1}{4} R(h^2 - 2h^{\alpha\beta} h_{\alpha\beta})\right) . \tag{6.5.61}$$

This theory is covariant with respect to the soft background $g_{s\mu\nu}$, which is used to raise and lower indices. Therefore, the covariant derivative ∇_μ as well as the purely-soft Ricci and Riemann tensors $R_{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}$ are computed from $g_{s\mu\nu}$. If one now performs a split $g_{s\mu\nu} = \eta_{\mu\nu} + s_{\mu\nu}$ in (6.5.61), one would find at leading order in this weak-field expansion, $\mathcal{O}(s^0)$, precisely the bilinear terms of the purely-collinear theory (5.4.7). The same holds true for the higher-order Lagrangians as well. Once the λ expansion is performed, note that the Lagrangian will simplify drastically, since the purely-soft tensors are strongly suppressed in λ.

Since we are interested in the graviton Lagrangian only to $\mathcal{O}(\lambda)$, we do not need to compute the full trilinear Lagrangian $\mathcal{L}_{hhh}$ in this background field formalism. Instead, we can use the full-theory result (4.3.11) and perform a minimal substitution, as explained below.

Inserting the redefinitions of the collinear fields (6.4.10), and expressing the theory covariant with respect to the residual background field $\hat{g}_{s\mu\nu}$, one finds

$$\mathcal{L}_{hh} = \sqrt{-\hat{g}_s}\,\bigg(\frac{1}{2}\nabla_\mu\hat{\mathfrak{h}}_{\alpha\beta}\nabla^\mu\hat{\mathfrak{h}}^{\alpha\beta} - \frac{1}{2}\nabla_\mu\hat{\mathfrak{h}}\nabla^\mu\hat{\mathfrak{h}} + \nabla_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\nabla_\beta\hat{\mathfrak{h}} - \nabla_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\nabla_\mu\hat{\mathfrak{h}}^\mu_\beta$$
$$- 4R_{\alpha\beta}\hat{\mathfrak{h}}^{\alpha\mu}\hat{\mathfrak{h}}^\beta_\mu + 2R_{\alpha\beta\mu\nu}\hat{\mathfrak{h}}^{\alpha\mu}\hat{\mathfrak{h}}^{\beta\nu} + R_{\alpha\beta}\hat{\mathfrak{h}}\hat{\mathfrak{h}}^{\alpha\beta} - \frac{1}{4}R\,(\hat{\mathfrak{h}}^2 - 2\hat{\mathfrak{h}}^{\alpha\beta}\hat{\mathfrak{h}}_{\alpha\beta})\bigg)\,. \tag{6.5.62}$$

This formulation is now covariant with respect to the residual metric $\hat{g}_{s\mu\nu}$, which is used to raise and lower indices. Furthermore, the Riemann and Ricci tensors as well as the covariant derivative ∇_μ are now constructed from the residual metric. Furthermore, note that this theory contains terms that are strongly suppressed in λ, despite appearing this early in the h-expansion. For example, the last three terms contribute starting at $\mathcal{O}(\lambda^4)$.

We now drop these three terms, as well as the contribution due to the covariant derivative, since ${\Gamma^\mu}_{\alpha\beta} \sim \lambda^4$. This yields the Lagrangian

$$\mathcal{L}_{hh} = \sqrt{-\hat{g}_s}\bigg(\frac{1}{2}\hat{g}_s^{\mu\nu}\partial_\mu\hat{\mathfrak{h}}_{\alpha\beta}\partial_\nu\hat{\mathfrak{h}}^{\alpha\beta} - \frac{1}{2}\hat{g}_s^{\mu\nu}\partial_\mu\hat{\mathfrak{h}}\partial_\nu\hat{\mathfrak{h}} + \partial_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\partial_\beta\hat{\mathfrak{h}} - \partial_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\partial_\mu\hat{\mathfrak{h}}^\mu_\beta\bigg) + \mathcal{O}(\lambda^3)\,, \tag{6.5.63}$$

where indices are now contracted with the residual background metric $\hat{g}_{s\mu\nu}$. Next, we can introduce the soft-covariant derivative n_-D_s (6.3.64) to find

$$\mathcal{L}_{hh} = \sqrt{-\hat{g}_s}\bigg(\frac{1}{2}\partial_+\hat{\mathfrak{h}}_{\alpha\beta}D_-\hat{\mathfrak{h}}^{\alpha\beta} + \frac{1}{2}\partial_{\mu_\perp}\hat{\mathfrak{h}}_{\alpha\beta}\partial^{\mu_\perp}\hat{\mathfrak{h}}^{\alpha\beta} - \frac{1}{2}\partial_+\hat{\mathfrak{h}}D_-\hat{\mathfrak{h}} - \frac{1}{2}\partial_{\mu_\perp}\hat{\mathfrak{h}}\partial^{\mu_\perp}\hat{\mathfrak{h}}$$
$$+ \partial_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\partial_\beta\hat{\mathfrak{h}} - \partial_\alpha\hat{\mathfrak{h}}^{\alpha\beta}\partial_\mu\hat{\mathfrak{h}}^\mu_\beta\bigg) + \mathcal{O}(\lambda^3)\,. \tag{6.5.64}$$

For the trilinear Lagrangian $\mathcal{L}_{hhh}$ we can employ a trick to save some work. First, note that the purely-collinear theory is equivalent to the full-theory result (4.3.11). Since the cubic interactions already scale as $\mathcal{O}(\lambda^3)$, we only need to take into account the additional leading-power soft-collinear interactions in this Lagrangian. This is achieved by replacing all ∂_- with the covariant D_-. Using this trick, one can avoid computing the full covariant expression with respect to a generic background, where most terms will be power-suppressed. The expanded Lagrangians homogeneous in λ and in collinear light-cone gauge are then given by

$$\mathcal{L}^{(0)}_{\rm EH} = \frac{1}{2}\partial_\mu\hat{h}_{\alpha_\perp\beta_\perp}\partial^\mu\hat{h}^{\alpha_\perp\beta_\perp} - \frac{1}{2}\partial_\mu\hat{h}\partial^\mu\hat{h}$$
$$+ \left(\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp\beta_\perp}\partial_{\beta_\perp}\hat{h} + \frac{1}{2}\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp -}\partial_+\hat{h} + \frac{1}{2}\partial_+\hat{h}^{-\beta_\perp}\partial_{\beta_\perp}\hat{h} + \frac{1}{4}\partial_+\hat{h}_{--}\partial_+\hat{h}\right)$$
$$- \left(\partial_{\alpha_\perp}\hat{h}^{\alpha_\perp\mu_\perp}\partial^{\beta_\perp}\hat{h}_{\beta_\perp\mu_\perp} + \partial_+\hat{h}^{-\mu_\perp}\partial^{\beta_\perp}\hat{h}_{\beta_\perp\mu_\perp} + \frac{1}{4}\partial_+\hat{h}^{-\mu_\perp}\partial_+\hat{h}_{-\mu_\perp}\right)$$
$$- \frac{\kappa}{8}s_{--}\partial_+\hat{h}_{\alpha\beta}\partial_+\hat{h}^{\alpha\beta} + \frac{\kappa}{8}s_{--}\partial_+\hat{h}\partial_+\hat{h}\,, \tag{6.5.65}$$

$$\mathcal{L}^{(1)}_{\rm EH} = -\frac{\kappa}{2}\hat{h}^{\alpha\beta}\Big(\hat{h}\partial_\alpha\partial_\beta\hat{h} + 2\partial_\mu\hat{h}^{\mu\nu}\partial_\nu\hat{h}_{\alpha\beta} + \partial_\alpha\hat{h}_{\mu\nu}\partial_\beta\hat{h}^{\mu\nu} + \hat{h}_{\alpha\beta}\partial_\mu\partial^\mu\hat{h} + 2\partial^\mu\hat{h}_{\mu\alpha}\partial_\beta\hat{h} \tag{6.5.66}$$
$$+ \partial_\alpha\hat{h}_{\beta\mu}\partial^\mu\hat{h} - \hat{h}\partial^\mu\partial_\alpha\hat{h}_{\beta\mu} - \hat{h}^\mu_\alpha\partial_\nu\partial^\nu\hat{h}_{\mu\beta} + 2\partial_\mu\hat{h}_{\beta\nu}\partial^\nu\hat{h}^\mu_\alpha + 4\partial^\mu\partial^\nu\hat{h}_{\alpha\nu}\hat{h}_{\beta\nu}\Big)$$
$$- \frac{\kappa}{4}\hat{h}\partial_\mu\hat{h}\partial^\mu\hat{h} + \frac{\kappa}{4}\hat{h}\partial_\alpha\hat{h}_{\mu\nu}\partial^\alpha\hat{h}^{\mu\nu} - \frac{\kappa}{4}s_{-\mu_\perp}\partial^{\mu_\perp}\hat{h}^{\alpha\beta}\partial_+\hat{h}_{\alpha\beta} + \frac{\kappa}{4}s_{-\mu_\perp}\partial^{\mu_\perp}\hat{h}\partial_+\hat{h}$$
$$- \frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha-}\right]x^\alpha_\perp\partial_+\hat{h}_{\mu\nu}\partial_+\hat{h}^{\mu\nu} + \frac{\kappa}{8}\left[\partial_\alpha s_{--} - \partial_- s_{\alpha-}\right]x^\alpha_\perp\partial_+\hat{h}\partial_+\hat{h}$$
$$+ \frac{\kappa}{2}\hat{h}^{\alpha\beta}\left(\hat{h}_{\alpha\beta}s_{--}\partial_+^2\hat{h} - \hat{h}^\mu_\alpha s_{--}\partial_+^2\hat{h}_{\mu\beta}\right) + \frac{\kappa}{32}\hat{h}s_{--}\partial_+\hat{h}\partial_+\hat{h} - \frac{\kappa}{32}\hat{h}s_{--}\partial_+\hat{h}_{\mu\nu}\partial_+\hat{h}^{\mu\nu}\,.$$

To keep the result somewhat compact, we did not split the result into light-cone components and kept the four-vector notation in $\mathcal{L}^{(1)}_{\rm EH}$. These Lagrangians could now be further simplified by employing the graviton equations of motion to eliminate the trace $\hat{h}$ as well as $\hat{h}_{\mu-}$.

6.6 Operator Basis

The minimal operator basis in the gravitational case is very similar to the gauge theory one presented in Section 3.5, once a few subtleties regarding gauge-invariance are taken into account. Therefore, we focus on the differences in the gravitational setting, provide a list of gauge-invariant building blocks and refer to the relevant gauge-theory and scalar sections for the general details of the N-jet operator in Sections 2.6 and 3.5.

The generic N-jet operator in gauge-theory (3.5.1) takes the form

$$\mathcal{J}(0) = \int [dt]_N \, \widetilde{C}(t_{i_1}, t_{i_2}, \dots) J_s(0) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \dots)\,, \tag{6.6.1}$$

where $[dt]_N = \prod_{i_k} dt_{i_k}$. We use the same conventions as before, so J_i are the i-collinear operators and J_s are soft gauge-covariant operators. All appearing operators must be invariant under the collinear gauge symmetries and covariant under the soft one. The collinear gauge invariance is achieved in a straightforward fashion by using the gauge-invariant building blocks in the collinear current J_i, explicitly given down below in (6.6.3). Soft fields are automatically collinear gauge-invariant. For soft gauge-covariance, the first subtlety appears. In gravity, the gauge symmetry corresponds to diffeomorphisms. Therefore, any operator considered in gravitational scattering should be defined in a manifestly translation-invariant form to be gauge-invariant. The N-jet operators as defined in (6.6.1) are located at $x = 0$, however, and transform under translations. Thus these objects are not yet gauge-invariant. To alleviate this problem, note that one can translate the entire operator to point x and then integrate over all x. This yields the gauge-invariant current

$$\mathcal{J} = \int d^4x \, T_x \mathcal{J}(0) T_x^{-1}\,. \tag{6.6.2}$$

Once a matrix element including this current is evaluated, the translation operator $T_x = e^{ix\hat{p}}$ and the space-time integral yield the momentum-conserving δ-function. Therefore, it is possible to simply work in terms of the standard N-jet operator (6.6.1) evaluated at $x = 0$ and impose momentum conservation by hand, in analogy to imposing colour-conservation in gauge-theory.

On a related note, one might wonder why there is no $\sqrt{g}$ in (6.6.2) to form the invariant measure. This is related to the underlying hard physics. The situation that we consider is a graviton propagating on top of flat Minkowski space with metric tensor $\eta_{\mu\nu}$. Correspondingly, the soft gauge transformations are *global* transformations evaluated at $x = 0$, where the scattering takes place. The invariant measure for this specific space-time is simply d^4x, and simply corresponds to hard momentum and angular momentum conservation. If one were to consider a different space-time as fundamental, equipped with a non-trivial background metric, the invariant measure would change. If the current were put at x_0 instead of $x = 0$, one would also need to adapt the multipole expansion and expand about $(x - (x_0 + x_-)$ instead of $(x - x_-)$. Therefore, one would find precisely the same soft-collinear physics, irrespective of the point of the hard scattering.

Finally, we explicitly provide the operators that can appear as building blocks in the N-jet operator. For the collinear sector, we use the collinear gauge-invariant building blocks (6.4.5), (6.4.6) given by

$$\hat{\chi}_c = \left[W_c^{-1}\hat{\varphi}_c\right]\,, \quad \hat{\mathfrak{h}}_{\mu\nu} = W^\alpha{}_\mu W^\beta{}_\nu \left[W_c^{-1}\hat{h}_{\alpha\beta}\right] + \left(W^\alpha{}_\mu W^\beta{}_\nu \left[W_c^{-1}\hat{g}_{s\alpha\beta}\right] - \hat{g}_{s\mu\nu}\right)\,. \tag{6.6.3}$$

Note that the scalar field counts as $\hat{\chi}_c \sim \mathcal{O}(\lambda)$. For the graviton, only the transverse components $\mathfrak{h}_{\perp\perp} \sim \mathcal{O}(\lambda)$, which correspond to the physical degrees of freedom, are relevant. To see this, first note that the building blocks satisfy the light-cone gauge condition $\hat{\mathfrak{h}}_{\mu+} = 0$, which eliminates the large components $h_{++} \sim \mathcal{O}(\lambda^{-1})$ and $h_{\perp+} \sim \mathcal{O}(\lambda^0)$, similar to QCD. Next, note that we can use the graviton equations of motion (5.4.6) to eliminate the power-suppressed components

$\hat{\mathfrak{h}}_{\perp -}$ and $\hat{\mathfrak{h}}_{--}$ order-by-order in λ in favour of the transverse components $\hat{\mathfrak{h}}_{\perp\perp}$. In summary, the collinear building blocks consist only of

$$J_i^{A0}(t_i) \in \left\{ \hat{\chi}_{c_i}(t_i n_{i+}) \, , \hat{\mathfrak{h}}_{i\perp\perp}(t_i n_{i+}) \right\} \, . \tag{6.6.4}$$

Comparing this to QCD (3.5.4), one immediately sees the similarity. Also here, the gauge-invariant scalar field and the physical polarisations of the gluon, $\mathcal{A}_{c\perp}$ are the building blocks, while $n_+\mathcal{A}_c = 0$ and $n_-\mathcal{A}_c$ is eliminated using the equations of motion.

Subleading operators are constructed in the exact same fashion as in QCD, namely by (i) adding transverse $\partial_\perp$ derivatives and by (ii) combining multiple building blocks of the same collinear sector.

Soft building blocks must be soft gauge-covariant. Similar to gauge theory, one can use the collinear equations of motion of the matter and gluon fields to eliminate $n_- D_s$ in favour of the other collinear and soft building blocks. Therefore, the first purely-soft building block one can use corresponds to the Riemann tensor, which appears at next-to-next-to-soft order $\mathcal{O}(\lambda^6)$.

6.7 Extension to Fermions

In this section, we give a quick summary of how to extend this construction to incorporate fields with half-integer spin, at the example of the Dirac fermion. The full construction is explained in detail in [64] and will be part of the upcoming [63] in collaboration with M. Beneke and D. Schwienbacher.

The main complexity arises due to the additional local Lorentz symmetry, which results in the introduction of two additional Wilson lines.

The local Lorentz symmetry features the spin-connection (4.4.37) which behaves like an ordinary gauge field. This allows us to define a Wilson line in the standard way

$$W \equiv \mathbf{P}exp \left[i \int_y^z dx^\mu \Omega_\mu(x(s)) \right] \, , \tag{6.7.1}$$

that transforms like an ordinary Wilson line under local Lorentz transformations

$$W \to D\left(\Lambda(z)\right) W D^{-1}\left(\Lambda(y)\right) \, . \tag{6.7.2}$$

The first new Wilson line is the analogue of the R Wilson line and takes the form

$$V_s = \mathbf{P}exp \left(+i \int_{x_1}^{x_2} dy^\mu [R^\nu{}_\mu R^{-1} \Omega_{s\mu}](y) \right) , \tag{6.7.3}$$

where $\Omega_{s\mu}$ is the soft spin connection, containing only soft fields. It transforms under a local Lorentz transformation as

$$V_s \longrightarrow \left[R^{-1} D_s \right] \left(\Lambda(x_2) \right) \check{V}_s \left[R^{-1} D_s \right] \left(\Lambda(x_1) \right) \, . \tag{6.7.4}$$

The condition

$$V_s{}^{-1} \left[R^{-1} \psi \right] \longrightarrow D\left(\Lambda(x_-) \right) V_s{}^{-1} \left[R^{-1} \psi \right] \tag{6.7.5}$$

fixes the upper boundary as $x_2 = x$, while for the lower boundary one requires

$$x_1{}^\mu(x) + \theta_s{}^\rho \partial_\rho x_1{}^\mu(x) = x_-{}^\mu \, , \tag{6.7.6}$$

which originates from the condition

$$[R^{-1} \omega_s{}^{ab}(x_1)] = \omega_s{}^{ab}(x_1) + \theta_s{}^\mu \frac{\partial x_1{}^\rho}{\partial x^\mu} \frac{\partial}{\partial x_1{}^\rho} \omega_s{}^{ab}(x_1) + \dots$$

$$
\begin{aligned}
&= \omega_s{}^{ab}\,(x_1{}^\mu + \theta_s{}^\rho \partial_\rho x_1{}^\mu(x)) \\
&\stackrel{!}{=} \omega_s{}^{ab}(x_-)\,.
\end{aligned} \tag{6.7.7}
$$

This equation is solved by

$$
x_1{}^\mu(x) = x\_{}^\mu + \frac{1}{4}\,(x - x_-)^a\, s^+{}_a n\_{}^\mu + \mathcal{O}(\lambda^4)\,. \tag{6.7.8}
$$

The soft LLT-Wilson line, acting on fermions, is then given by

$$
V_s = \mathbf{P}exp\Big(+i\int_0^1 ds\,(x - x_1)^\mu\,[R^\nu{}_\mu R^{-1}\Omega_{s\mu}]\,(x_1 + (x - x_1)\,s) + \mathcal{O}(\lambda^2)\Big)\,, \tag{6.7.9}
$$

where we use $\sigma^{ab} = \frac{i}{4}\left[\gamma^a, \gamma^b\right]$. The soft vierbein in the corresponding fixed-line gauge is determined by evaluating

$$
\check{e}_s{}^a{}_\mu(x) = V_{sb}{}^a R^\nu{}_\mu\left[R^{-1}e_s{}^b{}_\nu(x)\right]\,, \tag{6.7.10}
$$

and by definition, it transforms under soft Lorentz transformations as

$$
\check{e}_s{}^a{}_\mu(x) \to \Lambda_s{}^a{}_b(x_-)\check{e}_s{}^b{}_\mu(x)\,. \tag{6.7.11}
$$

In addition, for the fermionic case, we require a LLT Wilson line in order to define fields that are not only GCT-invariant but also manifestly LLT-invariant. This Wilson line should be diffeomorphism-invariant and thus we use the manifestly GCT-invariant quantity:

$$
\tilde{\Omega}_\mu(x) = \left[W^\rho{}_\mu W_c^{-1}\hat{\Omega}_\rho\right](x)\,, \tag{6.7.12}
$$

as our spin-connection. We can then define the collinear LLT Wilson-line as

$$
V_c(x) = \mathbf{P}exp\Big(+i\int_{-\infty}^0 ds' n^\mu{}_+\left[W^\rho{}_\mu W_c^{-1}\hat{\Omega}_\rho\right](x)\Big)\,. \tag{6.7.13}
$$

Note that $\hat{\Omega}_\rho$ is constructed from $\hat{e}^a_\mu = \hat{e}_s{}^a{}_\mu + \hat{E}_s{}^{\rho a}\hat{e}_{c\rho\mu}$ e.g. once again from the expansion of the vierbein about a soft background. Like the GCT Wilson line before, this object is not purely-collinear, since the soft background enters in the combination $n^\mu{}_+\left[W^\rho{}_\mu W_c^{-1}\hat{\Omega}_\rho\right](x)$. According to (6.7.2), this Wilson-line transforms under LLT as:

$$
V_c(x) \longrightarrow [W_c^{-1}D(\Lambda(x))]V(x)[W_c^{-1}D^{-1}](\Lambda(-\infty) = [W_c^{-1}D](\Lambda(x))V(x)\,. \tag{6.7.14}
$$

With these ingredients, one can perform redefinitions of LLT-tensors and construct the Lagrangian for half-integer fields. The main idea is to construct fields that are LLT-tensors but transform with the homogeneous symmetry $\Lambda(x_-)$. Under GCT, these fields then correspond to scalar fields, and we can employ the scalar construction of the previous sections. As the explicit expressions are excessively long, we refer for the construction of the Dirac Lagrangian to [63,64].

Chapter 7

Soft Theorems in the Lagrangian

As an application of the effective theory, we reconsider the soft theorems both in gauge theory and gravity. In the explicit calculation in Section 3.6, we have seen a number of accidental cancellations between the different contributions. This raises the question if the Feynman rules corresponding to single-soft emission can be cast into a simpler form. This could give new insights into the form and structure of the soft theorem.

In this section, we try to understand what the effective theory can tell us about the soft theorem. In particular: why there are two or three terms in gauge theory and gravity, respectively, and no further terms? Why does the angular momentum operator appear in both theorems, and what is its interpretation? To achieve this, we directly manipulate the interaction vertex at the Lagrangian level, employing only standard EFT tools, like integration by parts or applications of equations of motion. This allows us to restate the content of the soft theorem in terms of an operatorial statement. The following section is based closely on [48] by the author in collaboration with M. Beneke and R. Szafron.

7.1 Summary of Non-radiative Matching

The non-radiative matching is discussed in detail in Section 3.6.2, and we can directly use the results from there.

In summary, the non-radiative full-theory amplitude must be expanded in λ as

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{O}(\lambda^2)\,, \tag{7.1.1}$$

where the individual terms are given by the Taylor-expansion

$$\mathcal{A}^{(0)} = \mathcal{A}\Big|_{p_i^\mu = n_{i+}p_i\, n_{i-}^\mu/2}\,, \tag{7.1.2}$$

$$\mathcal{A}^{(1)} = p_{i\perp}^\mu \left(\frac{\partial}{\partial p_{i\perp}^\mu}\mathcal{A}\right)\Big|_{p_i^\mu = n_{i+}p_i\, n_{i-}^\mu/2}\,. \tag{7.1.3}$$

On the SCET side, this is matched to the N-jet operators. In order to simplify the notation, we now denote the N-jet operators by $\hat{\mathcal{A}}^{(n)}$, where n denotes the suppression in λ. At leading-power, the matching condition then reads

$$\begin{aligned}\mathcal{A}^{(0)} &= \langle p_1,\ldots,p_N|\hat{\mathcal{A}}^{(0)}|0\rangle \\ &= \int [dt]_N\, e^{i\sum_i n_{i+}p_i\, t_i}\widetilde{C}^{A0}(t_1,\ldots t_N) \equiv C^{A0}(n_{1+}p_1,\ldots,n_{N+}p_N)\,.\end{aligned} \tag{7.1.4}$$

This intuitive notation indicates that the matrix element of the operator $\hat{A}^{(n)}$ reproduces the n-th term $\mathcal{A}^{(n)}$ in the Taylor expansion of full-theory amplitude $\mathcal{A}$.

7.2 Scalar QCD

As the first example, we consider the emission of a soft gluon from scalar external legs. It is crucial to recall that in SCET QCD, as discussed in Section 3.5, there are no purely-soft operators

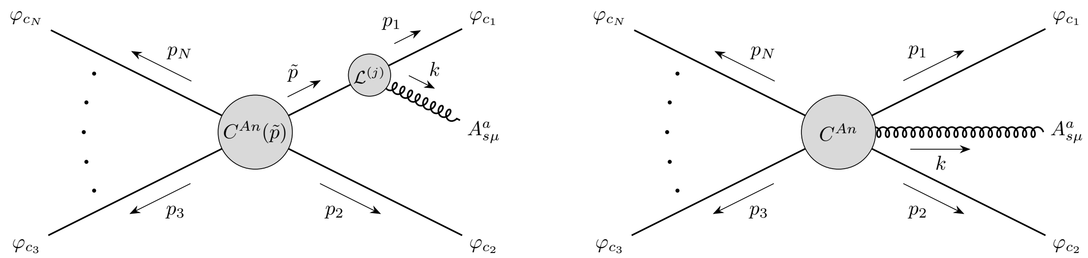


Figure 7.1: The two process-types contributing to single-soft emission in SCET. The first class are insertions of the soft-collinear Lagrangian and corresponds to emissions from the external legs. These processes are universal since they only depend on the properties of the legs. The second type, emission directly from the hard vertex, corresponds to the addition of a soft-covariant building block in the N-jet operator.

available to add to the N-jet operator until the field-strength tensor at $\mathcal{O}(\lambda^4)$, corresponding to next-to-next-to soft order. This type of emission corresponds to emission directly from the non-radiative amplitude, depicted on the left in Fig. 7.1. Adding such a building block means that one also modifies the matching coefficient, and thus this type of contribution is *process-dependent*, as the value of this coefficient can only be determined by an explicit matching computation to the soft-emission process. The absence of such a contribution implies that there are *no process-dependent terms* at the leading and next-to-soft order, corresponding to $\mathcal{O}(\lambda^0)$ to $\mathcal{O}(\lambda^2)$.

Since these processes are absent in SCET, this means that all contributions to soft emission must stem from Lagrangian insertions via time-ordered product operators.

$$i\int d^4x\, T\{J^{Ak}(t_i), \mathcal{L}_i^{(n)}(x)\} \sim \mathcal{O}(\lambda^{k+n})\,. \tag{7.2.1}$$

Intuitively, these contributions correspond to emissions from the external legs, as depicted on the right in Fig. 7.1 This type of emission does not depend on the non-radiative amplitude, but only on the properties of the leg (particle species, momentum, charge, spin, ...) where the Lagrangian is inserted. In particular, this insertion does not come with an independent matching coefficient and is completely determined already from non-radiative matching. These processes start to contribute at leading-power via interactions through the soft-covariant derivative.

Let us stress that this simple property of the operator basis, which follows directly from soft gauge-covariance, already implies *universality of soft emissions* at the leading and next-to-soft order in gauge theory. No explicit computation is necessary to conclude that the LBK amplitude has two universal terms that are independent of the hard source, it is a direct consequence of the effective gauge symmetry.

7.2.1 General Assumptions and Basic Features

Having understood this we can begin with the derivation of the soft theorem at the Lagrangian level. The assumptions are the same as in the previous explicit derivation in Section 3.6.1. We are interested in a single soft-emission process, with only a single energetic particle in each direction. Moreover, we consider for the *radiative amplitude* a coordinate system where each reference vector n_{i-}^μ is aligned with its corresponding momentum such that $p_{i\perp} = 0$ and

$$p_i^\mu = n_{i+}p_i\frac{n_{i-}^\mu}{2}\,. \tag{7.2.2}$$

We want to derive the soft theorem directly from the Lagrangian. To this end, whenever we manipulate the interaction vertex, we take it to be nested inside the relevant matrix element

$$\langle p, k|\, i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(n)}, \mathcal{L}_\chi^{(k)}\right\}|0\rangle\ , \tag{7.2.3}$$

i.e. inside the tree-level matrix element with single soft emission, no additional collinear emissions, evaluated in the special reference frame where all external collinear particles satisfy $p_{i\perp} = 0$. However, to safe space, we will not write down the matrix element explicitly, but only manipulate the terms in $\mathcal{L}^{(i)}$ under this assumption. Whenever we drop terms using this understanding, we use the symbol $\widehat{=}$. For example, we can write external $p^\mu \,\widehat{=}\, n_{i+}p\frac{n_{i-}}{\mu}$, as all other components vanish in our reference frame.

Furthermore, since we consider only a single collinear particle and no additional collinear emissions in each direction, we can again set the collinear Wilson lines $W_c = 1$ and do not need to distinguish between the fields ϕ_c and the gauge-invariant building blocks χ_c.

Since we employ the all-outgoing convention and want ϕ_c particles in the external states, one can always view $\chi_c^\dagger$ as the *external* particle, where $p_\perp = 0$ applies, while χ_c might get contracted with the Lagrangian and can carry internal momenta.

Finally, let us formalise the discovery of the eikonal propagator in (3.6.23), which we call the "universal contraction" from now on.[1] It is defined as

$$\int d^4x\, e^{i\frac{1}{2}n_-k\, n_+x} \langle p|\ \chi_c^\dagger(0),\ \chi_c^\dagger(x)\ [in_+\partial\chi_c(x)]\ |0\rangle = \frac{in_+p}{2p\cdot k} = \frac{i}{n_-k}\,, \tag{7.2.4}$$

and precisely corresponds to the eikonal propagator. Note the additional factor $in_+\partial$ acting on the $\chi_c(x)$ field. This is the position-space analogue of the propagator $\frac{in_{i+}\tilde{p}}{\tilde{p}^2}$, and ensures that both n_-x and $x_\perp^\mu$ do not act on this term, as

$$\int d^4x\, e^{i\frac{1}{2}n_-k\, n_+x}\, n_-x\ \langle p|\ \chi_c^\dagger(0),\ \chi_c^\dagger(x)\ [in_+\partial\chi_c(x)]\ |0\rangle = 0, \tag{7.2.5}$$

$$\int d^4x\, e^{i\frac{1}{2}n_-k\, n_+x}\, x_\perp^\mu\ \langle p|\ \chi_c^\dagger(0),\ \chi_c^\dagger(x)\ [in_+\partial\chi_c(x)]\ |0\rangle = 0\,. \tag{7.2.6}$$

This is most easily checked in momentum space, where the explicit x turn into derivatives with respect to the momentum p.

Most of the following manipulations boil down to re-arranging the Lagrangian to yield this manifest form of the universal contraction and applying all derivatives to the external state $\chi_c^\dagger$, where the simple kinematics apply. This ensures that the factors of $x_\perp$ and n_-x, which correspond to derivatives with respect to momenta, only act on the non-radiative amplitude, and not on the eikonal propagator, as we observe in the soft theorem.

7.2.2 The LBK Derivation

With all these notions and assumptions clarified, we can begin with manipulating the Lagrangians. For a single leg, there are three non-vanishing contributions to the soft-emission amplitude, as computed in Section 3.6, corresponding to the time-ordered products

$$\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(0)}\right\}\,, \tag{7.2.7}$$

$$\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(2)}\right\}\,, \tag{7.2.8}$$

$$\int d^4x\, T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(1)}\right\}\,. \tag{7.2.9}$$

The first one must give rise to the leading-power eikonal term, while the second and third one must combine to yield the subleading term. This means that we also have to explicitly verify that there is no $\mathcal{O}(\lambda)$ contribution from $\mathcal{L}^{(1)}$. The relevant scalar-gluon interaction terms in the Lagrangian are given by (3.6.14) – (3.6.16).

[1] This universal contraction arises differently for fields of integer and half-integer spin, due to different normalisation of the kinetic terms. In theories with a single-derivative kinetic term, it follows directly from the effective propagator.

Leading-power Contribution

The leading-power contribution stems from the time-ordered product (3.6.6). In the Lagrangian, the relevant interaction term reads

$$\mathcal{L}^{(0)}_{\text{int}} = \frac{g}{2} n_- A_s(x_-) \left(\phi_c^\dagger t^a i n_+ \partial \phi_c - i n_+ \partial \phi_c^\dagger t^a \phi_c \right) . \tag{7.2.10}$$

Here, we first replace $\phi_c \to \chi_c$ and express the interaction in terms of the gauge-invariant building block. This has an effect only on interactions featuring collinear gluons, so we can neglect the additional contributions. Next, one integrates by parts in the second term to obtain the universal contraction (7.2.4). one finds

$$\mathcal{L}^{(0)}_{\text{eikonal}} = g n_- A_s^a \chi_c^\dagger t^a i n_+ \partial \chi_c \, . \tag{7.2.11}$$

This immediately reproduces the leading-power term in the soft theorem (3.6.23). Written in this form, the structure of this term becomes manifest. The eikonal propagator stems from the universal contraction (7.2.4), and the numerator is a simple consequence of the minimal coupling to the homogeneous background field $n_- A_s(x_-)$ via the soft-covariant derivative $n_- D_s$ in the effective theory.

Next-to-soft Term

For the next-to-soft term, we require the sub- and subsubleading Lagrangian interactions from $\mathcal{L}^{(1)}_\chi$ and $\mathcal{L}^{(2)}$, which we write as

$$\mathcal{L}^{(1)}_\chi = \frac{1}{2} x_\perp^\mu \, n_+ j_a \, n_-^\nu g F^{s\,a}_{\mu\nu} \, , \tag{7.2.12}$$

$$\mathcal{L}^{(2)}_\chi \mathrel{\widehat{=}} \mathcal{L}^{(2a)}_\chi + \mathcal{L}^{(2b)}_\chi + \mathcal{L}^{(2c)}_\chi \, , \tag{7.2.13}$$

with

$$\mathcal{L}^{(2a)}_\chi = \frac{1}{2} x_\perp^\nu j_a^{\mu_\perp} g F^{s\,a}_{\nu\mu} \, , \tag{7.2.14}$$

$$\mathcal{L}^{(2b)}_\chi = \frac{1}{4} n_- x \, n_+^\mu \, n_+ j_a \, n_-^\nu g F^{s\,a}_{\mu\nu} \, , \tag{7.2.15}$$

$$\mathcal{L}^{(2c)}_\chi = \frac{1}{4} x_\perp^\mu x_{\perp\rho} \, n_+ j_a \, n_-^\nu \, \mathrm{tr} \left(\left[D_s^\rho , g F^s_{\mu\nu} \right] t^a \right) . \tag{7.2.16}$$

Here, we introduced the (linear) Noether current j_a^μ (3.4.6)

$$j_a^\mu = \chi_c^\dagger \, t^a \, i \partial^\mu \chi_c + \left[i \partial^\mu \chi_c \right]^\dagger \, t^a \, \chi_c \, . \tag{7.2.17}$$

First, we have to verify that $\mathcal{L}^{(1)}_\chi$ does not contribute at $\mathcal{O}(\lambda)$ via the time-ordered product

$$i \int d^4 x \, T \left\{ \hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(1)}_\chi \right\} \mathrel{\widehat{=}} 0 \, . \tag{7.2.18}$$

In the explicit computation (3.6.24), this term was proportional to $p_\perp$ and vanished as a consequence of $p_\perp = 0$.

Indeed, rewriting the Lagrangian $\mathcal{L}^{(1)}_\chi$ using integration by parts, one obtains

$$\mathcal{L}^{(1)}_\chi = \chi_c^\dagger \, x_\perp^\mu n_-^\nu g F^s_{\mu\nu} i n_+ \partial \chi_c \, . \tag{7.2.19}$$

Since the time time-ordered product with the $A0$ current (7.2.18) does not contain any transverse momenta, and the $x_\perp$ does not act on the universal contraction 7.2.6, one immediately finds that this contribution must vanish. This agrees with the explicit result (3.6.24), where the derivative of the eikonal propagator was evaluated explicitly.

At the sub-subleading, or next-to-soft power, there are two possible contributions, from the time-ordered products $T\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(2)}\}$ and $T\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(1)}\}$, corresponding to (3.6.7) and (3.6.8). These two terms must contribute very specifically to the soft theorem.

Notice that due to our choice of external momenta, the angular momentum in momentum space simplifies to

$$L_i^{\mu\nu} = \frac{1}{4} n_{i+}^{[\mu} n_{i-}^{\nu]} n_{i+} p_i n_{i-}^{\alpha} \frac{\partial}{p_i^{\alpha}} + \frac{1}{2} n_{i+} p_i n_{i-}^{[\nu} \frac{\partial}{\partial p_{i\perp\mu]}} \,, \tag{7.2.20}$$

which implies the position-space representation

$$L^{\mu\nu} = x^{[\mu}\partial^{\nu]} = \underbrace{\frac{1}{4} n_+^{[\mu} n_-^{\nu]}\, n_- \cdot x\, n_+ \cdot \partial}_{L_{+-}^{\mu\nu}} + \underbrace{\frac{1}{2} x_\perp^{[\mu} n_-^{\nu]}\, n_+ \cdot \partial}_{L_{\perp+}^{\mu\nu}} \,. \tag{7.2.21}$$

Next, observe that the $L_{\perp+}$ operator contains an explicit $x_\perp$, or equivalently $\frac{\partial}{\partial p_\perp}$. Therefore, it must hit a $\partial_\perp$ in order to yield a non-vanishing contribution. This implies that the mixed transverse-longitudinal term must stem from $\mathcal{L}^{(1)}$, as it contributes through the time-ordered product (3.6.8) which contains $\partial_\perp$.

On the other hand, the longitudinal term L_{+-} contains only $n_- x$, or equivalently, $\frac{\partial}{\partial n_{i+} p}$. Therefore, it must appear in a combination that *does not* contain external $p_\perp$, since these cannot be eliminated.

In summary, we find that the longitudinal angular momentum L_{+-} must be located inside $\mathcal{L}^{(2)}$, where it contributes via (3.6.7), while the mixed-longitudinal term must be part of $\mathcal{L}^{(1)}$, where it appears via (3.6.8), since it requires the explicit $\partial_\perp$ to yield a non-vanishing contribution.

Consider first $\mathcal{L}_\chi^{(2)}$ given in (7.2.14)–(7.2.16). Writing out the Noether current, we find

$$\mathcal{L}^{(2a)} = \frac{1}{2} x_\perp^\nu j_a^{\mu_\perp} ig F_{\nu\mu}^{s\,a} \left[\hat{\phi}_c^\dagger t^a \overset{\leftrightarrow}{\partial}_{\mu_\perp} \hat{\phi}_c\right] . \tag{7.2.22}$$

First, integrate by parts to have $\partial_\perp$ act on the external $\phi_c^\dagger$, where it can be dropped using the kinematic assumption $p_\perp = 0$. This integration yields

$$\mathcal{L}_\chi^{(2a)} = [i\partial_{\perp\mu}\chi_c]^\dagger x_\perp^\nu g F_{\nu\mu}^s \chi_c + \frac{1}{2} i \chi_c^\dagger \eta_\perp^{\mu\nu} g F_{\nu\mu}^s \chi_c \,, \tag{7.2.23}$$

and as anticipated the first term can be dropped since it is proportional to the external $p_\perp = 0$. Note that the second term also vanishes, since the antisymmetric $F_{\nu\mu}^s$ is contracted with the symmetric $\eta_\perp^{\mu\nu}$. Thus we conclude

$$\mathcal{L}_\chi^{(2a)}(x) \mathrel{\widehat{=}} 0 \,. \tag{7.2.24}$$

For the second term, $\mathcal{L}_\chi^{(2b)}$, we have

$$\mathcal{L}_\chi^{(2b)} = \frac{1}{4} n_- x n_+^\mu n_-^\nu ig F_{\mu\nu}^{s\,a} \left[\hat{\phi}_c^\dagger t^a n_+ \overset{\leftrightarrow}{\partial} \hat{\phi}_c\right] . \tag{7.2.25}$$

First, we integrate $n_+\partial$ to obtain the universal contraction. This yields

$$\mathcal{L}_\chi^{(2b)} = \frac{1}{2} \chi_c^\dagger\, n_- x n_+^\mu n_-^\nu g F_{\mu\nu}^s i n_+ \partial \chi_c + \frac{1}{2} i \chi_c^\dagger n_+^\mu n_-^\nu g F_{\mu\nu}^s \chi_c \,, \tag{7.2.26}$$

where we notice that the first term already contains structures reminding of the longitudinal angular momentum, contracted with the field-strength tensor. In fact, this first term

$$\mathcal{L}_\chi^{(2b)} \supset \frac{1}{2} \chi_c^\dagger ig F_{\mu\nu}^s\, n_- x n_+^\mu n_-^\nu n_+ \partial \chi_c = \chi_c^\dagger ig F_{\mu\nu}^s\, L_{+-}^{\mu\nu} \chi_c \,, \tag{7.2.27}$$

is precisely the full longitudinal component of the angular momentum. This means that the second term in (7.2.26) is left-over and must cancel in the end.

This is alleviated by looking at the contribution from $\mathcal{L}_\chi^{(2c)}$, which takes the form

$$\mathcal{L}_\chi^{(2c)} = \frac{1}{4} x_\perp^\mu x_{\perp\rho} n_-^\nu \left[\partial^\rho igF^a_{s\mu\nu}\right] \left[\hat{\phi}_c^\dagger t^a n_+ \overleftrightarrow{\partial} \hat{\phi}_c\right] . \tag{7.2.28}$$

First, we split the $x_\perp^\mu x_{\perp\rho}$ term into a traceless and a trace part, as

$$x_\perp^\mu x_{\perp\rho} = [x_\perp^\mu x_{\perp\rho} - \frac{1}{2} x_\perp^2 \delta^\mu_{\perp\rho}] + \frac{1}{2} x_\perp^2 \delta^\mu_{\perp\rho} . \tag{7.2.29}$$

Note that the traceless term cannot contribute if the external $p_\perp = 0$, and it can be dropped. This leaves us with

$$\mathcal{L}_\chi^{(2c)} = \frac{1}{8} x_\perp^2 n_-^\nu \left[\partial_\perp^\mu igF^a_{s\mu\nu}\right] \left[\hat{\phi}_c^\dagger t^a n_+ \overleftrightarrow{\partial} \hat{\phi}_c\right] . \tag{7.2.30}$$

We would like to move the derivative away from the field-strength tensor, but this derivative is evaluated *before* setting $x = x_-$ in the field-strength tensor, and moving it requires one to be very careful with the coordinate arguments. Therefore, we first employ the leading (linear) gluon equation of motion[2]

$$0 \mathrel{\widehat{=}} \partial^\mu F^a_{\mu\nu} = \partial_\perp^\mu F^a_{s\mu\nu} + \frac{1}{2} n_+\partial n_-^\mu F_{s\mu\nu} + \frac{1}{2} n_-\partial n_+^\mu F^a_{s\mu\nu} \tag{7.2.31}$$

to rewrite the transverse derivative as

$$n_-^\nu \partial_\perp^\mu F^a_{s\mu\nu} \mathrel{\widehat{=}} -\frac{1}{2} n_-^\nu n_+^\mu n_-\partial F^a_{s\mu\nu} , \tag{7.2.32}$$

where we used that $n_-^\mu n_-^\nu F^a_{s\mu\nu} = 0$ by symmetry. This yields

$$\mathcal{L}_\chi^{(2c)} = -\frac{1}{16} x_\perp^2 n_-^\nu n_+^\mu \left[n_-\partial igF^a_{s\mu\nu}\right] \left[\hat{\phi}_c^\dagger t^a n_+ \overleftrightarrow{\partial} \hat{\phi}_c\right] , \tag{7.2.33}$$

and now the $n_-\partial$ can be integrated by parts as usual, since the field-strength tensor is evaluated at x_-. This integration yields

$$\begin{aligned}
\mathcal{L}_\chi^{(2c)} &= \frac{1}{16} x_\perp^2 n_-^\nu n_+^\mu igF^a_{s\mu\nu} \Big(n_-\partial\hat{\phi}_c^\dagger t^a n_+\partial\hat{\phi}_c + \hat{\phi}_c^\dagger t^a n_-\partial n_+\partial\hat{\phi}_c - n_-\partial n_+\partial\hat{\phi}_c^\dagger t^a \hat{\phi}_c - n_+\partial\hat{\phi}_c^\dagger t^a n_-\partial\hat{\phi}_c \Big) \\
&\mathrel{\widehat{=}} \frac{1}{16} x_\perp^2 n_-^\nu n_+^\mu igF^a_{s\mu\nu} \left(+\hat{\phi}_c^\dagger t^a n_-\partial n_+\partial\hat{\phi}_c - n_+\partial\hat{\phi}_c^\dagger t^a n_-\partial\hat{\phi}_c \right) \\
&\mathrel{\widehat{=}} \frac{1}{8} x_\perp^2 n_-^\nu n_+^\mu igF^a_{s\mu\nu} \left(\hat{\phi}_c^\dagger t^a n_-\partial n_+\partial\hat{\phi}_c \right) ,
\end{aligned} \tag{7.2.34}$$

where we used that the external momentum $n_-p = 0$, thus $n_-\partial\phi^\dagger \mathrel{\widehat{=}} 0$ in the second line and integrated by parts in the third line. Next, to simplify the $n_-\partial$, employ the leading-power equation of motion of the collinear scalar field

$$n_-\partial n_+\partial\phi_c \mathrel{\widehat{=}} -\partial_\perp^2 \phi_c , \tag{7.2.35}$$

to obtain

$$\begin{aligned}
\mathcal{L}_\chi^{(2c)} &\mathrel{\widehat{=}} -\frac{1}{8} x_\perp^2 n_-^\nu n_+^\mu igF^a_{s\mu\nu} \left[\hat{\phi}_c^\dagger t^a \partial_\perp^2 \hat{\phi}_c\right] \\
&\mathrel{\widehat{=}} -\frac{1}{2} n_-^\nu n_+^\mu igF^a_{s\mu\nu} \left[\hat{\phi}_c^\dagger t^a \hat{\phi}_c\right] ,
\end{aligned} \tag{7.2.36}$$

[2] Using only the linear equation of motion is justified since we are not interested in higher-order terms in A_s, which correspond to multi-gluon emission.

where we integrated by parts the $\partial_\perp^2$ as

$$x_\perp^2 \partial_\perp^2 = \partial_\perp^2 x_\perp^2 + 4\partial_\perp \cdot x_\perp + 4 \mathrel{\widehat{=}} 4\,, \tag{7.2.37}$$

since $\partial_\perp$ on the left of $x_\perp$ yields an external $p_\perp \mathrel{\widehat{=}} 0$.

Observe that the term (7.2.36) precisely cancels out the second term in (7.2.26). This is the Lagrangian manifestation of the intricate cancellations we found in the previous calculation when employing the on-shell conditions (3.6.36). In the Lagrangian version, these correspond to the equations of motion of matter and gauge particles.

To conclude, we find that the sub-subleading Lagrangian can be written as

$$\mathcal{L}_\chi^{(2)} \mathrel{\widehat{=}} \mathcal{L}^{(2)}_{\text{orbital}} \equiv \frac{1}{2}\chi_c^\dagger\, n_- x\, n_+^\mu n_-^\nu g F^s_{\mu\nu}\, in_+\partial\chi_c = igF^{s\,a}_{\mu\nu}\chi_c^\dagger t^a\, L^{\mu\nu}_{+-}\,\chi_c\,. \tag{7.2.38}$$

and see that the sub-subleading Lagrangian precisely contains the longitudinal components of the angular momentum operator.

Next, we investigate the subleading Lagrangian (7.2.12). At $\mathcal{O}(\lambda)$, it yielded a vanishing contribution in the matrix element with the leading-power current, since one could set $X_\perp = 0$. However, at the next-to-soft order, this term now appears in the time-ordered product with the subleading current $\hat{\mathcal{A}}^{(1)}$, which contains $\partial_\perp$. This can lead to a non-vanishing contribution.

More precisely, the subleading Lagrangian is given by

$$\mathcal{L}^{(1)} = igF^{s\,a}_{\mu\nu}\phi_c^\dagger t^a L^{\mu\nu}_{\perp+}\phi_c\,, \tag{7.2.39}$$

and contains the transverse-longitudinal component of the angular momentum (7.2.21), with explicit $x_\perp$. There is a non-vanishing contribution in the time-ordered product when the transverse derivative $\partial_\perp$ in the subleading operator $\hat{\mathcal{A}}^{(1)}$ acts on this explicit $x_\perp$ in the Lagrangian. The contribution from $T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(1)}\right\}$ is given by

$$\begin{aligned} i\int d^4x\; T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(1)}\right\} &= i\int d^4x\; T\left\{\int dt\, \widetilde{C}^{A1}_\rho(t)\; i\partial_\perp^\rho\chi_c^\dagger, \chi_c^\dagger\left[x_\perp^\mu n_-^\nu g F^s_{\mu\nu}\right] in_+\partial\chi_c\right\} \\ &= -gt_i^a\frac{k_\mu\varepsilon^a_\nu(k)}{n_-k}n_-^{[\nu}\frac{\partial}{\partial p_{\perp\mu]}}\mathcal{A}^{(1)}\,. \end{aligned} \tag{7.2.40}$$

It is easiest to directly verify this step in momentum-space, following (3.6.43). However, this result can also be understood intuitively. First, note that the derivative $\partial_\perp$ corresponds to the internal momentum $\tilde{p}$, which satisfies $\tilde{p}_\perp \neq 0$. The explicit $x_\perp$ corresponds to a derivative with respect to this internal momentum $\tilde{p}$, which is carried by the Fourier-transform of the scalar χ_c, of $\partial_\perp$ and of the matching coefficient $\widetilde{C}^{A1}$. However, the $x_\perp$ does *not* act on the universal contraction $in_+\partial\chi_c$, since it does not depend on $\tilde{p}_\perp$. Similarly, it does not act on C^{A1}, since this also only depends on $n_{i+}\tilde{p}$. It can only act on $\partial_\perp^\mu$, which yields simply δ^μ_ρ. Then, the time-ordered product can be evaluated in standard fashion. Finally, identify the coefficient $C^{A1\mu}$ with the derivative $\frac{\partial}{\partial p_{\perp\mu}}\mathcal{A}^{(1)}$ of the non-radiative amplitude, using the explicit matching condition (2.7.14)

$$\mathcal{A}^{(1)} = -p^\mu_{j\perp}C^{A1}_{j\,\mu}(n_{1+}p_1,\ldots,n_{N+}p_N)\,. \tag{7.2.41}$$

This immediately yields (7.2.40). In conclusion, we find an operatorial statement also at the next-to-soft order, as

$$\begin{aligned} \mathcal{A}^{(2)}_{\text{rad}} &\mathrel{\widehat{=}} \sum_i i\int d^4x\; T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}^{(1)}_{i,\chi}\right\} + i\int d^4x\; T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(2)}_{i,\,\text{orbital}}\right\} \\ &= i\sum_i\int d^4x\; T\left\{\hat{\mathcal{A}}, \chi_{c_i}^\dagger t_i^a\, L_i^{\mu\nu} igF^{s\,a}_{\mu\nu}\chi_{c_i}\right\}\,. \end{aligned} \tag{7.2.42}$$

where in the second line, we combined the source and Lagrangian terms using (7.2.18) and $\int d^4x\ T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(2)}\right\} \mathrel{\widehat{=}} 0$. The sum $\hat{\mathcal{A}}^{(0)} + \hat{\mathcal{A}}^{(1)} = \hat{\mathcal{A}} + \mathcal{O}(\lambda^2)$ corresponds the non-radiative amplitude expanded up to order λ. This result (7.2.42) is completely equivalent to the soft theorem and can be viewed as a new result. The amplitude for a single soft-emission process is completely captured by the time-ordered product of the non-radiative amplitude with the Lagrangian interaction vertices.

7.3 Extension to Fermions

Next, we investigate how the derivation changes if one considers fermionic matter instead. This is interesting for two reasons: first, the Lagrangian is linear in the derivatives, not quadratic as the scalar one. This has implications for the propagator of the fermion field, and we will see that the universal contraction from before (7.2.4) now simply corresponds to the normal propagator, simplifying the derivation. Second, the LBK amplitude features the full angular momentum, not just the orbital part. Therefore, one should also be able to identify how the spin term arises in this Lagrangian discussion.

7.3.1 General Properties

For details regarding the power-counting, Lagrangian discussion and non-radiative matching, we refer to the previous Section 3.7. We summarise the key insights of this section: The light-cone reference vectors can be used to construct projection operators. Then, the full-theory spinor field is split as $\psi_c = \xi_c + \eta_c$, where ξ_c satisfies

$$\frac{\not{n}_-\not{n}_+}{4}\xi_c = \xi_c\,, \tag{7.3.1}$$

and η_c is subleading compared to ξ_c. Therefore, the effective theory only employs ξ_c while η_c is integrated out. The component ξ_c has the universal contraction

$$\int d^4x e^{i\frac{1}{2}n_-k\, n_+x}\,\langle p|\ \bar{\xi}_c(0),\ \bar{\xi}_c(x)\ \frac{\not{n}_+}{2}\xi_c(x)\ |0\rangle = \bar{\xi}_c(p)\frac{in_+p}{2p\cdot k} = \bar{\xi}_c(p)\frac{\not{n}_+\not{n}_-}{4}\frac{i}{n_-k}\,, \tag{7.3.2}$$

which is the projection onto the ξ_c component of the standard propagator. In addition, during matching computations, the full-theory spinor $\bar{u}(p_i)$ must be expanded as

$$\bar{u}(p_i) = \bar{\xi}_{c_i}(p_i)\left(1 - \frac{\not{p}_{i\perp}}{n_{i+}p_i}\frac{\not{n}_{i+}}{2}\right)\,. \tag{7.3.3}$$

This is the effect of the subleading spinor η_c, and leads to a modified RPI-relation for the $A1$ matching coefficient

$$\begin{aligned} C_i^{A1\,\mu}(n_{1+}p_1,\dots,n_{N+}p_N) &= \left[-\frac{\gamma^\mu_{i\perp}}{n_{i+}p_i}\frac{\not{n}_{i+}}{2} - \sum_{j\neq i}\frac{2n^\mu_{j-}}{n_{i-}\cdot n_{j-}}\frac{\partial}{\partial n_{i+}p_i}\right]C^{A0}(n_{1+}p_1,\dots,n_{N+}p_N) \\ &\equiv C^{A1\mu}_{i,\,\text{spin}}(n_{1+}p_1,\dots,n_{N+}p_N) + C^{A1\mu}_{i,\,\text{orbit}}(n_{1+}p_1,\dots,n_{N+}p_N)\,, \end{aligned} \tag{7.3.4}$$

which now contains a spin-dependent term. This relation is highly relevant for the angular momentum operator, analogous to the orbital part that was also critical in the scalar derivation. We also perform this split for the subleading N-jet operator as

$$\hat{\mathcal{A}}^{(1)} = \hat{\mathcal{A}}^{(1)}_{\text{orbit}} + \hat{\mathcal{A}}^{(1)}_{\text{spin}}\,. \tag{7.3.5}$$

The actual computation now proceeds along very similar lines as in the scalar case. In particular, the leading-power term follows immediately, without any computation. Therefore, we restrict our discussion to the next-to-soft term, which is the more interesting contribution.

Next-to-soft Term

The subleading Lagrangian takes the form [40]

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}_c \left(x_\perp^\mu n_-^\nu \, W_c \, gF^s_{\mu\nu} W_c^\dagger \right) \frac{\not{n}_+}{2} \xi_c \, , \tag{7.3.6}$$

$$\begin{aligned} \mathcal{L}_\xi^{(2)} &= \frac{1}{2} \bar{\xi}_c \left(n_- x \, n_+^\mu n_-^\nu \, W_c \, gF^s_{\mu\nu} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu W_c [D_s^\rho, gF^s_{\mu\nu}] W_c^\dagger \right) \frac{\not{n}_+}{2} \xi_c \\ &+ \frac{1}{2} \bar{\xi}_c \left(i\not{D}_\perp \frac{1}{in_+ D} x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF^s_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu \, W_c \, gF^s_{\mu\nu} W_c^\dagger \frac{1}{in_+ D} i\not{D}_\perp \right) \frac{\not{n}_+}{2} \xi_c \, . \end{aligned} \tag{7.3.7}$$

As before, we split the subsubleading Lagrangian as

$$\mathcal{L}_\xi^{(1)} \mathrel{\widehat{=}} \bar{\xi}_c \left(x_\perp^\mu n_-^\nu \, gF^s_{\mu\nu} \right) \frac{\not{n}_+}{2} \xi_c \, , \tag{7.3.8}$$

$$\mathcal{L}_\xi^{(2)} \mathrel{\widehat{=}} \mathcal{L}_\xi^{(2a)} + \mathcal{L}_\xi^{(2b)} + \mathcal{L}_\xi^{(2c)} + \mathcal{L}_\xi^{(2s)} \, , \tag{7.3.9}$$

where the labels follow the scalar decomposition (7.2.12)

$$\begin{aligned} \mathcal{L}_\xi^{(2a)} &= \bar{\xi}_c x_\perp^\mu gF^s_{\mu\nu} \frac{i\partial_\perp^\nu}{in_+\partial} \frac{\not{n}_+}{2} \xi_c \, , \\ \mathcal{L}_\xi^{(2b)} &= \frac{1}{2} \bar{\xi}_c (n_- x) \, n_+^\mu n_-^\nu \; gF^s_{\mu\nu} \frac{\not{n}_+}{2} \xi_c \, , \\ \mathcal{L}_\xi^{(2c)} &= \frac{1}{2} \bar{\xi}_c x_\perp^\mu x_{\perp\rho} n_-^\nu \, [D_s^\rho, gF^s_{\mu\nu}] \frac{\not{n}_+}{2} \xi_c \, . \end{aligned} \tag{7.3.10}$$

In addition to these terms, fermions feature a new spin-dependent term

$$\mathcal{L}_\xi^{(2s)} = \bar{\xi}_c g \Sigma_\perp^{\mu\nu} iF^s_{\mu\nu} \frac{1}{in_+\partial} \frac{\not{n}_+}{2} \xi_c \, . \tag{7.3.11}$$

Here, the spin operator $\Sigma_{\mu\nu}$ is decomposed in analogy to the orbital angular momentum (7.2.21) into its light-cone components as

$$\Sigma^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu] \mathrel{\widehat{=}} \underbrace{\frac{1}{4} \, [\gamma_\perp^\mu, \gamma_\perp^\nu]}_{\Sigma_\perp^{\mu\nu}} + \underbrace{\left(-\frac{1}{2} \frac{\not{n}_+}{2} \gamma_\perp^{[\mu} n_-^{\nu]} \right)}_{\Sigma_{\perp +}^{\mu\nu}} + \underbrace{\left(-\frac{1}{4} n_+^{[\mu} n_-^{\nu]} \right)}_{\Sigma_{+-}^{\mu\nu}} \, . \tag{7.3.12}$$

Note that after $\widehat{=}$, a number of terms were dropped which vanish due to the projection property (7.3.1) and the kinematics in the reference frame $p_\perp^\mu = 0$, similar to the terms in (7.2.21).

The derivation of the orbital part is completely analogous to the scalar case. One again finds that $\mathcal{L}^{(2a)} \mathrel{\widehat{=}} 0$ after integration by parts, and that $\mathcal{L}^{(2b)}$ is already in the correct form and contains the longitudinal part of the orbital angular momentum L_{+-} once it is evaluated in a matrix element, similar to (7.2.38). However, there is one small difference compared to the scalar case that is worth examining in more detail: the angular momentum term for the scalar field comes in the form (7.2.38)

$$\mathcal{L}^{(2)}_{\text{orbital}} \equiv \frac{1}{2} \chi_c^\dagger \, n_- x \, n_+^\mu n_-^\nu gF^s_{\mu\nu} \, in_+\partial \chi_c = igF^{s\,a}_{\mu\nu} \chi_c^\dagger t^a \, L_{+-}^{\mu\nu} \, \chi_c \, , \tag{7.3.13}$$

where one can directly read off the angular momentum in the Lagrangian. In the fermionic theory, however, (7.3.10) takes a different form

$$\mathcal{L}_\xi^{(2b)} = \frac{1}{2} \bar{\xi}_c (n_- x) \, n_+^\mu n_-^\nu \; gF^s_{\mu\nu} \frac{\not{n}_+}{2} \xi_c = \bar{\xi}_c \, igF^s_{\mu\nu} L_{+-}^{\mu\nu} \frac{1}{in_+\partial} \frac{\not{n}_+}{2} \xi_c \, , \tag{7.3.14}$$

and one notices that to identify L_{+-} directly, one needs to introduce an inverse derivative. This is due to the different propagators and resulting universal contractions for the scalar and spinor fields. Therefore, the angular momentum appearing in the LBK amplitude should not be thought of in the same way as the charge appearing in the first term. It is not a "charge" operator that appears in conjunction with the eikonal propagator. Rather, its appearance is a consequence of the coupling to the antisymmetric field-strength tensor and the form of the eikonal propagator, which combine into the orbital angular momentum. We will see below that the spin part also manifestly appears in this "non-local" form.[3]

With this being said, note that $\mathcal{L}^{(2b)}$ already completely reproduces the orbital angular momentum stemming from the time-ordered product of $\mathcal{L}^{(2)}$ with $\mathcal{J}^{(0)}$. Therefore, let us investigate what happens to $\mathcal{L}^{(2c)}$, which was required for some cancellations in the scalar case. It is given by

$$\mathcal{L}_\xi^{(2c)} = \frac{1}{2}\,\overline{\xi}_c x_\perp^\mu x_{\perp\rho} n_-^\nu \left[D_s^\rho, gF_{\mu\nu}^s\right]\frac{\not{n}_+}{2}\xi_c\,. \tag{7.3.15}$$

One can completely follow the manipulations of the scalar counterpart (7.2.28). First, one simplifies $x_\perp^\mu x_\perp^\rho$ by splitting these into a traceless and a trace part (7.2.29), dropping the traceless combination. Next, one employs the gluon equations of motion (7.2.31) to change $\partial_\perp$ into $n_-\partial$ acting on the field-strength tensor. Finally, one integrates by part and uses the collinear equations of motion to simplify the result. This culminates in

$$\mathcal{L}_\xi^{(2c)} \mathrel{\widehat{=}} -i\frac{1}{2}\overline{\xi}_c n_+^\mu n_-^\nu gF_{\mu\nu}^s \frac{1}{in_+\partial}\frac{\not{n}_+}{2}\xi_c\,, \tag{7.3.16}$$

and one can see that this term precisely corresponds to the *longitudinal* components of the spin operator $\Sigma_{+-} = -\frac{1}{4}n_+^{[\mu}n_-^{\nu]}$ contracted with the field-strength tensor.

The additional spin-dependent term $\mathcal{L}^{(2s)}$ (7.3.11) already contains the purely-transversal components $\Sigma_\perp$ of the spin term. Therefore, the subsubleading Lagrangian of a fermion can be recast in the form

$$\mathcal{L}_\xi^{(2)} \mathrel{\widehat{=}} \mathcal{L}_{\text{orbit}}^{(2)} + \mathcal{L}_{\text{spin}}^{(2)}\,, \tag{7.3.17}$$

where

$$\mathcal{L}_{\text{orbit}}^{(2)} = igF_{\mu\nu}^s\overline{\xi}_c L_{+-}^{\mu\nu}\frac{1}{in_+\partial}\frac{\not{n}_+}{2}\xi_c\,, \tag{7.3.18}$$

$$\mathcal{L}_{\text{spin}}^{(2)} = igF_{\mu\nu}^s\overline{\xi}_c\left(\Sigma_\perp^{\mu\nu} + \Sigma_{+-}^{\mu\nu}\right)\frac{1}{in_+\partial}\frac{\not{n}_+}{2}\xi_c\,. \tag{7.3.19}$$

Observe that the orbital part is completely analogous to the corresponding scalar term (up to the different form of the universal contraction), while the fermion now features explicit spin-dependent terms, where the spin operator $\Sigma^{\mu\nu}$ is manifest.

Similar to the scalar case, the mixed transverse-longitudinal components are missing. For the orbital part, these terms cannot stem from the time-ordered product with the leading-power $\hat{\mathcal{A}}^{(0)}$, since these terms require explicit $\partial_\perp$. The mixed transverse-longitudinal orbital part thus follows from the Lagrangian $\mathcal{L}_\xi^{(1)}$ (7.3.6) in combination with the orbital part of the $A1$ current (7.3.4) in the exact same fashion as in the scalar case (7.2.40), and the orbital angular momentum terms arise as

$$\begin{aligned}
&i\int d^4x\, T\left\{\hat{\mathcal{A}}_{\text{orbit}}^{(1)}, \mathcal{L}_\xi^{(1)}\right\} + i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\xi^{(2a)} + \mathcal{L}_\xi^{(2b)}\right\}\\
&= i\int d^4x\, T\left\{\hat{\mathcal{A}}_{\text{orbit}}, \overline{\xi}_c\frac{\not{n}_+}{2}L^{\mu\nu}igF_{\mu\nu}^s\frac{1}{in_+\partial}\xi_c\right\}.
\end{aligned} \tag{7.3.20}$$

[3] Later, we consider a vector field that has a similar universal contraction to the scalar (7.2.4). Here, we will see that both the orbital and spin part appear without non-locality like in the scalar case, due to the different propagator structure.

The same is actually true for the spin-part: the transverse-longitudinal components vanish when acting on ξ_c due to the projection property (7.3.1), and must act on the subleading spinor η_c to yield a non-vanishing result. To see this, first note that the leading-power term of the amplitude $\mathcal{A}^{(0)}$ comes with external spinor ξ_c. Therefore, it inherits the projection property (7.3.1), namely

$$\bar{\xi}_c \mathcal{A}^{(0)} = \bar{\xi}_c \frac{\slashed{n}_+ \slashed{n}_-}{4} \mathcal{A}^{(0)} \,. \tag{7.3.21}$$

Next, observe that the mixed transverse-longitudinal component $\Sigma^{\mu\nu}_{\perp+}$, as defined in (7.3.12), is projected out by this condition, since

$$\Sigma^{\mu\nu}_{\perp+} \frac{\slashed{n}_+ \slashed{n}_-}{4} \propto \frac{\slashed{n}_+}{2} \frac{\slashed{n}_+ \slashed{n}_-}{4} = 0 \,. \tag{7.3.22}$$

Therefore, the component $\Sigma_{\perp+}$ cannot appear in combination with $\mathcal{A}^{(0)}$. However, the subleading amplitude $\mathcal{A}^{(1)}$ contains one term that is proportional to the subleading spinor η_c, and since this component has the opposite projection property

$$\frac{\slashed{n}_+ \slashed{n}_-}{4} \eta_c = \eta_c \,, \tag{7.3.23}$$

the component $\Sigma_{\perp+}$ *can* appear in conjunction with $\mathcal{A}^{(1)}$, if this other projection is present. This term with the other projection property is precisely the term that stems from the spin-term in the $A1$ coefficient (7.3.4). We now show how the Lagrangian $\mathcal{L}^{(1)}_{\xi}$ (7.3.6) gives rise to the correct mixed-longitudinal terms in combination with the $A1$ matching coefficient. To this end, we evaluate its time-ordered product with the spin-term of (7.3.4), where we find

$$\begin{aligned} T\left\{ \mathcal{L}^{(1)}_{\xi}, -\bar{\xi}_c \frac{i\overleftarrow{\slashed{\partial}}_\perp}{in_+\overleftarrow{\partial}} \frac{\slashed{n}_+}{2} C^{A0} \right\} &\mathrel{\widehat{=}} T\left\{ \bar{\xi}_c \left(x^\mu_\perp n^\nu_- \, gF^s_{\mu\nu} \right) \frac{\slashed{n}_+}{2} \frac{i\slashed{\partial}_\perp}{in_+\partial} \xi_c \,, \bar{\xi}_c \frac{\slashed{n}_+}{2} C^{A0} \right\} \\ &\mathrel{\widehat{=}} T\left\{ giF^s_{\mu\nu} \bar{\xi}_c \, \Sigma^{\mu\nu}_{\perp+} \frac{1}{in_+\partial} \xi_c \,, \bar{\xi}_c \frac{\slashed{n}_+}{2} C^{A0} \right\} \,. \end{aligned} \tag{7.3.24}$$

Note again that $\sigma_{\perp+}$ does not vanish due to the additional $\slashed{n}_+$ in the contraction. This term is the component of the full propagator that is proportional to the subleading spinor η_c, and the mixed transverse-longitudinal component is not projected out. In summary, $\mathcal{L}^{(1)}_{\xi}$ with the spin-term of the $A1$-current (7.3.4) and $\mathcal{L}^{(2)}_{\text{spin}}$ (7.3.19) with the leading-power current then reproduce the full spin-term of the LBK amplitude as

$$\begin{aligned} & i \int d^4x \, T\left\{ \hat{\mathcal{A}}^{(1)}_{\text{spin}}, \mathcal{L}^{(1)}_{\xi} \right\} + i \int d^4x \, T\left\{ \hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(2)}_{\text{spin}} \right\} \\ &\mathrel{\widehat{=}} i \int d^4x \bar{\xi}_c(x) \, gF^s_{\mu\nu} \left(\Sigma^{\mu\nu}_{\perp+} \frac{1}{in_+\partial} \xi_c \, \bar{\xi}_c \frac{\slashed{n}_+}{2} + \left(\Sigma^{\mu\nu}_{+-} + \Sigma^{\mu\nu}_{\perp} \right) \frac{1}{in_+\partial} \frac{\slashed{n}_+}{2} \xi_c \, \bar{\xi}_c \right) C^{A0}(n_+p) \\ &= i \int d^4x \bar{\xi}_c(x) \, giF^s_{\mu\nu} \Sigma^{\mu\nu} \left(\frac{1}{in_+\partial} \xi_c \, \bar{\xi}_c \frac{\slashed{n}_+}{2} + \frac{1}{in_+\partial} \frac{\slashed{n}_+}{2} \xi_c \, \bar{\xi}_c \right) \mathcal{A} \,. \end{aligned} \tag{7.3.25}$$

To rewrite this result, we identified the matching coefficient C^{A0} with the non-radiative amplitude 7.1.4, using the kinematics to drop the subleading contributions in terms that do not contain derivatives acting on the amplitude. One can write the full $\Sigma^{\mu\nu}$ since the missing components are projected out using (7.3.1) and (7.3.22). The term inside the bracket actually corresponds to the full-spinor propagator, meaning that it contains components proportional to both the leading ξ_c and subleading η_c spinor, due to the different position of $\slashed{n}_+$ in both terms. After one evaluates the matrix element, this is simply the standard eikonal propagator without any projection operator, as one finds in the LBK theorem (1.1.9).

7.4 Extension to Vectorbosons

This section serves as a short summary of the derivation of the respective result for spin-1 fields. Here, aspects from both previous derivations, scalar as well as fermions, are relevant. Namely, the universal contraction and generic form of the interactions look similar to the scalar field, while the spin term arises in the same fashion as for the fermionic theory. This gives an indication that the soft theorem can be derived for a generic spin-s field in a similar style as for fermion and vector fields. In the following, we now consider a massive complex vector field V_c^μ, which forms some representation of $SU(N)$, i.e. it couples to the gluon. This theory should only be viewed as a convenient toy model, which is employed due to the strong formal similarities to the scalar field considered before in Section 7.2, and should not be thought of as a fundamental, UV complete theory. Most importantly, this vector field should come with its own gauge symmetry to form a consistent theory. However, this gauge symmetry is not relevant for the soft gluon emission, and it serves no purpose to construct the full consistent theory here. Therefore, we assume that the theory is either gauge-fixed in some physical gauge, or that a construction similar to the Wilson lines employed in the gauge-invariant building blocks (3.5.4) is used, and we do not discuss the gauge symmetry further. One example of such a physical gauge is light-cone gauge $n_+V_c = 0$. Such a choice is convenient since here, one manifestly finds that the large component $n_+V_c \sim 1$ is not relevant for the operator basis, and the transverse components $V_\perp$ scale as $V_{c\perp} \sim \lambda$ like the matter fields ϕ_c and ξ_c from before. In addition, the missing component $n_-V_c \sim \lambda^2$ can be integrated out in the same fashion as the subleading spinor η_c before, using the equation of motion

$$n_-\mathcal{V}_c = -\frac{2}{in_+\partial} i\partial_{\perp\alpha}\mathcal{V}^\alpha_{c\perp} + \mathcal{O}(\mathcal{V}_c^2)\,. \tag{7.4.1}$$

Relevant for the following discussion is only the transformation of the vector field under the action of the gauge group of the gluon. Therefore, in the following, we assume that the operator basis employs the gauge-invariant building block $\mathcal{V}_c$. The result is completely equivalent to any other physical gauge-fixing, and working with the building block means that one does not need to fix a gauge in the Lagrangian. The vector field and the building block are then related to linear order as

$$\mathcal{V}_{c\mu} = V_{c\mu} - \partial_\mu \frac{n_+V_c}{n_+\partial} + \dots\,. \tag{7.4.2}$$

7.4.1 Basics and Non-radiative Matching

For the non-radiative matching, we can directly transfer the results of the previous sections. The leading-power matching (7.1.4) corresponds to

$$\begin{aligned}
\mathcal{A}^{(0)} &= \varepsilon^*_{\alpha_1}(p_1)\dots\varepsilon^*_{\alpha_N}(p_N)\mathcal{A}^{(0)\alpha_1\dots\alpha_N}\\
&= \langle p_1,\dots,p_N|\hat{\mathcal{A}}^{(0)}|0\rangle\\
&= \tilde{\varepsilon}^*_{c_1\alpha_{1\perp}}(p_1)\dots\tilde{\varepsilon}^*_{c_N\alpha_{N\perp}}(p_N)\int [dt]_N\, \eta_\perp^{\alpha_1\beta_1}\dots\eta_\perp^{\alpha_N\beta_N}\left(\widetilde{C}^{A0}\right)_{\beta_1\dots\beta_N} e^{i\sum_i n_{i+}p_i\, t_i}\\
&\equiv \tilde{\varepsilon}^*_{c_1\alpha_{1\perp}}(p_1)\dots\tilde{\varepsilon}^*_{c_N\alpha_{N\perp}}(p_N)\eta_\perp^{\alpha_1\beta_1}\dots\eta_\perp^{\alpha_N\beta_N}\left(C^{A0}\right)_{\beta_1\dots\beta_N}\,.
\end{aligned} \tag{7.4.3}$$

This situation looks very different to the previous conditions since only the $\varepsilon_\perp$ appears. It is actually surprisingly similar to the fermionic matching condition, but now the projections are more explicit. First note that the full-theory amplitude $\mathcal{A}^{(0)}$ carries the full polarisation tensor ε_α, which has a full Lorentz index $\alpha = 0,\dots,3$. In the operator basis, however, one employs the transverse components $\mathcal{V}_{c\perp}$, which are restricted to transverse components only. The corresponding polarisation tensor is denoted by $\tilde{\varepsilon}_{c_i\alpha_{i\perp}}$, and is determined from (7.4.2) as

$$\tilde{\varepsilon}_{c_i\mu_\perp}(k) = \varepsilon_{\mu_\perp}(k) - k_{\mu_\perp}\frac{n_{i+}\varepsilon(k)}{n_{i+}k}\,. \tag{7.4.4}$$

This should be viewed in spirit as equivalent to the leading-power spinor polarisation, which differs from the full-theory one (4.4.31) by the projection property (7.3.1). The polarisation (7.4.4) is the full-theory polarisation projected onto the transverse components. In addition, one absorbs the $n_+\varepsilon_c$ polarisations in the same object. This does not affect the amplitude, since it is in practice equivalent to a gauge transformation.

The stripped full-theory amplitude $\mathcal{A}^{\alpha_1\ldots\alpha_n}$ itself is now also a Lorentz tensor, which has one index for each external vectorial leg. Since the matching coefficients correspond to the (stripped) amplitude, they must receive the Lorentz indices as well, as we indicate by the brackets. At leading order, all indices must be transverse, but at subleading order, the contributions of n_-V will contribute, as one can check explicitly in (7.4.7) below. Therefore, it is a useful choice to *not* restrict the indices of the coefficients like C^{A0} to be purely transverse. Instead, we employ the convention that C^{A0} carries a full Lorentz index, and the contraction with the effective polarisation tensor $\tilde{\varepsilon}_\perp$ leads to a restriction onto transverse components. In summary, the leading-power matching again states that the (stripped) full-theory amplitude simply corresponds to the leading-power matching coefficient

$$\left(C^{A0}\right)_{\beta_1\ldots\beta_N} = \mathcal{A}^{(0)}_{\beta_1\ldots\beta_N}\,. \tag{7.4.5}$$

For the subleading matching, one must be able to obtain the contributions due to $n_-\mathcal{V}_c$, in the same way as for the subleading spinor in the fermionic case. Using the equations of motion (7.4.1), one can relate the polarisation vectors as

$$n_{i-}\tilde{\varepsilon}_{c_i}(k) = -\frac{2}{n_{i+}k}\,k_\perp^\alpha\tilde{\varepsilon}_{c_i\alpha\perp}\,, \tag{7.4.6}$$

which is the analogue of the spinor relation (7.3.3). This condition then relates the subleading matching coefficient to the leading-power one in the same way as in the fermionic case (7.3.4). Namely, one finds

$$\begin{aligned}\left(C_i^{A1\,\mu}\right)_{\beta_1\ldots\beta_i\ldots\beta_N} &= \left[-\frac{\eta^\mu_{\perp\beta_i}}{n_{i+}p_i}n_+^{\rho_i} - \sum_{j\neq i}\eta^{\rho_i}_{\perp\beta_i}\frac{2n^\mu_{j-}}{n_{i-}\cdot n_{j-}}\frac{\partial}{\partial n_{i+}p_i}\right]\left(C^{A0}\right)_{\beta_1\ldots\rho_i\ldots\beta_N}\\ &\equiv \left(C^{A1\mu}_{i,\,\text{spin}}\right)_{\beta_1\ldots\beta_i\ldots\beta_N} + \left(C^{A1\mu}_{i,\,\text{orbit}}\right)_{\beta_1\ldots\beta_i\ldots\beta_N}\,,\end{aligned} \tag{7.4.7}$$

where one can again identify the orbital part, which is already present in the scalar coefficient (2.7.16), as well as a spin-part, which is due to the subleading polarisation tensor. Note that the Lorentz structure should be read as $\left(C^{A1\mu}\right)_{\ldots\alpha_i\ldots} i\partial_{\perp\mu}\mathcal{V}^{\alpha_i\perp}_{c_i}$, where the first μ is the one that is contracted with the $\partial_\perp$ in the current. Like for the fermion (7.3.5), we split this subleading current into two terms $\hat{\mathcal{A}}^{(1)}_{\text{orbit}}$ $\hat{\mathcal{A}}^{(1)}_{\text{spin}}$.

Since the vector field has a quadratic kinetic term, like the scalar field, the universal contraction is similar to the scalar 7.2.4 and again contains the large derivative $n_+\partial$. We define it as

$$\int d^4x e^{i\frac{1}{2}n_-k\,n_+x}\langle p|\,V^\dagger_{c\alpha}(0),\;V^\dagger_{c\nu}(x)\;n_+\partial V_{c\beta}(x)\;|0\rangle = \varepsilon^*_\nu(p)\frac{-i\eta_{\alpha\beta}}{n_-k}\,, \tag{7.4.8}$$

where we assumed Feynman gauge for the V_c propagator for simplicity.

We now drastically simplify the notation by removing all Lorentz indices that will be contracted with polarisation vectors. Only the ones relevant for the current constructions, e.g. contractions of $C^{A1\mu}\partial_{\perp\mu}$ will be kept explicit, while we write $\left[C^{A0}\right]^{\alpha_1\ldots\alpha_N} \equiv C^{A0}$ and understand that this object is a rank N Lorentz tensor.

7.4.2 Soft Theorem

The derivation of the soft theorem is now completely analogous to the scalar and fermionic case. In order to shorten the Lagrangian expressions, it is convenient to introduce the (linear) Noether current[4] as

$$j_\mu^a \equiv \left[i\partial_\mu V_{c\alpha}^\dagger\right] t^a V_c^\alpha - V_{c\alpha}^\dagger t^a i\partial_\mu V_c^\alpha + 2V_{c\alpha}^\dagger t^a i\partial^\alpha V_{c\mu} - 2\left[i\partial^\alpha V_{c\mu}^\dagger\right] t^a V_{c\alpha}\,. \tag{7.4.9}$$

Comparing this to the scalar current (7.2.17), note that there are two new terms, namely the last two, which will contribute to the spin term. The first two terms give rise to the orbital angular momentum just as in the scalar case.

To construct the Lagrangian, one can follow the procedure outlined in Section 3.4 and one obtains [48]

$$\mathcal{L}_V = \mathcal{L}_V^{(0)} + \mathcal{L}_V^{(1)} + \mathcal{L}_V^{(2)}\,, \tag{7.4.10}$$

where the individual terms are given by

$$\mathcal{L}_V^{(0)} = -\frac{1}{2}\operatorname{tr}\left((\partial_\mu V_{c\nu} - \partial_\nu V_{c\mu})^\dagger(\partial^\mu V_c^\nu - \partial^\nu V_c^\mu)\right) + \frac{1}{2}gn_- A_s^a n_+ j^a + \mathcal{O}(g^2)\,, \tag{7.4.11}$$

$$\mathcal{L}_V^{(1)} = \frac{1}{2}x_\perp^\mu\, n_+ j_a\, n_-^\nu gF_{\mu\nu}^{s\;a} \tag{7.4.12}$$

$$\mathcal{L}_V^{(2)} \mathrel{\widehat{=}} \mathcal{L}_V^{(2a)} + \mathcal{L}_V^{(2b)} + \mathcal{L}_V^{(2c)}\,. \tag{7.4.13}$$

Here, we introduced the same Lagrangian decomposition as in the scalar and fermionic cases (7.2.12), (7.2.13) and (7.3.8), (7.3.9), which read

$$\mathcal{L}_V^{(2a)} = \frac{1}{2}x_\perp^\nu j_a^{\mu_\perp} gF_{\nu\mu}^{s\;a}\,, \tag{7.4.14}$$

$$\mathcal{L}_V^{(2b)} = \frac{1}{4}n_- x\, n_+^\mu\, n_+ j_a\, n_-^\nu gF_{\mu\nu}^{s\;a}\,, \tag{7.4.15}$$

$$\mathcal{L}_V^{(2c)} = \frac{1}{4}x_\perp^\mu x_{\perp\rho}\, n_+ j_a\, n_-^\nu \operatorname{tr}\left(\left[D_s^\rho, gF_{\mu\nu}^s\right] t^a\right)\,. \tag{7.4.16}$$

In the fermionic theory, there was an explicit spin-dependent term that contains $\Sigma_\perp^{\mu\nu}$ in the Lagrangian, which seems to be absent here. However, note that in both the scalar and fermionic theory, we found $\mathcal{L}^{(2a)} \mathrel{\widehat{=}} 0$, whereas in the vectorial theory this contribution is non-zero. Instead, the spin-operator $\Sigma^{\mu\nu}$ hides in this term. For spin-1 representations, it is defined as

$$(\Sigma^{\mu\nu})^{\alpha\beta} = \left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}\right)\,. \tag{7.4.17}$$

Therefore, we rewrite the term $\mathcal{L}_V^{(2a)}$ as

$$\mathcal{L}_V^{(2a)} \mathrel{\widehat{=}} -igF_{\mu\nu}^{s\;a} V_{c\alpha}^\dagger t^a V_{c\beta}\left(\eta_\perp^{\alpha\mu}\eta_\perp^{\beta\nu} - \eta_\perp^{\alpha\nu}\eta_\perp^{\beta\mu}\right) = -igF_{\mu\nu}^{s\;a} V_{c\alpha}^\dagger t^a V_{c\beta}(\Sigma_\perp^{\mu\nu})^{\alpha\beta}\,. \tag{7.4.18}$$

Here, we integrated by parts and used the equations of motion as well as the transversality condition $\partial_\mu V^\mu \mathrel{\widehat{=}} 0$, and the spin-operator Σ for the vectorial representation is decomposed in the light-cone components as

$$(\Sigma^{\mu\nu})^{\alpha\beta} \mathrel{\widehat{=}} \underbrace{\eta_\perp^{\alpha\mu}\eta_\perp^{\beta\nu} - \eta_\perp^{\alpha\nu}\eta_\perp^{\beta\mu}}_{(\Sigma_\perp^{\mu\nu})^{\alpha\beta}} + \underbrace{\frac{1}{2}\eta_\perp^{\alpha[\mu}n_-^{\nu]}n_+^\beta - \eta_\perp^{\beta[\mu}n_-^{\nu]}n_+^\alpha}_{(\Sigma_{\perp+}^{\mu\nu})^{\alpha\beta}} + \underbrace{\frac{n_+^{[\mu}n_-^{\nu]}}{4}(n_-^\alpha n_+^\beta - n_+^\alpha n_-^\beta)}_{(\Sigma_{+-}^{\mu\nu})^{\alpha\beta}}\,. \tag{7.4.19}$$

Note that this again only contains the transverse, mixed transverse-longitudinal and longitudinal components, since all other components will drop out using our kinematic assumptions and projection properties of the operator basis.

[4]It is enough to consider the linear current since we are interested in tree-level single soft emission processes.

Next, we consider the term $\mathcal{L}_V^{(2c)}$, where we follow the same computation as for the respective scalar (7.2.36) and fermionic case (7.3.16) to obtain

$$\mathcal{L}_V^{(2c)} \mathrel{\widehat{=}} \frac{1}{2} igF_{+-}^{s\,a} V_c^{\dagger\alpha} t^a V_{c\alpha} \,. \tag{7.4.20}$$

Like for the scalar case (7.3.16), this term cancels out a contribution in $\mathcal{L}_V^{(2b)}$, and one finds for the sum

$$\mathcal{L}_V^{(2b)} + \mathcal{L}_V^{(2c)} \mathrel{\widehat{=}} -\frac{1}{2} n_- x g F_{+-}^{s\,a} \left(V_{c\alpha}^\dagger t^a in_+\partial V_c^\alpha \right) - \frac{1}{2} igF_{+-}^{s\,a} \left(V_{c-}^\dagger t^a V_{c+} - V_{c+}^\dagger t^a V_{c-} \right) . \tag{7.4.21}$$

Here, the first term has the same form as the respective orbital piece in the scalar derivation, and it precisely yields the longitudinal component L_{+-} of the orbital angular momentum. The second term can be further manipulated as

$$\begin{aligned} -\frac{1}{2} igF_{+-}^{s\,a} \left(n_- V_c^\dagger t^a n_+ V_c - n_+ V_c^\dagger t^a n_- V_c \right) &= -\frac{1}{4} n_+^{[\mu} n_-^{\nu]} \left(n_-^\alpha n_+^\beta - n_+^\alpha n_-^\beta \right) V_{c\alpha}^\dagger t^a V_{c\beta} igF_{\mu\nu}^{s\,a} \\ &= -(\Sigma_{+-}^{\mu\nu})^{\alpha\beta} igF_{\mu\nu}^{s\,a} V_{c\alpha}^\dagger t^a V_{c\beta} \,, \end{aligned} \tag{7.4.22}$$

and contains the longitudinal part of the spin term. In summary, the subsubleading Lagrangian takes the form

$$\mathcal{L}_V^{(2)} \mathrel{\widehat{=}} -\frac{1}{2} n_- x n_+^\mu n_-^\nu g F_{\mu\nu}^{s\,a} \left(V_{c\alpha}^\dagger t^a in_+\partial V_c^\alpha \right) - igF_{\mu\nu}^{s\,a} V_{c\alpha}^\dagger t^a V_{c\beta} \left((\Sigma_\perp^{\mu\nu})^{\alpha\beta} + (\Sigma_{+-}^{\mu\nu})^{\alpha\beta} \right) . \tag{7.4.23}$$

The first term corresponds to the longitudinal part of the orbital angular momentum like in the scalar case (7.2.38), while the second term explicitly contains the longitudinal and transverse parts of the spin angular momentum, as in the fermionic case (7.3.18).

As in the previous discussions, the transverse-longitudinal terms are missing, since they must come with the $A1$ operator due to their kinematic and projection properties. To this end, manipulate the subleading Lagrangian $\mathcal{L}_V^{(1)}$ to yield

$$\begin{aligned} \mathcal{L}_V^{(1)} &\mathrel{\widehat{=}} -x_\perp^\mu n_-^\nu g F_{\mu\nu}^{s\,a} V_{c\alpha}^\dagger t^a in_+\partial V_c^\alpha - ign_-^\nu F_{\mu_\perp\nu}^{s\,a} \left(V_c^{\dagger\mu_\perp} t^a n_+ V_c - n_+ V_c^\dagger t^a V_c^{\mu_\perp} \right) \\ &\mathrel{\widehat{=}} -x_\perp^\mu n_-^\nu g F_{\mu\nu}^{s\,a} V_{c\alpha}^\dagger t^a in_+\partial V_c^\alpha \,, \end{aligned} \tag{7.4.24}$$

which now yields a non-vanishing time-ordered product with the $A1$ current, which contains an explicit $\partial_\perp$ and n_+^μ. Here, we dropped the second term in the first line of (7.4.24). It cannot contribute in the time-ordered product $T\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_V^{(1)}\}$ with the leading-power current, since this comes with polarisation tensor $\tilde{\varepsilon}_\perp$ and the contraction with n_+^μ vanishes. Furthermore, it does not contain an explicit $x_\perp$, so its time-ordered product with $\hat{\mathcal{A}}^{(1)}$ is proportional to $p_\perp \mathrel{\widehat{=}} 0$. Therefore, one can neglect the second term.

The first term, on the other hand, has a striking resemblance to the scalar $\mathcal{L}_\chi^{(1)}$ (7.2.12) and fermionic $\mathcal{L}_\xi^{(1)}$ (7.3.8) results, and indeed the time-ordered product with the orbital piece $\hat{\mathcal{A}}_{\text{orbit}}^{(1)}$ precisely reproduces the mixed transverse-longitudinal component of the orbital angular momentum. The missing spin-term then arises similar to the fermionic result (7.3.24) from the spin-term $\hat{\mathcal{A}}_{\text{spin}}^{(1)}$ in the $A1$ current.

In summary, one can recast the vector Lagrangian in the same form as the scalar and fermionic one, as

$$\mathcal{L}_V \mathrel{\widehat{=}} \mathcal{L}_{\text{kinetic}}^{(0)} + \mathcal{L}_{\text{eikonal}}^{(0)} + \mathcal{L}_V^{(1)} + \mathcal{L}_{\text{orbit}}^{(2)} + \mathcal{L}_{\text{spin}}^{(2)} \,, \tag{7.4.25}$$

where the individual terms correspond to

$$\mathcal{L}_{\text{eikonal}}^{(0)} = -gn_- A^a n_+ V_{c\alpha}^\dagger t^a in_+\partial V_c^\alpha \,, \tag{7.4.26}$$

$$\mathcal{L}_V^{(1)} = -igF^{s\,a}_{\mu\nu} V^\dagger_{c\alpha} t^a L^{\mu\nu}_{\perp+} V^\alpha_c \tag{7.4.27}$$

$$\mathcal{L}_{\text{orbit}}^{(2)} = -igF^{s\,a}_{\mu\nu} V^\dagger_{c\alpha} t^a L^{\mu\nu}_{+-} V^\alpha_c \tag{7.4.28}$$

$$\mathcal{L}_{\text{spin}}^{(2)} = -igF^{s\,a}_{\mu\nu} V^\dagger_{c\alpha} t^a V_{c\beta} \left((\Sigma^{\mu\nu}_\perp)^{\alpha\beta} + (\Sigma^{\mu\nu}_{+-})^{\alpha\beta} \right) . \tag{7.4.29}$$

This result has a striking formal resemblance to the fermionic result (7.3.18), (7.3.19).

Combining both contributions from the $A0$ and $A1$ currents, the subleading soft-emission term is given by

$$\begin{aligned} &i\int d^4x\, T\left\{ \hat{\mathcal{A}}^{(1)}_{\text{orbit}} + \hat{\mathcal{A}}^{(1)}_{\text{spin}}, \mathcal{L}^{(1)}_V \right\} + i\int d^4x\, T\left\{ \hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(2)}_{\text{orbit}} + \mathcal{L}^{(2)}_{\text{spin}} \right\} \\ &= i\int d^4x\, T\left\{ \hat{\mathcal{A}}, -V^\dagger_{c\alpha}\left(\eta^{\alpha\beta} L^{\mu\nu} + (\Sigma^{\mu\nu})^{\alpha\beta} \right) igF^{s\,a}_{\mu\nu}\, V_\beta \right\} , \end{aligned} \tag{7.4.30}$$

and we recover the same structure as for the scalar (7.2.42) and fermionic (7.3.25) case. This seems to suggest that this soft-theorem derivation at the operatorial level can be generalised in a straightforward fashion to matter fields of arbitrary spin, simply by writing down the minimal-coupling terms and performing the "linear SCET" construction, as we did for scalar, fermion and vector fields. However, we will not pursue this idea further in the following.

7.5 Soft Graviton Theorem

As we have verified in gauge theory, the soft-emission interaction vertex in the Lagrangian can be cast into a form that directly corresponds to the LBK amplitude, including the angular momentum operator. We now wish to perform the analogous computation for the gravitational Lagrangian, up to the next-to-next-to-soft order, i.e. the third term of the soft theorem.

The existence of three universal terms in gravity follows already from the soft gauge symmetry, before performing any computations. Similar to gauge theory, the covariant derivative can be eliminated by equations of motion [47, 50], and the first valid soft-graviton building block is the analogue of the field-strength, the Riemann tensor $R^\mu{}_{\nu\alpha\beta} \sim \lambda^6$, and thus there are *three* terms that stem from the Lagrangian interaction. We will give an explicit derivation of all three terms, including the next-to-next-to-soft one, and investigate in detail what the effective theory tells us about the nature of these terms.

For a better comparison to the previous sections, we consider a *complex* scalar field coupled to gravity. This merely changes a few symmetry factors and normalisations in the Lagrangian, but allows us to directly transfer the notation of the previous section, including the universal contraction (7.2.4) as well as the non-radiative matching to this section.

Schematically, up to second order in the multipole expansion, the single soft-emission terms in the Lagrangian take the form

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\left[n_+\partial\chi_c\right]^\dagger n_-\partial\chi_c + \frac{1}{2}\left[n_-\partial\chi_c\right]^\dagger n_+\partial\chi_c + \left[\partial_{\mu_\perp}\chi_c^\dagger\right]\partial^{\mu_\perp}\chi_c \\ &- \frac{\kappa}{4}s_{-\mu}T^{\mu+} - \frac{\kappa}{4}\left[\partial_{[\mu}s_{\nu]-}\right](x-x_-)^\mu T^{\nu+} - \frac{1}{8}x^\alpha_\perp x^\beta_\perp R^s_{\alpha-\beta-}T_{++} + \mathcal{O}(x^3) , \end{aligned} \tag{7.5.1}$$

where $T_{\mu\nu}$ denotes the standard energy-momentum tensor

$$T_{\mu\nu} = \left[\partial_\mu\chi_c\right]^\dagger \partial_\nu\chi_c + \left[\partial_\nu\chi_c\right]^\dagger \partial_\mu\chi_c - \eta_{\mu\nu}\left[\partial_\alpha\chi_c\right]^\dagger \partial^\alpha\chi_c\,. \tag{7.5.2}$$

This version of the Lagrangian is not homogeneous in λ, but it already resembles the form of the soft theorem. As first observation, note that $T^{\mu+}$ contains a $n_+\partial$ that forms the universal contraction (7.2.4). Therefore, the first *two* interaction terms in the second line of (7.5.1) correspond to *eikonal* terms in the soft theorem. The first interaction term, proportional to $s_{\mu-}\partial^\mu$, is the one that corresponds to the leading term in the soft theorem

$$\varepsilon_{\mu-}\, p^\mu \,\frac{n_+p}{p\cdot k} , \tag{7.5.3}$$

and we note that the first p^μ corresponds to the coupling ∂^μ, while the second n_+p is due to the universal contraction inside $T^{\mu+}$. The polarisation tensor $\varepsilon_{\mu\nu}$ is the polarisation of $s_{\mu-}(x_-)$.

The second interaction term in (7.5.1) has a similar structure, but the "gauge field" is no longer $s_{\mu-}$ but rather $\partial_{[\mu}s_{\nu]-}$, and the coupling is no longer to the momentum via $T^{\mu+}$, but to the angular momentum density

$$\mathcal{J}^{\alpha\beta\mu} = (x-x_-)^\alpha T^{\beta\mu} - (x-x_-)^\beta T^{\alpha\mu} \,. \tag{7.5.4}$$

This term corresponds to the subleading term of the soft theorem

$$k_\rho \varepsilon_{\mu-} J^{\rho\mu} \frac{n_+p}{p\cdot k} \,. \tag{7.5.5}$$

Here, note that the first combination $k_\rho\varepsilon_{\mu-}$ can be viewed as the polarisation corresponding to the gauge field $\partial_{[\mu}s_{\nu]-}$. The angular momentum $J^{\rho\mu}$ appears as a generator, like the momentum p^μ or the gauge-charge t^a before, and the term comes with an eikonal propagator. Viewed from this angle, one immediately realises that this term is structurally very different from the subleading term in gauge theory. This was already anticipated in the classical derivation in (1.1.19), since this term is not manifestly gauge-invariant. From the EFT, we see that this term is simply the eikonal emission term for the second, independent gauge field $\partial_{[\mu}s_{\nu]-}$. Similar to gauge theory, we will find that in the special reference frame, this term counts as $\mathcal{O}(\lambda^2)$.

Let us stress that these first two terms should be viewed as the analogue of the eikonal term in gauge theory, since these terms can be traced back to the two independent gauge fields in the gravitational covariant derivative.

In the subsubleading Lagrangian, there are a number of Riemann tensor terms, similar to the field-strength tensor terms in the subleading QCD Lagrangian. These terms yield the next-to-next-to soft term

$$\frac{1}{2}\varepsilon_{\mu\nu}k_\rho k_\sigma J^{\rho\mu}\frac{J^{\sigma\nu}}{p\cdot k} \,, \tag{7.5.6}$$

which counts as $\mathcal{O}(\lambda^4)$. In this term, we will find that one factor $J^{\mu\nu}$ corresponds to the charge, while the second angular momentum arises in the same way as in the subleading term in QCD: In the following, we give an explicit derivation of these observations. The derivation proceeds in the same way as in the gauge-theory section (7.2). We again consider the terms order-by-order in the power-counting and manipulate the Lagrangian to yield the desired form.

However, since we are now interested in the next-to-next-to soft order, we require the non-radiative matching in one order higher. The relevant $A2$-operator contains two transverse derivatives[5]

$$J^{A2\mu\nu}_{\partial^2\chi_i^\dagger} = i\partial^\mu_{i\perp} i\partial^\nu_{i\perp}\chi_i^\dagger(t_i n_{i+}) \,, \tag{7.5.7}$$

and simply corresponds to the second term of the Taylor expansion of the non-radiative amplitude

$$\mathcal{A}^{(2)} = p^\mu_{i\perp}p^\nu_{i\perp}\left(\frac{\partial^2}{\partial p^\mu_{i\perp}\partial p^\nu_{i\perp}}\mathcal{A}\right)\Bigg|_{p_i^\mu = n_{i+}p_i\, n^\mu_{i-}/2} \,. \tag{7.5.8}$$

After identifying all matching coefficients with the respective terms in the Taylor series, one can write the expansion of the non-radiative amplitude as

$$\mathcal{A} = C^{A0}(n_+p) + p^\mu_{i\perp}C^{A1}_{i,\mu}(n_+p) + p^\mu_{i\perp}p^\nu_{i\perp}C^{A2}_{i,\mu\nu}(n_+p) + \mathcal{O}(\lambda^3) \,. \tag{7.5.9}$$

Furthermore, we require the fully expanded form of the Lagrangian, where each term is homogeneous in λ. In this form, the SCET gravity Lagrangian corresponds to a power-series [47]

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} \,, \tag{7.5.10}$$

[5] Note that $n_-\partial$ does not appear in the operator basis, since it can be expressed in terms of $\partial_\perp$ via $n_-\partial = -\frac{\partial_\perp^2}{n_+\partial}$.

where the single-soft emission terms are given by

$$\mathcal{L}^{(0)} = -\frac{\kappa}{8} s_{--} T_{++}\,, \tag{7.5.11}$$

$$\mathcal{L}^{(1)} = -\frac{\kappa}{4} s_{-\mu_\perp} T^{\mu_\perp}{}_{+} - \frac{\kappa}{8}\left[\partial_{[\mu} s_{\nu]-}\right] n_-^\nu x_\perp^\mu\, T_{++}\,, \tag{7.5.12}$$

$$\begin{aligned}\mathcal{L}^{(2)} &= -\frac{\kappa}{2}\left[\partial_{[\mu} s_{\nu]-}\right] x_\perp^\mu T^{\nu_\perp}{}_{+} - \frac{\kappa}{16}\left[\partial_{[\mu} s_{\nu]-}\right] n_+^\mu n_-^\nu n_- x\, T_{++}\\ &\quad - \frac{1}{8} x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-} T_{++} - \frac{\kappa}{8} s_{+-}\, T_{+-}\,,\end{aligned} \tag{7.5.13}$$

$$\begin{aligned}\mathcal{L}^{(3)} &= -\frac{1}{8} x_\perp^\alpha n_- x n_+^\beta R^s_{\alpha-\beta-} T_{++} - \frac{1}{24} x_\perp^\alpha x_\perp^\beta x_\perp^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] T_{++}\\ &\quad - \frac{1}{8}\left[\partial_{[\mu} s_{\nu]-}\right] n_+^\nu n_- x T^{\mu_\perp}{}_{+} - \frac{1}{8}\left[\partial_{[\mu} s_{\nu]-}\right] x_\perp^\mu n_+^\nu T_{+-} - \frac{1}{3} x_\perp^\alpha x_\perp^\beta R^s_{\alpha\mu_\perp\beta-} T^{\mu_\perp}{}_{+}\,,\end{aligned} \tag{7.5.14}$$

$$\begin{aligned}\mathcal{L}^{(4)} &= -\frac{1}{32}(n_- x)^2 R^s_{+-+-} T_{++} - \frac{1}{24} x_\perp^\alpha n_- x n_+^\beta x_\perp^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] T_{++}\\ &\quad - \frac{1}{48} x_\perp^\alpha x_\perp^\beta n_- x n_+^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] T_{++} - \frac{1}{96} x_\perp^\alpha x_\perp^\beta x_\perp^\rho x_\perp^\sigma \left[\partial_\rho \partial_\sigma R^s_{\alpha-\beta-}\right] T_{++}\\ &\quad - \frac{1}{6} x_\perp^\alpha n_- x n_+^\beta (R^s_{\alpha\mu_\perp\beta-} + R^s_{\beta\mu_\perp\alpha-}) T^{\mu_\perp}{}_{+}\\ &\quad - \frac{1}{16} x_\perp^\alpha x_\perp^\beta x_\perp^\nu \left[\partial_\nu R^s_{\alpha\mu_\perp\beta-}\right] T^{\mu_\perp}{}_{+} - \frac{1}{6} x_\perp^\alpha x_\perp^\beta R^s_{\alpha\mu_\perp\beta\nu_\perp} T^{\mu_\perp\nu_\perp}\\ &\quad + \frac{1}{12} x_\perp^\alpha x_\perp^\beta R^s_{\alpha+\beta-} T_{+-}\,.\end{aligned} \tag{7.5.15}$$

7.5.1 Leading-power Term

The leading-power term stems from $T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(0)}\right\}$, and can be read off directly from the Lagrangian after a single integration by parts to write the term proportional to the universal contraction (7.2.4). It reads

$$\mathcal{L}^{(0)}_{\text{eikonal}} = \frac{\kappa}{4} s_{--}\, \chi_c^\dagger n_+\partial\, n_+\partial \chi_c\,. \tag{7.5.16}$$

It is instructive to compare this leading-power Lagrangian to the gauge-theory result (7.2.11), where one sees that the colour generator t^a is now replaced by a derivative $n_+\partial$, while the second $n_+\partial$ is the one that corresponds to the universal contraction. Here we already see a manifestation of the colour-kinematics duality [37,65,66], as the leading-power Lagrangians can be obtained by replacing $gt^a \to \kappa n_+\partial$ in the Lagrangian. The operatorial statement then reads

$$\begin{aligned}\mathcal{A}^{(0)}_{\text{rad}} &\mathrel{\hat{=}} i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(0)}_{\text{eikonal}}\right\} && (7.5.17)\\ &= i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, -\frac{\kappa}{4}\chi_c^\dagger\, s_{--} in_+\partial\, in_+\partial\chi_c\right\}. && (7.5.18)\end{aligned}$$

This expression yields the leading eikonal term, where $n_+\partial s_{--}(x_-)$ corresponds to $\varepsilon_{--}n_+p$ in the soft theorem since the derivative only acts on the collinear field χ_c.

7.5.2 Next-to-soft Term

Next, we consider the subleading-power terms. First, note that there is no contribution at $\mathcal{O}(\lambda)$. To see this, observe that the subleading Lagrangian $\mathcal{L}_\chi^{(1)}$ takes the form

$$\mathcal{L}^{(1)} = -\frac{\kappa}{2} s_{-\mu_\perp} \left[\partial_\perp^\mu \chi_c\right]^\dagger n_+\partial\chi_c - \frac{\kappa}{4}\left[\partial_{[\mu} s_{\nu]-}\right] x_\perp^\mu n_-^\nu \left[n_+\partial\chi_c\right]^\dagger n_+\partial\chi_c\,, \tag{7.5.19}$$

after integration by parts. For the time-ordered product with the leading-power current, one then finds

$$T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(1)}\right\} \mathrel{\hat{=}} 0\,, \tag{7.5.20}$$

since it depends on external $p_\perp \mathrel{\widehat{=}} 0$.

The subleading term of the soft theorem at next-to-soft order reads

$$\frac{\kappa}{2} \frac{\varepsilon_{\mu\nu}(k)\, p^\mu k_\rho L^{\nu\rho}}{p \cdot k} \mathcal{A}\,, \tag{7.5.21}$$

and it must be reproduced by the two time-ordered products

$$\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}_\chi^{(2)}\right\}\,, \tag{7.5.22}$$

$$\int d^4x\, T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}_\chi^{(1)}\right\}\,. \tag{7.5.23}$$

Here, the angular momentum is the orbital piece which is given explicitly in (7.2.21).

Like in gauge theory (7.2.12), we decompose the Lagrangian into similar structures as

$$\mathcal{L}^{(2a)} = -\frac{\kappa}{4}\left[\partial_\mu s_{\nu-} - \partial_\nu s_{\mu-}\right] x_\perp^\mu \left(\left[\partial_\perp^\nu \chi_c\right]^\dagger n_+\partial\chi_c + \left[n_+\partial\chi_c\right]^\dagger \partial_\perp^\nu \chi_c\right)\,, \tag{7.5.24}$$

$$\mathcal{L}^{(2b)} = -\frac{\kappa}{8}\left[\partial_\mu s_{\nu-}\right] n_+^{[\mu} n_-^{\nu]} n_- x \left[n_+\partial\chi_c\right]^\dagger n_+\partial\chi_c\,, \tag{7.5.25}$$

$$\mathcal{L}^{(2c)} = -\frac{1}{4} x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-} \left[n_+\partial\chi_c\right]^\dagger n_+\partial\chi_c\,, \tag{7.5.26}$$

$$\mathcal{L}^{(2d)} = \frac{\kappa}{4} s_{+-} \left[\partial_\perp^\alpha \chi_c\right]^\dagger \partial_{\perp\alpha}\chi_c\,. \tag{7.5.27}$$

As before, $\mathcal{L}^{(2a)}$ vanishes on-shell. Observe that after integration by parts, one can set external $p_\perp = 0$ and finds

$$\mathcal{L}^{(2a)} \mathrel{\widehat{=}} -\frac{\kappa}{4}\left[\partial_{[\mu} s_{\nu]-}\right] \eta_\perp^{\mu\nu} \chi_c^\dagger n_+\partial\chi_c \mathrel{\widehat{=}} 0\,, \tag{7.5.28}$$

since the symmetric $\eta_\perp$ is contracted with antisymmetric $\partial_{[\mu} s_{\nu]-}$.

The next term $\mathcal{L}^{(2b)}$ is already in the right form and one can identify the longitudinal component of the orbital angular momentum

$$\mathcal{L}^{(2b)} \mathrel{\widehat{=}} \frac{\kappa}{8}\left[\partial_\mu s_{\nu-}\right] n_+^{[\mu} n_-^{\nu]} \chi_c^\dagger \left[n_+\partial n_- x n_+\partial\chi_c\right] = \frac{\kappa}{2}\left[\partial_\mu s_{\nu-}\right] \chi_c^\dagger \overleftarrow{L}_{+-}^{\mu\nu} n_+\partial\chi_c\,. \tag{7.5.29}$$

We will comment on the peculiar structure of this term below.

The next term, $\mathcal{L}^{(2c)}$ can be manipulated using the same steps as in gauge theory. First, we decompose the two $x_\perp$ using (7.2.29) and drop the traceless term. This yields

$$\mathcal{L}^{(2c)} \mathrel{\widehat{=}} -\frac{1}{8} x_\perp^2 \eta_\perp^{\alpha\beta} R^s{}_{\alpha_\perp-\beta_\perp-} \left[n_+\partial\chi_c^\dagger\right] n_+\partial\chi_c\,. \tag{7.5.30}$$

Note that while we write the Riemann tensor here, the relevant contribution is only the term *linear* in $s_{\mu\nu}$. Next, note that one can rewrite $\eta_\perp^{\alpha\beta} R^s{}_{\alpha_\perp-\beta_\perp-} = \eta^{\alpha\beta} R^s{}_{\alpha-\beta-}$ and then use the leading-power, sourceless equation of motion $R^s_{--} = 0$ to find that this term vanishes. In the amplitude, this is equivalent to using the transverse-traceless condition for the polarisation tensor $\varepsilon_{\mu\nu}$.

In summary, one finds for the subleading Lagrangians $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$

$$\mathcal{L}^{(1)} \mathrel{\widehat{=}} \frac{\kappa}{2}\left[\partial_\mu s_{\nu-}\right] \chi_c^\dagger \overleftarrow{L}_{+\perp}^{\mu\nu} n_+\partial\chi_c\,, \tag{7.5.31}$$

$$\mathcal{L}^{(2)}_{\text{orbital}} \mathrel{\widehat{=}} \frac{\kappa}{2}\left[\partial_\mu s_{\nu-}\right] \chi_c^\dagger \overleftarrow{L}_{+-}^{\mu\nu} n_+\partial\chi_c\,, \tag{7.5.32}$$

where the orbital angular momentum is given in (7.2.21), and we defined the left-acting version as

$$\overleftarrow{L}_{\mu\nu} = \overleftarrow{\partial}_{[\mu} x_{\nu]}\,, \tag{7.5.33}$$

with the convention $\chi\overleftarrow{\partial}_\mu = -\left[\partial_\mu\chi\right]$. Observe that $L^{\mu\nu}_{+-}$ appears in the combination $n_+\partial n_- x$, i.e. as an operator acting to the left on the *external* field. One can interpret this term as a minimal coupling of the gauge field $\partial_{[\mu}s_{\nu]-}$ to the angular momentum $L^{\mu\nu}_{+-}$, which appears with the universal contraction (7.2.4) to yield another eikonal term. It does *not* appear in the same combination as in gauge-theory (7.2.13), where the angular momentum contained a part of the eikonal contraction. Instead, the angular momentum really appears as an analogue of the colour generator t^a and mimics the structure of the eikonal term. This is another manifestation of the *two* independent gauge fields in SCET gravity – one encounters *two eikonal terms* in the soft theorem.

Using these results, the operatorial statement of the subleading term then corresponds to

$$\begin{aligned}\hat{\mathcal{A}}^{(2)}_{\rm rad} &\mathrel{\widehat{=}} i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}^{(1)}\right\} + i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(2)}_{\rm orbital}\right\}\\ &= \int d^4x\, T\left\{\hat{\mathcal{A}}, \frac{\kappa}{2}\chi_c^\dagger \overleftarrow{L}^{\mu\nu}\left[\partial_\mu s_{\nu-}\right] in_+\partial\chi_c\right\}.\end{aligned} \tag{7.5.34}$$

7.5.3 Sub-subleading / Next-to-next-to-soft Term

At the sub-subleading order, we first have to consider a possible contribution at $\mathcal{O}(\lambda^3)$, which does not exist in the soft theorem in the special reference frame where $p_\perp = 0$. Such a term would stem from either $T\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(3)}\}$, $T\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}^{(2)}\}$, $T\{\hat{\mathcal{A}}^{(2)}, \mathcal{L}^{(1)}\}$ or $T\{\hat{\mathcal{A}}^{(3)}, \mathcal{L}^{(0)}\}$. However, it is easy to see that all these contributions are proportional to $p_\perp = 0$, and thus vanish.

Therefore, we can proceed to the discussion of the $\mathcal{O}(\lambda^4)$ contribution, where the sub-subleading term reads

$$\frac{\kappa}{4}\frac{\varepsilon_{\mu\nu}(k)k_\rho k_\sigma L^{\rho\mu}L^{\sigma\nu}}{p\cdot k}\mathcal{A}\,. \tag{7.5.35}$$

First, there is a notational subtlety: In the final on-shell amplitude, the angular momentum operators are taken to act *only on the amplitude* and not on each other [32]. Therefore, it is useful to define a left- and right-acting angular momentum operator as

$$\overleftarrow{L}_{\mu\rho}\overrightarrow{L}_{\nu\sigma} \equiv \left(\overleftarrow{\partial}_{[\mu}x_{\rho]}\right)\left(x_{[\nu}\overrightarrow{\partial}_{\sigma]}\right). \tag{7.5.36}$$

This will turn out to be the Lagrangian equivalent of angular momenta that only act on the amplitude. Decomposing this operator into its light-cone components, one finds four non-vanishing terms

$$\begin{aligned}\overleftarrow{L}^{\mu\rho}\overrightarrow{L}^{\nu\sigma} &= \frac{1}{16}n_+^{[\mu}n_-^{\rho]}n_+^{[\nu}n_-^{\sigma]}\, n_+\overleftarrow{\partial}n_-x n_-x n_+\partial + \frac{1}{4}x_\perp^{[\mu}n_-^{\rho]}x_\perp^{[\nu}n_-^{\sigma]}n_+\overleftarrow{\partial}n_+\partial\\ &+\frac{1}{8}n_+^{[\nu}n_-^{\sigma]}\, x_\perp^{[\mu}n_-^{\rho]}n_+\overleftarrow{\partial}n_-x n_+\partial + \frac{1}{8}n_+^{[\mu}n_-^{\rho]}\, x_\perp^{[\nu}n_-^{\sigma]}n_+\overleftarrow{\partial}n_-x n_+\partial\,.\end{aligned} \tag{7.5.37}$$

These terms must appear in the Lagrangian, or in combinations of Lagrangian and subleading currents. Next, we want to relate the expression in the amplitude (7.5.35) to the Riemann tensor and the angular momentum operators. The linear part of the Riemann tensor reads

$$R^s_{\mu\alpha\nu\beta} = -\frac{\kappa}{2}\left(k_\nu k_\alpha\varepsilon_{\mu\beta} + k_\mu k_\beta\varepsilon_{\nu\alpha} - k_\alpha k_\beta\varepsilon_{\mu\nu} - k_\mu k_\nu\varepsilon_{\alpha\beta}\right) + \mathcal{O}(\varepsilon^2)\,, \tag{7.5.38}$$

and one can use this to rewrite the polarisation tensor contracted with the angular momenta as

$$\begin{aligned}\kappa\varepsilon^{\mu\nu}(k)k^\rho k^\sigma\overleftarrow{L}_{\rho\mu}\overrightarrow{L}_{\sigma\nu} &= -\frac{1}{8}R^s_{+-+-}n_+\partial(n_-x)^2 n_+\partial - \frac{1}{2}R^s_{\mu-+-}n_+\partial x_\perp^\mu n_-x n_+\partial\\ &- \frac{1}{2}R^s_{\mu-\nu-}n_+\partial x_\perp^\mu x_\perp^\nu n_+\partial\,.\end{aligned} \tag{7.5.39}$$

In this contraction, only three independent contributions remain, and we can distinguish them from their explicit x-dependence. This dependence allows us to determine from which time-ordered products the individual contributions arise:

- The first term contains $(n_-x)^2$, and contains no explicit $x_\perp$. Therefore, any time-ordered product that comes with $\partial_\perp$ will vanish due to $p_\perp = 0$. It must stem from $T\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(4)}\}$, and therefore this first term must be part of $\mathcal{L}^{(4)}$.
- The second term contains one $x_\perp$. Therefore, it must appear in combination with *one* $\partial_\perp$ in the current, otherwise, it yields a vanishing contribution. The second term can only contribute through $T\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}^{(3)}\}$, and thus it must be part of $\mathcal{L}^{(3)}$.
- The third term contains two factors of $x_\perp$, and therefore it must come in combination with precisely *two* transverse derivatives $\partial_\perp$ to yield a non-vanishing contribution. It must stem from $T\{\hat{\mathcal{A}}^{(2)}, \mathcal{L}^{(2)}\}$, and thus appear in $\mathcal{L}^{(2)}$.

In the subsequent explicit computation, we show how these terms arise in $\mathcal{L}^{(4)}$ to $\mathcal{L}^{(2)}$. To make the derivation more accessible, we already introduce the result, so the reader knows what to expect. We will find that the Lagrangians can be recast as

$$\mathcal{L}^{(2)}_{\text{orbital}} = \frac{1}{4} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\alpha\mu}_{\perp+} L^{\beta\nu}_{\perp+} \chi_c \,, \tag{7.5.40}$$

$$\mathcal{L}^{(3)}_{\text{orbital}} = \frac{1}{2} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\alpha\mu}_{\perp+} L^{\beta\nu}_{+-} \chi_c \tag{7.5.41}$$

$$\mathcal{L}^{(4)}_{\text{orbital}} = \frac{1}{4} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\mu\alpha}_{+-} L^{\nu\beta}_{+-} \chi_c \,, \tag{7.5.42}$$

and therefore that also the sub-subleading term can be cast into an operatorial statement

$$\begin{aligned}
\hat{\mathcal{A}}^{(4)} &\mathrel{\widehat{=}} i \int d^4x\, T\left\{\hat{\mathcal{A}}^{(2)}, \mathcal{L}^{(2)}_\xi\right\} + i \int d^4x\, T\left\{\hat{\mathcal{A}}^{(1)}, \mathcal{L}^{(3)}_\xi\right\} + i \int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(4)}_\xi\right\} \\
&= i \int d^4x\, T\left\{\hat{\mathcal{A}}, \frac{1}{4} \chi_c^\dagger \overleftarrow{L}^{\mu\nu} \overrightarrow{L}^{\alpha\beta} R^s_{\mu\alpha\nu\beta} \chi_c\right\} \\
&\mathrel{\widehat{=}} i \int d^4x\, T\left\{\hat{\mathcal{A}}, \frac{1}{4} \chi_c^\dagger L^{\mu\nu} L^{\alpha\beta} R^s_{\mu\alpha\nu\beta} \chi_c\right\} .
\end{aligned} \tag{7.5.43}$$

To get the last line, we made use of on-shell properties and equations of motion following [32] to bring the angular momentum operators in the standard form.

Contribution from $T\{\hat{\mathcal{A}}^{(0)}, \mathcal{L}^{(4)}\}$

We split the Lagrangian $\mathcal{L}^{(4)}$ (7.5.15) into 5 different pieces, labelled by $\mathcal{L}^{(4a)}$ through $\mathcal{L}^{(4f)}$, and discuss these pieces separately.

The first term reads

$$\mathcal{L}^{(4a)} = -\frac{1}{16} (n_-x)^2 R^s_{+-+-} \left[n_+\partial\chi_c\right]^\dagger n_+\partial\chi_c \,, \tag{7.5.44}$$

and this term is already the correct contribution to the longitudinal-longitudinal piece of the sub-subleading term. Therefore, we must show in the following that *all remaining terms* of the Lagrangian $\mathcal{L}^{(4)}$ cancel out or vanish when $p_\perp = 0$.

The second piece $\mathcal{L}^{(4b)}$ is given by

$$\begin{aligned}
\mathcal{L}^{(4b)} = &-\frac{1}{12} x_\perp^\alpha n_-x\, n_+^\beta x_\perp^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] \left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c \\
&-\frac{1}{24} x_\perp^\alpha x_\perp^\beta n_-x n_+^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] \left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c \,.
\end{aligned} \tag{7.5.45}$$

Again we decompose the two factors of $x_\perp$ and drop the traceless combination. Next, we use $\partial^\mu R^s_{\mu-\nu-} = 0$ as well as the property $x_\perp^2 \partial_\perp^2 = 4 + \dots$ to rewrite the first piece as

$$-\frac{1}{12} n_-x\, n_+^\beta x_\perp^\alpha x_\perp^\nu \left[\partial_\nu R^s_{\alpha-\beta-}\right] \left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c = -\frac{1}{24} x_\perp^2 n_-x \left[\partial^{\alpha_\perp} R^s_{\alpha_\perp-+-}\right] \left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c$$

$$
\begin{aligned}
&= -\frac{1}{24}\, x_\perp^2 n_- x \Big[\partial^\alpha R^s_{\alpha-+-} - \frac{1}{2}\partial_- R^s_{+-+-}\Big]\left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c \\
&= \frac{1}{48} n_- x \left[\partial_- R_{+-+-}\right]\left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c = -\frac{1}{48} n_- x R_{+-+-}\left[\partial_+\chi_c\right]^\dagger \left[\partial_-\partial_+\chi_c\right] \\
&= \frac{1}{48} x_\perp^2 n_- x R_{+-+-}\left[\partial_+\chi_c\right]^\dagger \Big[\partial_\perp^2\chi_c\Big] = \frac{1}{12} n_- x R_{+-+-}\left[\partial_+\chi_c\right]^\dagger \chi_c\,,
\end{aligned}
\tag{7.5.46}
$$

Doing the same for the second term yields

$$
-\frac{1}{24} x_\perp^\alpha x_\perp^\beta n_- x \Big[\partial_+ R^s_{\alpha-\beta-}\Big]\left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c = -\frac{1}{48} x_\perp^2 n_- x \left[\partial_+ R^s_{--}\right]\left[\partial_+\chi_c\right]^\dagger \partial_+\chi_c = 0\,. \tag{7.5.47}
$$

Note that this term vanishes since the leading-power sourceless equation of motion reads $R^s_{--} = 0$. In summary, the first piece $\mathcal{L}^{(4b)}$ contributes as

$$
\mathcal{L}^{(4b)} \mathrel{\widehat{=}} \frac{1}{12} n_- x R^s_{+-+-}\left[\partial_+\chi_c\right]^\dagger \chi_c\,. \tag{7.5.48}
$$

The third part $\mathcal{L}^{(4c)}$ is defined as

$$
\mathcal{L}^{(4c)} = -\frac{1}{48} x_\perp^\alpha x_\perp^\beta x_\perp^\rho x_\perp^\sigma \Big[\partial_\rho\partial_\sigma R^s_{\alpha-\beta-}\Big]\left[n_+\partial\chi_c\right]^\dagger n_+\partial\chi_c\,. \tag{7.5.49}
$$

We now decompose the four factors of $x_\perp$ as

$$
x_\perp^\alpha x_\perp^\beta x_\perp^\rho x_\perp^\sigma \mathrel{\widehat{=}} \frac{1}{8} x_\perp^4 (\eta_\perp^{\alpha\beta}\eta_\perp^{\rho\sigma} + \eta_\perp^{\alpha\rho}\eta_\perp^{\beta\sigma} + \eta_\perp^{\alpha\sigma}\eta_\perp^{\beta\rho})\,, \tag{7.5.50}
$$

where we already dropped all non-contributing terms. This $x_\perp^4$ term contributes as

$$
x_\perp^4 \partial_\perp^4 \mathrel{\widehat{=}} 64 + (\partial_\perp x_\perp \ldots)\,. \tag{7.5.51}
$$

In addition, we require the scalar equation of motion

$$
n_-\partial\chi_c \mathrel{\widehat{=}} -\frac{\partial_\perp^2}{n_+\partial}\chi_c\,. \tag{7.5.52}
$$

Then, $\mathcal{L}^{(4c)}$ simplifies to

$$
\mathcal{L}^{(4c)} \mathrel{\widehat{=}} \frac{1}{12} R^s_{+-+-}\, \chi_c^\dagger \chi_c\,. \tag{7.5.53}
$$

The fourth term $\mathcal{L}^{(4d)}$ is chosen to be

$$
\mathcal{L}^{(4d)} = -\frac{1}{6} x_\perp^\alpha n_- x n_+^\beta (R^s_{\alpha\mu_\perp\beta-} + R^s_{\beta\mu_\perp\alpha-}) \Big(\left[\partial^{\mu_\perp}\chi_c\right]^\dagger \partial_+\chi_c + \left[n_+\partial\chi_c\right]^\dagger \partial^{\mu_\perp}\chi_c\Big)\,. \tag{7.5.54}
$$

Note that the first term in the bracket yields a vanishing contribution once $p_\perp^\mu = 0$ is employed. The second term must be integrated by parts and reads

$$
\mathcal{L}^{(4d)} = -\frac{1}{12} n_- x R^s_{+-+-}\left[n_+\partial\chi_c\right]^\dagger \chi_c\,, \tag{7.5.55}
$$

which cancels the contribution (7.5.48) from $\mathcal{L}^{(4b)}$. The fifth term $\mathcal{L}^{(4e)}$ reads

$$
\mathcal{L}^{(4e)} = \frac{1}{12} x_\perp^\alpha x_\perp^\beta \eta_\perp^{\mu\nu} R^s{}_{\mu_\perp\alpha\nu_\perp\beta}\left[n_+\partial\chi_c\right]^\dagger n_-\partial\chi_c\,. \tag{7.5.56}
$$

Here, after decomposing the two factors of $x_\perp$, using the scalar equations of motion and performing an integration by parts, one obtains

$$
\mathcal{L}^{(4e)} \mathrel{\widehat{=}} -\frac{1}{12} R^s_{+-+-}\chi_c^\dagger\chi_c\,, \tag{7.5.57}
$$

which cancels the term (7.5.53) in $\mathcal{L}^{(4c)}$. Finally, the remaining terms are denoted by $\mathcal{L}^{(4f)}$ and are given by

$$\begin{aligned}\mathcal{L}^{(4f)} = &-\frac{1}{16}x_\perp^\alpha x_\perp^\beta x_\perp^\nu \left[\partial_\nu R^s_{\alpha\mu_\perp\beta-}\right]\left([\partial^{\mu_\perp}\chi_c]^\dagger\, n_+\partial\chi_c + [n_+\partial\chi_c]^\dagger\,\partial^{\mu_\perp}\chi_c\right)\\ &-\frac{1}{6}x_\perp^\alpha x_\perp^\beta R^s_{\alpha\mu_\perp\beta\nu_\perp}\,[\partial^{\mu_\perp}\chi_c]^\dagger\,\partial^{\nu_\perp}\chi_c + \frac{1}{6}x_\perp^\alpha x_\perp^\beta R^s_{\alpha+\beta-}\left[\partial_{\mu_\perp}\chi_c^\dagger\right]\partial^{\mu_\perp}\chi_c\\ &+\frac{1}{12}x_\perp^\alpha x_\perp^\beta R^{s\mu_\perp}{}_{\alpha\mu_\perp\beta}\,[\partial_{\nu_\perp}\chi_c]^\dagger\,\partial^{\nu_\perp}\chi_c\,.\end{aligned} \tag{7.5.58}$$

First note that the last three terms in (7.5.58) are proportional to $p_\perp = 0$, and thus yield no contribution. The first term is also vanishing. One can see this by performing integration by parts and decomposing the remaining $x_\perp$ tensor structures. In short, one finds

$$\mathcal{L}^{(4f)} \mathrel{\hat{=}} 0\,. \tag{7.5.59}$$

In summary, the time-ordered product $T\,\{\hat{\mathcal{A}}^{(0)}\,,\mathcal{L}^{(4)}\}$ is completely determined by the first term $\mathcal{L}^{(4a)}$ (7.5.44), where one finds

$$\hat{\mathcal{A}}^{(4)} \supset i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(0)}\,,\frac{\kappa}{16}\chi_c^\dagger n_+\partial\left((n_-x)^2 R^s_{+-+-}n_+\partial\chi_c\right)\right\}\,. \tag{7.5.60}$$

Contribution from $T\,\{\hat{\mathcal{A}}^{(1)}\,,\mathcal{L}^{(3)}\}$

The next term $\mathcal{L}^{(3)}$ (7.5.14) must contribute via $T\,\{\hat{\mathcal{A}}^{(1)}\,,\mathcal{L}^{(3)}\}$ to yield a non-vanishing contribution. Here, $\hat{\mathcal{A}}^{(1)}$ contains *one* transverse derivative. This time, we decompose the Lagrangian into three pieces.

The first part

$$\mathcal{L}^{(3a)} = -\frac{1}{4}x_\perp^\alpha n_-x n_+^\beta R^s_{\alpha-\beta-}\,[n_+\partial\chi_c]^\dagger\, n_+\partial\chi_c\,, \tag{7.5.61}$$

is already in the right form compared to the second term of (7.5.39). We have to verify that all remaining terms cancel out. The second part is defined as

$$\begin{aligned}\mathcal{L}^{(3b)} = &-\frac{1}{12}x_\perp^\alpha x_\perp^\beta x_\perp^\nu\left[\partial_\nu R^s_{\alpha-\beta-}\right][n_+\partial\chi_c]^\dagger\, n_+\partial\chi_c\\ &-\frac{1}{3}x_\perp^\alpha x_\perp^\beta R^s_{\alpha\mu_\perp\beta-}\left([\partial^{\mu_\perp}\chi_c]^\dagger\, n_+\partial\chi_c + [n_+\partial\chi_c]^\dagger\,\partial^{\mu_\perp}\chi_c\right)\,,\end{aligned} \tag{7.5.62}$$

and it does not contribute to the time-ordered product. For the first term, one finds

$$-\frac{1}{12}x_\perp^\alpha x_\perp^\beta x_\perp^\nu\left[\partial_\nu R^s_{\alpha-\beta-}\right]C^{A1\mu_\perp}\,[n_+\partial\chi_c]^\dagger\,[\partial_{\mu_\perp}n_+\partial\chi_c] \mathrel{\hat{=}} -\frac{1}{6}R^s_{+-\beta_\perp-}C^{A1\beta_\perp}\,[n_+\partial\chi_c]^\dagger\,\chi_c\,, \tag{7.5.63}$$

while the second one enters as

$$-\frac{1}{3}x_\perp^\alpha x_\perp^\beta R^s_{\alpha\mu_\perp\beta-}C^{A1\rho_\perp}\,[n_+\partial\chi_c]^\dagger\,[\partial^{\mu_\perp}\partial^{\rho_\perp}\chi_c] \mathrel{\hat{=}} \frac{1}{6}R^s_{+-\alpha_\perp-}C^{A1\alpha_\perp}\,[n_+\partial\chi_c]^\dagger\,\chi_c\,, \tag{7.5.64}$$

and both contributions cancel out. In short, $\mathcal{L}^{(3b)} \mathrel{\hat{=}} 0$. All remaining terms in $\mathcal{L}^{(3)}$ are easily seen to vanish when setting $p_\perp = 0$. To summarise, $\mathcal{L}^{(3)}$ has only one non-vanishing term (7.5.61) which contributes as

$$\hat{\mathcal{A}}^{(4)} \supset i\int d^4x\, T\left\{\hat{\mathcal{A}}^{(1)}\,,\frac{1}{4}\chi_c^\dagger n_+\partial(x_\perp^\alpha n_-x n_+^\beta R^s_{\alpha-\beta-}n_+\chi_c)\right\}\,. \tag{7.5.65}$$

Contribution from $T\{\hat{\mathcal{A}}^{(2)}, \mathcal{L}^{(2)}\}$

Finally, we consider $\mathcal{L}^{(2)}$, which can only contribute via $T\{\hat{\mathcal{A}}^{(2)}, \mathcal{L}^{(2)}\}$, where $\hat{\mathcal{A}}^{(2)}$ contains *two* transverse derivatives. There is only one non-vanishing contribution in $\mathcal{L}^{(2)}$ (7.5.13), namely

$$\mathcal{L}^{(2c)} = -\frac{1}{8} x_\perp^\alpha x_\perp^\beta R^s_{\alpha-\beta-} \left[\partial_+ \chi_c\right]^\dagger \partial_+ \chi_c \,. \tag{7.5.66}$$

This term contains two factors of $x_\perp$, which are eaten by the two transverse derivatives in $\hat{\mathcal{A}}^{(2)}$. All other terms in $\mathcal{L}^{(2)}$ give a vanishing contribution once one sets $p_\perp = 0$.

In summary, by using only equations of motion, integration by parts and the reference frame where $p_\perp = 0$, the single soft-emission Lagrangian terms simplify drastically and yield the simple contributions

$$\mathcal{L}^{(2)}_{\text{orbital}} = \frac{1}{4} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\alpha\mu}_{\perp+} L^{\beta\nu}_{\perp+} \chi_c \,, \tag{7.5.67}$$

$$\mathcal{L}^{(3)}_{\text{orbital}} = \frac{1}{2} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\alpha\mu}_{\perp+} L^{\beta\nu}_{+-} \chi_c \tag{7.5.68}$$

$$\mathcal{L}^{(4)}_{\text{orbital}} = \frac{1}{4} R^s_{\alpha\mu\beta\nu} \chi_c^\dagger \overleftarrow{L}^{\mu\alpha}_{+-} L^{\nu\beta}_{+-} \chi_c \,. \tag{7.5.69}$$

To summarise, we have explicitly derived an operatorial version of the soft theorem that includes all three terms in gravity. The key insight of this derivation is that in gravity, there are *two eikonal terms* (7.5.18) and (7.5.34) instead of the single eikonal term encountered in gauge theory (7.2.11). These two terms can be traced back directly to the two soft gauge fields $s_{\alpha-}(x_-)$ and $\partial_{[\alpha} s_{\beta]-}(x_-)$ which form the soft-covariant derivative in the effective theory. The first gauge field $s_{\alpha-}(x_-)$ couples to the momentum P^μ, and its gauge symmetry is related to the local translations along the collinear light-cone. The second one, $\partial_{[\alpha} s_{\beta]-}(x_-)$, couples to the angular momentum density of the scalar field, and its gauge transformation corresponds to local Lorentz transformations evaluated on the light-cone.

Similar to gauge theory, this covariant derivative is directly responsible for the eikonal terms. Therefore, there are now two eikonal terms in the soft theorem. The first one takes the form

$$\varepsilon_{\mu-} p^\mu \frac{n_+ p}{p \cdot k} \,, \tag{7.5.70}$$

and the explicit momentum p^μ appears instead of the gauge generator t^a. The second gauge field $\partial_{[\alpha} s_{\beta]-}(x_-)$ couples to the angular momentum density of the scalar field. This results in the second eikonal term explicitly containing the angular momentum as

$$k_\rho \varepsilon_{\mu-} J^{\rho\mu} \frac{n_+ p}{p \cdot k} \,. \tag{7.5.71}$$

Here, one notices that the first term is precisely the polarisation corresponding to the gauge field $\partial_{[\alpha} s_{\beta]-}(x_-)$, while the angular momentum $J^{\mu\nu}$ appears instead of the generator t^a.

This also explains why the first two terms of the soft theorem are not manifestly gauge-invariant by themselves. The eikonal term in gauge theory requires charge conservation. In gravity, the analogue of charge conservation is momentum and angular momentum conservation for the first and second gauge field, respectively. Therefore, the first term in the soft theorem is gauge-invariant only after momentum conservation is imposed, and likewise for the second term.

Besides this covariant derivative, the remaining soft-collinear interactions are expressed via the (gauge-invariant) Riemann tensor and its derivatives, evaluated at x_-. The first term corresponds to the well-known coupling to the quadrupole moment. The Riemann tensor is also the first possible soft gauge-invariant building block that can be used in the operator basis. However, it contains two derivatives, one more than the field-strength tensor, and is therefore suppressed by another power in the soft momentum. In the soft theorem, this manifests itself in the existence of a third, sub-subleading term, which can be expressed in terms of the Riemann tensor due to the quadrupole interaction in the Lagrangian. This third term corresponds to the second term in gauge theory, which stems from the dipole interaction. There are no further universal terms since the operators that contain a soft Riemann tensor can carry process-dependent matching coefficients, i.e. they are not determined by matching to the non-radiative amplitude.

As a final remark, note that the two factors of angular momenta that appear in the sub-subleading soft theorem have a different origin. One factor arises due to the coupling in the Lagrangian, where the angular momentum density appears. The other factor is related to the symmetries of the Riemann tensor and the eikonal propagator, in a similar fashion to the angular momentum that appears in the subleading term in gauge theory due to the symmetries of $F^s_{\mu\nu}$. Therefore, the form of all three terms of the gravitational soft theorem (as well as the two terms in gauge theory) can be directly linked to the properties and structure of the soft-collinear effective theory.

7.6 Loop Corrections to the Soft Theorem

While the soft theorem holds true for a generic non-radiative amplitude, it is only valid if the soft emission is a tree-level process. For gauge theory, already the leading-order eikonal term is affected by loop corrections. The gravitational corrections, however, have a very peculiar form [33]: The leading term is *not modified* by loop corrections. The subleading term only receives one-loop corrections, and the sub-subleading term is only affected by one- and two-loop contributions. Higher-loop contributions cannot affect the soft theorem.

This was first observed in [33], where amplitude techniques were used to derive this result. Since we now have access to the full soft-collinear effective theory of gravity, which in particular describes soft and collinear loops to any order, it is an interesting question if the effective theory provides any further insights into this curious feature. We shall see in the following that these loop corrections are an immediate consequence of the power-counting in the effective theory. This discussion is based on [48] and the upcoming [67].

In SCET, the loop corrections can arise from three different regions of loop momenta, the hard, soft and collinear regions. Since the hard modes are already integrated out, these contributions are part of the matching coefficients $\widetilde{C}^X(t_i)$ and thus of the non-radiative amplitude. Therefore, a hard loop never affects the soft theorem and we can restrict our discussion to soft and collinear loops, arising from the modes that are described by the effective Lagrangian.

There are two major differences between gravity and gauge-theory in both the soft and collinear sector [47], and can ultimately be traced back to the fact that both the purely-collinear as well as the purely-soft theory are equivalent to the full theory, which is a weak-field expansion with dimensionful coupling.

i) There are no leading-power interactions in the purely-collinear sector. This is related to the fact that the λ expansion precisely corresponds to the weak-field expansion and causes the absence of collinear divergences. The first purely-collinear graviton-interaction appears at $\mathcal{O}(\lambda)$.

ii) There are no leading-power interactions in the purely-soft sector. The weak-field expansion agrees with the λ^2 expansion, and the first purely-soft interaction vertices start at $\mathcal{O}(\lambda^2)$.

Intuitively, this corresponds to the fact that collinear gravity is expanded in collinear momenta $p_\perp \sim \lambda$, while soft gravity is expanded in soft momenta $k_s \sim \lambda^2$. Only soft-collinear interactions exist at leading-power, mediated through the soft-covariant derivative. These leading-power interactions are *eikonal*, and one can make use of soft exponentiation [1].

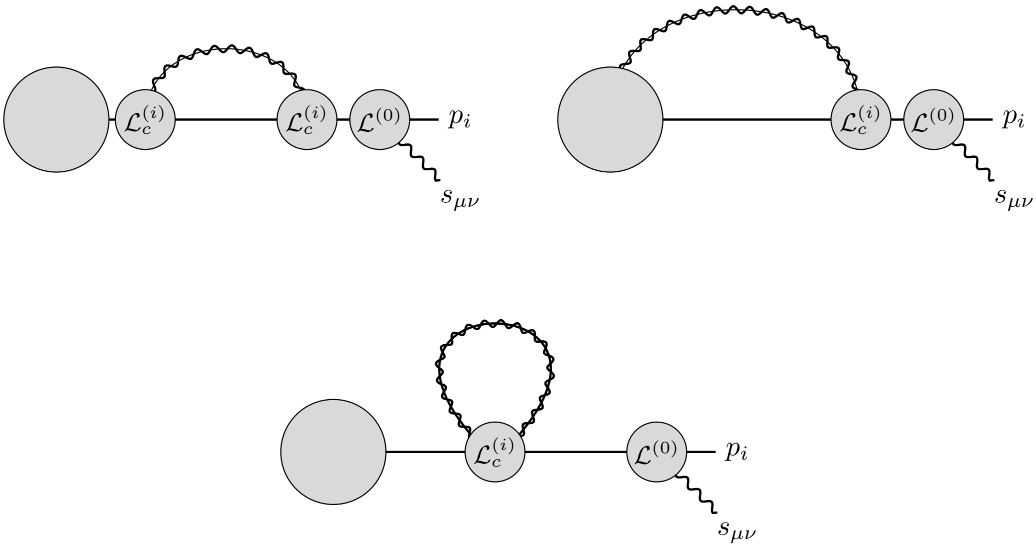


Figure 7.2: Diagram classes providing collinear one-loop corrections to the soft emission process. The loop must be attached either to the leg by collinear interactions, or to the hard vertex by adding a collinear building block. Collinear interaction vertices and building blocks are suppressed by at least $\mathcal{O}(\lambda)$, resulting in a suppression of λ^2 for each collinear loop.

Collinear Loops

We begin with the collinear sector. First, recall that there are no interactions between different collinear sectors. Therefore, it is enough to consider loop corrections to a single i-collinear leg. There are two ways to add such a collinear loop to the soft-emission process, depicted in Fig. 7.2. One can connect both ends of the loop to the i-collinear leg, or one can attach one end to the leg, and one leg to the hard source.[6] In the first case, both ends must be connected by a purely-collinear vertex, which is suppressed at least by $\mathcal{O}(\lambda)$. Alternatively, one can make use of higher-point graviton vertex, but these also come with higher λ-suppression, since the order in h agrees with the order in λ. In the second case, one must add another i-collinear building block to the N-jet operator. These building blocks start at $\mathcal{O}(\lambda)$ with the transverse graviton $\mathfrak{h}_{\perp\perp}$. In addition, to connect the leg to the external line, one must again use a collinear vertex. Therefore, such a process is also suppressed at least by $\mathcal{O}(\lambda^2)$. To summarise, a collinear one-loop contribution must be suppressed by at least $\mathcal{O}(\lambda^2)$, and therefore it cannot modify the leading-power term. It can, however, affect the subleading term. This discussion generalises in a straightforward fashion to two loops and further. Ultimately, each added collinear loop comes with a suppressing factor of $\mathcal{O}(\lambda^2)$. In conclusion, the leading term can never be affected by collinear loops the subleading term is collinear 1-loop exact and the sub-subleading term is collinear 2-loop exact.

Soft Loops

Next, consider adding soft loops. These soft loops are more complicated since here the theory contains a leading-power soft-collinear interaction. A soft loop can either be attached to a single collinear leg, or it can connect two different legs. This connection can happen at $\mathcal{O}(\lambda^0)$ using the

[6]If one attaches the collinear loop to the external soft graviton, the new intermediate graviton is a collinear one and situation is the same as the first type.

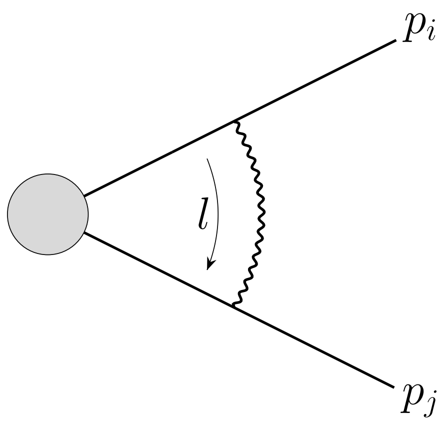


Figure 7.3: A soft loop connecting two different collinear legs. If the external legs are on-shell, the integral is scaleless and vanishes.

leading-power vertex. However, all these contributions vanish unless the soft graviton is *directly connected* to the soft graviton in the loop via a *purely-soft* interaction, which suppresses the process by $\mathcal{O}(\lambda^2)$.[7] This is because of the eikonal interactions in the effective theory. Due to the light-front multipole expansion, soft momenta k^μ in soft-collinear interactions always appear as $n_{i-}k\frac{n_{i+}^\mu}{2}$ in the momentum-conserving δ-function and yield the eikonal propagators of collinear fields. For a soft loop connected to two different collinear legs, as depicted in Fig. 7.3, the contribution schematically takes the form

$$I \quad \propto \quad \int \frac{d^dl}{(2\pi)^d}\frac{1}{p_i^2+n_{i+}p_in_{i-}l+i0}\frac{1}{p_j^2-n_{j+}p_jn_{j-}l+i0}\frac{1}{l^2+i0}\,. \tag{7.6.1}$$

If the external legs are on-shell (like for the soft theorem), we can further simplify $p_i^2=p_j^2=0$ and find

$$I \propto \int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2+i0}\frac{1}{n_{i-}l+i0}\frac{1}{n_{j-}l+i0}\,, \tag{7.6.2}$$

which is a scaleless integral, and thus the contribution vanishes in dimensional regularisation. Now consider adding the soft emission to one of the collinear legs. There are two corresponding diagrams, one where the graviton is emitted before the loop, and one after, the first two diagrams in Fig. 7.4. The sum of both contributions factorises using the eikonal identities, and we can always consider the graviton outside of the loop, in the sense that the external graviton momentum k only appears in one external eikonal propagator as

$$\frac{1}{n_{i-}l+i0}\to\frac{1}{n_{i-}(l+k)+i0}\,. \tag{7.6.3}$$

However, from this object, one cannot form a soft invariant, and the loop integral (7.6.2) remains scaleless even if k is present in a collinear propagator.

Only if the full soft momentum is injected into the loop integral (7.6.2) as

$$I \propto \int \frac{d^dl}{(2\pi)^d}\frac{1}{(l+k)^2+i0}\frac{1}{n_{i-}l+i0}\frac{1}{n_{j-}l+i0}\,, \tag{7.6.4}$$

one finds that the contribution is no longer scaleless and thus non-vanishing in general.[8] This is the third type of diagram in Fig. 7.4. However, this requires a *purely-soft* vertex, since only in these vertices one has full soft-momentum conservation $\delta^{(4)}(l^\mu+k^\mu+\ldots)$.

[7] A similar discussion of soft loops in gauge theory can be found in the Appendix of [68].
[8] One can form a soft invariant as $(n_{i-}k)(n_{j-}k)\,(n_{i+}n_{j+})$.

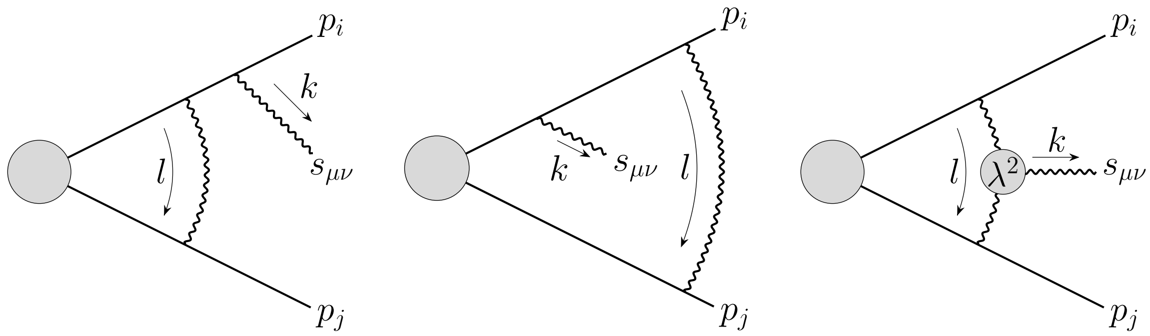


Figure 7.4: Diagram classes containing soft loops that contribute to the soft-emission process at one-loop order. The first two diagrams appear at leading power in λ. Due to the multipole expansion, these interactions factorise using the eikonal identities. The loop is scaleless and vanishes. The third type of diagram is non-vanishing since the soft momentum k is injected directly into the soft loop. This requires a purely-soft vertex which gives at least λ^2 suppression.

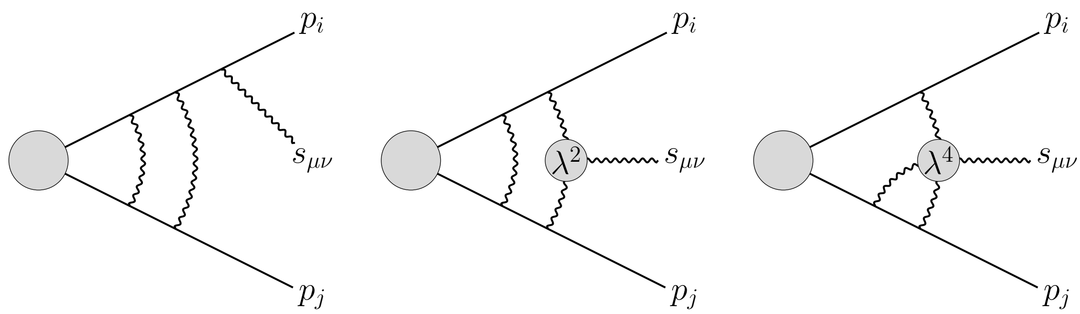


Figure 7.5: Examples of soft two-loop diagrams. The first type is relevant if the vertices connecting to the collinear legs are $\mathcal{O}(\lambda^0)$ or $\mathcal{O}(\lambda)$. In this case, one can use eikonal identities to see that the loops are scaleless and vanish. The same is true for the second type of diagrams. In the third class, all loops are connected to the soft emission via purely-soft vertices, either directly as depicted here, or via multiple vertices. This type of diagram is non-vanishing but suppressed by at least λ^4.

In conclusion, to find a non-vanishing contribution, the soft graviton must be connected to the soft graviton in the loop via a purely-soft interaction. Therefore, soft one-loop corrections cannot affect the leading-term of the soft theorem, but they can modify the subleading one.

Soft Two-loop Contribution

Next, let us proceed to two-loop corrections. For collinear two-loop contributions, we can use the previous argument to see that they must come with at least $\mathcal{O}(\lambda^4)$ suppression. Moreover, we do not need to consider mixed soft-collinear loops, since collinear loops are always at least $\mathcal{O}(\lambda^2)$ suppressed, and a soft one-loop contribution is also suppressed by $\mathcal{O}(\lambda^2)$.

Therefore, it remains to check the purely-soft two-loop corrections. There are a number of different topologies that can contribute, as depicted in Fig. 7.5.

The simplest one is the situation where the loops are connected to the soft emission by two purely-soft vertices, or by a purely-soft four-point vertex. These contributions are suppressed by $\mathcal{O}(\lambda^4)$ due to the soft power-counting, and such corrections can only modify the sub-subleading factor as we claim. No further argument is needed.

Therefore, the more interesting situation arises when the loops are connected to the energetic lines using the leading-power interactions, as this is a two-loop correction that naively scales only as $\mathcal{O}(\lambda^2)$. These interactions are eikonal, and we can again use the eikonal identities to argue that the loops factorise into a soft loop *without* soft emission, and a soft loop *with* soft emission. More precisely, when summing over all permutations of these diagrams, we find that the amplitude for soft graviton emission is multiplied by a factor corresponding to the eikonal factor [1]

$$\mathcal{A}(\{p_i\}) \prod_{n=1}^{\infty} \left(\sum_{i=1}^{N} \frac{\kappa}{2} \frac{p_i^{\mu_n} p_j^{\nu_n}}{p_i \cdot q_n + i\varepsilon} \right) . \tag{7.6.5}$$

This is soft exponentiation at work. In this factor (7.6.5), we now take one of the momenta as the emitted graviton, while the others are connected into loops. Due to this factorisation, the soft loops cannot depend on the external momentum k and are scaleless.

The next dangerous topology arises when the soft loops do not couple via the leading-power interaction but via the $\mathcal{O}(\lambda)$-interaction from $\mathcal{L}^{(1)}$. This can in principle yield a contribution that modifies the subleading factor at two-loop order. First, note that the interactions in $\mathcal{L}^{(1)}$ come either with a single $p_\perp$ or a single $x_\perp$. Therefore, by choosing a frame where $p_\perp^\mu = 0$, one can eliminate all single-insertions of such vertices. Consequently, one must use the vertex twice to construct an invariant object. This is now $\mathcal{O}(\lambda^2)$ suppressed. If the second loop comes only with leading-power interactions, we can construct a two-loop contribution that scales as $\mathcal{O}(\lambda^2)$. This takes the form of the first two diagrams in Fig. 7.5, where two leading-power vertices are replaced by $\mathcal{L}^{(1)}$. However, an explicit computation shows that for this two-graviton emission, summing over all permutation again yields eikonal propagators, even for the subleading interaction vertex. Therefore, the two loops again factorise and become scaleless.

In summary, any purely-soft loop is scaleless and vanishes unless the external soft momentum is injected directly into the loop, i.e. unless all soft loops are connected by purely-soft interaction vertices. Each such interaction vertex gives $\mathcal{O}(\lambda^2)$ suppression, and therefore each soft loop, if non-vanishing, comes effectively with a power-suppression by λ^2.

Therefore, adding a loop, regardless if soft or collinear, always amounts to a suppression by at least $\mathcal{O}(\lambda^2)$. This implies that the leading term is not modified by loop corrections, while the subleading term is one-loop exact, and the sub-subleading term is two-loop exact, as claimed in the beginning of the section.

The argumentation solely relied on the EFT power-counting, which provides the suppression of the interaction vertices, as well as the multipole expansion, which is responsible for the eikonal propagators that were paramount in the previous discussion.

Chapter 8

Scalar Field in deSitter Space-time

In this final section, we consider a different application of the EFT methodology, by investigating the physics of a light (or massless) scalar field in deSitter space-time.

While QFT in Minkowski space is well-understood, and even infrared divergences pose no conceptual problems anymore, the situation in deSitter space is quite different. This seems surprising at first, since deSitter space is one of the maximally-symmetric space-times, and one would naively expect that this is the simplest possible generalisation of flat-space QFT. However, for light and massless fields, the correlation functions computed in deSitter space-time suffer from large secular divergences at late times. Moreover, for free, massless scalar fields, a deSitter-invariant vacuum state cannot be defined [69].

This section aims to be self-contained, so we provide a brief discussion of the properties of deSitter space and a scalar field in this space-time. This is by now standard textbook material, and we refer for details to the excellent [70–72]. Throughout this section, we employ the in-in formalism when computing correlation functions, we mainly follow [73].

8.1 Basic Definitions and Mode Functions

DeSitter space is one of the maximally-symmetric Lorentzian manifolds. It corresponds to a standard Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with positive spatial curvature, and it arises naturally as the vacuum solution of the Einstein field equations with a positive cosmological constant.

In planar coordinates, the metric tensor is defined via the line-element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\boldsymbol{x}^2 = a^2(\eta)(\mathrm{d}\eta^2 - \mathrm{d}\boldsymbol{x}^2)\,, \tag{8.1.1}$$

where t denotes proper time, η is conformal time and the scale factor a is expressed in terms of the (constant) Hubble parameter H as

$$a(t) = e^{Ht}\,, \quad a(\eta) = -\frac{1}{H\eta}\,. \tag{8.1.2}$$

It is important to note that with this definition, the time coordinate has the standard range $t \in (-\infty, \infty)$, while conformal time is *negative*, $\eta \in (-\infty, 0]$. In addition, note that this set of coordinates only covers half of the deSitter space-time. However, this is enough for our purposes. The half that is covered corresponds to the region between the big bang at $t = -\infty$ and any future point in time. The other half of the space-time would correspond to the region before the big bang, passing through a singularity. This region is not of physical interest to us.

Next, consider a free, massive, minimally-coupled scalar field in deSitter space-time. The action is given by (4.3.1) and reads

$$S_0[\phi] = \frac{1}{2}\int d^4x\,\sqrt{-g}\,\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right)\,. \tag{8.1.3}$$

Inserting the metric tensor (8.1.1), one obtains for the free action

$$S_0[\phi] = \frac{1}{2}\int d^4x\,a^3(t)\left(\dot{\phi}^2 - \frac{1}{a^2(t)}(\partial_i\phi)^2 - m^2\phi^2\right)\,, \tag{8.1.4}$$

where $\dot{\phi} \equiv \partial_t \phi$. The free equation of motion reads

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\partial_i^2}{a^2(t)}\phi + m^2\phi = 0\,, \tag{8.1.5}$$

and differs from the flat-space equation by the friction term $3H\dot{\phi}$.

In FLRW space-times, it is convenient to work in conformal time and introduce a redefined scalar field $\chi(\eta, \boldsymbol{x})$ as

$$\phi(\eta, \boldsymbol{x}) = \frac{\chi(\eta, \boldsymbol{x})}{a(\eta)}\,. \tag{8.1.6}$$

Expressed in terms of this field, the free action (8.1.4) reads

$$S_0[\chi] = \frac{1}{2}\int d^3x\, d\eta\, \left(\partial_\mu \chi\, \partial^\mu \chi - \left(m^2 - 2H^2\right) a^2(\eta)\chi^2\right)\,, \tag{8.1.7}$$

and is formally equivalent to a free scalar field in flat space which has a *time-dependent* effective mass term

$$m_{\text{eff}}^2(\eta) = m^2 - 2(a(\eta)H)^2\,. \tag{8.1.8}$$

Super-horizon and Sub-horizon Modes

At this point, one can make an interesting observation: deSitter space comes with a physical scale $d \sim \frac{1}{H}$, related to the space-time curvature $R = 12H^2$. This scale can be used to define two regions of momenta. The first one, $k\,|\eta| > 1$ is called sub-horizon. Modes in this region have short wavelengths $k \to \infty$ or are evaluated at early times $|\eta| \to \infty$, and can be thought of as the analogue to the hard region of momenta. Since the physical wavelength of these modes is shorter than the curvature scale, these modes should not be affected by the gravitational background. Indeed, inserting the redefined field χ in the equation of motion (8.1.5), one finds

$$\chi_{\boldsymbol{k}}'' + \boldsymbol{k}^2\left[1 + \frac{1}{(k\eta)^2}\left(\frac{m^2}{H^2} - 2\right)\right]\chi_{\boldsymbol{k}} = 0\,. \tag{8.1.9}$$

For the limit $k\,|\eta| \to \infty$, one can drop the second term in the square bracket and the equation simplifies to

$$\chi_{\boldsymbol{k}}'' + \boldsymbol{k}^2\chi_{\boldsymbol{k}} = 0\,. \tag{8.1.10}$$

This is solved by standard plane-wave solutions $\chi_{\boldsymbol{k}} \sim e^{ik\eta}$, and no trace of the gravitational background can be found.

The second region $k\,|\eta| < 1$ is called super-horizon. These modes capture the long-wavelength $k \to 0$ or late-time $\eta \to 0$ behaviour, and their evolution is strongly affected by the gravitational background. It is more convenient to work in terms of the original field $\phi_{\boldsymbol{k}}(t)$, where the equation of motion (8.1.5) for $k \to 0$ reads

$$\ddot{\phi}_{\boldsymbol{k}} + 3H\dot{\phi}_{\boldsymbol{k}} + m^2\phi = 0\,. \tag{8.1.11}$$

Inserting the ansatz

$$\phi_{\boldsymbol{k}}(t) = (aH)^{-\frac{3}{2}+\nu}\,\varphi_{\boldsymbol{k}}(t)\,, \tag{8.1.12}$$

the equation of motion becomes

$$\ddot{\varphi}_{\boldsymbol{k}} + 2H\nu\dot{\varphi}_{\boldsymbol{k}} + \left(\nu^2 - \frac{9}{4} + \frac{m^2}{H^2}\right)\varphi_{\boldsymbol{k}} = 0\,. \tag{8.1.13}$$

The parameter ν is determined to be

$$\nu = \pm\sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,, \tag{8.1.14}$$

and the equation of motion for the residual field $\varphi_{\boldsymbol{k}}$ simply reads

$$\ddot{\varphi}_{\boldsymbol{k}} + 2H\nu\dot{\varphi}_{\boldsymbol{k}} = 0\,, \tag{8.1.15}$$

i.e. one finds that $\varphi_{\boldsymbol{k}}(t) = \varphi_{\boldsymbol{k}}$ is constant at leading power. For the case of a light mass, $\frac{m^2}{H^2} < \frac{9}{4}$, note that the solution with $+\nu$ decays more slowly than the one with $-\nu$ as time increases. The modes are usually called "growing" ($+\nu$) and "decaying" ($-\nu$), and the growing mode is the one that dominates any late-time correlator.

Field Quantisation

The field quantisation now proceeds along very similar lines to the flat-space analogue. First, one introduces a *spatial* mode decomposition $\chi(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\chi_{\boldsymbol{k}}(\eta)$, then one determines the mode functions $\chi_{\boldsymbol{k}}(\eta)$ from the equation of motion. For deSitter space, the respective mode functions are given by

$$\chi_{\boldsymbol{k}}(\eta) = \sqrt{k\,|\eta|}\,\left(A_k J_\nu(k\,|\eta|) + B_k Y_\nu(k\,|\eta|)\right)\,, \tag{8.1.16}$$

where $k = |\boldsymbol{k}|$ and ν is given in (8.1.14). The functions $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind, respectively, and the parameters A_k, B_k are determined by the choice of a vacuum. In the following, we employ the standard Bunch-Davies vacuum [74], which is defined by the condition that in the infinite past $\eta \to -\infty$, the mode functions should approximate the flat-space modes as

$$\lim_{\eta\to-\infty}\chi_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}}e^{ik\eta}\,. \tag{8.1.17}$$

This fixes the coefficients A_k, B_k, and one finds

$$\chi_{\boldsymbol{k}}(\eta) = \frac{\sqrt{\pi\,|\eta|}}{2}H_\nu^{(2)}(k\,|\eta|)\,, \tag{8.1.18}$$

where $H_\nu^{(2)}(x)$ is the Hankel function of the second kind

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)\,. \tag{8.1.19}$$

For the original scalar field $\phi(\eta, \boldsymbol{x})$, the mode functions correspond to

$$\phi_{\boldsymbol{k}}(\eta) = \frac{\sqrt{\pi}H\,|\eta|^{\frac{3}{2}}}{2}\left(J_\nu(k\,|\eta|) - iY_\nu(k\,|\eta|)\right) \equiv \frac{\sqrt{\pi}H\,|\eta|^{\frac{3}{2}}}{2}H_\nu^{(2)}(k\,|\eta|)\,. \tag{8.1.20}$$

Of particular interest is the massless scalar field, where $\nu = \frac{3}{2}$. Here, the mode functions take the simple form

$$\chi_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}}e^{-ik|\eta|+i\pi}\left(1 - \frac{i}{k\,|\eta|}\right)\,. \tag{8.1.21}$$

Next, one introduces creation and annihilation operators $a^\dagger_{\boldsymbol{k}}$, $a_{\boldsymbol{k}}$, as

$$\phi(t, \boldsymbol{x}) = \int\frac{d^3k}{(2\pi)^3}\left(e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_{\boldsymbol{k}}(t)a_{\boldsymbol{k}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi^*_{\boldsymbol{k}}(t)a^\dagger_{\boldsymbol{k}}\right)\,, \tag{8.1.22}$$

and imposes the equal-time canonical commutation relations

$$[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\,, \quad [\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = [\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = 0\,, \tag{8.1.23}$$

from which the commutation relations of the creation and annihilation operators follow

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q})\,, \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{8.1.24}$$

Using the mode expansion and the commutation relations, one can compute the two-point function (also called the Wightman function) $G^+(x, x')$ analytically for generic ν as

$$\begin{aligned} G^+(x,x') &= \frac{1}{a(\eta)a(\eta')}\sqrt{\frac{\pi\,|\eta|}{2}}\sqrt{\frac{\pi|\eta'|}{2}}\int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}H_\nu^{(2)}(k\,|\eta|)H_\nu^{(2)*}(k|\eta'|) \\ &= \frac{H^2}{16\pi}\frac{\frac{1}{4}-\nu^2}{\cos\pi\nu}\,{}_2F_1\left(\frac{3}{2}+\nu,+\frac{3}{2}-\nu;2;1+\frac{(\Delta\eta-i\varepsilon)^2-\Delta\boldsymbol{x}^2}{4\eta\eta'}\right) \end{aligned} \tag{8.1.25}$$

in terms of the hypergeometric function ${}_2F_1(a, b; c; z)$. However, a problem arises for the massless field: here, as $\nu \to \frac{3}{2}$, the hypergeometric function is finite and ${}_2F_1 \to 1$, but the factor $\frac{1}{\cos\pi\nu} \to \infty$, and thus the Wightman function is not well-defined.

This divergence can be traced back to the long-wavelength modes. Recall that the massless mode functions (8.1.21) contain a term that is singular as $k\,|\eta| \to 0$. In the computation of $G^+(x, x')$, this manifests itself as

$$\begin{aligned} G^+(x,x') = &\frac{1}{a(\eta)a(\eta')}\sqrt{\frac{\pi\,|\eta|}{2}}\sqrt{\frac{\pi|\eta'|}{2}}\int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}e^{-ik(|\eta|-|\eta'|)} \\ &\times\frac{1}{2k}\left(1-\frac{i}{k\,|\eta|+\frac{i}{k|\eta'|}+\frac{1}{k^2|\eta||\eta'|}}\right). \end{aligned} \tag{8.1.26}$$

The last term in the integral leads to a logarithmic divergence, and thus the two-point function is ill-defined. This signals that the massless, minimally coupled field, without any self-interactions, does not have a deSitter-invariant vacuum state [69].

This seems to be disconcerting news at first. However, there are hints that a scalar field with self-interaction dynamically cures this pathological behaviour [75–79]. For example, turning on a $\lambda\phi^4$-interaction seems to generate a dynamical mass $m^2 \sim \sqrt{\lambda}H^2$, proportional to $\sqrt{\lambda}$. This indicates a non-perturbative effect.

8.2 IR Dynamics of deSitter: Approaches

The infrared structure and the dynamics of light and massless scalar field in deSitter space have been studied in great detail, and multiple promising approaches have been identified [75–95]. These approaches provide key insights into the physics of super-horizon modes. In the following, we try to familiarise ourselves with the underlying concepts of two such approaches in detail, the stochastic inflation [75, 76] and the Euclidean deSitter theory [77, 78].

8.2.1 Stochastic Inflation

The most prevalent approach is the framework of stochastic inflation [75, 76]. The main idea is to treat the long-wavelength part, the super-horizon modes, as a classical stochastic field that is susceptible to a noise term due to the effects of sub-horizon modes. We follow [76] for the general derivation.

Consider a scalar field ϕ with action

$$S = \int d^Dx\,\sqrt{-g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right), \tag{8.2.1}$$

where $V(\phi)$ is some potential, later taken to be $\frac{\lambda}{4}\phi^4$, defined in $D = d + 1$ dimensions.

We want to consider the long-wavelength (super-horizon) modes as a coarse-grained stochastic variable which is sourced by the short-wavelength modes. These UV modes are taken to satisfy a Gaussian noise term. This is achieved by splitting the field as

$$\phi(t,\boldsymbol{x}) = \phi(t,\boldsymbol{x}) + \delta\phi(t,\boldsymbol{x}) \tag{8.2.2}$$

$$= \bar{\phi}(t,\boldsymbol{x}) + \int \frac{d^dk}{(2\pi)^d}\, \theta(k - \varepsilon a(t)H) \left(\phi_{\boldsymbol{k}}(t) a_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \phi^*_{\boldsymbol{k}}(t) a^\dagger_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right) . \tag{8.2.3}$$

Here, $\varepsilon \ll 1$ and $\bar{\phi}(t,\boldsymbol{x})$ corresponds to the long-wavelength part. The short-wavelength modes are split-off and treated as a perturbation $\delta\phi(t,\boldsymbol{x})$. The sub-horizon modes $\phi_{\boldsymbol{k}}(t)$ are assumed to satisfy the *free* massless equation of motion

$$\ddot{\phi}_{\boldsymbol{k}}(t) + 3H\dot{\phi}_{\boldsymbol{k}}(t) + \frac{\boldsymbol{k}^2}{a^2(t)}\phi_{\boldsymbol{k}}(t) = 0\,, \tag{8.2.4}$$

which, in turn, generates purely Gaussian correlators.

From the equation of motion $\Box\phi + V'(\phi) = 0$, where $'$ denotes the derivative with respect to ϕ, and neglecting second time-derivatives as well as gradient terms of $\bar{\phi}$, one obtains a Langevin equation for the long-wavelength part

$$\dot{\bar{\phi}}(t,\boldsymbol{x}) = -\frac{1}{3H}V'(\bar{\phi}) + f(t,\boldsymbol{x})\,, \tag{8.2.5}$$

where $f(t,\boldsymbol{x})$ is given by

$$f(t,\boldsymbol{x}) = \varepsilon a(t) H^2 \int \frac{d^dk}{(2\pi)^d}\delta(k - \varepsilon a(t)H) \left(\phi_{\boldsymbol{k}}(t) a_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + \phi^*_{\boldsymbol{k}}(t) a^\dagger_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\right) . \tag{8.2.6}$$

This term is a Gaussian noise term and satisfies

$$\langle f(t,\boldsymbol{x})\rangle = 0\,, \quad \langle f(t,\boldsymbol{x}) f(t',\boldsymbol{x})\rangle = \frac{H^3}{4\pi^2}\delta(t-t')\,. \tag{8.2.7}$$

One can rescale the field and potential to rewrite the Langevin equation (8.2.5) as

$$\dot{\varphi} + V' = \xi\,, \tag{8.2.8}$$

where ξ now satisfies $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. The associated Fokker-Planck (FP) equation [96,97] for the one-point probability distribution function (PDF) $P(\varphi,t)$ reads

$$\partial_t P(\varphi,t) = \frac{\partial}{\partial\varphi}\Big[V'(\varphi)P(\varphi,t)\Big] + \frac{\partial^2}{\partial\varphi^2}P(\varphi,t)\,. \tag{8.2.9}$$

This FP equation (8.2.9) can further be reduced to an eigenvalue problem. To see this, one introduces the reduced PDF

$$P(\varphi,t) = e^{-V(\varphi)}p(\varphi,t)\,, \tag{8.2.10}$$

where the equilibrium solution $e^{-V(\varphi)}$ is factorised. Inserting (8.2.10) in (8.2.9), one obtains

$$\frac{\partial}{\partial t}p(\varphi,t) = \frac{1}{2}\Delta_\varphi p(\varphi,t) - W(\varphi)p(\varphi,t)\,, \tag{8.2.11}$$

with the "potential" $W(\varphi) = \frac{1}{2}\left((V'(\varphi))^2 - \Delta_\varphi V(\varphi)\right)$. Here, $\Delta_\varphi \equiv \frac{\partial^2}{\partial\varphi^2}$. By introducing a mode decomposition

$$p(\varphi,t) = \sum_{n=0}^{\infty} a_n \Phi_n(\varphi) e^{-\Lambda_n(t-t_0)}\,, \tag{8.2.12}$$

one can equivalently formulate (8.2.11) as an eigenvalue equation

$$-\frac{1}{2}\Delta_\varphi\Phi_n + W\Phi_n = \Lambda_n\Phi_n\,. \tag{8.2.13}$$

The Λ_n are the so-called relaxation eigenvalues. Note that the equilibrium solution, satisfying

$$\frac{\partial}{\partial t}P_{\text{eq}}(\varphi) = 0\,, \tag{8.2.14}$$

always exists (for any sensible potential) and has the corresponding eigenvalue $\Lambda_0 = 0$. The other eigenvalues Λ_n for $n \geq 1$ are positive. Thus any probability distribution $P(\varphi, t)$ approaches the equilibrium solution $P_{\mathrm{eq}}(\varphi)$ as $t \to \infty$.

For the specific case of the quartic interaction $V(\varphi) = \frac{\lambda}{4}\varphi^4$, one finds, after re-introducing all constants, the equilibrium solution

$$P_{\mathrm{eq}}(\varphi) = \frac{2}{H\Gamma(\frac{1}{4})}\left(\frac{2\pi^2\lambda}{3}\right)^{\frac{1}{4}} \exp\left[-\frac{2\lambda\pi^2\varphi^4}{3H^4}\right] . \tag{8.2.15}$$

The other relaxation eigenvalues Λ_n for $n \geq 1$ cannot be obtained analytically and must be computed numerically.

The equal-time, equal-position two-point function of a massless mode in the soft limit can then be computed using the equilibrium solution as

$$\langle\bar{\phi}^2(t,\boldsymbol{x})\rangle = \frac{2}{H\Gamma(\frac{1}{4})}\left(\frac{2\pi^2\lambda}{3}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} d\varphi\, \varphi^2 \exp\left(-\frac{2\lambda\pi^2\varphi^4}{3H^4}\right) = \sqrt{\frac{3}{2\pi^2}}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\frac{H^2}{\sqrt{\lambda}} , \tag{8.2.16}$$

and one notices that this is a well-defined two-point function. The massless scalar field dynamically generates a mass term due to the self-interaction. However, the two-point function depends not on λ but $\sqrt{\lambda}$. This signals a non-perturbative origin of this dynamic mass term.

The key insight is that the pathological infrared behaviour is cured dynamically, in a non-perturbative fashion. In this formalism, it is possible to define a two-point function, which now contains a dynamic mass term, and correspondingly there is a deSitter-invariant vacuum state.

The main shortcoming of this approach is that the introduction of the stochastic noise term, which, in EFT terms, should be an effect of integrating out hard modes, is rather ad-hoc, and there is no consistent way to extend this discussion to higher powers beyond the white-noise approximation. This calls for more systematic frameworks.

8.2.2 Euclidean deSitter

The generation of the dynamic mass term is a non-perturbative effect that arises due to the dynamics of the soft modes. This notion can be made precise in Euclidean deSitter space, which was first observed in [77] and later investigated systematically also at subleading order in [78]. For the $O(N)$ model in the large-N limit, additional results and insights are available in [86–89]. The main difference to the Lorentzian counterpart is that the Euclidean version is a compact space-time. This implies that the mode decomposition is not continuous but discrete, with each mode carrying a quantised momentum. In particular, there is only a single zero mode. Physically, this means that the infinite pile-up of soft modes is not possible. In the following, we provide a quick introduction and show how the dynamic mass generation arises in this approach, then discuss the main problems that still persist.

Lorentzian deSitter space in global coordinates is characterised by the line-element

$$ds^2 = dt^2 - \frac{1}{H^2}\cosh^2(Ht)d\Omega_{d-1}^2 . \tag{8.2.17}$$

To reach the Euclidean counterpart, one substitutes

$$t = -\frac{i}{H}\left(\theta - \frac{\pi}{2}\right) , \tag{8.2.18}$$

where θ is an angular variable and taken to be $2\pi i/H$-periodic, which results in

$$ds^2 = -\frac{1}{H^2}\left(d\theta^2 + \sin^2(\theta)d\Omega_{d-1}^2\right) . \tag{8.2.19}$$

This is simply the metric of a sphere of radius $1/H$, and thus it is a compact space-time. Therefore, the mode expansion is discrete. Specifically, a scalar field defined on this sphere can be expanded as

$$\phi(x) = \sum_{\boldsymbol{L}} \tilde{\phi}_{\boldsymbol{L}} Y_{\boldsymbol{L}}(x)\,, \tag{8.2.20}$$

where $\boldsymbol{L} = (L, L_{d-1}, \ldots, L_1)$ are the generalised angular momentum eigenvalues, $Y_{\boldsymbol{L}}$ are the hyperspherical harmonics [98, 99], and we order the eigenvalues as $L \geq L_{d-1} \geq \cdots \geq L_2 \geq |L_1|$. The sum over $\boldsymbol{L}$ is defined as

$$\sum_{\boldsymbol{L}} = \sum_{L=0}^{\infty} \sum_{L_{d-1}=0}^{L} \cdots \sum_{L_2=0}^{L_3} \sum_{L_1=-L_2}^{L_2}\,. \tag{8.2.21}$$

The action of the scalar field takes the form (8.1.3)

$$S = \int_{S^d} d^d x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right)\,, \tag{8.2.22}$$

with corresponding equations of motion

$$\frac{1}{\sqrt{g}} \partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \phi \right] + m^2 \phi^2 + \frac{\lambda}{6}\phi^3 = 0\,. \tag{8.2.23}$$

Next, note that the hyperspherical harmonics are the eigenfunctions of the hyperspherical Laplacian, satisfying

$$\frac{1}{\sqrt{g}} \partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu Y_{\boldsymbol{L}}(x) \right] = -H^2 L(L+d-1) Y_{\boldsymbol{L}}(x)\,, \tag{8.2.24}$$

and are normalised as

$$\int_{S^d} d^d x \sqrt{g}\, Y^*_{\boldsymbol{L}}(x) Y_{\boldsymbol{L}'}(x) = \frac{1}{H^d} \delta_{\boldsymbol{L}\boldsymbol{L}'}\,. \tag{8.2.25}$$

Inserting the mode decomposition (8.2.20) as well as (8.2.24), (8.2.25), one finds for the action

$$S[\phi] = S^0[\phi] + S_{\text{int}}[\phi] = \sum_{\boldsymbol{L}} \frac{H^2 L(L+d-1) + m^2}{H^d} \left| \tilde{\phi}_{\boldsymbol{L}} \right|^2 + S_{\text{int}}[\phi]\,. \tag{8.2.26}$$

The two-point function follows from its standard derivation and reads [78]

$$\langle \phi(x)\phi(y) \rangle = \frac{\int \mathcal{D}[\phi] \phi(x)\phi(y)^{-S^0}}{\int \mathcal{D}[\phi] e^{-S^0}} = \sum_{\boldsymbol{L}} \frac{H^d Y_{\boldsymbol{L}}(x) Y^*_{\boldsymbol{L}}(y)}{H^2 L(L+d-1) + m^2}\,. \tag{8.2.27}$$

One can perform the sum over $\boldsymbol{L}$ to precisely recover the Wick-rotated version of (8.1.25) [78, 100]. In the massless limit $m \to 0$, the two-point function diverges, as expected. One can immediately see that this is due to the contribution of the zero mode $\tilde{\phi}_0 \equiv \varphi$, while any other mode with $L \neq 0$ yields a finite contribution. It was observed by [77] that the zero mode must be treated non-perturbatively. Namely, for a massless scalar field with quartic self-interaction, the zero mode becomes strongly coupled and one needs to take into account an infinite number of diagrams involving the zero mode at each order in the perturbative expansion. However, the merit of Euclidean deSitter is that such a treatment is in fact possible. In [78], a systematic framework was developed, which we introduce in the following.

The key insight is that the zero-mode interaction term $\lambda\varphi^4$ should be treated as part of the "free" action in the path integral. Since the zero mode is constant, one can perform these integrations analytically. Then, one finds that φ is of order $\lambda^{-\frac{1}{4}}$, a clear indication of a non-perturbative effect. This counting gives rise to a new perturbative expansion that is now organised in powers of $\sqrt{\lambda}$, as one also finds in the stochastic approach [75, 76].

More precisely, one decomposes the field as

$$\phi(x) = \varphi + \hat{\phi}(x)\,, \tag{8.2.28}$$

where φ is the (constant) zero mode and $\hat{\phi}(x)$ are the modes with non-zero L. Next, one explicitly treats the φ self-interaction as part of the free action, i.e. one replaces the naive free action in (8.2.26) by

$$S_0[\phi] = \lambda\varphi^4 \left|Y_0\right|^2 + \sum_{\boldsymbol{L}\neq 0} L(L+d-1)\left|\tilde{\phi}_{\boldsymbol{L}}\right|^2 . \tag{8.2.29}$$

The generating functional then takes the form

$$Z J_0, \hat{J} = \exp\left(-S_{\rm int}\left[\frac{\delta}{\delta J_0}, \frac{\delta}{\delta \hat{J}}\right]\right) Z_0[J_0]\hat{Z}[\hat{J}]\,, \tag{8.2.30}$$

where the interaction terms consist of

$$S_{\rm int}[\varphi,\hat{\phi}] = \frac{\lambda}{4!}\int d^dx\sqrt{g}\,\left(\hat{\phi}^4 + 4\varphi\hat{\phi}^3 + 6\varphi^2\hat{\phi}^2\right) . \tag{8.2.31}$$

It is convenient to introduce the short-hand notations

$$\tilde{\lambda} \equiv \frac{V_d\lambda}{4!}\,, \quad \tilde{J}_0 \equiv V_d J_0\,, \tag{8.2.32}$$

where V_d is the finite volume of the compact Euclidean deSitter space. It can be computed as

$$V_d = \int d^dx\sqrt{g} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})H^d}\,. \tag{8.2.33}$$

The zero-mode generating functional only consists of the φ^4 interaction term and the source and can be computed analytically as

$$\begin{aligned}
Z_0[J_0] &= N_0\int \mathcal{D}[\varphi]\exp\left[-\int d^dx\sqrt{g}\,\left(\frac{\lambda}{4!}\varphi^4 + J_0\varphi_0\right)\right] \\
&= \frac{\tilde{\lambda}^{\frac{1}{4}}}{2\Gamma(\frac{5}{4})}\int_{-\infty}^{\infty} d\varphi\,\exp\left(-\tilde{\lambda}\varphi^4 - \tilde{J}_0\varphi\right) \\
&= {}_0F_2\left(\frac{1}{2},\frac{3}{4};\frac{\tilde{J}_0^4}{256\tilde{\lambda}}\right) + \frac{\Gamma(\frac{3}{4})\tilde{J}_0^2}{2\Gamma(\frac{1}{4})\sqrt{\tilde{\lambda}}}\,{}_0F_2\left(\frac{5}{4},\frac{3}{2};\frac{\tilde{J}_0^4}{256\tilde{\lambda}}\right) .
\end{aligned} \tag{8.2.34}$$

Again let us stress that the self-interaction term $\lambda\varphi^4$ in the zero-mode generating functional is not expanded, and is thus treated non-perturbatively.

For the non-zero mode, one finds the more conventional generating functional

$$\begin{aligned}
\hat{Z}[\hat{J}] &= \hat{N}\int \mathcal{D}[\hat{\phi}]\exp\left[-\int d^dx\sqrt{g}\,\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} + \hat{J}\hat{\phi}\right)\right] \\
&= \exp\left(\frac{1}{2}\int d^dx\int d^dy\,\sqrt{g(x)g(y)}\hat{J}(x)\hat{G}_{\rm free}(x,y)\hat{J}(y)\right)\,,
\end{aligned} \tag{8.2.35}$$

where $\hat{G}_{\rm free}$ denotes the non-zero mode propagator.

With this non-perturbative treatment of the zero mode, one can again compute the two-point function. Now, one finds

$$\langle\phi(x)\phi(y)\rangle = \frac{H^{d-2}Y_0^2}{\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\sqrt{\tilde{\lambda}} + m^2} + \sum_{\boldsymbol{L}\neq 0}\frac{H^d Y_{\boldsymbol{L}}(x)Y^*_{\boldsymbol{L}}(y)}{H^2L(L+d-1)+m^2}\,, \tag{8.2.36}$$

and in the massless limit this becomes finite, as

$$\lim_{m\to 0}\langle\phi(x)\phi(y)\rangle = \frac{H^{d-2}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})\sqrt{\tilde{\lambda}}}\,. \tag{8.2.37}$$

This corresponds to a dynamical mass term

$$m^2_{\mathrm{dyn}} = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\sqrt{\frac{\Gamma(\frac{d+1}{2})}{2\pi^{\frac{d+1}{2}}}}\sqrt{\frac{\lambda}{4!}}\,, \tag{8.2.38}$$

which depends on $\sqrt{\lambda}$, as was observed in the stochastic approach.

The main advantage of this formalism is that it is cast in the language of a standard QFT, and, once the zero-mode is treated non-perturbatively, any object of interest has a well-defined perturbative expansion in $\sqrt{\lambda}$, so the results can be extended to any desired order. However, the form of the perturbative expansion is not yet suited to obtain the results of stochastic inflation. In general, an infinite number of diagrams must be resummed to all orders to recover the non-perturbative results, like the two-point function of stochastic inflation. In the large-N limit, this problem is circumvented by the expansion in $\frac{1}{N}$, which allows one to perform these resummations [86–89]. For the case $N = 1$, however, it remains unclear how a systematic resummation can be performed [101].

8.2.3 Schrödinger Formalism

A rather orthogonal approach [79] relies on the Schrödinger formalism of QFT [102–104]. In this approach, one splits the field into a long-wavelength and a short-wavelength part similar to the stochastic approach. This approach mimics the Euclidean approach in so far as the long-wavelength modes are to be treated non-perturbatively The advantage of this framework is that one can consistently compute higher-order corrections to the stochastic approach, and NLO corrections to the Fokker-Planck equation were computed for the first time in this formalism [79], and were verified by two independent computations using a hybrid open EFT [91, 92] and Soft deSitter Effective Theory [94, 95]. However, the formalism is quite non-standard and it is not a traditional effective field theory. In particular, the non-perturbative treatment of the long-wavelength modes using wave-functional methods is very different from an EFT that treats the soft modes. In addition, it requires a multitude of expansion parameters and this causes the construction to seem somewhat ad-hoc.

8.3 Method of Regions in deSitter Space

It is clear that the pathological behaviour of the massless scalar field is due to the soft super-horizon modes and their accumulation at late times. Ideally, if one is able to construct an effective theory that describes the dynamics of these super-horizon modes, this effective theory should also provide a way to directly obtain the previous results like the stochastic equation of motion or the non-perturbative dynamical mass.

Therefore, we now approach this problem with the standard EFT mindset. Modern effective theories, like SCET discussed earlier, rely on the fact that one can separate the contributions of different modes to a scattering amplitude using the method of regions [51]. The effective Lagrangian then describes the dynamics of modes that form only a part of these regions, while the contributions due to other regions (usually the hard one) must arise from additional objects that are fixed through a matching computation. A first step in this direction is to investigate if the method of regions holds for correlators in deSitter space, and if the full-theory result is completely reproduced by the sum of super- and sub-horizon contributions.

As a simple example, we consider a self-interacting $\lambda\phi^4$ theory, and more specifically the late-time limit $aH \to \infty$ of the tree-level trispectrum $\langle\phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\phi(\boldsymbol{k}_4)\rangle$, which can be computed as

$$\langle\phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\phi(\boldsymbol{k}_4)\rangle = i\frac{\lambda}{4!}\int^t dt_1\, d^3x\, a^3(t)\left\langle\left[\phi^4(\boldsymbol{x},t_1),\phi(\boldsymbol{k}_1)\ldots\phi(\boldsymbol{k}_4)\right]\right\rangle, \tag{8.3.1}$$

where all fields $\phi(\boldsymbol{k})$ are evaluated at time t and we have already split off the momentum-conserving δ-function $(2\pi)^3\delta^{(3)}(\sum \boldsymbol{k}_i)$ from the correlator.

For the tree-level computation, one can directly employ the massless mode functions using (8.1.20) or (8.1.21), which take the form

$$\phi(\boldsymbol{k},\eta) = \frac{H}{\sqrt{2k^3}}(1-ik\eta)e^{ik\eta}\,. \tag{8.3.2}$$

The trispectrum is then given by

$$\langle\phi_1\ldots\phi_4\rangle = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}\int^\eta \frac{d\eta'}{(\eta')^4}\mathrm{Im}\left[e^{-ik_t(\eta-\eta')}\prod_{j=1}^4(1+ik_j\eta)(1-ik_j\eta')\right], \tag{8.3.3}$$

and is readily computed to be [94]

$$\begin{aligned}\langle\ldots\rangle = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}&\left[\frac{1}{3}\left(\sum_i k_i^3\right)\left(\log\frac{k_t}{aH}+\gamma_E+\frac{1}{3}-2\right)-\frac{k_1k_2k_3k_4}{k_t}-\frac{1}{9}k_t^3\right.\\ &\left.+2\sum_{i<j<l}k_ik_jk_l+\frac{1}{3}k_t\left(\sum k_i^2-\sum_{i<j}k_ik_j\right)\right],\end{aligned} \tag{8.3.4}$$

where we defined the total momentum $k_t = k_1+k_2+k_3+k_4$, used $\eta = -\frac{1}{aH}$ and dropped terms that vanish in the soft limit $aH \to \infty$.

We now want to reproduce this result using the method of regions [51] applied to the time integral. The basic procedure works as follows: we split the time integral $\int^\eta d\eta'$ by introducing a cut-off Λ as

$$\int_\infty^\eta d\eta' f(\eta',\eta) = \int_\infty^\Lambda d\eta_1 f(\eta',\eta) + \int_\Lambda^\eta d\eta' f(\eta',\eta)\,. \tag{8.3.5}$$

The first integral corresponds to the "hard" region, where $k|\eta'| \sim 1$ and should arise due to the sub-horizon modes. The second integral lies in the "soft" region and should contain the contributions of the super-horizon modes. Therefore, we now perform the respective expansions of the integrands. Since we take the soft limit of the external momenta, the function $f(\eta',\eta)$ should be expanded in the soft limit $k\,|\eta| \to 0$ for both integrals, and we keep only the leading terms.

In the first integral, one should use the asymptotic behaviour $f(\eta',\eta)$ as $k|\eta'| \to \infty$, which corresponds to the sub-horizon modes. In the second integral, η' is soft, so one performs the same soft expansion of $f(\eta',\eta)$ as for η previously. If the integrands are then homogeneous, one can drop the explicit cut-off Λ and integrate over the entire range by introducing an analytic regulator. This procedure of introducing an analytic regulator instead of an explicit Λ is justified since adding the missing range of integration corresponds to a scaleless integral, which vanishes when regulating analytically. However, now the previously finite integrals become divergent. In the first integral, this leads to an infrared divergence, while the second integral will become ultraviolet divergent. In the final result – the sum of both terms – these divergences must cancel out and, if no other regions are present, one should recover the full result (8.3.4). We now verify this explicitly.

Hard Region

First, we consider the hard region of the trispectrum (8.3.3). We take the late-time limit for the external variable η, as $k_j\eta \to 0$ for all k, and expand. For the η' integration, we are interested in the region $k_i\eta' \sim 1$, effectively setting $\eta \to 0$ for the integration limit. This introduces a divergence as $\eta' \to 0$, which we regulate using an *analytic regulator* α in the η'-integral. Namely, we replace

$$\int^{\eta} d\eta'(-\eta')^{-4} \to \int^{0} d\eta'(-\eta')^{-4}(-\eta'\tilde{\mu})^{2\alpha}\,, \tag{8.3.6}$$

where we introduced the factorisation scale $\tilde{\mu} = \mu e^{\gamma_E}$. The correlator (8.3.3) is then modified as

$$\langle\phi_1\dots\phi_4\rangle|_{\text{hard}} = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}\tilde{\mu}^{2\alpha}\int^{0}\frac{d\eta'}{(-\eta')^{4-2\alpha}}\text{Im}\left[e^{-ik_t\eta}e^{ik_t\eta'}\prod_{j=1}^{4}(1+ik_j\eta)(1-ik_j\eta')\right]. \tag{8.3.7}$$

The products inside the integral are evaluated as

$$\prod_j(1-ik_j\eta') = 1 - i\eta' k_t - (\eta')^2 a + i(\eta')^3 b + (\eta')^4 c\,, \tag{8.3.8}$$

$$\prod_j(1+ik_j\eta) = 1 + ik_t\eta - \eta^2 a - i\eta^3 b + \eta^4 c\,, \tag{8.3.9}$$

where we introduced

$$a = \sum_{i<j} k_ik_j\,, \quad b = \sum_{i<j<l} k_ik_jk_l\,, \quad c = k_1k_2k_3k_4\,. \tag{8.3.10}$$

Keeping only the leading order for $\eta \to 0$, and evaluating the time-integral using

$$\int^{0} dx x^{-a}e^{ikx} = -(-ik)^{a-1}\Gamma(1-a)\,, \tag{8.3.11}$$

one finds

$$\begin{aligned} I &= \tilde{\mu}^{2\alpha}\int^{0} d\eta'(\eta')^{-4+2\alpha}e^{ik_t\eta'}\left(1 - i\eta' k_t - (\eta')^2 a + i(\eta')^3 b + (\eta')^4 c\right) \\ &= -\tilde{\mu}^{2\alpha}\bigg((-ik_t)^{3-2\alpha}\left(\Gamma(-3+2\alpha) + \Gamma(-2+2\alpha)\right) - a(-ik_t)^{1-2\alpha}\Gamma(-1+2\alpha) \\ &\quad + ib(-ik_t)^{-2\alpha}\Gamma(2\alpha) + \frac{ic}{k_t}\bigg)\,. \end{aligned} \tag{8.3.12}$$

Expanding the Γ-functions in I for $\alpha \to 0$ yields

$$\begin{aligned} I = -\tilde{\mu}^{2\alpha}\bigg[& ik_t^3(1-2\alpha\ln k_t)\left(\frac{1}{6\alpha} - \frac{1}{3}\gamma_E + \frac{4}{9}\right) + iak_t(1-2\alpha\ln k_t)\left(-\frac{1}{2\alpha} + \gamma_E - 1\right) \\ & + ib(1-2\alpha\ln k_t)\left(\frac{1}{2\alpha} - \gamma_E\right) + \frac{ic}{k_t}\bigg] + \mathcal{O}(\alpha)\,. \end{aligned} \tag{8.3.13}$$

Since only the imaginary part enters the result, this simplifies to

$$\text{Im}I = \tilde{\mu}^{2\alpha}\left(\frac{1}{3}k_t^3 - ak_t + b\right)\left(-\frac{1}{2\alpha} + \ln k_t + \gamma_E\right) - \left(\frac{4}{9}k_t^3 - ak_t + \frac{c}{k_t}\right). \tag{8.3.14}$$

Inserting $\tilde{\mu}$ and performing the α-expansion, one finds the scale μ in the logarithm while any factors of γ_E are eliminated. This results in

$$\text{Im}I = \left(\frac{1}{3}k_t^3 - ak_t + b\right)\left(-\frac{1}{2\alpha} + \ln\frac{k_t}{\mu}\right) - \left(\frac{4}{9}k_t^3 - ak_t + \frac{c}{k_t}\right). \tag{8.3.15}$$

Next, use

$$\frac{1}{3}k_t^3 - ak_t + b = \frac{1}{3}k_t^3 - k_t\sum_{i<j}k_ik_j + \sum_{i<j<l}k_ik_jk_l = \frac{1}{3}\sum_i k_i^3\,, \tag{8.3.16}$$

to rewrite ImI as

$$\mathrm{Im}I = \left(-\frac{1}{2\alpha} + \ln\frac{k_t}{\mu}\right)\left(\sum_i k_i^3\right) - \frac{1}{9}k_t^3 - \frac{1}{3}k_t^3 + ak_t - \frac{c}{k_t}\,. \tag{8.3.17}$$

One can add and subtract $\frac{2}{3}\sum_i k_i^3$ to re-arrange the finite parts as

$$-\frac{1}{3}k_t^3 + ak_t + \frac{2}{3}\sum_i k_i^3 = \sum_i k_i^2 - \frac{1}{3}ak_t + 2b\,, \tag{8.3.18}$$

where we used the relation

$$k_t^3 - 2ak_t = k_t\sum_i k_i^2\,. \tag{8.3.19}$$

Finally, one obtains

$$\begin{aligned}\mathrm{Im}I = &\left(\frac{1}{3}\sum_i k_i^3\right)\left(-\frac{1}{2\alpha} + \ln\frac{k_t}{\mu} - 2\right) - \frac{1}{9}k_t^3 + \frac{1}{3}k_t\left(\sum_i k_i^2 - \sum_{i<j}k_ik_j\right)\\ &+ 2\sum_{i<j<l}k_ik_jk_l - \frac{k_1k_2k_3k_4}{k_t}\,.\end{aligned} \tag{8.3.20}$$

Keeping track of the prefactors, the contribution of the hard region to the trispectrum is

$$\begin{aligned}\langle\phi_1\dots\phi_4\rangle|_{\text{hard}} = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}&\left[\left(\frac{1}{3}\sum_i k_i^3\right)\left(-\frac{1}{2\alpha} + \ln\frac{k_t}{\mu} - 2\right)\right.\\ &\left.-\frac{1}{9}k_t^3 + \frac{1}{3}k_t\left(\sum_i k_i^2 - \sum_{i<j}k_ik_j\right) + 2\sum_{i<j<l}k_ik_jk_l - \frac{k_1k_2k_3k_4}{k_t}\right].\end{aligned} \tag{8.3.21}$$

Soft Region

Next, we compute the soft region of the correlator (8.3.3). Again we start from the integral

$$I = \int^\eta \frac{d\eta'}{(\eta')^4}\mathrm{Im}\left[e^{-ik_t(\eta-\eta')}\prod_{j=1}^4(1+ik_j\eta)(1-ik_j\eta')\right]\,, \tag{8.3.22}$$

but this time we take both η and η' to be soft, i.e. we directly expand the integrand for $k\eta \to 0$ as well as $k\eta' \to 0$. This time, the integral will become UV divergent as $|\eta| \to \infty$, and must again be regularised. We choose the same analytic regulator (8.3.6) as for the hard region. The integral then reads

$$I = \int^\eta d\eta'\,(-\eta')^{-4+2\alpha}(\tilde\mu)^{2\alpha}\,\mathrm{Im}\left[-\frac{i}{3}\sum_i k_i^3(\eta^3 - \eta'^3) + \mathcal{O}((k_i\eta)^4)\right]\,, \tag{8.3.23}$$

and is readily computed as

$$\begin{aligned}I &= -\frac{1}{3}\sum_i k_i^3\left(\frac{(-\eta\tilde\mu)^{2\alpha}}{2\alpha-3} - \frac{(-\eta\tilde\mu)^{2\alpha}}{2\alpha}\right) + \mathcal{O}(k\eta)\\ &= \frac{1}{3}\sum_i k_i^3\left(\frac{1}{2\alpha} + \ln\frac{\mu}{aH} + \gamma_E + \frac{1}{3}\right) + \mathcal{O}(\alpha, k\eta)\,.\end{aligned} \tag{8.3.24}$$

The leading-order soft contribution to the trispectrum then amounts to

$$\langle\phi_1\dots\phi_4\rangle|_{\text{soft}} = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}\left[\left(\frac{1}{3}\sum_i k_i^3\right)\left(\frac{1}{2\alpha} + \ln\frac{\mu}{aH} + \gamma_E + \frac{1}{3}\right)\right]. \tag{8.3.25}$$

Sum of Hard and Soft Regions

We now sum both regions to verify that we reproduce the full result. Indeed, taking the sum of (8.3.21) and (8.3.25), one immediately notices that the poles in α as well as the dependence on the factorisation scale μ precisely cancel. The sum of both terms reads

$$\begin{aligned}
&\langle\phi(\boldsymbol{k}_1)\ldots\phi(\boldsymbol{k}_4)\rangle|_{\text{hard}} + \langle\phi(\boldsymbol{k}_1)\ldots\phi(\boldsymbol{k}_4)\rangle|_{\text{soft}} \\
&\quad= \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}\left[\frac{1}{3}\left(\sum_i k_i^3\right)\left(\log\frac{k_t}{aH} + \gamma_E + \frac{1}{3} - 2\right) - \frac{k_1k_2k_3k_4}{k_t} - \frac{1}{9}k_t^3\right. \\
&\qquad\left. + 2\sum_{i<j<l} k_ik_jk_l + \frac{1}{3}k_t\left(\sum k_i^2 - \sum_{i<j} k_ik_j\right)\right], \qquad (8.3.26)
\end{aligned}$$

which precisely agrees with (8.3.4). As we can see, we completely recover the full-theory result, and all divergences cancel out. This computation verifies that the method of regions works for the time-integrals in the computation of correlation functions in deSitter space. While it is not checked explicitly in this work, it also seems to be consistent in multi-loop integrals including momentum integrals [101].

8.4 Soft deSitter Effective Theory

Now that we have explicitly verified that the method of regions can be applied for correlation functions in deSitter space, we can begin the effective theory derivation and present the framework of Soft deSitter effective theory [94, 95]. The idea is to construct an effective theory that contains only the long-wavelength super-horizon modes by integrating out the sub-horizon modes. While the idea seems straightforward, we will see that this theory differs from an ordinary flat-space EFT in some aspects. The more rigorous derivation presented in the following differs from the original one in [94] and is part of a project by the author in collaboration with M. Beneke and A. F. Sanfilippo that is still in active development [105].

8.4.1 Soft Modes

As we have discussed in Section 8.1, the long-wavelength part of the scalar field is described by two modes, the growing and decaying modes. They are characterised by the parameter ν in (8.1.14), and the super-horizon field can be decomposed as[1] [94]

$$\phi_s(\mathtt{t},\boldsymbol{x}) = H\left[(a(\mathtt{t})H)^{-\frac{3}{2}+\nu}\,\varphi_+(\mathtt{t},\boldsymbol{x}) + (a(\mathtt{t})H)^{-\frac{3}{2}-\nu}\,\varphi_-(\mathtt{t},\boldsymbol{x})\right]. \qquad (8.4.1)$$

As hard scale we take the time-dependent cut-off $\Lambda(\mathtt{t}) = a(\mathtt{t})H$, and the power-counting parameter is the momentum compared to this scale

$$\lambda \sim \frac{k}{aH}. \qquad (8.4.2)$$

The full-theory Lagrangian is the minimally-coupled scalar field (8.1.4) with a four-point interaction[2]

$$S[\phi] = \int d^3x \int d\mathtt{t}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\kappa}{4!}\phi^4\right). \qquad (8.4.3)$$

Naively, the effective theory is now obtained by inserting the split

$$\phi = \phi_s + \phi_h \qquad (8.4.4)$$

[1] In the following we define the time-parameter as $\mathtt{t}' \equiv Ht$, absorbing the factor H.

[2] We now denote the coupling constant by κ instead of λ in order to not confuse it with the power-counting parameter.

in the Lagrangian, and integrating out the hard mode ϕ_h. However, note that from the soft limit of the equation of motion (8.1.15), the leading-order equations for $\varphi_\pm$ read

$$\dot{\varphi}_\pm = 0\,. \tag{8.4.5}$$

This means that the effective theory is described by a Lagrangian that contains *first-order* time derivatives, compared to the second-order Lagrangian (8.4.3). In order to derive the correct effective theory, we first have to understand this transition from a second-derivative theory to a first-derivative theory. A similar transition takes place in the effective description of a non-relativistic scalar field [106–108]. We consider this situation first.

8.4.2 Intuition from Non-relativistic Effective Theory

As it turns out, the effective theory describing soft modes of a scalar field in deSitter space-time shares many formal similarities with the effective theory of a non-relativistic scalar field. Intuitively, one can see this from the equation of motion (8.1.5) for the redefined field χ, which looks like the one of a flat-space scalar field with time-dependent mass term $m_{\text{eff}}(\eta)$ (8.1.8). For the soft mode, the dynamics are heavily constrained by this mass term.

Therefore, we first consider the non-relativistic scalar field with *constant* mass term, then generalise the construction to deSitter space. The most important step in this construction is that we want to go from a second-order equation of motion, that contains *second* time derivatives, to a first-order one, where only a *single* time-derivative is present. This transition contains some subtleties and the construction is more transparent in the non-relativistic derivation.

In the following, we go over a systematic construction for the non-relativistic effective action of a massive scalar field in Minkowski space-time. The discussion uses input from [106–108].

We consider a free real scalar field ϕ with action given by

$$S_\phi = \int \mathrm{d}^4x\, \frac{1}{2}\partial_\mu\phi\,,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\,. \tag{8.4.6}$$

The free theory has the equation of motion

$$(\Box + m^2)\phi(t,\boldsymbol{x}) = 0 \Rightarrow \ddot{\tilde{\phi}}(t,\boldsymbol{p}) + (p^2+m^2)\tilde{\phi}(t,\boldsymbol{p}) = 0\,, \tag{8.4.7}$$

which is solved in momentum-space by plane waves of the form

$$\tilde{\phi}(t,\boldsymbol{p}) \sim e^{ipx}\,, \quad p^2 = m^2\,. \tag{8.4.8}$$

Non-relativistic Limit $k \to 0$ and Ambiguities

We consider the non-relativistic limit, i.e. momenta $p^\mu \sim mv^\mu + k^\mu$, where $k \ll m$ and $v^2 = 1$, e.g. $v^\mu = \delta^\mu_0$. In this limit, the plane-wave is dominated by modes $e^{\pm imt}$, and we make the ansatz

$$\phi(x,t) = e^{-imt}\varphi(x,t) + e^{imt}\varphi^*(x,t)\,, \tag{8.4.9}$$

where φ^* is complex-conjugate to φ. The field φ should describe a slowly-varying mode, and obey a first-order differential equation. However, the description contains too many degrees of freedom, since we employ this $1 \to 2$ parameterisation $\phi \to (\varphi,\varphi^*)$. This manifests itself in the form of a gauge symmetry

$$\varphi(x,t) \to \varphi(x,t) + ie^{imt}\psi(x,t)\,, \tag{8.4.10}$$

with a real scalar field $\psi(x,t)$, where we find

$$\phi(x,t) = e^{-imt}\varphi + e^{imt}\varphi^*$$

$$\begin{aligned}
&\to e^{-imt}(\varphi + ie^{imt}\psi) + e^{imt}(\varphi^* - ie^{-imt}\psi^*) \\
&= \phi(x,t) + i(\psi - \psi^*) \\
&= \phi(x,t)\,,
\end{aligned} \tag{8.4.11}$$

since $\psi = \psi^*$. To fix this gauge symmetry, we need to impose a constraint. To find a suitable one, consider the equation of motion in the limit $k \to 0$, which reads

$$\ddot{\phi} + m^2\phi = 0\,. \tag{8.4.12}$$

Inserting the ansatz (8.4.9), we obtain

$$\begin{aligned}
\ddot{\phi} + m^2\phi &= e^{-imt}(\ddot{\varphi} - 2im\dot{\varphi} - m^2\varphi) + e^{imt}(\ddot{\varphi}^* + 2im\dot{\varphi}^* - m^2\varphi^*) + m^2(e^{-imt}\varphi + e^{imt}\varphi^*) \\
&= e^{-imt}(\ddot{\varphi} - 2im\dot{\varphi}) + e^{imt}(\ddot{\varphi}^* + 2im\dot{\varphi}^*)\,,
\end{aligned} \tag{8.4.13}$$

One possible constraint that gives first-order differential equations is to impose

$$e^{imt}\dot{\varphi} + e^{-imt}\dot{\varphi}^* = 0\,, \tag{8.4.14}$$

as this yields

$$\begin{aligned}
\dot{\phi} &= -im(e^{-imt}\varphi - e^{imt}\varphi^*)\,, \\
\ddot{\phi} &= -im(e^{-imt}\dot{\varphi} - e^{imt}\dot{\varphi}^*) - m^2(e^{-imt}\varphi + e^{imt}\varphi)\,,
\end{aligned} \tag{8.4.15}$$

and the equation of motion becomes

$$\ddot{\phi} + m^2\phi = -im(e^{-imt}\dot{\varphi} - e^{imt}\dot{\varphi}) = 0\,, \tag{8.4.16}$$

or, using the condition again,

$$\dot{\varphi} = 0 + \mathcal{O}\left(\frac{1}{m}\right)\,, \tag{8.4.17}$$

as we would expect for the leading-order behaviour if the time-dependence is really dominated by e^{-imt}. We see that there is a choice of fixing the gauge symmetry that leads to a first-order differential equation for φ and gives the correct leading behaviour in the limit $k \to 0$.

Finite k and Rapidly-oscillating Modes

Next, we consider non-zero k, where the equation (8.4.12) takes the form

$$\ddot{\phi} - \partial^2\phi + m^2\phi = 0\,, \tag{8.4.18}$$

or, inserting (8.4.9) and the constraint (8.4.14),

$$-ime^{-imt}\dot{\varphi} + ime^{imt}\dot{\varphi}^* - \partial^2(e^{-imt}\varphi + e^{imt}\varphi^*) = 0\,. \tag{8.4.19}$$

We can use the constraint again to eliminate the $\dot{\varphi}^*$ term to get

$$-i\dot{\varphi} - \frac{\partial^2}{2m}\varphi - \frac{\partial^2}{2m}e^{2imt}\varphi^* = 0\,. \tag{8.4.20}$$

Ignoring the rapidly-oscillating term $\sim e^{2imt}$, we recover the leading-order Schrödinger equation for φ. However, we want to remove these types of terms systematically.

To do this, we perform an expansion in Fourier modes

$$\varphi = \sum_n e^{nimt}\varphi_n\,, \tag{8.4.21}$$

$$\varphi^* = \sum_n e^{-nimt}\varphi_n^* \,, \tag{8.4.22}$$

insert the expansion in (8.4.20) and contract it with a basis function $\langle e_n|$ from the left, extracting the n-th mode, by defining it such that

$$\langle e_n|\,\varphi = \varphi_n \,, \tag{8.4.23}$$

e.g. via using orthonormality of the Fourier modes with respect to the L^2 inner product.

This leads us to the mode equation for φ^*

$$\begin{aligned} 0 &= i(-inm\varphi_n^* + \dot{\varphi}_n^*) - \frac{\partial^2}{2m}\varphi_n^* - \frac{\partial^2}{2m}\varphi_{n-2} \\ &= -nm\varphi_n^* + \dot{\varphi}_n^* - \frac{\partial^2}{2m}\varphi_n^* - \frac{\partial^2}{2m}\varphi_{n-2} \,. \end{aligned} \tag{8.4.24}$$

Power-counting

At this point, we have to discuss the power-counting. From the non-relativistic expansion, we have the small 3-momenta $\boldsymbol{k}$ compared to the mass scale, so we have

$$\lambda = \frac{\partial^2\varphi}{m^2} \ll 1 \,. \tag{8.4.25}$$

Next, the modes φ are assumed to be slowly varying in time, so we have additionally

$$\varepsilon = \frac{\dot{\varphi}}{m\varphi} \ll 1 \,. \tag{8.4.26}$$

We now expand all modes in these two parameters, in the sense

$$\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)} + \dots \,, \quad n \neq 0 \,, \tag{8.4.27}$$

where the expansion starts at $\mathcal{O}(\lambda, \varepsilon)$, since from the leading equation of motion

$$\dot{\varphi} = 0 \,, \tag{8.4.28}$$

we find $\varphi^{(0)} = \varphi_0^{(0)}$, as it is constant. This now allows us to expand the modes φ_n, and use (8.4.24) to eliminate φ_n systematically to only keep φ_0 in the equation of motion.

Eliminating the Non-zero Modes

We begin with (8.4.24)

$$0 = -nm\varphi_n^* + \dot{\varphi}_n^* - \frac{\partial^2}{2m}\varphi_n^* - \frac{\partial^2}{2m}\varphi_{n-2} \,. \tag{8.4.29}$$

At $\mathcal{O}(\lambda^0, \varepsilon^0)$, only φ_0 is non-vanishing, so the first correction is due to the $n = 2$ mode, where we have

$$\begin{aligned} 0 &= -2m\varphi_2^{(1)*} + \dot{\varphi}_2^{(1)*} - \frac{\partial^2}{2m}\varphi_2^{(1)*} - \frac{\partial^2}{2m}\varphi_0^{(0)} \\ &= -2m\varphi_2^{(1)*} - \frac{\partial^2}{2m}\varphi_0^{(0)} + \mathcal{O}(\varepsilon^2, \lambda^2) \,, \end{aligned} \tag{8.4.30}$$

and we find

$$\varphi_2^{(1)*} = -\frac{\partial^2}{4m^2}\varphi_0^{(0)} \,. \tag{8.4.31}$$

We now insert this in (8.4.24) for $n = 0$, which reads

$$\begin{aligned} 0 &= -i\dot{\varphi}_0 - \frac{\partial^2}{2m}\varphi_0 - \frac{\partial^2}{2m}\varphi_2^{(1)*} \\ &= -i\dot{\varphi}_0 - \frac{\partial^2}{2m}\varphi_0 + \frac{\partial^4}{8m^3}\varphi_0 + \mathcal{O}(\varepsilon^2, \lambda^2)\,, \end{aligned} \tag{8.4.32}$$

and we see that we recover the first relativistic correction of the kinetic energy. We can use this approach to systematically compute the full non-relativistic equation of motion. Since the theory is first-order in the time-derivative, we can then find the Lagrangian from the equation of motion

$$-i\dot{\varphi}_0 = D\varphi \tag{8.4.33}$$

by introducing the conjugate momentum φ^* and writing

$$\mathcal{L} = \varphi^*(i\partial_t - D)\varphi\,. \tag{8.4.34}$$

In this theory, as is usual for a first-derivative theory, we now obtain the equation of motion for φ by varying the action with respect to φ^*.

Let us summarise the key insights of this derivation. We factorised the leading-order time-dependence of the effective mode (8.4.9), and expressed the real scalar field ϕ in terms of two variables φ, φ^*. This description now features one additional scalar degree of freedom, which introduces an additional gauge symmetry. We employ (8.4.14) to fix it. In turn, the equation of motion now becomes linear in the time-derivative. Using the Fourier decomposition, we managed to systematically eliminate the rapidly-oscillating terms and recovered the full non-relativistic expansion of the scalar equation of motion. Finally, the Lagrangian is obtained by multiplying the equation of motion with the conjugate momentum φ^*.

8.4.3 Naive Derivation of the Soft deSitter Lagrangian

The previous derivation, while correct, made heavy use of concepts like “rapidly-oscillating terms” that must be systematically eliminated using equations of motion. While this construction indeed works perfectly fine for the case of a non-relativistic scalar field, the deSitter scenario is problematic, as we explain in the following.

In deSitter space, for massive fields with large mass, we have an additional expansion parameter $\varepsilon_H = \frac{H}{m}$, and the discussion is similar to the non-relativistic case. However, we are interested in the case of *light* masses, where $\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$ is no longer complex but a real number. This means that the mode functions are not plane-wave exponentials e^{imt}, but rather decaying solutions $e^{-\alpha t}$. In this case, the IR behaviour is dominated by the growing and decaying modes, and we introduce the analogue of Eq. (8.4.9) as

$$\phi(x,t) = H\left((aH)^{-\alpha}\varphi_+ + (aH)^{-\beta}\varphi_-\right)\,, \tag{8.4.35}$$

where $\alpha = \frac{3}{2} - \nu$ and $\beta = \frac{3}{2} + \nu$. We again have a $1 \to 2$ parameterisation, and we fix the gauge by imposing the analogue of (8.4.14)

$$(aH)^{-\alpha}\dot{\varphi}_+ + (aH)^{-\beta}\dot{\varphi}_- = 0\,. \tag{8.4.36}$$

This results in the time-derivatives

$$\begin{aligned} \dot{\phi} &= H\left(-\alpha(aH)^{-\alpha}\varphi_+ - \beta(aH)^{-\beta}\varphi_-\right)\,, \\ \ddot{\phi} &= H\left(\alpha^2(aH)^{-\alpha}\varphi_+ - \alpha(aH)^{-\alpha}\dot{\varphi}_+ + \beta^2(aH)^{-\beta}\varphi_- - \beta(aH)^{-\beta}\dot{\varphi}_-\right) \end{aligned}$$

$$= H\left(2\nu(aH)^{-\alpha}\dot{\varphi}_+ + \alpha^2(aH)^{-\alpha}\varphi_+ + \beta^2(aH)^{-\beta}\varphi_-\,,\right) \tag{8.4.37}$$

and plugging these into the equation of motion for ϕ for $k = 0$,

$$\ddot{\phi} + 3\dot{\phi} + \frac{m^2}{H^2}\,, \tag{8.4.38}$$

we find

$$\begin{aligned} 0 &= H\left(2\nu(aH)^{-\alpha}\dot{\varphi}_+ + \alpha^2(aH)^{-\alpha}\varphi_+ + \beta^2(aH)^{-\beta}\varphi_-\right) \\ &\quad + 3H\left(-\alpha(aH)^{-\alpha}\varphi_+ - \beta(aH)^{-\beta}\varphi_-\right) + \frac{m^2}{H^2}H\left((aH)^{-\alpha}\varphi_+ + (aH)^{-\beta}\varphi_-\right) \\ &= 2\nu\dot{\varphi}_+\,, \end{aligned} \tag{8.4.39}$$

where we used $\alpha^2 - 3\alpha + \frac{m^2}{H^2} = 0$ and $\beta^2 - 3\beta + \frac{m^2}{H^2} = 0$. This shows that for $k = 0$, we capture the correct leading behaviour.

For finite momenta, the equations of motion read

$$0 = 2\nu\dot{\varphi}_+ - \frac{\partial^2}{(aH)^2}\varphi_+ - \frac{\partial^2}{(aH)^2}(aH)^{-2\nu}\varphi_-\,, \tag{8.4.40}$$

$$0 = -2\nu\dot{\varphi}_- - \frac{\partial^2}{(aH)^2}\varphi_- - \frac{\partial^2}{(aH)^2}(aH)^{2\nu}\varphi_+\,, \tag{8.4.41}$$

and we see a similar mode mixing as in the non-relativistic case (8.4.20), but this time it appears as $(aH)^{-2\nu}$. Note that $(aH)^{-2\nu} \sim e^{-\nu t}$, so it is equivalent to having a complex frequency νt instead of imt in the exponent. The complex analogue of the Fourier transformation is the Laplace transformation, where the functions are expanded in terms of exponentials as

$$f = \sum_n e^{-n\nu t} f_n\,. \tag{8.4.42}$$

We choose a modification of this, where we expand the field as

$$\varphi_+ = \sum_n (aH)^{-\nu n}\varphi_{+,n}\,, \tag{8.4.43}$$

$$\varphi_- = \sum_n (aH)^{\nu n}\varphi_{-,n}\,. \tag{8.4.44}$$

While these exponentials are also dense and form a basis of $L^2[0,\infty]$, they are not orthogonal. However, we can still construct bras and a suitable inner product $\langle e_n|$ to obtain

$$\langle e_n|\,\varphi_+ = \varphi_{+,n}\,, \tag{8.4.45}$$

e.g. by using complex integration and the inverse Laplace transform. As these $\langle e_n|$ are just a tool to help with the construction, no one forces us to use standard inner products, and we can for now argue that this is a good mode expansion for our fields.

If we accept this expansion in modes with complex frequencies, we find similar behaviour to the non-relativistic case. The leading contribution stems from $n = 0$ modes, and from the equation for φ_- we find for mode n

$$0 = -2\nu(n\nu\varphi_{-,n} + \dot{\varphi}_{-,n}) - \frac{\partial^2}{(aH)^2}\varphi_{-,n} - \frac{\partial^2}{(aH)^2}\varphi_{+,n-2}\,. \tag{8.4.46}$$

However, here we distinguish *by hand* that $(aH)^{-2}$ is part of the gradient ∂_i^2 to form a deSitter-covariant expression, and *not* part of the mode $\varphi_{-,n}$. Strictly speaking, this is not justified at this point. However, we recover the correct equation of motion if we use this trick. Therefore, this approach is *not* rigorous and should only be viewed as a "back of the envelope" derivation.

With this out of the way, let us proceed like in the non-relativistic case, by assuming that we have small parameters

$$\lambda = \frac{\partial^2}{(aH)^2}\,, \quad \varepsilon = \frac{\dot{\varphi}}{\varphi}\,, \tag{8.4.47}$$

for small momenta and slowly varying fields φ. We can then expand the modes

$$\varphi_n = \varphi_n^{(1)} + \varphi_n^{(2)} + \dots\,, \tag{8.4.48}$$

where only φ_0 contains the constant $\mathcal{O}(1)$ contribution. We can proceed to systematically remove the non-zero modes by using their equations of motion.

We use the equation for φ_- for $n = 2$, where we find

$$\begin{aligned} 0 &= -2\nu(2\nu\varphi_{-,2} + \dot{\varphi}_{-,2}) - \frac{\partial^2}{(aH)^2}\varphi_{-,2} - \frac{\partial^2}{(aH)^2}\varphi_{+,0} \\ 4\nu^2\varphi_{-,2} &= -\frac{\partial^2}{(aH)^2}\varphi_{+,0}\,. \end{aligned} \tag{8.4.49}$$

Inserting this in the equation for $\varphi_{+,0}$ yields

$$\begin{aligned} 0 &= 2\nu\dot{\varphi}_{+,0} - \frac{\partial^2}{(aH)^2}\varphi_{+,0} - \frac{\partial^2}{(aH)^2}\varphi_{-,2} \\ &= 2\nu\dot{\varphi}_{+,0} - \frac{\partial^2}{(aH)^2}\varphi_{+,0} + \frac{\partial^4}{4\nu^2(aH)^4}\varphi_{+,0} + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{8.4.50}$$

which agrees with the equation of motion obtained in [94].

Again, we can use this to construct the Lagrangian as

$$\mathcal{L} = \varphi_-\left(-2\nu\dot{\varphi}_+ + \frac{\partial^2}{(aH)^2}\varphi_+ - \frac{\partial^4}{4\nu^2(aH)^4}\varphi_+\right) + \mathcal{O}(\lambda^5)\,, \tag{8.4.51}$$

which also agrees with [94]. Here, we introduced the conjugate momentum φ_-, and the equation of motion for φ_+ is obtained by variation with respect to φ_-. Note, however, that the modes φ_+ and φ_- *differ* in their effective power-counting, as we explain later.

Let us stress that this derivation heavily relied on two assumptions that cannot be justified. First, one had to assume that such expansion in complex modes is sensible, and second, one has to pick out certain modes by hand, in order to find the correct scale-factors in front of the derivatives. Therefore, we wish to find a more systematic and rigorous construction.

8.4.4 A Rigorous Derivation

As we have seen, the hand-wavy derivation of the previous section reproduces the Lagrangian encountered in [94], using the same methods that yield the Lagrangian for a non-relativistic scalar field. However, in this approach, the analogue of the rapidly-oscillating modes had to be identified by hand and thus this should not be viewed as a rigorous derivation. In the following section, we summarise how this construction can be made rigorous using the canonical Hamiltonian formalism and non-local field redefinitions, following [101, 106].

The Non-relativistic Scalar Field

For the non-relativistic scalar field, the main issue is to consistently go from a single scalar field ϕ to two fields φ, φ^*. This introduces a gauge symmetry in the naive derivation (8.4.10), that must be fixed by imposing some constraint (8.4.14). However, in the end, we found that φ^* takes the role of the conjugate momentum of the field φ. Therefore, in the canonical approach, one considers the change of variables $\phi, \pi \to \varphi, \varphi^*$, where one explicitly includes the conjugate momentum π. One begins with the ansatz

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2m}} \left(e^{-imt} D(t) \varphi + e^{imt} D^*(t) \varphi^* \right) \\ \pi &= -i \sqrt{\frac{m}{2}} \left(e^{-imt} P(t) \varphi - e^{imt} P^*(t) \varphi^* \right) . \end{aligned} \tag{8.4.52}$$

The old Hamiltonian $H = \int d^3x \, \mathcal{H}$ expressed in terms of ϕ, π is linked to the new one, with variables φ, φ^* by a canonical transformation

$$\tilde{\mathcal{H}}[\varphi, \varphi^*] = \mathcal{H}[\varphi, \varphi^*] + \frac{\partial F}{\partial t} , \tag{8.4.53}$$

such that

$$\pi[\phi, \varphi] = \frac{\partial F}{\partial \phi} , \quad i\varphi^*[\phi, \varphi] = -\frac{\partial F}{\partial \varphi} . \tag{8.4.54}$$

One can now determine $F[\varphi, \phi, t]$ to be [101, 106]

$$F[\varphi, \phi, t] = i \left(\frac{m}{2} \phi \frac{P^*}{D^*} \phi - \sqrt{2m} e^{-imt} \phi \frac{1}{D^*} \varphi + \frac{1}{2} e^{-2imt} \varphi \frac{D}{D^*} \varphi \right) . \tag{8.4.55}$$

The new Hamiltonian is then given by

$$\begin{aligned} \tilde{\mathcal{H}} = \frac{m}{4} \Bigg(& e^{-2imt} \varphi D^2 \left[-\frac{i}{m} \frac{\partial}{\partial t} \left(\frac{P}{D} \right) - \left(\frac{P}{D} \right)^2 + 1 - \frac{\partial_i^2}{m^2} \right] \varphi \\ & + 2\varphi^* \left[P^* P + D^* D \left(1 - \frac{\partial_i^2}{m^2} \right) - 2 - \frac{i}{2m} \left(P^* \dot{D} - \dot{P}^* D + \dot{P} D^* - P \dot{D}^* \right) \right] \varphi \\ & + e^{2imt} \varphi^* D^{*2} \left[\frac{i}{m} \frac{\partial}{\partial t} \left(\frac{P^*}{D^*} \right) - \left(\frac{P^*}{D^*} \right)^2 + 1 - \frac{\partial_i^2}{m^2} \right] \varphi^* \Bigg) . \end{aligned} \tag{8.4.56}$$

This Hamilton should take the form $\tilde{H} = \varphi^* E \varphi$, where only mixed terms survive. This is equivalent to integrating out the rapidly-oscillating modes in the previous approach. The condition leads to a differential equation for the combination $f \equiv \frac{P}{D}$ that reads

$$-\frac{i}{m} \frac{\partial f}{\partial t} - f^2 + 1 - \frac{\partial_i^2}{m^2} = 0 , \tag{8.4.57}$$

which has the solution

$$f(t) = -i \sqrt{1 - \frac{\partial_i^2}{m^2}} \tan \left((mt + C) \sqrt{1 - \frac{\partial_i^2}{m^2}} \right) , \quad C = \text{const.} \tag{8.4.58}$$

This solution fixes the coefficients as

$$\frac{P}{D} = \frac{P^*}{D^*} = \sqrt{1 - \frac{\partial_i^2}{m^2}} . \tag{8.4.59}$$

It is possible for both P and D to have a residual time-dependence [101], but we do not entertain this idea here. Instead, we fix

$$P = \left[1 - \frac{\partial_i^2}{m^2}\right]^{\frac{1}{4}} \,, \quad D = \left[1 - \frac{\partial_i^2}{m^2}\right]^{-\frac{1}{4}} \,, \tag{8.4.60}$$

and the Hamiltonian becomes

$$\tilde{\mathcal{H}} = m\varphi^* \left[\sqrt{1 - \frac{\partial_i^2}{m^2}} - 1\right] \varphi \,. \tag{8.4.61}$$

Expanding this closed expression for $\partial_i^2 \ll m^2$ precisely yields the non-relativistic expansion that was computed before.

8.4.5 Application to deSitter Space

We now apply this canonical formalism to the light scalar field in deSitter space. The super-horizon modes φ_+, φ_- take the role of φ, φ^* in the non-relativistic derivation. We summarise the explicit and very lengthy derivation in [101].

We begin by specifying the scalar field and its conjugate momentum as

$$\begin{aligned} \phi &= H\left[(aH)^{-\alpha}\varphi_+ + (aH)^{-\beta}\varphi_-\right] \,, \\ \pi &= -a^3 H\left[\alpha(aH)^{-\alpha}\varphi_+ + \beta(aH)^{-\beta}\varphi_-\right] \,, \end{aligned} \tag{8.4.62}$$

which incorporates the constraint (8.4.36). By enforcing the equation of motion of the conjugate momentum

$$\dot{\pi} = a^3 H \left(\frac{\partial_i^2}{(aH)^2}\phi - \frac{m^2}{H^2}\phi\right) \,, \tag{8.4.63}$$

one finds

$$\dot{\varphi}_\pm = \pm\frac{1}{2\nu}\frac{\partial^2}{(aH)^2}\left(\varphi_\pm + (aH)^{\mp 2\nu}\varphi_\mp\right) \,, \tag{8.4.64}$$

where the mode-mixing is again present. We now wish to perform a canonical transformation that removes these mode-mixing terms. Like in the flat-space scenario, we achieve this by performing a non-local field redefinition. We begin with the more general definition

$$\begin{aligned} \phi &= H\left[(aH)^{-\alpha}D_+\varphi_+ + (aH)^{-\beta}D_-\varphi_-\right] \,, \\ \pi &= -a^3 H\left[\alpha(aH)^{-\alpha}P_+\varphi_+ + \beta(aH)^{-\beta}P_-\varphi_-\right] \,. \end{aligned} \tag{8.4.65}$$

This leads to a canonical transformation [101]

$$F[\phi, \varphi_+, t] = -\frac{\alpha^3\beta}{2}\phi\frac{P_-}{D_-}\phi + \frac{2\nu(aH)^\beta}{H^2}\phi\frac{1}{D_-}\varphi_+ - \frac{\nu(aH)^{2\nu}}{H}\varphi_+\frac{D_+}{D_-}\varphi_+ \,, \tag{8.4.66}$$

bearing strong resemblance to the flat-space counterpart (8.4.54). One can now compute the Hamiltonian expressed in terms of φ_+, φ_-, where one still needs to remove the additional "oscillating" terms to obtain a Hamiltonian of the form $\mathcal{H} = \varphi_+(\dots)\varphi_-$. This yields a differential equation for the combination $f_\pm \equiv -(\frac{3}{2} \mp \nu)\frac{P_\pm}{D_\pm}$, which reads

$$f_\pm^2 + 3f_\pm + \dot{f}_\pm - \left(\frac{\partial^2}{(aH)^2} + \nu^2 - \frac{9}{4}\right) = 0 \,. \tag{8.4.67}$$

This differential equation now has the lengthy solution [101]

$$f_{\pm} = -\frac{1}{2}\frac{C_{\pm}\Gamma(1-\nu)\left[3I_{-\nu}(X) + X\left(I_{-1-\nu}(X) + I_{1-\nu}(X)\right)\right]}{C_{\pm}\Gamma(1-\nu)I_{-\nu}(X) + i^{2\nu}\Gamma(1+\nu)I_{\nu}(X)} + \frac{i^{2\nu}\Gamma(1+\nu)\left[3I_{\nu}(X) + X\left(I_{\nu+1}(X) + I_{\nu-1}(X)\right)\right]}{C_{\pm}\Gamma(1-\nu)I_{-\nu}(X) + i^{2\nu}\Gamma(1+\nu)I_{\nu}(X)}\,, \tag{8.4.68}$$

where $X \equiv \sqrt{\frac{\partial^2}{(aH)^2}}$. Here, $I_\nu(z)$ is the modified Bessel function of the first kind

$$I_\nu(z) = i^{-\nu} J_\nu(iz)\,. \tag{8.4.69}$$

Like in the flat-space scenario, this combination does not uniquely fix $D_\pm$ and $P_\pm$, and there is still a residual freedom to add a function $g_\pm(X^2, t)$. We will not pursue this further and fix the function to yield the desired form of the Lagrangian density. After a cumbersome computation [101], one then arrives at

$$D_+(\nu) = {}_0F_1\left(1-\nu; \frac{X^2}{4}\right)\,, \quad D_-(\nu) = {}_0F_1\left(1+\nu; \frac{X^2}{4}\right)\,, \tag{8.4.70}$$

as well as

$$P_+(\nu) = {}_0F_1\left(-\nu; \frac{X^2}{4}\right) + \frac{\beta X^2}{4\alpha\nu(\nu-1)}\,{}_0F_1\left(2-\nu; \frac{X^2}{4}\right)\,,$$
$$P_-(\nu) = {}_0F_1\left(\nu; \frac{X^2}{4}\right) + \frac{\beta X^2}{4\alpha\nu(\nu+1)}\,{}_0F_1\left(2+\nu; \frac{X^2}{4}\right)\,, \tag{8.4.71}$$

expressed in terms of the confluent hypergeometric function

$${}_0F_1\left(a; z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!(a)_k}\,, \quad (a)_n = \prod_{k=0}^{n-1}(a-k)\,. \tag{8.4.72}$$

Using these redefinitions, the final Lagrangian takes the form

$$\mathcal{L} = -\nu(\dot{\varphi}_+\varphi_- - \varphi_+\dot{\varphi}_-) + 2\nu^2\varphi_+\left[1 - \sqrt{1 + \frac{\partial^2}{(\nu aH)^2}}\right]\varphi_-\,. \tag{8.4.73}$$

Performing the expansion for $\partial^2 \ll (aH)^2$ precisely reproduces the Lagrangian (8.4.51) of the light scalar field obtained in the naive derivation. This approach can be viewed as a rigorous derivation of the free Soft deSitter action.

8.4.6 The Soft deSitter Lagrangian

Let us summarise the result of the previous section: the free Lagrangian describing the super-horizon modes in a deSitter space-time takes the form (8.4.73). We can now expand this in the small parameter $\partial^2/(aH)^2$ to obtain the free Lagrangian and subleading gradient terms. At the leading order, we obtain the free action

$$S_{\text{free}} = \int d^3x\, d\mathtt{t}\; -\nu\left(\dot{\varphi}_+\varphi_- - \varphi_+\dot{\varphi}_-\right)\,. \tag{8.4.74}$$

From this action, one can now assign a power-counting to the modes as

$$\varphi_+ \sim \lambda^\alpha\,, \quad \varphi_- \sim \lambda^\beta\,. \tag{8.4.75}$$

In addition, we have to include the effect of the self-interaction terms. Since we integrate out the hard modes, we expect that their effect is realised by an infinite tower of interaction terms, organised by the power-counting parameter λ, and a generic term takes the form

$$S_{\text{int}} = \int d^3x\, d\mathtt{t}\ (aH)^{3-n\alpha-n\beta} \frac{c_{n,m}}{n!m!} \varphi_+^n \varphi_-^m\,, \tag{8.4.76}$$

with generic integer n, m. There is one subtlety in these interactions, concerning terms of the form

$$\mathcal{L}_{\text{int},+} = (aH)^{3-n\alpha}\, \varphi_+^n\,. \tag{8.4.77}$$

In the massless limit $\alpha \to 0$, these terms become *super-leading* as λ^0 compared to the kinetic term $\varphi_+\dot{\varphi}_- \sim \lambda^3$, and it seems that these terms spoil the power-counting. However, these terms are an artefact of our first-order formulation.

To be precise, note that the fields φ_+ and φ_- are conjugate variables in our formulation. This manifests itself in the commutation relations [94][3]

$$[\varphi_+(t, \boldsymbol{x}), \varphi_-(t, \boldsymbol{y})] = \frac{i}{2\nu}\delta(\boldsymbol{x} - \boldsymbol{y})\,. \tag{8.4.78}$$

Therefore, in any correlation function, the super-leading interactions φ_+^n can only have an effect if the correlator contains *at least one* φ_-. Then, these interactions precisely produce leading-power contributions. We will see their effect in action in Section 8.5, when we compute the trispectrum.

At the moment it is unclear how one can reformulate the Lagrangian or the field basis to absorb these super-leading interaction terms. In [94], a power-counting violating field redefinition was suggested. However, in later computations [95], these redefinitions had to be reversed to yield the correct results, so this approach does not seem promising.

Derivative Interactions

In principle, the effective Lagrangian can also contain derivative interactions, both time derivative and gradient terms, of the form

$$\mathcal{L}_{\text{int}} \supset \frac{c_{n,m}}{n!m!} \left(\frac{d}{dt}\right)^r \left(\frac{\partial_i}{aH}\right)^{2s} \left((aH)^\alpha \varphi_+\right)^n \left((aH)^\beta \varphi_-\right)^m\,, \tag{8.4.79}$$

where the time-derivative terms are not suppressed in the power-counting parameter λ. For these terms, note that the equation of motion relates terms containing a single time derivative $\dot{\varphi}_\pm$ to terms containing spatial derivatives $\partial^2/(aH)^2\varphi_\pm$ plus higher-order corrections. Therefore, one can eliminate all time-derivative terms in these interactions and trade them for gradient terms that come with power-suppression. The most general interaction term then reads

$$\mathcal{L}_{\text{int}} \supset \frac{c^s_{n,m}}{n!m!} \left(\frac{\partial_i}{aH}\right)^{2s} \left((aH)^\alpha \varphi_+\right)^n \left((aH)^\beta \varphi_-\right)^m\,. \tag{8.4.80}$$

The precise structure of the subleading Lagrangian, as well as any symmetry relations between the different terms due to the residual Lagrangian symmetries or RPI-like symmetries, is beyond the scope of this introductory chapter. Instead, we will only consider the *leading-power* Lagrangian in the following, which is given by

$$\mathcal{L} = -\nu\left(\dot{\varphi}_+\varphi_- - \varphi_+\dot{\varphi}_-\right) - \frac{c_{4,0}}{4!}(aH)^4\varphi_+^4 - \frac{c_{3,1}}{3!}\varphi_+^3\varphi_- + \mathcal{O}(\lambda)\,. \tag{8.4.81}$$

[3] We find a different sign here, because we employ the opposite sign-convention in the commutation relations of creation and annihilation operators compared to [94].

8.4.7 Gaussian Initial Conditions

Soft deSitter effective theory features a new type of matching conditions compared to conventional EFT. The effective Lagrangian contains coefficients $c_{n,m}$, which can be fixed in principle already in a top-down approach by relating them to the full-theory interaction $\kappa\phi^4$ and systematically inserting redefinitions and taking care of the hard modes. This is similar to the SCET Lagrangian, where all couplings are readily expressed in terms of the full-theory coupling constants already during the Lagrangian construction. One might therefore wonder if a matching computation is necessary, and if so, why. The reason why matching is required is formally very similar to SCET. Here, we found that the effective theory contains new objects, the N-jet operators, whose coefficients can only be determined by a matching computation.

In soft deSitter effective theory, there exist also a new class of objects, *initial conditions* for the different correlation functions. The effective theory requires a matching computation to specify these initial conditions, while the Lagrangian interactions then correspond to subleading corrections of these correlations.

Physically, the Lagrangian describes only the evolution and dynamics of the soft modes, which is in particular Gaussian. Therefore, any higher-point correlator immediately factorises into Gaussian two-point correlations. However, the full theory has non-Gaussian correlations due to hard interactions. This effect must be included by matching.

As an example, consider the two-point function. In the full theory, it is given by

$$\langle\phi(t,\boldsymbol{x})\phi(t,\boldsymbol{y})\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\langle\phi(t,\boldsymbol{k})\phi(t,\boldsymbol{k})\rangle\,, \tag{8.4.82}$$

and the momentum-space correlator can be computed in terms of the mode function (8.1.20) as

$$\langle\phi(t,\boldsymbol{k})\phi(t,\boldsymbol{k})\rangle = \frac{\pi H^2}{4(aH)^3}\left|H_\nu^{(2)}\left(\frac{k}{aH}\right)\right|^2 . \tag{8.4.83}$$

For small momenta $k/aH \to 0$, the leading contributions to this correlation read

$$\begin{aligned}\langle\phi(t,\boldsymbol{k})\phi(t,\boldsymbol{k})\rangle \approx \frac{\pi H^2}{4}\Bigg[&\left(\frac{2^\nu\Gamma(\nu)}{\pi}\right)^2\frac{(aH)^{-2\alpha}}{k^{2\nu}} + \left(\frac{1}{2^\nu\Gamma(1+\nu)\sin(\pi\nu)}\right)^2\frac{(aH)^{-2\beta}}{k^{-2\nu}}\\ &-\frac{2\cot(\pi\nu)}{\pi\nu}(aH)^{-3}\Bigg]\,.\end{aligned} \tag{8.4.84}$$

On the EFT side, the correlation function is given by inserting the decomposition (8.4.1)

$$\begin{aligned}\langle\phi(t,\boldsymbol{k})\phi(t,\boldsymbol{k})\rangle = H^2\Big[&(aH)^{-2\alpha}\langle\varphi_+(t,\boldsymbol{k})\varphi_+(t,\boldsymbol{k})\rangle + (aH)^{-2\beta}\langle\varphi_-(t,\boldsymbol{k})\varphi_-(t,\boldsymbol{k})\rangle\\ &+(aH)^{-3}\left[\langle\varphi_+(t,\boldsymbol{k})\varphi_-(t,\boldsymbol{k})\rangle + \langle\varphi_-(t,\boldsymbol{k})\varphi_+(t,\boldsymbol{k})\rangle\right]\Big]\,.\end{aligned} \tag{8.4.85}$$

Comparing the coefficients of $(aH)^x$, one finds the correlations

$$\langle\varphi_+(t,\boldsymbol{k})\varphi_+(t,\boldsymbol{k})\rangle = \frac{2^{2\nu-2}\Gamma(\nu)^2}{\pi}k^{-2\nu}\,, \tag{8.4.86}$$

$$\langle\varphi_-(t,\boldsymbol{k})\varphi_-(t,\boldsymbol{k})\rangle = \frac{\pi}{2^{\nu+2}\Gamma(1+\nu)^2\sin^2(\pi\nu)}k^{2\nu}\,, \tag{8.4.87}$$

$$\langle\varphi_+(t,\boldsymbol{k})\varphi_-(t,\boldsymbol{k})\rangle + \langle\varphi_-(t,\boldsymbol{k})\varphi_+(t,\boldsymbol{k})\rangle = -\frac{\cot(\pi\nu)}{2\nu}\,, \tag{8.4.88}$$

from which one can determine

$$\langle\varphi_+(t,\boldsymbol{k})\varphi_-(t,\boldsymbol{k})\rangle = \frac{-ie^{i\pi\alpha}}{4\nu\cos(\pi\beta)}\,, \tag{8.4.89}$$

$$\langle\varphi_-(t,\boldsymbol{k})\varphi_+(t,\boldsymbol{k})\rangle = \frac{ie^{i\pi\alpha}}{4\nu\cos(\pi\beta)}\,, \tag{8.4.90}$$

compatible with the commutation relations

$$[\varphi_+(t,\boldsymbol{x}),\varphi_-(t,\boldsymbol{y})] = \frac{i}{2\nu}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,. \tag{8.4.91}$$

A similar computation is necessary for all other correlations, in principle at any loop order, if the initial condition is corrected by hard loops. For the tree-level trispectrum, we compute this explicitly below. For more details on this, we refer to the exposition in [94, 95].

8.5 The Scalar Trispectrum in SdSET

As a final application for the effective theory, we perform the matching computation to the tree-level trispectrum at $\mathcal{O}(\kappa)$. The full-theory result was computed in Section 8.3, where we also explicitly obtained the hard and soft regions. The expectation is that the effective field theory, using the Lagrangian interactions, precisely reproduces the soft region of the correlator, while the hard region must be determined explicitly by a matching computation. Similar to the two-point function, we can anticipate that the hard region will appear in the form of an initial condition for the trispectrum.

Soft-deSitter Correlation

The relevant interaction Hamiltonian is given by

$$H_{\text{int}} = \int d^3x\,\left(\frac{c_{4,0}}{4!}(aH)^3\varphi_+^4 + \frac{c_{3,1}}{3!}\varphi_+^3\varphi_-\right) + \mathcal{O}(\lambda)\,, \tag{8.5.1}$$

and the trispectrum in the effective theory takes the form

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = H^4\Big(\langle\varphi_{+,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\rangle + (aH)^{-3}\left(\langle\varphi_{-,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\rangle + \text{perm}\right)\Big) + \mathcal{O}(\lambda)\,, \tag{8.5.2}$$

where $\phi_i \equiv \phi(t,\boldsymbol{x}_i)$, $\varphi_{\pm,i} \equiv \varphi_\pm(t,\boldsymbol{x}_i)$. Here, we have included terms with a single φ_- to account for the super-leading interaction φ_+^4. Only correlators with a single φ_- are contributing, since we can insert the super-leading interaction only once at $\mathcal{O}(\kappa)$. Perm denotes the permutations of the φ_- field.

For the first term, we have to evaluate

$$\langle\varphi_{+,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\rangle = i\int d^3x\,d\mathrm{t}\,\left(\frac{\tilde{\mu}}{aH}\right)^{2\alpha}\left\langle\left[\frac{c_{3,1}}{3!}\varphi_+^3(\boldsymbol{x})\varphi_-(\boldsymbol{x}),\varphi_{+,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\right]\right\rangle\,, \tag{8.5.3}$$

where we regularised the time integral using an analytic regulator and introduced the factorisation scale $\tilde{\mu} = \mu e^{\gamma_E}$. The commutator is evaluated using the canonical commutation relations (8.4.91). This yields

$$\int d^3x\,\left[c_{3,1}\varphi_+^3(\boldsymbol{x})\varphi_-(\boldsymbol{x}),\varphi_{+,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\right] = -\frac{i}{3}c_{3,1}\left(\varphi_+^3(\boldsymbol{x}_1)\varphi_{+,2}\varphi_{+,3}\varphi_{+,4} + \text{perm}\right)\,, \tag{8.5.4}$$

where perm denotes terms where the positions $\boldsymbol{x}_i$ are permutated. The correlator is computed in a straightforward fashion since the soft correlators are purely Gaussian and one can use

$$\langle\varphi_{+,1}^3\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\rangle = 3!\langle\varphi_{+,1}\varphi_{+,2}\rangle\langle\varphi_{+,1}\varphi_{+,3}\rangle\langle\varphi_{+,1}\varphi_{+,4}\rangle\,, \tag{8.5.5}$$

which is readily evaluated using the initial condition (8.4.86)

$$\langle\varphi_{+,1}\varphi_{+,2}\rangle = \int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\frac{1}{2k_1^3}\,. \tag{8.5.6}$$

Thus, one finds for the momentum-space correlator

$$\langle\varphi_+(\mathtt{t},\boldsymbol{k}_1)\varphi_+(\mathtt{t},\boldsymbol{k}_2)\varphi_+(\mathtt{t},\boldsymbol{k}_3)\varphi_+(\mathtt{t},\boldsymbol{k}_4)\rangle = \frac{c_{3,1}}{8(k_1k_2k_3k_4)^3}\sum_i \frac{k_i^3}{3}\left(\frac{1}{2\alpha}+\log\frac{\mu}{aH}+\gamma_E\right), \quad (8.5.7)$$

where $\sum_i k_i^3$ arises from pulling out the denominator $(k_1k_2k_3k_4)^{-3}$ in each term.

The second contribution arises due to the superleading interaction in the Lagrangian, where we find

$$(\langle\varphi_{-,1}\varphi_{+,2}\varphi_{+,3}\varphi_{+,4}\rangle + \text{perm}) = (aH)^3\frac{c_{4,0}}{24(k_1k_2k_3k_4)^3}\sum_i\frac{k_i^3}{3}. \quad (8.5.8)$$

The full Soft deSitter correlator then reads

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = H^4\left(\frac{c_{3,1}}{8(k_1k_2k_3k_4)^3}\sum_i\frac{k_i^3}{3}\left(\frac{1}{2\alpha}+\log\frac{\mu}{aH}+\gamma_E\right)+\frac{c_{4,0}}{24(k_1k_2k_3k_4)^3}\sum_i\frac{k_i^3}{3}\right) \quad (8.5.9)$$

Comparing this to the full-theory result (8.3.4), one finds that this is precisely the soft region (8.3.25) with $c_{3,1} = c_{4,0} = \kappa$.

Initial Condition

Therefore, the hard region (8.3.21) is missing in the effective theory and must be implemented by an initial condition for the four-point correlator. Specifically, it reads

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle_{IC} = \frac{\lambda H^4}{8(k_1k_2k_3k_4)^3}\Bigg[&\left(\frac{1}{3}\sum_i k_i^3\right)\left(-\frac{1}{2\alpha}+\ln\frac{k_t}{\mu}-2\right) \\ &-\frac{1}{9}k_t^3+\frac{1}{3}k_t\left(\sum_i k_i^2-\sum_{i<j}k_ik_j\right)+2\sum_{i<j<l}k_ik_jk_l-\frac{k_1k_2k_3k_4}{k_t}\Bigg].\end{aligned} \quad (8.5.10)$$

Note that these non-Gaussian initial conditions are not specific to SdSET but appear more generally in inflationary effective field theories. These initial conditions can be implemented directly in the Schwinger-Keldysh generating functional, where they correspond to new types of operators with non-local interaction vertices. This form is useful for a systematic investigation of subleading-power effects, and the construction is explained in detail in [109]. For our purposes, the implementation as initial condition for each correlator is sufficient.

8.6 Conclusion

In this section, we have touched on the dynamics of a light scalar field in a deSitter space-time. We have explained the pathological behaviour of the free, massless scalar field, where a two-point function and corresponding vacuum state cannot be defined. Then, we have turned our attention towards promising ideas that can partly resolve these problems for an interacting scalar field, encountering non-perturbative physics. One of the most recent entries is Soft deSitter Effective Theory [94,95], which aims to incorporate EFT methodology into this discussion. After verifying that the method of regions can be applied to deSitter correlation functions, we have explained in detail how the free action can be derived, and how it is related to the non-relativistic scalar field, expanding on the discussion in [94]. Then, we have explicitly performed a matching computation for the trispectrum, where we found that the Lagrangian precisely reproduces the soft region of the time-integral, while the hard region must be included as a non-Gaussian initial condition [94, 95, 109].

The next step in this program is to rigorously understand how the effective theory encodes the non-perturbative effects, like dynamic mass generation. In [94,95], a trick is employed that relates the renormalisation group equations (RGE) of the effective theory to the Fokker-Planck equation of stochastic inflation. However, it seems strange that the dynamics of a correlation function follow from the RGE, and not from a quantum field theoretic object like the quantum effective action. Moreover, the derivation presented in [94] lacks rigour and seems quite ad-hoc. Therefore, this property needs to be investigated and must be well-understood before one can truly claim that this effective theory resolves the infrared problems. This is part of an ongoing investigation [105].

Chapter 9

Conclusion and Outlook

In this thesis, we presented the systematic construction of the soft-collinear effective theory of gravity beyond leading power. Starting from the simple example of a purely-scalar theory, we explained how the soft-collinear effective theory can be obtained to any desired order in the power-counting parameter and discussed the appearance, necessity and properties of the N-jet operators, obtaining a minimal set of building blocks. We increased the complexity at each step by first allowing for gauge symmetries, which needed to be modified to account for the multipole expansion, and finally considering gravitational interactions. Throughout the derivation, the guiding principle was the effective gauge symmetry which lies at the heart of this discussion. By exploiting covariance with respect to an emergent homogeneous background symmetry, the Lagrangian can be constructed to all orders in the power-counting parameter – both in the case of gauge theory as well as gravity.

The key insight besides the technical details and the full effective Lagrangian is the new intuition in the gravitational sector. We found that the soft-collinear effective theory is covariant with respect to an emergent background field that contains not one but *two* independent gauge fields, one linked to local translations – restricted to the classical trajectory of the energetic particles – and one linked to local Lorentz transformations on this light-cone. The effective theory can then be constructed in a straightforward fashion as long as each object is covariant with respect to this background. The soft gauge symmetry also severely restricts the possible operator basis of the N-jet operators, and immediately implies the soft theorem with three universal terms in gravity, and two universal terms in gauge theory.

In the gravitational soft theorem itself, we found that the first two terms should be viewed as *eikonal* terms and are related to the effective gauge fields. They are of the same origin as the first (eikonal) term in gauge theory. The sub-subleading term, on the other hand, is the exact analogue of the subleading term in gauge theory. In this way, the effective theory provided a new interpretation and explanation for both the form and the number of universal terms in the soft theorem, both in the case of gravity and gauge theory, linking them directly to the soft gauge symmetry. We managed to identify these terms already at the Lagrangian level and restated the soft theorem as an operatorial statement.

Since the soft theorem is only a tree-level process, we made use of the properties of the effective theory to investigate its loop corrections. Using only the power-counting and eikonal identities due to the multipole expansion, we found that in gravity, the leading-power term is never affected by loop corrections, the next-to-leading power term is at most affected by one-loop, and the next-to-next-to-leading power term at most by two-loop contributions, reproducing the result in [32] in the effective theory.

Single soft-emission processes are the simplest type one can consider in SCET, and the effective theory automatically provides the rules to generalise these computations to multiple soft emissions, even accounting for quantum corrections. This approach complements the insights gained from spinor-helicty [29] or the double-copy [37, 65, 66] methods employed in the investigation of scattering amplitudes. It would be interesting to further explore the connection of SCET to the large field of asymptotic symmetries [34, 46, 110–112]. Here, the soft theorems follow from Ward identities of these asymptotic symmetries. It would be worthwhile to relate these symmetries to the effective theory, strengthening the interconnections in this triangle of effective theories, scattering amplitudes and asymptotic symmetries.

The framework of SCET gravity now allows for a systematic investigation of soft and collinear gravity beyond leading power. This is interesting in particular for the collinear sector, where interactions begin at next-to-leading power. While the actual technical form of the Lagrangian as well as most expressions are a lot more complicated and lengthy than the corresponding ones in gauge-theory, the structure of interactions is, surprisingly, simpler than gauge theory. For example, there are no leading-power collinear interactions and thus no collinear divergences in gravity. Furthermore, both the purely-collinear and purely-soft interactions are power-suppressed when considering higher-point interactions. This implies that gravitational loops, if they are not of soft-collinear origin, are in general power-suppressed and thus there often are only a finite number of loop corrections to a given object. For example, the analogue of the collinear function in gravity would be 1 at leading-power, since there are no interactions. However, at $\mathcal{O}(\lambda)$, it is not possible to form any loops, since these require multiple collinear interactions and are thus power-suppressed. Therefore, one can immediately conclude that the N-th order collinear function is also N-loop exact. This is in stark contrast to gauge theory, where these objects are modified by arbitrary loop corrections at any order.

Consequently, the effective theory provides a starting point for a systematic derivation of factorisation theorems in gravity and comparing these with the results in QCD. While these considerations are not as interesting for the phenomenological community, they could provide new insights into the relation between gauge theory and gravity, and how these theories are connected in the infrared.

To close this discussion, let us go back to the very beginning, where we quoted Weinberg [1]: *"It would be difficult to pretend that the gravitational infrared divergence problem is very urgent. My reasons for now attacking this question are: (1) Because I can. [...] (2) Because something might go wrong and this would be interesting. Unfortunately, nothing does go wrong."*
We attacked this problem with a similar motivation, because *we could*, well-knowing that collinear divergences are absent and indeed *nothing does go wrong* in gravity, because nothing *can* go wrong if low-energy gravity is a consistent quantum theory. However, the effective theory has to take a very intricate form to incorporate leading-power soft interactions – which are related to soft divergences – while not allowing collinear divergences. This form is constrained by the power-counting and the effective gauge symmetry, and while the purely-collinear sector appears close in form to the usual weak-field gravity, the soft-collinear one emerges very differently and closely resembles a gauge theory. This result, we would say, *is* interesting.

Appendix

A List of Useful Gauge-transformation Identities

This appendix from [47] summarises a number of key identities that are used throughout the derivation. These properties are derived from the corresponding properties one uses in the passive point of view, where one considers the explicit action of the coordinate transformation $x \to x + \varepsilon(x)$. In the following, we denote $\varepsilon(x) \equiv \varepsilon$ and $U(x) \equiv U$, suppressing the arguments. We have

- the gauge transformation U (4.2.5),
$$U = 1 - \varepsilon^\alpha \partial_\alpha + \frac{1}{2}\varepsilon^\alpha \varepsilon^\beta \partial_\alpha \partial_\beta + \varepsilon^\alpha \partial_\alpha \varepsilon^\beta \partial_\beta + \mathcal{O}(\varepsilon^3)\,, \tag{A.0.1}$$
- its inverse U^{-1},
$$U^{-1} = 1 + \varepsilon^\alpha \partial_\alpha + \frac{1}{2}\varepsilon^\alpha \varepsilon^\beta \partial_\alpha \partial_\beta + \mathcal{O}(\varepsilon^3)\,, \tag{A.0.2}$$
- the Jacobian matrix $U^\mu{}_\alpha$,
$$U^\mu{}_\alpha = \delta^\mu_\alpha + \partial_\alpha \varepsilon^\mu\,, \tag{A.0.3}$$
- the inverse Jacobian matrix $U_\alpha{}^\mu$,
$$U_\alpha{}^\mu = \delta^\mu_\alpha - \partial_\alpha \varepsilon^\mu + \partial_\alpha \varepsilon^\beta \partial_\beta \varepsilon^\mu + \mathcal{O}(\varepsilon^3)\,, \tag{A.0.4}$$
- the Jacobian determinant $\det(\underline{U})$ of the Jacobian $[\underline{U}]^\mu{}_\alpha = U^\mu{}_\alpha$,
$$\det(\underline{U}) = 1 + \partial_\alpha \varepsilon^\alpha + \frac{1}{2}\partial_\alpha \varepsilon^\alpha \partial_\beta \varepsilon^\beta - \frac{1}{2}\partial_\alpha \varepsilon^\beta \partial_\beta \varepsilon^\alpha + \mathcal{O}(\varepsilon^3)\,, \tag{A.0.5}$$
- the inverse Jacobian determinant $\det\left(\underline{U}^{-1}\right)$,
$$\det\left(\underline{U}^{-1}\right) = 1 - \partial_\alpha \varepsilon^\alpha + \frac{1}{2}\partial_\alpha \varepsilon^\alpha \partial_\beta \varepsilon^\beta + \frac{1}{2}\partial_\alpha \varepsilon^\beta \partial_\beta \varepsilon^\alpha + \mathcal{O}(\varepsilon^3)\,. \tag{A.0.6}$$

These objects satisfy a number of useful identities, that we employ in the following.

- They are inverse with respect to each other, i.e.
$$UU^{-1} = 1\,, \quad U^\mu{}_\alpha U_\mu{}^\nu = \delta^\nu_\alpha\,, \quad \det(\underline{U})\det\left(\underline{U}^{-1}\right) = 1\,. \tag{A.0.7}$$
- We can move translation and inverse translation past derivatives,
$$[\partial_\mu U\phi] = UU_\mu{}^\alpha \partial_\alpha \phi\,, \tag{A.0.8}$$
$$\left[\partial_\mu U^{-1}\phi\right] = U^\alpha{}_\mu U^{-1}\partial_\alpha \phi\,, \tag{A.0.9}$$

  which is consistent with the gauge transformation of a covariant vector.

- There is a "product rule" for the translation operator

$$[U\phi\psi] = U\phi\psi U^{-1} = U\phi U^{-1}U\psi U^{-1} = [U\phi]\,[U\psi]\;, \tag{A.0.10}$$

  and the same holds for U^{-1}.

- Scalar densities $\sqrt{-g}\phi$ are gauge-invariant up to total derivatives. These transform as

$$\int d^4x\,\sqrt{-g}\phi \to \int d^4x\,\sqrt{-g}\phi\,. \tag{A.0.11}$$

  With the gauge transformation of the metric determinant

$$\sqrt{-g} \to U\det\!\left(\underline{U}^{-1}\right)\sqrt{-g}\,, \tag{A.0.12}$$

  one can infer

$$\int d^4x\,U\det\!\left(\underline{U}^{-1}\right)\sqrt{-g}\phi = \int d^4x\,\sqrt{-g}\phi\,, \tag{A.0.13}$$

  which is the active point of view of the invariance (A.0.11).

- There is an integration by parts identity based on (A.0.11). Moving the inverse translation $U(x)$ from one term to another via integration by parts, we generate U^{-1} and a determinant, namely

$$\phi\,U\psi = \det(\underline{U})\left[U^{-1}\phi\right]\psi\,. \tag{A.0.14}$$

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