The geometry and cohomology of Newton strata in Shimura varieties

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Attachments

Core publications

[20] l-adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients

[21] On G-isoshtukas over function fields

[22] Irreducible components of affine Deligne-Lusztig varieties

Other publications

[14] On the generalisation of cohomology with compact support to nonfinite type schemes

[15] On the geometry of affine Deligne-Lusztig varieties for quasi-split groups

[16] The almost product structure of Newton strata in the deformation space of a Barsotti-Tate group with crystalline Tate tensors

[19] Point counting on Igusa varieties

[23] Finiteness properties of affine Deligne-Lusztig varieties.

1. Overview

One of the principal goals of number theory is to understand the structure of a global field F, that is a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_p(t)$, and its integers. While the topic has been studied with elementary methods for millenia, the modern approach to this problem is to use the tools of other areas of mathematics, e.g. commutative algebra, representation theory, complex analysis or probability theory. The formulation of number theoretic problems in terms of these areas and the development of specific tools have created several branches within the field of number theory. Unfortunately one is mostly limited to the tools of only one of these areas, as getting from one branch to another is usually a highly non-trivial exercise, if possible at all. We have seen that when successful, transforming a problem in one branch of number theory into another may carry enormous rewards. Famous examples are Deligne's proof of Ramanujan's conjecture on the growth of the Fourier coefficients of the modular form $\Delta(z) = \prod_{n>0}(1-e^{2\pi i n z})^{24}$ and Wiles' et al. proof of Fermat's conjecture, by associating a twodimensional representation of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ to a modular form (when certain conditions are satisfied) and vice versa, respectively.

A half century ago, Langlands created an ambitious web of conjectures relating representations of the absolute Galois group of F with automorphic representations, which can be seen as a generalisation of modular forms ([42]). Roughly speaking, it conjectures that for any reductive algebraic group G (e.g. GL_n) there exists a bijection between a certain class of automorphic representations of the group $\mathsf{G}(\mathbb{A}_\mathsf{F})$ and L-parameters, that is certain equivalence classes of morphisms of the absolute Galois group of F to $\hat{\mathsf{G}}(\mathbb{C})$, where $\hat{\mathsf{G}}$ denotes the Langlands dual group of G. Even though much progress has been made to date, our understanding of the Langlands correspondence is still quite limited. The most promising approach to prove the conjecture is to study certain geometric objects with a lot of symmetry, Shimura varieties if $\mathrm{char}\,\mathsf{F} = 0$ and moduli spaces of global $\mathscr{G}$-shtukas if $\mathrm{char}\,\mathsf{F} = p$, and realise the correspondence in their cohomology. To explain, we focus on Shimura varieties. For this we fix a Shimura datum (G, X), that is a linear algebraic group G is a reductive linear algebraic group over $\mathbb{Q}$ and X a conjugacy class of morphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to \mathsf{G}_\mathbb{R}$ satisfying certain conditions (see e.g. [9, § 2.1]). For every small enough open subgroup $\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)$, we obtain a quasi-projective variety $\mathsf{Sh}_\mathsf{K}(\mathsf{G},\mathsf{X})$ defined over a certain finite extension E of $\mathbb{Q}$, called the reflex field. Moreover, for every pair $\mathsf{K}_1 \subset \mathsf{K}_0$ we obtain a canonical projection $\mathsf{Sh}_{\mathsf{K}_1}(\mathsf{G},\mathsf{X}) \twoheadrightarrow \mathsf{Sh}_{\mathsf{K}_0}(\mathsf{G},\mathsf{X})$ making the family $(\mathsf{Sh}_\mathsf{K}(\mathsf{G},\mathsf{X}))$ a projective system. It is naturally equipped with commuting actions of the absolute Galois group of E and the adelic group $\mathsf{G}(\mathbb{A}_f)$. In particular both groups act on the cohomology groups of the "infinite level Shimura variety" $\mathsf{Sh}(\mathsf{G},\mathsf{X}) \coloneqq \varprojlim_K \mathsf{Sh}_K(\mathsf{G},\mathsf{X})$. These are conjectured to decompose in terms of certain automorphic representations and their L-parameters, inducing the Langlands correspondence.

Similarly, one expects that there should exist a local analogue of Shimura varieties whose cohomology decomposes according to the Langlands correspondence for local fields and Jacquet–Langlands correspondences; cf. [64]. It is natural to expect that there should be a relationship between the cohomology of Shimura varieties and "local Shimura varieties" that encodes the local-global compatibility of the Langlands correspondence. For compact Shimura varieties associated to an PEL Shimura datum unramified at a chosen prime p, such a cohomological formula was obtained by Mantovan ([46],[47],[48]) built upon the work of Harris and Taylor on the special case of Shimura varieties of Harris–Taylor type ([24]).

In general not much is known about the geometry of Shimura varieties, due to the fact that their construction is rather implicit. One of the main tools to study the geometry (and cohomology) of Shimura varieties is the Newton stratification. We will discuss how the Newton stratification is defined in the general context and present our results from [19] concerning Newton strata over arbitrary schemes in chapter 2. In chapter 3, we restrain ourselves to the Newton stratification of Shimura varieties and discuss the results of [20] building apon the previous results in [16], [17]. One of the main results of this article is that we can further decompose the Newton strata of a Shimura variety into two parts, the Igusa variety and the Rapoport-Zink space, which will play the part of the "local Shimura variety". The cohomology of (the rigid analytic fibre of) Rapoport–Zink spaces carries purely local information, while the cohomology of Igusa towers carries global information. As an application of this decomposition, we can formulate a generalisation of Mantovan's formula, which expresses the cohomology of Shimura varieties in a way that nicely separates the purely local contribution from the remaining global information. More precisely, this formula relates the compactly supported cohomology groups of the infinite level Shimura variety with the cohomology Igusa variety and the rigid analytic fibre of Rapoport-Zink spaces. Since the first two geometric objects are no longer varieties as they are not of finite type, we use ad-hoc definitions of their compactly supported cohomology group. A general formalism to define cohomology groups with compact support for such schemes alongside is constructed in [14], which is discused in chapter 5. This allows us to give a more natural description of Mantovan's formula and in fact an ad-hoc version of part of this formalism was used in the proof of the formula. Being the analogue of Shimura varieties over local fields, Rapoport-Zink spaces are fascinating objects that are widely studied. Their underlying reduced scheme is called an affine Deligne-Lusztig variety, which can be expressed by purely group theoretical means. In the fourth chapter, we present the results of [23],[22] and [18] on their geometry. As a consequence we also derive further geometric properties of Newton strata of Shimura varieties. Lastly, we consider the moduli space of global $\mathscr{G}$-shtukas, which is the function field analogue of a Shimura variety. The underlying variety can be decomposed according to the isogeny class of the universal $\mathscr{G}$-shtuka, which is a refinement of the decomposition into Newton strata. Each of these "slices" was parametrised in explicit group theoretical by a work of Arasteh Rad and Hartl. In the last chapter, we present the parametrisation of isogeny classes of global $\mathscr{G}$-shutukas constructed in [21]; thus completing the parametrisation of points of the underlying variety of the moduli space of global $\mathscr{G}$-shtukas.

2. General definitions and results for the Newton stratification

2.1. Background

Let F denote a local field of residue characteristic p with ring of integer O_F, uniformiser ϖ and field of fractions k_F. We fix an algebraic closure k of k_F and let $\breve{F}$ the completion of the maximal unramified extension of F. We denote by $\sigma \in \mathrm{Gal}(\breve{F}/F) = \mathrm{Gal}(k/k_F)$ the Frobenius automorphism. If $\mathrm{char}\, F = 0$ we will require any k_F-algebra and k_F-scheme in this section to be perfect without further mention. To ensure that we do not leave this category accidentally, the setup for $\mathrm{char}\, F = 0$ is less general then for $\mathrm{char}\, F = p$. If $\mathrm{char}\, F = p$, we will write "torsor" instead of "torsor for the fpqc-topology". If $\mathrm{char}\, F = p$, we will write "torsor" instead of "torsor for the proétale-topology". We fix a flat affine group scheme G over O_F.

Depending on F, we define for any k_F-algebra R

$$D_R \coloneqq \begin{cases} R[\![\varpi]\!] & \text{if } \mathrm{char}\, F = p \\ W(R) \otimes_{\mathbb{Z}_p} O_F & \text{if } \mathrm{char}\, F = 0 \end{cases}$$

$$D_R^* \coloneqq \begin{cases} R(\!(\varpi)\!) & \text{if } \mathrm{char}\, F = p \\ W(R) \otimes_{\mathbb{Z}_p} F & \text{if } \mathrm{char}\, F = 0. \end{cases}$$

The positive loop group and the loop group are defined as the group-valued functors on (perfect) k_F-algebras given by

$$L^+G(R) \coloneqq G(D_R)$$
$$LG(R) \coloneqq G(D_R^*).$$

The functors L^+G and LG are representable by an affine scheme and an ind-scheme, respectively (see e.g. [61, § 1.a],[73, Prop. 1.1]). If $\mathrm{char}\, F = 0$ they are perfect by definition as we only allowed perfect test objects. For any L^+G-torsor $\mathcal{G}$, we denote by $\mathcal{LG} \coloneqq LG \times^{L^+G} \mathcal{G}$ the associated LG-torsor.

Definition 2.1.1. Let S be a k_F-scheme. A local G-isoshtuka over S is a pair $(\mathcal{H}, \varphi)$ where $\mathcal{H}$ is an LG-torsor over S and $\varphi\colon \sigma^*\mathcal{H} \to \mathcal{H}$ an isomorphism. Similarly, a local G-shtuka over S is a pair $(\mathcal{G}, \varphi)$, where $\mathcal{G}$ is an L^+G-torsor over S and and isomorphism $\varphi\colon \sigma^*\mathcal{LG} \to \mathcal{LG}$.

Example 2.1.2. Let us consider the case that $G = \mathrm{GL}_n$ and (for simplicity) that $S =$

Spec R is affine. By [25, § 4] the groupoid of GL_n-shtukas is equivalent to the groupoid of local shtukas (also called F-crystals if $F = \mathbb{Q}_p$), that is pairs (M, φ) where M is a locally free D_R-module of rank n and $\varphi\colon \sigma^* M[\varpi^{-1}] \xrightarrow{\sim} M[\varpi^{-1}]$. By a result of Gabber, the Dieudonné functor defines an equivalence of categories between the categories of Barsotti-Tate groups over R and the full subcategory of F-crystals (M, φ) satisfying $M \subset \varphi(\sigma^* M) \subset pM$ (see also [43, Thm. 6.4]), allowing us to view the groupoid of Barsotti-Tate groups of height n as a full subcategory of the groupoid of GL_n-shtukas. Similarly, Barsotti-Tate groups of height n with additional structure such as polarisation, endomorphisms and crystalline Tate tensors correspond to local G-shtukas, where $G \subset \mathrm{GL}_n$ is a linear algebraic subgroup determined by the type of additional structure (see for example [62, Rmk. 2.4], [20, Pf. of Cor. 4.12]).

If $S = \operatorname{Spec} C$ is the spectrum of an algebraically closed field, any LG-torsor is trivial and hence every local G-isoshtuka is isomorphic to $(LG, b\sigma)$ for some $b \in G(D_C^*)$. Note that b is only determined up to σ-conjugacy; we denote by $[b] \coloneqq \{g^{-1} b \sigma(g) \mid g \in G(C)\}$ its σ-conjugacy class. We hence obtain a bijection between the isomorphism classes of local G-isoshtukas over C and σ-conjugacy classes $B(F, G)(C)$ in $G(D_C^*)$. By [62, Thm. 1.1], this set does not depend on C and will simply be denoted by $B(F, G)$ or $B(G)$ if F is understood. By a result of Kottwitz ([39]) every $[b] \in B(G)$ is uniquely determined by the following two invariants. The Newton point is a rational conjugacy class of quasi-cocharacters $\bar\nu([b])\colon \mathbb{D}_{\overline{F}} \to G_{\overline{F}}$, where $\mathbb{D}$ denotes the diagonalisable group over F with character group $\mathbb{Q}$. The Kottwitz point $\bar\kappa([b])$ is an element in the Galois coinvariants $\pi_1(G_F)_{\mathrm{Gal}(\overline{F}, F)}$ of Borovoi's fundamental group. Moreover, a partial order on $B(G)$ is defined as follows. Let G^* be the quasi-split inner form of G_F and let $S \subset B \subset G^*$ be a maximal split torus and Borel subgroup. We denote by T the centraliser of S, which is a maximal torus of G^*. Then there exists a unique representative $\bar\nu([b]) \in X_*(S)_{\mathbb{Q},\mathrm{dom}}$ of $\bar\nu([b])$. We write $[b'] \leq [b]$ iff $\bar\nu([b']) \leq \bar\nu([b])$ and $\bar\kappa([b']) = \bar\kappa([b])$. Lastly, for any $b \in G(\breve F)$ we denote by J_b the linear algebraic group over F whose R valued points are given by

$$J_b(R) = \{g \in G(R \hat\otimes_F \breve F) \mid gb = b\sigma(g)\}.$$

In particular, $J_b(F)$ equals the automorphism group of $(LG, b\sigma)$.

Example 2.1.3. For $G = \mathrm{GL}_n$ we may choose S to be the diagonal torus and B to be the Borel subgroup of upper triangular matrices, which yields a canonical identification $X_*(S)_{\mathbb{Q},\mathrm{dom}} \cong \{(\lambda_i) \in \mathbb{Q}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$. Classically $\bar\nu([b])$ is regarded as the polygon obtained as the graph of continuous piecewise linear function $[0, n] \to \mathbb{R}$ of slope λ_i over the interval $(i-1, i)$. By the Dieudonné-Manin classification the Newton point determines the isoshtuka uniquely and the image of $\bar\nu$ are the polygons, whose break points and end point have integral coordinates ([10],[45], see also [75, § 6]). The polygons obtained from Barsotti-Tate groups via the identification in Example 2.1.2 are the polygons whose slopes are contained in the interval $[0, 1]$.

Given a local G-isoshtuka $\underline{\mathcal{H}}$ over a k_F-scheme S and $[b] \in B(G)$, it is a classical result that the geometric points $\bar{s}$ of S such that $\mathcal{H}_{\bar{s}} \cong (LG, b\sigma)$ form a locally closed subscheme $S^{[b]}$ ([62, Thm. 3.6], see also Proposition 2.3.1 below for a stronger result); more precisely $S^{\leq [b]} := \bigcup_{[b'] \leq [b]} S^{[b']}$ is closed in S. The decomposition $S = \bigcup_{[b] \in B(G)} S^{[b]}$ is called the Newton stratification on S. If $S = S^{[b]}$ for some b, we say that $\underline{\mathcal{H}}$ (or a local shtuka $\underline{\mathcal{G}}$ if $\underline{\mathcal{H}} = \mathcal{L}\underline{\mathcal{G}}$) has constant isogeny class.

2.2. Constancy and rigidity results

Let $\underline{\mathcal{H}}$ be an G-isoshtuka over a k_F-scheme S having constant isogeny class. Since we only know that $\underline{\mathcal{H}}$ is "point-wise" isomorphic to some $(LG, b\sigma)$, this does not imply that $\underline{\mathcal{H}} \cong (LG, b\sigma)$. However, there exists a surjective purely inseparable morphism $S' \to S$ such that $\mathcal{H}_{S'}$ is isomorphic to $(LG, b\sigma)$ locally for the proétale topology on S'. More precisely, we have the following result.

Theorem 2.2.1 ([19, Thm. 1.2])**.** *Let $\underline{\mathcal{H}}$ be a G-isoshtuka over a perfect k_F-scheme S with constant isogeny class $[b]$. Then there exists a profinite étale cover $S' \to S$ such that $\underline{\mathcal{H}}_{S'} \cong (LG, b\sigma)_{S'}$. In other words the functor*

$$X^b_{\underline{\mathcal{H}}} \colon S' \to \mathrm{Isom}((LG, b\sigma)_{S'}, \underline{\mathcal{H}}_{S'})$$

on perfect S-schemes is represented by a $J_b(F)$-torsor.

Since many properties are local for the proétale topology, this statement allows to reduce from isoshtukas with constant isogeny class to isoshtukas defined over k_F for many problems. As a corollary, we obtain Tate's theorem for local G-shtukas without further conditions on G or the base scheme.

Proposition 2.2.2 (Tate's isogeny theorem for local G-shtukas, [19, Prop. 1.3])**.** *Let S be an integral normal scheme over k_F with generic point $\eta = \operatorname{Spec} K$. Let $\underline{\mathcal{G}}_1, \underline{\mathcal{G}}_2$ be two local G-shtukas with constant Newton point. Then the restriction map*

$$\mathrm{Isom}(\mathcal{L}\underline{\mathcal{G}}_1, \mathcal{L}\underline{\mathcal{G}}_2) \to \mathrm{Isom}(\mathcal{L}\underline{\mathcal{G}}_{1,\eta}, \mathcal{L}\underline{\mathcal{G}}_{2,\eta})$$

is a bijection, which identifies $\mathrm{Isom}(\underline{\mathcal{G}}_1, \underline{\mathcal{G}}_2)$ *with* $\mathrm{Isom}(\underline{\mathcal{G}}_{1,\eta}, \underline{\mathcal{G}}_{2,\eta})$.

Tate originally proved this theorem for Barsotti-Tate groups over a discrete valuation ring of characteristic 0 ([66]). Since then, his theorem was subsequently extended to more general cases in characteristic p. The most recent ones are Neupert's result [54, Thm. 2.7.6] for $\operatorname{char} F = p$, which assumes that S is Noetherian, and the result of Caraiani and Scholze

[7, Rmk. 4.2.7] for F-isocrystals (i.e. $F = \mathbb{Q}_p$ and $G = \mathrm{GL}_n$), where they removed the Noetherian hypothesis from the earlier result of Berthelot[3].

2.3. The Newton stratification

The behaviour of general local G-isoshtukas is a lot more complicated. Therefore a popular approach is to study the restriction of the G-isoshtuka to the Newton strata, which has constant isogeny class by definition, alongside with the Newton stratification instead.

An important property of the Newton stratification is the purity property, i.e. that its strata (and even certain unions of strata) are affine if the base scheme is affine. A weaker statement, that $S^{[b]} \subset S^{\leq [b]}$ is of pure codimension one or empty, was proved by de Jong and Oort for F-crystals over a local Noetherian ring ([8]). This statement was generalised by Yang in [70], where she considers the union $S^{[b]_i}$ of Newton strata contained in $S^{\leq [b]}$ whose associated Newton polygon (as defined in Example 2.1.2) contains a fixed break point $(i, \bar{\nu}([b])(i))$ of $\bar{\nu}([b])$. This setup is more general, as one can obtain the Newton stratum $S^{[b]}$ by intersecting above subschemes for all break points of $\bar{\nu}([b])$. Yang proves that if S is locally Noetherian then the complement of $S^{[b]_i}$ in the closed stratum $S^{\leq [b]} := \bigcup_{[b'] \leq [b]} S^{[b]}$ is of pure codimension one or empty. A similar statement was proven by Vasiu, who proved that for any F-crystal over an $\mathbb{F}_p$-scheme S the embedding $S^{[b]} \hookrightarrow S$ is affine. These statements were generalised to F-isocystals with additional G-structure in [16] using the notion of breakpoints for arbitrary reductive groups as defined in [68]. By generalising Vasiu's approach, Viehmann proved that for a local G-shtuka over an integral Noetherian scheme S for a split reductive group G the subschemes $S^{[b]_i}$ are affine over S. Using our results above, we can remove the restrains from above results.

Proposition 2.3.1 ([19, Prop. 1.4], see also [16, Prop. 1] for the reduction step to $G = \mathrm{GL}_n$). *Let $\underline{\mathcal{H}}$ be a local G-isoshutka over an k_F-scheme S. Then the immersion $S^{[b]_i} \hookrightarrow S$ is an affine morphism.*

As a consequence, the following result of Viehmann, which in its original version has purity as a prerequisite, always holds for the Newton stratification.

Corollary 2.3.2 ([69, Lemma 5.12]). *Let $\underline{\mathcal{H}}$ be an isoshtuka over an irreducible scheme S. Denote by $[b_\eta] \in B(G)$ the σ-conjugacy class associated to its generic point. Let $[b_0] < [b_\eta]$ such that $S^{[b_0]} \subset S$ is nonempty and such that the codimension of every irreducible component is at least the length $\ell[b_0, b_\eta]$ of any maximal chain $[b_0] < \cdots < [b_\eta]$ in $B(G)$. Then we have for any $[b'] \in B(G)$ with $[b_0] \leq [b'] \leq [b_\eta]$*

1. $S^{[b_0]} \subset \overline{S^{[b']}}$, *in particular* $S^{[b']}$ *is non-empty.*

2. *Every irreducible component of* $S^{[b_0]}$ *is of codimension* $\ell[b_0, b_\eta]$ *in* S.

3. The product structure of Newton strata in Shimura varieties of Hodge type

3.1. Newton strata in the Siegel modular variety

In order to explain the general result for Shimura varieties of Hodge type, we first consider the Siegel moduli space. For $\mathsf{K}'^p \subset \mathsf{G}(\mathbb{A}_f^p)$ small enough compact open we denote by $\mathscr{S}'$ the $\mathbb{Z}_p$-scheme solving the moduli problem mapping an $\mathbb{Z}_p$-scheme S to isomorphism classes of tuple (A, λ, η), where

- A is a projective Abelian scheme of dimension $2g$ over S,
- $\lambda\colon A \to A^\vee$ is a principal polarisation and
- $\eta\colon \mathbb{A}_f^{p,2g} \stackrel{\sim}{\to} V_{\mathbb{A}_f^p}(A)$ is a K^p-orbit of isomorphisms of $\pi_1(S, \overline{s})$-sets, identifying the standard symplectic form on the left hand side with a scalar of the form given by λ on the right hand side.

This functor is representable by [53]. Moreover the scheme $\mathscr{S}'_{\mathsf{K}'^p}$ is the canonical integral model of $\mathsf{Sh}_{\mathsf{K}'}(\mathrm{GSp}_{2g}, \mathsf{S}^\pm)$, where $\mathsf{K}' = \mathsf{K}'_p\mathsf{K}'^p$ with $\mathsf{K}'_p \subset \mathsf{GSp}_{2g}(\mathbb{Q}_p)$ is hyperspecial (see e.g. [38]). We note that if $g \in \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ and $\mathsf{K}_1'^p, \mathsf{K}_2'^p \subset \mathsf{GSp}_{2g}(\mathbb{A}_f^p)$ with $g^{-1}\mathsf{K}_1'^p g \subset \mathsf{K}_2'^p$ then precomposing η with g induces a morphism $\mathscr{S}'_{\mathsf{K}_1'^p} \to \mathscr{S}'_{\mathsf{K}_2'^p}$. In other words, the Hecke action of $\mathsf{GSp}_{2g}(\mathbb{A}_f^p)$ on the tower $(\mathsf{Sh}_{\mathsf{K}'_p\mathsf{K}'^p}(\mathsf{GSp}_{2g}, \mathsf{S}^\pm))_{\mathsf{K}'^p}$, where K'^p runs through all small enough compact open subgroups of $\mathsf{GSp}_{2g}(\mathbb{A}_f^p)$, extends to $(\mathscr{S}'_{\mathsf{K}^p})_{\mathsf{K}'^p}$.

We denote by $(\mathscr{A}', \lambda')$ the universal principally polarised Abelian scheme over $\mathscr{S}'_{\mathsf{K}'^p}$. We denote the Newton stratification of $\mathscr{S}'_{\mathsf{K}'^p,\mathbb{F}_p}$ given by the isogeny class of $(\mathscr{A}'[p^\infty], \lambda'|_{\mathscr{A}'[p^\infty]})$ over geometric points of $\mathscr{S}'_{\mathsf{K}'^p,\mathbb{F}_p}$ by $\mathscr{S}'_{\mathsf{K}'^p,\mathbb{F}_p} = \bigcup \mathscr{S}'^{[b]}_{\mathsf{K}'^p}$. Alternatively, this is the Newton stratification defined by the local GSp_{2g}-shtuka corresponding to $(\mathscr{A}'[p^\infty], \lambda'|_{\mathscr{A}'[p^\infty]})$ over the perfection of $\mathscr{S}'_{\mathbb{F}_p}$ by the construction in Example 2.1.2. These Newton strata can be further decomposed due to the work of Oort ([60]), Mantovan ([47]) and Caraiani-Scholze ([7]). For this we fix $b \in \mathrm{GSp}_{2g}(\breve{\mathbb{Q}}_p)$ and consider a polarised decent Barsotti-Tate group $(\mathbb{X}_b, \lambda_b)$ over $\mathbb{F}_p$ whose isogeny class corresponds to $[b]$. By Dieudonné theory the group of self-quasi-isogenies of $\mathbb{X}_b$ can be canonically identified with

$$J'_b(\mathbb{Q}_p) = \{g \in \mathrm{GSp}_{2g}(\breve{\mathbb{Q}}_p) \mid gb = b\sigma(g)\}.$$

Rather than studying $\mathscr{S}'^{[b]}_{\mathsf{K}'^p}$ directly, we consider the functor $\mathfrak{X}'^b_{\mathsf{K}'^p}$ of schemes over $\mathrm{Spf}\,\breve{\mathbb{Z}}_p$ given by defining $\mathfrak{X}'^b_{\mathsf{K}'^p}(S)$ as the set of isomorphism classes of tuples (A, λ, η, ρ), where

- $(A, \lambda, \eta) \in \mathscr{S}'(S)$ is a polarised Abelian scheme with K^p-level structure,

- $\rho\colon (\mathbb{X}_b, \lambda_b)_{\bar{S}} \dashrightarrow (A[p^\infty], \lambda)_{\bar{S}}$ is a quasi-isogeny respecting the polarisations.

Here and in the following we write $\bar{S}$ for $S \times_{\operatorname{Spf}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$. The functor $\mathfrak{X}^b_{\mathsf{K}_p}$ is representable by a formal scheme ([7, Lem 4.3.12], see also below). By varying K'^p, we obtain a projective system $(\mathfrak{X}'^b_{\mathsf{K}'^p})_{\mathsf{K}'^p}$, which is equipped with three commuting group actions. The group $J'_b(\mathbb{Q}_p)$ acts on $\mathfrak{X}'^b_{\mathsf{K}'^p}$ by precomposition. Moreover the Hecke action of $\operatorname{GSp}_{2g}(\mathbb{A}^p_f)$ on $(\mathscr{S}'_{\mathsf{K}'^p})_{\mathsf{K}'^p}$ canonically lifts to the tower $(\mathfrak{X}'^b_{\mathsf{K}'^p})_{\mathsf{K}'^p}$. Lastly, we have a semilinear action of the Weil group $W_{\mathbb{Q}_p}$ on $\mathfrak{X}'^b_{\mathsf{K}'^p}$. Since $\mathfrak{X}'^b_{\mathsf{K}'^p}$ is a formal $\breve{\mathbb{Z}}_p$-scheme, the action of the inertia group will be trivial. Thus the $W_{\mathbb{Q}_p}$-action is given by a Weil descent datum, that is a σ-semilinear isomorphism $\alpha\colon \mathfrak{X}'^b_{\mathsf{K}'^p} \xrightarrow{\sim} \mathfrak{X}'^b_{\mathsf{K}'^p}$ defining the action of the Frobenius (cf. [65, Def. 3.45]). The Weil descent datum for $\mathfrak{X}'^b_{\mathsf{K}'^p}$ is given by $(A, \lambda, \eta, \rho) \mapsto (A, \lambda, \eta, \rho \circ F^{-1})$, where F denotes the Frobenius morphism. Note that the $\mathsf{G}(\mathbb{A}^p_f)$-, $J'_b(\mathbb{Q}_p)$ and $W_{\mathbb{Q}_p}$-action commute with each other. The following construction allows us to split $\mathfrak{X}'^b_{\mathsf{K}'^p}$ into two factors.

Construction 3.1.1. Let (A, λ) be a polarised Abelian variety over a scheme S and let $\rho\colon (A[p^\infty], \lambda) \to (X', \lambda')$ be a quasi-isogeny. If S is quasi-compact, we choose $N > 0$ big enough such that $p^N\rho$ is an isogeny and we denote by ρ_A the quasi-isogeny of Abelian varieties $(A, \lambda) \overset{\cdot p^{-N}}{\dashrightarrow} A/\ker(p^N\rho) =: (A_\rho, \lambda_\rho)$ and by $i_\rho\colon (X', \lambda') \xrightarrow{\sim} (A_\rho, \lambda_\rho)$ the canonical isomorphism. We note that this construction is independent on N. If S is not quasi-compact, we construct ρ_A and i_ρ by executing the above construction over a cover of S by quasi-compact open subschemes and applying the glueing lemma.

The first component of the decomposition of the Newton stratum is the Igusa variety, parametrising polarised Abelian schemes with level structure with an isomorphism of the Barsotti-Tate group with $\mathbb{X}_b$, i.e. its special fibre is defined by

$$\operatorname{Ig}'^b_{\mathsf{K}^p}(S) = \{(A, \lambda, \eta, \iota) \mid (A, \lambda, \eta) \in \mathscr{S}'(S), \iota\colon (\mathbb{X}_b, \lambda_b)_S \xrightarrow{\sim} (A[p^\infty], \lambda|_{A[p^\infty]})\}.$$

Then $\operatorname{Ig}'^b_{\mathsf{K}^p}$ is represented by a perfect $\mathbb{F}_p$-scheme ([7, Prop. 4.3.3,4.3.8]). In particular, it has a unique flat lift $\mathfrak{Ig}'^b_{\mathsf{K}}$ over $\operatorname{Spf}\mathbb{Z}_p$, which represents the functor $S \mapsto \operatorname{Ig}'^b_{\mathsf{K}^p}(\bar{S})$. The $\mathsf{GSp}_{2g}(\mathbb{A}^p_f)$-action extends canonically to the tower $(\mathfrak{Ig}'^b_{\mathsf{K}^p})_{\mathsf{K}^p}$. Moreover $\mathfrak{Ig}'^b_{\mathsf{K}^p}$ is equipped with a $J_{b'}(\mathbb{Q}_p)$-action, given by $j.(A, \lambda, \eta, \iota) = (A_{j\circ\iota^{-1}}, \lambda_{j\circ\iota^{-1}}, \eta, i_{j\circ\iota^{-1}})$ and the twisted $W_{\mathbb{Q}_p}$-action induced by the Weil decent datum given by the absolute Frobenius on $\mathfrak{Ig}'^b_{\mathsf{K}^p}$.

The second part is the Rapoport-Zink space, parametrising quasi-isogenies of Barsotti-Tate groups with domain $\mathbb{X}_b$, i.e.

$$\mathfrak{M}'^b(S) = \{\rho\colon (\mathbb{X}_b, \lambda_b)_{\bar{S}} \dashrightarrow (X, \lambda)_{\bar{S}} \text{ quasi-isogeny }\}.$$

The functor $\mathfrak{M}'^b$ is represented by a p-adic formal scheme ([65, Thm. 2.16]). We define the trivial action of $\mathsf{GSp}_{2g}(\mathbb{A}^p_f)$ on $\mathfrak{M}'^b$, the $J'_b(\mathbb{Q}_p)$-action by precomposition and the twisted $W_{\mathbb{Q}_p}$-action analogously to $\mathfrak{X}'^b_{\mathsf{K}^p}$.

Given two points $(A, \lambda, \eta, \iota) \in \mathrm{Ig}^{b'}(S)$ and $(X, \lambda_X, \rho) \in \mathfrak{M}^{b'}(S)$, the quasi-isogeny $\rho \circ i^{-1}\colon (A[p^\infty], \lambda) \dashrightarrow (X, \lambda_X)$ induces a quasi-isogeny $(A, \lambda, \eta) \dashrightarrow (A', \lambda', \eta')$ (compatible with the additional structure). Hence we obtain a point $(A', \lambda', \eta', \rho) \in \mathfrak{X}^{b'}(S)$. The corresponding morphism $\pi'\colon \mathfrak{Ig}'^{b}_{\mathsf{K}'^p} \times \mathfrak{M}'^{b} \to \mathfrak{X}'^{b}_{\mathsf{K}'^p}$ is an isomorphism by [7, Prop. 4.3.13] and is $\mathsf{G}(\mathbb{A}_f^p)$-, $J'_b(\mathbb{Q}_p)$ and $W_{\mathbb{Q}_p}$-equivariant. In particular $\mathfrak{X}'^{b}$ is represented by a p-adic formal scheme.

3.2. Newton strata in the deformation space of a Barsotti-Tate group with crystalline Tate tensors

As a first step, we explain how the product structure works on a locally in the case of good reduction. The formal neighbourhood of a closed point in the canonical model of a Shimura variety of Hodge type is isomorphic to a certain closed subspace of a deformation space of a Barsotti-Tate group ([34, § 2.3]). We briefly explain its general construction.

We fix a reductive group G over $\mathbb{Z}_p$, a σ-conjugacy class $[b]$ in $G(\widehat{\mathbb{Q}}_p^{\mathrm{nr}})$ and a cocharacter μ of G such that $(G, b, -\mu)$ is an integral local Shimura datum of Hodge type in the sense of [31, Def. 2.5.10]. In particular, there exists a finite free $\mathbb{Z}_p$-module M and a faithful representation $i\colon G \hookrightarrow \mathrm{GL}(M)$ such that μ acts with weights 0 and 1 on M and a family $\underline{s}$ of tensors of M such that G is the stabiliser of $\underline{s}$. Let $(X, \underline{t})$ be a Barsotti-Tate group with crystalline Tate tensors over k such that there exists an isomorphism of the Dieudonné module of X with $M \otimes_{\mathbb{Z}_p} W$ identifying its Frobenius with an element of the σ-conjugacy class $i(\mathbf{b})$ and the crystalline Tate tensors with $\underline{s} \otimes 1$. We denote by $\mathfrak{Def}(X, \underline{t})$ the formal subscheme of the deformation space $\mathfrak{Def}(X)$ of X cut out by i, whose power series-valued points can be regarded as those deformations where the crystalline Tate tensors $\underline{t}$ extend to the whole group ([12], [52]).

Let $\mathrm{Def}(X, \underline{t})$ denote the algebraisation of $\mathfrak{Def}(X, \underline{t})$. The "universal deformation" of $(\underline{X}, \underline{t})$ algebraises to a Barsotti-Tate group with crystalline Tate tensors $(\mathscr{X}^{\mathrm{univ}}\underline{t}^{\mathrm{univ}})$ over $\mathrm{Def}(X, \underline{t})$ by Messing's algebraisation result for Barsotti-Tate groups (see e.g. [29, Rmk. 2.3.5 (c)]). Denote by $[b_0]$ the isogeny class of $(X, \underline{s})$ and by $\mathrm{N}_G(X) \coloneqq \mathrm{Def}(X, \underline{t})^{[b_0]}$ the (unique) minimal Newton stratum. Similar to above we can decompose $\mathfrak{N}_G$ into the central leaf $\mathrm{C}_G(X) \subset \mathrm{N}_G(X)$, defined as the locus where the fibre of $(\mathscr{X}^{\mathrm{univ}}, \underline{t}^{\mathrm{univ}})$ over the geometric points is *isomorphic* to $(X, \underline{t})$, and the isogeny leaf, defined as the maximal reduced subscheme of $\mathrm{N}_G(X)$ such that the restriction $(\mathscr{X}^{\mathrm{univ}}, \underline{t}^{\mathrm{univ}})|_{\mathrm{I}_G(X)}$ is isogenous to $(X, \underline{t})_{\mathrm{I}_G(X)}$. By the rigidity of quasi-isogenies, there exists a unique quasi-isogeny $(X, \underline{t})_{\mathrm{I}_G(X)} \dashrightarrow (\mathscr{X}^{\mathrm{univ}}, \underline{t}^{\mathrm{univ}})|_{\mathfrak{I}_G(X)}$ whose restriction to the closed point is the identity. Similarly, there exists an isomorphism $j_\infty\colon (X, \underline{s})_{\mathrm{C}_G(X)^{\mathrm{perf}}} \to \mathscr{X}^{\mathrm{univ}}|_{\mathrm{C}_G(X)^{\mathrm{perf}}}$ by [16, Lemma 8].

Theorem 3.2.1 ([16, § 4]). *There exists a canonical universal homeomorphism*

$\pi\colon \mathrm{C}_G(X)^{\mathrm{perf}} \times \mathrm{I}_G(X) \to \mathrm{N}_G(X)$ *such that* $\pi^*(\mathscr{X}^{\mathrm{univ}}, \underline{t}^{\mathrm{univ}})$ *is isomorphic to the pullback of* $(\mathscr{X}^{\mathrm{univ}}, \underline{t}^{\mathrm{univ}})|_{\mathrm{I}_G(X)}$ *along the canonical projection.*

Remark 3.2.2. The above result was afterwards generalised to the case that G is a parahoric scheme by Wansu Kim in [32]. Kim realised that $\mathfrak{Def}(X, \underline{t})$ should be considered to parametrise deformations of $\mathrm{id}_X\colon X \to X$ as quasi-isogeny (respecting crystalline Tate tensors) with domain $(X, \underline{t})$ instead of the deformations of $(X, \underline{t})$. These notions can be identified canonically if the test objects are local Artinean rings, but the latter description allows to extend the test objects to zero-dimensional local rings. If we denote by $\mathfrak{C}_G(X)$ the formal neighbourhood of the closed point in $\mathrm{C}_G(X)$, Kim showed that $\mathfrak{C}_G(X)^{\mathrm{perf}}$ parametrises the self-quasi-isogenies of X in $\mathfrak{Def}(X, \underline{t})$ and that π is the restriction of the morphism

$$\mathfrak{C}_G(X)^{\mathrm{perf}} \times \mathfrak{Def}(X, \underline{t}) \to \mathfrak{Def}(X, \underline{t}), (j, \rho) \mapsto \rho \circ j.$$

3.3. Newton strata in Shimura varieties of Hodge type

Let (G, X) be a Shimura datum of Hodge type an let $\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset \mathsf{G}(\mathbb{A}_f)$ be a small enough compact open subgroup with $\mathsf{K}_p \subset \mathsf{G}(\mathbb{Q}_p)$ is parahoric. We denote by G the corresponding parahoric groups scheme over $\mathbb{Z}_p$. Under some mild conditions one can find $\mathsf{K}' = \mathsf{K}'_p\mathsf{K}'^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f)$ small enough compact open with hyperspecial K'_p and an embedding $(\mathsf{G}, \mathsf{X}) \hookrightarrow (\mathsf{GSp}_{2g}, \mathsf{S}^{\pm})$ identifying $\mathsf{K} = \mathsf{K}' \cap \mathsf{G}(\mathbb{A}_f)$. Then the induced morphism $\mathsf{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X}) \to \mathsf{Sh}_{\mathsf{K}'}(\mathsf{GSp}_{2g}, \mathsf{S}^{\pm})$ is a closed immersion. We fix a prime $v|p$ of the Shimura field E of (G, X) and denote by E the v-adic completion of E.

In [33], Kisin and Pappas defined the canonical model $\mathscr{S}_{\mathsf{K}^p}$ of $\mathsf{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})$ as the normalisation of $\mathsf{Sh}_{\mathsf{K}}(\mathsf{G}, \mathsf{X})$ in the Siegel moduli space $\mathscr{S}'_{\mathsf{K}'^p} \times O_E$ assuming some mild conditions and extended the $\mathsf{G}(\mathbb{A}_f^p)$-Hecke action to the tower $(\mathscr{S}_{\mathsf{K}^p})_{\mathsf{K}^p}$. Moreover, for every point $x \in \mathscr{S}(\overline{\mathbb{F}}_p)$ they constructed crystalline Tate tensors $\underline{t}_x \subset\in \mathbb{D}(\mathscr{A}'_x[p^\infty])^{\otimes}$ on the Dieudonne-module of the fibre of the universal Abelian scheme over x. We denote by $\mathscr{A}$ the pullback of $\mathscr{A}'$ to $\mathscr{S}$. In order to associate a local G-shtuka to $\underline{\mathscr{A}}[p^\infty]$ we need to interpolate the $\underline{t}_x$.

Theorem 3.3.1 ([20, Cor. 4.11])**.** *There exists a family* $\underline{t} \subset \mathbb{D}(\mathscr{A}_{\mathscr{S}^{\mathrm{perf}}_{\overline{\mathbb{F}}_p}})$ *of crystalline Tate tensors on the Dieudonné module of the pullback of* $\mathscr{A}[p^\infty]$ *to the perfection* $\mathscr{S}^{\mathrm{perf}}_{\mathsf{K}^p, \overline{\mathbb{F}}_p}$ *of the geometric special fibre such that for every* $x \in \mathscr{S}(\overline{\mathbb{F}}_p)$ *their restriction to* $\mathbb{D}(\mathscr{A}[p^\infty]_x)$ *coincides with* $\underline{t}_x$.

Now a construction similar to Example 2.1.2 associates to $(\underline{\mathscr{A}}[p^\infty]_{\mathscr{S}^{\mathrm{perf}}_{\mathsf{K}^p, \overline{\mathbb{F}}_p}}, \underline{t})$ a local G-shtuka $\underline{\mathcal{G}}$ ([20, Cor. 4.12]). We denote by $\mathscr{S}_{\mathsf{K}^p, \overline{\mathbb{F}}_p} = \bigcup \mathscr{S}^{[b]}_{\mathsf{K}^p}$ the Newton stratification induced by $\underline{\mathcal{G}}$. As in the case of the Siegel modular variety, the Newton strata can be further decomposed into a Rapoport-Zink space and an Igusa variety. For this we fix $b \in G(\breve{\mathbb{Q}}_p)$ and a decent polarised Barsotti-Tate group $(\mathbb{X}_b, \lambda_b, \underline{t}_b)$ over $\overline{\mathbb{F}}_p$ with crystalline Tate tensors whose isogeny class correponds to $[b]$. By Dieudonné theory its group of self-quasi-isogenies is identified with

$$J_b(F) = \{g \in G(\breve{F}) \mid gb = b\sigma(g)\}.$$

The Igusa variety $\mathfrak{Ig}^b$ is defined as the closed formal subscheme of $\mathfrak{Ig}'^b \times'_{\mathscr{S}} \mathscr{S}$ cut out by the condition that the universal isomorphism $i\colon \mathbb{X}_{b'} \xrightarrow{\sim} \mathscr{A}[p^\infty]_{\mathfrak{Ig}^{b'},\mathrm{red}}$ identifies crystalline Tate tensors on both sides. The canonical morphism $\mathfrak{Ig}^b \to \mathfrak{Ig}'^b$ is a closed immersion and $\mathfrak{Ig}^b$ is stable under the action of $\mathsf{G}(\mathbb{A}_f^p) \times J_b(\mathbb{Q}_p) \times W_E$ ([20, Cor. 6.3]). The definition of $\mathfrak{M}^b$ is more involved and needs a bit more background.

Background on Affine Deligne Lusztig varieties

We denote by $\mathscr{F}\ell_G = [L\mathsf{G}_{\mathbb{Q}_p}/L^+\mathsf{G}_{\mathbb{Q}_p}]$ the Witt vector flag variety. It is represented by a limit of $\mathbb{F}_p$-schemes perfectly of finite type ([73]). In particular, we have that $\mathrm{Gr}_G(\overline{\mathbb{F}}_p) = G(\breve{\mathbb{Q}}_p)/\breve{\mathsf{K}}_p$, where $\breve{\mathsf{K}}_p \subset G(\breve{\mathbb{Q}}_p)$ is the parahoric subgroup given by extension of scalars of K_p. Let $\{\mu\}$ be the conjugacy class of cocharacters $\mathbb{G}_{m,\mathbb{C}} \to \mathsf{G}_{\mathbb{C}}$ associated to X. We denote by $\mathrm{Adm}(\mu) \subset \tilde{W}$ the μ-admissible subset inside the extended affine Weyl group over $\breve{\mathbb{Q}}_p$ as defined in [63, (3.4)]. We define the affine Deligne-Lusztig variety $X_{\sigma(\mu)}(b)$ as the closed perfect subscheme of Gr_G with points

$$X_{\sigma(\mu)}(b)(\overline{\mathbb{F}}_p) \coloneqq \{g\breve{\mathsf{K}}_p \in G(\breve{\mathbb{Q}}_p)/\breve{\mathsf{K}}_p \mid g^{-1}b\sigma(g) \in \bigcup_{w \in \tilde{W}} \breve{\mathsf{K}}_p w \breve{\mathsf{K}}_p\}$$

If we apply this construction to the Siegel moduli space, it simplifies to

$$X_{\mu'}(b)(\overline{\mathbb{F}}_p) \coloneqq \{g\breve{\mathsf{K}}'_p \in \mathrm{GSp}_{2g}(\breve{\mathbb{Q}}_p)/\breve{\mathsf{K}}'_p \mid g^{-1}b\sigma(g) \in \breve{\mathsf{K}}'_p\mu'(p)\breve{\mathsf{K}}'_p\}$$

where $\mu'\colon \mathbb{G}_m \to \mathrm{GSp}_{2g}$ maps an element $t \in \mathbb{G}_m$ to the diagonal matrix with entries $1, \dots, 1, t, \dots, t$ where each entry occurs g times.

By [73, Prop. 3.11] the Dieudonné functor over perfect rings induces an isomorphism $\mathfrak{M}'^{b,\mathrm{perf}}_{\overline{\mathbb{F}}_p} \cong X_{\mu'}(b)$. Denote by $\rho\colon (\mathbb{X}_b, \lambda_b) \to (\mathscr{X}', \lambda')$ the restriction of the universal quasi-isogeny to $X_{\sigma(\mu)}(b)$. By [20, Lemma 5.5] $\underline{t}^{ADLV} \coloneqq \rho_* \underline{t}_b$ are crystalline Tate tensors on $\mathscr{X}'$. In order to define $\mathfrak{M}^b$, we need the following compatibility criterion.

Axiom 3.3.2. We fix a point $\tilde{z} \in \mathfrak{Ig}^b(\overline{\mathbb{F}}_p)$. We require that the composition of

$$X_{\sigma(\mu)} \hookrightarrow X_\mu(b) \xrightarrow{\pi(\tilde{z},\cdot)} \mathscr{S}'$$

factors through the closure of $\mathsf{Sh}_\mathsf{K}(\mathsf{G},\mathsf{X})$ in $\mathscr{S}'$. Moreover, there exists a unique lift on $\overline{\mathbb{F}}_p$-points $f_{\tilde{z}}\colon X_{\sigma(\mu)}(b)(\overline{\mathbb{F}}_p) \to \mathscr{S}(\overline{\mathbb{F}}_p)$ such that for every $P \in X_{\sigma(\mu)}(b)(\overline{\mathbb{F}}_p)$ we have

$\underline{t}_P^{ADLV} = (f_{\tilde{z}}^* \underline{t})_P$

This axiom has been proven in the case that K_p is absolutely special ([67, Prop. A.4.3]) and in the case the G is residually split or that b is basic ([71, Prop. 6.4]).

Theorem 3.3.3 ([20, Prop. 5.10,Lem. 5.12,5.14])**.** *Assume that Axiom 3.3.2 is satisfied. Then there exists a unique closed formal subscheme $\mathfrak{M}^b \subset \mathfrak{M}'^b$ satisfying the following conditions.*

1. *Its restriction to the perfection of the underlying reduced subscheme coincides with $X_{\sigma(\mu)}(b) \subset X_{\mu'}(b)$.*
2. *For any $y \in X_{\sigma(\mu)}(b)(\mathbb{F}_p)$, the canonical isomorphism of the formal neighbourhood $\mathfrak{M}'^{b,\wedge}_y \cong \mathfrak{Def}(\mathscr{X}', \lambda')$ identifies $\mathfrak{M}^b$ with $\mathfrak{Def}(\mathscr{X}', \lambda', \underline{t}_y^{ADLV})$.*

Moreover, the $J'_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$-action on $\mathfrak{M}'^b$ restricts to a $J_b(\mathbb{Q}_p) \times W_E$-action on $\mathfrak{M}^b$.

We can now formulate the main result about the product structure of Newton strata in $\mathscr{S}$. We denote by $\mathfrak{X}^b$ the image of $\mathrm{Ig}^b \times \mathfrak{M}^b$ in $\mathfrak{X}'^b$.

Theorem 3.3.4 ([20, Thm. 6.8])**.** *The restriction $\pi'|_{\mathfrak{X}^b_{\mathsf{K}^p}} := \mathfrak{X}'^b_{\mathsf{K}'^p} \to \mathscr{S}'_{\mathsf{K}'^p}$ has a unique lift $\pi \colon \mathfrak{X}^b_{\mathsf{K}'^p} \to \mathscr{S}_{\mathsf{K}'^p}$. The perfection of the underlying reduced subscheme is the closed subscheme of $\mathscr{X}^b_{\mathsf{K}^p}$, cut out by the condition that the universal quasi-isogeny $\mathbb{X}_b \dashrightarrow \mathscr{A}[p^\infty]$ identifies $\underline{t}_b$ with $\underline{t}$.*

In particular, we can describe the geometry of the Newton stratum $\mathscr{S}^b_{\mathsf{K}^p}$ by the geometry of $X_\mu(b)$ and $\mathfrak{Ig}'^b_{\mathsf{K}^p,\overline{\mathbb{F}}_p}$ together with the $J_b(\mathbb{Q}_p)$-action.

3.4. Mantovan's formula

We can translate the geometric statement of Theorem 3.3.4 into a statement about the cohomology of $\mathsf{Sh}(\mathsf{G}, \mathsf{X})$. For this we need to assume that the following statement holds.

Axiom 3.4.1. Let $\mathsf{K} = \mathsf{K}^p\mathsf{K}_p$ be as above and denote by $\mathsf{K}_p(m) = \ker(G(\mathbb{Z}_p) \to G(\mathbb{Z}_p/p^m))$ and $\mathsf{K}(p^m) = \mathsf{K}^p\mathsf{K}_p(m)$. Denote by $\mathscr{S}_{m,\mathsf{K}^p}$ the normalisation of $\mathscr{S}_{\mathsf{K}_p}$ in $\mathsf{Sh}_{\mathsf{K}(m)}(\mathsf{G}, \mathsf{X})$, Then for any automorphic l-adic étale sheaf $\mathscr{L}_\xi$, the canonical morphism

$$\mathrm{H}^i_c(\mathscr{S}_{\mathsf{K}^p,m,\overline{\mathbb{F}}_p}, \mathrm{R}\Psi\mathscr{L}_\xi) \to \mathrm{H}^i_c(\mathsf{Sh}_{\mathsf{K}(m)}(\mathsf{G}, \mathsf{X}), \mathscr{L}_\xi)$$

is an isomorphism.

While this was not proven in full generality, it is reasonable to expect it to be proven in the near future. It was proven by Lan and Stroh under the assumption that a "good compactification" for $\mathscr{S}_m$ exist in [41, Cor. 5.20]. Following the reasoning of the proof of Mantovan's formula for PEL Shimura varieties in their paper, we see that it suffices to prove the isomorphy (respectively, the existence of good compatifications) for the more standard choice of the integral model given by the relative normalisation of the integral model of the Shimura variety with parahoric level structure (instead of the integral models constructed via Drinfeld level structure as in Mantovan's papers [46, 47]). This was done in the PEL case in [40] (see also [41, Prop. 2.2]).

In order to formulate the main result, we need one further ingredient. Let $\mathcal{M}^b := (\mathfrak{M}^b)^{\mathrm{ad}}_{\breve{E}}$ the adic generic fibre. Analogous to the classical case of Rapoport-Zink spaces with hyperspecial level structure we get a Galois tower $(\mathcal{M}^b_K)_{K \subset \mathsf{K}_p}$ over $\mathcal{M}^b$. We define Mantovan's functor

$$\mathrm{Mant}_{b,\mu}(-) := \sum_{i=0}^{2d} (-1)^i \varinjlim_{K} \mathrm{Ext}^{-2d+i}_{J_b(\mathbb{Q}_p)}(\mathrm{R}\Gamma_c(\mathcal{M}^b_K, \overline{\mathbb{Q}}_l), -)(-d),$$

from the Grothendieck group of smooth $J_b(\mathbb{Q}_p)$-representations over $\overline{\mathbb{Q}}_l$ to the Grothendieck group of smooth representations of $\mathsf{G}(\mathbb{A}_f) \times W_E$ over $\overline{\mathbb{Q}}_l$.

We use the following ad-hoc definitions to define the cohomology groups of the schemes $\mathsf{Sh}(\mathsf{G},\mathsf{X}) = \varprojlim_{\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)} \mathsf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})$ and $\mathrm{Ig}^b := \varprojlim_{\mathsf{K} \subset \mathsf{G}(\mathbb{A}_f)} \mathrm{Ig}^b_{\mathsf{K}^p}$. For this we note that $\mathrm{Ig}^b_{\mathsf{K}^p}$ can be written as limit of finite level Igusa varieties $\mathrm{Ig}^b_{\mathsf{K}^p,m}$, which are of finite type. We define

$$H^i_c(\mathsf{Sh}(\mathsf{G},\mathsf{X}), -) := \varinjlim_{\mathsf{K}} \mathrm{H}^i_c(\mathsf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X}), -)$$
$$H^i_c(\mathrm{Ig}^b, -) := \varinjlim_{\mathsf{K}^p,m} \mathrm{H}^i_c(\mathrm{Ig}^b_{\mathsf{K}^p,m}, -)$$

Theorem 3.4.2 (cf. [20, Cor. 6.2.9])**.** *Assume there exists at least one Kisin-Pappas integral model for* (G,X) *such that Axiom 3.3.2 and Axiom 3.4.1 hold. Then for any l-adic automorphic sheaf* $\mathscr{L}_\xi$ *we have the following equality of virtual smooth representations of* $\mathsf{G}(\mathbb{A}_f) \times W_E$ *over* $\overline{\mathbb{Q}}_l$*:*

$$\sum_i (-1)^i H^i_c(\mathsf{Sh}(\mathsf{G},\mathsf{X})_{\overline{\breve{E}}}, \mathscr{L}_\xi) = \sum_{b \in B(G,\{\mu\})} \sum_j (-1)^j \,\mathrm{Mant}_{b,\mu}(H^j_c(\mathrm{Ig}^b, \mathscr{L}_\xi)).$$

4. Geometry of affine Deligne-Lusztig varieties

4.1. Basic definitions

In this chapter we work in the same setup and use the same notation as in chapter 2. We denote by $K = L^+G(k) = G(O_{\breve{F}}) \subset G(\breve{F})$ the group of integral points. The quotient $[LG/L^+G]$ is representable by an inductive limit of finite type schemes ($\mathrm{char}\, F = p$) resp. of perfectly of finite type schemes ($\mathrm{char}\, F = 0$); see [61, Thm. 1.4],[5, Cor. 9.6]. We denote by $\mathscr{F}\ell_G$ its base change to k. Since the underlying topological space of $\mathscr{F}\ell_G$ is Jacobson, any locally closed subscheme is uniquely determined by its k-valued points. We note $\mathscr{F}\ell_G(k) = G(L)/K$.

To define affine Deligne-Lusztig varieties we fix an element $b \in G(\breve{F})$ and a locally closed subscheme Z of the loop group LG which is stable under L^+G-σ-conjugation. The associated affine Deligne-Lusztig variety is defined as the locally closed sub-ind-scheme of $\mathscr{F}\ell_G$ given by

$$X_Z(b)(k) = \{gK \in G(\breve{F})/K \mid g^{-1}b\sigma(g) \in Z\}.$$

By construction, we have a $J_b(F)$-action on $X_Z(b)$ given by left multiplication.

Theorem 4.1.1 ([23, Thm. 1.2])**.** *Assume that Z is quasi-compact.*

1. *$X_Z(b)$ is a scheme locally of finite type over k.*
2. *The action of $J_b(F)$ on the irreducible components of $X_Z(b)$ has finitely many orbits.*

This theorem is motivated to by fact that they are the underlying reduced subscheme of Rapoport-Zink spaces and to the general expectation for the arithmetic case that (at least in the minuscule case) affine Deligne-Lusztig varieties are the reduction modulo p of integral models of local Shimura varieties. Their cohomology is conjectured to decompose according to the local Langlands and Jacquet-Langlands correspondences. In order to be able to apply the usual methods, one needs the cohomology groups to be finitely generated $J_b(F)$-representations, and thus the "infinite level" cohomology groups to be admissible. This follows from the above theorem by a formal argument once the integral model is constructed (see for example [49, Thm. 4.4], [64, Prop. 6.1]).

Many particular cases of the theorem have been considered before. For the particular case of affine Deligne-Lusztig varieties arising as the underlying reduced subscheme of a Rapoport-Zink moduli space of p-divisible groups with additional structure of PEL type, questions as in Theorem 4.1.1 have been considered by several people. A recent general theorem along these lines is shown by Mieda [49]. Also, the (rare) cases where an affine Deligne-Lusztig variety is even of finite type have been classified, compare [13, Prop. 4.13].

4.2. Open questions regarding the geometry

The most interesting case is the case that G is a parahoric group scheme and that Z is a union of K-double cosets. There has been much progress in studying the geometry of these affine Deligne-Lusztig varieties in the recent years. Motivated by their applications in Shimura varieties and the Langlands program one has studied the following basic questions regarding affine Deligne Lusztig varieties with parahoric level K:

- What is their dimension ?
- What are their connected components ?
- What are their irreducible components ?

In [26], He defined the virtual dimension and proved that it is an upper bound in the case that G is quasi-split and splits over a tamely ramified extension. Recently Milicevic and Viehmann ([50]), and He ([27]) have proven that equality holds in many cases, but it is known that the formula does not hold in general.

If b is HN-irreducible it has been conjectured by Zhou that the connected components of $X_Z(b)$ with $Z = \bigcup_{w \in \mathrm{Adm}(\mu)} KwK$are the intersection of the affine Deligne-Lusztig variety with the connected components of the affine flag variety ([71, Conj. 5.4]). This statement would also imply an explicit (but more involved) description of the connected components in the case that b is not HN-irreducible. This statement has been proven for hyperspecial K by Nie ([59]) and for basic b or residually split G_F by He and Zhou ([28]).

For hyperspecial G, the set $\mathrm{Irr}\, X_{K\mu(\varpi)K}(b)$ of top-dimensional irreducible components of $X_{K\mu(\varpi)K}(b)$ has been shown to be in canonical bijection with the Mirkovic-Vilonen basis $\mathbb{B}_\mu(\lambda)$ of a certain weight space $V_\mu(\lambda)$ of the highest weight representation of the dual group $\hat{G}$ in [58], [72], which was originally conjectured by Chen and Zhu. Moreover Zhou and Zhu deduced the numerical version of the conjecture, i.e. they showed that both sets have the same cardinality, for any very special K by a formal argument from the hyperspecial case. The variety $X_{K\mu(\varpi)K}(b)$ is known to equidimensional for most cases so that one may ignore the "top-dimensional" requirement (see Thm 4.3.1 below).

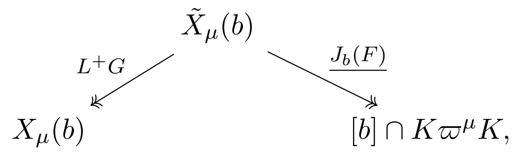
4.3. Affine Deligne-Lusztig varieties with absolutely special level

For general parahoric level structure and the above questions are still widely open. We will therefore focus on the case that K is absolutely special. In particular, we assume that G is quasi-split. We fix a special F-torus $S \subset G_F$, i.e. S is a maximal $\breve{F}$-split torus and contains a maximal F-split torus. Moreover, let $T = Z_{G_F}(S)$ be its centralizer, which is a maximal F-torus, and let $B = T \cdot U$ be a (rational) Borel subgroup. We denote by $\mathcal{A} = X_*(T)_{\mathrm{Gal}(\overline{F}, \breve{F}), \mathbb{Q}}$ the apartment of $T_{\breve{F}}$ in the Bruhat-Tits building of $G_{\breve{F}}$ and denote by I and K the parahoric subgroups fixing the base alcove in the dominant Weyl chamber and the point 0, respectively. We fix $\mu \in X_*(T)_{\mathrm{Gal}(\overline{F}, \breve{F})}$ and denote by $\varpi^\mu \in \mathscr{F}\ell_T(k) \subset \mathscr{F}\ell_G(k)$ the corresponding element. We abbreviate $X_\mu(b) = X_{K\varpi^\mu K}(b)$.

Before stating our main result, we present a slightly different viewpoint on the $X_\mu(b)$ and the $J_b(F)$-actions. We consider σ-conjugacy class $[b] \subset G(L)$ as the k-valued points of a reduced sub-ind-scheme in LG (also denoted $[b]$). We denote by $\mathcal{G}_{[b],\mu}$ the "universal local $\mathscr{K}$-shtuka over $K\varpi^\mu K \cap [b]$", i.e. whose fibre over b' equals $(L^+\mathscr{K}, b'\sigma)$. Then the local model map

$$\ell_b \colon \tilde{X}_\mu(b) \coloneqq \{g \in LG \mid g^{-1}b\sigma(g) \in K\varpi^\mu K\} \to [b] \cap K\varpi^\mu K$$
$$g \mapsto g^{-1}b\sigma(g)$$

canonically identifies $\tilde{X}_\mu(b)$ with the moduli space of quasi-isogenies $(L^+G, b\sigma) \to \mathcal{G}_{[b],\mu}$. Thus the morphism is the composition of a proétale $\underline{J_b(F)}$-torsor and a universal homeomorphism . We obtain a diagram



where an arrow labeled by a group H means that it defines an H-torsor over a scheme universally homeomorphic to the target of the arrow. Since L^+G is irreducible, we deduce that $J_b(F)$-orbits of connected/irreducible components of $X_\mu(b)$ are canonically identified with the respective components of $[b] \cap K\varpi^\mu K$.

Theorem 4.3.1 ([15, Thm. 1.1,Cor. 7.4,7.9])**.** *1.* *The dimension of $X_\mu(b)$ equals the virtual dimension*

$$d_G(\mu, b) \coloneqq \langle \rho, \mu - \nu \rangle - \frac{1}{2}\operatorname{def}(b),$$

where $\operatorname{def}(b) \coloneqq \operatorname{rk}_F(G) - \operatorname{rk}_F(J_b)$ *equals the rank of defect of* b*.*

2. *There exists a canonical bijection $J_b(F)\backslash \operatorname{Irr} X_\mu(b) \overset{1:1}{\to} \mathbb{B}_\mu(\lambda([b]))$ where $\lambda([b]) \in X_*(S)$ denotes the best integral approximation of the Newton point.*

3. *If (b, μ) is HN-irreducible, then the connected components of $X_\mu(b)$ are precisely the intersection of $X_\mu(b)$ with the connected components of $\mathscr{F}\ell_G$.*

4. *Assume that* $\operatorname{char} F = p$ *or that $F = \mathbb{Q}_p$ and (G, μ) is induced by a Shimura datum of Hodge type* (G, X)*. Then $X_\mu(b)$ is equidimensional.*

The first two points are proven by reducing to the case that $[b]$ is superbasic and μ is minuscule. In this case one can write $X_\mu(b)$ as locally finite disjoint union of affine spaces stable under $J_b(F)$-action. In particular, the dimension of $X_\mu(b)$ equals the maximal dimension of an affine space occuring in this decomposition and the $J_b(F)$-orbits of irreducible components are in canonical bijection with the $J_b(F)$-orbits of top-dimensional affine spaces. Thus the statistics of the dimension of the affine spaces is a central point of this proof and is one of the main results of [22]. Studying this decomposition in detail, we can give an explicit description of the bijection in (2) . Denote by M the (unique) Levi subgroup such that $[b] \cap M(L)$ is superbasic and $P = MN$ the corresponding standard parabolic subgroup of G. We assume that $b \in M(\breve{F})$ normalises T and Iwahori $I_M := M(\breve{F}) \cap I$. For $C \in \operatorname{Irr} X_\mu(b)$ denote by $\lambda_C \in X_*(T)_{\operatorname{Gal}(\overline{F}/\breve{F})}$ the cocharacter such that C is generically contained in $N(L) I_M \varpi^{\lambda_C} K$ and let w_C the shortest element in the Weyl group of M such that $w_C^{-1}(\lambda_C)$ is M-dominant. Moreover we denote by $\tilde{C} \subset LG$ the preimage of C.

Proposition 4.3.2 ([15, Thm. 1.2])**.** *Denote by $Z_C \subset U\varpi^{\tilde{\lambda}} K \cap K\mu K$ (for a certain $\tilde{\lambda} \in X_*(T)_I$ restricting to $\lambda([b])$) be the Mirkovic-Vilonen cycle corresponding to the image of C in $\mathbb{B}_\mu(\lambda([b]))$. Then Z_C is determined by the following properties.*

1. $\tilde{\lambda} = b\sigma(\lambda) - \lambda$.
2. *The set $\ell_b(\tilde{C} \cap N(L) I_M \varpi^{\lambda_C}) \cdot K$ is an open dense subset of $w_C(Z_C)$.*

4.4. Application to Shimura varieties

As a consequence of the results above, we obtain the following application to Shimura varieties. Let (G, X) be a Shimura datum of Hodge type such that $\mathsf{G}_{\mathbb{Q}_p}$ is quasi-split, $\mathsf{K} = K \cdot \mathsf{K}^p \subset G(\mathbb{A}_f)$ a small enough compact open subgroup whose level K at p is as above. We denote by μ the dominant representative in the conjugacy class of cocharacters of G associated to X.

Theorem 4.4.1 ([15, Thm. 1.3], see also [16, Thm. 2] for hyperspecial K)**.** *For any $[b] \in B(\mathsf{G}_{\mathbb{Q}_p}, \mu)$, we have*

1. *$\mathscr{S}^{[b]}_{\mathsf{K}^p}$ is of pure dimension $\langle \rho, \mu + \nu([b]) \rangle - \frac{1}{2} \operatorname{def}_G(b)$.*
2. $\overline{\mathscr{S}^{[b]}_{\mathsf{K}^p}} = \bigcup_{[b'] \leq [b]} \mathscr{S}^{[b']}_{\mathsf{K}^p}$.

Sketch of proof. This theorem is direct consequence of the purity result combined with the almost product structure proven above. As a consequence of Corollary 2.3.2, it suffices to show that $\langle \rho, \mu + \nu([b]) \rangle - \frac{1}{2} \operatorname{def}_G(b)$. By [20, Thm. 4.10], we have that for every

$x \in \mathscr{S}_{\mathsf{K}^p}(\overline{\mathbb{F}}_p)^{[b]}$ the Serre-Tate isomorphism $\mathscr{S}^{\wedge}_{\mathsf{K}^p,x} \cong \mathfrak{Def}(\mathscr{A}[p^\infty], \underline{t}_x)$ respects the Newton stratification. As a consequence of the product structure constructed in § 3.2, we obtain

$$\dim_x \mathscr{S}^{[b]}_{\mathsf{K}^p} = \dim \mathrm{C}_G(\mathscr{A}_x[p^\infty], \underline{t}_x) + \dim \mathrm{I}_G(\mathscr{A}_x[p^\infty])$$

We have $\dim \mathrm{C}_G(X) = \langle 2\rho, \nu([b])\rangle$ by [32, Cor. 5.3.1] (see also [16, Prop. 5] in the hyperspecial case). As a consequence of their respective moduli descriptions, $\mathrm{I}_G(\mathscr{A}_x[p^\infty])^{\mathrm{perf}}$ is a formal neighbourhood in $X_\mu(b)$ and hence

$$\dim \mathrm{I}_G(\mathscr{A}_x[p^\infty])^{\mathrm{perf}} = \dim \mathrm{I}_G(\mathscr{A}_x[p^\infty])^{\mathrm{perf}} \leq \dim X_\mu(b) = \langle \rho, \mu - \nu([b])\rangle - \frac{1}{2}\operatorname{def}(b),$$

finishing the proof of above theorem. □

5. Compact support cohomology of “infinite type” schemes

5.1. Schemes of finite expansion

In the last years there has been a trend in arithmetic algebraic geometry to work directly with geometric objects which are not of finite type. The most popular example are Scholze’s perfectoid spaces, which form a family of “infinite” objects such that any analytic adic space over $\mathbb{Z}_p$ has a proétale cover which is a perfectoid space. The most important instances of perfectoid spaces, from the view of the Langlands program, are (adic) infinite level Shimura varieties and their local analogues. For schemes this development has taken place on a smaller scale; examples are the work about perfect schemes with application to the affine flag variety of Bhatt and Scholze ([5]), as well as their work on proétale morphisms ([4]).

In the previous sections we worked with such “infinite type” schemes such as affine Deligne Lusztig varieties in mixed characteristic, which are perfections of schemes locally of finite type, and the Igusa variety and implicitly the infinite level Shimura variety $\mathsf{Sh}(\mathsf{G},\mathsf{X}) := \varprojlim_{\mathsf{K}} \mathsf{Sh}_{\mathsf{K}}(\mathsf{G},\mathsf{X})$, which are profinite étale covers over schemes (perfectly) of finite type.

All examples above are contained in a well-behaved larger class of morphisms, called morphisms locally of finite expansion. Their definition and behaviour is similar to those of finite type morphisms.

Definition 5.1.1. Let R be a ring and let A be an R-algebra. A family $(a_i)_{i\in I}$ of elements in A is called quasi-generating system of A, if A is integral over $R[a_i \mid i \in I]$; the a_i are called quasi-generators. If there exists a finite quasi-generating system of A, we say that A is of finite expansion over R.

Definition 5.1.2 ([14, Def./Cor. 1.4])**.** We call a morphism $f\colon X \to Y$ of schemes locally of finite expansion if the following equivalent conditions are satisfied.

1. For every affine open subscheme $V \subset Y$ and every affine open subscheme $U \subset f^{-1}(V)$, the $\mathscr{O}_Y(V)$-algebra $\mathscr{O}_X(U)$ is of finite expansion.
2. There exists a covering $Y = \bigcup V_i$ by open affine subschemes $V_i \cong \operatorname{Spec} R_i$ and a covering $f^{-1}(V_i) = \bigcup U_{i,j}$ by open affine subschemes $U_{i,j} \cong \operatorname{Spec} A_{i,j}$ such that for all i, j the R_i-algebra $A_{i,j}$ is of finite expansion.

We say that a morphism $f\colon X \to Y$ is of finite expansion if it is locally of finite expansion and quasi-compact.

The property of morphism of finite expansion is local and satisfies the usual permanence properties (cf. [14, Cor. 1.5]). Most importantly, they are precisely the compactifiable morphisms in the following sense.

Theorem 5.1.3 ([14, Thm. 1.17,Cor. 1.9]). *Let $f\colon X \to Y$ be a separated morphism between qcqs schemes. Then f is of finite expansion if and only if it can be written as composition $f = \bar{f} \circ j$ where $j\colon X \to \bar{X}$ is an open embedding and $\bar{f}\colon \bar{X} \to Y$ is separated and universally closed.*

5.2. Cohomology of compact support

For applications in the Langlands program, we are interested in the cohomology groups with compact support of schemes of finite expansion over an algebraically closed field, e.g. tge infinite level Shimura variety. The classical construction only defines cohomology of compact support (or more generally the direct image with compact support of a morphism) for morphisms of finite type.

Following the general framework of Deligne ([1]), we define the higher direct image of f as

$$Rf_! \coloneqq Rp_* \circ j_!.$$

We show that this functor is well defined and has a right adjoint $Rf^!$ such that the sextuple $(Rf_*, f^*, Rf_!, Rf^!, R\operatorname{Hom}, \otimes_{\breve{F}})$ satisfies the axioms of Grothendieck's six operations. More precisely, we prove the following results.

Theorem 5.2.1 ([14, Thm. 1.2]). *Let S be a scheme and $\mathscr{A}$ be a torsion sheaf on the étale site of S. Let (S) denote the category whose objects are qcqs schemes over S and whose morphisms are separated morphism of schemes of finite expansion.*

1. *There is an essentially unique way to assign to every morphism $f\colon X \to Y$ a functor $Rf_!\colon D(X, \mathscr{A}_X) \to D(Y, \mathscr{A}_Y)$ such that $Rf_! = Rf_*$ if f is universally closed, $Rf_! = f_!$ if f is an open immersion and there is a collection of isomorphisms $R(f \circ g)_! \cong Rf_! \circ Rg_!$ satisfying the usual cocycle condition.*
2. *The functor $Rf_!$ has a partial right adjoint $Rf^!\colon D^+(Y, \mathscr{A}_Y) \to D^+(X, \mathscr{A}_X)$. The formalism of Grothendieck's six operations for morphisms of finite type extends to (S).*

Example 5.2.2. If $S = \operatorname{Spec} k$, we may simply write $R^i f_! = \mathrm{H}^i_c$ is called the i-th cohomology group with compact support. For $X = \mathsf{Sh}(\mathsf{G}, \mathsf{X})$ and $X = \mathrm{Ig}^b$, the construction above coincides with the limit construction in § 3.4 applied to torsion coefficients.

For separated $\mathbb{C}$-schemes of finite expansion the above notion of $Rf_!$ coincides with the topological direct image functor with proper support. More precisely, the following generalisation of Artin's comparison theorem holds.

Theorem 5.2.3 ([14, Thm. 1.3])**.** *Let $f\colon X \to Y$ be a morphism of separated $\mathbb{C}$-schemes of finite expansion, $\mathscr{F} \in D^+_{\mathrm{tors}}(X, \mathbb{Z}_X)$ and denote by $(-)_{\mathrm{an}}$ the analytification functor. Then there exists a natural isomorphism $(Rf_!\mathscr{F})_{\mathrm{an}} \cong Rf_{\mathrm{an},!}\mathscr{F}_{\mathrm{an}}$.*

6. Isogeny classes of global G-shtukas

6.1. Background

The function field analogue of an Abelian variety with additional structure is a global $\mathscr{G}$-shtuka. More precisely, let C be a curve (i.e. a geometrically integral smooth projective scheme of dimension 1) over $\mathbb{F}_q$ and let F be its field of rational functions. We fix a smooth affine group $\mathscr{G}$ over C whose generic fibre G is a reductive linear algebraic group.

Definition 6.1.1. Let S be an $\mathbb{F}_q$-scheme. We denote $\sigma \coloneqq (\mathrm{id} \times \mathrm{Frob}_q) \colon C \times_{\mathbb{F}_q} S \to C \times_{\mathbb{F}_q} S$.

1. A global $\mathscr{G}$-shtuka with n paws over S is a tuple $(x_1, \dots, x_n, \mathscr{V}, \tau)$ where
   - $x_i \colon S \to C$ are morphisms of $\mathbb{F}_q$-schemes,
   - $\mathscr{V}$ is a $\mathscr{G}$-bundle on $C \times_{\mathbb{F}_q} S$ and
   - $\tau \colon \sigma^*\mathscr{V}|_{S \setminus \bigcup_{i=1}^n \Gamma(x_i)} \xrightarrow{\sim} \mathscr{V}|_{S \setminus \bigcup_{i=1}^n \Gamma(x_i)}$ where $\Gamma(x_i) \subset C \times_{\mathbb{F}_q} S$ denotes the graph of x_i.
2. A quasi-isogeny between two global $\mathscr{G}$-shtukas $(x_1, \dots, x_n, \mathscr{V}, \tau) \dashrightarrow (x_1', \dots, x_n', \mathscr{V}', \tau')$ is a birational $C \times S$-map $\varphi \colon \mathscr{V} \dashrightarrow \mathscr{V}'$ such that $\tau' \circ \varphi = \sigma^*\varphi \circ \tau$. This defines an equivalence relation on the set of global G-shtukas over S, whose equivalence classes are called isogeny classes.

Let $\breve{\mathsf{F}} \coloneqq \mathsf{F} \otimes_{\mathbb{F}_q} k$. It is easy to see that the restriction to the generic point defines a bijection between the isogeny classes of $\mathscr{G}$-shtukas over k and pairs (V, τ) where V is a G-torsor over $\breve{\mathsf{F}}$ and $\tau \colon \sigma^*\mathsf{V} \xrightarrow{\sim} \mathsf{V}$ is an isomorphism. Since $\breve{\mathsf{F}}$ has cohomological dimension one by Tsen's theorem, any G-torsor V is trivial by [6, § 8.6]. Choosing a trivialisation $\mathsf{V} \cong \mathsf{G}_{\breve{\mathsf{F}}}$, the morphism τ gets identified with the morphism $\mathsf{b} \circ \sigma$ for some $\mathsf{b} \in \mathsf{G}(\breve{\mathsf{F}})$. Every other trivialisation of V can be obtained by postcomposing the above isomorphism with an element $\mathsf{g} \in \mathsf{G}(\breve{\mathsf{F}})$, thus replacing b by $\mathsf{g}\mathsf{b}\sigma(\mathsf{g}^{-1})$. Hence this construction yields a natural bijection between the isomorphism classes of G-isoshtukas over k and the set of σ-conjugacy classes in $\mathsf{G}(\breve{\mathsf{F}})$. We denote the latter by $B(\mathsf{F}, \mathsf{G})$.

6.2. Classifying σ-conjugacy classes over function fields

Following the strategy of Kottwitz' work [37],[39] on σ-conjugacy classes over p-adic fields and his construction of $B(F, G)$ for local and global fields in terms of Galois gerbs in [36], we describe its elements via two invariants ν_{G} and κ_{G} on $B(\mathsf{F}, \mathsf{G})$. This work is independent of the classification of Drinfeld's classification of φ-spaces ([11], see also [44]), which corresponds to the case $\mathsf{G} = \mathsf{GL}_n$.

Let us give more details on ν_{G} and κ_{G}. For any finite field extension E/F, we denote by $\mathrm{Div}(\mathsf{E})$ the free abelian group generated by the set of places in E and let

$$\mathrm{Div}(\mathsf{E})_0 = \left\{\sum n_y \cdot y \in \mathrm{Div}(\mathsf{E}) \mid \sum n_y = 0\right\}.^1$$

For every finite extension E'/E, we obtain a homomorphism $\mathrm{Div}(\mathsf{E})_0 \to \mathrm{Div}(\mathsf{E}')_0$ given by $x \mapsto \sum_{x'|x}[\mathsf{E}'_{x'} : \mathsf{E}_x]x'$. We define $\mathrm{Div}(\mathsf{F}^s)_0 = \varinjlim \mathrm{Div}(\mathsf{E})_0$, where E runs through all finite separable extensions of F and let $\mathbb{D}_{\mathsf{F}}$ be the diagonalisable group over F with character group $\mathrm{Div}(\mathsf{F}^s)_0$. Moreover we fix F'/F Galois such that G splits over F'. In [21] we construct invariants

$$\begin{aligned} \bar{\kappa}_{\mathsf{G}} &\colon B(\mathsf{F},\mathsf{G}) \to (\pi_1(\mathsf{G}) \otimes \mathrm{Div}(\mathsf{F}')_0)_{\mathrm{Gal}(\mathsf{F}'/\mathsf{F})} \\ \bar{\nu}_{\mathsf{G}} &\colon B(\mathsf{F},\mathsf{G}) \to \left(\mathrm{Hom}_{\breve{\mathsf{F}}}(\mathbb{D}_{\mathsf{F}}, \mathsf{G})/\mathsf{G}(\breve{\mathsf{F}})\right)^{\sigma}, \end{aligned}$$

which we call the Kottwitz map and the Newton map, respectively.

Interestingly, the set $B(\mathsf{F},\mathsf{G})$ shares a lot of properties with its analogue over local fields. We show in [21] that $\bar{\nu}_{\mathsf{G}}(\mathbf{b})$ is trivial if and only if $\mathbf{b}$ lies in the image of $\mathrm{H}^1(\mathsf{F},\mathsf{G}) \hookrightarrow B(\mathsf{F},\mathsf{G})$. More generally, we call $\mathbf{b} \in B(\mathsf{F},\mathsf{G})$ basic, if $\bar{\nu}_{\mathsf{G}}(\mathbf{b})$ factors through the center of G. The following theorem classifies basic σ-conjugacy classes. In particular, this gives a complete description of $B(\mathsf{F},\mathsf{G})$ when G is a torus.

Theorem 6.2.1 ([21, Thm. 1.2])**.** *The Kottwitz map induces an isomorphism*

$$B(\mathsf{F},\mathsf{G})_b \xrightarrow{\sim} (\pi_1(\mathsf{G}) \otimes \mathrm{Div}(\mathsf{F}')_0)_{\mathrm{Gal}(\mathsf{F}'/\mathsf{F})}.$$

To obtain a description of the whole $B(\mathsf{F},\mathsf{G})$ by its invariants, we proceed as follows. Since there exists a simple combinatorial description how $B(\mathsf{F},\mathsf{G})$ behaves under ad-isomorphisms, we may reduce to the case that G is of adjoint type (cf. [21, Prop. 5.6]. In particular, the quasi-split inner form G^* of G is an (extended) pure inner form. From that we deduce that $B(\mathsf{F},\mathsf{G}) \cong B(\mathsf{F},\mathsf{G}^*)$ ([21, Lem. 5.3]). Thus it suffices to describe $B(\mathsf{F},\mathsf{G})$ for quasi-split G. In this case, we can reduce to the theorem above since every σ-conjugacy class in G is induced by a σ-conjugacy class of an F-torus in G. More precisely, we get the following result.

Theorem 6.2.2 ([21, Thm. 1.3])**.** *Let* G *be a reductive group.*

[1] Note that this definition is different from the subgroup of degree zero divisors $\mathrm{Div}^0(\mathsf{E}) = \left\{\sum n_y \cdot y \in \mathrm{Div}(\mathsf{E}) \mid \sum n_y \cdot \deg(y) = 0\right\}$. This is due to the fact that we are actually considering the Galois coinvariants of $\mathrm{Div}^0(\breve{\mathsf{F}})$ where every place has degree one.

1. *Every* $\mathbf{b} \in B(\mathsf{F}, \mathsf{G})$ *is uniquely determined by its invariants* $\bar{\kappa}_{\mathsf{G}}(\mathbf{b})$ *and* $\bar{\nu}_{\mathsf{G}}(\mathbf{b})$.
2. *If* G *is quasi-split, the canonical map*

$$\bigcup_{\substack{\mathsf{T} \subset \mathsf{G} \\ \text{max. } \mathsf{F}-\text{torus}}} B(\mathsf{F}, \mathsf{T}) \to B(\mathsf{F}, \mathsf{G})$$

is surjective.

The second part of statement can be seen as an analogue for the moduli space of global $\mathscr{G}$-shtukas to the statement that every isogeny class in the special fibre of a Shimura variety contains a point that can be lifted to a CM-point. A proof of the latter statement for Shimura varieties of PEL-type was first sketched in a letter of Langlands to Rapoport, and was proven in differing generality by Milne [51], Zink [74], Kottwitz [38], Kisin [35] and most recently by Zhou [71] for Shimura varieties of Hodge type with paraholic level structure at p given that certain group theoretic conditions are satisfied.

These results about $B(\mathsf{F}, \mathsf{G})$ provide useful tools to study points in the special fibre of moduli space of $\mathscr{G}$-shtukas. There is an extremely long list of previous results on 'point-counting' on the moduli space of $\mathscr{G}$-shtukas, most of which partition points by isogeny classes. For the most recent results applicable beyond inner forms of GL_n, see [2, 55, 56, 57].

Another natural question to ask is whether the pointed set of σ-conjugacy classes $B(\mathsf{F}, \mathsf{G})$ and the pointed set of Galois gerbs constructed by Kottwitz in [36] are the same, i.e. whether there exists a canonical isomorphism of functors. This question is proven to have a positive answer in an upcoming work of Iakovenko [30].

6.3. Comparison with σ-conjugacy classes over local fields

Let x be a closed point of C and let $F = \mathsf{F}_x$ be the completion of F at x and let $\breve{F} = D_{\overline{k_F}^*}$ denote its maximal unramified extension. Then every σ-conjugacy class in $\mathsf{G}(\breve{\mathsf{F}})$ induces a σ-conjugacy class in $\mathsf{G}(\breve{F})$, whose Newton and Kottwitz point can be calculated from their global counterparts.

Firstly, we give a more explicit description of $\mathrm{Div}(\mathsf{F}^s)_0$. For every finite field extension E/F we denote by C_{E} the curve with function field E and by $|C_{\mathsf{E}}|$ the set of its closed points, equipped with the discrete topology. We identify

$$\mathrm{Div}(\mathsf{E}) \cong \{\varphi \in C_c^\infty(|C_{\mathsf{E}}|, \mathbb{Q}) \mid \forall y \in |C_E| : \varphi(y) \in \frac{1}{[\mathsf{E}_y : \mathsf{F}_y]} \cdot \mathbb{Z}\}$$
$$D = \sum a_y \cdot y \mapsto (\varphi_D : y \mapsto \frac{a_y}{[\mathsf{E}_y : \mathsf{F}_y]}).$$

Note that with respect to this identification $\varphi_{f^*_{\mathsf{E}'/\mathsf{E}}(D)}$ is precisely the composition of φ_D with the canonical morphism $f_{\mathsf{E}'/\mathsf{E}}\colon C_{\mathsf{E}'} \to C_{\mathsf{E}}$ for any finite extension E'/E and $D \in \mathrm{Div}(\mathsf{E})$. Hence we can take the limit to obtain $\mathrm{Div}(\mathsf{F}^s) := \varinjlim \mathrm{Div}(\mathsf{E}) \cong C_c^\infty(|C_{\mathsf{F}^s}|, \mathbb{Q})$, where $|C_{\mathsf{F}^s}| = \varprojlim |C_{\mathsf{E}}|$ is equipped with the limit topology. To describe $\mathrm{Div}(\mathsf{F}^s)_0$, we let μ be the Borel measure on $|C_{\mathsf{F}^s}|$ such that its restriction to $f^{-1}_{\mathsf{F}^s,\mathsf{F}}(\{x\}) = \varprojlim f^{-1}_{\mathsf{E}/\mathsf{F}}(\{x\})$ equals the limit of the uniform probability measures on $f^{-1}_{\mathsf{E}/\mathsf{F}}(\{x\})$. Then the above isomorphism identifies

$$\mathrm{Div}(\mathsf{F})_0 \cong \{\varphi \in C_c^\infty(|C_{\mathsf{F}^s}|, \mathbb{Q}) \mid \int \varphi \, d\mu = 0\}.$$

Now fix a point $x \in |C|$ and an embedding of separable closures $\mathsf{F}^s \hookrightarrow \mathsf{F}_x{}^s$, determining a place $y \in |C_{\mathsf{F}^s}|$. The evaluation map $\mathrm{Div}(\mathsf{F})_0 \to \mathbb{Q}, D \mapsto \varphi_D(y)$ induces a morphism of tori $\iota_x\colon \mathbb{D} \to \mathbb{D}_{\mathsf{F}}$ defined over $\mathsf{F}_x{}^s$.

Proposition 6.3.1 ([21, Prop. 1.1])**.** *The map* $\mathsf{b} \mapsto \mathsf{b} \cdot \sigma(\mathsf{b}) \cdots \sigma^{\deg(x)-1}(\mathsf{b})$ *induces a map* $N_x\colon B(\mathsf{F}, \mathsf{G}) \to B(F_x, \mathsf{G})$*. Moreover, we have*

$$\bar{\nu}(N_x([\mathsf{b}])) = \bar{\nu}_{\mathsf{G}}([\mathsf{b}]) \circ \iota_x,$$
$$\bar{\kappa}([N_x(\mathsf{b})]) = \mathrm{loc}_x(\bar{\kappa}_{\mathsf{G}}([\mathbf{b}]))$$

where the terms on the left hand side are the Newton point and Kottwitz point for the local field F_x *as described in section 2.1 and* loc_x *denotes the composition of*

$$(\mathrm{Div}(\mathsf{F}')_0 \otimes \pi_1(\mathsf{G}))_{\mathrm{Gal}(\mathsf{F}'/\mathsf{F})} \xrightarrow{\text{can. proj.}} (\bigoplus_{x'|x} \mathbb{Z} \cdot x' \otimes X_*(\mathsf{G}))_{\mathrm{Gal}(\mathsf{F}'/\mathsf{F})} \cong \pi_1(\mathsf{G})_{\mathrm{Gal}(\mathsf{F}'_y/F_x)}$$

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