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Many visits to a single site by a transient random walk in random environment

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Abstract

We consider a transient random walk on $\mathbb{Z}$ in random environment, and study the almost sure asymptotics of the supremum of its local time. Our main result states that if the random walk has zero speed, there is a (random) sequence of sites and a (random) sequence of times such that the walk spends a positive fraction of the times at these sites. This was known for a recurrent random walk in random environment (Random Walk in Random and Non-Random Environments, World Scientific, Singapore, 1990; Stochastic Process. Appl. 76 (1998) 231). Our method of proof is different and relies on the connection of random walk in random environment with branching processes in random environment used in Kesten et al. (Compositio Math. 30 (1975) 145).
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1. Introduction and statement of the results

Let $\omega=(\omega_x)_{x\in\mathbb{Z}}$ be a collection of i.i.d. random variables taking values in $[0,1]$ and let μ be the distribution of ω. For each $\omega\in\Omega=[0,1]^{\mathbb{Z}}$, we define the random walk in random environment (RWRE) as the time-homogeneous Markov chain taking values in $\mathbb{Z}$, with transition probabilities $P_\omega[X_{n+1}=x+1\mid X_n=x]=\omega_{X_n}=1-P_\omega[X_{n+1}=x-1\mid X_n=x]$,

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and $X_0 = 0$. We equip Ω with its Borel σ-field $\mathscr{F}$ and $\mathbb{Z}^{\mathbb{N}}$ with its Borel σ-field $\mathscr{G}$. The distribution of $(\omega, (X_n))$ is the probability measure $\mathbb{P}$ on $\Omega \times \mathbb{Z}^{\mathbb{N}}$ defined by $\mathbb{P}[F \times G] = \int_F P_\omega[G]\mu(\mathrm{d}\omega)$, $F \in \mathscr{F}$, $G \in \mathscr{G}$. Let $\rho_0 = \rho_0(\omega) := (1 - \omega_0)/\omega_0$. We will always assume that

$$-\infty < \int \log \rho_0 \, \mu(\mathrm{d}\omega) < 0 \tag{1.1}$$

and that there is $\kappa > 0$ such that

$$\int \rho_0^\kappa \, \mu(\mathrm{d}\omega) = 1. \tag{1.2}$$

Assumption (1.2) implies that the RWRE has "mixed drifts", i.e., $\frac{1}{2}$ is in the convex hull of the support of ω_0. In particular, (1.2) and (1.1) imply that $\mathrm{Var}(\omega_0) > 0$, i.e., the environment is non-deterministic. Assumption (1.1) implies that the RWRE (X_n) is transient to the right, i.e., $X_n \to \infty$, $\mathbb{P}$-a.s., (see Solomon, 1975).

We further assume that

$$\int \rho_0^\kappa (\log^+ \rho_0) \mu(\mathrm{d}\omega) < \infty \tag{1.3}$$

and that the distribution of $\log \rho_0$ is non-arithmetic, meaning that the group generated by $\mathrm{supp}(\log \rho_0)$ is dense in $\mathbb{R}$.

Let

$$\xi(n,x) := |\{0 \leqslant i \leqslant n \colon X_i = x\}|,$$

$$\xi^*(n) = \xi^*(n)(\omega, (X_i)) := \sup_{x \in \mathbb{Z}} \xi(n,x).$$

In words, $\xi^*(n)$ records the maximal number of visits the RWRE can pay to a single site in the first n steps. If $0 < \kappa \leqslant 1$, the random walk has zero speed, i.e., $X_n/n \to 0$, $\mathbb{P}$-a.s. (Solomon, 1975). Our main result is the following.

Theorem 1.1. *Assume $0 < \kappa \leqslant 1$. There is a constant $c > 0$ such that*

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n} = c, \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

Identity (1.4) was proved by Révész (1990) (see also Shi, 1998) for a recurrent RWRE, i.e., if $\int (\log \rho_0)\mu(\mathrm{d}\omega) = 0$. This is in complete contrast with the case of usual random walk (in non-random environment) for which $\xi^*(n)$ is at most $\mathcal{O}(\sqrt{n \log \log n})$ a.s., whether the walk is transient or recurrent. It is interesting to note that $\xi^*(n)$ can be as large as a constant multiple of n (at least along a random subsequence), even though the RWRE is transient.

If $\kappa > 1$, the RWRE has (strictly) positive, deterministic speed (Solomon, 1975):

$$\lim_{n\to\infty} \frac{X_n}{n} = v > 0, \quad \mathbb{P}\text{-a.s.}, \tag{1.5}$$

where v is given by $(1-\langle\rho_0\rangle)/(1+\langle\rho_0\rangle)$, with $\langle\rho_0\rangle := \int \rho_0(\omega)\mu(\mathrm{d}\omega)$. In this regime, we prove that $\xi^*(n)/n \to 0$ $\mathbb{P}$-a.s. We actually give some accurate information about how this quantity goes to 0.

Theorem 1.2. *Assume* $\kappa > 1$. *For any positive and non-decreasing sequence* (a_n), *we have*

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa} a_n} = \begin{cases} 0 \\ \infty \end{cases} \mathbb{P}\text{-a.s.} \Leftrightarrow \sum_n \frac{1}{n(a_n)^\kappa} \begin{cases} < \infty, \\ = \infty. \end{cases}$$

As a consequence, $\mathbb{P}$-*almost surely*,

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa}(\log n)^\gamma} = \begin{cases} 0 & \text{if } \gamma > 1/\kappa, \\ \infty & \text{otherwise.} \end{cases}$$

In particular,

$$\lim_{n\to\infty} \frac{\xi^*(n)}{n} = 0, \quad \mathbb{P}\text{-a.s.}$$

The rest of the paper is organized as follows. In Section 2, we present some probability estimates of the maximum local time of RWRE stopped at the first hitting time. These estimates are obtained by exploiting a relationship used in Kesten et al. (1975) between RWRE and branching processes in random environment, and will be used in Section 3 to prove Theorems 1.1 and 1.2. Finally, Section 4 is devoted to some further remarks and open questions.

2. Probability estimates

Under condition (1.1), the RWRE is transient to the right (Solomon, 1975), so the hitting times

$$T_m := \inf\{n: X_n = m\}$$

are finite, $\mathbb{P}$-a.s., for any $m \geqslant 1$. The proofs of Theorems 1.1 and 1.2 are based on some probability estimates for $\xi^*(T_m)$, stated as follows. For the sake of clarity, these estimates are formulated in three distinct lemmas, which together cover all the possible cases for κ.

Lemma 2.1. *Let* $0 < \kappa < 1$. *There exist constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\inf_{m\geqslant 1} \mathbb{P}[\xi^*(T_m) \geqslant c_1 T_m] \geqslant c_2. \tag{2.1}$$

Lemma 2.2. *Assume $\kappa = 1$. There exist constants $c_3 > 0$ and $c_4 > 0$ such that for all $m \geqslant 2$,*

$$\mathbb{P}[\xi^*(T_m) \geqslant c_3\, T_m] \geqslant \frac{c_4}{\log m}. \tag{2.2}$$

Lemma 2.3. *Whenever $\kappa > 0$, there exist constants $c_5 > 0$ and $c_6 > 0$ such that for all $m \geqslant 1$ and $\lambda \geqslant m^{1/\kappa}$,*

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \geqslant \frac{c_5 m}{\lambda^\kappa} - e^{-c_6 m}. \tag{2.3}$$

To prove these lemmas, we use the following representation of the hitting times T_m in terms of branching process in random environment (BPRE), which goes back to Kozlov (1973), and was used in Kesten et al. (1975) (see also Dembo et al., 1996). Let D_i^m be the number of steps of (X_n) from $i+1$ to i before the first visit to m. We interpret the steps from i to $i-1$ which take place after a step from $i+1$ to i and before the next visit to $i+1$ as the children of this step. Represent each step from $i+1$ to i before the first visit to m, as a vertex at level i, and draw edges between the parents and its children, $i = 0, 1, \ldots, m-2$. Put m additional vertices at levels $0, 1, \ldots, m-1$ which represent the first visits to $0, 1, \ldots, m-1$, and connect the additional vertex at level i to those vertices at level $i-1$ which have no parent ($i = 0, 1, \ldots, m-1$). This gives a bijection of paths of a walk between times 0 and T_m and sequences of m finite trees, rooted at levels i, ($i = 0, 1, \ldots, m-1$). Finally, join the roots of this m trees with special edges. The m trees correspond to the excursions down from i before the first visit to $i+1$, $0 \leqslant i \leqslant m-1$, and the special edges correspond to the first steps from i to $i+1$. Let U_j^m be the total number of all vertices in the m trees at level j, $-\infty < j \leqslant m$. In particular, $U_m^m = 0$ and $U_{m-1}^m = 1$. Then $U_j^m = D_j^m$ for $j = -1, -2, -3, \ldots$ and $U_j^m = D_j^m + 1$ for $j = 0, 1, \ldots, m-1$. All edges in the m trees connecting a vertex on level j and a vertex on level $j-1$ correspond to two steps of the random walk (one step down from j to $j-1$ and one step up from $j-1$ to j), all special edges correspond to one step of the random walk, the first step from i to $i+1$, ($i = 0, 1, \ldots, m-1$).

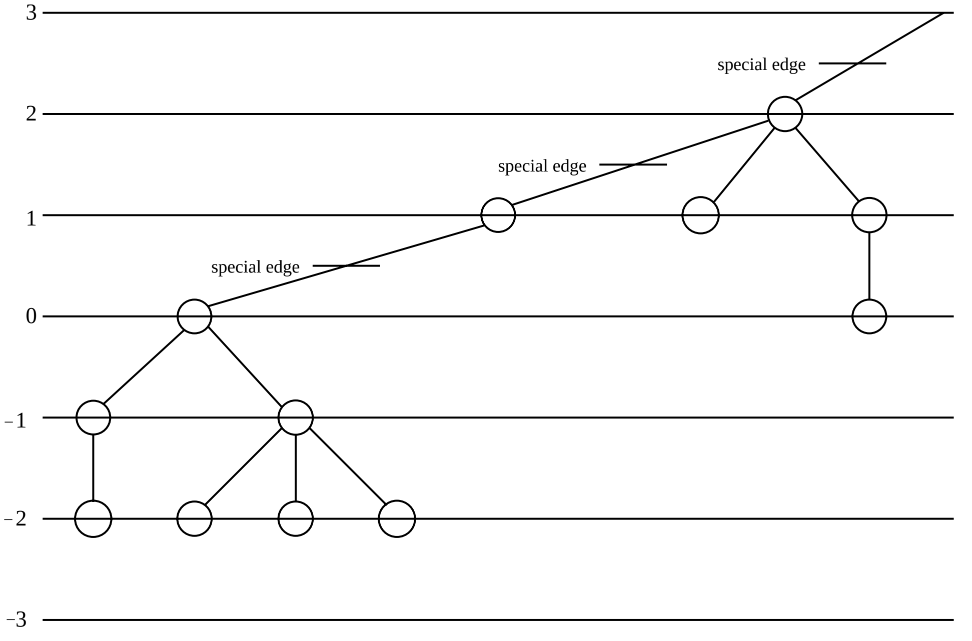
Accordingly,

$$T_m = -m + 2 \sum_{i=-\infty}^{m} U_i^m \tag{2.4}$$

and $\xi^*(T_m) = \max_{-\infty < i < m} (U_i^m + U_{i-1}^m - I_{\{i>0\}})$, so that

$$\max_{-\infty < i \leqslant m} U_i^m \leqslant \xi^*(T_m) \leqslant 2 \max_{-\infty < i \leqslant m} U_i^m. \tag{2.5}$$

An example with $m=3$ is illustrated in the following figure.



Of course, the random walk path can be reconstructed from the tree by walking along the edges and visiting all the leaves (vertices without children) in lexicographical order, i.e., from the left to the right. The reader may check that the tree in the figure corresponds to the following path: $X_0=0$, $X_1=-1$, $X_2=-2$, $X_3=-1$, $X_4=0$, $X_5=-1$, $X_6=-2$, $X_7=-1$, $X_8=-2$, $X_9=-1$, $X_{10}=-2$, $X_{11}=-1$, $X_{12}=0$, $X_{13}=1$, $X_{14}=2$, $X_{15}=1$, $X_{16}=2$, $X_{17}=1$, $X_{18}=0$, $X_{19}=1$, $X_{20}=2$, $X_{21}=3$. We have $U^3_{-2}=4, U^3_{-1}=2, U^3_0=2, U^3_1=3, U^3_2=1$, $T_3=21$ and $\xi^*(T_3)=\xi(T_3,-1)=6$.

For $i=0,1,\dots,m-2$, the conditional distribution of U_i^m-1, given $U^m_{i+1}, U^m_{i+2},\dots,U^m_{m-1}=1$ under P_ω is the distribution of the sum of U^m_{i+1} i.i.d. random variables $Y_1,Y_2,\dots$ with geometric distribution

$$P[Y_1=n]=\omega_{i+1}(1-\omega_{i+1})^n, \quad n=0,1,2,3,\dots\ . \tag{2.6}$$

The distribution of $U^m_{m-1}=1, U^m_{m-2},\dots,U^m_1,U^m_0$ under $\mathbb{P}$ is the distribution of the first m generations of a branching process in random environment with one immigrant in each of these m generations, and with branching law given by (2.6), where the random environment (ω_i) is i.i.d. Let $Z_1=1,Z_2,Z_3,\dots$ be a BPRE with i.i.d. environment (ω_i) and one immigrant in each generation. Then there is a coupling of $U^m_{m-1},U^m_{m-2},U^m_{m-3},\dots$ and $Z_1=1,Z_2,Z_3,\dots,$ such that

$$U^m_{m-j}=Z_j \quad \text{for } j=1,2,\dots,m \quad \text{and} \quad U^m_{m-j}\leqslant Z_j \quad \text{for } j=m+1,m+2,\dots\ . \tag{2.7}$$

In the following, we will always assume that the joint distribution of $(U^m_{m-1}, U^m_{m-2}, U^m_{m-3}, \ldots)$ and $(Z_1, Z_2, Z_3, \ldots)$ is given by this coupling. We introduce the stopping times $v_0 = 1$, $v_{j+1} := \min\{\ell > v_j\colon Z_\ell = 1\}$. The stopping times v_j are times when all offspring from previous generations has died out and the branching process starts again from one particle, namely the new immigrant. Then the random variables $v_{j+1} - v_j$ as well as the random variables $\sum_{v_j \leqslant \ell < v_{j+1}} Z_\ell$, $j = 1, 2, \ldots$ are i.i.d. It is known (see Lemma 2 in Kesten et al., 1975) that

$$E[\mathrm{e}^{a v_1}] < \infty \quad \text{for some } a > 0. \tag{2.8}$$

Before proceeding to prove Lemmas 2.1–2.3, we need a preliminary estimate for a BPRE with immigration.

Lemma 2.4. *Let $0 < \kappa \leqslant 2$. There exist constants $c_7 > 0$ and $c_8 > 0$ such that, for all x large enough,*

$$P\left[\max_{1 \leqslant i < v_1} Z_i > x,\ \sum_{i=1}^{v_1 - 1} Z_i < c_7 x\right] \geqslant \frac{c_8}{x^\kappa}. \tag{2.9}$$

Proof. Let $\bar{Z}_1 = 1$, $\bar{Z}_2$, $\bar{Z}_3, \ldots$ be a BPRE with i.i.d. environment (ω_i), without immigration, and with the same branching law given by (2.6). For any $\kappa > 0$, it is shown in Afanasyev (2001) that for the maximum of such a subcritical BPRE satisfying (1.2),

$$P\left[\max_{1 \leqslant i < \infty} \bar{Z}_i > x\right] \sim \bar{K} x^{-\kappa}, \quad x \to \infty, \tag{2.10}$$

where $\bar{K} > 0$ is a constant. Since there is a coupling of $Z_1 = 1, Z_2, Z_3, \ldots$ and $\bar{Z}_1 = 1, \bar{Z}_2, \bar{Z}_3, \ldots$ such that $\bar{Z}_i \leqslant Z_i$ for all i, we conclude that

$$P\left[\max_{1 \leqslant i < v_1} Z_i > x\right] \geqslant K x^{-\kappa}, \quad x \to \infty, \tag{2.11}$$

where $K > 0$ is a constant. On the other hand, according to Lemma 6 of Kesten et al. (1975), if $0 < \kappa \leqslant 2$, there exists a constant $\tilde{K}$ such that

$$P\left[\sum_{i=1}^{v_1 - 1} Z_i \geqslant x\right] \sim \tilde{K} x^{-\kappa}, \quad x \to \infty. \tag{2.12}$$

Since

$$\left[\max_{1 \leqslant i < v_1} Z_i > x,\ \sum_{i=1}^{v_1 - 1} Z_i < c_7 x\right] \geqslant P\left[\max_{1 \leqslant i < v_1} Z_i > x\right] - P\left[\sum_{i=1}^{v_1 - 1} Z_i \geqslant c_7 x\right],$$

and in view of (2.11) and (2.12), we can choose c_7 sufficiently large such that

$$P\left[\max_{1 \leqslant i < v_1} Z_i > x, \sum_{i=1}^{v_1 - 1} Z_i < c_7 x\right] \geqslant \frac{c_8}{x^\kappa}$$

for some constant $c_8 > 0$. $\square$

The rest of the section is devoted to the proofs of Lemmas 2.1–2.3.

Proof of Lemma 2.1. It follows from (2.4) and (2.5) that

$$\mathbb{P}\left[\xi^*(T_m) \geqslant \frac{c_9}{2}\, T_m\right] \geqslant \mathbb{P}\left[\max_{-\infty<i\leqslant m} U_i^m \geqslant c_9 \sum_{i=-\infty}^{m} U_i^m\right].$$

Let $(M_n, V_n) := \left(\max_{v_{n-1}\leqslant i<v_n} Z_i, \sum_{i=v_{n-1}}^{v_n-1} Z_i\right)$, $n = 1, 2, \ldots$. Then, $(M_n, V_n)_{n=1,2,\ldots}$ are i.i.d. random variables with the distribution of $(\max_{1\leqslant i<v_1} Z_i, \sum_{i=1}^{v_1-1} Z_i)$. Let

$$\varrho_m := \max\{j\colon v_j \leqslant m\}, \tag{2.13}$$

(notation: $\varrho_m := 0$ if $v_1 > m$) and $r := m - v_{\varrho_m}$. We note for further reference that, using (2.7)

$$\sum_{i=-\infty}^{r} U_i^m \leqslant \sum_{i=v_{\varrho_m}}^{v_{\varrho_m+1}-1} Z_i = V_{\varrho_m+1}. \tag{2.14}$$

Then,

$$\begin{aligned}
&\mathbb{P}\left[\max_{-\infty<i\leqslant m} U_i^m \geqslant c_9 \sum_{i=-\infty}^{m} U_i^m\right]\\
&\quad = P\left[\max_{1\leqslant i\leqslant \varrho_m} M_i + \max_{-\infty\leqslant i\leqslant r} U_i^m \geqslant c_9\left(\sum_{i=1}^{\varrho_m} V_i + \sum_{i=-\infty}^{r} U_i^m\right)\right]\\
&\quad \geqslant P\left[\max_{1\leqslant i\leqslant \varrho_m} M_i \geqslant c_9 \sum_{i=1}^{\varrho_m+1} V_i\right],
\end{aligned}$$

where we used (2.14) in the last inequality.

Fix $\varepsilon \in (0,1)$. Let $N := m/E[v_1]$ (we know $E[v_1] < \infty$ thanks to (2.8)). We have

$$\begin{aligned}
&\mathbb{P}\left[\xi^*(T_m) \geqslant \frac{c_9}{2}\, T_m\right]\\
&\quad \geqslant P\left[\max_{1\leqslant i\leqslant \varrho_m} M_i \geqslant c_9 \sum_{i=1}^{\varrho_m+1} V_i\right]\\
&\quad \geqslant P\left[\max_{1\leqslant i\leqslant (1-\varepsilon)N} M_i \geqslant c_9 \sum_{i=1}^{(1+\varepsilon)N+1} V_i\right] - P[\varrho_m < (1-\varepsilon)N] - P[\varrho_m > (1+\varepsilon)N].
\end{aligned}$$

(Where we omit integer parts for simplicity.) For any n, $\{\varrho_m \geqslant n\} = \{v_n \leqslant m\} = \{\sum_{i=1}^{n}(v_i - v_{i-1}) \leqslant m - 1\}$ (with $v_0 := 1$). By (2.8) and Chernoff's theorem, there exists $c_{10} = c_{10}(\varepsilon)$ such that

$$P[\varrho_m < (1-\varepsilon)N] + P[\varrho_m > (1+\varepsilon)N] \leqslant \mathrm{e}^{-c_{10}m}. \tag{2.15}$$

Accordingly,

$$\mathbb{P}\left[\xi^*(T_m) \geqslant \frac{c_9}{2}\, T_m\right] \geqslant P\left[\max_{1\leqslant i\leqslant(1-\varepsilon)N} M_i \geqslant c_9 \sum_{i=1}^{(1+\varepsilon)N+1} V_i\right] - \mathrm{e}^{-c_{10}m}. \tag{2.16}$$

Consider the probability term on the right-hand side. We choose c_9 such that

$$0 < \frac{1}{c_9} - c_7 < 1,$$

where c_7 is the constant introduced in (2.9). (For example, $c_9 := 2/(1+2c_7)$ will do the job.) For any $c_{11} > 0$ and $b_N > 0$,

$$\begin{aligned} &P\left[\max_{1\leqslant i\leqslant(1-\varepsilon)N} M_i \geqslant c_9 \sum_{i=1}^{(1+\varepsilon)N+1} V_i\right] \\ &\quad\geqslant P\left[\max_{1\leqslant i\leqslant(1-\varepsilon)N} M_i \geqslant c_{11} b_N,\ \sum_{i=1}^{(1+\varepsilon)N+1} V_i < \frac{c_{11}}{c_9}\, b_N\right] \\ &\quad\geqslant P\left[\bigcup_{i=1}^{(1-\varepsilon)N} A_i\right], \end{aligned}$$

where, for $i \leqslant (1-\varepsilon)N$,

$$A_i := \left\{ M_i \geqslant c_{11} b_N, V_i < c_7 c_{11} b_N, \sum_{1\leqslant j\leqslant(1+\varepsilon)N+1, j\neq i} V_j < \left(\frac{c_{11}}{c_9} - c_7 c_{11}\right) b_N \right\}.$$

Since $M_i \leqslant V_i$, the events A_i are pairwise disjoint events (this is where the condition $1/c_9 - c_7 < 1$ comes in). Hence

$$P\left[\max_{1\leqslant i\leqslant(1-\varepsilon)N} M_i \geqslant c_9 \sum_{i=1}^{(1+\varepsilon)N+1} V_i\right] \geqslant P\left[\bigcup_{i=1}^{(1-\varepsilon)N} A_i\right] = \sum_{i=1}^{(1-\varepsilon)N} P[A_i] = (1-\varepsilon)N\, P[A_1].$$

By independence,

$$\begin{aligned} P[A_1] &= P[M_1 \geqslant c_{11} b_N, V_1 \leqslant c_7 c_{11} b_N] \times P\left[\sum_{j=2}^{(1+\varepsilon)N+1} V_j < \left(\frac{c_{11}}{c_9} - c_7 c_{11}\right) b_N\right] \\ &:= p_1(N) \times p_2(N), \end{aligned}$$

with obvious notation. Plugging this into (2.16) yields that

$$\mathbb{P}\left[\xi^*(T_m) \geqslant \frac{c_9}{2}\, T_m\right] \geqslant (1-\varepsilon)N\, p_1(N)p_2(N) - \mathrm{e}^{-c_{10}m}. \tag{2.17}$$

We note that (2.17) holds whenever $\kappa > 0$.

We now estimate $p_1(N)$ and $p_2(N)$. We assume from now on $\kappa < 1$, and choose $b_N := N^{1/\kappa}$. By virtue of Lemma 2.4, for N large enough,

$$p_1(N) = P[M_1 \geqslant c_{11}N^{1/\kappa}, V_1 \leqslant c_7c_{11}N^{1/\kappa}] \geqslant \frac{c_8}{(c_{11}N^{1/\kappa})^\kappa} = \frac{c_8}{(c_{11})^\kappa}\,\frac{1}{N}. \tag{2.18}$$

On the other hand,

$$p_2(N) = P\left[\sum_{j=1}^{(1+\varepsilon)N+1} V_j < \left(\frac{c_{11}}{c_9} - c_7c_{11}\right) N^{1/\kappa}\right].$$

According to (2.12), $P[V_1 > x] \sim \tilde{K}x^{-\kappa}$ (for $x \to \infty$); thus by Theorem XVII.5.3 of Feller (1971), the distribution of $n^{-1/\kappa}\sum_{j=1}^n V_j$ converges (as $n \to \infty$) to a (completely asymmetric) stable distribution of index κ. Therefore, for c_{11} large enough,

$$\lim_{m\to\infty} P\left[\sum_{j=1}^{(1+\varepsilon)N+1} V_j < \left(\frac{c_{11}}{c_9} - c_7c_{11}\right) N^{1/\kappa}\right] > 0$$

so that

$$p_2(N) \geqslant c_{12} \tag{2.19}$$

uniformly in m. Combining (2.17)–(2.19) yields Lemma 2.1 with $c_1 := c_9/2$. □

Proof of Lemma 2.2. We apply again estimate (2.17) which is valid whenever $\kappa > 0$. Assume $\kappa = 1$, and we choose for this situation $b_N := N\log N$. By means of Lemma 2.4, we have

$$p_1(N) = P[M_1 \geqslant c_{11}N\log N,\ V_1 \leqslant c_7c_{11}N\log N] \geqslant \frac{c_8}{c_{11}}\,\frac{1}{N\log N}. \tag{2.20}$$

On the other hand,

$$p_2(N) = P\left[\sum_{j=1}^{(1+\varepsilon)N+1} V_j < \left(\frac{c_{11}}{c_9} - c_7c_{11}\right) N\log N\right].$$

According to (2.12), $P[V_1 > x] \sim \tilde{K}x^{-1}$ (for $x \to \infty$). It follows from Aaronson and Denker (1998, p. 402) that there exists a constant $A > 0$ (whose value is explicitly known) such that the distribution of $n^{-1}(\sum_{j=1}^n V_j - An\log n)$ converges to a (completely asymmetric) Cauchy distribution. In particular, this implies that $(n\log n)^{-1}\sum_{j=1}^n V_j$ converges to A in distribution, thus in probability. If we choose c_{11} large enough such that $c_{11}/c_9 - c_7c_{11} > (1+\varepsilon)A$, then $p_2(N) \underset{m\to\infty}{\longrightarrow} 1$. In particular, $p_2(N) \geqslant c_{13}$. This estimate, together with (2.20) and (2.17), yields Lemma 2.2. □

Proof of Lemma 2.3. We keep the same notation as in the proof of Lemma 2.1. Using (2.5), we have

$$\begin{aligned}
\mathbb{P}[\xi^*(T_m) \geqslant \lambda] &\geqslant \mathbb{P}\left[\max_{-\infty<i\leqslant m} U_i^m \geqslant \lambda\right] \\
&\geqslant P\left[\max_{1\leqslant i\leqslant \varrho_m} M_i \geqslant \lambda\right] \\
&\geqslant P\left[\max_{1\leqslant i\leqslant (1-\varepsilon)N} M_i \geqslant \lambda\right] - \mathrm{e}^{-c_{10}m},
\end{aligned}$$

the last inequality following from the fact that $P[\varrho_m < (1-\varepsilon)N] \leqslant \mathrm{e}^{-c_{10}m}$ (this is a consequence of (2.15)). Recall that (M_i) is a sequence of i.i.d. random variables having the same distribution as $\max_{1\leqslant i<v_1} Z_i$. Therefore,

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \geqslant 1 - \left\{1 - P\left[\max_{1\leqslant i<v_1} Z_i \geqslant \lambda\right]\right\}^{(1-\varepsilon)N} - \mathrm{e}^{-c_{10}m}.$$

In view of (2.11), we deduce that there exists $c_{14} > 0$ such that,

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \geqslant 1 - \left\{1 - \frac{c_{14}}{\lambda^\kappa}\right\}^{(1-\varepsilon)N} - \mathrm{e}^{-c_{10}m}$$

and, using the inequality $1-(1-x)^n \geqslant nx(1-\mathrm{e}^{-d})/d$ for $nx \leqslant d$, we conclude that for $\lambda \geqslant m^{1/\kappa}$,

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \geqslant \frac{c_{15}(1-\varepsilon)N}{\lambda^\kappa} - \mathrm{e}^{-c_{10}m}$$

and this completes the proof of Lemma 2.3. □

3. Proofs of Theorems 1.1 and 1.2

The proofs of the theorems are based on a 0–1 law for the maximum local time, stated as follows:

Proposition 3.1. *Let $\varphi : \mathbb{N} \to \mathbb{R}_+$ be such that $\varphi(n) \to \infty$ for $n \to \infty$. Then*

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{\varphi(n)} = \text{const} \in [0,\infty], \quad \mathbb{P}\text{-a.s.}$$

Proof of Proposition 3.1. The proof is carried out in three steps.

Step 1: Define the tail field of (X_m),

$$\mathscr{A}^* := \bigcap_n \sigma(X_n, X_{n+1}, X_{n+2}, \ldots).$$

We claim that, for μ-a.a. ω, P_ω is 0–1 on $\mathscr{A}^*$.

To prove this, let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables with the uniform distribution in $[0,1]$ which are independent of $(\omega_x)_{x\in\mathbb{Z}}$. We claim that, for each ω there is a sequence of measurable functions $(f_n) = (f_n(\omega))$ such that

$$(X_i)_{0\leqslant i\leqslant n} = f_n(Y_1, Y_2, \ldots, Y_n).$$

In fact, one can define f_n recursively: Given $(X_i)_{0\leqslant i\leqslant n-1}$ and f_{n-1},

$$X_n := \begin{cases} X_{n-1}+1 & \text{if } Y_n \leqslant \omega_{X_{n-1}}, \\ X_{n-1}-1 & \text{otherwise.} \end{cases}$$

Let A be the set of all ω such that $(X_n) \to \infty$, P_ω-a.s. Due to (1.1), $\mu(A)=1$ (Solomon, 1975). Recall $T_m := \inf\{n \geqslant 1 : X_n = m\}$. For $\omega \in A$,

$$\mathscr{A}^* \subseteq \bigcap_m \sigma(X_{T_m}, X_{T_{m+1}}, X_{T_{m+2}}, \ldots) \cap \sigma(\{T_m < \infty\}) \subseteq \bigcap_n \sigma(Y_n, Y_{n+1}, Y_{n+2}, \ldots) := \mathscr{B}^*.$$

Since P_ω is 0–1 on $\mathscr{B}^*$ (Kolmogorov's 0–1 law), we conclude that for $\omega \in A$, P_ω is 0–1 on $\mathscr{A}^*$.

Step 2: Let $g(\omega,(X_n)) := \limsup_{n\to\infty}(\xi^*(n)/\varphi(n))$. We check that for $\omega \in A$, $g(\omega,\cdot)$ is measurable with respect to $\mathscr{A}^*$: since $T_m < \infty$, P_ω-a.s. and due to the Markov property, $g(\omega,\cdot)$ does not depend on the finite path $\{X_j : j \leqslant T_m\}$. Therefore, we can apply what we have proved in Step 1 to conclude that, for μ-a.a. ω, $g(\omega) := g(\omega,(X_n))$ is constant for P_ω-a.a. realizations of (X_n).

Step 3: Let $\theta : \Omega \to \Omega$ (recalling that $\Omega = (0,1)^{\mathbb{Z}}$) denote the shift transformation: $\theta\omega \in \Omega$ is defined by $(\theta\omega)_x = \omega_{x+1}$, $x \in \mathbb{Z}$. Note that A is invariant under θ. We have

$$\xi(n,x)(\omega,(X_j)) = \xi(n\wedge T_1,x)(\omega,(X_j)) + \xi(n-(n\wedge T_1),x)(\theta\omega,(X_{T_1+j})). \qquad (3.1)$$

Hence

$$\xi^*(n)(\omega,(X_j)) \leqslant \sup_{x\in\mathbb{Z}} \xi(n\wedge T_1,x)(\omega,(X_j)) + \sup_{x\in\mathbb{Z}} \xi(n-(n\wedge T_1),x)(\theta\omega,(X_{T_1+j})).$$

Since $\varphi(n) \to \infty$ and, for $\omega \in A$, $T_1 < \infty$, P_ω-a.s., dividing by $\varphi(n)$ and letting $n \to \infty$ implies that $g(\omega) \leqslant g(\theta\omega)$ for $\omega \in A$. On the other hand, (3.1) implies

$$\xi^*(n)(\omega,(X_j)) \geqslant \sup_{x\in\mathbb{Z}} \xi(n-(n\wedge T_1),x)(\theta\omega,(X_{T_1+j})).$$

Dividing by $\varphi(n)$ and letting $n \to \infty$ yields $g(\omega) \geqslant g(\theta\omega)$ for $\omega \in A$. We have shown that $g(\omega) = g(\theta\omega)$ for μ-a.a. ω and since θ is ergodic with respect to μ, we conclude that $g(\omega)$ is a constant (with possible values 0 or ∞) for μ-a.a. ω. □

Remark 3.2. The same arguments apply to $\liminf_{n\to\infty} \xi^*(n)/\varphi(n)$.

Before proceeding to prove Theorems 1.1 and 1.2, we need two preliminary probability estimates.

Lemma 3.3. *Whenever* $\kappa > 0$, *there exists a constant* $c_{16} > 0$ *such that for any* $m \geqslant 1$,

$$\mathbb{P}[T_{-m} < \infty] \leqslant \exp(-c_{16} m). \qquad (3.2)$$

Proof. In case $\kappa > 1$, Lemma 3.3 is an immediate consequence of Lemma 2.2 in Dembo et al. (1996). The proof of Lemma 3.3 given here holds for all $\kappa > 0$.

Let $\rho_i = \rho_i(\omega) = (1-\omega_i)/\omega_i$. According to Solomon (1975), formula (1.3), or (Dembo et al., 1996, formula (5)),

$$P_\omega[T_{-m} < \infty] = \frac{\sum_{i=0}^{\infty} \prod_{j=-(m-1)}^{i} \rho_j}{1 + \sum_{i=-(m-1)}^{\infty} \prod_{j=-(m-1)}^{i} \rho_j}. \tag{3.3}$$

We will show that $\mathbb{P}[T_{-m} < \infty] = \int P_\omega[T_{-m} < \infty]\mu(\mathrm{d}\omega)$ decays exponentially in m. Let $0 < c < -\int \log \rho_0(\omega)\mu(\mathrm{d}\omega)$ and define

$$A_m := \bigcup_{i=0}^{\infty} \{\rho_{-(m-1)} \cdots \rho_i \geqslant \mathrm{e}^{-(m+i)c}\}.$$

Using Cramér's theorem, we have

$$\mu[\rho_{-(m-1)} \cdots \rho_i \geqslant \mathrm{e}^{-(m+i)c}] = \mu\left[\frac{1}{m+i}\sum_{j=1}^{m+i} \log \rho_j \geqslant -c\right]$$
$$\leqslant \mathrm{e}^{-(m+i)I(c)},$$

where I is the Cramér rate function. Note that $I(x) > 0$ for $x > \int \log \rho_0(\omega)\mu(\mathrm{d}\omega)$, since the logarithmic moment generating function of $\log \rho_0(\omega)$ is finite in a neighbourhood of 0 due to (1.2). We conclude that

$$\mu[A_m] \leqslant \mathrm{e}^{-mI(c)} \sum_{i=0}^{\infty} \mathrm{e}^{-iI(c)}. \tag{3.4}$$

Now,

$$\mathbb{P}[T_{-m} < \infty] = \int_{A_m} P_\omega[T_{-m} < \infty]\mu(\mathrm{d}\omega) + \int_{A_m^c} P_\omega[T_{-m} < \infty]\mu(\mathrm{d}\omega)$$
$$\leqslant \mu[A_m] + \sum_{i=0}^{\infty} \mathrm{e}^{-(m+i)c}, \tag{3.5}$$

where we used (3.3) for the last inequality. We see from (3.5) and (3.4) that we can find a constant c_{17} such that for all sufficiently large m, say $m \geqslant m_0$,

$$\mathbb{P}[T_{-m} < \infty] \leqslant \exp(-c_{17} m).$$

Due to (1.1), the inequality also holds for $m < m_0$ if we take c_{17} small enough, and this proves Lemma 3.3. □

Remark 3.4. The precise exponential rate of decay of $\mathbb{P}[T_{-m} < \infty]$ can be determined along the lines of Comets et al. (2000), but we do not need it here.

Lemma 3.5. *Let $\kappa > 0$. There exists a constant $c_{18} > 0$ such that for any $\lambda \geqslant 2$,*

$$\mathbb{P}\left[\max_{x \leqslant 0} \xi(\infty, x) \geqslant \lambda\right] \leqslant c_{18} \frac{(\log \lambda)^{\kappa+2}}{\lambda^{\kappa}}.$$

Proof. Let $m \geqslant 1$. Then

$$\mathbb{P}\left[\max_{x \leqslant 0} \xi(\infty, x) \geqslant \lambda\right] \leqslant \mathbb{P}[T_{-m} < \infty] + \mathbb{P}\left[\max_{-m \leqslant x \leqslant 0} \xi(\infty, x) \geqslant \lambda\right]$$
$$\leqslant \exp(-c_{16} m) + (m+1)\mathbb{P}[\xi(\infty, 0) \geqslant \lambda], \tag{3.6}$$

the last inequality being a consequence of Lemma 3.3.

To estimate $\mathbb{P}[\xi(\infty, 0) \geqslant \lambda]$, we use the representation of excursions as BPRE. More precisely, for any $\ell \in [1, \lambda] \cap \mathbb{Z}$, the probability of having more than λ visits to 0 is less than or equal to the probability of having either more than λ/ℓ visits to 0 in one of the ℓ excursions between i and $i+1$, $0 \leqslant i \leqslant \ell - 1$ or to return to 0, starting from ℓ. Recall that $(\bar{Z}_i)$ denotes a BPRE without immigration as in the proof of Lemma 2.4. Let $\bar{v}_1 := \min\{\ell > 1 \colon \bar{Z}_\ell = 0\}$ be the extinction time of $(\bar{Z}_i)$. Hence, using the coupling of $(U_i^1)_{i=0,-1,-2,\ldots}$ and $(\bar{Z}_i)_{i=1,2,\ldots}$, i.e., (2.7) with $m = 1$,

$$\mathbb{P}[\xi(\infty, 0) \geqslant \lambda] \leqslant \ell\, P\left[\max_{1 \leqslant i < \bar{v}_1} \bar{Z}_i \geqslant \frac{\lambda}{2\ell}\right] + \mathbb{P}[T_{-\ell} < \infty].$$

According to (2.10), for some constant $c_{19} > 0$,

$$P\left[\max_{1 \leqslant i < \bar{v}_1} \bar{Z}_i \geqslant \frac{\lambda}{2\ell}\right] \leqslant \frac{c_{19}\, \ell^{\kappa}}{\lambda^{\kappa}},$$

whereas by Lemma 3.3, $\mathbb{P}[T_{-\ell} < \infty] \leqslant \exp(-c_{16}\,\ell)$. Hence

$$\mathbb{P}[\xi(\infty, 0) \geqslant \lambda] \leqslant \frac{c_{19}\, \ell^{\kappa+1}}{\lambda^{\kappa}} + \exp(-c_{16}\,\ell).$$

Choosing $\ell := (\kappa/c_{16}) \log \lambda$, we see that for some $c_{20} > 0$ and all large λ,

$$\mathbb{P}[\xi(\infty, 0) \geqslant \lambda] \leqslant c_{20} \frac{(\log \lambda)^{\kappa+1}}{\lambda^{\kappa}}.$$

Plugging this into (3.6), and taking $m := (\kappa/c_{16}) \log \lambda$, we obtain that for some $c_{21} > 0$ and all large λ,

$$\mathbb{P}\left[\max_{x \leqslant 0} \xi(\infty, x) \geqslant \lambda\right] \leqslant c_{21} \frac{(\log \lambda)^{\kappa+2}}{\lambda^{\kappa}}. \tag{3.7}$$

Choosing c_{21} large enough, (3.7) holds for all $\lambda \geqslant 2$. □

We are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We first treat the case $\kappa < 1$. In this case, we have

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\xi^*(n)}{n} \geqslant c_1\right] \geqslant \mathbb{P}\left[\bigcap_n \bigcup_{m\geqslant n} \{\xi^*(T_m) \geqslant c_1\, T_m\}\right]$$

$$= \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{m\geqslant n} \{\xi^*(T_m) \geqslant c_1\, T_m\}\right]$$

$$\geqslant \limsup_{m\to\infty} \mathbb{P}[\xi^*(T_m) \geqslant c_1\, T_m]$$

$$> 0,$$

where we used Lemma 2.1 for the last inequality, and together with Proposition 3.1, (1.4) follows.

It remains to study the situation when $\kappa = 1$. Let $m_j := j^j$ and $n_j \in (m_{j-1}, m_j) \cap \mathbb{Z}$ such that $n_j/m_j \to 0$. Recall that (X_n) denotes the position of the RWRE, and let

$$E_j := \left\{ \min_{n\in[T_{n_j}, T_{m_j}]\cap\mathbb{Z}} X_n > m_{j-1},\ \max_{x\in(n_j,m_j]\cap\mathbb{Z}} \xi(T_{m_j}, x) \geqslant c_3\, T_{m_j} \right\},$$

where c_3 is the positive constant in (2.2). Since

$$\max_{x\in(n_j,m_j]} \xi(T_{m_j}, x) = \max_{x\in(n_j,m_j]} (\xi(T_{m_j}, x) - \xi(T_{n_j}, x)),$$

the events (E_j) are independent. We now show that $\sum_j \mathbb{P}[E_j] = \infty$. Let $N := m_j - n_j$, $M := n_j - m_{j-1}$. Then

$$\mathbb{P}[E_j] = \mathbb{P}\left[T_{-M} > T_N,\ \sup_{x>0} \xi(T_N, x) \geqslant c_3\, T_N\right]$$

$$\geqslant \mathbb{P}\left[\sup_{x\in\mathbb{Z}} \xi(T_N, x) \geqslant c_3\, T_N\right] - \mathbb{P}[T_{-M} < T_N] - \mathbb{P}\left[\sup_{x\leqslant 0} \xi(T_N, x) \geqslant c_3 T_N\right]$$

$$\geqslant \mathbb{P}\left[\sup_{x\in\mathbb{Z}} \xi(T_N, x) \geqslant c_3\, T_N\right] - \mathbb{P}[T_{-M} < \infty] - \mathbb{P}\left[\sup_{x\leqslant 0} \xi(\infty, x) \geqslant c_3\, N\right]. \tag{3.8}$$

In light of (2.2), we have

$$\mathbb{P}\left[\sup_{x\in\mathbb{Z}} \xi(T_N, x) \geqslant c_3\, T_N\right] \geqslant \frac{c_4}{\log N}.$$

On the other hand, by means of Lemmas 3.3 and 3.5, respectively, we have (recalling that $\kappa = 1$)

$$\mathbb{P}[T_{-M} < \infty] \leqslant \exp(-c_{16}\, M),$$

$$\mathbb{P}\left[\sup_{x\leqslant 0} \xi(\infty, x) \geqslant c_3\, N\right] \leqslant c_{22}\, \frac{(\log N)^3}{N}.$$

Plugging these three estimates into (3.8) gives

$$\mathbb{P}[E_j] \geqslant \frac{c_4}{\log N} - \exp(-c_{16} M) - c_{22} \frac{(\log N)^3}{N}.$$

It is possible to choose (n_j) such that $M \geqslant \sqrt{N}$ so that, for some constant $c_{23} > 0$ and all large N,

$$\mathbb{P}[E_j] \geqslant \frac{c_{23}}{\log N} \geqslant \frac{c_{23}}{\log m_j}.$$

Hence $\sum_j \mathbb{P}[E_j] = \infty$ and the Borel–Cantelli lemma implies that $\mathbb{P}$-a.s. there are infinitely many j such that

$$\sup_x \xi(T_{m_j}, x) \geqslant c_3 \, T_{m_j},$$

implying (1.4) in the case $\kappa = 1$. Theorem 1.1 is proved. □

We now turn to the proof of Theorem 1.2. For the sake of clarity, its two parts are proved separately, namely,

$$\sum_n \frac{1}{n(a_n)^\kappa} = \infty \Rightarrow \limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa} a_n} = \infty, \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

$$\sum_n \frac{1}{n(a_n)^\kappa} < \infty \Rightarrow \lim_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa} a_n} = 0, \quad \mathbb{P}\text{-a.s.} \tag{3.10}$$

Proof of Theorem 1.2 (Part (3.9)).
Assume $\kappa > 1$, and let (a_n) be a positive and non-decreasing sequence such that

$$\sum_n \frac{1}{n\,(a_n)^\kappa} = \infty. \tag{3.11}$$

Without loss of generality, we can assume that $a_n \to \infty$ (for $n \to \infty$).
By Solomon (1975), in our case (i.e., $\kappa > 1$), there exists $c_{24} > 0$ such that

$$\lim_{m\to\infty} \frac{T_m}{m} = c_{24}, \quad \mathbb{P}\text{-a.s.} \tag{3.12}$$

(In fact, comparing to (1.5), $c_{24} = 1/v$, but we do not need this here.) We introduce the new sequence

$$\tilde{a}_n := \min\{a_{\lfloor 2c_{24} n\rfloor},\ \lfloor 2c_{24} n\rfloor\}. \tag{3.13}$$

Clearly, $(\tilde{a}_n)$ is again positive and non-decreasing, and since $\tilde{a}_n \leqslant a_{\lfloor 2c_{24} n\rfloor}$, we also have

$$\sum_n \frac{1}{n\,(\tilde{a}_n)^\kappa} = \infty. \tag{3.14}$$

The proof of (3.9) is in the same spirit as that of Theorem 1.1 in the case $\kappa = 1$. Indeed, let $m_j := 2^j$ and $n_j \in (m_{j-1}, m_j) \cap \mathbb{Z}$ such that $n_j/m_j \to 0$. Let $\lambda \geqslant (m_j)^{1/\kappa}$ and consider

$$F_j := \left\{ \min_{n \in [T_{n_j}, T_{m_j}] \cap \mathbb{Z}} X_n > m_{j-1}, \ \max_{x \in (n_j, m_j] \cap \mathbb{Z}} \xi(T_{m_j}, x) \geqslant \lambda \right\}.$$

The events (F_j) are independent. Moreover, by writing $N := m_j - n_j$ and $M := n_j - m_{j-1}$ as before, we have

$$\mathbb{P}[F_j] = \mathbb{P}\left[T_{-M} > T_N, \ \sup_{x>0} \xi(T_N, x) \geqslant \lambda \right]$$

$$\geqslant \mathbb{P}\left[\sup_{x \in \mathbb{Z}} \xi(T_N, x) \geqslant \lambda \right] - \mathbb{P}[T_{-M} < \infty] - \mathbb{P}\left[\sup_{x \leqslant 0} \xi(\infty, x) \geqslant \lambda \right].$$

Applying Lemmas 2.3, 3.3 and 3.5, respectively, to the three probability expressions on the right-hand side, we obtain.

$$\mathbb{P}[F_j] \geqslant \frac{c_5 N}{\lambda^\kappa} - \exp(-c_6 N) - \exp(-c_{16} M) - c_{18} \frac{(\log \lambda)^{\kappa+2}}{\lambda^\kappa}.$$

We can choose (n_j) such that $N \geqslant m_j/2$ and $M \geqslant (m_j)^{1/2}$. Take $\lambda = (m_j)^{1/\kappa} \tilde{a}_{m_j}$, where $(\tilde{a}_n)$ is the sequence defined in (3.13). Taking into account the fact that $\tilde{a}_n \leqslant 2c_{24} n$, we obtain, for all large j,

$$\mathbb{P}[F_j] \geqslant \frac{c_{25}}{(\tilde{a}_{m_j})^\kappa} - \exp(-c_{26} (m_j)^{1/2}),$$

which in view of (3.14) implies $\sum_j \mathbb{P}[F_j] = \infty$ (we note that (3.14) guarantees $\sum_j (\tilde{a}_{m_j})^{-\kappa} = \infty$). Since (F_j) is a sequence of independent events, an application of the Borel–Cantelli lemma yields that $\mathbb{P}$-a.s. there are infinitely many j such that $\xi^*(T_{m_j}) \geqslant (m_j)^{1/\kappa} \tilde{a}_{m_j}$. A fortiori,

$$\limsup_{m \to \infty} \frac{\xi^*(T_m)}{m^{1/\kappa} \tilde{a}_m} \geqslant 1, \quad \mathbb{P}\text{-a.s.}$$

In light of (3.12), we have, $\mathbb{P}$-a.s.,

$$\limsup_{n \to \infty} \frac{\xi^*(n)}{n^{1/\kappa} \min\{a_n, n\}} \geqslant \limsup_{m \to \infty} \frac{\xi^*(T_m)}{(T_m)^{1/\kappa} \min\{a_{T_m}, T_m\}}$$

$$\geqslant \limsup_{m \to \infty} \frac{m^{1/\kappa} \tilde{a}_m}{(c_{24} m)^{1/\kappa} \min\{a_{\lfloor 2c_{24} m \rfloor}, \lfloor 2c_{24} m \rfloor\}}$$

$$= \frac{1}{(c_{24})^{1/\kappa}} > 0.$$

On the other hand, by definition, $\xi^*(n) \leqslant n$, hence $\lim_{n\to\infty} \xi^*(n)/n^{(1/\kappa)+1} = 0$. Thus, we have proved that

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa} a_n} > 0, \quad \mathbb{P}\text{-a.s.}$$

Since replacing a_n by any constant multiple of a_n does not change the outcome of the integral test, this yields (3.9). □

Proof of Theorem 1.2 (Part (3.10)).

Assume $\kappa > 1$, and let (a_n) be a positive and non-decreasing sequence such that

$$\sum_n \frac{1}{n(a_n)^\kappa} < \infty. \tag{3.15}$$

In particular, a_n goes to ∞ as $n \to \infty$.

We use again the representation in (2.5) to see that for any $\lambda > 0$,

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \leqslant \mathbb{P}\left[\max_{-\infty < i \leqslant m} U_i^m \geqslant \frac{\lambda}{2}\right] \leqslant P\left[\max_{1 \leqslant i \leqslant 1+\varrho_m} M_i \geqslant \frac{\lambda}{2}\right],$$

where (M_i) is defined as in the proof of Lemma 2.1 and hence is sequence of i.i.d. random variables having the same distribution as $\max_{1 \leqslant i < v_1} Z_i$, and $\varrho_m := \max\{j:\ v_j \leqslant m\}$ as in (2.13). Let $N := m/E[v_1]$ as in the proof of Lemma 2.1, and fix $\varepsilon \in (0,1)$. Since $\mathbb{P}[\varrho_m > (1+\varepsilon)N] \leqslant \mathrm{e}^{-c_{10}m}$ (see (2.15)), we have

$$\begin{aligned}\mathbb{P}[\xi^*(T_m) \geqslant \lambda] &\leqslant P\left[\max_{1 \leqslant i \leqslant 1+(1+\varepsilon)N} M_i \geqslant \frac{\lambda}{2}\right] + \mathrm{e}^{-c_{10}m} \\ &\leqslant \{1+(1+\varepsilon)N\}\, P\left[M_1 \geqslant \frac{\lambda}{2}\right] + \mathrm{e}^{-c_{10}m}.\end{aligned}$$

By (2.11), there exists a constant c_{27} such that $P[M_1 \geqslant \lambda/2] \leqslant c_{27}/\lambda^\kappa$. Accordingly,

$$\mathbb{P}[\xi^*(T_m) \geqslant \lambda] \leqslant \frac{c_{28} m}{\lambda^\kappa} + \mathrm{e}^{-c_{10}m}.$$

We consider the subsequence $m_j := 2^j$, and choose $\lambda := (m_{j-1})^{1/\kappa} a_{m_{j-1}}$ to see that

$$\mathbb{P}[\xi^*(T_{m_j}) \geqslant (m_{j-1})^{1/\kappa} a_{m_{j-1}}] \leqslant \frac{2c_{28}}{(a_{m_{j-1}})^\kappa} + \mathrm{e}^{-c_{10}m_j},$$

which is summable in j by means of (3.15). By the Borel–Cantelli lemma, $\mathbb{P}$-almost surely for all large j, $\xi^*(T_{m_j}) < (m_{j-1})^{1/\kappa} a_{m_{j-1}}$. For $n \in [m_{j-1}, m_j]$, we have

$$\xi^*(n) \leqslant \xi^*(T_n) \leqslant \xi^*(T_{m_j}) < (m_{j-1})^{1/\kappa} a_{m_{j-1}} \leqslant n^{1/\kappa} a_n.$$

Thus,

$$\limsup_{n\to\infty} \frac{\xi^*(n)}{n^{1/\kappa} a_n} \leqslant 1, \quad \mathbb{P}\text{-a.s.}$$

Since replacing a_n by any constant multiple of a_n does not change the outcome of the integral test, this yields (3.10) and completes the proof of Theorem 1.2. □

4. Remarks and open questions

This final section is devoted to some further remarks and open questions.

(1) The assumptions on the distribution of ω_0 may be relaxed; we took them to cover the assumptions of Kesten et al. (1975). In Afanasyev (2001), only (1.1) and (1.2) are needed.

(2) The present paper gives an accurate description of the "lim sup" asymptotics of $\xi^*(n)$ for transient RWRE. What about the "lim inf" asymptotics? Can we say something about $\liminf_{n\to\infty} \xi^*(n)/\varphi(n)$ with some appropriate choices of $\varphi(n)$?

(3) Intuitively, it is clear that $\xi^*(n)$ for transient RWRE would be (stochastically) smaller than for recurrent RWRE. Does our Theorem 1.1 imply the corresponding result for recurrent RWRE, which was originally proved by Révész (1990)? Or do our methods yield a proof for the recurrent case? Proposition 3.1 remains true for recurrent RWRE.

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