EINSTEIN RELATION AND LINEAR RESPONSE IN ONE–DIMENSIONAL MOTT VARIABLE–RANGE HOPPING

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Abstract. We consider one-dimensional Mott variable-range hopping with a bias, and prove the linear response as well as the Einstein relation, under an assumption on the exponential moments of the distances between neighboring points. In a previous paper [12] we gave conditions on ballisticity, and proved that in the ballistic case the environment viewed from the particle approaches, for almost any initial environment, a given steady state which is absolutely continuous with respect to the original law of the environment. Here, we show that this bias–dependent steady state has a derivative at zero in terms of the bias (linear response), and use this result to get the Einstein relation. Our approach is new: instead of using e.g. perturbation theory or regeneration times, we show that the Radon-Nikodym derivative of the bias–dependent steady state with respect to the equilibrium state in the unbiased case satisfies an L^p-bound, $p > 2$, uniformly for small bias. This L^p-bound yields, by a general argument not involving our specific model, the statement about the linear response.

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1. Introduction

Mott variable–range hopping is a transport mechanism introduced by N.F. Mott [37, 38, 39, 40, 41, 44] to model the phonon–assisted electron transport in disordered solids in the regime of strong Anderson localisation (e.g. doped semiconductors and doped organic semiconductors).

In the case of doped semiconductors, atoms of some other material, called *impurities*, are introduced into the solid at random locations $\{x_i\}$. One can associate to each impurity a random variable E_i called *energy mark*, the E_i's taking value in some finite interval $[-A, A]$. Due to the strong Anderson localisation, a single conduction electron is well described by a quantum wave–function localized around some impurity x_i and E_i is the associated energy in the ground state (to simplify the discussion we refer to spinless electrons). In Mott variable–range hopping an electron localized around x_i jumps (by quantum tunneling) to another impurity site x_k, when x_k is not occupied by any other electron, with probability rate

$$C(\beta)\exp\Big\{-\frac{2}{\xi}|x_i-x_k|-\beta\{E_k-E_i\}_+\Big\}. \tag{1}$$

Above, β is the inverse temperature, ξ is the localization length, $\{v\}_+ := \max\{v, 0\}$ and the positive prefactor $C(\beta)$ has a β–dependence which is negligible w.r.t. the exponential decay in (1). Treating the localized electrons as classical particles, the description is then given by an exclusion process on the sites $\{x_i\}$, with the above jump rates (1) when

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the exclusion constraint is satisfied. Calling η a generic configuration in $\{0,1\}^{\{x_i\}}$, it then follows that the disordered Bernoulli distribution μ on $\{0,1\}^{\{x_i\}}$ such that $\mu(\eta_i) = \frac{e^{-\beta(E_i-\gamma)}}{1+e^{-\beta(E_i-\gamma)}}$ is reversible for the exclusion process. The chemical potential γ is determined by the density of conduction electrons; equivalently - as usually done in the physical temperature - we take $\gamma = 0$ at the cost of translating the energy (i.e. we take the Fermi energy level equal to zero).

We point out that the mathematical analysis of such an exclusion process is very demanding from a technical viewpoint due to site disorder. We refer to [13, 42] for the derivation of the hydrodynamic limit when the impurities are localized at the sites of $\mathbb{Z}^d$ and hopping is only between nearest–neighbor sites (from a physical viewpoint, the nearest–neighbor assumption leads to a good approximation of Mott variable–range hopping at not very small temperature). Due to the these technical difficulties, in the physical literature, in the regime of low density of conduction electrons the above exclusion process on $\{x_i\}$ is then approximated by independent continuous time random walks (hence one focuses on a single random walk), with probability rate $r_{i,k}$ for a jump from x_i to $x_k \neq x_i$ given by (1) times $\mu(\eta_{x_i} = 1\,, \eta_{x_k} = 0)$. Note that the last factor encodes the exclusion constraint. The validity of this low density approximation has been indeed proved for the exclusion process with nearest–neighbor jumps on $\mathbb{Z}^d$ (cf. [42, Thm. 1]).

It is simple to check (cf. [1, Eq. (3.7)]) that in the physically interesting low temperature regime (i.e. for large β) the resulting jump rate of the random walk behaves as

$$r_{i,k} \approx C(\beta) \exp\Big\{ -\frac{2}{\xi}|x_i - x_k| - \frac{\beta}{2}\big(|E_i| + |E_k| + |E_i - E_k|\big)\Big\}\,. \tag{2}$$

In conclusion, considering the above approximations, Mott variable–range hopping consists of a random walk $(\mathbb{Y}_t)$ in a random spatial and energetic environment given by $\{x_i\}$ and $\{E_i\}$ with jump rates (2). We will consider here also a generalization of the above jump rates (see eq. (6) below).

The name variable–range hopping comes from the possibility of arbitrarily long jumps, which are facilitated (when β is large) if energetically convenient. Indeed, it has been proved that long jumps contribute to most of the transport in dimension $d \geq 2$ [14, 15] but not in dimension $d = 1$ [7]. The physical counterpart of this feature is the anomalous behavior of conductivity at low temperature for $d \geq 2$ [41, 44], which has motivated the introduction of Mott variable–range hopping. Indeed, for an isotropic medium, the conductivity $\sigma(\beta)$ is a multiple of the identity matrix and vanishes as $\beta \to \infty$ as a stretched β–exponential:

$$\sigma(\beta) \sim \exp\big\{-c\,\beta^{\frac{\alpha+1}{\alpha+1+d}}\big\}\mathbb{I} \tag{3}$$

if the energy marks are i.i.d. random variables with $P(|E_i| \in [E, E+dE]) = c(\alpha)E^\alpha dE$ (these are the physically relevant energy distributions). On the other hand, in dimension $d = 1$, the conductivity exhibits an Arrenhius–type decay (similarly to the nearest–neighbor case):

$$\sigma(\beta) \sim \exp\big\{-c\,\beta\big\}\,. \tag{4}$$

The decay (3) has been derived by heuristic arguments by Mott, Efros, Shklovskii (see [41, 44] and references therein), afterwards refined by arguments involving random resistor networks and percolation [1, 36]. The decay (4) has been derived by Kurkijärvi in terms of resistor networks [26]. A rigorous derivation of upper and lower bounds in agreement with (3) and (4) has been achieved in [14, 15] for $d \geq 2$ and in [7] for $d = 1$. Strictly speaking, in [7, 14, 15] it has been shown that the above random walk satisfies an invariance principle and the asymptotic diffusion matrix $D(\beta)$ satisfies lower and upper bounds in agreement with the asymptotics in the r.h.s. of (3) and (4). Assuming the validity of the Einstein relation, i.e. $\sigma(\beta) = \beta D(\beta)$, the same asymptotic is valid for the conductivity itself. We point out that, in dimension $d = 1$, considering shift–stationary and shift–ergodic point processes $\{x_i\}$ containing the origin, the above result on $D(\beta)$ holds if $\mathbb{E}\big[e^{Z_0}\big] < \infty$ where $Z_0 = x_1 - x_0$, x_1 being the first point right to $x_0 := 0$ (cf. [7, Thm. 1.1]). When $\mathbb{E}\big[e^{Z_0}\big] = \infty$ the random walk is subdiffusive, i.e. $D(\beta) = 0$ (cf. [7, Thm. 1.2]).

The present work has two main results: Considering the above Mott variable–range hopping (also with more general jump rates) we develop the linear response theory and derive the Einstein relation. As a byproduct, the latter, together with [7] completes the rigorous proof of (4). The presence of the external field of intensity λ is modelled by inserting the term $\lambda\beta(x_k - x_i)$ into the exponent in (2). For simplicity of notation, and without loss of generality, we assume that the localization length ξ equals 2. Then, to have a well–defined random walk, one has to take $|\lambda|\beta < 1$. As shown in [12, Thm. 1, Thm. 2], if $\lambda \neq 0$ and $\mathbb{E}[e^{(1-|\lambda|\beta)Z_0}] < \infty$, then the random walk is ballistic (i.e. it has a strictly positive/negative asymptotic velocity) and moreover the environment viewed from the walker admits an ergodic invariant distribution $\mathbb{Q}_\lambda$ mutually absolutely continuous w.r.t. the original law $\mathbb{P}$ of the environment. Strictly speaking, the last statement is referred to the discrete–time version $(Y_n)_{n\geq 0}$ of the original continuous–time Mott random walk $(\mathbb{Y}_t)_{t\geq 0}$ (anyway, the latter can be obtained by a random time change from the former, which allows to extend asymptotic results from Y_n to $\mathbb{Y}_t$). For $\lambda = 0$ the result is still true with $\mathbb{Q}_0$ having an explicit form and being reversible for the environment viewed from the walker.

The ergodicity of $\mathbb{Q}_\lambda$ and its mutual absolute continuity w.r.t. $\mathbb{P}$, together with Birkhoff's ergodic theorem, imply in particular that, for any bounded measurable function f,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\omega_n) = \mathbb{Q}_\lambda[f] \qquad \text{a.s.} \tag{5}$$

for $\mathbb{P}$–almost any environment ω, where ω_n denotes the environment viewed from Y_n. Above, $\mathbb{Q}_\lambda[f]$ denotes the expectation of f w.r.t. $\mathbb{Q}_\lambda$. In what follows, under the assumption that $\mathbb{E}[e^{pZ_0}] < \infty$, we show that the map $(-1, 1) \ni \lambda \mapsto \mathbb{Q}_\lambda[f] \in \mathbb{R}$ is continuous if $p \geq 2$ (see Theorem 2) and that it is derivable at $\lambda = 0$ if $p > 2$ and f belongs to a precise H_{-1} space (see Theorem 3). The derivative can moreover be expressed both in terms of the covariance of suitable additive functionals and in terms of potential forms (the first representation is related to the Kipnis–Varadhan theory of additive functionals [21], the second one to homogenization theory [25, 34]). We point out that similar issues concerning the behavior of the asymptotic steady state (characterized by (5)) for random walks in random environments have been addressed in [18] and [35]. Finally, in Theorem 4 we state the continuity in λ of the asymptotic velocity of (Y_n) and of $(\mathbb{Y}_t)$ and the Einstein relation.

Two main technical difficulties lie behind linear response and Einstein relation: Typically, in the biased case, the asymptotic steady state is not known explicitly and a limited information on the speed of convergence to the steady state is available. A weaker form of the Einstein relation, which is often used as a starting point, was proved in [29]. Since then, the analysis of the Einstein relation, the steady states and the linear response for random walks in static/dynamic random environments have been addressed in [2, 3, 17, 18, 19, 23, 24, 27, 28, 30, 31, 33, 35] (the list is not exhaustive). The approach used here is different from the previous works: Although the distribution $\mathbb{Q}_\lambda$ is not explicit, by refining the analysis of [12] we prove that the Radon–Nikodym derivative

$\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}$ belongs to $L^p(\mathbb{Q}_0)$ if $\mathbb{E}\big[\mathrm{e}^{pZ_0}\big] < \infty$ for some $p \geq 2$ (see Theorem 1). This result has been possible since $\mathbb{Q}_\lambda$ is indeed the weak limit as $\rho \to \infty$ of the asymptotic steady state of the environment viewed from a ρ–cutoff version of (Y_n), for which only jumps between the first ρ neighbors are admitted. For the last ρ–parametrized asymptotic steady state it is possible to express the Radon–Nikodym derivative w.r.t. $\mathbb{P}$ by a regeneration times method developed already by Comets and Popov in [9] for random walks on $\mathbb{Z}$ with long jumps. This method is therefore very model–dependent. On the other hand, having the above bound on $\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}$, one can derive Theorems 2, 3 and 4 by a general method that could be applied in other contexts as well.

Outline of the paper: In Section 2 we describe the model, recall some previous results and present our main theorems (Theorems 1, 2, 3 and 4). Sections 3 and 4 are devoted to the proof of Theorem 1. Theorem 2 is proved in Section 5. The proof of Theorem 3 is split between Sections 6 and 7. The proof of Theorem 4 is split between Sections 8 and 9. Finally, in the Appendices A, B and C we collect some technical results and proofs.

2. Models and main results

One-dimensional Mott random walk is a random walk in a random environment. The environment ω is given by a double–sided sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$, with $Z_k \in (0, +\infty)$ and $E_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$. We denote by $\Omega = ((0, +\infty) \times \mathbb{R})^{\mathbb{Z}}$ the set of all environments. Let $\mathbb{P}$ be a probability on Ω, standing for the law of the environment, and let $\mathbb{E}$ be the associated expectation. Given $\ell \in \mathbb{Z}$, we define the shifted environment $\tau_\ell\omega$ as $\tau_\ell\omega := (Z_{k+\ell}, E_{k+\ell})_{k\in\mathbb{Z}}$. From now on, with a slight abuse of notation, we will denote by Z_k, E_k also the random variables on $(\Omega, \mathbb{P})$ such that $(Z_k(\omega), E_k(\omega))$ is the k–th coordinate of the environment ω.

Our main assumptions on the environment are the following:

- (A1) The random sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$ is stationary and ergodic with respect to shifts;
- (A2) $\mathbb{E}[Z_0]$ is finite;
- (A3) $\mathbb{P}(\omega = \tau_\ell\omega) = 0$ for all $\ell \in \mathbb{Z}$;
- (A4) There exists $d > 0$ such that $\mathbb{P}(Z_0 \geq d) = 1$.

The random environment can be thought of as a marked random point process [10, 16]. Indeed, we can associate to the the double–sided sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$ the point process $\{x_k\}_{k\in\mathbb{Z}}$ such that $x_0 = 0$ and $x_{k+1} = x_k + Z_k$, marking each point x_k with the value E_k. We introduce the map $\psi : \{x_k\} \to \mathbb{Z}$ defined as $\psi(x_k) = k$.

Given the environment ω and $\lambda \in [0, 1)$ we define the *continuous–time Mott random walk* $(\mathbb{Y}^\lambda_t)_{t\geq 0}$ as the random walk on $\{x_k\}_{k\in\mathbb{Z}}$ starting at $x_0 = 0$ with probability rate for a jump from x_i to $x_k \neq x_i$ given by

$$r^\lambda_{i,k}(\omega) := \exp\{-|x_i - x_k| + \lambda(x_k - x_i) + u(E_i, E_k)\}\,, \tag{6}$$

with $u(\cdot, \cdot)$ a symmetric bounded continuous function. It is convenient to set $r^\lambda_{i,i}(\omega) := 0$. To have a well–defined random walk one needs to restrict to $|\lambda| < 1$, and without loss of generality we assume $\lambda \in [0, 1)$.

We then define the *discrete–time Mott random walk* $(Y^\lambda_n)_{n\geq 0}$ (n varies in $\mathbb{N} := \{0, 1, \dots\}$) as the jump process associated to $(\mathbb{Y}^\lambda_t)$. In particular it is a random walk on $\{x_k\}_{k\in\mathbb{Z}}$ starting at $x_0 = 0$ with probability for a jump from x_i to x_k given by

$$p^\lambda_{i,k}(\omega) := \frac{r^\lambda_{i,k}(\omega)}{\sum_{j\in\mathbb{Z}} r^\lambda_{i,j}(\omega)}\,. \tag{7}$$

Note that $p^\lambda_{0,0}\equiv 0$. We denote by φ_λ the local drift of the random walk (Y^λ_n), i.e.

$$\varphi_\lambda(\omega):=\sum_{k\in\mathbb{Z}} x_k\, p^\lambda_{0,k}(\omega)\,. \tag{8}$$

Warning 2.1. *When $\lambda=0$ we usually omit the index λ from the notation, writing simply $\mathbb{Y}_t$, Y_n, $r_{i,k}(\omega)$, $p_{i,k}(\omega)$, $\varphi(\omega)$.*

We now recall some results under the assumption that $\lambda\in(0,1)$ and $\mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0}\big]<+\infty$ (cf. [12, Thm. 1 and Thm. 2]). The asymptotic velocities

$$v_Y(\lambda):=\lim_{n\to\infty}\frac{Y^\lambda_n}{n}\qquad v_{\mathbb{Y}}(\lambda):=\lim_{t\to\infty}\frac{\mathbb{Y}^\lambda_t}{t} \tag{9}$$

exist a.s. and for $\mathbb{P}$–almost all realizations of the environment ω. The above asymptotic velocities are deterministic and do not depend on ω, they are finite and strictly positive. The *environment viewed from the discrete-time random walk* (Y^λ_n), i.e. the process $\big(\tau_{\psi(Y^\lambda_n)}\omega\big)_{n\geq 0}$, admits a unique invariant and ergodic distribution $\mathbb{Q}_\lambda$ which is absolutely continuous w.r.t. $\mathbb{P}$ (in [12] uniqueness is not discussed: Since invariant ergodic distributions are mutually singular, $\mathbb{Q}_\lambda$ is the unique distribution fulfilling the above properties). Moreover, $\mathbb{Q}_\lambda$ and $\mathbb{P}$ are mutually absolutely continuous. Finally (see also Appendix A) the asymptotic velocities $v_Y(\lambda)$ and $v_{\mathbb{Y}}(\lambda)$ can be expressed as

$$v_Y(\lambda)=\mathbb{Q}_\lambda\big[\varphi_\lambda\big]\quad\text{and}\quad v_{\mathbb{Y}}(\lambda)=\frac{v_Y(\lambda)}{\mathbb{Q}_\lambda\Big[1/(\sum_{k\in\mathbb{Z}} r^\lambda_{0,k}(\omega))\Big]}\,. \tag{10}$$

We recall some results concerning the unperturbed random walk (Y_n) (i.e. with $\lambda=0$). In this case the asymptotic velocities in (9) still exist a.s. and for $\mathbb{P}$–almost all realizations of the environment ω, but they are zero: $v_Y(0)=v_{\mathbb{Y}}(0)=0$ (cf. [12, Remark 2.1]). Moreover, the environment viewed from the walker (Y_n) has reversible measure $\mathbb{Q}_0$ defined as

$$\mathbb{Q}_0(d\omega)=\frac{\pi(\omega)}{\mathbb{E}[\pi]}\mathbb{P}(d\omega)\,,\qquad \pi(\omega):=\sum_{k\in\mathbb{Z}} r_{0,k}(\omega)\,. \tag{11}$$

It is known (cf. [7, Sec. 2]) that, when $\mathbb{E}\big[\mathrm{e}^{Z_0}\big]<\infty$, for $\mathbb{P}$–almost all the realizations of the environment ω the random walk (Y_n) starting at the origin converges, under diffusive rescaling, to a Brownian motion with positive diffusion coefficient given by

$$D_Y=\inf_{g\in L^\infty(\mathbb{Q}_0)}\mathbb{Q}_0\Big[\sum_{i\in\mathbb{Z}} p_{0,i}\big(x_i+\nabla_i g\big)^2\Big]\,, \tag{12}$$

where $\nabla_i g(\omega):=g(\tau_i\omega)-g(\omega)$ (note that, since $\mathbb{Q}_0$ and $\mathbb{P}$ are mutually absolutely continuous, in formula (1.14) in [7] one can replace $L^\infty(\mathbb{P})$ by $L^\infty(\mathbb{Q}_0)$). Similarly (cf. [7, Thm. 1.1]) $(\mathbb{Y}_t)$ satisfies a quenched functional CLT with diffusion coefficient

$$D_{\mathbb{Y}}=\mathbb{E}[\pi]\,D_Y\,. \tag{13}$$

In order to present our results we need to introduce the symmetric non–negative operator $-\mathbb{L}_0: L^2(\mathbb{Q}_0)\to L^2(\mathbb{Q}_0)$ with $\mathbb{L}_0$ defined as

$$\mathbb{L}_0 f(\omega)=\sum_{k\in\mathbb{Z}} p_{0,k}(\omega)\big[f(\tau_k\omega)-f(\omega)\big]\,. \tag{14}$$

We recall some basic facts on the spaces H_1 and H_{-1} associated to the operator $\mathbb{L}_0$ (cf. [11, 21, 22]). In what follows we denote the scalar product in $L^2(\mathbb{Q}_0)$ by $\langle\cdot,\cdot\rangle$. The H_1 space is given by the completion of $L^2(\mathbb{Q}_0)$ endowed with the scalar product $\langle f,g\rangle_1:=$

$\langle f, -\mathbb{L}_0 g\rangle$ and H_{-1} will denote the space dual to H_1. In particular, $f \in L^2(\mathbb{Q}_0)$ belongs to H_{-1} if and only if there exists a constant $C > 0$ such that $|\langle f, g\rangle| \leq C\langle g, -\mathbb{L}_0 g\rangle^{1/2}$ for any $g \in L^2(\mathbb{Q}_0)$. Note that $\mathbb{Q}_0(f) = 0$ for any $f \in L^2(\mathbb{Q}_0) \cap H_{-1}$. Equivalently, denoting by $\mathrm{e}_f(dx)$ the spectral measure associated to f and the operator $-\mathbb{L}_0$ (see e.g. [43]), $f \in L^2(\mathbb{Q}_0)$ belongs to H_{-1} if and only $\int_{[0,\infty)} \frac{1}{x}\mathrm{e}_f(dx) < \infty$.

We can now present our main results. Although having a technical flavour, the following theorem is indeed our starting point for the investigation of the continuity in λ and the linear response at $\lambda = 0$ of the system, as explained in the introduction:

Theorem 1. *Fix $\lambda_* \in (0,1)$ and suppose that $\mathbb{E}\big[\mathrm{e}^{pZ_0}\big] < +\infty$ for some $p \geq 2$. Then, it holds*

$$\sup_{\lambda\in[0,\lambda_*]} \Big\| \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} \Big\|_{L^p(\mathbb{Q}_0)} < \infty\,. \tag{15}$$

Our next result concerns the continuity in λ of the expectation $\mathbb{Q}_\lambda(f)$.

Theorem 2. *Suppose that $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ for some $p \geq 2$ and let q be the conjugate exponent of p, i.e. q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in L^q(\mathbb{Q}_0)$ and $\lambda \in [0,1)$, it holds that $f \in L^1(\mathbb{Q}_\lambda)$ and the map*

$$[0,1) \ni \lambda \mapsto \mathbb{Q}_\lambda(f) \in \mathbb{R} \tag{16}$$

is continuous.

We point out that, for what concerns linear response at $\lambda = 0$, only the continuity of the map (16) at $\lambda = 0$ plays some role. Anyway, our techniques allow to prove continuity of the map (16) beyond the linear response regime.

Our next result concerns the derivative at $\lambda = 0$ of the map $\lambda \mapsto \mathbb{Q}_\lambda(f)$ for functions $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$. This derivative can be represented both as a suitable expectation involving a square integrable form and as a covariance. To describe these representations we fix some notation starting with the square integrable forms.

We consider the space $\Omega \times \mathbb{Z}$ endowed with the measure M defined by

$$M(u) = \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k} u(\cdot, k)\Big]\,, \qquad \forall u : \Omega \times \mathbb{Z} \to \mathbb{R} \qquad \text{Borel, bounded}\,.$$

A generic Borel function $u : \Omega \times \mathbb{Z} \to \mathbb{R}$ will be called a *form*. $L^2(\Omega \times \mathbb{Z}, M)$ is known as the space of *square integrable forms*. Below, we will shorten the notation writing simply $L^2(M)$, and in general $L^p(M)$ for p–integrable forms. Given a function $g = g(\omega)$ we define

$$\nabla g(\omega, k) := g(\tau_k\omega) - g(\omega)\,. \tag{17}$$

If $g \in L^2(\mathbb{Q}_0)$ then $\nabla g \in L^2(M)$ (this follows from the identity $\mathbb{Q}_0[\sum_k p_{0,k} g(\tau_k\cdot)^2] = \mathbb{Q}_0[g^2]$ due to the stationarity of $\mathbb{Q}_0$). The closure in M of the subspace $\{\nabla g \,:\, g \in L^2(\mathbb{Q}_0)\}$ forms the set of the so called *potential forms* (the orthogonal subspace is given by the so called *solenoidal forms*). Take again $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$ and, given $\varepsilon > 0$, define $g^f_\varepsilon \in L^2(\mathbb{Q}_0)$ as the unique solution of the equation

$$(\varepsilon - \mathbb{L}_0) g^f_\varepsilon = f\,. \tag{18}$$

As discussed in Section 6, as ε goes to zero the family of potential forms ∇g^f_ε converges in $L^2(M)$ to a potential form h^f:

$$h^f = \lim_{\varepsilon\downarrow 0} \nabla g^f_\varepsilon \qquad \text{in } L^2(M)\,. \tag{19}$$

We now fix the notation that will allow us to state the second representation of $\partial_{\lambda=0}\mathbb{Q}_\lambda(f)$ in terms of covariances. To this aim we write (ω_n) for the environment viewed form the unperturbed walker (Y_n), i.e. $\omega_n := \tau_{\psi(Y_n)}\omega$ where ω denotes the initial environment (recall that $\psi(x_i) = i$). Take now $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$. Due to [21, Cor. 1.5] and Wold theorem, starting the process (ω_n) with distribution $\mathbb{Q}_0$, we have the following weak convergence of 2d random vectors

$$\frac{1}{\sqrt{n}}\Big(\sum_{j=0}^{n-1} f(\omega_j), \sum_{j=0}^{n-1}\varphi(\omega_j)\Big) \overset{n\to\infty}{\to} (N^f, N^\varphi) \tag{20}$$

for a suitable 2d gaussian vector (N^f, N^φ) (with possibly degenerate diffusion matrix). We recall that φ denotes the local drift when $\lambda = 0$ (cf. (8) and Warning 2.1).

We can now state our next main result:

Theorem 3. *Suppose* $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ *for some* $p > 2$. *Then, for any* $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$, *the map* $\lambda \mapsto \mathbb{Q}_\lambda(f)$ *is differentiable at* $\lambda = 0$. *Moreover it holds*

$$\partial_{\lambda=0}\mathbb{Q}_\lambda(f) = \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}(x_k - \varphi)h^f(\cdot, k)\Big] \tag{21}$$

$$= -\mathrm{Cov}(N^f, N^\varphi)\,. \tag{22}$$

Starting from the above theorems one can derive the continuity of the velocity and the Einstein relation between velocity and diffusion coefficient both for (Y_n) and for $(\mathbb{Y}_t)$:

Theorem 4. *The following holds:*

(i) *If* $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$, *then* $v_Y(\lambda)$ *and* $v_{\mathbb{Y}}(\lambda)$ *are continuous functions of* λ;
(ii) *If* $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ *for some* $p > 2$, *then the Einstein relation is fulfilled, i.e.*

$$\partial_{\lambda=0}v_Y(\lambda) = D_Y \qquad \text{and} \qquad \partial_{\lambda=0}v_{\mathbb{Y}}(\lambda) = D_{\mathbb{Y}}\,. \tag{23}$$

Remark 2.2. *We point out that in general the velocities* $v_Y(\lambda)$ *and* $v_{\mathbb{Y}}(\lambda)$ *can have discontinuities. See* [12, Ex. 2 in Sec. 2] *for an example.*

If we make explicit the temperature dependence in the jump rates (6) we would have

$$r^\lambda_{i,k}(\omega) := \exp\{-|x_i - x_k| + \lambda\beta(x_k - x_i) + \beta u(E_i, E_k)\}\,,$$

where λ is the strength of the external field. Then equation (23) takes the more familiar (from a physical viewpoint) form

$$\partial_{\lambda=0}v_Y(\lambda, \beta) = \beta D_Y(\beta) \text{ and } \partial_{\lambda=0}v_{\mathbb{Y}}(\lambda, \beta) = \beta D_{\mathbb{Y}}(\beta)\,.$$

Remark 2.3. *In our treatment, and in particular in Theorems 2, 3 and 4, we have restricted our analysis to* $\lambda \in [0, 1)$. *One can easily extend the above results to* $\lambda \in (-1, 1)$. *Indeed, by taking a space reflection w.r.t. the origin, the resulting random environment still satisfies the main assumptions (A1),...,(A4) and the same exponential moment bounds as the original enviroment, while random walks with negative bias become random walks with positive bias. Hence, after taking a space reflection w.r.t. the origin, one can apply the above theorems to study continuity for* $\lambda \in (-1, 0]$ *and derivability from the left at* $\lambda = 0$. *Noting that the left derivatives at* $\lambda = 0$ *in Theorem 3 and 4 equal the right derivatives at* $\lambda = 0$, *one recovers that the claims in Theorems 2, 3 and 4 remain valid with* $\lambda \in (-1, 1)$.

3. Proof of Theorem 1

It is convenient to introduce the following notation for $i, j \in \mathbb{Z}$:

$$c^\lambda_{i,j}(\omega) := \begin{cases} \mathrm{e}^{-|x_j - x_i| + \lambda(x_i + x_j) + u(E_i, E_j)} & \text{if } i \neq j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{24}$$

The above $c^\lambda_{i,j}(\omega)$ can be thought of as the conductance associated to the edge $\{i, j\}$ and indeed the perturbed walk (Y^λ_n) is a random walk among the above conductances, since $p^\lambda_{i,j}(\omega) = c^\lambda_{i,j}(\omega)/\sum_{k\in\mathbb{Z}} c^\lambda_{i,k}(\omega)$.

The proof of Theorem 1 is an almost direct consequence of the following lemma:

Lemma 3.1. *Fix $\lambda_* \in (0, 1)$. Then there exist positive constants K, K_* such that, given $\lambda \in (0, \lambda_*]$, the Radon–Nikodym derivative $\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}$ satisfies*

$$\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}(\omega) \leq K\lambda g(\omega, \lambda)\,, \tag{25}$$

where

$$g(\omega, \lambda) := K_0(c^\lambda_{-1,0} + c^\lambda_{0,1}) \sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_j + (1-\lambda)(x_{j+1} - x_j)}\,. \tag{26}$$

The proof of Lemma 3.1 requires a fine analysis of Mott random walk $(Y^\lambda_n)_{n\geq 0}$. We postpone it to the next section. Here we show how to derive Theorem 1 from Lemma 3.1.

Proof of Theorem 1. It is enough to consider the case $\lambda \neq 0$. The constants c, C, C_*, C' appearing below are to be thought independent from $\lambda \in (0, \lambda_*]$ (they can depend on λ_*). By (11) and Lemma 3.1 we can write

$$\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} = \frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}\frac{d\mathbb{P}}{d\mathbb{Q}_0} = \frac{\mathbb{E}[\pi]}{\pi}\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \leq \lambda C_* \frac{c^\lambda_{-1,0} + c^\lambda_{0,1}}{\pi} \sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_j + (1-\lambda)(x_{j+1} - x_j)}\,. \tag{27}$$

Since (recall the bounded function u in (6))

$$\frac{c^\lambda_{-1,0} + c^\lambda_{0,1}}{\pi}\mathrm{e}^{-2\|u\|_\infty} \leq \frac{\mathrm{e}^{-|x_{-1}| + \lambda x_{-1}} + \mathrm{e}^{-x_1 + \lambda x_1}}{\sum_{k\neq 0} \mathrm{e}^{-|x_k|}} \leq 1 + \mathrm{e}^{\lambda x_1} \leq 2\mathrm{e}^{\lambda x_1}\,,$$

$$\mathrm{e}^{\lambda x_1} \sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_j + (1-\lambda)(x_{j+1} - x_j)} \leq \mathrm{e}^{x_1} + \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda x_j + (1-\lambda)(x_{j+1} - x_j)} \leq \mathrm{e}^{Z_0} + \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda d j + Z_j}\,,$$

from (27) we get $\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0} \leq 2\mathrm{e}^{2\|u\|_\infty} C_* \lambda \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda d j + Z_j}$. As a consequence, to conclude it is enough to prove that

$$\mathbb{Q}_0\Big[\big(\sum_{j=0}^{\infty} \mathrm{e}^{-\lambda d j + Z_j}\big)^p\Big] \leq C/\lambda^p \tag{28}$$

for some constant C. To this aim let q be the conjugate exponent such that $1/p + 1/q = 1$. By the Hölder inequality we can bound

$$\sum_{j=0}^{\infty} \mathrm{e}^{-\lambda d j + Z_j} \leq \Big(\sum_{j=0}^{\infty} \mathrm{e}^{-\frac{\lambda d q}{2} j}\Big)^{\frac{1}{q}} \Big(\sum_{j=0}^{\infty} \mathrm{e}^{-\frac{\lambda d p}{2} j + p Z_j}\Big)^{\frac{1}{p}} = \Big(1 - \mathrm{e}^{-\frac{\lambda d q}{2}}\Big)^{-\frac{1}{q}} \Big(\sum_{j=0}^{\infty} \mathrm{e}^{-\frac{\lambda d p}{2} j + p Z_j}\Big)^{\frac{1}{p}}\,.$$

By using the above bound in (28) we get

$$\mathbb{Q}_0\Big[\big(\sum_{j=1}^{\infty} \mathrm{e}^{-\lambda dj+Z_j}\big)^p\Big] \le \Big(1-\mathrm{e}^{-\frac{\lambda dq}{2}}\Big)^{-\frac{p}{q}}\Big(1-\mathrm{e}^{-\frac{\lambda dp}{2}}\Big)^{-1}\mathbb{Q}_0[\mathrm{e}^{pZ_0}]$$
$$\le (C'\lambda)^{-\frac{p}{q}}(C'\lambda)^{-1}\mathbb{Q}_0[\mathrm{e}^{pZ_0}] = C\lambda^{-\frac{p}{q}-1} = C\lambda^{-p}\,,$$

thus implying (28). □

4. Proof of Lemma 3.1

In the first part of the section we will improve a bound obtained in [12], see Proposition 4.4 below. This result will be essential to the proof of Lemma 3.1 (which will be carried out in Subsection 4.1).

In the rest of this section $\lambda \in (0,\lambda_*]$ is fixed once and for all and is omitted from the notation. In particular, we write (Y_n) for the biased discrete–time Mott random walk (Y_n^λ) and we write $c_{i,j}(\omega)$ instead of $c_{i,j}^\lambda(\omega)$ (cf. (24)). As in [12], it will be convenient to consider the ψ–projection of (Y_n) on the integers. We call (X_n) the discrete-time random walk on $\mathbb{Z}$ such that $X_n = \psi(Y_n)$. As already pointed out, the probability for a jump of X_n from i to k is given by (7) which equals $\frac{c_{i,k}}{\sum_{j\in\mathbb{Z}} c_{i,j}}$.

We further introduce a truncated version of (X_n). We set $\mathbb{N}_+ := \{1,2,3,\dots\}$. For $\rho \in \mathbb{N}_+ \cup \{+\infty\}$ we call (X_n^ρ) the discrete-time random walk with jumping probabilities from i to j given by

$$\begin{cases} c_{i,j}(\omega)/\sum_{k\in\mathbb{Z}} c_{i,k}(\omega), & \text{if } 0<|i-j|\le\rho\,,\\ 0 & \text{if } |i-j|>\rho\,,\\ 1-\sum\limits_{j:|j-i|\le\rho} c_{i,j}(\omega)/\sum_{k\in\mathbb{Z}} c_{i,k}(\omega) & \text{if } i=j\,. \end{cases} \tag{29}$$

Clearly the case $\rho=\infty$ corresponds to the random walk (X_n). We write $P_i^{\omega,\rho}$ for the law of (X_n^ρ) starting at point $i\in\mathbb{Z}$ and $E_i^{\omega,\rho}$ for the associated expectation. In order to make the notation lighter, inside $P_i^{\omega,\rho}(\cdot)$ and $E_i^{\omega,\rho}[\cdot]$ we will sometimes write X_n instead of X_n^ρ, when there will be no possibility of misunderstanding.

Call

$$T_i^\rho := \inf\{n\ge 0:\ X_n^\rho \ge i\} \tag{30}$$

the first time the ρ-truncated random walk jumps over point $i\in\mathbb{Z}$ (also for $T_\cdot^\rho$ we will drop the ρ super-index inside $P_i^{\omega,\rho}(\cdot)$ and $E_i^{\omega,\rho}[\cdot]$). A fundamental fact (cf. [12, Lemma 3.16]) is the following: One can find a positive $\varepsilon=\varepsilon(\lambda_*)$ independent from ρ, ω and $\lambda\in(0,\lambda_*]$ such that

$$P_k^{\omega,\rho}(X_{T_i}=i)\ge 2\varepsilon \qquad \forall k<i,\, \forall \rho\in\mathbb{N}_+\cup\{\infty\}\,. \tag{31}$$

Remark 4.1. *In* [12, Rem. 3.2] *it is stated that all constants K's and the constat ε appearing in* [12, Sec. 3] *can be taken independent of λ if λ e.g. varies in* $[0,1/2)$. *As the reader can easily check the same still holds as λ varies in* $[0,\lambda_*]$ *for any fixed λ_* in* $(0,1)$ *(note that the above constants will depend on λ_*).*

Given a subset $A\subset\mathbb{Z}$ we define τ_A as the hitting time of the subset A, i.e. τ_A is the first nonnegative time for which the random walk is in A. For A,B disjoint subsets of $\mathbb{Z}$, we define the effective ρ–conductance between A and B as

$$C_{\text{eff}}^\rho(A,B) := \min\Big\{\sum_{i<j:\,|i-j|\le\rho} c_{i,j}(f(j)-f(i))^2 :\ f|_A=0,\, f|_B=1\Big\}\,. \tag{32}$$

The following technical fact provides a crucial estimate for the proof of Lemma 3.1:

Lemma 4.2. *For all $k \in \{1, ..., \rho - 1\}$,*

$$P_k^{\omega,\rho}(\tau_0 < \tau_{[\rho,\infty)}) \geq 2\varepsilon^3 \frac{C_{\text{eff}}^{\rho}(k, (-\infty, 0])}{C_{\text{eff}}^{\rho}(k, (-\infty, 0] \cup [\rho, \infty))} \,.$$

Proof. For simplicity we will call $A := (-\infty, 0]$ and $B := [\rho, \infty)$. First of all notice that $P_k^{\omega,\rho}(\tau_0 < \tau_{[\rho,\infty)}) \geq 2\varepsilon P_k^{\omega,\rho}(\tau_A < \tau_B)$. In fact,

$$\begin{aligned} P_k^{\omega,\rho}(\tau_0 < \tau_B) &= \sum_{j \leq 0} P_k^{\omega,\rho}(\tau_0 < \tau_B | \, \tau_A < \tau_B, \, X_{\tau_A} = j) P_k^{\omega,\rho}(\tau_A < \tau_B, \, X_{\tau_A} = j) \\ &= \sum_{j \leq 0} P_j^{\omega,\rho}(\tau_0 < \tau_B) P_k^{\omega,\rho}(\tau_A < \tau_B, \, X_{\tau_A} = j) \geq 2\varepsilon P_k^{\omega,\rho}(\tau_A < \tau_B), \end{aligned} \tag{33}$$

where in the last line we have used that $P_j^{\omega,\rho}(\tau_0 < \tau_B) \geq P_j^{\omega,\rho}(X_{T_0} = 0) \geq 2\varepsilon$, which follows from (31). We can therefore focus on $P_k^{\omega,\rho}(\tau_A < \tau_B)$.

We consider now the following reduced Markov chain (X'_n) starting at k. Given $\omega \in \Omega$, (X'_n) is the random walk on the state space $\{0, ..., \rho\}$ with conductances $c'_{i,j} = c'_{i,j}(\omega)$ defined by requiring that $c'_{i,j} = c'_{j,i}$ and that

$$c'_{i,j} := \begin{cases} c_{i,j} & \text{if } i, j \in \{1, ..., \rho - 1\} \,, \; i \neq j \,, \\ \sum_{m:\, i-\rho \leq m \leq 0} c_{i,m} & \text{if } i \in \{1, ..., \rho - 1\} \,, \; j = 0 \,, \\ \sum_{m:\, \rho \leq m \leq i+\rho} c_{i,m} & \text{if } i \in \{1, ..., \rho - 1\} \,, \; j = \rho \,, \\ 0 & \text{if } i = j \,. \end{cases}$$

We recall that, by definition, the probability for a transition from i to j in $\{0, 1, \ldots, \rho\}$ equals $c'_{i,j}/\pi'(i)$ where $\pi'(i) = \sum_{j: 0 \leq j \leq \rho} c'_{i,j}$. Note that π' is a reversible measure for (X'_n).

By a suitable coupling on an enlarged probability space (the probability of which will be denoted again by $P_k^{\omega,\rho}$) it holds

$$P_k^{\omega,\rho}(\tau_A < \tau_B) = P_k^{\omega,\rho}(\tau'_0 < \tau'_\rho), \tag{34}$$

where τ'_j is the first time (X'_n) hits point j. In fact, starting at k, if we ignore the times when (X_n^ρ) does not move, (X'_n) and (X_n^ρ) can be coupled in a way that guarantees that $X_n^\rho = X'_n$ until the moment when X'_n touches 0 or ρ. More precisely, one can couple the two random walks to have that $(X'_n \,:\, 0 \leq n \leq \min\{\tau'_0, \tau'_\rho\})$ equals the sequence of different visited sites of the path $(\phi(X_n) \,:\, 0 \leq n \leq \min\{\tau_A, \tau_B\})$, where $\phi : \mathbb{Z} \to \{0, 1, \ldots, \rho\}$ is defined as $\phi(i) := 0$ for $i \leq 0$, $\phi(i) = \rho$ for $i \geq \rho$ and $\phi(i) = i$ otherwise. The advantage of the above reduction is to have to deal now with a finite graph, so that we will be able to use classical results for resistor networks.

As in [4, proof of Fact 2], we call $t_0 = 0$ and t_i the i-th time the walk (X'_n) returns to the starting point k. We call the interval $[t_{i-1}, t_i]$ the i-th excursion. For a set $D \subset \{0, ..., \rho\}$ we call $V(i, D)$ the event that (X'_n) visits the set D during the i-th excursion. We also call $\bar{V}(i, D)$ the event that set D has been visited for the first time in the i-th excursion. Noticing now that the excursions are i.i.d., we can compute

$$\begin{aligned} P_k^{\omega,\rho}(\tau'_0 < \tau'_\rho) &= \sum_{i=1}^{\infty} P_k^{\omega,\rho}\big(\tau'_0 < \tau'_\rho | \, \bar{V}(i, \{0, \rho\})\big) P_k^{\omega,\rho}(\bar{V}(i, \{0, \rho\})) \\ &= P_k^{\omega,\rho}\big(\tau'_0 < \tau'_\rho | \, \bar{V}(1, \{0, \rho\})\big) = P_k^{\omega,\rho}\big(\tau'_0 < \tau'_\rho | \, V(1, \{0, \rho\})\big), \end{aligned} \tag{35}$$

so that we are left to estimate the probability that 0 is visited before ρ knowing that at least one of the two has been visisted during the first excursion. We see that

$$\begin{aligned} P_k^{\omega,\rho}\big(\tau_0' < \tau_\rho' \big| V(1,\{0,\rho\})\big) &\geq P_k^{\omega,\rho}\big(V(1,0)\cap V^c(1,\rho)\big| V(1,\{0,\rho\})\big) \\ &= \frac{P_k^{\omega,\rho}(V(1,0)\cap V^c(1,\rho))}{P_k^{\omega,\rho}(V(1,\{0,\rho\}))}, \end{aligned} \tag{36}$$

where $V^c(1,\rho)$ is the event that the walk does not visit ρ during the first excursion. We claim that

$$P_k^{\omega,\rho}(V(1,0)\cap V^c(1,\rho)) \geq \varepsilon^2\, P_k^{\omega,\rho}(V(1,0)). \tag{37}$$

To see why this is true, we first of all simplify the notation by setting $V(0) := V(1,0)$ and $V(\rho) := V(1,\rho)$ and write

$$P_k^{\omega,\rho}(V(0)\cap V^c(\rho)) = \alpha\, P_k^{\omega,\rho}(V(0)) \tag{38}$$

with

$$\alpha := \frac{P_k^{\omega,\rho}(V(0)\cap V^c(\rho))}{P_k^{\omega,\rho}(V(0))} = \frac{R_1}{R_1+R_2+R_3} \tag{39}$$

with $R_1 := P_k^{\omega,\rho}(V(0)\cap V^c(\rho))$, $R_2 := P_k^{\omega,\rho}(V(0)\cap V(\rho),\, \tau_0' < \tau_\rho')$, $R_3 := P_k^{\omega,\rho}(V(0)\cap V(\rho),\, \tau_\rho' < \tau_0')$.

The proof of (37) is then based on the following bounds:

$$R_3 \leq \big(R_1+R_2\big)/(2\varepsilon)\,, \tag{40}$$

$$R_2 \leq R_1/(2\varepsilon)\,. \tag{41}$$

Before proving (40) and (41) we explain how to derive (37) and conclude the proof of Lemma 4.2.

Trivially, (40) and (41) imply that

$$\alpha \geq \frac{R_1}{(R_1+R_2)(1+\frac{1}{2\varepsilon})} \geq \frac{1}{(1+\frac{1}{2\varepsilon})}\cdot\frac{R_1}{R_1+\frac{1}{2\varepsilon}R_1} = \frac{1}{(1+\frac{1}{2\varepsilon})^2} \geq \varepsilon^2\,.$$

In the last bound we have used that $\varepsilon \leq 1/2$ (cf. (31)). This together with (38) gives (37).

Putting now (37) into (36), (36) into (35), we get

$$P_k^{\omega,\rho}(\tau_0' < \tau_\rho') \geq \varepsilon^2 \frac{P_k^{\omega,\rho}(V(1,0))}{P_k^{\omega,\rho}(V(1,\{0,\rho\}))}\,, \tag{42}$$

where we have restored the notation $V(i,D)$ for the event of having a visit to set D during excursion i. By a well-known formula (see, e.g., formula (5) in [4]) we know that for each $D\subset\{0,...,\rho\}$

$$P_k^{\omega,\rho}(V(i,D)) = \frac{C'_{\rm eff}(k,D)}{\pi'(k)}, \tag{43}$$

where $C'_{\rm eff}(k,D)$ denotes the effective conductance between k and D in the reduced model. More precisely, given disjoint subsets E,F in $\{0,1,\dots,\rho\}$, we define

$$C'_{\rm eff}(E,F) := \min\Big\{ \sum_{i,j\,:\,0\leq i<j\leq\rho} c'_{i,j}(f(j)-f(i))^2 \,:\, f|_E = 0,\ f|_F = 1\Big\}\,. \tag{44}$$

As a byproduct of (42) and (43) we get

$$P_k^{\omega,\rho}(\tau_0' < \tau_\rho') \geq \varepsilon^2 \frac{C'_{\rm eff}(k,0)}{C'_{\rm eff}(k,\{0,\rho\})} \geq \varepsilon^2 \frac{C^\rho_{\rm eff}(k,(-\infty,0])}{C^\rho_{\rm eff}(k,(-\infty,0]\cup[\rho,\infty))}. \tag{45}$$

Let us explain the last bound. Given a function $f : \{0, 1, \dots, \rho\} \to \mathbb{R}$ and calling $\bar f$ its extension on $\mathbb{Z}$ such that $\bar f(z) = f(0)$ for all $z \le 0$ and $\bar f(z) = f(\rho)$ for all $z \ge \rho$, it holds

$$\sum_{i,j\,:\,0\le i<j\le\rho} c'_{i,j}(f(j)-f(i))^2 = \sum_{i<j\,:\,|i-j|\le\rho} c_{i,j}(\bar f(j)-\bar f(i))^2\,. \tag{46}$$

As a consequence, by comparing the variational definitions of effective conductances given in (32) and (44), one gets that $C'_{\text{eff}}(k,0) \ge C^{\rho}_{\text{eff}}(k,(-\infty,0])$ and $C'_{\text{eff}}(k,\{0,\rho\}) = C^{\rho}_{\text{eff}}(k,(-\infty,0]\cup[\rho,\infty))$, thus implying the last bound in (45). Having (45), we finally use (33) and (34) to get the lemma.

We are left with the proof of (40) and (41).

• *Proof of* (40). We define $\tau_0(1) := \tau'_0$, $\tau_\rho(1) := \tau'_\rho$ and $\tau_{0,\rho}(1) := \min\{\tau_0(1), \tau_\rho(1)\}$. We also define $x_* := X'_{\tau_{0,\rho}(1)}$ (note that x_* equals 0 or ρ). Then, iteratively, for all $j \ge 1$ we define (see Figure 1)

$$\begin{aligned}
\tau_{0,\rho}(j+1) &:= \inf\{n : \, n > \tau_0(j),\, n > \tau_\rho(j),\, X'_n = x_*\}\\
\tau_0(j+1) &:= \begin{cases} \tau_{0,\rho}(j+1) & \text{if } x_* = 0\\ \inf\{n : \, X_n = 0,\, n > \tau_{0,\rho}(j+1)\} & \text{if } x_* = \rho \end{cases}\\
\tau_\rho(j+1) &:= \begin{cases} \inf\{n : \, X_n = \rho,\, n > \tau_{0,\rho}(j+1)\} & \text{if } x_* = 0\\ \tau_{0,\rho}(j+1) & \text{if } x_* = \rho\,. \end{cases}
\end{aligned}$$

Notice that, almost surely, either $\tau_0(1) < \tau_\rho(1) < \tau_0(2) < \tau_\rho(2) < \dots$ or $\tau_\rho(1) < \tau_0(1) < \tau_\rho(2) < \tau_0(2) < \dots$. We also define τ_k^+ as the first time the random walk started in k returns to k. Notice that all the $\tau_\cdot(\cdot)$'s and τ_k^+ are stopping times. We decompose

$$R_3 := P_k^{\omega,\rho}(V(0)\cap V(\rho),\, \tau'_\rho < \tau'_0) = P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \tau_k^+) = \sum_{i=1}^{\infty} A_i + \sum_{i=2}^{\infty} B_i\,, \tag{47}$$

where

$$\begin{aligned}
A_i &:= P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \tau_\rho(2) < \cdots < \tau_0(i) < \tau_k^+ < \tau_\rho(i+1))\\
B_i &:= P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \tau_\rho(2) < \cdots < \tau_\rho(i) < \tau_k^+ < \tau_0(i))\,.
\end{aligned}$$

We first focus on the terms of the form A_i.

Claim 4.3. *It holds*

$$A_i \le \frac{1}{2\varepsilon} C_i, \qquad i \ge 1 \tag{48}$$

$$B_i \le \frac{1}{2\varepsilon} D_{i-1}, \qquad i \ge 2 \tag{49}$$

where

$$\begin{aligned}
C_i &:= P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_k^+ < \tau_\rho(i)),\\
D_i &:= P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_\rho(i) < \tau_k^+ < \tau_0(i+1)).
\end{aligned}$$

Proof of the Claim. We start with (48). By reversibility (just decompose the event on all the possible trajectories of the random walk and then use the detailed balance equations, see Figure 1),

$$A_i = P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_\rho(i) < \tau_k^+ < \tau_0(i+1)). \tag{50}$$

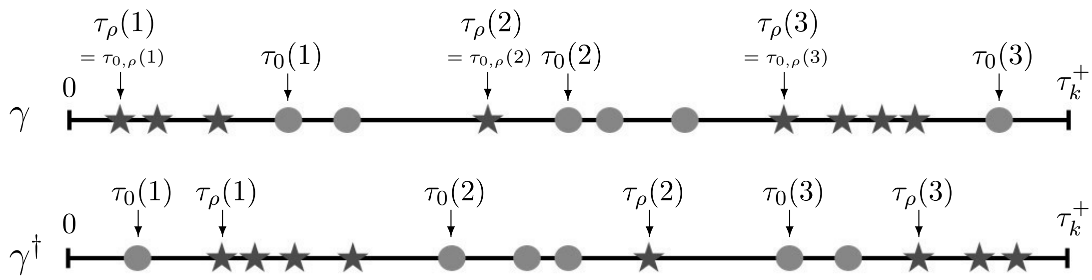

FIGURE 1. *γ corresponds to the trajectory of an excursion from k to k associated to the probability A_i, $i = 3$. $\gamma^\dagger$ is the time-reversed trajectory. Balls denote times when the random walk hits* 0*, while stars denote times when it hits ρ.*

On the other hand, we have

$$\begin{aligned} C_i &\geq P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_k^+ ,\, \xi(0,i)) \\ &= P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_k^+)\, P_0^{\omega,\rho}(X'_{T_k^\rho} = k) \\ &\geq 2\varepsilon\, P_k^{\omega,\rho}(\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_k^+) \;\geq 2\varepsilon\, A_i, \end{aligned} \tag{51}$$

where the event $\xi(0,i)$ is defined as $\xi(0,i) := \{$the first time after $\tau_0(i)$ that the random walk tries to overjump the point k, it actually lands on $k\}$. The first inequality is trivial. For the second line, we can apply the strong Markov property at the stopping time $\tau_0(i)$ observing that the event $\{\tau_0(1) < \tau_\rho(1) < \cdots < \tau_0(i) < \tau_k^+\}$ is in the σ-algebra generated by the process up to time $\tau_0(i)$. Finally, for the last line we first notice that (31) is also valid for the random walk (X'_n) and then use (50). This gives (48).

We move to the proof of (49). Clearly

$$B_i \leq P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \cdots < \tau_0(i-1) < \tau_k^+)\,. \tag{52}$$

On the other hand

$$\begin{aligned} D_{i-1} &= P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \cdots < \tau_0(i-1) < \tau_k^+ < \tau_\rho(i)) \\ &\geq P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \cdots < \tau_0(i-1) < \tau_k^+ ,\, \xi(0,i-1)) \\ &\geq 2\varepsilon\, P_k^{\omega,\rho}(\tau_\rho(1) < \tau_0(1) < \cdots < \tau_0(i-1) < \tau_k^+)\,, \end{aligned} \tag{53}$$

where for the first line we have used again the reversibility of the process, in the second and third line we have used the same arguments as for the proof of (48). (52) and (53) together show that $B_i \leq \frac{1}{2\varepsilon} D_{i-1}$. □

We come back to (47). Thanks to the above claim, we have

$$\begin{aligned} R_3 &\leq \frac{1}{2\varepsilon}\Big(C_1 + \sum_{i=2}^{\infty} C_i + \sum_{i=1}^{\infty} D_i\Big) \\ &= \frac{1}{2\varepsilon}\Big(P_k^{\omega,\rho}(V(0)\cap V^c(\rho)) + P_k^{\omega,\rho}(V(0)\cap V(\rho),\, \tau'_0 < \tau'_\rho)\Big) = \frac{1}{2\varepsilon}\big(R_1 + R_2\big) \end{aligned}$$

as we wished, since $C_1 = P_k^{\omega,\rho}(V(0)\cap V^c(\rho))$ and $\sum_{i=2}^{\infty} C_i + \sum_{i=1}^{\infty} D_i$ is a decomposition of the probability of the event $\{V(0)\cap V(\rho),\, \tau'_0 < \tau'_\rho\}$ in a similar fashion as in (47).

• *Proof of* (41). We notice that

$$\begin{aligned} R_1 = P_k^{\omega,\rho}(V(0)\cap V^c(\rho)) &= P_k^{\omega,\rho}(\tau_0(1)<\tau_k^+<\tau_\rho(1)) \\ &\ge P_k^{\omega,\rho}(\tau_0(1)<\tau_k^+,\, \tau_0(1)<\tau_\rho(1),\, \xi(0,1)) \\ &\ge 2\varepsilon P_k^{\omega,\rho}(\tau_0(1)<\tau_k^+,\, \tau_0(1)<\tau_\rho(1)) \\ &\ge 2\varepsilon P_k^{\omega,\rho}(V(0)\cap V(\rho),\, \tau_0'<\tau_\rho') = 2\varepsilon\, R_2\,, \end{aligned}$$

where we have used the event $\xi(0,1)$ introduced in the proof of Claim 4.3 and the same argument based on (31) therein. □

Having Lemma 4.2 we can prove the following lower bound on the expected value of T_ρ, which refines that of Lemma 4.3 in [12]:

Proposition 4.4. *Fix $\lambda_* \in (0,1)$. Then there exist constants $C_1, C_2 > 0$, independent of $\lambda \in (0,\lambda_*]$ and of $\rho \in \mathbb{N}_+ \cup \{+\infty\}$, such that*

$$\mathbb{E}E_0^{\omega,\rho}[T_\rho] \ge C_1\frac{\rho}{\lambda} - C_2\frac{1}{\lambda^2}\,.$$

Proof. Formula (3.22) in [6] reads in our case as

$$E_0^{\omega,\rho}[T_\rho] = \frac{1}{C_{\rm eff}^\rho(0,[\rho,\infty))}\sum_{k<\rho}\Big(\sum_{j\in\mathbb{Z}} c_{k,j}\Big)P_k^{\omega,\rho}(\tau_0<\tau_{[\rho,\infty)})\,,$$

where $k\mapsto \sum_{j\in\mathbb{Z}} c_{k,j}$ is a reversible measure for the ρ-truncated random walk for each ρ. Hence,

$$\begin{aligned} E_0^{\omega,\rho}[T_\rho] &\ge \frac{1}{C_{\rm eff}^\rho(0,[\rho,\infty))}\sum_{0<k<\rho}\Big(\sum_{j\in\mathbb{Z}} c_{k,j}\Big)P_k^{\omega,\rho}(\tau_0<\tau_{[\rho,\infty)}) \\ &\ge C_3\sum_{0<k<\rho} c_{k,k+1}\frac{C_{\rm eff}^1(k,(-\infty,0])}{C_{\rm eff}^1(0,[\rho,\infty))C_{\rm eff}^1(k,(-\infty,0]\cup[\rho,\infty))}\,, \end{aligned} \tag{54}$$

where C_3 is a strictly positive constant independent of ρ, ω and λ as λ varies in $(0,\lambda_*]$ (as all the constants of the form C_i that will appear in what follows). For the last line in (54) we have used Lemma 4.2 and the bounds

$$C_{\rm eff}^1(A,B) \le C_{\rm eff}^\rho(A,B) \le c\cdot C_{\rm eff}^1(A,B),$$

for some universal constant $c \ge 1$. The above bounds follow from [12, Prop. 3.4]. The fact that c can be taken uniformly in $\lambda \in [0,\lambda_*]$ follows from [12, Rem. 3.2] and Remark 4.1.

Writing for simplicity $c_j := c_{j,j+1}$, we explicitly calculate

$$\begin{aligned} &\frac{C_{\rm eff}^1(k,(-\infty,0])}{C_{\rm eff}^1(0,[\rho,\infty))C_{\rm eff}^1(k,(-\infty,0]\cup[\rho,\infty))} \\ &\qquad= \frac{\big(\sum_{j=0}^{k-1}\frac{1}{c_j}\big)^{-1}}{\big(\sum_{j=0}^{\rho-1}\frac{1}{c_j}\big)^{-1}\Big(\big(\sum_{j=k}^{\rho-1}\frac{1}{c_j}\Big)^{-1}+\big(\sum_{j=0}^{k-1}\frac{1}{c_j}\big)^{-1}\Big)} = \sum_{j=k}^{\rho-1}\frac{1}{c_j}\,. \end{aligned} \tag{55}$$

Therefore, by taking the expectation w.r.t. the environment in (54), we obtain

$$
\begin{aligned}
\mathbb{E}E_0^{\omega,\rho}[T_\rho] &\geq C_3\,\mathbb{E}\Big[\sum_{0<k<\rho} c_k \sum_{j=k}^{\rho-1}\frac{1}{c_j}\Big] \\
&\geq C_3 \mathrm{e}^{-2\|u\|_\infty}\,\mathbb{E}\Big[\sum_{0<k<\rho}\mathrm{e}^{-(1-\lambda)Z_k+2\lambda(Z_0+\ldots+Z_{k-1})}\sum_{j=k}^{\rho-1}\mathrm{e}^{(1-\lambda)Z_j-2\lambda(Z_0+\ldots+Z_{j-1})}\Big] \\
&\geq C_4\Big(\rho+\sum_{0<k<\rho}\sum_{j=k+1}^{\rho-1}\mathbb{E}[\mathrm{e}^{-(1-\lambda)Z_k-2\lambda(Z_k+\ldots+Z_{j-1})}]\Big) \\
&\geq C_5\sum_{0<k<\rho}\sum_{j=k+1}^{\rho-1}\mathrm{e}^{-2\lambda\mathbb{E}[Z_0](j-k)}\,, \qquad (56)
\end{aligned}
$$

where in the third line ρ comes from the case $j = k$ and in the last line we have used Jensen's inequality and the fact that $\mathrm{e}^{-(1-\lambda)\mathbb{E}[Z_0]}$ is bigger than a constant independent from λ. We call now $A := \mathrm{e}^{-2\lambda\mathbb{E}[Z_0]} < 1$ and calculate

$$
\begin{aligned}
\sum_{0<k<\rho}\sum_{j=k+1}^{\rho-1}A^{j-k} &= \sum_{0<k<\rho}\sum_{m=1}^{\rho-k-1}A^m = \sum_{0<k<\rho}\frac{A-A^{\rho-k}}{1-A} = \sum_{0<k<\rho}\frac{A-A^k}{1-A} \\
&= (\rho-1)\frac{A}{1-A}-\frac{A-A^\rho}{(1-A)^2} \geq C_6\Big(\frac{\rho}{1-A}-\frac{1}{(1-A)^2}\Big)\,.
\end{aligned}
$$

We can then continue the chain of inequalities of (56):

$$
\mathbb{E}E_0^{\omega,\rho}[T_\rho] \geq C_7\Big(\frac{\rho}{1-A}-\frac{1}{(1-A)^2}\Big) \geq C_1\,\frac{\rho}{\lambda}-C_2\,\frac{1}{\lambda^2}\,,
$$

which is the statement of the proposition. Here we have used the fact that

$$
0 < \inf_{\lambda\in(0,\lambda_*]}\frac{\lambda}{1-\mathrm{e}^{-2\lambda\mathbb{E}[Z_0]}} < \sup_{\lambda\in(0,\lambda_*]}\frac{\lambda}{1-\mathrm{e}^{-2\lambda\mathbb{E}[Z_0]}} < +\infty\,,
$$

which follows from the fact that the the function $\frac{\lambda}{1-A} = \frac{\lambda}{1-\mathrm{e}^{-2\lambda\mathbb{E}[Z_0]}}$ can be extended to a continuous strictly positive function on the compact interval $[0,\lambda_*]$. □

4.1. **Proof of Lemma 3.1.** With Proposition 4.4 we can finally prove Lemma 3.1. We first stress that below all constants of type C, K can depend on λ_*, but do not depend on the chosen parameter $\lambda \in (0,\lambda_*]$. We recall that, in [12], for a given $\rho \in \mathbb{N}\cup\{+\infty\}$, one calls $\mathbb{Q}^\rho$ the asymptotic invariant distribution for the environment viewed from the ρ–truncated random walk (X_n^ρ), when an external drift of intensity λ (here implicit in the notation) is applied (the case $\rho = \infty$ corresponds again to the random walk (X_n) without cut-off, and $\mathbb{Q}^\infty = \mathbb{Q}_\lambda$). In [12] it is shown that $\mathbb{Q}^\rho$ is absolutely continuous to $\mathbb{P}$. In order to describe the Radon–Nikodym derivative $\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}$ we have to introduce an auxiliary process. We let $\zeta = (\zeta_1,\zeta_2,\ldots)$ be a sequence of i.i.d. Bernoulli random variables of parameter ε, where ε is the same appearing in (31). We call P the law of ζ and E the relative expectation. As detailed in [12, Sec. 4] adapting a construction in [9], one can couple ζ and the random walk (X_n^ρ) so that if $\zeta_j = 1$ for some $j \in \mathbb{N}$, then $X^\rho_{T^\rho_{j\rho}} = j\rho$ (see (30)). In [12, Eq. (46) and Eq. (47)] one has the precise construction of the quenched probability $P_0^{\omega,\rho,\zeta}$ for the random walk once the sequence ζ has been fixed. $E_0^{\omega,\rho,\zeta}$ is the associated expectation. The Radon–Nikodym derivative for the environment viewed from the ρ-truncated walk w.r.t. the original measure of the environment $\mathbb{P}$ is given by (cf. [12, Eq. (63)])

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) = \frac{1}{\mathbb{E}E[E_0^{\omega,\zeta,\rho}[T_{\ell_1\rho}]]} \sum_{k\in\mathbb{Z}} EE_0^{\tau_{-k}\omega,\zeta,\rho}\left[N_{T_{\ell_1\rho}}(k)\right]. \tag{57}$$

Above, given a generic integer $n \geq 0$, $N_n(k)$ denotes the time spent at k by the random walk up to time n, i.e. $N_n(k) = \sum_{r=0}^n \mathbb{1}(X_r^\rho = k)$.

Due to [12, Eq. (50)] we have $E[E_0^{\omega,\zeta,\rho}[T_{\ell_1\rho}]] \geq \varepsilon E_0^{\omega,\rho}[T_\rho]$, thus implying that

$$\mathbb{E}E[E_0^{\omega,\zeta,\rho}[T_{\ell_1\rho}]] \geq \varepsilon \mathbb{E}\big[E_0^{\omega,\rho}[T_\rho]\big]. \tag{58}$$

We set

$$K_1(\rho,\lambda) := \frac{C_1\varepsilon}{\lambda} - \frac{C_2\varepsilon}{\rho\lambda^2}. \tag{59}$$

Then, by combining Proposition 4.4 with (57) and (58), when $K_1(\rho,\lambda) > 0$ we have

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) \leq \frac{1}{K_1(\rho,\lambda)\rho} \sum_{k\in\mathbb{Z}} EE_0^{\tau_{-k}\omega,\zeta,\rho}\left[N_{T_{\ell_1\rho}}(k)\right].$$

The above estimate can be rewritten as

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) \leq \frac{H_+(\omega) + H_-(\omega)}{K_1(\rho,\lambda)\rho}, \tag{60}$$

where (as in [12, Eq. (67)]) we have defined

$$H_+(\omega) := \sum_{k>0} EE_0^{\tau_{-k}\omega,\zeta,\rho}\left[N_{T_{\ell_1\rho}}(k)\right], \qquad H_-(\omega) := \sum_{k\leq 0} EE_0^{\tau_{-k}\omega,\zeta,\rho}\left[N_{T_{\ell_1\rho}}(k)\right].$$

Note that (60) equals [12, Eq. (67)] with the only difference that the constant K_1 in [12] is now replaced by $K_1(\rho,\lambda)$. The computations done in the proof of Prop. 5.4 in [12] show how to go from [12, Eq. (67)] to [12, Eq. (77)] by bounding $H_+(\omega)$ and $H_-(\omega)$, and these bounds do not involve the constant K_1 there. In particular, due to (60), the first line in [12, Eq. (77)] remains valid with K_1 replaced with $K_1(\rho,\lambda)$. In conclusion, since the function $g(\omega,\lambda)$ introduced in (26) equals the function $g_\omega(0)$ defined in [12, Prop. 3.11], we have:

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) \leq G_{\rho,\lambda}(\omega) := \frac{C'}{K_1(\rho,\lambda)}\Big(\frac{\pi^1(0)\sum_{k\leq 0} \mathrm{e}^{-2\lambda x_{-k}} F_*(\tau_{-k}\omega)}{\rho} + g(\omega,\lambda)\Big) \tag{61}$$

where the notation has the following meaning. As in [12] $\pi^1(0) := c_{-1,0} + c_{0,1}$ (recall that λ is understood and that in this section we write $c_{i,j}$ instead of $c_{i,j}^\lambda$). C' is a constant depending only on ε. Finally, F_* is the function defined in [12, Lemma 5.5], i.e.

$$F_*(\omega) := K_0 \sum_{i=0}^\infty (i+1)\mathrm{e}^{-2\lambda x_i + (1-\lambda)(x_{i+1}-x_i)}.$$

Note that the positive constant K_0 is independent of $\lambda \in (0,\lambda_*]$ and ρ (see [12, Rem. 3.2] and Remark 4.1). We have that $\lim_{\rho\to\infty} K_1(\rho,\lambda)\rho = \infty$ and $\lim_{\rho\to\infty} K_1(\rho,\lambda) = C_1\varepsilon/\lambda$. Hence, for any $\rho \geq \rho_0$ (the latter can depend on λ) it holds $K_1(\rho,\lambda) > 0$ and

$$0 \leq G_{\rho,\lambda}(\omega) \leq C_3\Big(\pi^1(0)\sum_{k\leq 0} \mathrm{e}^{-2\lambda x_{-k}} F_*(\tau_{-k}\omega) + g(\omega,\lambda)\Big), \tag{62}$$

$$\lim_{\rho\to\infty} G_{\rho,\lambda}(\omega) = C_4\lambda\, g(\omega,\lambda), \tag{63}$$

for suitable positive constants C_3, C_4 independent of ρ, λ. We claim that the r.h.s. of (62) is in $L^1(\mathbb{P})$. Indeed, $\pi^1(0)$ is bounded by an universal constant. The series appearing in (62) can be bounded from above by using the equivalent expression given by [12, Eq. (78)] together with the property $|x_k| \geq kd$. In this way one easily gets that the series is in $L^1(\mathbb{P})$. Finally, $g(\omega, \lambda) \in L^1(\mathbb{P})$ due to [12, Lemma 3.12]. By the above claim, (62), (63) and the dominated convergence theorem, we conclude that $G_{\rho,\lambda}(\omega)$ converges to $C_4 \lambda g(\omega, \lambda)$ in $L^1(\mathbb{P})$. Take now a bounded positive continuous function h on Ω. Since $\mathbb{Q}^\rho$ weakly converges to $\mathbb{Q}^\infty = \mathbb{Q}_\lambda$ as $\rho \to \infty$ (cf. [12, Prop. 5.3]), by (61) and the above observations we get

$$\mathbb{E}\Big[\frac{d\mathbb{Q}^\infty}{d\mathbb{P}} h\Big] = \mathbb{Q}^\infty[h] = \lim_{\rho\to\infty} \mathbb{Q}^\rho[h] = \lim_{\rho\to\infty} \mathbb{E}\Big[\frac{d\mathbb{Q}^\rho}{d\mathbb{P}} h\Big] \leq \lim_{\rho\to\infty} \mathbb{E}[G_{\rho,\lambda}(\omega) h] = \mathbb{E}[C_4 \lambda g(\omega, \lambda) h]\,.$$

The above bound trivially implies (25).

5. Proof of Theorem 2

Warning 5.1. *In the previous section, in order to make more transparent the comparison with the formulas in* [12]*, we used the convention to omit λ from the index of several objects. From now on we drop this convention and we come back to the notation introduced in Sections 2 and 3.*

Take $f \in L^q(\mathbb{Q}_0)$, p and q be as in Theorem 2. The fact that $f \in L^1(\mathbb{Q}_\lambda)$ is a simple consequence of the Hölder inequality and Theorem 1. Indeed we can bound

$$\mathbb{Q}_\lambda(|f|) = \mathbb{Q}_0\big(|f| \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\big) \leq \|f\|_{L^q(\mathbb{Q}_0)} \|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\|_{L^p(\mathbb{Q}_0)} < \infty\,.$$

The proof of the continuity of the map $\lambda \mapsto \mathbb{Q}_\lambda(f)$ is more subtle and uses two main tools. One tool comes from functional analysis and is given by the following proposition (we postpone the proof to Appendix C):

Lemma 5.2. *Let I be a finite interval of the real line and let $\lambda_0 \in I$. Let Q_λ, $\lambda \in I$, be probability measures on some measurable space $(\Theta, \mathcal{F})$. Let L_λ, $\lambda \in I$, be a family of operators defined on a common subset $\mathcal{C}$ of $L^2(Q_{\lambda_0})$, i.e. $L_\lambda : \mathcal{C} \subset L^2(Q_{\lambda_0}) \to L^2(Q_{\lambda_0})$. We assume the following hypotheses:*

- (H1) *$Q_\lambda \ll Q_{\lambda_0}$ and $\sup_{\lambda\in I} \|\rho_\lambda\|_{L^2(Q_{\lambda_0})} < \infty$, where $\rho_\lambda := \frac{dQ_\lambda}{dQ_{\lambda_0}}$;*
- (H2) *if Q is a probability measure on $(\Omega, \mathcal{F})$ such that $Q \ll Q_{\lambda_0}$, $\frac{dQ}{dQ_{\lambda_0}} \in L^2(Q_{\lambda_0})$ and $Q(L_{\lambda_0} f) = 0$ for all $f \in \mathcal{C}$, then $Q = Q_{\lambda_0}$;*
- (H3) *$Q_\lambda(L_\lambda f) = 0$ for all $\lambda \in I$ and $f \in \mathcal{C}$;*
- (H4) *$\lim_{\lambda\to\lambda_0} \|L_\lambda f - L_{\lambda_0} f\|_{L^2(Q_{\lambda_0})} = 0$ for all $f \in \mathcal{C}$.*

Then ρ_λ converges to ρ_{λ_0} in the weak topology of $L^2(Q_{\lambda_0})$, and

$$\lim_{\lambda\to\lambda_0} Q_\lambda(f) = Q_{\lambda_0}(f)\,, \qquad \forall f \in L^2(Q_{\lambda_0})\,. \tag{64}$$

We point out that, in the above lemma, $f \in L^1(Q_\lambda)$ if $f \in L^2(Q_{\lambda_0})$, hence the expectation $Q_\lambda(f)$ in the l.h.s. of (64) is well–defined. Indeed, since $\frac{dQ_\lambda}{dQ_{\lambda_0}} \in L^2(Q_{\lambda_0})$, it is enough to apply the Cauchy–Schwarz inequality.

In order to apply the above lemma with $\lambda_0 \in [0, 1)$, $I := [\lambda_0 - \delta, \lambda_0 + \delta] \subset (0, 1)$, $\Theta := \Omega$ and $Q_\lambda := \mathbb{Q}_\lambda$ to get the continuity of the map $\lambda \mapsto \mathbb{Q}_\lambda(f)$ at λ_0, we need an upper bound of the norm $\|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}}\|_{L^2(\mathbb{Q}_{\lambda_0})}$ uniformly in λ as λ varies in a neighborhood of λ_0 (the above mentioned second tool). In the special case $\lambda_0 = 0$ this uniform upper bound is provided by Theorem 1. For $\lambda_0 > 0$, this bound is stated in the following lemma:

Lemma 5.3. *Suppose that* $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$. *Fix* $\lambda_0 \in (0,1)$ *and* $\delta > 0$ *such that* $[\lambda_0 - \delta, \lambda_0 + \delta] \subset (0,1)$. *Then we have*

$$\sup_{\lambda:\, |\lambda-\lambda_0|\leq\delta} \Big\| \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}} \Big\|_{L^2(\mathbb{Q}_{\lambda_0})} < \infty\,. \tag{65}$$

Proof. In what follows, we restrict to $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. We recall that all $\mathbb{Q}_\lambda$'s are mutually absolutely continuous w.r.t. $\mathbb{P}$ [12, Thm. 2]. As a consequence, $\mathbb{Q}_\lambda \ll \mathbb{Q}_{\lambda_0}$ and moreover we can write

$$\begin{aligned} \Big\| \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}} \Big\|^2_{L^2(\mathbb{Q}_{\lambda_0})} &= \mathbb{Q}_{\lambda_0}\Big[\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}} \frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}} \Big] = \mathbb{Q}_\lambda\Big[\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_{\lambda_0}} \Big] = \mathbb{Q}_\lambda\Big[\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \Big(\frac{d\mathbb{Q}_{\lambda_0}}{d\mathbb{P}} \Big)^{-1} \Big] \\ &= \mathbb{E}\Big[\Big(\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \Big)^2 \Big(\frac{d\mathbb{Q}_{\lambda_0}}{d\mathbb{P}} \Big)^{-1} \Big]\,. \end{aligned} \tag{66}$$

Due to (25) and assumption (A4) we can bound $\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}} \leq 2K_0 \sum_{j=0}^\infty \mathrm{e}^{-cdj+Z_j}$ for suitable positive constants K_0 and c depending only on λ_0 and δ (note that $c^\lambda_{-1,0}, c^\lambda_{0,1}$ are bounded by a universal constant from above). On the other hand [12, Thm. 2] provides the bound $\frac{d\mathbb{Q}_{\lambda_0}}{d\mathbb{P}} \geq \gamma$, for some strictly positive constant γ depending on λ_0. By combining the above bounds with (66), to get (65) it is enough to prove that $\mathbb{E}\Big[(\sum_{j=0}^\infty \mathrm{e}^{-cdj+Z_j})^2 \Big] < \infty$. By expanding the square, the last estimate can be easily checked since $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$. ☐

The next step is then to apply Lemma 5.2 (with the support of Theorem 1 and Lemma 5.3) to get the continuity of the map $\lambda \mapsto \mathbb{Q}_\lambda(f)$ for $f \in L^2(\mathbb{Q}_0)$. To this aim, given a bounded Borel function f on Ω, we define $\mathbb{L}_\lambda f$ as

$$\mathbb{L}_\lambda f(\omega) = \sum_{k\in\mathbb{Z}} p^\lambda_{0,k}(\omega) \big[f(\tau_k \omega) - f(\omega) \big]\,. \tag{67}$$

Trivially, $\mathbb{L}_\lambda f \in L^2(\mathbb{Q}_\lambda)$. We now consider Lemma 5.2 with $\Theta := \Omega$, $Q_\lambda := \mathbb{Q}_\lambda$, $I := [\lambda_0 - \delta, \lambda_0 + \delta] \subset (0,1)$, $\mathcal{C}$ being the set of Borel bounded functions on Ω and with L_λ defined as the above operator $\mathbb{L}_\lambda$ restricted to $\mathcal{C}$. As an application we get:

Lemma 5.4. *Suppose that* $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$. *Then for any bounded measurable function* $f : \Omega \to \mathbb{R}$ *and for any* $\lambda_0 \in [0,1)$, *it holds*

$$\lim_{\lambda\to\lambda_0} \mathbb{Q}_\lambda(f) = \mathbb{Q}_{\lambda_0}(f)\,. \tag{68}$$

Proof. Since bounded measurable functions are in $L^2(\mathbb{Q}_{\lambda_0})$, due to (64), to get (68) we only need to check the hypotheses of Lemma 5.2 with Θ, Q_λ, I, $\mathcal{C}$ and L_λ defined as above.

Hypothesis (H1) is satisfied due to Theorem 1 and Lemma 5.3. Let us check (H2). Suppose that $\mathbb{Q}$ is a probability on the environment space Ω satisfying the properties listed in (H2). Since $\mathcal{C}$ is dense in $L^2(\mathbb{Q}_{\lambda_0})$ and $\mathbb{Q}(\mathbb{L}_{\lambda_0} f) = 0$ for any $f \in \mathcal{C}$, $\mathbb{Q}$ is an invariant distribution for the process $(\omega^{\lambda_0}_n)$, defined as $\omega^{\lambda_0}_n := \tau_k \omega$ where $k = \psi(Y^{\lambda_0}_n)$ (the environment viewed from the walker). We now want to use that $\mathbb{Q} \ll \mathbb{Q}_{\lambda_0}$ to deduce that $\mathbb{Q} = \mathbb{Q}_{\lambda_0}$. To this aim we denote by $\mathbb{P}^{\lambda_0}_\nu$ the law of the process $(\omega^{\lambda_0}_n)$ starting with distribution ν and by $\mathbb{E}^{\lambda_0}_\nu$ the associated expectation. If $\nu = \delta_\omega$ we simply write $\mathbb{P}^{\lambda_0}_\omega$ and

$\mathbb{E}_\omega^{\lambda_0}$. We take $f : \Omega \to \mathbb{R}$ to be any bounded measurable function. By the invariance of $\mathbb{Q}$ we have

$$\mathbb{Q}[f] = \mathbb{E}_{\mathbb{Q}}^{\lambda_0}\Big[\frac{1}{n}\sum_{j=0}^{n-1} f(\omega_j^{\lambda_0})\Big] = \mathbb{Q}[F_n]\,, \tag{69}$$

where $F_n(\omega) := \mathbb{E}_\omega^{\lambda_0}\Big[\frac{1}{n}\sum_{j=0}^{n-1} f(\omega_j^{\lambda_0})\Big]$. Now, since $\mathbb{Q}_{\lambda_0}$ is ergodic, we know that for

$$A := \Big\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(\omega_j^{\lambda_0}) = \mathbb{Q}_{\lambda_0}[f] \quad \mathbb{P}_\omega^{\lambda_0} - a.s.\Big\}$$

we have $\mathbb{Q}_{\lambda_0}[A] = 1$. Since the map $(\omega_j^{\lambda_0})_{j\geq 0} \to \frac{1}{n}\sum_{j=0}^{n-1} f(\omega_j^{\lambda_0})$ is bounded by $\|f\|_\infty$, we can apply the dominated convergence theorem to obtain that, for each $\omega \in A$,

$$\lim_{n\to\infty} F_n(\omega) = \mathbb{Q}_{\lambda_0}[f]\,.$$

To conclude, we would like to apply again the dominated convergence theorem to analyze $\lim_{n\to\infty} \mathbb{Q}[F_n]$. We can do that since $|F_n(\omega)| \leq \|f\|_\infty$ and since $F_n(\omega) \to \mathbb{Q}_{\lambda_0}[f]$ for $\mathbb{Q}$–a.a. ω (because $\mathbb{Q} \ll \mathbb{Q}_{\lambda_0}$ and $\mathbb{Q}_{\lambda_0}(A) = 1$, thus implying that $\mathbb{Q}(A) = 1$). We then obtain that $\lim_{n\to\infty} \mathbb{Q}[F_n] = \mathbb{Q}_{\lambda_0}[f]$. By (69) we get $\mathbb{Q}[f] = \mathbb{Q}_{\lambda_0}[f]$. Since this is true for every f, we have $\mathbb{Q} = \mathbb{Q}_{\lambda_0}$.

(H3) follows from the fact that $\mathbb{Q}_\lambda$ is an invariant distribution for the process "environment viewed from the random walk Y_n^λ".

It remains to check (H4). Since $f \in \mathcal{C}$ is bounded, it is enough to have

$$\lim_{\lambda\to\lambda_0} \mathbb{Q}_{\lambda_0}\Big[\Big(\sum_{k\in\mathbb{Z}} |p_{0,k}^\lambda - p_{0,k}^{\lambda_0}|\Big)^2\Big] = 0\,. \tag{70}$$

To conclude we observe that, by writing $\mathbb{Q}_{\lambda_0}[\cdot] = \mathbb{Q}_0\big[\frac{d\mathbb{Q}_{\lambda_0}}{d\mathbb{Q}_0}\cdot\big]$, (70) follows from the Cauchy–Schwarz inequality, the fact that $\frac{d\mathbb{Q}_{\lambda_0}}{d\mathbb{Q}_0} \in L^2(\mathbb{Q}_0)$ and Lemma B.2 in Appendix B. □

As a byproduct of Theorem 1, Lemma 5.3 and Lemma 5.4 we can complete the proof of Theorem 2. To this aim we suppose the assumptions of Theorem 2 to be satisfied and we take $f \in L^q(\mathbb{Q}_0)$ and $\lambda_0 \in [0,1)$. We take $\lambda_* \in (\lambda_0, 1)$ and from now on we restrict to $\lambda \in [0, \lambda_*]$. Recall that at the beginning of this section we have proved that $f \in L^1(\mathbb{Q}_\lambda)$.

We want to show that $\mathbb{Q}_\lambda(f) \to \mathbb{Q}_{\lambda_0}(f)$ as $\lambda \to \lambda_0$. To this aim, given $M > 0$, we define $f_M(\omega)$ as M if $f(\omega) > M$, as $-M$ if $f(\omega) < -M$ and as $f(\omega)$ otherwise. We then can bound

$$\begin{aligned}|\mathbb{Q}_\lambda(f) - \mathbb{Q}_{\lambda_0}(f)| \leq |\mathbb{Q}_\lambda(f) - \mathbb{Q}_\lambda(f_M)| + |\mathbb{Q}_\lambda(f_M) - \mathbb{Q}_{\lambda_0}(f_M)|\\ + |\mathbb{Q}_{\lambda_0}(f_M) - \mathbb{Q}_{\lambda_0}(f)|\,.\end{aligned} \tag{71}$$

To conclude it is enough to show that the r.h.s. of (71) goes to zero when we take first the limit $\lambda \to \lambda_0$ and afterwards the limit $M \to \infty$. Due to Lemma 5.4 the second term in the r.h.s. of (71) goes to zero already as $\lambda \to \lambda_0$ since f_M is bounded. On the other hand, by the Hölder inequality, the first and third terms in the r.h.s. of (71) can be bounded by

$$\|f - f_M\|_{L^q(\mathbb{Q}_0)} \sup_{\zeta\in[0,\lambda_*]} \Big\|\frac{d\mathbb{Q}_\zeta}{d\mathbb{Q}_0}\Big\|_{L^p(\mathbb{Q}_0)}\,.$$

Note the independence from λ of the above expression. Since $f \in L^q(\mathbb{Q}_0)$, $\|f - f_M\|_{L^q(\mathbb{Q}_0)}$ goes to zero as $M \to \infty$ by the dominated convergence theorem, thus completing the proof.

6. Proof of Theorem 3 (first part)

In this section we prove the existence of $\partial_{\lambda=0}\mathbb{Q}_\lambda(f)$ and equation (21). As in the theorem, we suppose that $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ for some $p > 2$ and that $f \in H_{-1} \cap L^2(\mathbb{Q}_0)$. In what follows, q is the exponent conjugate to p, i.e. the value satisfying $p^{-1} + q^{-1} = 1$.

To simplify the notation we write here g_ε, h instead of the functions g_ε^f, h^f introduced in (18), (19), respectively. Recall that, given $\varepsilon > 0$, $g_\varepsilon \in L^2(\mathbb{Q}_0)$ is the solution of the equation $\varepsilon g_\varepsilon - \mathbb{L}_0 g_\varepsilon = f$. Since $L^2(\mathbb{Q}_0) \subset L^1(\mathbb{Q}_\lambda)$ (by Theorem 1 and the Cauchy–Schwarz inequality), the above identity on g_ε implies that $\mathbb{Q}_\lambda(f) = \varepsilon\mathbb{Q}_\lambda(g_\varepsilon) - \mathbb{Q}_\lambda(\mathbb{L}_0 g_\varepsilon)$. Using that $\mathbb{Q}_0(f) = 0$ since $f \in H_{-1}$, we can write

$$\frac{\mathbb{Q}_\lambda(f) - \mathbb{Q}_0(f)}{\lambda} = \frac{\varepsilon\mathbb{Q}_\lambda(g_\varepsilon)}{\lambda} - \frac{\mathbb{Q}_\lambda(\mathbb{L}_0 g_\varepsilon)}{\lambda}\,. \tag{72}$$

In what follows we will take first the limit $\varepsilon \to 0$ and afterwards the limit $\lambda \to 0$.

Since $f \in H_{-1}$ we can apply the results and estimates of [21]. In particular, it holds $\varepsilon\|g_\varepsilon\|^2_{L^2(\mathbb{Q}_0)} \to 0$ as $\varepsilon \to 0$ (see [21, Eq. (1.12)]) and, due to Theorem 1, we can bound

$$\big|\varepsilon\mathbb{Q}_\lambda(g_\varepsilon)\big| = \Big|\varepsilon\Big\langle\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}, g_\varepsilon\Big\rangle\Big| \le \varepsilon\|g_\varepsilon\|_{L^2(\mathbb{Q}_0)}\Big\|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\Big\|_{L^2(\mathbb{Q}_0)} \to 0 \text{ as } \varepsilon \to 0\,. \tag{73}$$

We recall that the scalar product in $L^2(\mathbb{Q}_0)$ is denoted by $\langle\cdot,\cdot\rangle$. As a consequence of (73), the first term in the r.h.s. of (72) is negligible as $\varepsilon \to 0$.

It remains to analyze the second term in the r.h.s. of (72). Recall the space $L^2(M)$ of square integrable forms introduced in Section 2 and recall (17).

Lemma 6.1. *Let $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ for some $p > 2$. Let $\hat q > 2$ be such that $\frac{1}{p} + \frac{1}{\hat q} = \frac{1}{2}$. Given a form v with $v(\cdot,0) \equiv 0$ and a square integrable form $w \in L^2(M)$, there exists $C > 0$ such that for all $\lambda \in (0,1/2)$ it holds*

$$\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}} |v(\cdot,k)w(\cdot,k)|\Big] \le C\,\|w\|_{L^2(M)}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}} p_{0,k}\Big|\frac{v(\cdot,k)}{p_{0,k}}\Big|^{\hat q}\Big]^{\frac{1}{\hat q}}\,. \tag{74}$$

Proof. We simply compute

$$\begin{aligned}
\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}} |v(\cdot,k)w(\cdot,k)|\Big] &= \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}} p_{0,k}\Big|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\frac{v(\cdot,k)}{p_{0,k}}\Big|\,\big|w(\cdot,k)\big|\Big]\\
&\le \|w\|_{L^2(M)}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}} p_{0,k}\Big(\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\Big)^2\Big(\frac{v(\cdot,k)}{p_{0,k}}\Big)^2\Big]^{1/2}\\
&\le \|w\|_{L^2(M)}\mathbb{Q}_0\Big[\Big(\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\Big)^p\Big]^{\frac{1}{p}}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}} p_{0,k}\Big|\frac{v(\cdot,k)}{p_{0,k}}\Big|^{\hat q}\Big]^{\frac{1}{\hat q}},
\end{aligned}$$

where for the second line we have used the Cauchy-Schwarz inequality with respect to the measure M, while for the second inequality we used the Hölder inequality again with respect to M and with exponents $p/2$ and $\hat q/2$, so that $(p/2)^{-1}+(\hat q/2)^{-1} = 1$ by hypothesis. We also have used the fact that $M[(\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0})^p] = \mathbb{Q}_0[(\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0})^p]$. To conclude it is enough to apply Theorem 1. □

Lemma 6.2. *Let $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ for some $p > 2$ and let $\hat q$ be as in Lemma 6.1. Then there exists a constant C not depending on $\lambda \in [0, \frac{1}{2\hat q})$ such that, for any form $w \in L^2(M)$, it* *holds*

$$\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\big|(p^\lambda_{0,k}-p_{0,k})w(\cdot,k)\big|\Big]\leq C\|w\|_{L^2(M)}\,.$$

Proof. In this proof the constants C, C' are positive, might vary from line to line and do not depend on the specific choice of $\lambda\in[0,\frac{1}{2\hat q})$. By applying Lemma 6.1 with $v(\cdot,k)=p^\lambda_{0,k}-p_{0,k}$ we already know that

$$\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\big|(p^\lambda_{0,k}-p_{0,k})w(\cdot,k)\big|\Big]\leq C\|w\|_{L^2(M)}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p_{0,k}\Big|\frac{p^\lambda_{0,k}}{p_{0,k}}-1\Big|^{\hat q}\Big]^{\frac{1}{\hat q}}\,. \tag{75}$$

Since for $a\geq 0$ it holds $|a-1|^q\leq |a|^q+1$, we can bound

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p_{0,k}\Big|\frac{p^\lambda_{0,k}}{p_{0,k}}-1\Big|^{\hat q}\Big]\leq 1+\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p_{0,k}\Big|\frac{p^\lambda_{0,k}}{p_{0,k}}\Big|^{\hat q}\Big]\,. \tag{76}$$

Since $\frac{p^\lambda_{0,k}}{p_{0,k}}\leq C\mathrm{e}^{\lambda(x_k+Z_{-1})}$ (see (110) in the Appendix for a proof of this fact), we can bound

$$\begin{aligned}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p_{0,k}\Big|\frac{p^\lambda_{0,k}}{p_{0,k}}\Big|^{\hat q}\Big]&\leq C\,\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}\mathrm{e}^{\lambda\hat q(|x_k|+Z_{-1})}\Big]\leq C\mathrm{e}^{\|u\|_\infty}\,\mathbb{E}\Big[\frac{d\mathbb{Q}_0}{d\mathbb{P}}\sum_{k\in\mathbb{Z}}\frac{\mathrm{e}^{-|x_k|}}{\pi}\mathrm{e}^{\lambda\hat q(|x_k|+Z_{-1})}\Big]\\&\leq C'\Big(\sum_{k\in\mathbb{Z}}\mathrm{e}^{-(1-\lambda\hat q)|k|d}\Big)\,\mathbb{E}\Big[\mathrm{e}^{\lambda\hat q Z_{-1}}\Big]\,,\end{aligned} \tag{77}$$

where for the last inequality we have used that $\frac{d\mathbb{Q}_0}{d\mathbb{P}}=\frac{\pi}{\mathbb{E}[\pi]}$ and that $|x_k|\geq|k|d$. Note that the last line in (77) is uniformly bounded for $\lambda\in(0,\frac{1}{2\hat q})$ (recall that $\mathbb{E}[\mathrm{e}^{pZ_0}]<\infty$). This bound together with (75) and (76) allows to conclude. □

Lemma 6.3. *Given $g\in L^2(\mathbb{Q}_0)$, the series $\sum_{k\in\mathbb{Z}}p^\lambda_{0,k}|g(\tau_k\cdot)-g(\cdot)|$ belongs to $L^1(\mathbb{Q}_\lambda)$. Defining, as in (67), $\mathbb{L}_\lambda g(\omega):=\sum_{k\in\mathbb{Z}}p^\lambda_{0,k}(g(\tau_k\cdot)-g(\cdot))$, we get that $\mathbb{L}_\lambda g\in L^1(\mathbb{Q}_\lambda)$ and $\mathbb{Q}_\lambda(\mathbb{L}_\lambda g)=0$.*

Proof. Recall that $\mathbb{Q}_\lambda$ is an invariant distribution for the environment viewed from the perturbed walker, i.e. for $(\tau_{Y^\lambda_n}\omega)_{n\geq 0}$. This implies that $\mathbb{Q}_\lambda\big[\sum_{k\in\mathbb{Z}}p^\lambda_{0,k}|g(\tau_k\cdot)|\big]=\mathbb{Q}_\lambda[|g|]<\infty$ (in the last bound we have used Theorem 1 to get $g\in L^1(\mathbb{Q}_\lambda)$). As a consequence, $\sum_{k\in\mathbb{Z}}p^\lambda_{0,k}|g(\tau_k\cdot)-g(\cdot)|$ belongs to $L^1(\mathbb{Q}_\lambda)$ and therefore $\mathbb{L}_\lambda g$ is a well–defined element of $L^1(\mathbb{Q}_\lambda)$. Finally, again by the invariance of $\mathbb{Q}_\lambda$, we have $\mathbb{Q}_\lambda[g]=\mathbb{Q}_\lambda\big[\sum_{k\in\mathbb{Z}}p^\lambda_{0,k}g(\tau_k\cdot)\big]$, which is equivalent to $\mathbb{Q}_\lambda(\mathbb{L}_\lambda g)=0$. □

By the above lemma $\mathbb{Q}_\lambda(\mathbb{L}_\lambda g_\varepsilon)$ is well–defined and equals zero. Hence we can write

$$-\mathbb{Q}_\lambda(\mathbb{L}_0 g_\varepsilon)=\mathbb{Q}_\lambda([\mathbb{L}_\lambda-\mathbb{L}_0]g_\varepsilon)=\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}(p^\lambda_{0,k}-p_{0,k})(g_\varepsilon(\tau_k\cdot)-g_\varepsilon)\Big]\,. \tag{78}$$

By [21, Eq. (1.11a)] we have that the sequence g_ε is Cauchy, as $\varepsilon\downarrow 0$, in the space H_1 referred to the operator $-\mathbb{L}_0$. In particular, we have

$$\lim_{\varepsilon_1,\varepsilon_2\downarrow 0}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}\Big((g_{\varepsilon_1}-g_{\varepsilon_2})(\tau_k\cdot)-(g_{\varepsilon_1}-g_{\varepsilon_2})\Big)^2\Big]=0\,. \tag{79}$$

(79) can be restated as follows: The family of quadratic forms $(\nabla g_\varepsilon)_{\varepsilon>0}$ is Cauchy in $L^2(M)$. As a consequence, we get that $\nabla g_\varepsilon\to h$ in $L^2(M)$ for some form $h\in L^2(M)$.

Finally, we point out that, due to Lemma 6.2, the expectation $\mathbb{Q}_\lambda[\sum_{k\in\mathbb{Z}}(p^\lambda_k - p^0_k)h(\cdot,k)]$ is well–defined.

Lemma 6.4. *It holds*

$$\lim_{\varepsilon\downarrow 0}\Big|\mathbb{Q}_\lambda(\mathbb{L}_0 g_\varepsilon) + \mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}(p^\lambda_{0,k} - p_{0_k})h(\cdot,k)\Big]\Big| = 0\,. \tag{80}$$

Proof. We set $w_\varepsilon = \nabla g_\varepsilon - h$. Due to (78) we only need to show that

$$\lim_{\varepsilon\downarrow 0}\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\big|(p^\lambda_k - p^0_k)w_\varepsilon(\cdot,k)\big|\Big] = 0\,. \tag{81}$$

By applying Lemma 6.2 and using that $\lim_{\varepsilon\to 0}\nabla g_\varepsilon = h$ in $L^2(M)$, we get the claim. □

Lemma 6.5. *It holds*

$$\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\Big(p^\lambda_{0,k} - p_{0,k}\Big)h(\cdot,k)\Big] \;= \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}h(\cdot,k)\Big]\,. \tag{82}$$

Proof. We can write

$$\begin{aligned}\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\frac{p^\lambda_{0,k} - p_{0,k}}{\lambda}h(\cdot,k)\Big] &= \mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}h(\cdot,k)\Big]\\ &\quad + \mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\big(\frac{p^\lambda_{0,k} - p_{0,k}}{\lambda} - \partial_{\lambda=0}p^\lambda_{0,k}\big)h(\cdot,k)\Big]\,.\end{aligned} \tag{83}$$

In the first part of the proof (Step 1) we show that the first term in the r.h.s. converges to the r.h.s. of (82), while in the second part (Step 2) we show that the second term in the r.h.s. goes to zero as $\lambda\to 0$.
Step 1. Due to Theorem 2 it is enough to show that $\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}h(\cdot,k)$ belongs to $L^q(\mathbb{Q}_0)$. Since $\partial_\lambda p^\lambda_{0,k} = p^\lambda_{0,k}(x_k - \varphi_\lambda)$, we can rewrite $\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}h(\cdot,k)$ as $\sum_{k\in\mathbb{Z}}p_{0,k}(x_k - \varphi)h(\cdot,k)$. Applying the Cauchy-Schwarz inequality we get

$$\Big\|\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)h(\cdot,k)\Big\|^q_{L^q(\mathbb{Q}_0)} \le \mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)^2\Big)^{q/2}\Big(\sum_{k\in\mathbb{Z}}p_{0,k}h(\cdot,k)^2\Big)^{q/2}\Big].$$

We choose now exponents $A := 2/q > 1$ and $B := 2/(2-q)$ such that $A^{-1}+B^{-1} = 1$ and apply the Hölder inequality to the previous display obtaining

$$\Big\|\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)h(\cdot,k)\Big\|^q_{L^q(\mathbb{Q}_0)} \le \mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)^2\Big)^{\frac{qB}{2}}\Big]^{1/B}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}h(\cdot,k)^2\Big]^{1/A}.$$

The second factor in the r.h.s. is bounded since $h\in L^2(M)$. For finishing Step 1 we are thus left to show that

$$\mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)^2\Big)^{\frac{qB}{2}}\Big] < \infty. \tag{84}$$

By the Cauchy–Schwarz inequality one has $\varphi^2 = (\sum_k p_{0k}x_k)^2 \le \sum_k p_{0k}x_k^2$ so that

$$\sum_{k\in\mathbb{Z}^d}p_{0,k}(x_k-\varphi)^2 \le 2\sum_{k\in\mathbb{Z}^d}p_{0,k}x_k^2 + 2\sum_{k\in\mathbb{Z}^d}p_{0,k}\varphi^2 \le 4\sum_{k\in\mathbb{Z}^d}p_{0,k}x_k^2\,. \tag{85}$$

Since $qB/2 = q/(2-q) > 1$, by the Hölder inequality we have

$$\Big(\sum_{k\in\mathbb{Z}^d} p_{0,k} x_k^2\Big)^{\frac{qB}{2}} \leq \sum_{k\in\mathbb{Z}^d} p_{0,k} x_k^{qB} \,. \tag{86}$$

At this point (84) follows from (85), (86) and (114) in Appendix B.

Step 2. By Taylor expansion with the Lagrange rest we can write

$$\frac{p^\lambda_{0,k}-p_{0,k}}{\lambda} - \partial_{\lambda=0} p^\lambda_{0,k} = \frac{\lambda}{2}\, \partial^2_{\lambda=\xi_k} p^\lambda_{0,k} \,, \tag{87}$$

where $\partial^2_{\lambda=\xi_k} p^\lambda_{0,k}$ denotes the second derivative of the function $\lambda \mapsto p^\lambda_{0,k}$ evaluated at some $\xi_k \in [0,\lambda]$. To prove that the second term in the r.h.s. of (83) is negligible as $\lambda \to 0$, it is therefore enough to show that, for some $\delta > 0$,

$$\sup_{\lambda\in[0,\delta]} \mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}} \big|v(\cdot,k)h(\cdot,k)\big|\Big] < \infty, \qquad v(\cdot,k) := \sup_{\xi_k\in[0,\delta]} |\partial^2_{\lambda=\xi_k} p^\lambda_{0,k}| \,. \tag{88}$$

By Lemma 6.1, since $h \in L^2(M)$, it is enough to show

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}} p_{0,k}\Big|\frac{v(\cdot,k)}{p_{0,k}}\Big|^{\hat q}\Big]^{\frac{1}{\hat q}} < \infty$$

where $\hat q > 2$ is such that $\frac{1}{p} + \frac{1}{\hat q} = \frac{1}{2}$. This follows from (117) in Lemma B.1 in Appendix B. □

By collecting Lemma 6.4, Lemma 6.5 and using that $\partial_{\lambda=0} p^\lambda_{0,k} = p_{0,k}(x_k - \varphi)$ we obtain

$$\lim_{\lambda\downarrow 0}\lim_{\varepsilon\downarrow 0} -\frac{\mathbb{Q}_\lambda(\mathbb{L}_0 g_\varepsilon)}{\lambda} = \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} \partial_{\lambda=0} p^\lambda_{0,k} h(\cdot,k)\Big] = \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}(x_k-\varphi)h(\cdot,k)\Big] \,. \tag{89}$$

This together with (72) and (73) gives that $\mathbb{Q}_\lambda[f]$ is derivable at $\lambda = 0$ and we obtain (21).

7. Proof of Theorem 3 (second part)

In this section we deal with the second identity in Theorem 3, that is, equation (22), and show how it can be derived from (21). Recall the process (ω_n) of the environment viewed from the unperturbed walker (Y_n) defined through $\omega_n = \tau_{Y_n}\omega$, where ω denotes the initial environment. Below we denote by $\|\cdot\|_{-1}$ the H_{-1} norm referred to the operator $-\mathbb{L}_0$ in $L^2(\mathbb{Q}_0)$ and by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{Q}_0)$.

Lemma 7.1. *For any $V \in H_{-1}\cap L^2(\mathbb{Q}_0)$, the sequence $\frac{1}{\sqrt n}\sum_{j=0}^{n-1} V(\omega_j)$ converges weakly as $n\to\infty$ to a Gaussian random variable with variance $\sigma^2 = 2\|V\|^2_{-1} - \|V\|^2_{L^2(\mathbb{Q}_0)}$.*

Proof. By [21, Cor. 1.5] we have that $\frac{1}{\sqrt n}\sum_{j=0}^{n-1} V(\omega_j)$ converges to a Gaussian random variable with variance given by (see [21, Eq. (1.1)])

$$\sigma^2 = \int_{[0,1]} \frac{1+\theta}{1-\theta}\, \mathfrak{m}_V(d\theta) < \infty \,,$$

where $\mathfrak{m}_V$ denotes the spectral measure of V associated to the symmetric operator S_0 on $L^2(\mathbb{Q}_0)$ defined as $S_0 f(\omega) := \sum_{k\in\mathbb{Z}} p_{0,k} f(\tau_k\omega)$. Since $-\mathbb{L}_0 = \mathbb{I} - S_0$, by spectral calculus we obtain

$$\sigma^2 = 2\int_{[0,1]} \frac{1}{1-\theta}\,\mathfrak{m}_V(d\theta) - \int_{[0,1]} \mathfrak{m}_V(d\theta) = 2\|V\|_{-1}^2 - \|V\|^2_{L^2(\mathbb{Q}_0)}\,. \qquad \square$$

Let $f \in H_{-1}\cap L^2(\mathbb{Q}_0)$ be as in Theorem 3. A direct consequence of the above lemma is that, for the gaussian variables N^f and N^φ considered in (20), it holds $Var(N^f) = 2\|f\|^2_{-1} - \|f\|^2_{L^2(\mathbb{Q}_0)}$, $Var(N^\varphi) = 2\|\varphi\|^2_{-1} - \|\varphi\|^2_{L^2(\mathbb{Q}_0)}$ and $Var(N^f+N^\varphi) = 2\|f+\varphi\|^2_{-1} - \|f+\varphi\|^2_{L^2(\mathbb{Q}_0)}$. By this we obtain a first formula for their covariance:

$$\begin{aligned} Cov(N^f, N^\varphi) &= \frac{1}{2}\big(Var(N^f+N^\varphi) - Var(N^f) - Var(N^\varphi)\big)\\ &= \|f+\varphi\|^2_{-1} - \|f\|^2_{-1} - \|\varphi\|^2_{-1} - \langle f,\varphi\rangle. \end{aligned} \tag{90}$$

We are now ready to show (22). In what follows, we write g_ε, h for the functions g^f_ε, h^f introduced in (18), (19), respectively. Recall by (21) that one has

$$\partial_{\lambda=0}\mathbb{Q}_\lambda(f) = \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k} x_k h(\cdot,k)\Big] - \mathbb{Q}_0\Big[\varphi\sum_{k\in\mathbb{Z}} p_{0,k} h(\cdot,k)\Big]. \tag{91}$$

We divide the proof into the two following claims, that together with (90) and (91) clearly imply (22).

Claim 7.2. *We have*

$$\mathbb{Q}_0\Big[\varphi\sum_{k\in\mathbb{Z}} p_{0,k} h(\cdot,k)\Big] = -\langle f,\varphi\rangle\,.$$

Claim 7.3. *We have*

$$-\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k} x_k h(\cdot,k)\Big] = \|f+\varphi\|^2_{-1} - \|f\|^2_{-1} - \|\varphi\|^2_{-1}\,.$$

Proof of Claim 7.2. We can write

$$\mathbb{Q}_0\Big[\varphi\sum_{k\in\mathbb{Z}} p_{0,k} h(\cdot,k)\Big] = \mathbb{Q}_0\Big[\varphi\sum_{k\in\mathbb{Z}} p_{0,k}\nabla g_\varepsilon(\cdot,k)\Big] + \mathbb{Q}_0\Big[\varphi\sum_{k\in\mathbb{Z}} p_{0,k}\big(h(\cdot,k) - \nabla g_\varepsilon(\cdot,k)\big)\Big]\,.$$

We denote by A_ε and B_ε the two terms in the r.h.s. of the above expression. We now show that, as $\varepsilon\downarrow 0$, $A_\varepsilon \to -\langle f,\varphi\rangle$ and $B_\varepsilon\to 0$, which gives the claim.

Since $(\varepsilon - \mathbb{L}_0)g_\varepsilon = f$, we have

$$A_\varepsilon = \mathbb{Q}_0[\varphi(\mathbb{L}_0 g_\varepsilon)] = \varepsilon\mathbb{Q}_0[\varphi g_\varepsilon] - \mathbb{Q}_0[\varphi f].$$

For the first summand we can bound

$$\big|\varepsilon\mathbb{Q}_0[\varphi g_\varepsilon]\big| \le \varepsilon\|\varphi\|_{L^2(\mathbb{Q}_0)}\|g_\varepsilon\|_{L^2(\mathbb{Q}_0)} \xrightarrow{\varepsilon\downarrow 0} 0$$

since, by [21, Eq. (1.12)], we know that $\varepsilon\|g_\varepsilon\|_{L^2(\mathbb{Q}_0)}\to 0$ as $\varepsilon\downarrow 0$. This implies $\lim_{\varepsilon\downarrow 0} A_\varepsilon = -\mathbb{Q}_0[\varphi f] = -\langle f,\varphi\rangle$.

Turning to B_ε, by (19) and the Cauchy–Schwarz inequality with respect to the measure M, we have

$$\begin{aligned} |B_\varepsilon| &\le \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}\varphi^2\Big]^{\frac{1}{2}}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}(h(\cdot,k)-\nabla g_\varepsilon(\cdot,k))^2\Big]^{\frac{1}{2}}\\ &= \|\varphi\|_{L^2(\mathbb{Q}_0)}\|h-\nabla g_\varepsilon\|_{L^2(M)} \xrightarrow{\varepsilon\downarrow 0} 0\,. \end{aligned} \qquad \square$$

Proof of Claim 7.3. First of all we notice that

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k h(\cdot,k)\Big] = \lim_{\varepsilon\downarrow 0}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k \nabla g_\varepsilon(\cdot,k)\Big]. \tag{92}$$

Indeed, by the Cauchy–Schwarz inequality and (19), it holds

$$\begin{aligned}\Big|\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k(h(\cdot,k)-\nabla g_\varepsilon(\cdot,k))\Big]\Big| &\le \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k^2\Big]^{\frac12}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}(h(\cdot,k)-\nabla g_\varepsilon(\cdot,k))^2\Big]^{\frac12}\\ &= \mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k^2\Big]^{\frac12}\|h-\nabla g_\varepsilon\|_{L^2(M)} \xrightarrow{\varepsilon\to 0} 0\,.\end{aligned}$$

The expectation in the r.h.s. of (92) can be rewritten as

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k(g_\varepsilon(\tau_k\cdot)-g_\varepsilon)\Big] = -2\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k g_\varepsilon\Big] = -2\mathbb{Q}_0[\varphi g_\varepsilon]. \tag{93}$$

To see why the first equality holds we just note that for each $k\in\mathbb{Z}$

$$\begin{aligned}\mathbb{Q}_0\Big[p_{0,k}x_k g_\varepsilon(\tau_k\cdot)\Big] &= \frac{1}{\mathbb{E}[\pi]}\mathbb{E}[r_{0,k}x_k g_\varepsilon(\tau_k\cdot)\Big] = \frac{1}{\mathbb{E}[\pi]}\mathbb{E}[r_{0,k}(\tau_{-k}\cdot)x_k(\tau_{-k}\cdot)g_\varepsilon(\cdot)\Big]\\ &= \frac{1}{\mathbb{E}[\pi]}\mathbb{E}[r_{0,-k}(\cdot)(-x_{-k}(\cdot))g_\varepsilon(\cdot)\Big] = -\mathbb{Q}_0[p_{0,-k}x_{-k}g_\varepsilon]\end{aligned}$$

where for the first equality we have used that $d\mathbb{Q}_0/d\mathbb{P} = \pi/\mathbb{E}[\pi]$ and for the second equality the translation invariance of $\mathbb{P}$. The first equality in (93) then follows by summing over all $k\in\mathbb{Z}$.

By putting (93) back into (92), we see that the proof of the claim is concluded if we can prove that

$$\lim_{\varepsilon\downarrow 0} 2\mathbb{Q}_0[\varphi g_\varepsilon] = \|f+\varphi\|_{-1}^2 - \|f\|_{-1}^2 - \|\varphi\|_{-1}^2. \tag{94}$$

Note that, by spectral calculus, the symmetric operators $(\varepsilon-\mathbb{L}_0)^{-1}$ and $(\varepsilon-\mathbb{L}_0)^{-1/2}$ are defined on the whole $L^2(\mathbb{Q}_0)$. Since moreover $(\varepsilon-\mathbb{L}_0)g_\varepsilon = f$, we have that

$$\begin{aligned}2\mathbb{Q}_0[\varphi g_\varepsilon] &= 2\mathbb{Q}_0[\varphi(\varepsilon-\mathbb{L}_0)^{-1}f]\\ &= 2\langle(\varepsilon-\mathbb{L}_0)^{-1/2}\varphi,(\varepsilon-\mathbb{L}_0)^{-1/2}f\rangle\\ &= \langle(\varepsilon-\mathbb{L}_0)^{-1/2}(\varphi+f),(\varepsilon-\mathbb{L}_0)^{-1/2}(\varphi+f)\rangle\\ &\qquad - \langle(\varepsilon-\mathbb{L}_0)^{-1/2}f,(\varepsilon-\mathbb{L}_0)^{-1/2}f\rangle - \langle(\varepsilon-\mathbb{L}_0)^{-1/2}\varphi,(\varepsilon-\mathbb{L}_0)^{-1/2}\varphi\rangle\\ &\xrightarrow{\varepsilon\downarrow 0} \|f+\varphi\|_{-1}^2 - \|f\|_{-1}^2 - \|\varphi\|_{-1}^2\,.\end{aligned}$$

The last limit follows from the observation that, for each $V\in H_{-1}\cap L^2(\mathbb{Q}_0)$, we have

$$\langle(\varepsilon-\mathbb{L}_0)^{-1/2}V,(\varepsilon-\mathbb{L}_0)^{-1/2}V\rangle \xrightarrow{\varepsilon\downarrow 0} \|V\|_{-1}^2.$$

Indeed, writing e_V for the spectral measure associated to V and $-\mathbb{L}_0$, it holds

$$\langle(\varepsilon-\mathbb{L}_0)^{-1/2}V,(\varepsilon-\mathbb{L}_0)^{-1/2}V\rangle = \int_{[0,\infty)}\frac{1}{\varepsilon+\theta}e_V(d\theta) \xrightarrow{\varepsilon\downarrow 0} \int_{[0,\infty)}\frac{1}{\theta}e_V(d\theta) = \|V\|_{-1}^2\,.$$

□

8. Proof of Theorem 4–(i)

We fix $\lambda_0 \in [0,1)$ and prove the continuity of $v_Y(\lambda)$, $v_{\mathbb{Y}}(\lambda)$ at λ_0. To this aim we take $\lambda_* \in (\lambda_0, 1)$ and restrict below to $\lambda \in [0, \lambda_*)$. The positive constants C, C' will depend on λ_* but not on the specific choice of λ, moreover they can change from line to line.

8.1. **Continuity of $v_Y(\lambda)$.** We first observe that $\lim_{\lambda\to\lambda_0} \pi^\lambda = \pi^{\lambda_0}$ $\mathbb{P}$–a.s., where $\pi^\lambda(\omega) := \sum_{k\in\mathbb{Z}} c^\lambda_{0,k}(\omega)$. Indeed, by Assumption (A4), we can bound $|c^\lambda_{0,k}| \leq C\mathrm{e}^{-(1-\lambda_*)d|k|}$, $\mathbb{P}$–a.s., and therefore the claim follows from dominated convergence applied to the counting measure on $\mathbb{Z}$.

Since $p^\lambda_{0,k} = c^\lambda_{0,k}/\pi^\lambda$ and $\pi^\lambda \to \pi^{\lambda_0}$, we obtain that

$$\lim_{\lambda\to\lambda_0} p^\lambda_{0,k} = p^{\lambda_0}_{0,k} \qquad \forall k \in \mathbb{Z}\,, \qquad \mathbb{P}\text{–a.s.}$$

Note that $\pi^\lambda \geq c^\lambda_{0,1} \geq C\mathrm{e}^{-Z_0}$. Using also that $\mathrm{e}^{-(1-\lambda_*)u}u \leq C\mathrm{e}^{-\frac{(1-\lambda_*)}{2}u}$ for all $u \geq 0$ and using Assumption (A4) we get

$$p^\lambda_{0,k}|x_k| \leq C\mathrm{e}^{Z_0}\mathrm{e}^{-|x_k|+\lambda x_k}|x_k| \leq C'\mathrm{e}^{Z_0}\mathrm{e}^{-\frac{(1-\lambda_*)}{2}|d|k} \qquad \mathbb{P}\text{–a.s.} \tag{95}$$

We claim that $\varphi_\lambda \in L^2(\mathbb{Q}_0)$ and that $\lim_{\lambda\to\lambda_0} \|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} = 0$. Indeed, by (95), we have that $|\varphi_\lambda| \leq C\mathrm{e}^{Z_0}$, $\mathbb{Q}_0$–a.s. Since $\mathbb{E}[\mathrm{e}^{2Z_0}] < \infty$, $\pi \leq C$ $\mathbb{P}$–a.s. and $\mathbb{Q}_0[\star] = \mathbb{E}[\pi]^{-1}\mathbb{E}[\pi\star]$, we have that $\mathrm{e}^{Z_0} \in L^2(\mathbb{Q}_0)$. To conclude the proof of our claim it is enough to apply the dominated convergence theorem to the measure $\mathbb{Q}_0$.

Since $\varphi_{\lambda_0} \in L^2(\mathbb{Q}_0)$, by Theorem 1 and the Cauchy–Schwarz inequality we derive that $\varphi_{\lambda_0} \in L^1(\mathbb{Q}_\lambda)$, in particular the expectation $\mathbb{Q}_\lambda[\varphi_{\lambda_0}]$ is well–defined. Due to (10) we can therefore write

$$v_Y(\lambda) - v_Y(\lambda_0) = \mathbb{Q}_\lambda[\varphi_\lambda] - \mathbb{Q}_{\lambda_0}[\varphi_{\lambda_0}] = \mathbb{Q}_\lambda[\varphi_\lambda - \varphi_{\lambda_0}] + \mathbb{Q}_\lambda[\varphi_{\lambda_0}] - \mathbb{Q}_{\lambda_0}[\varphi_{\lambda_0}]\,. \tag{96}$$

By Theorem 1, the Cauchy–Schwarz inequality and since $\lim_{\lambda\to\lambda_0} \|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} = 0$, we get for $\lambda \to \lambda_0$

$$\big|\mathbb{Q}_\lambda[\varphi_\lambda - \varphi_{\lambda_0}]\big| = \Big|\mathbb{Q}_0\Big[\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}(\varphi_\lambda - \varphi_{\lambda_0})\Big]\Big| \leq \Big\|\frac{d\mathbb{Q}_\lambda}{d\mathbb{Q}_0}\Big\|_{L^2(\mathbb{Q}_0)} \|\varphi_\lambda - \varphi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} \to 0\,. \tag{97}$$

Since we have proved that $\varphi_{\lambda_0} \in L^2(\mathbb{Q}_0)$, by Theorem 2 we get that $\lim_{\lambda\to\lambda_0} \mathbb{Q}_\lambda[\varphi_{\lambda_0}] = \mathbb{Q}_{\lambda_0}[\varphi_{\lambda_0}]$. By combining this last limit with (96) and (97), we conclude that $\lim_{\lambda\to\lambda_0} v_Y(\lambda) = v_Y(\lambda_0)$.

8.2. **Continuity of $v_{\mathbb{Y}}(\lambda)$.** Due to the continuity of $v_Y(\lambda)$ and due to (10), it is enough to prove that the map $\lambda \mapsto \mathbb{Q}_\lambda\big[1/\pi^\lambda\big]$ is continuous (note that $c^\lambda_{0,k} = r^\lambda_{0,k}$, thus implying that $\pi^\lambda = \sum_{k\in\mathbb{Z}} r^\lambda_{0,k}(\omega)$).

By the observations in the above subsection we have that $\lim_{\lambda\to\lambda_0} \pi^\lambda = \pi^{\lambda_0}$ $\mathbb{Q}_0$–a.s. and $1/\pi^\lambda \leq C\mathrm{e}^{Z_0} \in L^2(\mathbb{Q}_0)$. We get three main consequences (applying also Theorem 1 and the Cauchy–Schwarz inequality): (i) $1/\pi_\lambda \in L^2(\mathbb{Q}_0)$, (ii) $1/\pi_\lambda \in L^1(\mathbb{Q}_{\lambda_0})$ (hence the expectation $\mathbb{Q}_{\lambda_0}[1/\pi^\lambda]$ is well–defined) and (iii) $\lim_{\lambda\to\lambda_0} \|1/\pi_\lambda - 1/\pi_{\lambda_0}\|_{L^2(\mathbb{Q}_0)} = 0$. We then write

$$\mathbb{Q}_\lambda\big[1/\pi^\lambda\big] - \mathbb{Q}_{\lambda_0}\big[1/\pi^{\lambda_0}\big] = \mathbb{Q}_\lambda[1/\pi^\lambda - 1/\pi^{\lambda_0}] + \mathbb{Q}_\lambda[1/\pi^{\lambda_0}] - \mathbb{Q}_{\lambda_0}[1/\pi^{\lambda_0}]\,. \tag{98}$$

At this point, we can proceed as done for (96), replacing φ_λ by $1/\pi^\lambda$.

9. Proof of Theorem 4–(ii)

We recall that we denote by $\|\cdot\|_{-1}$ the H_{-1} norm referred to the operator $-\mathbb{L}_0$ in $L^2(\mathbb{Q}_0)$ and by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{Q}_0)$.

9.1. **Einstein relation for** (Y_n^λ)**.** Since $v_Y(\lambda)=\mathbb{Q}_\lambda[\varphi_\lambda]$ and $v_Y(0)=\mathbb{Q}_0[\varphi]=0$ we can write

$$\frac{v_Y(\lambda)-v_Y(0)}{\lambda}=\frac{v_Y(\lambda)}{\lambda}=\frac{\mathbb{Q}_\lambda[\varphi_\lambda]}{\lambda}=\mathbb{Q}_\lambda\Big[\frac{\varphi_\lambda-\varphi}{\lambda}\Big]+\frac{\mathbb{Q}_\lambda[\varphi]-\mathbb{Q}_0[\varphi]}{\lambda}\,. \tag{99}$$

Lemma 9.1. $\varphi\in H_{-1}$.

Proof. We need to show that there exists a constant $C>0$ such that for any $h\in L^2(\mathbb{Q}_0)$ it holds

$$\langle\varphi,h\rangle\leq C\langle h,-\mathbb{L}_0h\rangle^{1/2}\,.$$

The above bound is equivalent to

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}x_kp_{0,k}h\Big]\leq\frac{C}{\sqrt{2}}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}\big(h(\tau_k\cdot)-h\big)^2\Big]^{1/2},$$

which is equivalent to (cf. $C':=C\sqrt{\mathbb{E}[\pi]/2}$)

$$\mathbb{E}\Big[\sum_{k\in\mathbb{Z}}x_kc_{0,k}h\Big]\leq C'\mathbb{E}\Big[\sum_{k\in\mathbb{Z}}c_{0,k}\big(h(\tau_k\cdot)-h\big)^2\Big]^{1/2}. \tag{100}$$

Note that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\mathbb{E}\left[x_kc_{0,k}h\right]&=-\sum_{k\in\mathbb{Z}}\mathbb{E}\left[x_{-k}(\tau_k\cdot)c_{0,-k}(\tau_k\cdot)h\right]\\&=-\sum_{k\in\mathbb{Z}}\mathbb{E}\left[x_{-k}c_{0,-k}h(\tau_{-k}\cdot)\right]=-\sum_{k\in\mathbb{Z}}\mathbb{E}\left[x_kc_{0,k}h(\tau_k\cdot)\right].\end{aligned}$$

Indeed, in the first identity we have used that $c_{0,k}(\omega)=c_{0,-k}(\tau_k\omega)$ and $x_k(\omega)=-x_{-k}(\tau_k\omega)$, in the second one we have used the translation invariance of $\mathbb{P}$, in the third one we have replaced k by $-k$. By the above identity and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\text{l.h.s. of (100)}&=-\frac{1}{2}\sum_{k\in\mathbb{Z}}\mathbb{E}\big[c_{0,k}x_k\big(h(\tau_k\cdot)-h\big)\big]\\&\leq C''\Big(\sum_{k\in\mathbb{Z}}\mathbb{E}\big[c_{0,k}x_k^2\big]\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}}\mathbb{E}\Big[c_{0,k}[h(\tau_k\cdot)-h]^2\Big]\Big)^{1/2}.\end{aligned}$$

thus concluding the proof of (100). □

As a consequence of Lemma 9.1 and Theorem 3 we have (recall definition (19))

$$\lim_{\lambda\to0}\frac{\mathbb{Q}_\lambda[\varphi]-\mathbb{Q}_0[\varphi]}{\lambda}=\partial_{\lambda=0}\mathbb{Q}_\lambda(\varphi)=\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)h^\varphi\Big]. \tag{101}$$

Take $\delta>0$ small enough as in Lemma B.1 of Appendix B. Using (87) we can write, for $\lambda\in(0,\delta)$,

$$\mathbb{Q}_\lambda\Big[\frac{\varphi_\lambda-\varphi}{\lambda}\Big]=\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}\,x_k\Big]+\frac{\lambda}{2}\mathcal{E}(\lambda) \tag{102}$$

where $\mathcal{E}(\lambda)$ can be bounded as

$$\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\Big(\sup_{\zeta\in[0,\delta]}|\partial^2_{\lambda=\zeta}p^\lambda_{0,k}|\Big)|x_k|\Big]\leq\sup_{\xi\in[0,\delta]}\Big\|\frac{d\mathbb{Q}_\xi}{d\mathbb{Q}_0}\Big\|_{L^2(\mathbb{Q}_0)}\Big\|\sum_{k\in\mathbb{Z}}\Big(\sup_{\zeta\in[0,\delta]}|\partial^2_{\lambda=\zeta}p^\lambda_{0,k}|\Big)|x_k|\Big\|_{L^2(\mathbb{Q}_0)}.$$

Due to Theorem 1 and (116) in Lemma B.1 in the Appendix, the above λ–independent upper bound is finite. Hence $\sup_{\lambda\in[0,\delta]}|\mathcal{E}(\lambda)|<\infty$, thus implying that $\lim_{\lambda\downarrow 0}\lambda\mathcal{E}(\lambda)=0$. On the other hand, since by Lemma B.1 in the Appendix the function $\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}\,x_k$ belongs to $L^q(\mathbb{Q}_0)$, by Theorem 2 we get that

$$\lim_{\lambda\downarrow 0}\mathbb{Q}_\lambda\Big[\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}\,x_k\Big]=\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}\partial_{\lambda=0}p^\lambda_{0,k}\,x_k\Big]\,. \tag{103}$$

At this point, by using that $\partial_{\lambda=0}p^\lambda_{0,k}=p^\lambda_{0,k}(x_k-\varphi)$ and by combining (99), (101), (102), the limit $\lim_{\lambda\downarrow 0}\lambda\mathcal{E}(\lambda)=0$ and (103), we conclude that $v_Y(\lambda)$ is derivable at $\lambda=0$ and that

$$\partial_{\lambda=0}v_Y(\lambda)=\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)(x_k+h^\varphi)\Big]\,. \tag{104}$$

It remains to show that the last part of (104) equals D_Y. We manipulate (104) to obtain

$$\begin{aligned}\partial_{\lambda=0}v_Y(\lambda)&=\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}(x_k-\varphi)h^\varphi\Big]+\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}x_k^2\Big]-\|\varphi\|^2_{L^2(\mathbb{Q}_0)}\\&=-Var(N^\varphi)+\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}x_k^2\Big]-\|\varphi\|^2_{L^2(\mathbb{Q}_0)}\\&=-2\|\varphi\|^2_{-1}+\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}}p_{0,k}x_k^2\Big]=D_Y.\end{aligned}$$

For the second equality we have used the second part of Theorem 3 (i.e., equation (22)) with the function $f=\varphi$, for the third equality we have used Lemma 7.1 with $V=\varphi$ and finally the last line follows from [11, Thm. 2.1, Eq. (2.28)].

9.2. **Einstein relation for** $(\mathbb{Y}^\lambda_t)$**.** The continuous time process $\tau_{\mathbb{Y}^\lambda_t}\omega$ can be obtained by a suitable random time change from the discrete time process $\tau_{Y^\lambda_n}\omega$ as detailed in [12, Sec. 7]. By using this random time change and arguing as in the derivation of [11, Eq. (4.20)], we get that $D_{\mathbb{Y}}=\mathbb{E}[\pi]D_Y$, where π was defined in (11). Since we have just proved that $D_Y=\partial_{\lambda=0}v_Y(\lambda)$, to get the Einstein relation for $\mathbb{Y}^\lambda_t$ it is enough to show that $v_{\mathbb{Y}}(\lambda)$ is differentiable at $\lambda=0$ and moreover $\partial_{\lambda=0}v_{\mathbb{Y}}(\lambda)=\mathbb{E}[\pi]\partial_{\lambda=0}v_Y(\lambda)$. Since $v_{\mathbb{Y}}(0)=0$, thanks to (10) and since $\pi^\lambda=\sum_{k\in\mathbb{Z}}c^\lambda_{0,k}=\sum_{k\in\mathbb{Z}}r^\lambda_{0,k}$ (cf. Section 8.2), we can write

$$\partial_{\lambda=0}v_{\mathbb{Y}}(\lambda)=\lim_{\lambda\downarrow 0}\frac{v_{\mathbb{Y}}(\lambda)}{\lambda}=\lim_{\lambda\downarrow 0}\frac{v_Y(\lambda)}{\lambda}\frac{1}{\mathbb{Q}_\lambda[1/\pi^\lambda]}\,. \tag{105}$$

In Section 8.2 we have proved that the map $[0,1)\ni\lambda\mapsto\mathbb{Q}_\lambda[1/\pi^\lambda]\in\mathbb{R}$ is continuous. Hence, we have $\lim_{\lambda\downarrow 0}\mathbb{Q}_\lambda[1/\pi^\lambda]=\mathbb{Q}_0\big[1/\pi^{\lambda=0}\big]=\mathbb{E}[\pi]^{-1}$. On the other hand we have just proved that $\lim_{\lambda\downarrow 0}\frac{v_Y(\lambda)}{\lambda}=D_Y$. Coming back to (105) we conclude that $\partial_{\lambda=0}v_{\mathbb{Y}}(\lambda)=D_Y\mathbb{E}[\pi]^{-1}=D_{\mathbb{Y}}$.

Appendix A. Comments on (10)

Formula (10) for $v_{\mathbb{Y}}(\lambda)$ coincides with [12, Eq. (9)]. The expression for $v_Y(\lambda)$ given in [12, Eq. (10)] is slightly different from our identity $v_Y(\lambda)=\mathbb{Q}_\lambda\big[\varphi_\lambda\big]$ in (10), since [12, Eq. (10)] has been obtained from the asymptotic velocity of a third random walk (which is the discrete–time random walk on $\mathbb{Z}$ with probability for a jump from i to k given by (7)). Let us explain how to derive that $v_Y(\lambda)=\mathbb{Q}_\lambda\big[\varphi_\lambda\big]$. We consider the process (ω^λ_n), defined as $\omega^\lambda_n:=\tau_k\omega$ where $k\in\mathbb{Z}$ satisfies $x_k=Y^\lambda_n$. Note that, due to Assumption (A3), one recovers a.s. (Y^λ_n) as an additive functional of (ω^λ_n). More precisely, $Y^\lambda_n=\sum_{k=0}^{n-1}h(\omega^\lambda_k,\omega^\lambda_{k+1})$, where

$h(\omega,\omega') := x_i$ if $\omega' = \tau_i\omega$ for some i, and $h(\omega,\omega') := 0$ if ω' does not coincide with any translation of ω. Let us denote by $\mathbb{E}^\lambda_{\mathbb{Q}_\lambda}$ the expectation w.r.t. the process (ω^λ_n) starting with distribution $\mathbb{Q}_\lambda$. Then, using that $\mathbb{Q}_\lambda$ is an ergodic distribution for the process (ω^λ_n), by Birkhoff's ergodic theorem we get that $\lim_{n\to\infty}\frac{Y^\lambda_n}{n}$ exists a.s. for $\mathbb{Q}_\lambda$–a.a. initial configurations and equals $\mathbb{E}^\lambda_{\mathbb{Q}_\lambda}\big[h(\omega_0,\omega_1)\big] = \mathbb{Q}_\lambda[\varphi_\lambda]$. Since, as proven in [12], $\mathbb{Q}_\lambda$ and $\mathbb{P}$ are mutually absolutely continuous, we conclude that $\lim_{n\to\infty}\frac{Y^\lambda_n}{n} = \mathbb{Q}_\lambda[\varphi_\lambda]$ a.s. for $\mathbb{P}$–a.a. initial configurations.

Appendix B. Collected computations

Here we collect some basic estimates that are useful in several parts of the paper. In what follows, λ_* is a fixed value in $(0,1)$. All constants of the form K, C appearing below (possibly with some additional typographic character) have to be thought of as λ_*–dependent but uniform for all $\lambda\in[0,\lambda_*]$. Moreover, the above constants can change from line to line. Moreover, without further mention, we will restrict to ω such that $|x_k|\geq k|d|$. We recall that by Assumption (A4) this event has $\mathbb{P}$–probability one.

It is convenient to express the jump probabilities $p^\lambda_{0,k}(\omega)$ in terms of the conductances introduced in (24). Comparing with (7) we can write

$$p^\lambda_{0,k}(\omega) = \frac{c^\lambda_{0,k}(\omega)}{\pi^\lambda(\omega)}\,, \qquad \pi^\lambda(\omega) := \sum_{j\in\mathbb{Z}} c^\lambda_{0,j}(\omega)\,. \tag{106}$$

Note that $\pi^\lambda = \pi$ when $\lambda = 0$ (cf. (11)).

An easy calculation shows that

$$\partial_\lambda p^\lambda_{0,k} = p^\lambda_{0,k}(x_k - \varphi_\lambda) \tag{107}$$

$$\partial^2_\lambda p^\lambda_{0,k} = p^\lambda_{0,k}\Big(x_k^2 - 2x_k\varphi_\lambda + 2\varphi_\lambda^2 - \sum_{j\in\mathbb{Z}} p^\lambda_{0,j}x_j^2\Big)\,. \tag{108}$$

We also observe that, for some universal constant c, it holds

$$\big|\partial^2_\lambda p^\lambda_{0,k}\big| \leq c\, p^\lambda_{0,k}\Big(x_k^2 + \sum_{j\in\mathbb{Z}} p^\lambda_{0,j}x_j^2\Big)\,. \tag{109}$$

Indeed, by (108) we can bound

$$\big|\partial^2_\lambda p^\lambda_{0,k}\big| \leq c' p^\lambda_{0,k}\Big(x_k^2 + \varphi_\lambda^2 + \sum_{j\in\mathbb{Z}} p^\lambda_{0,j}x_j^2\Big)$$

for some universal constant c'. On the other hand, by the Cauchy–Schwarz inequality, $\varphi_\lambda^2 \leq \sum_{j\in\mathbb{Z}} p^\lambda_{0,j}x_j^2$. We also have that, for some finite constant $C>0$,

$$\frac{p^\lambda_{0,k}}{p_{0,k}} = \mathrm{e}^{\lambda x_k}\frac{\pi}{\pi^\lambda} \leq C\,\mathrm{e}^{\lambda(x_k+Z_{-1})} \qquad \forall k\in\mathbb{Z}\,, \qquad \forall\lambda\in[0,\lambda_*]\,. \tag{110}$$

This is true since $c^\lambda_{-1,0}+c^\lambda_{0,1}\leq\pi^\lambda\leq K(c^\lambda_{-1,0}+c^\lambda_{0,1})$ for some constant K (see [12, Rem. 3.2], [12, Lemma 3.6] and Remark 4.1), and therefore

$$\frac{\pi}{\pi^\lambda} \leq K'\frac{\mathrm{e}^{-Z_{-1}}+\mathrm{e}^{-Z_0}}{\mathrm{e}^{-(1+\lambda)Z_{-1}}+\mathrm{e}^{-(1-\lambda)Z_0}} \leq K'\Big(1+\frac{\mathrm{e}^{-Z_{-1}}}{\mathrm{e}^{-(1+\lambda)Z_{-1}}}\Big) \leq C\mathrm{e}^{\lambda Z_{-1}}\,. \tag{111}$$

Another bound which will be repeatedly used below is the following. For a fixed positive integer n, it holds

$$\sum_{k\in\mathbb{Z}} p^\lambda_{0,k}|x_k|^n \le C\frac{1}{\pi^\lambda}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-|x_k|+\lambda x_k}|x_k|^n \le \tilde C\frac{1}{\pi^\lambda}\,, \qquad \forall\lambda\in[0,\lambda_*] \tag{112}$$

($\tilde C$ depends on λ_* and n). Above we used that $\mathrm{e}^{-(1-\lambda_*)u}u^n \le C\mathrm{e}^{-(1-\lambda_*)u/2}$ for all $u\ge 0$ and that $|x_j|\ge dj$. As a consequence of (112) we get

$$|\varphi_\lambda|^n \le \sum_{k\in\mathbb{Z}} p^\lambda_{0,k}|x_k|^n \le \frac{C}{\pi^\lambda}\,, \qquad \forall\lambda\in[0,\lambda_*]\,. \tag{113}$$

Since $d\mathbb{Q}_0/d\mathbb{P} = \pi/\mathbb{E}[\pi]$, by (111), (112) and (113) we get

$$\mathbb{E}[\mathrm{e}^{Z_0}]<\infty \implies \sup_{\lambda\in[0,\lambda_*]}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p^\lambda_{0,k}|x_k|^n\Big]<\infty \text{ and } \sup_{\lambda\in[0,\lambda_*]}\mathbb{Q}_0\Big[|\varphi_\lambda|^n\Big]<\infty\,. \tag{114}$$

Lemma B.1. *Suppose $\mathbb{E}[\mathrm{e}^{pZ_0}]<\infty$ for some $p>2$, let $q>1$ be such that $p^{-1}+q^{-1}=1$ and let $\hat q>2$ be such that $p^{-1}+\hat q^{-1}=2^{-1}$. Then, for δ small enough, it holds*

$$\sum_{k\in\mathbb{Z}} |\partial_{\lambda=0}p^\lambda_{0,k}\cdot x_k| \in L^q(\mathbb{Q}_0)\subset L^1(\mathbb{Q}_\lambda)\,, \tag{115}$$

$$\sum_{k\in\mathbb{Z}} \Big(\sup_{\zeta\in[0,\delta]}|\partial^2_{\lambda=\zeta}p^\lambda_{0,k}|\Big)|x_k| \in L^2(\mathbb{Q}_0)\subset L^1(\mathbb{Q}_\lambda)\,, \tag{116}$$

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} (p_{0,k})^{1-\hat q}\Big(\sup_{\zeta\in[0,\delta]}|\partial^2_{\lambda=\zeta}p^\lambda_{0,k}|\Big)^{\hat q} \in L^1(\mathbb{Q}_0)\,. \tag{117}$$

Proof. Since $p>2$ we have $q\in(1,2)$, thus implying that $L^2(\mathbb{Q}_0)\subset L^q(\mathbb{Q}_0)$ by the Hölder inequality. To get the set inclusions stated in the lemma, it is therefore enough to check that $L^q(\mathbb{Q}_0)\subset L^1(\mathbb{Q}_\lambda)$. This can be easily checked by writing $\mathbb{Q}_\lambda[\star]=\mathbb{Q}_0[\star\cdot d\mathbb{Q}_\lambda/d\mathbb{Q}_0]$, using the Hölder inequality and then Theorem 1.

We call f_1, f_2 and f_3 the l.h.s. of (115), (116) and (117), respectively. For (115) we use (107) and the Cauchy-Schwarz inequality to bound

$$\|f_1\|^q_{L^q(\mathbb{Q}_0)} \le \mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}} p_{0,k}(x_k-\varphi)^2\Big)^{q/2}\Big(\sum_{k\in\mathbb{Z}} p_{0,k}x_k^2\Big)^{q/2}\Big].$$

As in the proof of Lemma 6.5 we take $A:=2/q>1$ (recall that $p>2$) and $B:=2/(2-q)$ (so that $A^{-1}+B^{-1}=1$) and use the Hölder inequality to further obtain

$$\|f_1\|^q_{L^q(\mathbb{Q}_0)} \le \mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}} p_{0,k}(x_k-\varphi)^2\Big)^{\frac{qB}{2}}\Big]^{1/B}\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}} p_{0,k}x_k^2\Big]^{1/A}.$$

The first term in the r.h.s. can be bounded as in (84), the second is bounded by (114).

We move to (116). To prove that $f_2\in L^2(\mathbb{Q}_0)$ we need to show that $\mathbb{E}[\pi f_2^2]<\infty$. We take δ small (the precise value will be stated at the end) and set $\lambda_*:=\delta$ (hence, our C–type constants below depend on δ but not on the specific $\lambda\in[0,\delta]$). We note that for all $\zeta\in[0,\delta]$ it holds

$$\begin{aligned}|\partial^2_{\lambda=\zeta}p^\lambda_{0,k}||x_k| &\le Cp^\zeta_{0,k}\Big(|x_k|^3+\big(\sum_{j\in\mathbb{Z}} p^\zeta_{0,j}x_j^2\big)^2\Big)\le Cp^\zeta_{0,k}\Big(|x_k|^3+\sum_{j\in\mathbb{Z}} p^\zeta_{0,j}x_j^4\Big)\\ &\le C'p_{0,k}\mathrm{e}^{\delta(|x_k|+Z_{-1})}\Big(|x_k|^3+\sum_{j\in\mathbb{Z}} p_{0,j}\mathrm{e}^{\delta(|x_j|+Z_{-1})}x_j^4\Big)\,.\end{aligned} \tag{118}$$

Indeed, the first inequality follows from (109) and the property that $|x_k| \geq d$ for $k \neq 0$ (as intermediate step bound the product $(\sum_j p^\zeta_{0,j} x_j^2)|x_k|$ by the sum of their squares). The second inequality follows from the Cauchy–Schwarz inequality, while the third inequality follows from (110).

Note that the last term of (118) depends only on δ. Hence, to prove that $\mathbb{E}[\pi f_2^2] < \infty$, we only need to show that (we use repeatedly the Cauchy–Schwarz inequality)

$$\mathbb{E}\Big[\pi \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{2\delta(|x_k|+Z_{-1})} |x_k|^6\Big] < \infty \tag{119}$$

$$\mathbb{E}\Big[\pi \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{2\delta(|x_k|+Z_{-1})} \sum_{j\in\mathbb{Z}} p_{0,j} \mathrm{e}^{2\delta(|x_j|+Z_{-1})} x_j^8\Big] < \infty\,. \tag{120}$$

We prove (120), the proof of (119) follows the same lines and it is even simpler. Using that $\mathrm{e}^{-(1-2\delta)u}(1+u^8) \leq C\mathrm{e}^{-u/2}$ for all $u \geq 0$ if we restrict to $\delta \leq 1/8$, we can bound the integrand in (120) by

$$\frac{C}{\pi} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\frac{|x_k|}{2}} \mathrm{e}^{2\delta Z_{-1}} \sum_{j\in\mathbb{Z}} \mathrm{e}^{-\frac{|x_j|}{2}} \mathrm{e}^{2\delta Z_{-1}}\,.$$

Since $|x_k| \geq d|k|$ and since $\pi \geq c_{-1,0} \geq C\mathrm{e}^{-(1+\delta)Z_{-1}}$, we conclude that the $\mathbb{P}$–expectation of (120) is finite if $\mathbb{E}[\mathrm{e}^{(1+5\delta)Z_{-1}}] < \infty$. By taking δ small enough, the last bound is satisfied due to the assumption $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$.

We move to (117). Again we need to prove that $\mathbb{E}[\pi f_3] < \infty$. Similarly to (118), by (109) and (110), we get

$$|\partial^2_{\lambda=\zeta} p^\lambda_{0,k}| \leq C p^\zeta_{0,k}\Big(|x_k|^2 + \sum_{j\in\mathbb{Z}} p^\zeta_{0,j} x_j^2\Big) \leq C' p_{0,k} \mathrm{e}^{\delta(|x_k|+Z_{-1})}\Big(|x_k|^2 + \sum_{j\in\mathbb{Z}} p_{0,j} \mathrm{e}^{\delta(|x_j|+Z_{-1})} x_j^2\Big)\,.$$

Then, using also that $(x+y)^{\hat q} \leq c(\hat q)(x^{\hat q} + y^{\hat q})$ for all $x, y \geq 0$ and the Hölder inequality,

$$f_3 \leq C \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{\hat q\delta(|x_k|+Z_{-1})} |x_k|^{2\hat q} + \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{\hat q\delta(|x_k|+Z_{-1})} \sum_{j\in\mathbb{Z}} p_{0,j} \mathrm{e}^{\hat q\delta(|x_j|+Z_{-1})} x_j^{2\hat q}\,. \tag{121}$$

At this point, we get that $\mathbb{E}[\pi f_3] < \infty$ if we prove

$$\mathbb{E}\Big[\pi \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{\hat q\delta(|x_k|+Z_{-1})} |x_k|^{2\hat q}\Big] < \infty\,, \tag{122}$$

$$\mathbb{E}\Big[\pi \sum_{k\in\mathbb{Z}} p_{0,k} \mathrm{e}^{\hat q\delta(|x_k|+Z_{-1})} \sum_{j\in\mathbb{Z}} p_{0,j} \mathrm{e}^{\hat q\delta(|x_j|+Z_{-1})} x_j^{2\hat q}\Big] < \infty\,. \tag{123}$$

The above bound can be proved by the same arguments adopted for (120) when δ is small enough. □

Lemma B.2. *Suppose* $\mathbb{E}[\mathrm{e}^{pZ_0}] < \infty$ *for some* $p > 1$. *Given* $\lambda_0 \in [0, 1)$*, it holds*

$$\lim_{\lambda\to\lambda_0} \mathbb{Q}_0\Big[\big(\sum_{k\in\mathbb{Z}} |p^\lambda_{0,k} - p^{\lambda_0}_{0,k}|\big)^4\Big] = 0\,. \tag{124}$$

Proof. We fix $\lambda_* \in (\lambda_0, 1)$. Recall that all constants of type C, K appearing in what follows can depend on λ_* but do not depend on the particular bias parameter taken in $[0, \lambda_*]$, and moreover can change from line to line. First of all we bound, by applying the Hölder inequality,

$$\mathbb{Q}_0\Big[\big(\sum_{k\in\mathbb{Z}}|p^{\lambda}_{0,k}-p^{\lambda_0}_{0,k}|\big)^4\Big]=\mathbb{Q}_0\Big[\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}}p^{\lambda_0}_{0,k}\Big|\frac{p^{\lambda}_{0,k}-p^{\lambda_0}_{0,k}}{p^{\lambda_0}_{0,k}}\Big|\Big)^4\Big]\leq\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p^{\lambda_0}_{0,k}\Big|\frac{p^{\lambda}_{0,k}-p^{\lambda_0}_{0,k}}{p^{\lambda_0}_{0,k}}\Big|^4\Big]. \tag{125}$$

By the Taylor expansion with the Lagrange rest at the first order and by (107) we have

$$p^{\lambda}_{0,k}-p^{\lambda_0}_{0,k}=(\lambda-\lambda_0)\,\partial_{\lambda=\xi_k}p^{\lambda}_{0,k}=(\lambda-\lambda_0)\,p^{\xi_k}_{0,k}(x_k-\varphi_{\xi_k}),$$

where ξ_k is some random value between λ_0 and λ depending on k, λ_0 and λ. Therefore we can continue from (125) and bound

$$\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}p^{\lambda_0}_{0,k}\Big|\frac{p^{\lambda}_{0,k}-p^{\lambda_0}_{0,k}}{p^{\lambda_0}_{0,k}}\Big|^4\Big]\leq C(\lambda-\lambda_0)^4\Big(\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^3}x_k^4\Big]+\mathbb{Q}_0\Big[\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^3}\varphi^4_{\xi_k}\Big]\Big). \tag{126}$$

Given $\delta>0$ small (the precise value of δ will be stated below) we set $U_\delta:=[\lambda_0-\delta,\lambda_0+\delta]$ and assume $U_\delta\subset[0,\lambda_*]$. If we show that both the $\mathbb{Q}_0$-expectations on the r.h.s. of (126) are finite uniformly in $\lambda\in U_\delta$, then we are done. To this aim we extend the bound in (110). Indeed, by the same arguments used for (110), we have for any $\lambda,\zeta\in[0,\lambda_*]$ and $k\in\mathbb{Z}$ that

$$\frac{p^{\lambda}_{0,k}}{p^{\zeta}_{0,k}}=\mathrm{e}^{(\lambda-\zeta)x_k}\frac{\pi^{\zeta}}{\pi^{\lambda}}\leq C\mathrm{e}^{(\lambda-\zeta)x_k}\frac{\mathrm{e}^{-(1-\zeta)Z_0}+\mathrm{e}^{-(1+\zeta)Z_{-1}}}{\mathrm{e}^{-(1-\lambda)Z_0}+\mathrm{e}^{-(1+\lambda)Z_{-1}}}\leq C\mathrm{e}^{|\lambda-\zeta|\cdot|x_k|}\Big[\mathrm{e}^{|\lambda-\zeta|Z_0}+\mathrm{e}^{|\lambda-\zeta|Z_{-1}}\Big] \tag{127}$$

(the above constant C does not depend on $k\in\mathbb{Z}$).

From now on we restrict to $\lambda\in U_\delta$ (thus implying that $\xi_k\in U_\delta$). Then by (127) we can bound

$$\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^4}x_k^4\leq C\mathrm{e}^{4\delta|x_k|}\big[\mathrm{e}^{4\delta Z_0}+\mathrm{e}^{4\delta Z_{-1}}\big]x_k^4\leq C'\mathrm{e}^{5\delta|x_k|}\big[\mathrm{e}^{4\delta Z_0}+\mathrm{e}^{4\delta Z_{-1}}\big]$$

(C' depends on δ). Hence we get (cf. (111))

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}}\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^3}x_k^4=\mathbb{E}[\pi]^{-1}\frac{\pi}{\pi^{\lambda_0}}c^{\lambda_0}_{0,k}\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^4}x_k^4\leq C'\mathrm{e}^{\lambda_0 Z_{-1}}\mathrm{e}^{-(1-\lambda_0-5\delta)|x_k|}\mathrm{e}^{4\delta Z_0+4\delta Z_{-1}}\,. \tag{128}$$

We assume δ so small that $\lambda_0+5\delta<1$. Using that $|x_k|\geq kd$, to prove that the first expectation in the r.h.s. of (126) is bounded uniformly in $\lambda\in U_\delta$ we only need to show that

$$\mathbb{E}\big[\mathrm{e}^{(\lambda_0+4\delta)Z_{-1}+3\delta Z_0}\big]<\infty\,. \tag{129}$$

Before explaining how to proceed we move to the second $\mathbb{Q}_0$-expectation on the last line of (126). Due to (113) and since $\pi^{\xi_k}\geq c^{\xi_k}_{0,1}\geq C\mathrm{e}^{-(1-\xi_k)Z_0}$, we have

$$\varphi^4_{\xi_k}\leq\frac{C}{\pi^{\xi_k}}\leq\tilde{C}\mathrm{e}^{(1-\lambda_0-\delta)Z_0}\,.$$

Reasoning as in (128) we get

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}}\frac{(p^{\xi_k}_{0,k})^4}{(p^{\lambda_0}_{0,k})^3}\varphi^4_{\xi_k}\leq C'\mathrm{e}^{\lambda_0 Z_{-1}}\mathrm{e}^{-(1-\lambda_0-4\delta)|x_k|}\mathrm{e}^{4\delta Z_0+4\delta Z_{-1}}\mathrm{e}^{(1-\lambda_0-\delta)Z_0}$$

and the second $\mathbb{Q}_0$-expectation on the last line of (126) is bounded uniformly in $\lambda \in U_\delta$ if we prove that

$$\mathbb{E}\big[e^{(\lambda_0+4\delta)Z_{-1}+(1-\lambda_0+3\delta)Z_0}\big] < \infty\,. \tag{130}$$

We explain how to get (130) (indeed, (130) implies (129)). By the Hölder inequality, given $a, b \geq 1$ with $a^{-1}+b^{-1}=1$, (130) is satisfied if the expectations $\mathbb{E}[e^{a(\lambda_0+4\delta)Z_{-1}}]$ and $\mathbb{E}[e^{b(1-\lambda_0+3\delta)Z_0}]$ are finite. To conclude we take $a := (\lambda_0+4\delta)^{-1}$ and therefore $b := (1-\lambda_0-4\delta)^{-1}$, and take δ small to have $b(1-\lambda_0+3\delta) \leq p$. At the end, it remains to invoke the bound $\mathbb{E}[e^{pZ_0}] < \infty$. □

Appendix C. Proof of Lemma 5.2

To simplify the notation, inside the proof we write $\|\cdot\|$ and $\langle \cdot,\cdot\rangle$ for the norm and the scalar product in $L^2(Q_0)$. Note that $\rho_0 \equiv 1$. Since $Q_\lambda(f) = Q_0(\rho_\lambda f) = \langle \rho_\lambda, f\rangle$, the L^2–weak convergence $\rho_\lambda \rightharpoonup \rho_0$ would imply (64). Hence, we only need to prove that $\rho_\lambda \rightharpoonup \rho_0$.

Suppose by contradiction that $\rho_\lambda \not\rightharpoonup \rho_0$. Then we can extract a sequence $\lambda_n \to \lambda_0$ such that $\rho_{\lambda_n} \notin U$, with U being a suitable open neighbourhood of ρ_0. Let $R := \sup_{\lambda\in I}\|\rho_\lambda\|$ and set $B(0,R) := \{f \in L^2(Q_0) : \|f\| \leq R\}$. Note that $R < \infty$ by (H1). By Kakutani's theorem the ball $B(0,R)$ is compact in the L^2–weak topology, hence the set $\{\rho_{\lambda_n}\}$ is relatively compact in the L^2–weak topology. As a consequence, at the cost of extracting a subsequence, we have that $\rho_{\lambda_n} \rightharpoonup \rho$ for some $\rho \in L^2(Q_0)$. Since $\rho_{\lambda_n} \notin U$, we also have that $\rho \neq \rho_0$. To get a contradiction, we prove that it must be $\rho = \rho_0$.

To this aim we first isolate some properties of ρ. For any function $f \in L^2(Q_0)$ with $f \geq 0$, we have $\langle \rho, f\rangle \geq 0$ (indeed $\langle \rho_n, f\rangle \geq 0$ since $\rho_n \geq 0$). As a consequence $\rho \geq 0$. Moreover $\langle \rho, 1\rangle = \lim_{n\to\infty}\langle \rho_n, 1\rangle = 1$. By the above properties $dQ := \rho dQ_0$ is a well–defined probability measure and $\frac{dQ}{dQ_0} \in L^2(Q_0)$. We claim that $Q(L_0 f) = 0$ for any $f \in \mathcal{C}$. By (H2), assuming our claim, we obtain that $Q = Q_0$, thus implying that $\rho = \rho_0$ and leading to the contradiction.

It remains to prove the claim. Note that for $f \in \mathcal{C}$

$$Q(L_0 f) = \langle \rho, L_0 f\rangle = \lim_{n\to\infty}\langle \rho_{\lambda_n}, L_0 f\rangle = \lim_{n\to\infty} Q_{\lambda_n}(L_0 f)\,. \tag{131}$$

Since $Q_{\lambda_n}(L_{\lambda_n} f) = 0$ by (H3), using assumptions (H1) and (H4) we can bound

$$\big|Q_{\lambda_n}(L_0 f)\big| = \big|Q_{\lambda_n}(L_0 f - L_{\lambda_n} f)\big| = \big|Q_0(\rho_{\lambda_n}(L_0 f - L_{\lambda_n} f))\big| \leq \|\rho_{\lambda_n}\|\,\|L_0 f - L_{\lambda_n} f\| \to 0 \tag{132}$$

as $n \to \infty$. As a byproduct of (131) and (132) we get that $Q(L_0 f) = 0$ for any $f \in \mathcal{C}$, thus proving our claim.

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