New Journal of Physics
The open-access journal for physics

Collective decoherence of cold atoms coupled to a Bose–Einstein condensate

M A Cirone[1], G De Chiara[2], G M Palma[3,5] and A Recati[4]

[1] Dipartimento di Scienze Fisiche ed Astronomiche, Università degli Studi di Palermo, via Archirafi 36, I-90123 Palermo, Italy
[2] Grup d'Òptica, Departament de Física, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain
[3] NEST—CNR—INFM and Dipartimento di Scienze Fisiche ed Astronomiche, Università degli Studi di Palermo, via Archirafi 36, I-90123 Palermo, Italy
[4] Dipartimento di Fisica, Università di Trento, CNR-INFM BEC Center, I-38050 Povo, Trento, Italy
and
Physik-Department, Technische Universität München, D-85748 Garching, Germany
E-mail: massimo.palma@fisica.unipa.it

New Journal of Physics **11** (2009) 103055 (19pp)
Received 28 July 2009
Published 27 October 2009
Online at http://www.njp.org/
doi:10.1088/1367-2630/11/10/103055

Abstract. We examine the time evolution of cold atoms (impurities) interacting with an environment consisting of a degenerate bosonic quantum gas. The impurity atoms differ from the environment atoms, being of a different species. This allows one to superimpose two independent trapping potentials, each being effective only on one atomic kind, while transparent to the other. When the environment is homogeneous and the impurities are confined in a potential consisting of a set of double wells, the system can be described in terms of an effective spin-boson model, where the occupation of the left or right well of each site represents the two (pseudo)-spin states. The irreversible dynamics of such system is here studied exactly, i.e. not in terms of a Markovian master equation. The dynamics of one and two impurities is remarkably different in respect of the standard decoherence of the spin-boson system. In particular, we show: (i) the appearance of coherence oscillations, (ii) the presence of super and subdecoherent states that differ from the standard ones of the

[5] Author to whom any correspondence should be addressed.

spin-boson model, and (iii) the persistence of coherence in the system at long times. We show that this behaviour is due to the fact that the pseudospins have an internal spatial structure. We argue that collective decoherence also prompts information about the correlation length of the environment. In a one-dimensional (1D) configuration, one can change even more strongly the qualitative behaviour of the dephasing just by tuning the interaction of the bath.

Contents

1. Introduction

The reasons for the great interest in the physics of ultracold atoms in recent years are manifold. On the one hand, experimentalists have reached an unprecedented control over the many-body atomic state with very stable optical potentials and by the use of Feshbach resonances, which allow one to change the scattering length of the atoms [1]. In this context, the tremendous experimental results that have been achieved include: the observation of the superfluid-Mott insulator transition for bosons [2], one-dimensional (1D) strongly interacting bosons in the Tonks–Girardeau regime [3] and Anderson localization [4, 5]. On the other hand, new experimental challenges come from different theoretical proposals for using this system for quantum information processing [6] and as a quantum simulator of condensed matter models (see for example [7]–[9] and references therein).

Not only can ultracold atoms simulate Hamiltonian systems, but such systems also offer a way to engineer non-classical environments. Thanks to the flexibility of quantum gases, a broad range of regimes of irreversible dynamics of open quantum systems and in particular of spin-boson systems can be explored [10]–[12].

In the present paper, we propose a new method by which an instance of the spin-boson model [13] can be realized with a suitable arrangement of interacting cold atoms. In particular, we analyse a system consisting of cold impurity atoms interacting with a degenerate quantum gas of a different atomic species. This setup makes possible the superposition of two independent trapping potentials, each being effective on one atomic species only, while transparent to the other. When the quantum gas is homogeneous and the impurities are confined in a potential composed of double wells, the system can be described in terms of an effective spin-boson model, where occupations of the left or right well represent the two (pseudo)-spin states. At variance with other setups, where the role of the pseudospin is played by the presence or absence of one particle in a trapping well [14], by the vibrational modes of a single well [15] or by internal electronic levels [12], in our case each pseudospin has a spatial dimension, namely the separation between the two minima of the impurity double well. This introduces an effective suppression of the decoherence due to low-frequency modes of the environment and leads to unusual and interesting phenomena, like oscillations of coherence at finite times and the survival of coherence at long times. Further novel features appear when one considers the irreversible collective decoherence of a systems of two impurities. In this case, we still predict the existence of subdecoherent and superdecoherent states, but with the interesting fact that their role is exactly the opposite from what one observes in conventional spin-boson systems. Further interesting features appear when one considers how the collective decoherence rates change as a function of the impurities' separation and the effects of dimensionality of the system.

In discussing our investigations, for the sake of simplicity we shall consider an experimental setup where the impurity atoms are trapped by a periodic (optical) lattice. We would like to stress, however, that our findings do not depend on the lattice properties (e.g. periodicity) but on the numerical results. Other setups, such as microtraps on atom chips or quantum dots, just to mention a few, can be equally envisaged.

2. The Hamiltonian

Our system is composed of a cold quantum gas of bosonic atoms and a sample of cold atoms separated from each other and immersed in the quantum gas. In presenting our investigations, we shall use the words 'reservoir', 'bath' and 'environment' as synonyms to indicate the quantum gas, since its properties are not the focus of the present paper.

The second-quantized form of the Hamiltonian of the impurities + bath system takes the form (see also [16])

$$\hat{H} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{B}} + \hat{H}_{\mathrm{AB}}, \tag{1}$$

where

$$\hat{H}_{\mathrm{A}} = \int \mathrm{d}^3x\ \hat{\Psi}^\dagger(\mathbf{x}) \left[\frac{\mathbf{p}_A^2}{2m_{\mathrm{A}}} + V_{\mathrm{A}}(\mathbf{x}) \right] \hat{\Psi}(\mathbf{x}) \tag{2}$$

is the Hamiltonian of atomic impurities, described by the field operator $\hat{\Psi}(\mathbf{x})$ in the trapping potential $V_{\mathrm{A}}(\mathbf{x})$, which creates a set of double wells of size $2L$ and separated by a distance $2D$, see figure 1,

$$\hat{H}_{\mathrm{B}} = \int \mathrm{d}^3x\ \hat{\Phi}^\dagger(\mathbf{x}) \left[\frac{\mathbf{p}^2}{2m_{\mathrm{B}}} + V_{\mathrm{B}}(\mathbf{x}) + \frac{g_{\mathrm{B}}}{2}\ \hat{\Phi}^\dagger(\mathbf{x})\hat{\Phi}(\mathbf{x}) \right] \hat{\Phi}(\mathbf{x}) \tag{3}$$

is the Hamiltonian of the bath, composed of $N \gg 1$ bosons, represented by the field operator $\hat{\Phi}(\mathbf{x})$ and confined by a trapping potential $V_{\mathrm{B}}(\mathbf{x})$ and $g_{\mathrm{B}} = 4\pi\hbar^2 a_{\mathrm{B}}/m_{\mathrm{B}}$ is the boson–boson coupling constant, with a_{B} the scattering length of the condensate atoms, and

$$\hat{H}_{\mathrm{AB}} = g_{\mathrm{AB}} \int \mathrm{d}^3x\ \hat{\Psi}^\dagger(\mathbf{x})\hat{\Phi}^\dagger(\mathbf{x})\hat{\Phi}(\mathbf{x})\hat{\Psi}(\mathbf{x}) \tag{4}$$

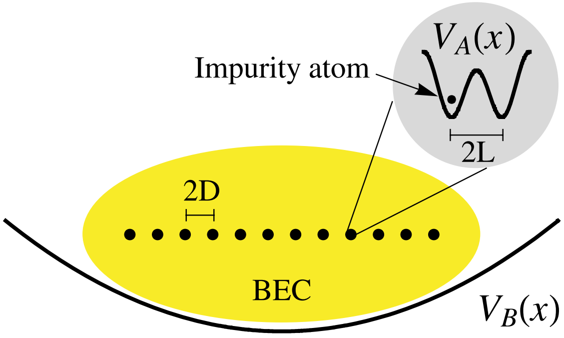

Figure 1. A Bose–Einstein condensate (yellow region) confined in a shallow harmonic trap $V_B(x)$ interacts with cold impurity atoms each of which is trapped in a double well potential $V_A(x)$ (grey circle). The distance between two wells in the same trap is 2L and the distance between adjacent traps is 2D.

describes the interactions between the impurities and the bath; here $g_{\rm AB} = 2\pi\hbar^2 a_{\rm AB}/m_{\rm AB}$ is the coupling constant of impurities–gas interaction, with $a_{\rm AB}$ the scattering length of the impurities–gas collisions and $m_{\rm AB} = m_{\rm A}m_{\rm B}/(m_{\rm A}+m_{\rm B})$ their reduced mass. Both impurity and bath atoms are described in the second-quantized formalism. The field operator of the atomic impurities

$$\hat{\Psi}(\mathbf{x}) = \sum_{i,p} \hat{a}_{i,p}\varphi_{i,p}(\mathbf{x}) \tag{5}$$

can be decomposed in terms of the real eigenstates $\varphi_{i,p}(\mathbf{x})$ of impurity atoms localized on the double well i of the potential $V_{\rm A}(x)$ in the $p^{\rm th}$ state, with energy $\hbar\omega_{i,p}$ and the corresponding annihilation operator $\hat{a}_{i,p}$. We assume that the wavefunctions of different double wells have a negligible common support, i.e. $\varphi_{i,p}(\mathbf{x})\varphi_{j\neq i,m}(\mathbf{x}) \simeq 0$ at any position $\mathbf{x}$.

We treat the gas of bosons following Bogoliubov's approach (see, for instance, [17]) and assuming a very shallow trapping potential $V_{\rm B}(\mathbf{x})$, such that the bosonic gas can be considered homogeneous. In the degenerate regime, the bosonic field can be decomposed as

$$\hat{\Phi}(\mathbf{x}) = \sqrt{N_0}\,\Phi_0(\mathbf{x}) + \delta\hat{\Phi}(\mathbf{x}) = \sqrt{N_0}\,\Phi_0(\mathbf{x}) + \sum_{\mathbf{k}}\left(u_{\mathbf{k}}(\mathbf{x})\hat{c}_{\mathbf{k}} - v^*_{\mathbf{k}}(\mathbf{x})\hat{c}^\dagger_{\mathbf{k}}\right), \tag{6}$$

where $\Phi_0(\mathbf{x})$ is the condensate wave function (or order parameter), $N_0 < N$ is the number of atoms in the condensate and $\hat{c}_{\mathbf{k}}$, $\hat{c}^\dagger_{\mathbf{k}}$ are the annihilation and creation operators of the Bogoliubov modes with momentum $\mathbf{k}$. For a homogeneous condensate $\Phi_0(\mathbf{x}) = 1/\sqrt{V}$, V being the volume. Its Bogoliubov modes

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + n_0 g_{\rm B}}{E_{\mathbf{k}}} + 1\right)}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}}, \tag{7}$$

$$v_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_{\mathbf{k}} + n_0 g_{\rm B}}{E_{\mathbf{k}}} - 1\right)}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \tag{8}$$

have energy

$$E_{\mathbf{k}} = \left[2\epsilon_{\mathbf{k}} n_0 g_{\mathrm{B}} + \epsilon_{\mathbf{k}}^2\right]^{1/2}, \tag{9}$$

where $\epsilon_{\mathbf{k}} = \hbar^2 k^2/(2m_{\mathrm{B}})$ and $n_0 = N_0/V$ is the condensate density. As one can see from (9), low-energy excitations have phonon-like (wave-like) spectrum, whereas high-energy excitations have particle-like spectrum. The condition for wave-like excitations is $\epsilon_{\mathbf{k}} \ll n_0 g_{\mathrm{B}}$, i.e. $k \ll 4\sqrt{\pi n_0 a_{\mathrm{B}}}$, or equivalently $k \ll 2m_{\mathrm{B}} c_{\mathrm{s}}/\hbar$, where $c_{\mathrm{s}} = \sqrt{n_0 g_{\mathrm{B}}/m}$ is the speed of sound at zero temperature. Note that $|u_{\mathbf{k}}| = 1/\sqrt{V}$ and $|v_{\mathbf{k}}| = 0$ describe the limiting case of $N \gg 1$ non-interacting bosons, each with energy $E_{\mathbf{k}} = \epsilon_{\mathbf{k}}$.

Inserting equations (5) and (6) into the Hamiltonian (1) we obtain

$$\hat{H}_{\mathrm{A}} = \sum_{i,p} \hbar\omega_{i,p} \hat{a}_{i,p}^{\dagger} \hat{a}_{i,p} \tag{10}$$

for the impurities,

$$\hat{H}_{\mathrm{B}} = H_{\mathrm{Cond}} + \hat{H}_{\mathrm{Bog}} \tag{11}$$

for the quantum gas, with

$$H_{\mathrm{Cond}} = N_0 \int \mathrm{d}^3x \; \Phi_0^*(\mathbf{x}) \left[\frac{\mathbf{p}^2}{2m_{\mathrm{B}}} + V^B(\mathbf{x}) + \frac{g_{\mathrm{B}}}{2} N_0 |\Phi_0(\mathbf{x})|^2\right] \Phi_0(\mathbf{x}) \tag{12}$$

for the condensate and

$$\hat{H}_{\mathrm{Bog}} = \sum_{\mathbf{k}} E_{\mathbf{k}} \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} \tag{13}$$

for the collective excitations (Bogoliubov modes) of energy $E_{\mathbf{k}}$ in the condensate, and

$$\begin{aligned} \hat{H}_{\mathrm{AB}} = g_{\mathrm{AB}} \sum_i \sum_{p,q} \hat{a}_{i,p}^{\dagger} \hat{a}_{i,q} \Bigg[& N_0 \int \mathrm{d}^3x \varphi_{i,p}(\mathbf{x}) \varphi_{i,q}(\mathbf{x}) |\Phi_0(\mathbf{x})|^2 \\ & + \sqrt{N_0} \sum_{\mathbf{k}} \hat{c}_{\mathbf{k}} \int \mathrm{d}^3x \varphi_{i,p}(\mathbf{x}) \varphi_{i,q}(\mathbf{x}) \left(\Phi_0^*(\mathbf{x}) u_{\mathbf{k}}(\mathbf{x}) - \Phi_0(\mathbf{x}) v_{\mathbf{k}}(\mathbf{x})\right) \\ & + \sqrt{N_0} \sum_{\mathbf{k}} \hat{c}_{\mathbf{k}}^{\dagger} \int \mathrm{d}^3x \varphi_{i,p}(\mathbf{x}) \varphi_{i,q}(\mathbf{x}) \left(\Phi_0(\mathbf{x}) u_{\mathbf{k}}^*(\mathbf{x}) - \Phi_0^*(\mathbf{x}) v_{\mathbf{k}}^*(\mathbf{x})\right) \Bigg] \end{aligned} \tag{14}$$

for the interaction Hamiltonian; the terms that are quadratic in the Bogoliubov excitation operators $\hat{c}, \hat{c}^{\dagger}$ give negligible contributions and have been omitted. The first term in (14) describes transitions between impurities' vibrational states due to the condensate, whereas the remaining terms describe similar transitions induced by the collective excitations in the condensate. In a homogeneous condensate, transitions between different vibrational eigenstates of the impurities induced by the condensate are suppressed, while all vibrational states $\varphi_{i,p}(\mathbf{x})$ get an energy shift $\delta\omega_{i,p}$,

$$g^{\mathrm{AB}} N_0 \int \mathrm{d}^3 x\, |\Phi_0|^2(\mathbf{x})\varphi_{i,p}(\mathbf{x})\varphi_{i,q}(\mathbf{x}) = \begin{cases} 0, & \text{for } p \neq q, \\ n_0 g^{\mathrm{AB}} \equiv \delta\omega_{i,p}, & \text{for } p = q \end{cases} \tag{15}$$

so the contribution of the first term in (14) can be included in the definition of $\omega_{i,p}$.

In the limit of deep, symmetric wells in each double well and separated by a high-energy barrier, the tunnelling between the adjacent wells is suppressed. In this regime, the ground states $\varphi_{i,\mathrm{L}}$ and $\varphi_{i,\mathrm{R}}$ of, respectively, the left and right wells of double well i are well separated in space with vanishing spatial overlap, their coupling to the excited states becomes negligible and the total Hamiltonian further simplifies into

$$\hat{H} = \sum_{i,} \sum_{p=\mathrm{L,R}} \hbar\omega_{i,p}\hat{n}^i_p + \sum_{\mathbf{k}} E_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} + \sum_i \hbar \sum_{p=\mathrm{L,R}} \sum_{\mathbf{k}} \left[\Omega^i_{p,\mathbf{k}}\hat{c}_{\mathbf{k}} + \Omega^{i*}_{p,\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}}\right]\hat{n}^i_p, \tag{16}$$

where we have defined the coupling frequencies

$$\Omega^i_{p,\mathbf{k}} \equiv \frac{g_{\mathrm{AB}}\sqrt{n_0}}{\hbar}(|u_{\mathbf{k}}| - |v_{\mathbf{k}}|)\int \mathrm{d}^3 x\, |\varphi_{i,p}(\mathbf{x})|^2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \tag{17}$$

and $\hat{n}^i_p \equiv \hat{a}^\dagger_{i,p}\hat{a}_{i,p}$ is the number operator of impurities in the double well i in the well $p = \mathrm{L, R}$.

We consider the case where each double well is occupied by at most one impurity atom. This allows us to describe the occupation of the left and right wells of each site in terms of pseudospin states. Introducing the Pauli operators as $\hat{n}^i_{\mathrm{L}} = (1 - \hat{\sigma}^i_z)/2$, $\hat{n}^i_{\mathrm{R}} = (1 + \hat{\sigma}^i_z)/2$, the Hamiltonian (16) takes the form of the independent boson model [18]

$$\hat{H} = \sum_{\mathbf{k}} E_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}} + \frac{\hbar}{2}\sum_{\mathbf{k}}\left\{\left[\sum_i \left(\Omega^i_{\mathrm{R},\mathbf{k}} - \Omega^i_{\mathrm{L},\mathbf{k}}\right)\hat{\sigma}^i_z + \sum_i \left(\Omega^i_{\mathrm{R},\mathbf{k}} + \Omega^i_{\mathrm{L},\mathbf{k}}\right)\right]\hat{c}_{\mathbf{k}} + \left[\sum_i \left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} - \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right)\hat{\sigma}^i_z + \sum_i \left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} + \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right)\right]\hat{c}^\dagger_{\mathbf{k}}\right\}, \tag{18}$$

where a constant energy shift has been omitted. We note that spin-boson systems with larger spin values can be realized in the same way with higher occupation of the double wells.

The effects due to quantum noise on coherent superpositions of states of a double well spin-boson Hamiltonian have been analysed in the Markovian regime. In [19]–[21] the effects of a cold atom reservoir has been analysed, while [22] has considered the effects of scattered photons, taking into account also the role of the inter-well separation. As we will show in the following section, for our system it is possible to carry out a full analysis of the impurity dynamics, going beyond the Markov approximation.

3. Exact reduced impurities dynamics

The dynamics due to the spin-boson Hamiltonian (18) is amenable to an exact analytical solution and is characterized by decoherence without dissipation [23]–[25]. The time-evolution operator

$\hat{U}(t) = \exp\left[-\mathrm{i}\hat{H}t/\hbar\right]$ corresponding to the Hamiltonian (18) can be factorized into a product of simpler exponential operators,

$$\hat{U}(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{\mathbf{k}} E_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}}t\right]$$
$$\times \exp\left[\sum_{\mathbf{k}}\left(\sum_i A^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}} - \sum_{\mathbf{k}}\left(\sum_i A^{i*}_{\mathbf{k}}(t)\hat{\sigma}^i_z + \alpha^*_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right]$$
$$\times \exp\left[\mathrm{i}\hbar^2\sum_{\mathbf{k}} f_{\mathbf{k}}(t)\Re\sum_{ij}\frac{\left(\Omega^i_{\mathrm{R},\mathbf{k}} - \Omega^i_{\mathrm{L},\mathbf{k}}\right)\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} - \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)}{4E^2_{\mathbf{k}}}\hat{\sigma}^i_z\hat{\sigma}^j_z\right]$$
$$\times \exp\left[\mathrm{i}\hbar^2\sum_{\mathbf{k}} f_{\mathbf{k}}(t)\Re\sum_{i}\frac{\left(\Omega^i_{\mathrm{R},\mathbf{k}} - \Omega^i_{\mathrm{L},\mathbf{k}}\right)\sum_j\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} + \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)}{2E^2_{\mathbf{k}}}\hat{\sigma}^i_z\right]$$
$$\times \exp\left[\mathrm{i}\hbar^2\sum_{\mathbf{k}} f_{\mathbf{k}}(t)\frac{\sum_i\left(\Omega^i_{\mathrm{R},\mathbf{k}} + \Omega^i_{\mathrm{L},\mathbf{k}}\right)\sum_j\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} + \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)}{4E^2_{\mathbf{k}}}\right], \tag{19}$$

where the functions

$$f_{\mathbf{k}}(t) = \frac{E_{\mathbf{k}}}{\hbar}t - \sin\frac{E_{\mathbf{k}}}{\hbar}t, \tag{20}$$

$$A^i_{\mathbf{k}}(t) = \frac{\hbar\left(1 - \mathrm{e}^{\mathrm{i}E_{\mathbf{k}}t/\hbar}\right)}{2E_{\mathbf{k}}}\left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} - \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right), \tag{21}$$

$$\alpha_{\mathbf{k}}(t) = \frac{\hbar\left(1 - \mathrm{e}^{\mathrm{i}E_{\mathbf{k}}t/\hbar}\right)}{2E_{\mathbf{k}}}\sum_i\left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} + \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right), \tag{22}$$

have been introduced for ease of notation. Details of the derivation of (19) for the time-evolution operator are given in appendix A. As in this paper, we are interested in the irreversible collective decoherence of the impurities we will focus our attention on the conditional displacement operator

$$\hat{U}_{\mathrm{D}}(t) = \prod_{\mathbf{k}}\hat{U}_{\mathbf{k},\mathrm{D}}(t), \tag{23}$$

$$\hat{U}_{\mathbf{k},\mathrm{D}}(t) \equiv \exp\left[\left(\sum_i A^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}} - \left(\sum_i A^{i*}_{\mathbf{k}}(t)\hat{\sigma}^i_z + \alpha^*_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right]. \tag{24}$$

Indeed this operator is the one responsible of the decoherence of impurities as it induces entanglement between them and the reservoir. Labelling the state of the impurities as $|\{n_p\}\rangle = |\{n_1, n_2, n_3, \ldots\}\rangle$ with $n_p = 0, 1$ denoting the presence of the atom, respectively, in the left or right well, the matrix elements of reduced density operator of the impurities are

$$\rho_{\{n_p\},\{m_p\}}(t) = \exp\left[-\Gamma_{\{n_i\},\{m_i\}}(t)\right]\rho_{\{n_p\},\{m_p\}}(0)\exp\left\{\mathrm{i}\Theta_{\{n_p\},\{m_p\}}(t)\right\}$$
$$\times\exp\left\{\mathrm{i}\Xi_{\{n_p\},\{m_p\}}(t)\right\}\exp\left\{\mathrm{i}\Delta_{\{n_p\},\{m_p\}}(t)\right\}. \tag{25}$$

Assuming that each mode of the bosonic environment is in a mixed state $\rho_{\mathbf{k}}$ at equilibrium at temperature T the decay exponent contains all the information concerning the time dependence of the decoherence process and takes the form

$$\Gamma_{\{n_i\},\{m_i\}}(t) = \hbar^2 \sum_{\mathbf{k}} \frac{\left(1-\cos\frac{E_{\mathbf{k}}}{\hbar}t\right)}{E_{\mathbf{k}}^2} \left|\sum_i [m_i - n_i]\left(\Omega^i_{\mathrm{R},\mathbf{k}} - \Omega^i_{\mathrm{L},\mathbf{k}}\right)\right|^2 \coth\frac{\beta E_{\mathbf{k}}}{2} \tag{26}$$

with $\beta = 1/K_{\mathrm{B}}T$. The phase factors $\Theta_{\{n_p\},\{m_p\}}(t)$, $\Xi_{\{n_p\},\{m_p\}}(t)$ and $\Delta_{\{n_p\},\{m_p\}}(t)$, whose specific form is given in appendix B, do not play any role in the decoherence [26]. They contain, however, interesting information on the effective coupling between the pseudospins induced by the condensate and will be analysed in a future paper [27].

4. Results for the decoherence

As mentioned in the introduction, we shall assume that the impurity atoms are trapped by an optical (super)lattice, whose form can be controlled and varied in time with great accuracy [28, 29]. The coupling frequencies $\Omega^i_{p,\mathbf{k}}$ are accordingly evaluated in appendix C assuming an optical lattice, with identical, double wells in each site, and deep trapping of impurity atoms in their wells, with identical confinement in each direction. Atomic wavefunctions can then be approximated by harmonic oscillator ground states of variance parameter $\sigma = \sqrt{\hbar/(m\omega)}$ [30], where ω is the corresponding harmonic frequency. As will be clear shortly, σ acts as a natural cutoff parameter, quenching the coupling with high-frequency modes.

Specifically, we consider ^{23}Na impurity atoms trapped in a far-detuned optical lattice and a ^{87}Rb condensate. The condensate density is $n_0 = 10^{20}\,\mathrm{m}^{-3}$, the lattice wavelength is $\lambda = 600\,\mathrm{nm}$, and we have taken $2L = \lambda/2$ and $D = 2L$. The depth of the optical lattice is described by the parameter $\alpha \equiv V_0/E_{\mathrm{R}}$, V_0 being the optical lattice potential maximum intensity and $E_{\mathrm{R}} = \hbar^2k^2/(2m)$ the recoil energy of impurity atoms in the lattice; in our evaluations we put $\alpha = 20$. Finally, we assume $a_{\mathrm{AB}} = 55a_0$ [31], where a_0 is the Bohr radius, for the scattering length of impurities–condensate mixtures. This parameter can be modified in laboratory with the help of Feshbach resonances.

4.1. Single impurity decoherence

We first examine the decoherence exponent of a single impurity

$$\Gamma_0(t) \equiv \Gamma_{\{0\},\{1\}}(t) \equiv \hbar^2 \sum_{\mathbf{k}} \frac{\left(1-\cos\frac{E_{\mathbf{k}}}{\hbar}t\right)}{E_{\mathbf{k}}^2} \coth\frac{\beta E_{\mathbf{k}}}{2} \left|\Omega^1_{\mathrm{R},\mathbf{k}} - \Omega^1_{\mathrm{L},\mathbf{k}}\right|^2 . \tag{27}$$

This quantity, which will be a useful benchmark in our analysis of the collective decoherence of impurity pairs, already shows interesting features. Assuming, from now on, that the condensate is at temperature $T = 0$, we obtain

$$\Gamma_0(t) = 8g^2_{\mathrm{AB}}n_0 \sum_{\mathbf{k}} (|u_{\mathbf{k}}| - |v_{\mathbf{k}}|)^2\, \mathrm{e}^{-k^2\sigma^2/2} \frac{\sin^2\frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}^2} \sin^2(\mathbf{k}\cdot\mathbf{L})\,. \tag{28}$$

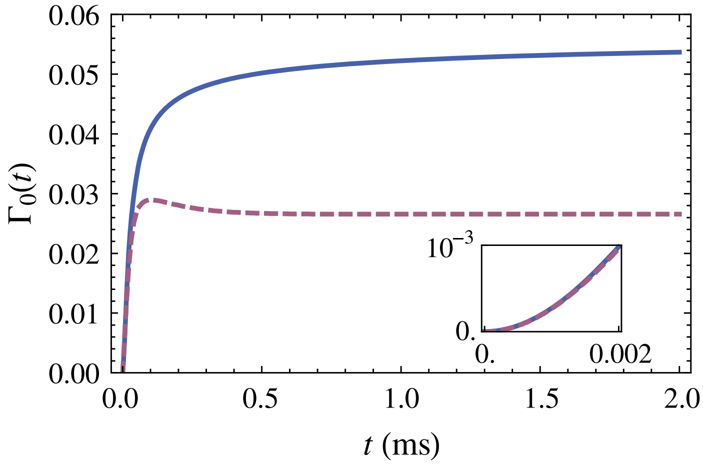

Figure 2. $\Gamma_0(t)$ versus time for a single impurity atom interacting with free bosons (solid line) and with a bosonic condensate (dashed line) in three dimensions. The inset shows $\Gamma_0(t)$ for very short times $0 \leqslant t \leqslant 2\,\mu$s.

We note the dependence of $\Gamma_0(t)$ on the length $\mathbf{L}$, where $2\mathbf{L}$ is the distance between two wells within each site. The presence of the factor $\sin^2(\mathbf{k}\cdot\mathbf{L})$ supresses the effect of the reservoir modes at small $\mathbf{k}$. This is clearly understandable: environment modes whose wavelength is longer than $\mathbf{L}$ cannot 'resolve' the spatially separated wells within each site. The consequences of this fact will be clear shortly. Replacing the sum over discrete modes to a continuum with the usual rule $V^{-1}\sum_{\mathbf{k}} \to (2\pi)^{-3}\int \mathrm{d}\mathbf{k}$, choosing x as azimuthal axis and using well-known relations for Bogoliubov modes [32], we finally obtain

$$\Gamma_0^{c}(t) = \frac{2g_{AB}^2 n_0}{\pi^2}\int_0^\infty \mathrm{d}k \left[k^2 \mathrm{e}^{-k^2\sigma^2/2} \frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}\,(\epsilon_{\mathbf{k}} + 2g_{\mathrm{B}}n_0)} \right] \left(1 - \frac{\sin 2kL}{2kL}\right). \tag{29}$$

The superscript c is to remind us that we are dealing with impurities interacting with a condensate. For the special case of a bath of non-interacting bosons $\Gamma_0^{n.i.}(t)$ is obtained from (29) simply imposing $g_{\mathrm{B}} = 0$ and $E_{\mathbf{k}} = \epsilon_{\mathbf{k}}$. Let us point out that the spectral density, which reads

$$J(\omega) \equiv \sum_{\mathbf{k}} |\Omega_{\mathrm{R},\mathbf{k}} - \Omega_{\mathrm{L},\mathbf{k}}|^2 \delta(\hbar\omega - E_{\mathbf{k}}), \tag{30}$$

has a nontrivial form, which at small frequencies, scales as $\omega^{\mathrm{d}+2}$ for the interacting case, where d is the dimensionality of the condensate, and as $\omega^{\mathrm{d}/2}$ for the non-interacting case. It is worth noticing that while the former case is always superohmic, the latter is subohmic, ohmic and superohmic depending on the dimensionality of the environment. Note that the high power in $J(\omega)$ is due to the fact that the bath has to 'resolve' the structure of the impurity, formally again the factor $\sin^2(\mathbf{k}\cdot\mathbf{L})$. Furthermore, as already pointed out, no *ad hoc* cutoff frequency ω_c needs to be inserted but appears naturally in the decaying exponential of variance σ in (29).

Figure 2 shows clearly that the impurity maintains much of its coherence at long times. Such survival is due to the above-mentioned suppressed effect of soft modes, which are responsible for the long time behaviour of $\Gamma_0(t)$, and is more pronounced when the environment consists of a condensate than in the case of a reservoir consisting of free bosons. This can be intuitively described in terms of greater 'stiffness' of the condensate whose Bogoliubov modes are less displaced by the coupled impurity. The condensate is even able to give some coherence back to the impurity, since $\Gamma_0^c(t)$ is not monotonic in time. Oscillations of coherence in spin-boson systems were predicted in [24] (and even earlier, in a different context, in [33]).

We can distinguish three stages in the dynamics of the Γ_0. In the first stage $\Gamma_0(t) \propto t^2$, as can be easily seen from a series expansion of (29). This very short stage, shown in the inset of figure 2, corresponds to coherent dynamics. The second stage corresponds to a Markovian behaviour, i.e. $\Gamma_0(t) \propto t$, and lasts a few tens of microseconds. Finally, in the third stage $\Gamma_0(t)$ saturates to a stationary value. This behaviour calls for particular caution in treating an environment of (free or interacting) bosons as a Markovian reservoir for atomic impurities immersed in it, which is clearly not the case in the present situation.

4.2. Collective decoherence of two impurities

Decoherence of quantum systems in a common environment is characterized by collective decoherence. It is well known that two spins interacting with the same bosonic reservoir with a spin-boson interaction Hamiltonian like the one discussed in this paper show sub- and superdecoherence [23]. Put simply, the decoherence rate of the two spins is not simply $2\Gamma_0(t)$ but, according to the initial state of the spins, much smaller or larger. In this final section of the present paper, we analyse the specific features of collective decoherence in our system.

For two pseudospins, three decoherence parameters appear in the density matrix elements independently of the exact form of the impurities' state. One is $\Gamma_0(t)$ and appears in elements such as $\rho_{0,0;0,1}(t)$, $\rho_{0,1;1,1}(t)$, etc which corresponds to individual dephasing of each impurity atom; two more parameters $\Gamma_1(t)$ and $\Gamma_2(t)$ appear in elements such as $|\rho_{0,0;1,1}(t)| = \exp[-\Gamma_1(t)]|\rho_{0,0;1,1}(0)|$ and $|\rho_{0,1;1,0}(t)| = \exp[-\Gamma_2(t)]|\rho_{0,1;1,0}(0)|$, and corresponds to decay of the coherences between states with the particles in the same or in the opposite side, respectively, of the double well. For two pseudospins at distance $2\mathbf{D} = 4L$, these two parameters are

$$\Gamma_1(t) \equiv \Gamma_{\{0,0\},\{1,1\}}(t) = \hbar^2 \sum_{\mathbf{k}} \frac{\left(1-\cos\frac{E_{\mathbf{k}}}{\hbar}t\right)}{E_{\mathbf{k}}^2} \coth\frac{\beta E_{\mathbf{k}}}{2} \left|\left(\Omega^1_{\mathrm{R},\mathbf{k}} - \Omega^1_{\mathrm{L},\mathbf{k}} + \Omega^2_{\mathrm{R},\mathbf{k}} - \Omega^2_{\mathrm{L},\mathbf{k}}\right)\right|^2$$

$$= 32 g_{\mathrm{AB}}^2 n_0 \sum_{\mathbf{k}} (|u_{\mathbf{k}}| - |v_{\mathbf{k}}|)^2\, \mathrm{e}^{-k^2\sigma^2/2} \frac{\sin^2\frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}^2} \sin^2(\mathbf{k}\cdot\mathbf{L}) \cos^2(\mathbf{k}\cdot\mathbf{D}), \quad (31)$$

$$\Gamma_2(t) \equiv \Gamma_{\{1,0\},\{0,1\}}(t) = \hbar^2 \sum_{\mathbf{k}} \frac{\left(1-\cos\frac{E_{\mathbf{k}}}{\hbar}t\right)}{E_{\mathbf{k}}^2} \coth\frac{\beta E_{\mathbf{k}}}{2} \left|\left(\Omega^1_{\mathrm{R},\mathbf{k}} - \Omega^1_{\mathrm{L},\mathbf{k}} - \Omega^2_{\mathrm{R},\mathbf{k}} + \Omega^2_{\mathrm{L},\mathbf{k}}\right)\right|^2$$

$$= 32 g_{\mathrm{AB}}^2 n_0 \sum_{\mathbf{k}} (|u_{\mathbf{k}}| - |v_{\mathbf{k}}|)^2\, \mathrm{e}^{-k^2\sigma^2/2} \frac{\sin^2\frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}^2} \sin^2(\mathbf{k}\cdot\mathbf{L}) \sin^2(\mathbf{k}\cdot\mathbf{D}). \quad (32)$$

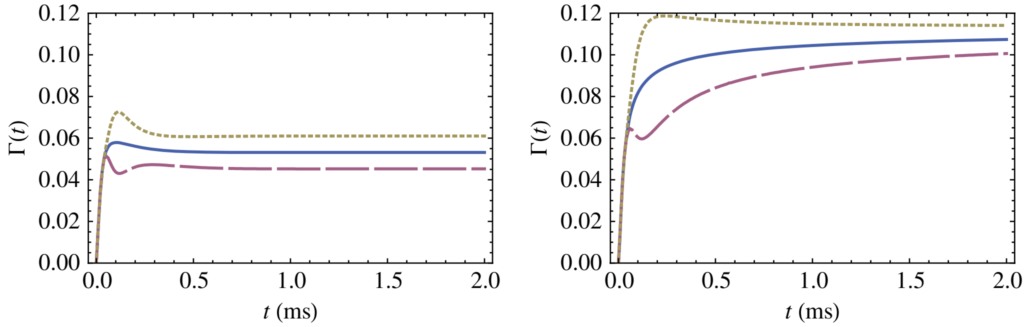

Figure 3. $\Gamma_1(t)$ (dashed line), $\Gamma_2(t)$ (dotted line), and $2\Gamma_0(t)$ (solid line) versus time for a pair of impurity atoms at a distance $2D = 4L$ (see text), immersed in a condensate (left) and in an environment of free bosons (right) in three dimensions.

Calculations similar to those performed for Γ_0 give for a condensate environment

$$\Gamma_1^{c}(t) = \frac{2g_{AB}^2 n_0}{\pi^2}\int_0^\infty dk\, k^2 e^{-k^2\sigma^2/2}\,\frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}\,(\epsilon_{\mathbf{k}}+2g_B n_0)} \times\left(2-2\frac{\sin 2kL}{2kL}+2\frac{\sin 2kD}{2kD}-\frac{\sin 2k(L+D)}{2k(L+D)}-\frac{\sin 2k(D-L)}{2k(D-L)}\right) \equiv 2\Gamma_0(t)-\delta^{c}(t), \tag{33}$$

$$\Gamma_2^{c}(t) = \frac{2g_{AB}^2 n_0}{\pi^2}\int_0^\infty dk\, k^2 e^{-k^2\sigma^2/2}\,\frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar}t}{E_{\mathbf{k}}\,(\epsilon_{\mathbf{k}}+2g_B n_0)} \times\left(2-2\frac{\sin 2kL}{2kL}-2\frac{\sin 2kD}{2kD}+\frac{\sin 2k(L+D)}{2k(L+D)}+\frac{\sin 2k(D-L)}{2k(D-L)}\right) \equiv 2\Gamma_0(t)+\delta^{c}(t). \tag{34}$$

In the above equations, it is easy to identify the term $\delta^c(t)$ which quantifies the deviation to the dechoherence exponent $2\Gamma_0$ typical of the decoherence of two impurities interacting with independent environments. Note that while Γ_0 depends only on $\mathbf{L}$, i.e. on the spatial size of the double well, δ depends nontrivially on $\mathbf{L} \pm \mathbf{D}$, i.e. on the distance between the impurities of different wells. As before the special case of a bath of non-interacting bosons $\Gamma_1^{n.i}(t)$, $\Gamma_2^{n.i.}(t)$ are obtained from the above equations (33) simply imposing $g_B = 0$ and $E_{\mathbf{k}} = \epsilon_{\mathbf{k}}$.

As in the case of single impurity decoherence the impurities do not loose all their coherence: Γ_1 and Γ_2 saturate to a stationary value that can be varied with the help of Feshbach resonances. Furthermore figure 3 shows that in a system of two impurities coherence oscillations appear, both for interacting and non-interacting bosons in the environment (even more pronounced oscillation are shown in figure 5). Such coherence revival is due to the collective nature of the coupling, as quantified by $\delta^c(t)$ ($\delta^{n.i.}(t)$ for free bosons). As shown

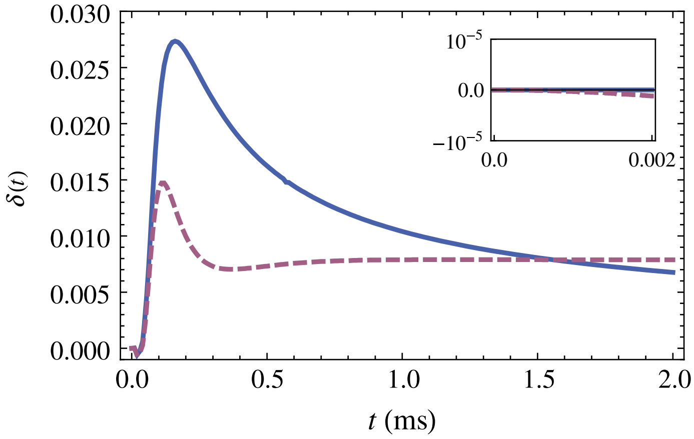

Figure 4. $\delta^c(t)$ (dashed line) and $\delta^{n.i.}(t)$ (solid line) versus time for a pair of impurity atoms in a 3D environment. The inset shows $\delta(t)$ for very short times $0 \leqslant t \leqslant 2\,\mu$s.

in figure 4 also the $\delta(t)$ are characterized by three different timescales comparable to those analysed for $\Gamma_0(t)$. In the first stage, the difference $|\delta(t)|$ is negligible, since the presence of each impurity cannot have modified yet the environment seen by the other one; in the second stage, corresponding to the Markovian dynamics, the difference $|\delta(t)|$ steadily grows up; and in the third stage it decreases, reaching a stationary value.

For a pair of impurities we observe super- and sub-decoherences; however, with a peculiarity which is characteristic of the system here considered. Indeed we observe sub-decoherence in $\Gamma_1 \equiv \Gamma_{\{0,0\},\{1,1\}}$ and super-decoherence with $\Gamma_2 \equiv \Gamma_{\{1,0\},\{0,1\}}$, at variance with what one observes in a standard spin-boson model, where their role would be exchanged [23]. This different behaviour is due to the particular configuration of our system: Γ_1 gets contribution from superpositions of the states $|0,0\rangle$ and $|1,1\rangle$, where the atoms sit in wells with identical distance, whereas the states $|0,1\rangle$ and $|1,0\rangle$, contributing to Γ_2, correspond to atoms sitting in wells with different separations.

Further insight on the features of the collective decoherence is gained by considering the decoherence of impurities sitting in sites which are at a larger distance than $2D = 4L = 600$ nm. In figure 5, we plot the decoherence exponents for impurities trapped in lattice sites at distances $2D = 8L$, $16L$ and $40L$, respectively. These plots suggest the following picture: initially the impurities decohere independently, as if they were each immersed in its own environment; at some later time, the environment correlations due to the impurities act back on them and give rise to oscillating deviations from $2\Gamma_0(t)$. The onset time of these oscillations depends on the separation: the larger the separation, the later the onset. On the other hand, the correlations become weaker as the distance increases and the oscillations become consequently smaller in amplitude. At large separation (here, approximately $40L$), the parameters Γ_1 and Γ_2 are hardly discernible from $2\Gamma_0$, since the environment correlations induced by the impurities vanish. Similar features in a related context are reported in [34]. In summary, $\Gamma_1(t)$ and $\Gamma_2(t)$ also prompt information about the correlation length of the environment.

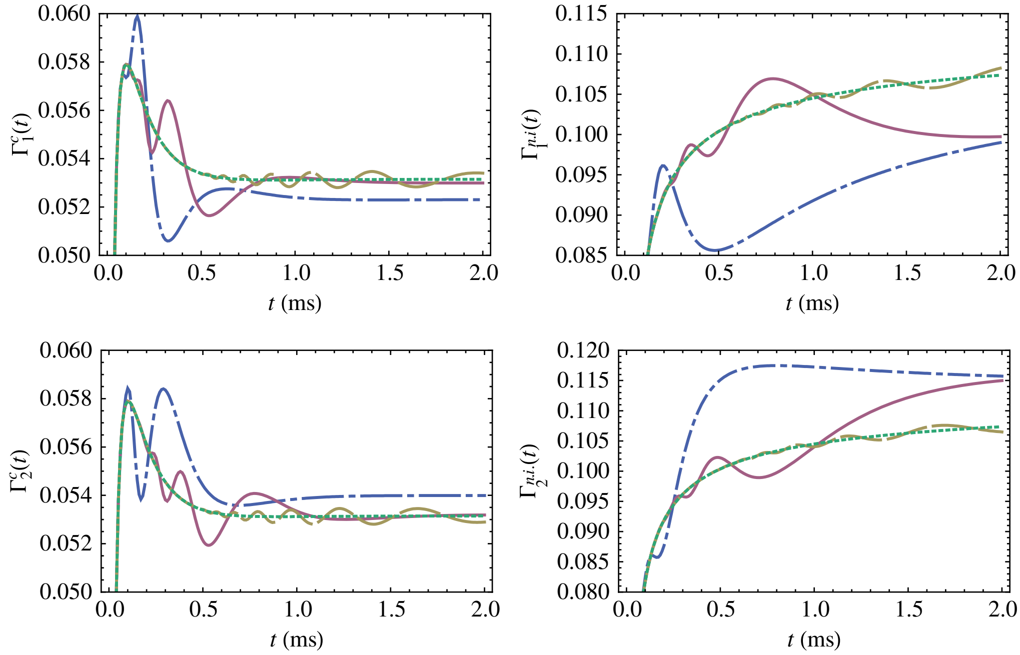

Figure 5. $\Gamma_1(t)$ (top) and $\Gamma_2(t)$ (bottom) versus time for a pair of impurity atoms interacting with a bosonic condensate (left) and with free bosons (right) in three dimensions for different distances between the impurities: $2D = 8L$ (dash-dotted line), $2D = 16L$ (solid line), and $2D = 40L$ (dashed line); $2\Gamma_0(t)$ (dotted line) is also shown for comparison.

4.3. Decoherence in one dimension

Finally, we examine the decoherence process in a 1D condensate. Since, as previously discussed, the spectral density (30) is superohmic for an interacting gas, but subohmic for a free Bose gas, we expect qualitative different results for the two cases, in contrast to the 3D case. The decay exponents in one dimension $\gamma(t)$ become

$$\gamma_0^c(t) = \frac{4g_{AB}^2 n_0}{\pi} \int_{-\infty}^{\infty} dk \left[e^{-k^2\sigma^2/2} \frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar} t}{E_{\mathbf{k}} (\epsilon_{\mathbf{k}} + 2g_B n_0)} \right] \sin^2 kL \tag{35}$$

for one impurity and

$$\gamma_1^c(t) = \frac{4g_{AB}^2 n_0}{\pi} \int_{-\infty}^{\infty} dk \left[e^{-k^2\sigma^2/2} \frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar} t}{E_{\mathbf{k}} (\epsilon_{\mathbf{k}} + 2g_B n_0)} \right] \sin^2(kL) \cos^2(kD)$$
$$\equiv 2\gamma_0(t) - \delta^c(t), \tag{36}$$

$$\gamma_2^c(t) = \frac{4g_{AB}^2 n_0}{\pi} \int_{-\infty}^{\infty} dk \left[e^{-k^2\sigma^2/2} \frac{\sin^2 \frac{E_{\mathbf{k}}}{2\hbar} t}{E_{\mathbf{k}} (\epsilon_{\mathbf{k}} + 2g_B n_0)} \right] \sin^2(kL) \sin^2(kD)$$
$$\equiv 2\gamma_0(t) + \delta^c(t) \tag{37}$$

for two impurities in a condensate. The behaviour of these parameters critically depends on the nature of the environment, see figure 6. In particular, decoherence in a 1D sample of free bosons

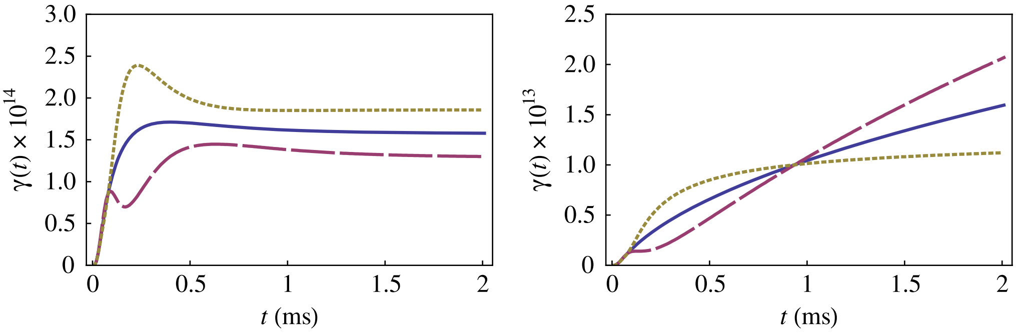

Figure 6. $\gamma_1(t)$ (dashed line), $\gamma_2(t)$ (dotted line), and $2\gamma_0(t)$ (solid line) versus time for a pair of impurity atoms immersed in a condensate (left) and in an environment of free bosons (right) in one dimension. The separation between two impurity atoms is $2D = 4L$.

becomes Markovian, in agreement with the naive expectation, due to its subohmic spectral density.

5. Conclusions

We have shown how a system of impurity atoms trapped in an array of double wells, interacting with a cold atomic gas, is described, in a suitable regime, by a spin-boson Hamiltonian. The specific nature of our system, in which the pseudospins associated with the presence of an impurity in the right/left well of each site have a spatial dimension, introduces peculiar features in the decoherence of a single impurity as well as in the collective decoherence, with the persistence of coherence at long times, the presence of coherence oscillations and counterintuitive super/subdecoherent states.

We have shown in particular that for a three-dimensional bath one never has a Markovian behaviour. A 1D bath is in this respect more interesting since one can go from a non-Markovian to a Markovian behaviour just by tuning the interaction of the bath.

As a final comment we would like to say a few words about the role of the quadratic terms in the Bogoliubov operators which we have neglected in our derivation of Hamiltonian (14). Although a detailed study of their effects is beyond the scope of the present paper, we would like to point out that their effects are negligible with respect to the linear terms we have analysed here. One can show that their inclusion amounts to taking into account elastic scattering of Bogoliubov particles, which is simply responsible of an energy shift, inelastic scattering processes and Bogoliubov pair creation and annihilation. In these two latter additional terms the length of wave vectors k that can play some role in the impurities' dynamics is limited from below by the finite size of the condensate and from above by cutoff parameter σ^{-1}. It can be shown that, in this frequency range, the coupling constants of the neglected processes are, for the values of parameters assumed in our analysis, three orders of magnitude smaller than the coupling constants $\hbar\Omega^i_{n,\mathbf{k}}$ of the linear terms. As a consequence, a rough estimate leads us to suppose that any possible relevant effect of the quadratic terms in the Hamiltonian would become apparent at timescales that are three orders of magnitude larger than those examined in this paper.

Acknowledgments

We acknowledge financial support from MIUR through the project PRIN 2006 'Collective decoherence in engineered reservoirs' and from EUROTECH S.p.A. GDC is supported by the Spanish Ministry of Science and Innovation through the program Juan de la Cierva. AR acknowledges support also by the EuroQUAM FerMix program.

Appendix A. Disentangling the time-evolution operator

The factorization of the time-evolution operator $\hat{U}(t) = \exp[-\mathrm{i}\hat{H}t/\hbar]$ is often an impossible task. When the Hamiltonian contains operators forming a Lie algebra the transformation of $\hat{U}(t)$ into a product of simpler exponential operators is however possible in some cases [35]. Here, we show a practical way to transform $\hat{U}(t)$, which we write as

$$\hat{U}(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{\mathbf{k}} E_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}}t\right]\exp\left[\sum_{\mathbf{k}}\left(\sum_i A^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}}\right] \times \exp\left[-\sum_{\mathbf{k}}\left(\sum_i B^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \beta_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right]\hat{U}_{\mathrm{R}}(t), \tag{A.1}$$

where $\hat{U}_{\mathrm{R}}(t)$ is to be determined, as well as the quantities $A^i_{\mathbf{k}}(t)$, $B^i_{\mathbf{k}}(t)$, $\alpha_{\mathbf{k}}(t)$ and $\beta_{\mathbf{k}}(t)$. Since at $t = 0$ the time-evolution operator $\hat{U}$ reduces to the identity operator, $A^i_{\mathbf{k}}(0) = B^i_{\mathbf{k}}(0) = \beta_{\mathbf{k}}(0) = \alpha_{\mathbf{k}}(0) = 0$. All unknown quantities can be found with the help of the relation

$$\hat{H} = \mathrm{i}\hbar\left[\mathrm{d}\hat{U}(t)/\mathrm{d}t\right]\hat{U}^{-1}(t), \tag{A.2}$$

which holds for any time-independent Hamiltonian and of the relation

$$\mathrm{e}^{\hat{X}}\hat{Y}\mathrm{e}^{-\hat{X}} = \hat{Y} + [\hat{X},\hat{Y}] + \tfrac{1}{2}[\hat{X},[\hat{X},\hat{Y}]] + \tfrac{1}{6}[\hat{X},[\hat{X},[\hat{X},\hat{Y}]]] + \cdots \tag{A.3}$$

for arbitrary operators $\hat{X}$ and $\hat{Y}$. After inserting the expression (A.1) for the time-evolution operator $\hat{U}(t)$ in the right-hand side of (A.2), a comparison with the Hamiltonian (18) leads to the expressions

$$A^i_{\mathbf{k}}(t) = \frac{\hbar\left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} - \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right)}{2E_{\mathbf{k}}}\left(1 - \mathrm{e}^{\mathrm{i}E_{\mathbf{k}}t/\hbar}\right), \quad B^i_{\mathbf{k}}(t) = A^{i*}_{\mathbf{k}}(t), \tag{A.4}$$

$$\alpha_{\mathbf{k}}(t) = \frac{\hbar\sum_i\left(\Omega^{i*}_{\mathrm{R},\mathbf{k}} + \Omega^{i*}_{\mathrm{L},\mathbf{k}}\right)}{2E_{\mathbf{k}}}\left(1 - \mathrm{e}^{\mathrm{i}E_{\mathbf{k}}t/\hbar}\right), \quad \beta_{\mathbf{k}}(t) = \alpha^*_{\mathbf{k}}(t) \tag{A.5}$$

for $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$, and to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{\mathrm{R}}(t) = -\sum_{\mathbf{k}}\left(\sum_i \dot{B}^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \dot{\beta}_{\mathbf{k}}(t)\right)\left(\sum_j A^j_{\mathbf{k}}(t)\hat{\sigma}^j_z + \alpha_{\mathbf{k}}(t)\right)\hat{U}_{\mathrm{R}}(t) \tag{A.6}$$

for the unknown exponential operator $\hat{U}_{\mathrm{R}}(t)$, which we write as

$$\hat{U}_{\mathrm{R}}(t) = \exp\left[-\sum_{\mathbf{k}}\left(\sum_{ij}\eta^{ij}_{\mathbf{k}}(t)\hat{\sigma}^i_z\hat{\sigma}^j_z + \sum_i \mu^i_{\mathbf{k}}(t)\hat{\sigma}^i_z + \epsilon_{\mathbf{k}}(t)\right)\right]. \tag{A.7}$$

A comparison with (A.6) gives

$$\dot{\eta}^{ij}_{\mathbf{k}}(t) = \dot{B}^{i}_{\mathbf{k}}(t)A^{j}_{\mathbf{k}}(t), \quad \dot{\epsilon}_{\mathbf{k}}(t) = \dot{\beta}_{\mathbf{k}}(t)\alpha_{\mathbf{k}}(t), \quad \dot{\mu}^{i}_{\mathbf{k}}(t) = \dot{B}^{i}_{\mathbf{k}}(t)\alpha_{\mathbf{k}}(t) + \dot{\beta}_{\mathbf{k}}(t)A^{i}_{\mathbf{k}}(t) \tag{A.8}$$

that is

$$\eta^{ij}_{\mathbf{k}}(t) = -\mathrm{i}\hbar \frac{(\Omega^{i}_{\mathrm{R},\mathbf{k}} - \Omega^{i}_{\mathrm{L},\mathbf{k}})\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} - \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)}{4E_{\mathbf{k}}} \left[t + \frac{\mathrm{i}\hbar}{E_{\mathbf{k}}}\left(1 - \mathrm{e}^{-\mathrm{i}E_{\mathbf{k}}t/\hbar}\right)\right], \tag{A.9}$$

$$\epsilon_{\mathbf{k}}(t) = -\mathrm{i}\hbar \frac{\sum_{ij}(\Omega^{i}_{\mathrm{R},\mathbf{k}} + \Omega^{i}_{\mathrm{L},\mathbf{k}})\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} + \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)}{4E_{\mathbf{k}}} \left[t + \frac{\mathrm{i}\hbar}{E_{\mathbf{k}}}\left(1 - \mathrm{e}^{-\mathrm{i}E_{\mathbf{k}}t/\hbar}\right)\right], \tag{A.10}$$

$$\mu^{i}_{\mathbf{k}}(t) = -\frac{\mathrm{i}\hbar}{2E_{\mathbf{k}}}\Re\left[\left(\Omega^{i}_{\mathrm{R},\mathbf{k}} - \Omega^{i}_{\mathrm{L},\mathbf{k}}\right)\sum_{j}\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}} + \Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)\right]\left[t + \frac{\mathrm{i}\hbar}{E_{\mathbf{k}}}\left(1 - \mathrm{e}^{-\mathrm{i}E_{\mathbf{k}}t/\hbar}\right)\right]. \tag{A.11}$$

Moreover, using Glauber's relation

$$\exp\left[\sum_{\mathbf{k}} g_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}\right]\exp\left[-\sum_{\mathbf{k}} g^{*}_{\mathbf{k}}\hat{c}_{\mathbf{k}}\right] = \exp\left[\sum_{\mathbf{k}}\left(g_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}} - g^{*}_{\mathbf{k}}\hat{c}_{\mathbf{k}}\right)\right]\exp\left[\frac{1}{2}\sum_{\mathbf{k}}|g_{\mathbf{k}}|^{2}\right] \tag{A.12}$$

the two exponentials linear in Bogoliubov operators can be merged into

$$\begin{aligned}
&\exp\left[\sum_{\mathbf{k}}\left(\sum_{i} A^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}}\right]\exp\left[-\sum_{\mathbf{k}}\left(\sum_{i} B^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \beta_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right] \\
&\quad = \exp\left\{\left[\sum_{\mathbf{k}}\left(\sum_{i} A^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}} - \sum_{\mathbf{k}}\left(\sum_{i} A^{i*}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha^{*}_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right]\right\} \\
&\quad\times \exp\left\{\frac{1}{2}\left[\sum_{\mathbf{k}}\left(\sum_{i} A^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha_{\mathbf{k}}(t)\right)\left(\sum_{j} A^{j*}_{\mathbf{k}}(t)\hat{\sigma}^{j}_{z} + \alpha^{*}_{\mathbf{k}}(t)\right)\right]\right\}
\end{aligned} \tag{A.13}$$

and the contribution of the last exponential can be included in $U_R(t)$. Performing some commutations where it is possible, the time-evolution operator becomes

$$\begin{aligned}
\hat{U}(t) &= \exp\left[-\frac{\mathrm{i}}{\hbar}\sum_{\mathbf{k}} E_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}}t\right]\exp\left[-\sum_{\mathbf{k}}\left(\sum_{ij}\eta^{ij}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z}\hat{\sigma}^{j}_{z} + \sum_{i}\mu^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \epsilon_{\mathbf{k}}(t)\right)\right] \\
&\quad\times \exp\left[\sum_{\mathbf{k}}\left(\sum_{i} A^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}} - \sum_{\mathbf{k}}\left(\sum_{i} B^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \beta_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right] \\
&\quad\times \exp\left\{\frac{1}{2}\left[\sum_{\mathbf{k}}\left(\sum_{i} A^{i}_{\mathbf{k}}(t)\hat{\sigma}^{i}_{z} + \alpha_{\mathbf{k}}(t)\right)\left(\sum_{j} A^{j*}_{\mathbf{k}}(t)\hat{\sigma}^{j}_{z} + \alpha^{*}_{\mathbf{k}}(t)\right)\right]\right\}.
\end{aligned} \tag{A.14}$$

Finally, the exponential operators that do not contain bath operators commute, so the time-evolution operator can be further modified into the final form (19).

Appendix B. Derivation of the dynamics of the impurities

The action of $\hat{U}_{\mathbf{k},\mathrm{D}}(t)$ on a pure state of the whole system is

$$\hat{U}_{\mathbf{k},\mathrm{D}}(t)|\{n_p\}\rangle\langle\{m_p\}|\otimes\rho_{\mathbf{k}}\hat{U}^{\dagger}_{\mathbf{k},\mathrm{D}}(t)=|\{n_p\}\rangle\langle\{m_p\}|\otimes$$
$$\times\exp\left[\left(-\sum_j A^j_{\mathbf{k}}(t)(-1)^{n_j}+\alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}}-\left(-\sum_j A^{j*}_{\mathbf{k}}(t)(-1)^{n_j}+\alpha^*_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right]\rho_{\mathbf{k}}$$
$$\times\exp\left[-\left(-\sum_j A^j_{\mathbf{k}}(t)(-1)^{m_j}+\alpha_{\mathbf{k}}(t)\right)\hat{c}^{\dagger}_{\mathbf{k}}+\left(-\sum_j A^{j*}_{\mathbf{k}}(t)(-1)^{m_j}+\alpha^*_{\mathbf{k}}(t)\right)\hat{c}_{\mathbf{k}}\right] \tag{B.1}$$

and the density matrix elements $\rho_{\{n_p\},\{m_p\}}(t)$ of the impurities are obtained by tracing over the bath,

$$\rho_{\{n_p\},\{m_p\}}(t)=\exp\left\{\mathrm{i}\Theta_{\{n_p\},\{m_p\}}(t)\right\}\exp\left\{\mathrm{i}\Xi_{\{n_p\},\{m_p\}}(t)\right\}\rho_{\{n_p\},\{m_p\}}(0)$$
$$\times\langle\{n_p\}|\prod_{\mathbf{k}}\mathrm{Tr}_{\mathrm{B},\mathbf{k}}\left\{\hat{U}_{\mathbf{k},\mathrm{D}}(t)|\{n_p\}\rangle\langle\{m_p\}|\otimes\rho_{\mathbf{k}}\hat{U}^{\dagger}_{\mathbf{k},\mathrm{D}}(t)\right\}|\{m_p\}\rangle, \tag{B.2}$$

where $\mathrm{Tr}_{\mathrm{B},\mathbf{k}}$ denotes the trace over each Bogoliubov mode of the environment and the phases

$$\Theta_{\{n_p\},\{m_p\}}(t)=\hbar^2\sum_{\mathbf{k}}\frac{f_{\mathbf{k}}(t)}{4E^2_{\mathbf{k}}}\sum_{ij}\Re\left(\Omega^i_{\mathrm{R},\mathbf{k}}-\Omega^i_{\mathrm{L},\mathbf{k}}\right)\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}}-\Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)\left[(-1)^{n_i+n_j}-(-1)^{m_i+m_j}\right], \tag{B.3}$$

$$\Xi_{\{n_p\},\{m_p\}}(t)=\hbar^2\sum_{\mathbf{k}}\frac{f_{\mathbf{k}}(t)}{E^2_{\mathbf{k}}}\Re\sum_j\left(\Omega^{j*}_{\mathrm{R},\mathbf{k}}+\Omega^{j*}_{\mathrm{L},\mathbf{k}}\right)\sum_i\left(\Omega^i_{\mathrm{R},\mathbf{k}}-\Omega^i_{\mathrm{L},\mathbf{k}}\right)(n_i-m_i) \tag{B.4}$$

come from the unitary operators in (19). Performing cyclic permutation inside the trace and using the identity $\exp(\hat{M})\exp(\hat{N})=\exp(\hat{M}+\hat{N})\exp[\hat{M},\hat{N}]/2$, which holds for operators $\hat{M}$ and $\hat{N}$ that commute with their commutator, the trace $\mathrm{Tr}_{\mathrm{B},\mathbf{k}}$ in (25) becomes

$$\exp\left[\mathrm{i}\Im\left(\sum_j A^j_{\mathbf{k}}(t)(-1)^{n_j}-\alpha_{\mathbf{k}}(t)\right)\left(\sum_j A^{j*}_{\mathbf{k}}(t)(-1)^{m_j}-\alpha^*_{\mathbf{k}}(t)\right)\right\}$$
$$\times\mathrm{Tr}_{\mathrm{B},\mathbf{k}}\left\{\exp\left[2\sum_i(n_i-m_i)\left(A^i_{\mathbf{k}}(t)\hat{c}^{\dagger}_{\mathbf{k}}-A^{i*}_{\mathbf{k}}(t)\hat{c}_{\mathbf{k}}\right)\right]\rho_{\mathbf{k}}\right\}$$
$$\equiv\exp\left\{\mathrm{i}\Delta_{\{n_p\},\{m_p\}}(t)\right\}\mathrm{Tr}_{\mathrm{B},\mathbf{k}}\left\{\exp\left[2\sum_i(n_i-m_i)\left(A^i_{\mathbf{k}}(t)\hat{c}^{\dagger}_{\mathbf{k}}-A^{i*}_{\mathbf{k}}(t)\hat{c}_{\mathbf{k}}\right)\right]\rho_{\mathbf{k}}\right\}. \tag{B.5}$$

The trace over the thermal bath of the displacement operators is well-known [23],

$$\mathrm{Tr}_{\mathrm{B},\mathbf{k}}\left[\exp\left\{g_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}}-g^*_{\mathbf{k}}\hat{c}_{\mathbf{k}}\right\}\rho_{\mathbf{k}}\right]=\exp\left\{-\frac{|g_{\mathbf{k}}|^2}{2}\coth\frac{\beta E_{\mathbf{k}}}{2}\right\}, \tag{B.6}$$

where $\beta=(k_{\mathrm{B}}T)^{-1}$, and leads to equation (25).

Appendix C. The coupling constant in a deep optical lattice

In a deep optical lattice, the ground state wavefunctions of each well can be approximated with those of harmonic oscillators,

$$\varphi_{i,N}(\mathbf{x}) = \frac{1}{\left[\pi^3 x_0^2 y_0^2 z_0^2\right]^{1/4}} \exp\left[-\frac{(x-x_{i,N})^2}{2x_0^2} - \frac{(y-y_{i,N})^2}{2y_0^2} - \frac{(z-z_{i,N})^2}{2z_0^2}\right]. \tag{C.1}$$

Here $N = \mathrm{L}, \mathrm{R}$, and $x_0 = \sqrt{\hbar/(m\omega_x)}$, $y_0 = \sqrt{\hbar/(m\omega_y)}$, and $z_0 = \sqrt{\hbar/(m\omega_z)}$, where the ω's are the trapping frequencies of the harmonic trap approximating the lattice potential at bottom of L and R wells of the lattice site i. The coupling frequencies (17) of the spin-boson model then become

$$\begin{aligned}\Omega^i_{n,\mathbf{k}} &= \frac{g_{\mathrm{AB}}\sqrt{n_0}}{\hbar}\left(|u_{\mathbf{k}}| - |v_{\mathbf{k}}|\right)\int \mathrm{d}^3x\, |\varphi_{i,\mathrm{L}}(\mathbf{x})|^2 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \\ &= \frac{g_{\mathrm{AB}}\sqrt{n_0}}{\hbar}\left(|u_{\mathbf{k}}| - |v_{\mathbf{k}}|\right)\mathrm{e}^{-k^2\sigma^2/4}\mathrm{e}^{\mathrm{i}k_x x_{i,n}}, \qquad n = \mathrm{L}, \mathrm{R}\end{aligned} \tag{C.2}$$

having assumed identical confinement in the three directions, $\sigma = x_0 = y_0 = z_0$.

References

[1] Bloch I 2005 *J. Phys.* B **38** S629
Bloch I 2005 *Nat. Phys.* **1** 23
Bloch I 2008 *Nature* **453** 1016

[2] Greiner M, Mandel O, Esslinger T, Hänsch T W and Bloch I 2002 *Nature* **415** 39

[3] Paredes B, Widera A, Murg V, Mandel O, Fölling S, Cirac I, Shlyapnikov G V, Hänsch T W and Bloch I 2004 *Nature* **429** 277

[4] Billy J *et al* 2008 *Nature* **453** 891

[5] Roati G, D'Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 *Nature* **453** 895

[6] Jaksch D, Briegel H-J, Cirac J I, Gardiner C W and Zoller P 1999 *Phys. Rev. Lett.* **82** 1975

[7] Jaksch D and Zoller P 2005 *Ann. Phys.* **315** 52

[8] Lewenstein M, Sanpera A, Ahufinger V, Damski B, Sen De A and Sen U 2007 *Adv. Phys.* **56** 243

[9] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80** 885

[10] Recati A, Fedichev P O, Zwerger W, von Delft J and Zoller P 2005 *Phys. Rev. Lett.* **94** 040404

[11] Griessner A, Daley A J, Clark S R, Jaksch D and Zoller P 2007 *New J. Phys.* **9** 44

[12] Klein A and Fleischhauer M 2005 *Phys. Rev.* A **71** 033605

[13] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P, Garg A and Zwerger W 1987 *Rev. Mod. Phys.* **59** 1

[14] Garcia-Ripoll J J and Cirac J I 2003 *New J. Phys.* **5** 76

[15] Charron E, Tiesinga E, Mies F and Williams C 2002 *Phys. Rev. Lett.* **88** 077901

[16] Klein A, Bruderer M, Clark S R and Jaksch D 2007 *New. J. Phys.* **9** 411

[17] Stringari S and Pitaevskii L P 2003 *Bose–Einstein Condensation* (Oxford: Oxford University Press)

[18] Mahan G D 1990 *Many-Particle Physics* (New York: Plenum)

[19] Micheli A, Jaksch D, Cirac I J and Zoller P 2003 *Phys. Rev.* A **67** 013607

[20] Dalvit D A R, Dziarmaga J and Zurek W H 2000 *Phys. Rev.* A **62** 013607

[21] Louis P J, Brydon P M R and Savage C M 2001 *Phys. Rev.* A **64** 053613

[22] Huang Y P and Moore M G 2006 *Phys. Rev.* A **73** 023606

[23] Palma G M, Suominen K-A and Ekert A K 1996 *Proc. Soc R.* A **452** 567
[24] Reina J H, Quiroga L and Johnson N F 2002 *Phys. Rev.* A **65** 032326
[25] Braun D 2002 *Phys. Rev. Lett.* **89** 277901
Braun D 2005 *Phys. Rev.* A **72** 062324
[26] Averin D V and Fazio R 2003 *JETP Lett.* **78** 1162
[27] DeChiara G, Cirone M A, Palma G M and Recati A 2009 in preparation
[28] Anderlini M *et al* 2007 *Nature* **452** 448
[29] Sebby-Strabley J, Anderlini M, Jessen P S and Porto J V 2006 *Phys. Rev.* A **73** 033605
[30] Jaksch D, Bruder C, Cirac J, Gardiner C and Zoller P 1998 *Phys. Rev. Lett.* **81** 3108
[31] Weiss S B, Bhattacharya M and Bigelow N P 2003 *Phys. Rev.* A **68** 042708
[32] Castin Y 2001 *Coherent Atomic Matter Waves (Les Houches Summer School Session LXXII)* ed R Kaiser, C Westbrook and F David, pp 1–136
see also http://arXiv:cond-mat/0105058
[33] Hu B L, Paz J P and Zhang Y 1992 *Phys. Rev.* D **45** 2843
[34] Doll R, Wubs M, Hänggi P and Kohler S 2006 *Europhys. Lett.* **76** 547
[35] Wei J and Norman E 1963 *J. Math. Phys.* **4** 575
Puri R R 2001 *Mathematical Methods of Quantum Optics* (Berlin: Springer)