SONDERFORSCHUNGSBEREICH 96

LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

BERICHTE
zur
ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

STUDIES ON THE RELIABILITY OF REDUNDANT STRUCTURAL SYSTEMS

STUDIEN ZUR ZUVERLÄSSIGKEIT VON REDUNDANTEN TRAGSYSTEMEN

Heft 58/1981

Hohenbichler

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SONDERFORSCHUNGSBEREICH 96 (SFB 96)

Der SFB 96 "Zuverlässigkeit von Bauwerken" ist eine Einrichtung der Technischen Universität München und der Deutschen Forschungsgemeinschaft. Der SFB hat sich die Entwicklung baustoff- und bauartenübergreifender Sicherheitssysteme auf wahrscheinlichkeitstheoretischer Grundlage zum Ziel gesetzt.

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PREFACE

The five studies in this report represent consecutive steps of the development of a reliability theory and numerical techniques for finite parallel structural systems. The work is part of the program of project B 9: "Reliability theory of structural systems". It is believed that many reliability problems can now be handled for which up to now only trivial solutions were available. It is also believed that the same methodology can successfully be applied to many other multi-state systems, e.g. for transportation and life line networks or operation schemes.

Munich, July 1981

The authors

VORWORT

Die fünf Studien in diesem Bericht stellen aufeinanderfolgende Schritte in der Entwicklung der Theorie und von numerischen Techniken der Zuverlässigkeit von redundanten Tragwerken dar. Sie sind Teil der Arbeiten in Teilprojekt B 9: "Zuverlässigkeitstheorie von Systemen". Die Ansätze erlauben die Lösung von einer ganzen Reihe von Zuverlässigkeitsproblemen, die bislang einer Behandlung nicht zugänglich waren. Es ist zu vermuten, daß die Methoden auch auf andere Systeme mit vielen Zuständen, z.B. für Verkehrs- und Versorgungssysteme oder Ablaufschemata mit Erfolg angewandt werden können.

München, Juli 1981

Die Autoren

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AN ORDER STATISTICS APPROACH TO PARALLEL STRUCTURAL SYSTEMS

Rüdiger Rackwitz and Michael Hohenbichler

ABSTRACT

The distribution of strength of systems of a number of elements in parallel under imposed uniform strain is investigated. Daniels asymptotic results are generalized towards systems with dependent elements. Using an order statistics approach, sharper results are obtained for small to medium systems. The models presented can be applied to a large number of types of material and construction, e.g. to concrete and masonry under tensile stresses, connections made of nails, dowels or rivets and, of course, all bundle-of-threads like structures. In particular, the formula given may be used in all cases where the deformation modulus of the elements is nearly constant, i.e. strength and ultimate strain are highly positively correlated.

CIVIL ENGINEERING ABSTRACT

The exact and asymptotic distribution of strength of systems of a number of elements with independent or dependent strength properties under imposed uniform strain is investigated.

Key Words: Brittle material; Parallel systems; Safety; Statistical Analysis; Strength models; Structural Reliability

Introduction

Modelling of uncertain phenomena jeopardizing structural reliability has to a certain extent concentrated on loading variables. Much less work has been done to establish suitable statistical models for the resistance of materials against those loadings. However, realistic models for both types of uncertainties are needed for a sound analysis of structural reliability. Their development is particularly important since serious world-wide activities are underway with the intention to create a new generation of probability-based design codes.

There appear to be only two limiting cases in which a satisfying statistical theory of strength of stressed zones of arbitrary shape and size exists. On the one hand, so-called "weakest link" materials, i.e. materials which fail to sustain the applied load as soon as its weakest "element" fails, have found some interesting solutions. The best known and most usable results are nevertheless asymptotic in various ways. Apart from the famous work of Weibull [17] for independent sequences of elemental strengths leading strictly to one of the three asymptotic extreme-value distributions some additional results are available for the strength of materials whose elemental properties can be modelled by continuous random processes or spatial fields and whose first-order distribution belongs to the Gaussian family (see e.g.[1,6,15]. Although, the references mentioned are primarily concerned with large value problems, the corresponding small value results can easily be derived.

Similarly, the extreme value results for exceptional loadings modelled as renewal processes have their small value counterpart when modelling "defects" [13]. Many of these results rely on the equivalence between extreme value and first-passage time statistics of random processes.

On the other hand, certain exact or asymptotic results exist for perfectly ductile materials. In this case, given the spatial correlation structure of elemental strengths, the random strength of cross-sections is obtained by integrating the random field over finite domains. Under certain regularity conditions such integrals converge to the normal distribution with known mean and variance [16].

Very few results are available for realistic types of material where "elements" form a redundant system and whose mechanical behaviour is neither ideally ductile nor ideally brittle. The earliest and still most prominent investigation for brittle materials is due to Daniels [2]. Daniels considered a system of discrete elements in a parallel arrangement (bundles) with equal load-sharing and whose elemental strengths are positivly, independently and identically distributed. Equal load-sharing implies perfectly elastic-brittle stress-strain behaviour. The ratio of ultimate elemental strength and strain is constant. Daniels derived an exact solution for the distribution function of system strength which, unfortunately, is less amenable to numerical applications. It is, nevertheless, the only known exact analytical result in this area. But, perhaps more important, he found an asymptotic result expressing that, if

$\lim_{x\to\infty} x\,(1-F(x)) = 0$ where $F(x)$ is the distribution function of positive elemental strength X, then, the distribution function of system strength $R_n = R(X_1,\ldots,X_n)$ tends with increasing n to the normal with known mean and variance. More general asymptotic results have been obtained only recently by relaxing the assumption of independence of elemental strength [7,8,14]

As a matter of fact though, convergence to the normal distribution is extremely slow even under ideal mechanical and stochastic conditions (brittleness, stochastic independence). Therefore, more accurate solutions are desireable for small to medium number of elements. Moreover, the mechanical basis needs to be made more realistic, i.e. allowance must be made for non-linear stress-strain behaviour of the "elements" and other load-sharing rules must be implemented. The only studies in this direction known to the authors are those by Phoenix et. al.[11] and by Kersken-Bradley [9] where the first mentioned reference studied the influence of a certain mechanical interaction of adjecent elements while the latter reference arrives at quite general results for arbitrary load-sharing rules and elemental stress-strain relationships in terms of first and second moments of system strength.

In the following, some further considerations are devoted to the case of small to medium numbers of elements and equal load-sharing among them. Also, a few important large system solutions are given. Others will be discussed in a separate paper [8] as will be done with respect to the important question of

statistical uncertainties about the distribution parameters of elemental strength [12]. This paper concentrates on the probabilistic aspects of strength models. No attempt is being made to verify theoretical results by experimental observations since suitable large sample data appear to be inexisting but, also extremely difficult to obtain.

Perfectly Elastic-Brittle Material-Large Systems

Denote by $X_1, X_2, \ldots, X_n$ the sequence of uncertain strength of (perfectly brittle) elements. Then, the resistance of the system can be given by

$$R_n = R_n(X_1, \ldots, X_n) = \max\{n\,\hat{X}_1, (n-1)\,\hat{X}_2, \ldots, \hat{X}_n\} \qquad (1)$$

where $\hat{X}_1 \le \hat{X}_2 \le \ldots \le \hat{X}_n$ are the order statistics of $X_1, X_2, \ldots, X_n$. If the elemental strengths are independent, then, following Daniels [2], a suitable version of a recursion formula for the distribution function of R_n is:

$$F_{R_n}(x) = P(R_n \le x)$$

$$=: S_n(x) = (-1)^{n+1} F^n\left(\frac{s}{n}\right) - \sum_{k=1}^{n-1} \left[(-1)^k \binom{n}{k} F^k\left(\frac{x}{k}\right) S_{n-k}(x)\right] \qquad (2)$$

Daniels's asymptotic result can be expressed by:

$$\lim_{n\to\infty} P\left(\frac{R_n - E_n}{D_n} \le x\right) = \phi(x) \qquad (3)$$

in which

$$E_n = n \cdot x_o (1-F(x_o))$$

$$D_n = x_o [n\, F(x_o)(1-F(x_o))]^{1/2}$$

and x_o the unique solution of

$$x(1-F(x)) = \max \qquad (4)$$

$\phi(.)$ is the standard normal integral. The convergence to the normal distribution is not surprising since for $x = x_o$ the

probability of a component having strength $X \ge x_o$ is $p = 1 - F(x_o)$ and, thus, the number of "active" elements having at least strength $X = x_o$ is binomially distributed with mean $n\,p$ and variance $n\,p\,(1-p)$. According to the central limit theorem in probability theory (version of de Moivre/Laplace) the binomial approaches the normal distribution for large n. It is also worth noting that eq.(4) obviously implies that the mean system strength $x_o\ n\ p$, i.e. the product of a level x_o and the proportion of active elements at that level, is maximized A slight improvement of the statement in eq.(3) can be reache by replacing the normal distribution by the exact binomial distribution. The mathematical and physical meaning of both 3) the value x_o obtained by eq.(4) and the normal asymptote eq.(3) will still be discussed furtheron.

In order to show that similar results can be obtained for dependent elemental strength values it is assumed that the sequence of strength values above and below a given value forms a stationary Markovian sequence. Define a new binary random variable Z_k as (see also Figure 1):

$$Z_k = \begin{cases} 0 & \text{if } X_k < x_o \\ 1 & \text{if } X_k \ge x_o \end{cases}$$

implying that, for example, the transition probability p_{o1} becomes

$$p_{o1} = P(Z_k = 1 \,|\, Z_{k-1} = 0) = P(X_k \ge x_o \,|\, X_{k-1} < x_o) \qquad (6)$$

For an arbitrary value of x_o, the system resistance can now be written as

$$R_n = x_o \sum_{k=1}^{n} Z_k \tag{7}$$

Its exact distribution function is given by Gabriel [5]. Now, for Markovian Z_k's and large n, we have [4], pp. 314

$$E_n = n\, x_o\, p_1 \tag{8}$$

and

$$\lim_{n\to\infty}\left(\frac{D_n^2}{n}\right) = x_o^2\, p_o\, p_1\, \frac{1+p_{oo}-p_{10}}{1-p_{oo}+p_{11}} \tag{9}$$

with $p_o = \frac{p_{10}}{p_{01}+p_{10}}$ and $p_1 = \frac{p_{01}}{p_{01}+p_{10}}$ the stationary probabilities of the sequence Z_k. The distribution of R_n tends to the normal according to the central limit theorem for Markovian sequence [4]. However, it is not easy to find a sequence $\{X_k\}$ for which the Markovian property of the corresponding sequence $\{Z_k\}$ holds. For the moment, we, therefore, are content with the fact that the tendency to the normal distribution is still maintained as long as the sequence $\{X_k\}$ has a certain dependence structure, specifically being asymptotically independent, and with the fact that in the same case the asymptotically correct level x_o is found from eq.(4) as well. The proofs are beyond the scope of this paper and are given in [8].

It remains to determine mean and variance of a system with known probability law and dependence structure of elemental strength values. As an example, assume the sequence $\{X_k\}$ to be a stationary Gaussian sequence with mean μ, standard deviation σ and auto-correlation coefficient function $\rho(t-s)$ fulfilling the conditions mentioned above. For the mean of R_n defined by eq.(7) and an arbitrary value x_o we clearly have

$$E[R_n] = n\, x_o[1-\phi(u_o)] \tag{10}$$

with $u_o = \frac{x_o-\mu}{\sigma}$. Since, for $t > s$

$$E[Z_t, Z_s] = P[X_t > x_o \cap X_s > x_o] \tag{11}$$

we have

$$E[R_n^2] = x_o^2 \sum_{s=1}^{n} \sum_{t=1}^{n} E[Z_t, Z_s] \tag{12}$$

and

$$D^2[R_n] = E[R_n^2] - E^2[R_n] \tag{13}$$

In the Gaussian case, eq.(11) becomes

$$E[Z_t, Z_s] = \phi(-u_o, -u_o; \rho(t-s)) \tag{14}$$

where $\phi(h,k;\rho)$ is the standard bivariate normal integral and, hence

$$E[R_n^2] = x_o^2 \sum_{t=1}^{n} \sum_{s=1}^{n} \phi(-u_o, -u_o; \rho(t-s))$$

$$= x_o^2\, n\left[\phi(-u_o) + 2 \sum_{\tau=1}^{n-1} \left(1 - \frac{\tau}{n}\right)\phi(-u_o, -u_o; \rho(\tau))\right] \tag{15}$$

where $\tau = t - s$ has been used. Inserting eq.(10) and (15) into (13) we obtain

$$D^2[R_n] = x_o^2\, n\left[\phi(-u_o) + 2 \sum_{\tau=1}^{n-1} \left(1 - \frac{\tau}{n}\right)\, \phi(-u_o, -u_o; \rho(\tau)) - n\, \phi^2(-u_o)\right] \tag{16}$$

The binormal probabilities may be evaluated exactly, e.g. by using the tables or expansions given in [10] or may be bounded by simple expressions involving univariate normal integrals

given by Ditlevsen*) [3]. The same concept allows an important generalization to continuous parameter elemental strength processes X(t) over an intervall [0,T] (Figure 2). In this case, the summation in eq.(7) is replaced by integration. It is then easy to see that the continuous equivalents to eqs.(10) and (15) are:

$$E[R_T] = x_o\, T[1-\phi(u_o)] \tag{17}$$

$$E[R_T^2] = x_o^2\, 2\, T \int_o^T (1-\frac{\tau}{T})\,\phi(-u_o,-u_o;\rho(\tau))\, d\tau \tag{18}$$

It is also straigthforward to formally expand these results to random strength fields over given domains A although the numerical part when determining the variance of R_A may require additional restrictions on the auto-correlation structure of the random field, e.g.isotropy, or on the shape of the domain A.

It is worthwhile to discuss briefly the foregoing formulae. In Figure 3 the exact distribution of system resistance is compared with the asymptote eq.(3) for different n and normally distributed elemental strength with $E[X] = \mu = 1$ and $D[X] = \sigma = 0.2$. In this case one determines $x_o = 0.78$ implying $p = 1 - F(x_o) = 0.8618$. As mentioned, convergence of the exact to the asymptotic distribution is slow. Convergence is expected to be even slower for dependent elemental strength values. On the other hand, the asymptote is always conservative in the sense that strength realizations in excess of x_o are neglected. Furthermore, looking at eq.(1), the asymptote essentially represents one dominating term which is the one

*) For example, a first estimate is obtained from $L \le \phi(-a,-a;\rho) \le U = 2\,L$ where $L = \phi(-a)\;\phi(-a\sqrt{\frac{1-\rho}{1+\rho}})$

where the proportion of active elements equals $n(1-F(x_o))$. One might suspect that given the level x_o according to eq.(4) the correct distribution of order statistics produces better results. This is, in fact, not true. On the contrary, it is rather the choice of an inappropriate level x_o, e.g. the one determined from eq.(4), which leads to the excessive conservatism of Daniels' asymptote.

Before investigating the possibilities of using an alternative to eq.(4) the effect of auto-correlation is studied in Figure 4 for the asymptote of a Gauß-Markov sequence. As expected, correlation is important in the asymptotic case and may even be more important for small systems.

Perfectly Elastic-Brittle Material-Small Systems

It is possible to improve Daniels' results substantially. Rewriting eq.(1) as

$$R_n = \max_{1\le k\le n} \{\ldots,(n-k+1)\hat{X}_k,\ldots\} \tag{19}$$

and making use of the fact, that one term usually dominates in eq.(19), the distribution function of R_n can be bounded by

$$P(R_n \le x) \le \min_{1\le k\le n} \{P((n-k+1)\ \hat{X}_k \le x)\} \tag{20}$$

In the independent case the right-hand term probabilities can be given exactly [4]

$$P((n-k+1)\ \hat{X}_k \le x) = F_{n,k}(x)$$

$$= \sum_{j=k}^{n} \binom{n}{j}\ F^j(\frac{x}{n-k+1})\ [1-F(\frac{x}{n-k+1})]^{n-j} \tag{21}$$

which, again, may be approximated by (see [4], pp. 447)

$$P((n-k+1)\,\hat{X}_k \le x) \approx$$

$$\phi\left(\frac{x-n\left(1-\frac{k}{n}+\frac{1}{n}\right)F^{-1}\left(\frac{k}{n}\right)}{\frac{n\left(1-\frac{k}{n}+\frac{1}{n}\right)}{f\left(F^{-1}\left(\frac{k}{n}\right)\right)}\sqrt{\frac{k}{n}\,\frac{\left(1-\frac{k}{n}\right)}{n^2}\,n}}\right)=\phi\left(\frac{\frac{x}{n-k+1}-F^{-1}\left(\frac{k}{n}\right)}{\frac{1}{f\left(F^{-1}\left(\frac{k}{n}\right)\right)}\sqrt{\frac{k(n-k)}{n^3}}}\right) \qquad (22)$$

with $F^{-1}(.)$ the inverse of F(.) and f(.) the probability density function of X. Note, that the solution to $x(1-F(x)) = \max$ is equivalent to solve $\frac{1-F(x)}{f(x)} = x$ and, therefore, with $x_o = F^{-1}(\frac{k}{n})$ we have $E_n(x_o) = n\,x_o(1-F(x_o)+\frac{1}{n})$ and $D_n(x_o) = \frac{1-F(x_o)+1/n}{f(x_o)}$ $(n\,F(x_o)[1-F(x_o)])^{1/2}$ which for large n approaches Daniels' result. Further, observe that eq.(21) reproduces the exact result for n = 1. The search for the optimal k must be done numerically. Figure 5 displays a numerical example which demonstrates the superiority of eq.(21) or eq.(22) above eq.(3). Unfortunately, as can be seen in Figure 3 (thin lines), convergence of eqs.(21) or (22) to eq.(3) is faster than convergence of eq.(3) to eq.(2).

Quite generally, it is concluded that the use of eq.(20) is exact for n = 1, yields an upper bound for medium n which is a sharper upper bound than eq.(3) to which it converges for large n. It is conjectured that the same is also true for dependent sequences or processes. However, exact order statistics distributions appear to be inexistent in these cases. Therefore, in first approximation one might assume that the distribution of the relevant order statistics is already close to the normal and, hence, replace eq.(20) by:

$$P(R \le x) \approx \min_{(x_o)} \{\phi(\frac{x-E(x_o)}{D(x_o)})\} \qquad (23)$$

where $E(x_o)$ and $D(x_o)$ are the mean and standard deviation of the normal approximation to the exact order statistics distribution. For example, formulae (10) and (16) or (17) and (18), respectively, may be used in eq.(23). For small discrete systems, the number k should be treated as a real number when searching for the minimum. The inequality sign in eq.(20) no longer holds for eq.(23) unless the conditions leading to Daniels' asymptote are approached. From figure 5 one can see that the normal approximation to the exact order statistics distribution is sufficiently exact for not too small n.

Extensive numerical studies with other distribution types for elemental strength and a large range of variabilities, which are not reproduced here, showed that eq.(21) generally approaches the normal distribution with fairly accurate standard deviation rather fast while it is only the mean which still remains at the conservative side. Therefore, further research might be directed towards the development of correction factors for the mean.

Summary and Conclusions

The distribution of strength of systems of a number of elements in parallel under imposed uniform strain is investigated. Daniels asymptotic results are generalized towards systems with dependent elements. Using an order statistics approach, sharper results are obtained for small to medium systems. The models presented can be applied to a large number of types of material and construction, e.g. to concrete and masonry under tensile stresses, connections made of nails, dowels or rivets and, of course, all bundle-of-threads like structures. In particular, the formula given may be used in all cases where the deformation modulus of the elements is nearly a constant, i.e. strength and ultimate strain are highly positively correlated.

Appendix I: References

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Appendix II: Notations

$D(X)$	= standard deviation of X
$E(X)$	= expectation (mean) of X
$F(.)$	= distribution function
$F^{-1}(.)$	= inverse distribution function
$f(.)$	= probability density function
$P(A)$	= probability of event A
R_n	= system strength (system of n elements)
X,Y,Z	= random variables
$\hat{X}_k, \hat{Y}_k$	= k-th order statistics of random variables
ρ	= correlation coefficient
$\phi(.)$	= standard normal probability function
$\phi^{-1}(.)$	= inverse normal probability function

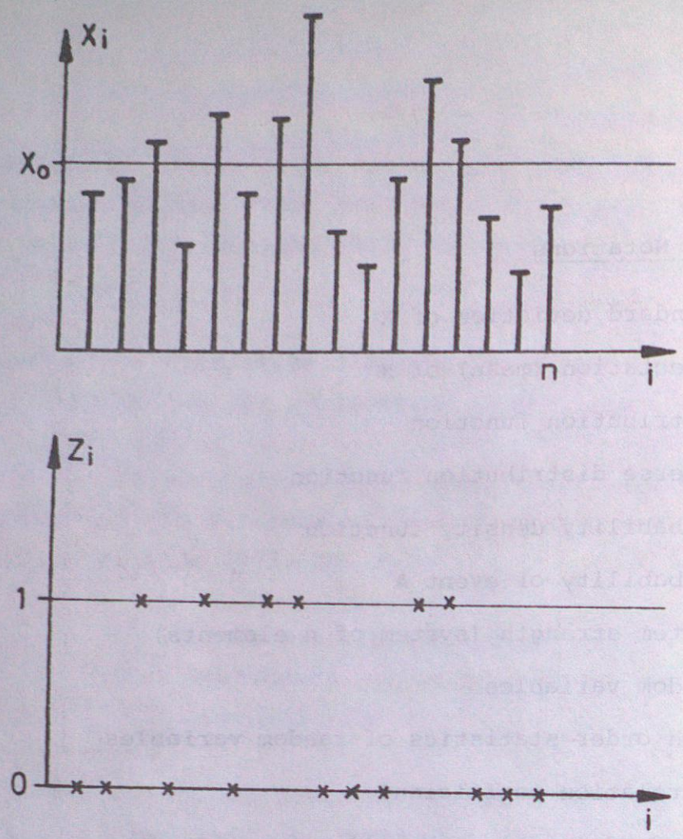

Figure 1: Discrete Daniels System

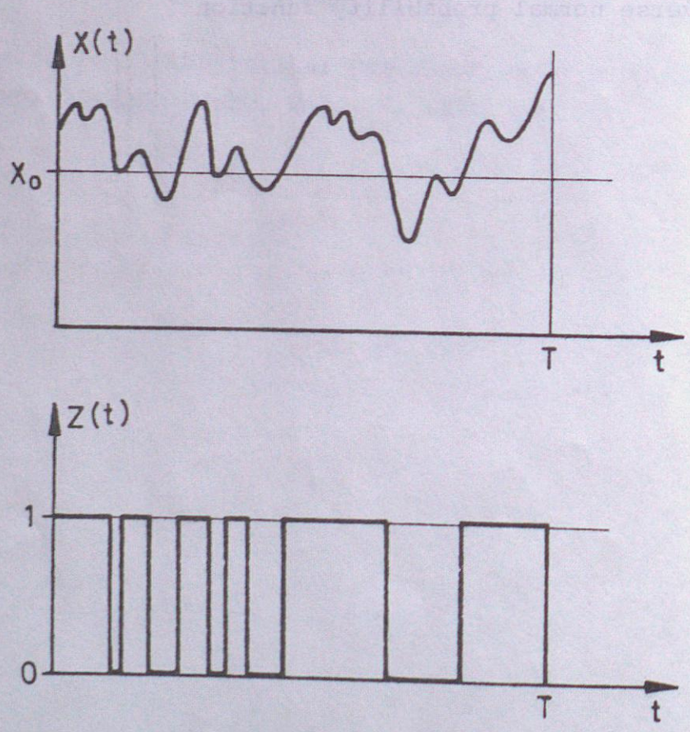

Figure 2: Continuous Daniels System

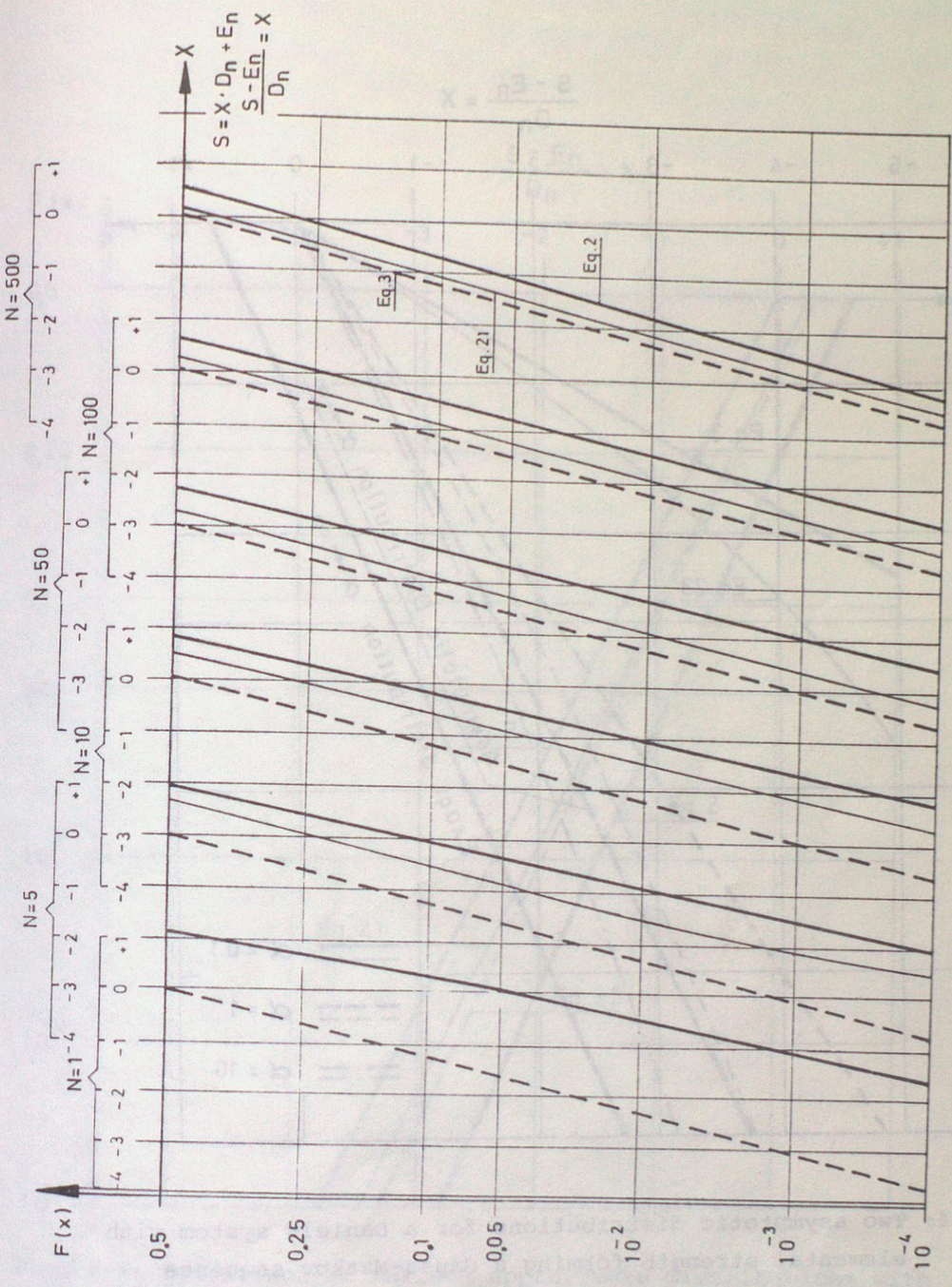

Figure 3: Exact and approximate distribution functions for Daniels systems (E_n and D_n from eq.3)

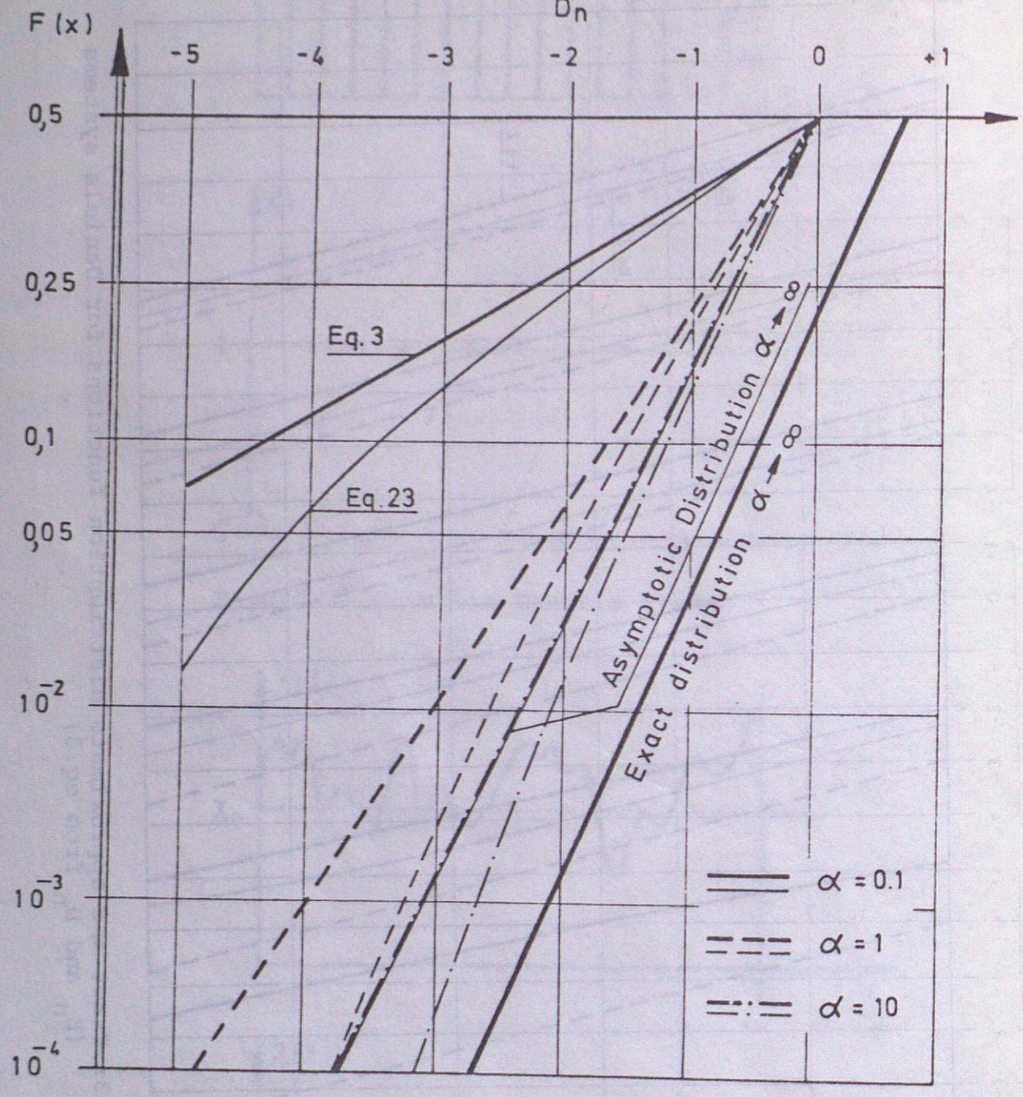

Figure 4: Two asymptotic distributions for a Daniels system with elemental strength forming a Gauss-Markov sequence with autocorrelation function $g(h) = e^{-\alpha h}$ (n = 100)

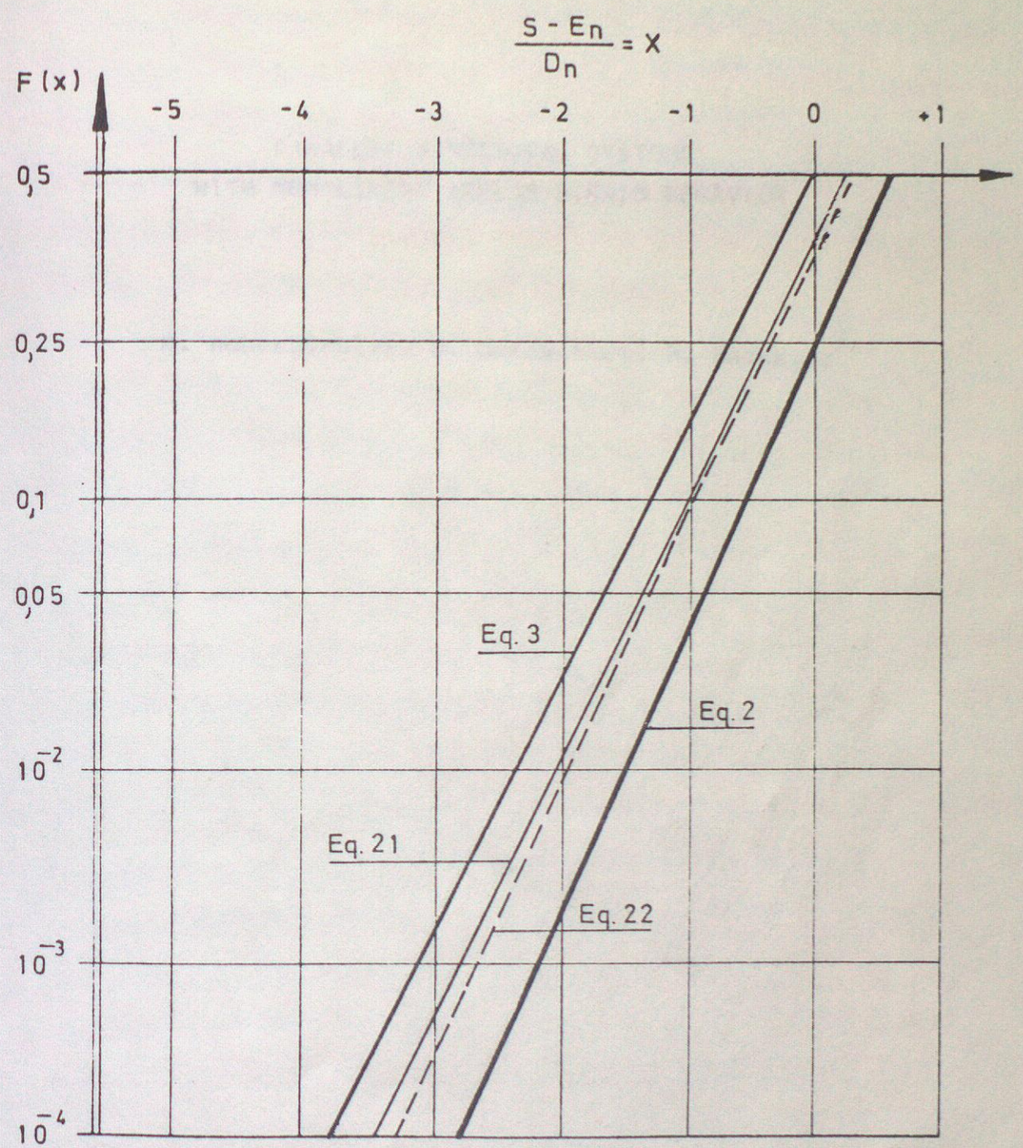

Figure 5: Comparison of various approximate distributions with exact distribution of the strength of a Daniels system (n = 100)

PARALLEL STRUCTURAL SYSTEMS WITH NON-LINEAR STRESS-STRAIN BEHAVIOR

M. Hohenbichler, S. Gollwitzer, R. Rackwitz

Introduction

Considerable effort is now being invested in the elaboration of a generation of probability-based structural design codes. They rest on a certain set of agreed stochastic models for the uncertain variables and a simplified probabilistic approach most frequently called "first-order reliability method". Relatively few studies have been directed towards the formulation of realistic models for the strength of materials, connections or cross-sections which necessarily include size effects. In fact, most studies on stochastic strength models have been limited to the statistical evaluation of tests of specified type and subsequent distribution curve fitting. As the type of distribution is concerned, two limiting cases have found special interest. On the one hand, so-called "weakest-link" models have been studied repeatedly. Their basis is the famous work of Weibull [1] yielding one of the asymptotic extreme-value distributions. On the other hand, materials whose "elements" behave perfectly ductile and form a parallel system have been treated in the context of integrals of random processes or fields over certain domains [2] yielding asymptotically a normal distribution of system strength. Very few results exist for materials whose elements are arranged in a parallel manner but do not behave perfectly plastic. The earliest and still most prominent work in this area is due to Daniels [3] who was able to derive the exact distribution of the strength of a bundle of threads with perfectly elastic-brittle behavior

and whose elemental strength values are positively, independently and identically distributed. This implies full positive correlation between strength and ultimate strain. More important, Daniels proved that the strength of the system is asymptotically normally distributed with a given mean and variance. This asymptote is always a conservative bound to the true distribution. Unfortunately, the exact distribution is numerically rather intractable if the number of elements grows large and convergence to the asymptote is extremely slow. While only recently some generalizations of the previous asymptotic results have been put forward for the case of dependent elemental strength or continuous strength fields [4 , 5 ,6] it has been shown that an order statistics approach generally yields sharper bounds, particularly for smaller systems [7]. In the following, this order statistics approach is applied to parallel systems whose elements have arbitrary non-linear stress-strain behavior. The order statistics approach is used in the context of modern first-order reliability methods [8] and is further improved by applying certain concepts in first-order system reliability [9]. The assumption of fully correlated strength and ultimate strain is relaxed. However, the condition of uniform imposed strain is still maintained as well as stochastic independence between elements.

Preliminaries

Assume the elements E_i in a given system of size n to follow different stress-strain laws which depend on an uncertain parameter vector $\underline{\theta}_i$, i.e. for the i-th element we have (compare figure 1):

$$S_i = S_i(\varepsilon) = S_{\underline{\theta}_i}(\varepsilon) = S(\varepsilon,\underline{\theta}_i) \text{ for } \varepsilon \geq 0 \qquad (1)$$

in which S denotes stress and ε denotes strain.

Then, the system strength under uniform imposed strain simply is (figure 1):

$$R_n = \max_{\varepsilon\geq 0}\{\sum_{i=1}^{n} S(\varepsilon,\underline{\theta}_i)\} \qquad (2)$$

An important special case is already worth mentioning. Let $S(\varepsilon,\underline{\theta}_i)$ be non-decreasing in ε. If a certain value ε_o exists where all curves $S(\varepsilon,\underline{\theta}_i)$ attain their maximum, then the i-th element strength is:

$$S(\varepsilon_o,\underline{\theta}_i) = \max_{\varepsilon\geq 0}\{S(\varepsilon,\underline{\theta}_i)\} = X_i \qquad (3)$$

and, therefore:

$$R_n = \sum_{i=1}^{n} \max_{\varepsilon\geq 0}\{S(\varepsilon,\underline{\theta}_i)\} = \sum_{i=1}^{n} X_i \qquad (4a)$$

The failure probability at a load level x becomes:

$$P(R_n \leq x) = P(\sum_{i=1}^{n} X_i \leq x) = P(\sum_{i=1}^{n} X_i - x \leq 0) \qquad (4b)$$

which, for arbitrary distribution functions of X may best be evaluated by using the well-known fast convolution techniques as given, for example, in [8]. For perfectly plastic materials

eq.(3) is formally true with $\varepsilon_o = \infty$ and eqs.(4) yield the expected result.

Unfortunately, there is no such simple solution in the general case. In using eq.(2) the failure probability may be written as:

$$P(R_n \leq x) = P(\max_{\varepsilon\geq 0}\{\sum_{i=1}^{n} S(\varepsilon,\underline{\theta}_i)\} - x \leq 0) \qquad (5)$$

where the max-term is most difficult to handle probabilistically. In order to demonstrate this, we assume a special perfectly elastic-brittle material with $X_i = Y_i$ the element ultimate strain and, therefore

$$S(\varepsilon,\underline{\theta}_i) = \begin{cases} \varepsilon & \text{for } 0 \leq \varepsilon \leq Y_i \\ 0 & \text{for } \varepsilon > Y_i \end{cases} \qquad (6)$$

Here, for example, $\underline{\theta}_i = (Y_i)$. It is easy to see that in this case

$$\max_{\varepsilon\geq 0}\{\sum_{i=1}^{n} S(\varepsilon,\underline{\theta}_i)\} \leq x \Longleftrightarrow \max_{A\subset I}\{\sum_{i\in A} X_i\} \leq x \qquad (7a)$$

or, equivalently

$$\{R_n \leq x\} = \bigcap_{A} \{\sum_{i\in A} X_i \leq x\} \qquad (7b)$$

where A runs through all $2^n - 1$ non-empty subsets of $I = \{1,\ldots,n\}$. Eqs.(7) essentially are the basis for the aforementioned exact result given by Daniels [3]. In a similar manner it is possible to formulate the system strength for less specialized though still brittle stress-strain relationships which, however, appear hardly be amenable to numerical evaluation. On the other hand

each event in braces in eq.(7b) may be denoted as a failure mode. But even the powerful techniques of first-order system reliability methods as developed e.g. by Ditlevsen [11] fail due to the enormous number of substantially contributing and partly highly correlated modes to be considered. Therefore, an alternative approach is chosen.

Let $X_1, X_2, \ldots, X_n$ be the sequence of random strength of brittle elements as before. Then, the resistance of the system with uniform imposed strain (i.e., in this special case, equal load sharing) can be given by

$$R_n = R_n(X_1, \ldots, X_n)$$

$$= \max_{1 \le k \le n} \{n \hat{X}_1, (n-1) \hat{X}_2, \ldots, \hat{X}_n\}$$

$$= \max_{1 \le k \le n} \{\ldots, (n-k+1) \hat{X}_k, \ldots\} \qquad (8)$$

where $\hat{X}_1 \le \hat{X}_2 \le \ldots \le \hat{X}_n$ are the order statistics of $X_1, X_2, \ldots, X_n$ and $k-1$ the number of elements which have failed at ultimate system strength. Fortunately, one term in eq.(8) usually dominates. Therefore, the distribution function of R_n can efficiently be bounded by:

$$F_R(x) = P(R_n \le x) \le \min_{1 \le k \le n} \{P((n-k+1) \hat{X}_k \le x\} \qquad (9)$$

Clearly, the optimum k minimizing eq.(9) depends on n and, in general, also on the level x, but $\frac{k}{n}$ converges to a well-known limit. Some exact and approximate formula are derived in [7] where it is also shown that eq.(9) is exact for n=1-systems and converges to Daniels' asymptote for large n.

General Formulation Using Order Statistics

For later convenience and without loss of generality, we specify two components of $\underline{\Theta}_i$ to be the elemental strength (= maximum stress) X_i and the corresponding ultimate strain Y_i so that eq.(1) can be written as:

$$S(\varepsilon, \underline{\Theta}_i) = X_i \, Q(\frac{\varepsilon}{Y_i}, \underline{\Pi}_i) \qquad (10)$$

where $\underline{\Pi}_i$ is the remaining parameter vector describing, for example, the type or the coefficients of the stress-strain relationship. Note that the auxiliary function $Q(t, \underline{\Pi}_i)$ has been introduced in a normalized form such that for each fixed $\underline{\Pi}_i$ it is non-decreasing $0 \le t \le 1$ and $Q(1, \underline{\Pi}_i) = 1$ and $Q(t, \underline{\Pi}_i) < 1$ in $t > 1$. Consequently, it is

$$X_i = S(Y_i, \underline{\Theta}_i) \qquad (11)$$

Further, we shall speak of brittle material if $Q(t, \underline{\Pi}_i) = 0$ for $1 < t < \infty$. If $Q(t, \underline{\Pi}_i)$ is decreasing in the latter interval we shall speak of almost brittle material (figure 1).

We first generalize the order statistics approach for arbitrary brittle stress strain relationships. Under uniform imposed strain the elements now fail according to the order statistics $\hat{Y}_1 \le \hat{Y}_2 \ldots \le \hat{Y}_n$ of ultimate strain (figure 2). The maximum system strength occurs for a sum of the stresses in the unbroken elements. Thus eq.(2) attains its maximum at one of the points $\varepsilon = Y_k$, and the condition "$S(\varepsilon, \underline{\Theta}_i) = 0$ for $\varepsilon > Y_i$" reduces eq.(2) to:

$$R_n = \max_{1\le k\le n} \sum_{i=1}^{n} S(Y_k,\underline{\Theta}_i) = \max_{1\le k\le n} \{ \sum_{i=k}^{n} S(\hat{Y}_k,\hat{\underline{\Theta}}_i)\} =$$

$$= \max_{1\le k\le n} \{\hat{X}_k + \sum_{i=k+1}^{n} \hat{X}_i \cdot Q(\frac{\hat{Y}_k}{\hat{Y}_i}, \hat{\underline{\Pi}}_i)\} \qquad (12)$$

The ordering indicated by ^ is always with respect to the Y_i's.

For "almost brittle" elements we have instead (figure 1b):

$$R_n \ge \max_{1\le k\le n} \{ \sum_{i=1}^{n} S(Y_k,\underline{\Theta}_i)\} = \max_{1\le k\le n} \{ \sum_{i=1}^{n} S(\hat{Y}_k,\hat{\underline{\Theta}}_i) =$$

$$= \max_{1\le k\le n} \{\hat{X}_k + \sum_{\substack{i=1\\ i\neq k}}^{n} \hat{X}_i\ Q(\frac{\hat{Y}_k}{\hat{Y}_i}, \hat{\underline{\Pi}}_i) \qquad (13)$$

The inequality sign in eq.(13) just indicates that depending on the functional form of $Q(t,\underline{\Pi}_i)$ beyond $t = 1$ there might exist a larger value of eq.(2) between the points $\hat{Y}_i$ at which R_n is solely considered.

In view of eq.(12), eq.(5) may now be written as

$$P(R_n \le x) = P(\max_k\{\hat{X}_k + \sum_i \hat{X}_i\ Q(\frac{\hat{Y}_k}{\hat{Y}_i}, \hat{\underline{\Pi}}_i)\} - x \le 0)$$

$$= P(\bigcap_k\{\hat{X}_k + \sum_i \hat{X}_i\ Q(\frac{\hat{Y}_k}{\hat{Y}_i}, \hat{\underline{\Pi}}_i) - x \le 0\})$$

$$\le \min_k\{P(\hat{X}_k + \sum_i \hat{X}_i\ Q(\frac{\hat{Y}_k}{\hat{Y}_i}, \hat{\underline{\Pi}}_i) - x \le 0) \qquad (14)$$

with "$\sum_i$" corresponding to eq.(12) and (13) and where the special version (10) of the stress-strain relationship is used. For almost brittle material the first equality in (14) becomes "$\le$".

In general the numerical effort for the evaluation of the probabilities in eq.(14) is still exorbitant if not prohibitive even for small systems and the most elementary joint distribution functions for $\underline{\Theta} = (Y,X,\underline{\Pi})$. However, the formulations of the failure events in the last three lines are exactly the ones required for a solution within first-order reliability methods [8]. To make this clear, we write eq.(14) with $\hat{\underline{X}} = (\hat{X}_1,\hat{X}_2,\ldots,\hat{X}_n)^T$, $\hat{\underline{Y}} = (\hat{Y}_1,\hat{Y}_2,\ldots,\hat{Y}_n)^T$ and $\hat{\underline{\underline{\Pi}}} = (\hat{\underline{\Pi}}_1,\hat{\underline{\Pi}}_2,\ldots,\hat{\underline{\Pi}}_n)^T$ as

$$P(R_n \le x) \le \min_k\{P(g_k(\hat{\underline{Y}},\hat{\underline{X}},\hat{\underline{\underline{\Pi}}},x) \le 0)\} \qquad (15a)$$

and, provided that a probability distribution transformation $F(\hat{\underline{Y}},\hat{\underline{X}},\hat{\underline{\underline{\Pi}}}) \to \phi(\underline{U},\underline{V},\underline{W})$ exists where $\underline{U}$, $\underline{V}$ and $\underline{W}$ are vectors and matrices of independent standard normal variates:

$$P(R_n \le x) \le \min_k\{P(\tilde{g}_k(\underline{U},\underline{V},\underline{\underline{W}},x) \le 0)\}$$

$$\approx \min_k\{\phi(-\beta_k)\} \qquad (15b)$$

in which β_k is the safety index of the k-th term to be determined in the usual way [8] and $\phi(.)$ the standard normal integral. Note that the approximation sign is only an indication the possible error introduced by the linearization at the Hasofer-Lind point [10] and which allows the simple calculation of the failure probabilities by $\phi(-\beta_k)$. As verified by a number of authors this error usually is negligible and, can further be diminished by polyhedric approximation as proposed in [11].

We need to transform $(\hat{\underline{Y}},\hat{\underline{X}},\hat{\underline{\underline{\Pi}}})$ into a standard normal vector using the transformation technique given in [8]. Assume an independent vector sequence $\{Y_i,X_i,\underline{\Pi}_i\}$ with joint distribution $F_{Y,X,\underline{\Pi}}(y,x,\underline{\pi})$ where X_i's and $\underline{\Pi}_i$'s are not necessarily independent of the Y_i's.

The joint distribution of the order statistics $(\hat{Y}_1,\ldots,\hat{Y}_n)$ can be given by a sequence of conditional distributions making use of the obvious fact that

$$F_{n,i}(y_i|y_1,\ldots,y_{i-1}) = P(\hat{Y}_i \le y_i|\hat{Y}_1 = y_1,\hat{Y}_2 = y_2,\ldots,\hat{Y}_{i-1} = y_{i-1})$$

$$= P(\hat{Y}_i \le y_i|\hat{Y}_{i-1} = y_{i-1}) = F_{n,i}(y_i|y_{i-1}) \quad (16)$$

for $y_1 \le y_2 \le \ldots \le y_i$. In particular

$$F_{n,i}(y_i|y_{i-1}) = \begin{cases} 0 \text{ for } y_i < y_{i-1} \\ 1 - (1 - \dfrac{F_Y(y_i)-F_Y(y_{i-1})}{1-F_Y(y_{i-1})})^{n-i+1} = \end{cases} \quad (17)$$

$$= 1 - (\frac{1-F_Y(y_i)}{1-F_Y(y_{i-1})})^{n-i+1} \quad \text{for } y_i \ge y_{i-1}$$

Hence, from

$$\Phi(u_1) = F_{n,1}(y_1) = 1 - (1-F_Y(y_1))^n \quad (18a)$$

$$\Phi(u_2) = F_{n,2}(y_2|y_1) \quad (18b')$$

$$\Phi(u_i) = F_{n,i}(y_i|y_1,\ldots,y_{i-1}) \quad (18i)$$

we obtain:

$$\hat{Y}_1 = F_{n,1}^{-1}(\Phi(U_1)) = F_Y^{-1}\{1-\Phi(-U_1)^{\frac{1}{n}}\} \quad (19a)$$

$$\hat{Y}_2 = F_{n,2}^{-1}(\Phi(U_2)|\hat{Y}_1) = F_Y^{-1}\{1-[1-F_Y(\hat{Y}_1)]\Phi(-U_2)^{\frac{1}{n-1}}\} =$$

$$= F_Y^{-1}\{1-\Phi(-U_1)^{\frac{1}{n}}\ \Phi(-U_2)^{\frac{1}{n-1}}\} \quad (19b)$$

$$\vdots$$

$$\hat{Y}_i = F_{n,i}^{-1}(\Phi(U_i)|\hat{Y}_{i-1}) = F_Y^{-1}\{1-[1-F_Y(\hat{Y}_{i-1})]\Phi(-U_i)^{\frac{1}{n-i+1}}\} =$$

$$= F_Y^{-1}\{1-\prod_{j=1}^{i}[\Phi(-U_j)^{\frac{1}{n-j+1}}]\} \quad (19i)$$

Next we have:

$$\hat{X}_i = F_X^{-1}[\phi(V_i)|\hat{Y}_i] \quad (20)$$

If, in particular, (X,Y) is binormally distributed, i.e.:

$$F_{XY}(x,y) = \phi(x,y;\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2,\rho) \quad (21)$$

it follows:

$$F_Y(y) = \phi(\frac{y-\mu_y}{\sigma_y}) \quad (21a)$$

$$F_X(x|y) = \phi(\frac{x-(\mu_x+\rho\cdot\frac{\sigma_x}{\sigma_y}(y-\mu_y))}{\sqrt{1-\rho^2}\ \sigma_X}) \quad (21b)$$

and, therefore, by substitution in eqs.(19) and (20):

$$\hat{Y}_i = \mu_y - \sigma_y\ \phi^{-1}\{\prod_{j=1}^{i}[\phi(-U_j)^{\frac{1}{n-j+1}}]\} \quad (22a)$$

$$\hat{X}_i = V_i\ \sqrt{(1-\rho^2)}\ \sigma_X + \mu_x + \rho\cdot\frac{\sigma_x}{\sigma_y}(\hat{Y}_i - \mu_y) \quad (22b)$$

Finally, the vector $\underline{W}_i = (W_{i,1},\ldots,W_{i,l})$ (the i-th row of the variable matrix $\underline{\underline{W}}$) is defined by:

$$\Phi(W_{i,1}) = F_{\Pi,1}(\hat{\Pi}_{i,1}|\hat{Y}_i,\hat{X}_i) \qquad (23.1a)$$

$$\hat{\Pi}_{i,1} = F^{-1}_{\Pi,1}(\Phi(W_{i,1})|\hat{Y}_i,\hat{X}_i) \qquad (23.1b)$$

$$\vdots$$

$$\Phi(W_{i,l}) = F_{\Pi,l}(\hat{\Pi}_{i,l}|\hat{Y}_i,\hat{X}_i,\hat{\Pi}_{i,1},\ldots,\hat{\Pi}_{i,l-1}) \qquad (23.1a)$$

$$\hat{\Pi}_{i,l} = F^{-1}_{\Pi,l}(\Phi(W_{,l.})|\hat{Y}_i,\hat{X}_i,\hat{\Pi}_{i,1},\ldots,\hat{\Pi}_{i,l-1}) \qquad (23.1b)$$

with $\underline{\hat{\Pi}}_i = (\hat{\Pi}_{i,1},\ldots,\hat{\Pi}_{i,l})$.

If $\underline{\Pi}$ is independent of X and Y, then $\hat{X}_i$ and $\hat{Y}_i$ do not appear in eqs.(24). For the special case that Π vanishes, eqs.(24) also vanish. If $\underline{\Pi}$ is constant or, more general, $\underline{\Pi} = \underline{T}(Y,X)$ is a function of Y and X, eqs.(24) reduce to

$$\underline{\hat{\Pi}}_i = \underline{T}(\hat{Y}_i,\hat{X}_i) \qquad (24)$$

These formulae, introduced in eq.(15a) provide the required formulation in the standard normal space. Since the algorithms proposed for finding β_k, e.g. the one proposed in [8], are rapidly converging in this case it is only a matter of computer storage capacity which limits the size of the system.

A "Failure-Mode" Improvement

As demonstrated in [7] the use of the bound (last line) in eq.(14) yields sufficiently accurate results for very small and very large systems. For medium systems the distributional bound is fairly conservative indicating that other than only the dominating term in eq.(14) have to be considered. However, it is possible to find closer bounds when using the second line of eq.(14) directly. In analogy to eq.(15b) we write in collecting $(\underline{u},\underline{v},\underline{w})_k$ in $\underline{\omega}_k$ (see [14]):

$$P(R_n \le x) = P(\cap(\tilde{g}_k(\underline{U},\underline{V},\underline{W},x) \le 0))$$

$$\approx P(\cap\{\underline{\alpha}_k\ \underline{\omega}_k \le -\beta_k\}) = P(\cap\{Z_k \le -\beta_k\}) \le P(\bigcap_{k=k_1}^{k_2}\{Z_k \le -\beta_k\}) =$$

$$= \int_{-\infty}^{-\underline{\beta}} \varphi(\underline{z};\underline{\underline{R}})\ d\underline{z}$$

$$\le \int_{-\infty}^{+\infty} \varphi(t) \prod_{k=k_1}^{k_2} \phi\left(\frac{-\beta_k - \sqrt{\rho_c}\,t}{\sqrt{1-\rho_c}}\right) dt \qquad (25)$$

in which $\underline{\alpha}_k$ is the vector of direction cosines of the approximating hyperplane of the k-th "mode", Z_k the corresponding standard normal safety margin, $\underline{\underline{R}}$ the matrix of correlation coefficients between safety margins, i.e. $\underline{\underline{R}} = (\rho[Z_i,Z_j])$ for all $k_1 \le i,\ j \le k_2$ with $\rho[Z_i,Z_j] = \underline{\alpha}_i^T\ \underline{\underline{I}}\ \underline{\alpha}_j$, $\varphi(.;\underline{\underline{R}})$ the standard multinormal density and $\rho_c = \max\{\rho[Z_i,Z_j] : k_1 \le i,\ j \le k_2\}$. The interval $[k_1,k_2]$ should include the mode k with the largest β_k. Within first order reliability the first three lines of eq.(25) are exact but no simple method is available for the evaluation of quadrant multinormal probabilities.

Therefore, the simple bound in the last line of eq.(25), involving only one-dimensional integration is used (see, for example, [14,15,16]).

It is worth noting that by some more involved considerations one can replace transformation (19) by

$$\begin{aligned}
\hat{Y}_{k_1} &= F^{-1}_{(k_1)}(\Phi(U_{k_1})) \\
\hat{Y}_{k_1+1} &= F^{-1}_{n,k_1+1}(\Phi(U_{k_1+1})|\hat{Y}_{k_1}) \\
&\vdots \\
\hat{Y}_{k_2} &= F^{-1}_{n,k_2}(\Phi(U_{k_2})|\hat{Y}_{k_2-1})
\end{aligned} \tag{26a}$$

with

$$F_{(k_1)}(y) = \sum_{i=k_1}^{n} \binom{n}{i} F_Y^i(y)\,(1-F_Y(y))^{n-1}$$

the distribution function of the k_1-th order statistic [12], and transform the remaining Y_i's as independent, truncated variates as follows:

$$Y_i = F_Y^{-1}(F_Y(\hat{Y}_{k_1}) \cdot \Phi(U_{k_i}) \qquad (i < k_y) \tag{26b}$$

$$Y_j = F_Y^{-1}(F_Y(\hat{Y}_{k_2}) + (1-F_Y(\hat{Y}_{k_2}))\cdot\Phi(U_{kj})) \quad (j > k_2) \tag{26c}$$

This transformation produces smoother failure regions and for large systems allows the application of the central limit theorem for the variates $(k < k_1)$ and $(k > k_2)$.

Discussion and Applications

A special case is studied first since it has an exact solution given by Daniels [3] and, therefore, can demonstrate the accuracy of the method. Assume the components to behave perfectly elastic-brittle, i.e.:

$$S_i(\varepsilon) = S(\varepsilon,\underline{\Theta}_i) = X_i \cdot Q(\frac{\varepsilon}{Y_i},\underline{\Pi}_i) = X_i\,\frac{\varepsilon}{Y_i} \qquad \text{for } 0 \le \varepsilon \le Y_i \tag{27}$$

together with $\rho[X_i,Y_i] = 1$ and $\mu_X/\mu_Y = \sigma_X/\sigma_Y$. Figure 3a demonstrates at an example that in the probability range of interest formula (15b) is slightly conservative and is improved by formula (25). Figure 3b shows the reliability of the system for different correlation coefficients ρ between X and Y (eq.(21)).

Next we demonstrate how the general formulations above can be employed to solve more realistic problems. In essence, this requires to define an appropriate Q-function and a corresponding $\underline{\Pi}$-parameter, as well as an appropriate stochastic model.

Example 1: Different Shapes of Stress-Strain curves

Let $\underline{\Pi} = (B)$ and

$$Q(t,\underline{\Pi}) = \begin{cases} t^B & \text{for } 0 \le z \le 1 \\ 0 & \text{for } z > 0 \end{cases} \tag{28a}$$

so that $\underline{\Theta}_i = (Y_i,X_i,B_i)$ and, hence,

$$S(\varepsilon,\underline{\Theta}_i) = \begin{cases} X_i\, Y_i^{-B_i}\, \varepsilon^{B_i} & \text{for } 0 \le \varepsilon \le Y_i \\ 0 & \text{for } \varepsilon > Y_i \end{cases} \tag{28b}$$

Example 2:

Let for each parameter $\underline{B}: h_{\underline{B}} = h_{\underline{B}}(z)$ $(0 \leq z < \infty)$ be a non-decreasing function and $\underline{\Pi} = (A, \underline{B})$, $(A > 0)$

$$Q(t, \underline{\Pi}) = \begin{cases} \frac{1}{h_{\underline{B}_i}(A)} \cdot h_{\underline{B}_i}(At) & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t \geq 1 \end{cases} \qquad (29a)$$

resp.

$$S(\varepsilon, \underline{\Theta}_i) = \begin{cases} \frac{X_i}{h_{\underline{B}}(A_i)} \cdot h_{\underline{B}}(A_i \frac{\varepsilon}{Y_i}) & \text{for } 0 \leq \varepsilon \leq Y_i \\ 0 & \text{for } \varepsilon > 1 \end{cases} \qquad (29a)$$

To illuminate the effect of the parameter A, we observe that for fixed $A_i, Y_i, \underline{B}_i$ the argument $z = A_i \cdot \frac{\varepsilon}{Y_i}$ becomes $0 \leq z \leq A_i$ if $0 \leq \varepsilon \leq Y_i$. Thus at the elements breaking-point Y_i the $h_{\underline{B}}$-function is just in its "phase" A_i, so that A_i determines the phase of the stress-strain relationship, in which rupture occurs while $\underline{B}$ determines is t type. For an illustration see figure 4 with a bilinear h-function ($\underline{B}$ vanishing) and the important cases $A_i = 1$, $A_i = Y_i$.

Example 3 ("Slip")

Some more effort is required if the elemental stress-strain relationships exhibit uncertain Slip for example by imperfect anchorage of the threads in a bundle or by unavoidable tolerances in screwed joints. Then, an additional random parameter Z may be introduced. If the Slip is independent of elemental stress-strain properties, it is convenient to write:

$\underline{\Theta} = (Y, X, \underline{\Pi})$, $\underline{\Pi} = (Z, \underline{B})$,

$$S(\varepsilon, \underline{\Theta}_i) = \begin{cases} 0 & \text{for } 0 \leq \varepsilon \leq Z_i \\ \frac{X_i}{h_{\underline{B}_i}(A_i)} \cdot h_{\underline{B}_i}(A_i \frac{\varepsilon - Z_i}{Y_i - Z_i}) & \text{for } Z_i \leq \varepsilon \leq Y_i \\ 0 & \text{for } \varepsilon > Y_i \end{cases} \qquad (30)$$

Note that Y_i is the ultimate strain of the i-th element including its Schlupf Z_i, so that $Y_{M,i} = Y_i - Z_i$ is the "material ultimate strain" (without slip). This is important to bear in mind since for the order-statistics of rupture naturally the variables Z_i must be taken into account.

In order to illustrate some essential points of the transformations eq.(19), (20), (23) resp. (22a), (22b), (23) we consider elements with binormally distributed "material values", i.e.

$$F_{YX}(Y_{M,i}, X_i) = \Phi(x, y; \mu_X, \mu_{M,Y}, \sigma_X, \sigma_{M,Y}, \rho_M) \qquad (31)$$

$\underline{B}$ vanishing, Z_i independent of $(X_i, Y_{M,i})$ and normally distributed with mean μ_z and variance σ_z^2.

Then, the variables $Y_i = Y_{M,i} + Z_i$ are mutually independent normal variables with mean and variance: $\mu_Y = \mu_{M,Y} + \mu_Z$, $\sigma_Y^2 = \sigma_{M,Y}^2 + \sigma_Z^2$ to be used in eqs.(22a) and (22b). The correlation $\rho = \rho[Y_i, X_i]$ in eq.(22b) yields

$$\rho = \rho_M \cdot \sigma_{M,Y} \cdot (\sigma_{M,Y}^2 + \sigma_Z^2)^{-\frac{1}{2}} \qquad (32)$$

Finally, the Z's have to be transformed into standard normal variates W according to eq.(23). We have to determine the condi-

tional distribution of Z conditioned on Y and X, where $Y = Y_M + Z$, i.e.

$$V[Z|Y,X] = V[Z|Y_M + Z, X] \tag{33}$$

where Y_M, X and Z are normal and Z independent of (Y_M, X). Abbreviating as $\tilde{X} = X - \mu_X$, $\tilde{Y}_M = Y_M - \mu_{M,Y}$, $\tilde{Z} = Z - \mu_Z$ and $\sigma_M = \sigma_{M,Y}$, the vector $(\tilde{Y}_M + \tilde{Z}, \tilde{X}, \tilde{Z})$ is tri-normally distributed with mean zero and covariance-matrix

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma_M^2 + \sigma_Z^2 & \rho_M \sigma_X \sigma_M & \sigma_Z^2 \\ \rho_M \sigma_X \sigma_M & \sigma_X^2 & 0 \\ \sigma_Z^2 & 0 & \sigma_Z^2 \end{pmatrix} \tag{34a}$$

With

$$\underline{\underline{A}} = \underline{\underline{\Sigma}}^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{34b}$$

the required distribution eq.(33) is normal with mean and variance

$$\mu_c = -\frac{1}{a_{33}}(a_{31}(\tilde{Y}_M + \tilde{Z}) + a_{32}\tilde{X})) + \mu_Z = a_1 \tilde{Y} + a_2 \tilde{X} + \mu_Z \tag{35a}$$

$$\sigma_c^2 = \frac{1}{a_{33}} = a_3^2 \tag{35b}$$

Therefore, eqs.(23) reduce to

$$\Phi(W_i) = F_Z(\hat{Z}_i|\hat{X}_i,\hat{Y}_i) = \Phi\left(\frac{\hat{Z}_i - a_1(\hat{Y}_i - \mu_Y) - a_2(\hat{X}_i - \mu_X) - \mu_Z}{a_3}\right) \tag{36a}$$

$$\hat{Z}_i = a_3 W_i + a_1(\hat{Y}_i - \mu_{M,Y} - \mu_Z) + a_2(\hat{X}_i - \mu_X) + \mu_Z \tag{36b}$$

Example 4: Defect Elements

Next, the case of unfastend or defect elements is discussed which otherwise would involve rather tedious numerical calculations. This example also illustrates the construction of a new parameter, if the original parameter is not of the form $(Y, X, \underline{\Pi})$.

Let $X_{M,i}$ and $Y_{M,i}$ the "material" strength and ultimate strain and $\underline{\Theta}_{M,i}$ the material parameter without reference to possible defects. The corresponding stress-strain relationship is:

$$S_M(\varepsilon, \underline{\Theta}_{M,i}) = \begin{cases} \frac{X_{M,i}}{h(A_i)} \cdot h\left(A_i \cdot \frac{\varepsilon}{Y_{M,i}}\right) & \text{for } 0 \le \varepsilon \le Y_{M,i} \\ 0 & \text{for } \varepsilon > Y_{M,i} \end{cases} \tag{37}$$

(compare example 2; for simplicity $\underline{\Theta}_M = (Y_M, X_M, A)$ with vanishing $\underline{B}$-Parameter, which does not essentially influence the following derivations).

Including the existence of defects into our considerations we may write:

$$\tilde{S}(\varepsilon, \underline{\tilde{\Theta}}_i) = Z_i \cdot S_M(\varepsilon, \underline{\Theta}_{M,i}) \tag{38}$$

where Z_i is an independent 0-1-variable (0 for defect, 1 else) and $\underline{\tilde{\Theta}} = (\underline{\Theta}_M, Z)$.

In case of $Z_i = 1$ we obviously have $Y_i = Y_{M,i}$. Since a stress-strain curve which is constantly zero may have its "Y-point" everywhere, we can also write $Y_i = Y_{M,i}$ if $Z_i = 0$.

Then from eq.(11)

$$X_i = \tilde{S}(Y_i, \underline{\tilde{\Theta}}_i) = Z_i \cdot S_M(Y_{M,i}, \underline{\Theta}_{M,i}) = Z_i \cdot X_{M,i} \tag{39}$$

It is now easily verified that

$$\tilde{S}(\varepsilon,\underline{\tilde{\Theta}}_i) = \tilde{S}(\varepsilon,(Y_{M,i},X_{M,i},A_i,Z_i)) =$$

$$= \begin{cases} \frac{X_i}{h(A_i)} \cdot h(A_i \quad \frac{\varepsilon}{Y_i}) & \text{for } 0 \leq \varepsilon \leq Y_i \\ 0 & \text{for } \varepsilon > Y_i \end{cases} \qquad (40)$$

$$= : S(\varepsilon,\underline{\Theta}_i)$$

where $\underline{\Theta}_i = (Y_i,X_i,A_i)$ is the final parameter.

With $F_{M,Y}(y)$, $F_{M,X}(x|y)$ and $F_{M,A}(a|y,x)$ being the set of (conditional) distribution functions of the original material parameters (Y_M,X_M,A), the necessary (conditional) distributions of the final parameters (Y,X,A) are

$$F_Y(y) = F_{M,Y}(y) \qquad (41a)$$

$$F_X(x|y) = p + (1-p)F_{M,X}(x|y) \qquad (41b)$$

(where $p = P(Z_i = 0)$, and since in the case $X_i = 0$ the stress-strain curve does not depend on A_i, we may also write:

$$F_A(a|y,x) = F_{M,A}(a|y,x) \qquad (41c)$$

For $x = 0$ the last equation is not true in a statistical sence, but it yields the exact failure probability in all cases where the "failure-condition" can be expressed in terms of $S(\varepsilon,\underline{\Theta})$ (see eqs.(12) and (14)). Eqs.(41) and (19), (20), (23) define the necessary transformations into the space of independent, standard normal variates. In particular, eqs.(41b) and (20) yield:

$$\hat{X}_i = \begin{cases} 0 & \text{for } \hat{W}_i < \Phi^{-1}(p) \\ F^{-1}_{M,X} \quad (\frac{\Phi(W_i)-p}{1-p} \mid \hat{Y}_i) & \text{for } \hat{W}_i \geq \Phi^{-1}(p) \end{cases} \qquad (42)$$

In a similar way, the "defect problem" can also be solved with another parameter, i.e.:

$$Y_i = Z_i \cdot Y_{M,i}$$

$$X_i = \tilde{S}(Y_i,\underline{\tilde{\Theta}}_i) = Z_i \; X_{M,i} \quad \text{(if } h(0) = 0\text{)}$$

$$\underline{\Theta}_i = (Y_i,X_i,A_i)$$

For $h(0) = 0$ as is usually the case, the corresponding distributions then are

$$F_Y(y) = p + (1-p) \; F_{M,Y}(y)$$

$$F_X(x|y) = F_{M,X}(x|y)$$

$$F_A(a|y,x) = F_{M,A}(a|y,x)$$

Example 5: Reliability of a temporary bridge support during construction

Finally, a practical, numerical example will be given. Assume that a cantilever beam must be supported temporarily by a row of steel columns. The supported superstructure can be taken as perfectly rigid. If the columns rest on single footing it is obvious that even very small unequal settlements can cause a drastic redistribution of column forces and hence partial or total collapse of the scaffolding. The vertical displacements of columns are third order effects. Therefore, one might avoid such dangerous situations in practice in controlling the column forces by jackets or the like (stress-controlled design). Here, we determine the reliability of the scaffolding structure under uniform imposed strain (rigid column foundation). Figure 5a displays some mechanical and geometrical data, figure 5b shows an example of the force-displacement relationship. The maximum load corresponds to the yield load. Second order effects are taken into account. Beyond the yield load it is assumed that a plastic hinge is formed at mid-span. Strain-hardening effects can still be neglected. For simplification the uncertainty of the force-displacement curve is entirely attributed to the independent random initial excentricity of the columns which is assumed to be truncated log-normally distributed with mean 2 cm and standard ceviation 1 cm. The truncation point has been chosen at e_u = 6 cm for obvious geometrical reasons. Figure 6 shows the distribution of the resistance of a structure consisting of 10 columns. The dashed line is obtained when perfect plasticity beyond the yield load is assumed.

The dotted line corresponds to perfectly brittle colum behaviour, i.e. no bearing capacity beyond the yield load. Table 1 collects the safety indices for all "failure modes" indicating that due to relatively high correlation between the modes (ρ = 0.998 to 1) and almost non-varying safety indices the mode with largest resistance is fairly representative for the system. Only little is achieved if more than the critical mode is considered. The authors conjecture that this is generally true for small parallel systems, perhaps with the exception of systems whose elements behave ideally elastic-brittle.

Summary and Conclusions

The reliability of parallel structural systems with independent, identically distributed elemental properties and arbitrary stress-strain relationships under imposed uniform deformation (strain) is investigated. Failure probabilities are estimated using the techniques of modern first order reliability methods. The necessary probability distribution transformations are derived. The system fails if none of the possible sets of elements in a system ordered according to the order statistics of ultimate elemental strain can carry the load. A simple upper probability bound is obtained by selecting only that set of elements with largest expected strength. An improved highly accurate upper bound can be derived by considering the entire family of sets of elements corresponding to the order statistics. Thus, a system of n elements has at most n survival modes to be considered. Some few generalizations, for example when (statistical) uncertainties about the distribution parameters of the elemental properties exist, when their is a probability of defect or inexisting elements or when elements share the load only after some initial deformation, are indicated. The method may be applied to bundles of threads, to joints connected by screws, bolts, nails or the like, to laminated timber beams under uniaxial loads and to a number of other structural (sub-)systems with equal or approximately equal laod-sharing. The order statistics approach may no more be feasible if the element characteristics are dependent between the elements.

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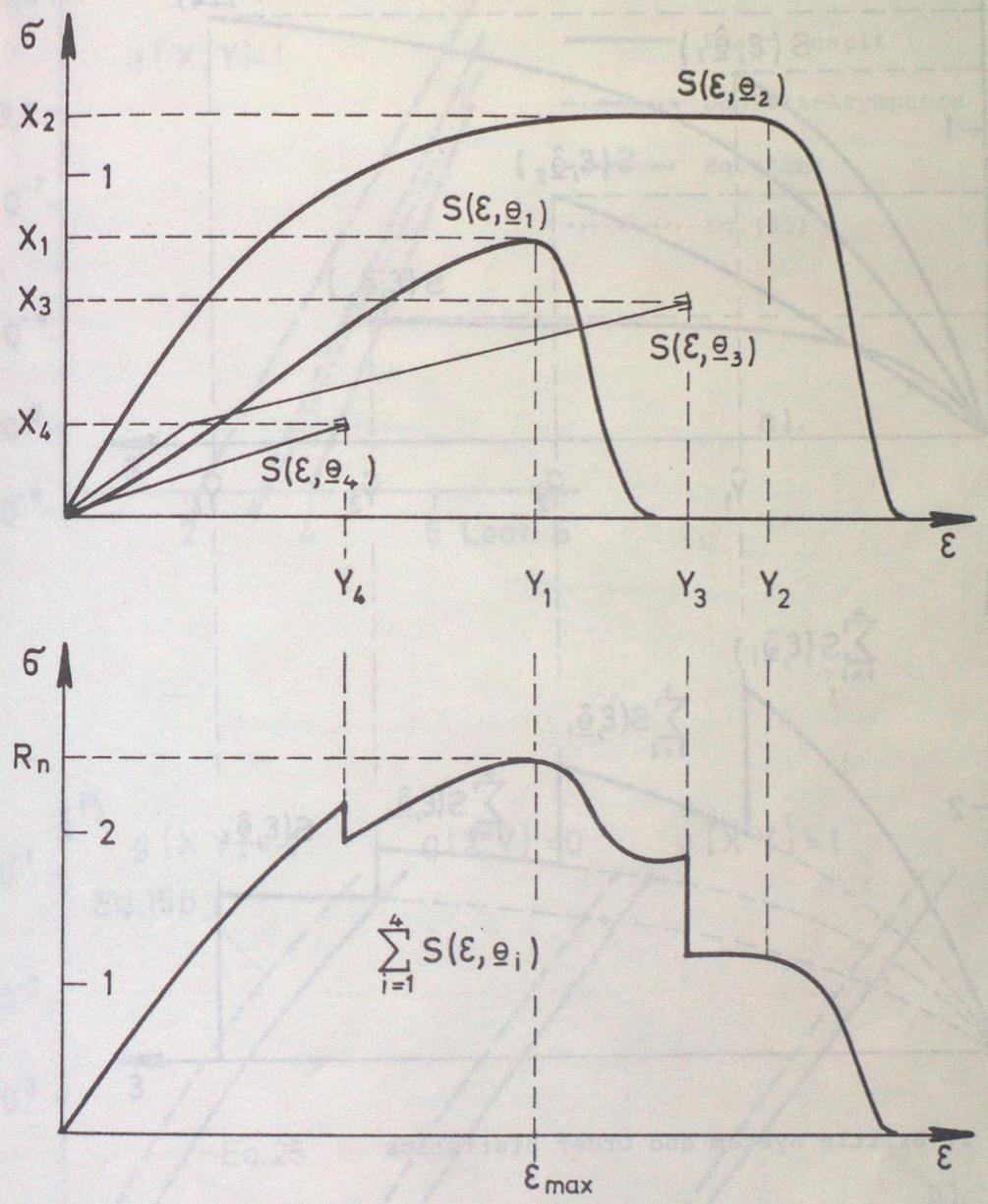


Figure 1: Examples of Stress-Strain Curves with Different Parameters and Resulting System Strength (Four System Components)

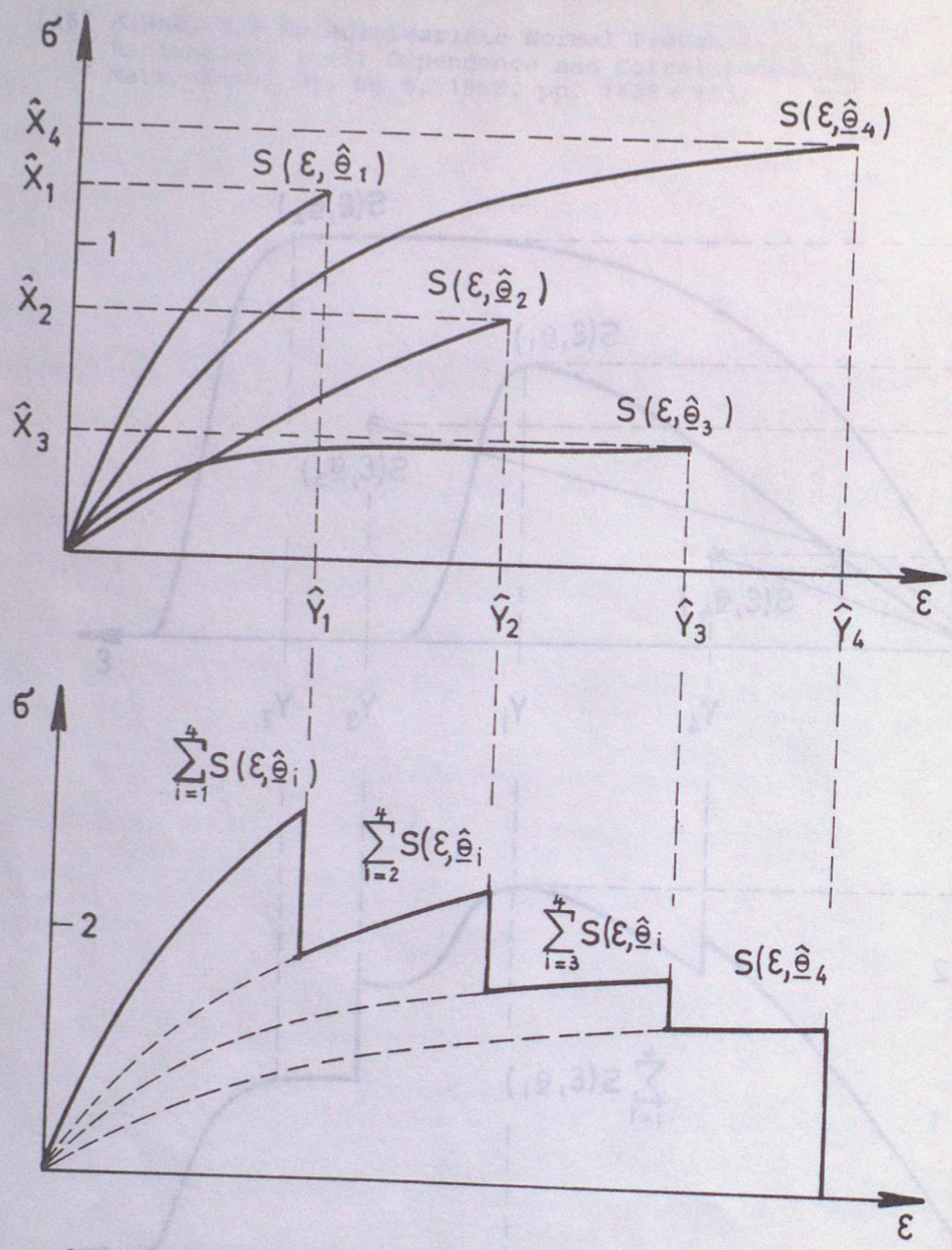
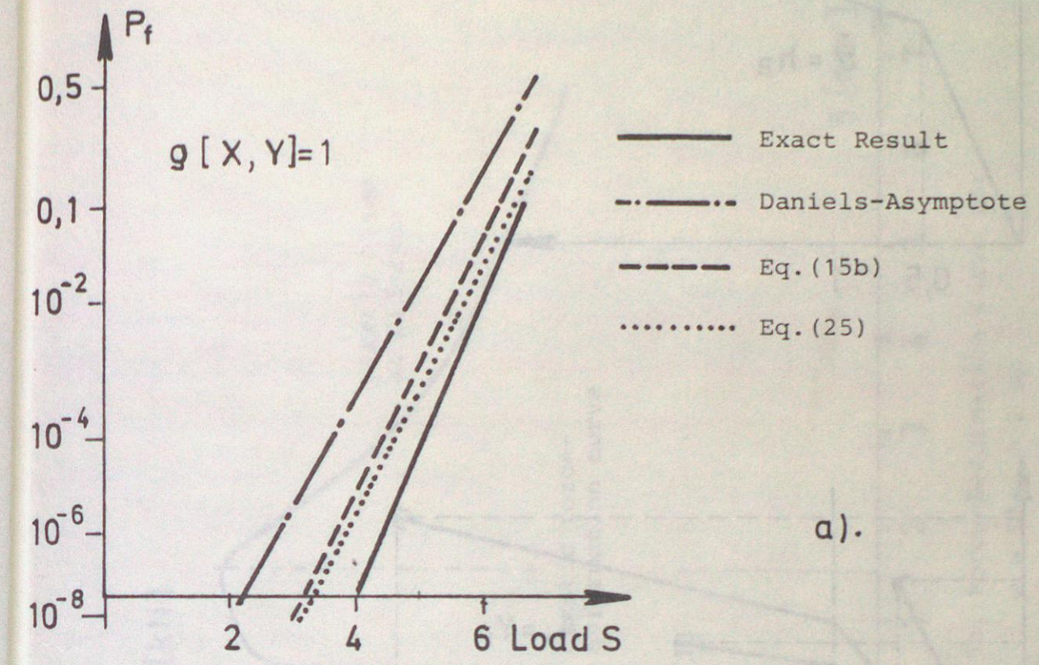


Figure 2: Brittle System and Order Statistics



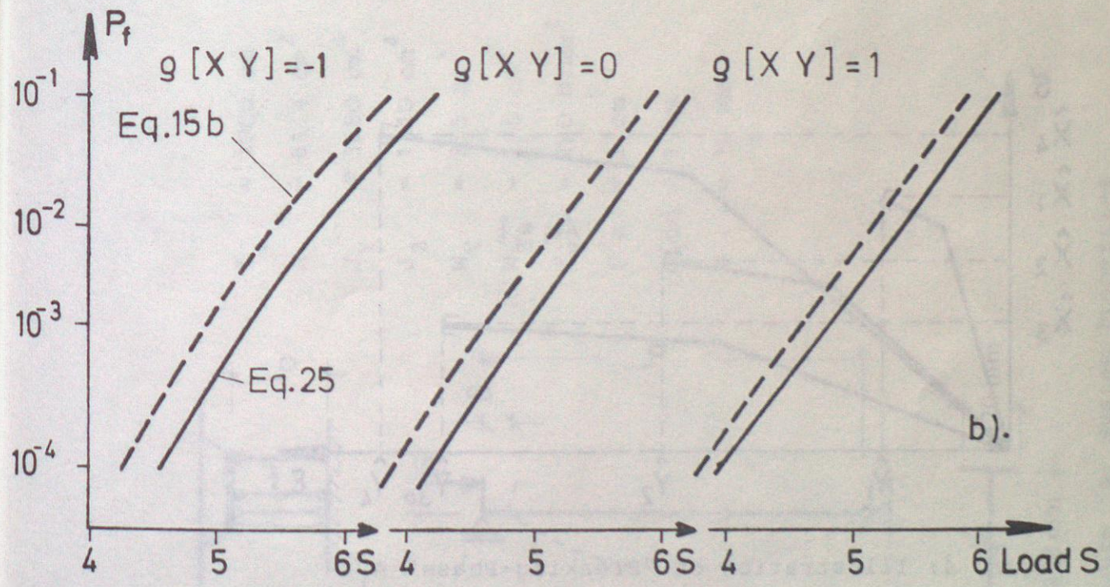


Figure 3: System with N = 10 Components (X,Y), Binormally Distributed, Mean = 1, Standard Deviation = 0.2

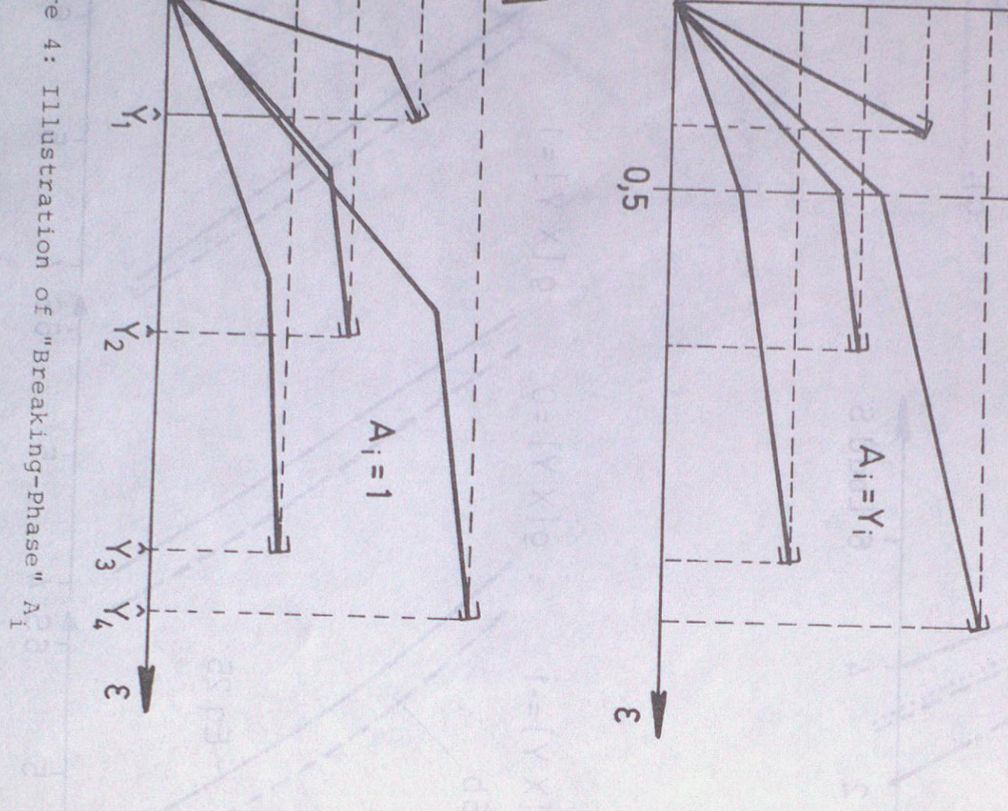
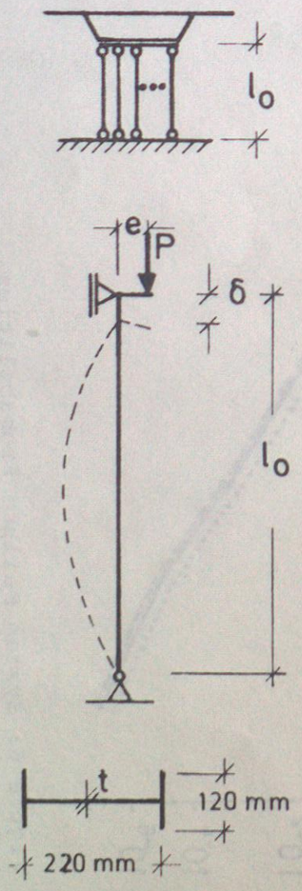


Figure 4: Illustration of "Breaking-Phase" A_1



l_o = 1000 cm
A = 61.4 cm^2
J_Y = 3680 cm^4
J_Z = 1410 cm^4
W_Y = 335 cm^3
W_Z = 235 cm^3
f_Y = 240 N/mm^2
E[e] = 2 cm
D[e] = 1 cm
t = 10 mm

Figure 5a: System Properties

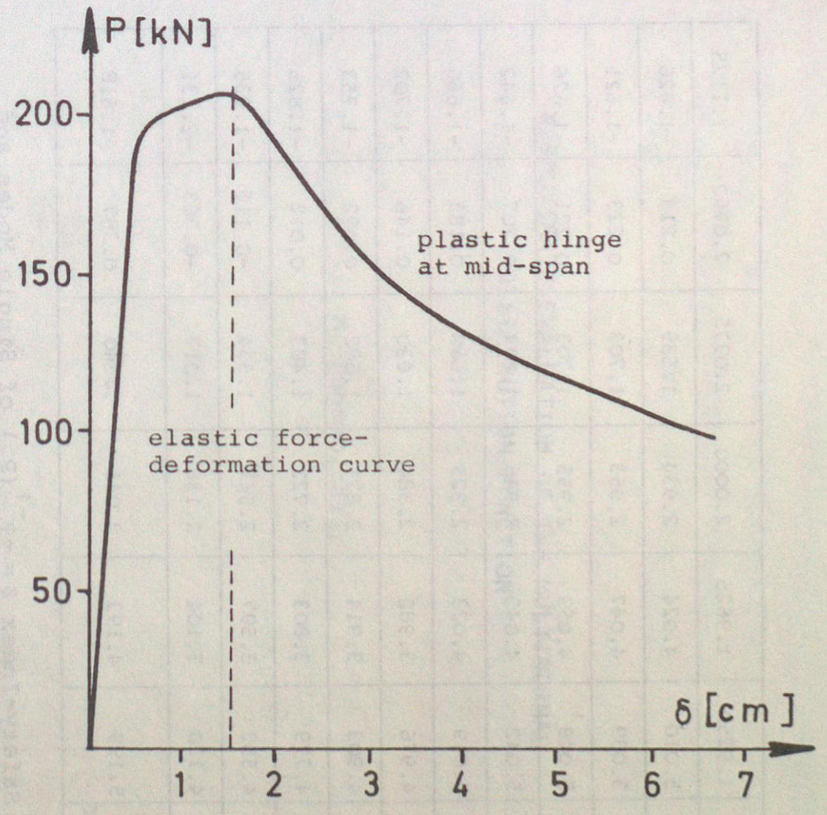


Figure 5b: Force-Deformation Curve for
e = E[e] = 2 cm

LOAD / MODE	1.9250	1.9625	2.0000	2.0375	2.0750	2.1125
K = 1	5.010	4.024	2.934	1.689	0.213	-1.626
K = 2	5.039	4.047	2.953	1.703	0.223	-1.621
K = 3	5.048	4.053	2.955	1.703	0.221	-1.626
K = 4	5.042	4.045	2.946	1.692	0.207	-1.642
K = 5	5.019	4.023	2.923	1.668	0.183	-1.666
K = 6	4.976	3.982	2.884	1.630	0.146	-1.702
K = 7	4.903	3.914	2.821	1.572	0.092	-1.753
K = 8	4.779	3.803	2.722	1.483	0.012	-1.825
K = 9	4.553	3.597	2.547	1.334	-0.116	-1.935
K = 10	4.120	3.104	2.136	1.017	-0.363	-2.131
FAILURE-MODE IMPROVEMENT	5.199	4.193	3.041	1.740	0.387	-1.618

Table 1: Safety-Index $\beta = -\Phi^{-1}(P_f)$ of Single Modes and System (10 Elements). Load in 10^3 KN

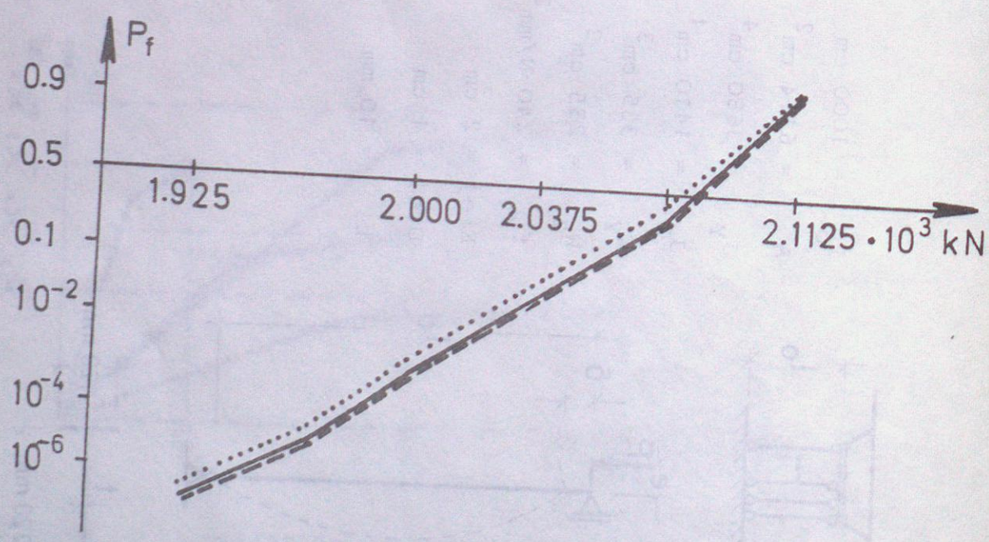


Figure 6: System Failure Probabilities

APPROXIMATE EVALUATION OF THE MULTINORMAL DISTRIBUTION FUNCTION

M. Hohenbichler

1. Introduction

Let $\underline{c} = (c_1,\ldots,c_n) \in \mathbb{R}^n$ and let $\underline{X} = (X_1,\ldots,X_n)$ be a standard-normal random vector with correlation coefficient matrix $\underline{\underline{R}} = (\rho_{ij} : 1 \leq i, j \leq n)$. The n-dimensional normal integral

$$(1) \quad P(\bigcap_{i=1}^{n} \{X_i \leq c_i\}) = \Phi_n(\underline{c};\underline{\underline{R}}) = \int_{-\infty}^{c_1} \cdots \int_{-\infty}^{c_n} \varphi_n(\underline{x};\underline{\underline{R}})\, dx_n \ldots dx_1$$

cannot be solved analytically. However, it has important applications. For example, it represents the failure probability of a redundant (parallel) system with failure events $\{X_i - c_i \leq 0\}$ or the survival probability of a series system with safe domains $\{X_i - c_i \leq 0\}$ with the X's denoting uncertain performance variables. Therefore, approximate solution methods or bounding techniques are highly desireable.

A survey of the available solution methods of eq.(1) is given in chapter 35 of reference [1]. The methods described in [1] generally require high-dimensional numerical integration and, thus, exponentially increasing computing time, so that their application is, in fact, limited to very few dimensions ($n \leq 5$, say) [2]. Less extensive methods exist only for special cases. For example, a formula of Dunnett and Sobel [3] reducing eq.(1) to a one-dimensional integral (eq.(26) in [1], chapter 35) is restricted to correlation coefficients $\rho_{ij} = \lambda_i \lambda_j$ $(i \neq j)$.

In the following a numerical solution method is proposed which is based on first-order reliability concepts [4,5]. The idea to apply first-order reliability techniques for the evaluation of multivariate normal probabilities has first been put forward by B. Fießler, who developed a first-order solution of the Dunnett-Sobel integral.

2. Method of Solution

Firstly, $\Phi_n(\underline{c};\underline{\underline{R}})$ is separated into two factors one involving a (n-1)-dimensional conditional probability, i.e.

$$(2) \quad \Phi_n(\underline{c};\underline{\underline{R}}) = P(\bigcap_{i=2}^{n} \{X_i \leq c_i\} \mid \{X_1 \leq c_1\}) \cdot P(X_1 \leq c_1)$$

and the vector $\underline{X}$ is expressed as a linear function of standard-normal, independent variables U_i:

$$(3) \quad \begin{aligned} X_1 &= U_1 = \alpha_{11} U_1 \\ X_2 &= \alpha_{21} U_1 + \alpha_{22} U_2 \qquad (\alpha_{21} = \rho_{21}) \\ &\vdots \\ X_n &= \alpha_{n1} U_1 + \ldots + \alpha_{nn} U_n \quad (\alpha_{n1} = \rho_{n1}) \end{aligned}$$

For the evaluation of α_{ij} see appendix A.

Suppose now $U_1 = X_1 \leq c_1$; the conditional distribution function of U_1 conditioned on $U_1 \leq c_1$ is

$$(4) \quad F(u \mid (-\infty, c_1]) = \frac{1}{\Phi(c_1)} \cdot \Phi(u) \qquad (u \leq c_1),$$

and, thus, the distribution of the variable $\tilde{U}_1$ defined by

$$(5) \quad \tilde{U}_1 := F^{-1}(\Phi(U_1) \mid (-\infty, c_1]) = \Phi^{-1}(\Phi(c_1)\, \Phi(U_1))$$

equals the conditional distribution of $U_1 \mid U_1 \leq c_1$. Since the U_i's are independent, the distribution of $(\tilde{X}_2,\ldots,\tilde{X}_n)$,

$$(6) \quad \tilde{X}_i := \alpha_{i1} \tilde{U}_1 + \sum_{j=2}^{i} \alpha_{ij} U_j \qquad (i \geq 2)$$

then equals the conditional distribution of $(X_2,\ldots,X_n) \mid U_1 \leq c_1$

$$(7) \quad P(\bigcap_{i=2}^{n} \{\tilde{X}_i \leq c_i\}) =$$

$$= \int P(\bigcap_{i=2}^{n} \{\tilde{X}_i \leq c_i\} \mid \tilde{U}_1 = \tilde{u})\, dF_{\tilde{U}_1}(\tilde{u}) =$$

$$= \int P(\bigcap_{i=2}^{n} \{\alpha_{i1}\tilde{u} + \sum_{j=2}^{i} \alpha_{ij} U_j \leq c_i\})\, dF_{\tilde{U}_1}(\tilde{u}) =$$

$$= \int \ldots\, dF(\tilde{u} \mid (-\infty, c_1]) =$$

$$= \int P(\bigcap_{i=2}^{n} \{\sum_{j=1}^{i} \alpha_{ij} U_j \leq c_i\} \mid U_1 = u)\, dF(u \mid (-\infty, c_1]) =$$

$$= P(\bigcap_{i=2}^{n} \{X_i \leq c_i\} \mid U_1 \leq c_1),$$

With (5) and (6) the last probability reads in the space of U_i-variables:

$$(8)\quad P(\bigcap_{i=2}^{n}\{X_i \le c_i\} \mid X_1 \le c_1) = P(\bigcap_{i=2}^{n}\{\tilde{X}_i \le c_i\}) =$$

$$= P(\bigcap_{i=2}^{n}\{\alpha_{i1}\Phi^{-1}(\Phi(c_1) \quad \Phi(U_1)) + \sum_{j=2}^{i}\alpha_{ij}\,U_j - c_i \le 0\}) =$$

$$=: P(\bigcap_{i=2}^{n}\{g_i(U_1,\ldots,U_n) \le 0\}).$$

Linearisation of the "modes" g_i in their "ß-points"

$$(9)\quad \{g_i(U_1,\ldots,U_n) \le 0\} \approx \{\sum_{j=1}^{n}\gamma_{ij}U_j + \beta_i \le 0\}$$

with the normalising condition $\sum_{j=1}^{n}\gamma_{ij}^2 = 1$ generates a new (n-1)-set of standardnormal variables:

$$(10)\quad X_i^{(2)} := \sum_{j=1}^{n}\gamma_{ij}\,U_j \qquad (2 \le i \le n).$$

Their correlation matrix is

$$(11)\quad \underline{\underline{R}}^{(2)} := (\rho_{ij}^{(2)} : 2 \le i,j \le n),$$

$$\rho_{ij}^{(2)} := \sum_{k=1}^{n}\gamma_{ik}\,\gamma_{jk}.$$

Thus, writing $c_i^{(2)} := -\beta_i$, eq.(8) can be approximated by

$$(12)\quad P(\bigcap_{i=2}^{n}\{\tilde{X}_i \le c_i\}) \approx P(\bigcap_{i=2}^{n}\{X_i^{(2)} \le c_i^{(2)}\}) = \Phi_{n-1}(\underline{c}^{(2)} ; \underline{\underline{R}}^{(2)})$$

indicating that the dimension of the integral (1) has been reduced by one:

$$(13)\quad \Phi_n(\underline{c} ; \underline{\underline{R}}) \approx \Phi(c_1)\ \Phi_{n-1}(\underline{c}^{(2)} ; \underline{\underline{R}}^{(2)}).$$

Repeated application of eq.(13) finally yields:

$$(14)\quad \Phi_n(\underline{c} ; \underline{\underline{R}}) \approx \Phi(c_1)\ \Phi(c_2^{(2)}) \cdot \ldots \cdot \Phi(c_n^{(n)}).$$

In non-degenerate cases in the l-th iteration n-l modes g_i have to be linearized, so that the total amount of computation time is roughly proportional to $n(n-1) \sim n^2$. This enables also solutions for higher dimensions.

For degenerate multinormal distributions, the case $\alpha_{i1} = 1$ may occur which implies $X_i = X_1 \le \min\{c_1, c_i\}$ and enables a faster solution (less than n factors in eq.(14).

3. Discussion and Modifications

To illustrate the accuracy of the method, Figure 1 presents some results for n = 1 % 50, $c_1 = c_2 = \ldots = c_n = c$ and $\rho_{ij} = \rho$ (equi-correlation). The ordinate is $\beta = -\Phi^{-1}(\Phi_n(\underline{c};\underline{\underline{R}}))$. It is emphasized, however, that the algorithm works for arbitrary $\underline{c}$ and $\underline{\underline{R}}$ and is not simplified by these assumptions. The exact result is obtained from Fießler's solution of the Dunnett-Sobel integral, whose error is negligible for this purpose as compared with numerical integration.

The crucial point of the proposed technique is the linearisation of the surfaces $\{g_i = 0\}$. The resulting "first-order error" in eq.(9) might be significant and, therefore, the total error of eq.(14) might increase disproportionally. This requires some additional remarks:

a) The functions g_i in eqs.(8), (9) are linear in all variables except the first. Thus, the first-order error essentially is the one of a two-dimensional problem, which usually is small.

b) In the case $c_i >> 0$, in the region of interest $\tilde{U}_1$ is very close to U_1 and, therefore, g_i is "almost" linear, so that in turn the first-order error is small. This accounts for the good result obtained for parallel systems with large failure probabilities resp. series systems with small failure probabilities (see also Figures 1b, 1c).

c) If the X_i's are zero-correlated, then $a_{i1} = 0$ $(i \ge 2)$ and, therefore, g_i is linear; for the X_i's being fully correlated, $X_1 = X_i$ and g_i is univariate. Hence, the result is exact for fully correlated and independent X_i's.

d) In the course of the iteration the correlations $\underline{\underline{R}}^{(1)}$ are decreasing with l while (for positive correlations) the $c_i^{(1)}$'s are significantly increasing. Both effects reduce the error in subsequent iterations (see b) and c)). This explains why the total error is surprisingly moderately increasing with the dimension n (see Figure 1).

e) For simplification, all of the c_i's are set equal to c ($c_i = c$ for all i) in Figure 1. This is the most unfavourable case (no dominating mode). Furthermore, all correlation coefficients ρ_{ij} are assumed equal ($\rho_{ij} = \rho$ for all i,j). This is again unfavourable, since with originally equal c_i's and mutually different ρ_{ij}'s some modes would dominate after the first iteration.

f) Following a suggestion of R. Rackwitz, an essential improvement is obtained in replacing β_i in eq.(9) by

$$(15)\quad \{g_i \leq 0\} \approx \{\sum_{j=1}^{n} \gamma_{ij} U_j + \beta_i^* \leq 0\}\ ,\ \beta_i^* := -\Phi^{-1}(p_i)$$

where p_i is a better estimate of $P(g_i \leq 0)$ obtained with higher order techniques. A very natural application of this idea originates from a deeper investigation of the general shapes of the g_i's. As already mentioned, g_i can be represented in a two-dimensional space of independent standard-normal variates U,V ($U = U_1$, $V = (1-\alpha_{i1}^2)^{-1/2} \cdot \sum_{j=2}^{i} \alpha_{ij} U_j$). Figure 2a shows a typical graph of g_i for positively correlated X_1 and X_i (i.e. $\rho_{i1} = \alpha_{i1} \geq 0$), Figure 2b for negative correlation. Two asymptotes for $\{g_i = 0\}$ are obvious. Namely, one obtains for

$$(16)\quad X_1 = U_1 = U\ ,$$

$$X_i = \alpha_{i1} U + \sqrt{1-\alpha_{i1}^2}\, V =: aU + bV \text{ and}$$

$$g_i(u,v) = a\,\Phi^{-1}(\Phi(c_1)\Phi(u)) + bv - c_i$$

the asymptotes

$$(17)\quad h_1(u,v) = au + bv - c_i = 0$$

$$h_2(u,v) = v - \frac{1}{b}(c_i - ac_1) = 0$$

These asymptotes together with the first-order linearisation eq.(9)

$$(18)\quad h_3(u,v) = \gamma_1 u + \gamma_2 v + \beta = 0$$

determine a polygonal approximation to $\{g_i = 0\}$. The probability

$$(19)\quad P(g_i \leq 0) \approx p_i := \begin{cases} 1 - P(\bigcap_{j=1}^{3} \{-h_j(U,V) \leq 0\}) & (\alpha_{i1} \leq 0) \\ P(\bigcap_{j=1}^{3} \{h_j(U,V) \leq 0\}) & (\alpha_{i1} \leq 0) \end{cases}$$

where $h_j(U,V)$ are again correlated normal variables, can in turn be evaluated with the proposed method (n = 3).

The resulting improvement is illustratd in Figure 1. It also enables a rough estimate of the error of eq.(14). Closer approximations can, in principle, be obtained by considering other polygonials. Better linearization strategies for the modes g_i are studied in a subsequent paper.

g) For probabilities close to one (e.g. when applying eq.(14) to series systems), eq.(14) is not suitable numerically since numbers very close to 1 can only be represented on computers up to a certain number of digits. Here (but also for smaller probabilities) use of logarithms produces much better results:

$$(14^*)\quad \log \Phi_n(\underline{c};\underline{\underline{R}}) \approx \log\Phi(c_1) + \log\Phi(c_2^{(2)}) + \ldots + \log\Phi(c_n^{(n)})$$

where $(\log\Phi)$ is directly computed. Also, instead of eq.(5),

$$(5^*)\quad \tilde{U}_1 = (\log\Phi)^{-1}\,(\log\Phi(c_1) + \log\Phi(U_1))$$

is numerically more convenient.

h) For $\rho_{i1} = \alpha_{i1} \geq 0$ eq. (9) can be completed by (compare figure 2):

(9a) $\{g_i(U_1,\ldots,U_n) \leq 0\} \supset \{\sum_{j=1}^{n} \gamma_{ij} U_j + \beta_i \leq 0\}$

This results in:

(13a) $\Phi_n(\underline{c};\underline{\underline{R}}) \geq \Phi(c_1)\ \Phi_{n-1}(\underline{c}^{(2)};\underline{\underline{R}}^{(2)})$

if $\alpha_{i1} \geq 0$ $(i = 2,\ldots,n)$.

Therefore, the final result is too small if in the course of the algorithm all α_{i1} remain equal or larger than zero. Consequently, the probability estimates are unconservative for parallel system (too small failure probability) and conservative for series systems (too small survival probability). The converse is true for the $\alpha_{i1} \leq 0$, which is of minor practical interest.

4. Summary and Conclusions

The method for the approximate determination of n-dimensional standardnormal variables with arbitrary correlation coefficient matrix over semi-cubes as proposed herein appears computationally superior to any other known analytical or numerical methods. It is particularly useful in system reliability but may also be applicable in many other statistical problems involving multi-dimensional integration. Certain refinements will be discussed in a separate paper.

5. References

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Appendix A

The α_{ij} can be determined recursively:

$$\alpha_{11} = 1$$

$$\rho_{21} = \alpha_{11}\alpha_{21} = \alpha_{21} \Rightarrow \alpha_{21} = \rho_{21}$$

$$\mathrm{var}(X_2) = \alpha_{21}^2 + \alpha_{22}^2 = 1 \Rightarrow \alpha_{22} = \sqrt{1-\rho_{21}^2}$$

$$\rho_{31} = \alpha_{11}\alpha_{31} = \alpha_{31} \Rightarrow \alpha_{31} = \rho_{31}$$

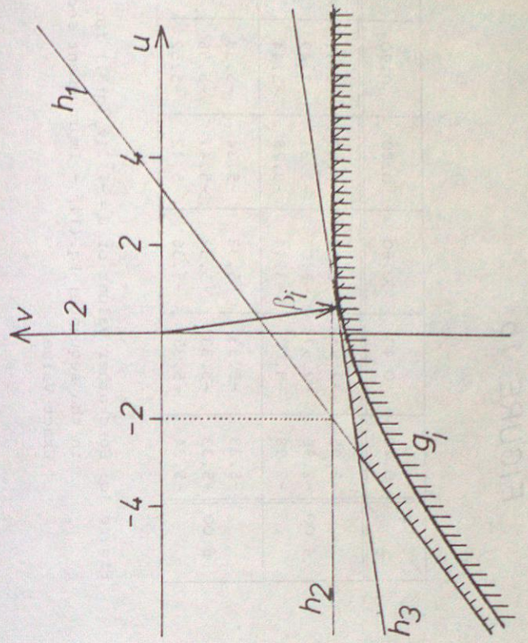
$$\rho_{32} = \alpha_{31}\alpha_{21} + \alpha_{32}\alpha_{22} \Rightarrow \alpha_{32} = \frac{\rho_{32}-\alpha_{31}\alpha_{21}}{\alpha_{22}}$$

$$\mathrm{Var}(X_3) = \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1 \Rightarrow \alpha_{33} = \sqrt{1-\alpha_{31}^2-\alpha_{32}^2}$$

etc.

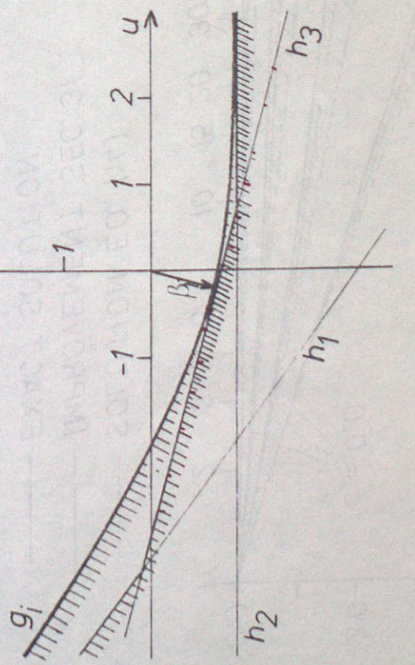
For degenerate distributions this algorithm is easily modified.

Eq.(3) is the Rosenblatt-Transformation of $(X_1, X_2, \ldots, X_n)$, eq.(6) together with (5) is the Rosenblatt-Transformation of $(X_2, \ldots, X_n)$ ([5]).



$c_1 = c_i = -2$, $\rho = -0.6$

FIGURE 2b



$c_1 = c_i = -2$, $\rho = +0.6$

FIGURE 2a

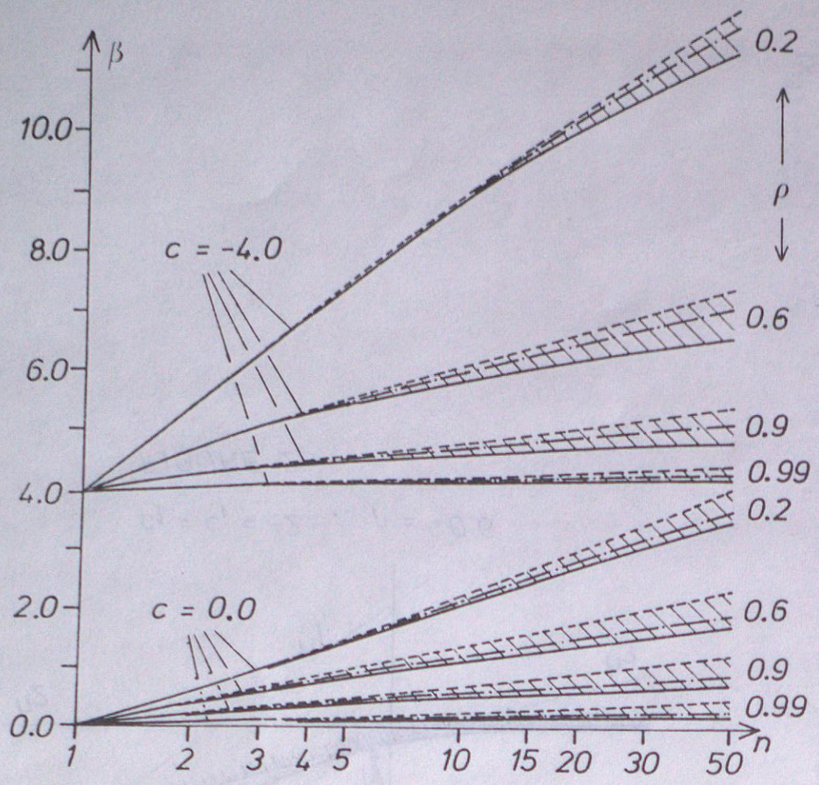


FIGURE 1a

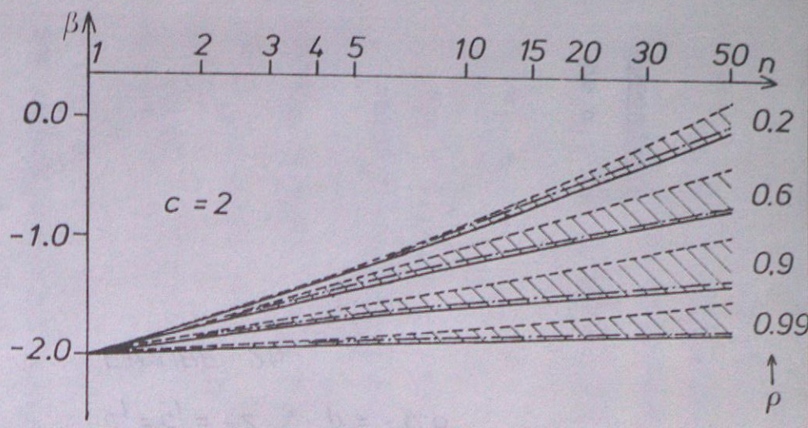


FIGURE 1b

c \ ρ	0.20	0.40	0.60	0.80	0.90	0.99
4.00	-2.95 -2.96 -2.98	-2.95 -2.97 -3.02	-2.98 -3.07 -3.11	-3.07 -3.31 -3.28	-3.19 -3.49 -3.44	-3.55 -3.80 -3.80
6.00	-5.33 -5.33 -5.34	-5.33 -5.33 -5.35	-5.33 -5.33 -5.36	-5.34 -5.41 -5.42	-5.38 -5.56 -5.52	-5.61 -5.82 -5.80

Figure 1c: Some other values of $\beta = -\phi^{-1}(\Phi_n(\underline{c};\underline{R}))$ for n = 50 in the sequence: Eq.(14) - Improvement sec. 3f - Exact Value

A BROWNIAN PROCESS MODEL IN STOCHASTIC ACTIVITY NETWORKS - AN APPLICATION OF FIRST-ORDER RELIABILITY METHODS

R. Rackwitz, S. Gollwitzer

1. Introduction

The representation of complex technical projects by activity networks has proven to be a useful tool in the efficient management of engineering tasks. Unfortunately, deterministic considerations quite frequently are of limited value since many input variables are uncertain beforehand and, therefore, total durations are generally underestimated, previously critical paths become subcritical, penalties may possibly burden certain parties in an unjustified manner and the total project costs can increase. At least in the state of planning those uncertainties should be taken into account but might be removed gradually by updating during the course of the job.

A number of proposals exists to handle activity networks with uncertain activity durations [1,2,8,9,10,12,13]. Most of them are restricted to relatively simple stochastic models for the uncertain activity durations. Others introduce significant simplications and approximations in order to overcome the problem of dependences of activities and/or activity paths. In principle, a probabilistic method for activity networks should be capable to give answers to the following two questions:

a) What is the probability distribution (or mean and variance) of the state of production in a given point in time so that, for example, the capital demand can be determined.

b) What is the probability distribution of the duration of partial or complete networks consisting of a number of sequences of various activities.

In the following, some concepts of modern first-order reliability methods will be applied to stochastic networks with emphasis to the last question. It will be demonstrated that these concepts enable a rather realistic and numerically accurate treatment of many network problems. In particular, the spectrum of possibilities of modelling the uncertain phenomena can be broadened considerably and, therefore, can give rise to efficient control strategies. In this respect, the problem of stochastic activity networks is not only a problem of organization and costs but also an ingredient for the assurance of the quality of technical facilities.

2. General Formulation for Stochastic Networks

Denote the i-th activity by A_i having uncertain duration T_i and the j-th path through a network by ω_j having uncertain duration T_j. Define a connectivity matrix $\underline{\gamma} = \{\gamma_{ij}\}$ with zero-one variables indicating whether the i-th activity is contained in path j or not. Then, the probability distribution of the duration of path j is

$$(1) \quad P(T_j \leq t) = P\Big(\sum_i \gamma_{ij}\, T_i \leq t\Big)$$

and, therefore, the probability distribution of project duration T becomes

$$(2) \quad P(T \leq t) = P(\max\{T_j\} \leq t) = P\Big(\bigcap_j \{T_j \leq t\}\Big) = P\Big(\bigcap_j \{T_j - t \leq 0\}\Big)$$

which, in general, is difficult to compute because of the many convolution operations required by eq.(1) and the complex dependence structure of the T_j's to be considered in eq.(2). However, let $\underline{T} = T_1,\ldots,T_n) = \underline{G}(\underline{U}) = G(U_1,\ldots,U_n)$ be a suitable transformation of the random, generally dependent vector $\underline{T}$ of activity durations into an independent standard normal vector $\underline{U}$, e.g. the one proposed in [6]:

$$\underline{T} = G(\underline{U}) = \begin{cases} T_1 = F_1^{-1}(\phi(U_1)) = G_1(U) & (3a) \\ T_2 = F_2^{-1}(\phi(U_2)\,|\,T_1) = G_2(\underline{U}) & (3b) \\ \vdots & \\ T_i = F_i^{-1}(\phi(U_i)\,|\,T_1,\ldots,T_{i-1}) = G_i(\underline{U}) & (3i) \\ \vdots & \\ T_n = F_n^{-1}(\phi(U_n)\,|\,T_1,\ldots,T_{n-1}) = G_n(\underline{U}) & (3n) \end{cases}$$

where $F_1(t_1) = P(T_1 \leq t_1)$, $F_i(t_i) = F_i(t_i|t_1,\ldots,t_{i-1}) = P(T_i \leq t_i | T_1 = t_1,\ldots,T_{i-1} = t_{i-1})$ and $\phi(.)$ the standard normal integral. Eq.(1) may then be written as

$$(4)\quad P(T_j \le t) = P(\sum_i \gamma_{ij}\, T_i \le t)$$
$$= P(\sum_i \gamma_{ij}\, G_i(\underline{U}) \le t)$$
$$= P(\sum_i \gamma_{ij}\, G_i(\underline{U}) - t \le 0)$$
$$\approx P(\underline{\alpha}_j\, \underline{U} + \beta_j \le 0)$$

where the surface $\sum_i \gamma_{ij}\, G_i(\underline{U}) - t = 0$ is replaced by its linearization $\underline{\alpha}_j \underline{U} + \beta_j = 0$ in the so-called "ß-point" (see [5,11]). $\underline{\alpha}_j$ is the vector of direction cosines and β_j the distance of the linearization to the coordinate origin (β is negative if $\sum_i \gamma_{ij}\, G_i(\underline{0}) - t < 0$). We note that $Z_j = \underline{\alpha}_j \underline{U}$ is again a standard normal variable. Two variables Z_p and Z_q have correlation coefficient $\rho_{pq} = \underline{\alpha}_p^T\, \underline{\alpha}_q$. Hence, the probability eq.(2) can be approximated by

$$(5)\quad P(T \le t) = P(\bigcap_j \{Z_j \le \beta_j\})$$
$$= \int_{-\infty}^{\underline{\beta}} f(\underline{z};\underline{\underline{R}})\, d\underline{z}$$

the standard multi-normal integral with correlation coefficient matrix $\underline{\underline{R}} = \{\rho_{pq}\}$ which may be bounded or closely estimated by using the algorithm given in [7].

As an example for questions of the first type mentioned before we study the production states of the system. For the moment, we concentrate only on states when one activity has been terminated. Let V_i be the deterministic volume produced in activity i and assume that it is unlikely that two or more activities end at the same point in time. Then, the expected total volume produced at time t_k is

$$(6)\quad E[V] = \sum_{i=1}^{n} V_i\, P(\bigcap_j \{\sum_\nu \gamma_{j\nu} T_\nu \le t_k\})$$

where the summation goes only over those activities which are necessary to accomplish activity i.

3. The Brownian Process Model for Duration of Activities

Various assumptions for the distribution of activity durations have been made; among them are the beta-, normal-, log-normal- and the gamma-distribution. Here, it is assumed that the distribution of durations is the first-passage time distribution of a Brownian process with linear drift (see Figure 1).

The Brownian process results quite naturally from the following considerations. Assume that at random points in time the production is modified by a random amount Y, so that $X = \sum_{i=1}^{N} Y_i$ is the total production after N such changes in production velocity. The state of production at time t therefore has mean $E[N] \cdot E[Y]$ and variance $E^2[Y]\, Var[N] + Var[Y]\, E[N]$ where N is the random number of changes in the time interval [0,t]. If, in particular, the changes occur according to a Poisson process with intensity ν and $\eta = E[Y]$ and $\xi^2 = Var[Y]$, the process has mean $\mu = E[X] = \eta\nu t$ and variance $\sigma^2 = Var[X] = (\eta^2 + \xi^2)\nu t$. By virtue of certain limiting operations the Brownian process is obtained. In order to assure that the production remains positive we require $\mu >> \sigma$. As known, the states of a Brownian process at any time can be described by the normal distribution. The first-passage distribution is the distribution of times until the process first crosses a certain level. Specifically, if the production process starts at x_o, its probability density of states x after time t is known to be [3]:

$$(7)\quad f(x,t|x_o) = \frac{1}{\sigma\sqrt{2\pi t}} \exp -[\frac{1}{2}(\frac{x-(x_o+\mu t)}{\sigma\sqrt{t}})^2]$$

The probability density of the time to the first crossing of a barrier at distance $V - x_o$ is [3]:

$$(8)\quad f(t|x_o,V) = \frac{V-x_o}{\sigma\sqrt{2\pi t^3}} \exp [-\frac{1}{2}(\frac{(V-x_o)-\mu t)}{\sigma\sqrt{t}})^2]$$

Here, V represents the total volume to be produced in order to accomplish a given task. In this case the time to the first passage has mean $E[T] = (V-x_o)/\mu$ and variance $Var[T] = (V-x_o)\sigma^2/\mu^3$. The distribution function must be determined by numerical integration or by use of formula 16, chapter 15 given in [14].

In order to make the model slightly more realistic, assume now that the process starts at $x_o = o$ with an initial mean "velocity" μ_o. The time target for completing the activity is T^*. The initial velocity generally is much smaller than V/T^*. The state of the process is observed at given points in time $t_1, t_2, \ldots$. At time t_1 the process state has distribution function

$$(9) \quad F_X(x_1,t_1) = \phi\left(\frac{x_1-\mu_o t_1}{\sqrt{t_1}\,\sigma}\right)$$

The probability of having completed the task up to $t = t_1$ is:

$$p_1 = P(T \leq t_1) = \int_0^{t_1} f_T(t|x_o = 0, V, \mu_o, \sigma)\, dt$$

If $x_1 < V$, the process continues. Given the state $X = x_1$ at $T = t_1$ the process mean is now changed into

$$(10) \quad \mu_1 = \frac{V - x_1}{T^* - t_1}$$

and, therefore, the distribution function at time $T = t_2$ is (compare also Figure 2):

$$(11) \quad F_X(x_2,t_2|x_1,t_1) = \phi\left[\frac{x_2-(\mu_1(t_2-t_1)+x_1)}{\sqrt{t_2-t_1}\,\sigma}\right]$$

The probability of having completed the task now is:

$$p_2 = P(t_1 < T \leq t_2) = \int_0^{t_2-t_1} f_T(t|x_1, V, \mu_1, \sigma)\, dt$$

Of course, eq.(10) represents only one of several alternative control strategies. The foregoing procedure is repeated until $T = t_k$. For a given probability level of first-passage times k is a random number. Hence, the density of the first-passage time in the last interval is:

$$(12) \quad f_T(t|x_k,V) = \left(1 - \sum_{i=1}^{k} p_i\right) \frac{V - x_k}{\sigma\sqrt{2\pi(t-t_k)^3}} \exp\left[-\frac{1}{2}\left(\frac{(V-x_k)-\mu_k(t-t_k)}{\sigma\sqrt{t-t_k}}\right)^2\right]$$

$$\text{for } t > t_k$$

Various refinements are possible, for example, a random observation error for x_i or a random delay time after which the new mean velocities become valid can be introduced. Other control strategies may be used. A possible modification is also to let sequential activities start at a given percentage of volume of the preceeding activity. In this case, total project duration is the intersection of all path durations and all individual activity durations smaller than the given value of t (see eq.(2)).

Furthermore, the parameters μ_o and σ may be treated as uncertain reflecting the particular conditions of that activity or of the entire job. It is seen that the sequence of conditional distributions has exactly the form required for the transformation eq.(3). and, therefore, the numerical calculations can be performed.

4. Examples

Figure 3 shows a network, which has been computed with different methods for the determination of the distribution of the project duration given in the literature [1,8,9]. In Figure 4 the results are compared with the proposed method and the Ditlevsen-Bounds [4]. (Note, that all individual activity durations are normally distributed.) It can be recognized that the method proposed herein falls well between the bounds according to [4] and is close to the simulation results [9]. For the same network one can conclude from Figure 5 that the distributional assumptions can be quite significant. In this figure the activity durations are assumed independently normally, log-normally and Brownian first-passage-time distributed with the same mean and standard deviation, respectively. Note, that the log-normal and Brownian assumption lead essentially to the same result and that there is a probability of less than 0.5 for the deterministically critical duration (i.e. all activities with their mean values). Figure 6 represents the network for a road construction project with 20 activities and 6 possible paths of which 5 are deterministically critical. In Figure 7 the distribution of total duration is shown for uncontrolled independent Brownian processes with the given means and standard deviations for the first-passage distributions and

for processes which start with the same parameters but are controlled at $t_1 = E[T_i]/2$ according to eq.(10). For comparison, the bounds obtained by the PERT technique are also given and shown to be considerably in error (without control, naturally).

5. Summary and Conclusions

Techniques of first-order system reliability are applied to activity networks. They enable the realistic modelling of individual activities as well as the treatment of complex dependencies between activities and paths. For very large networks it may be necessary to compromize between the details of modelling of the individual activity and the numerical effort such that the number of uncertain variables remains sufficiently small, e.g. less than 100, say. Further studies are intended to investigate the role of a number of generalizations and refinements.

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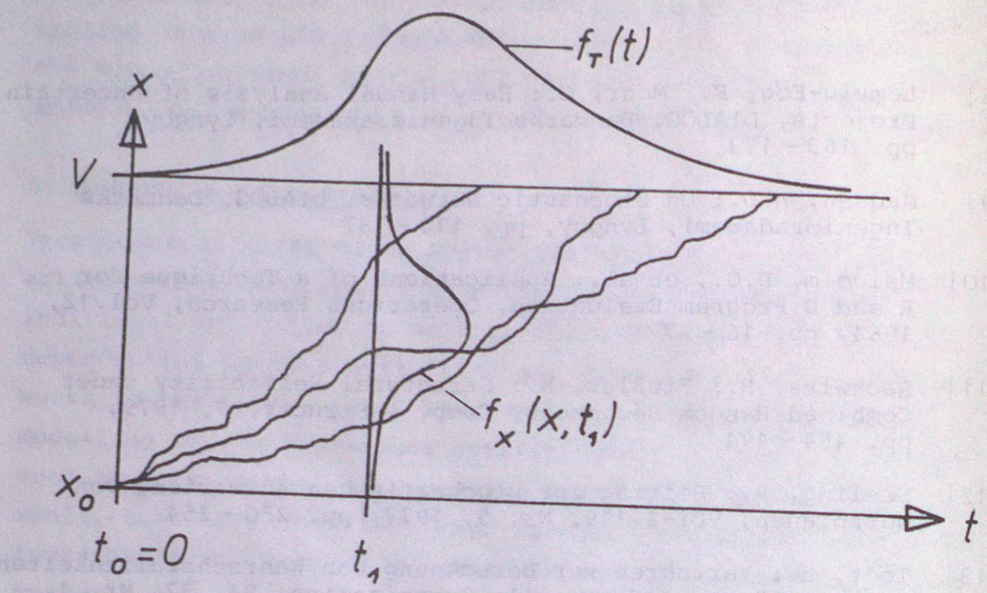


Figure 1

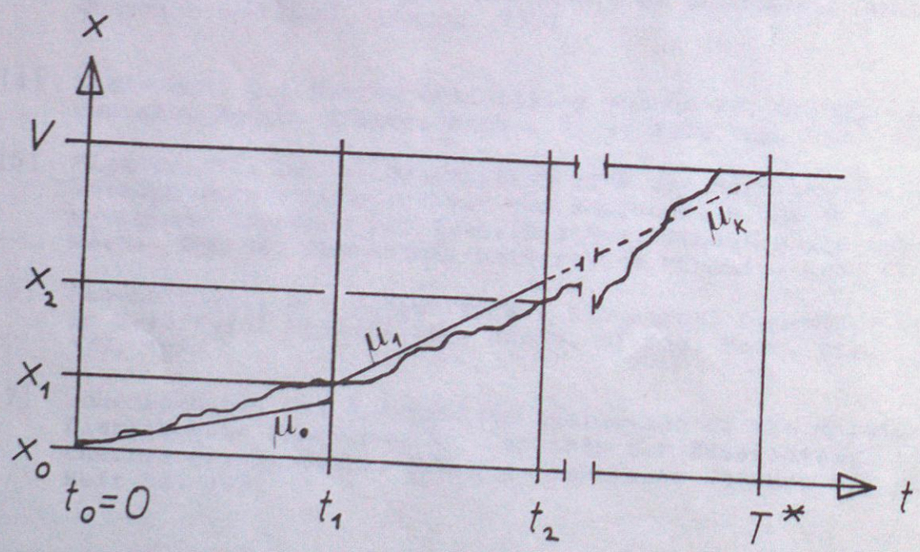


Figure 2

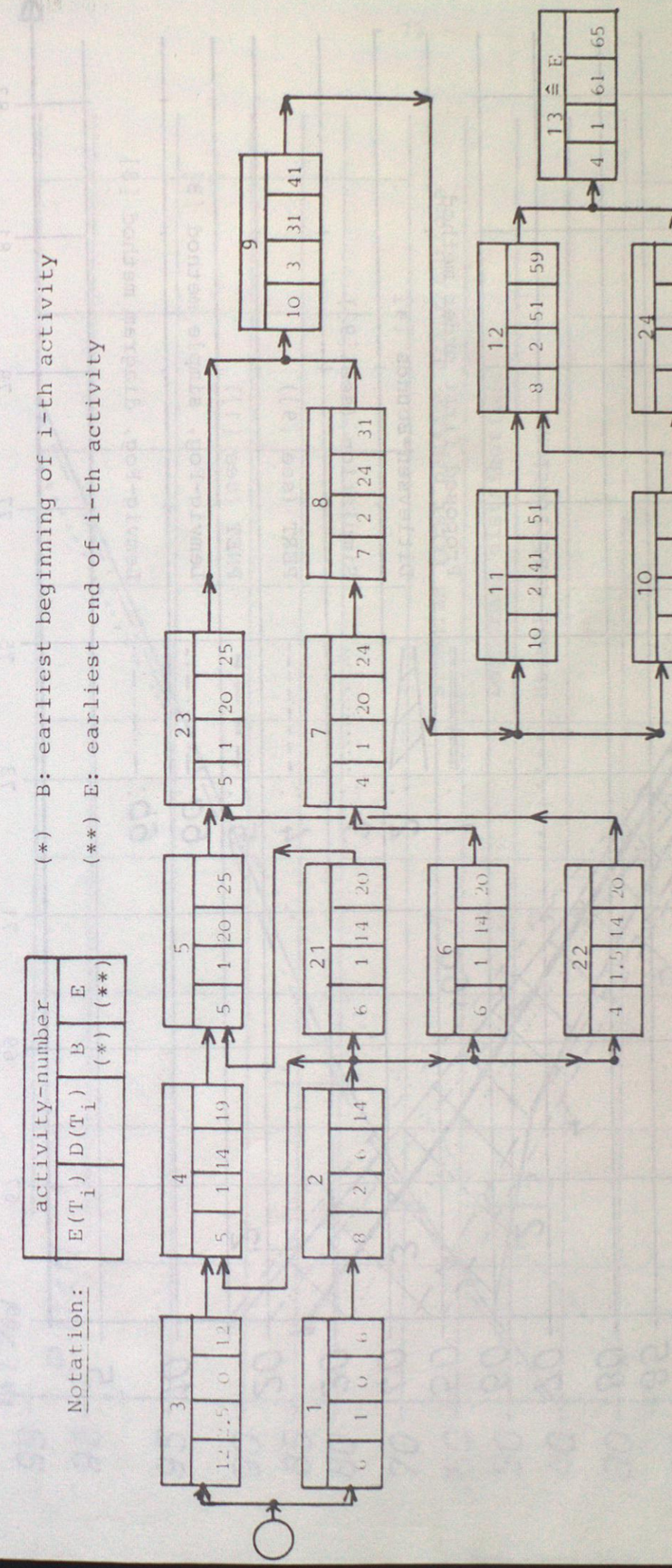


Figure 3: Network from [9]

Figure 4

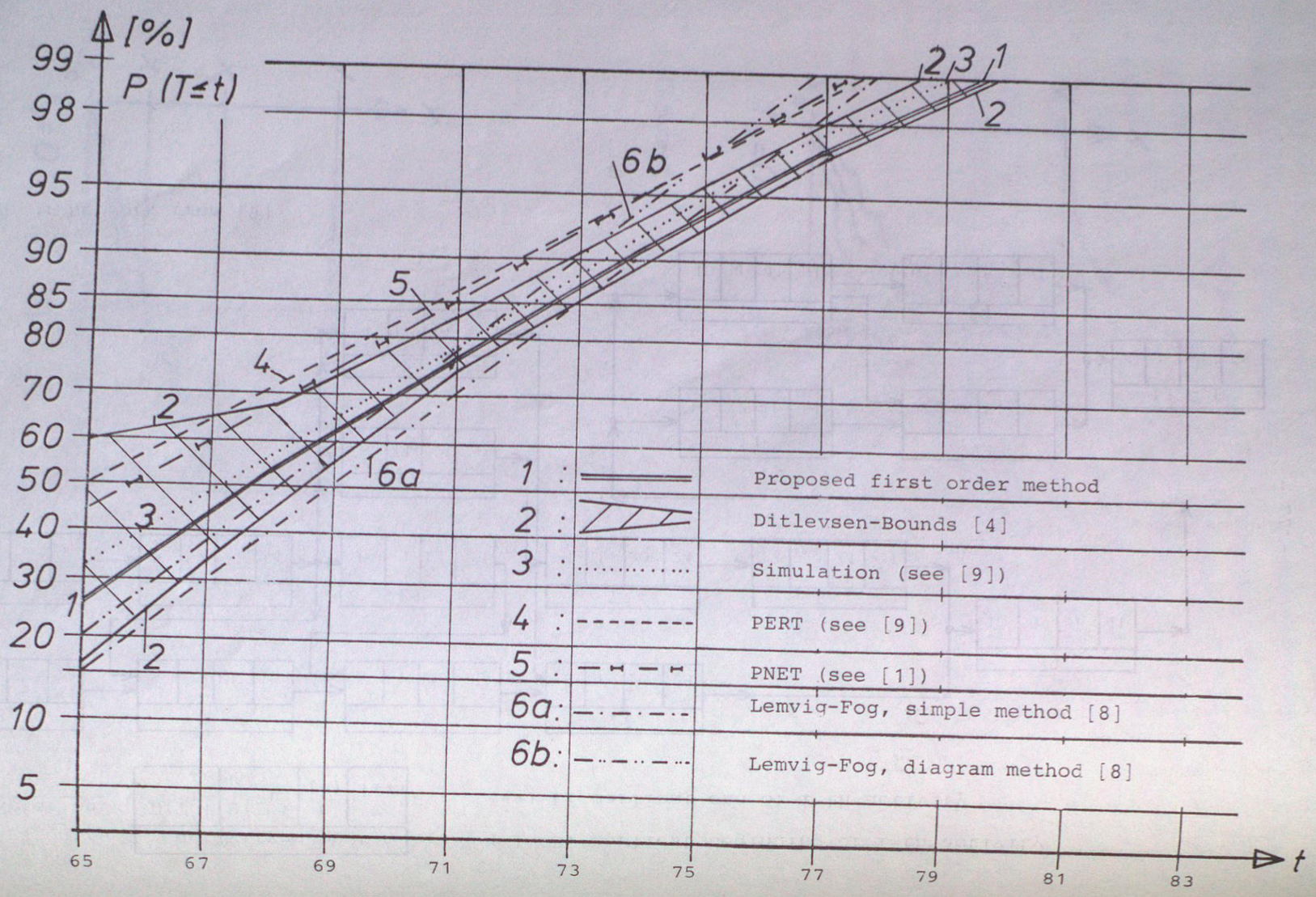
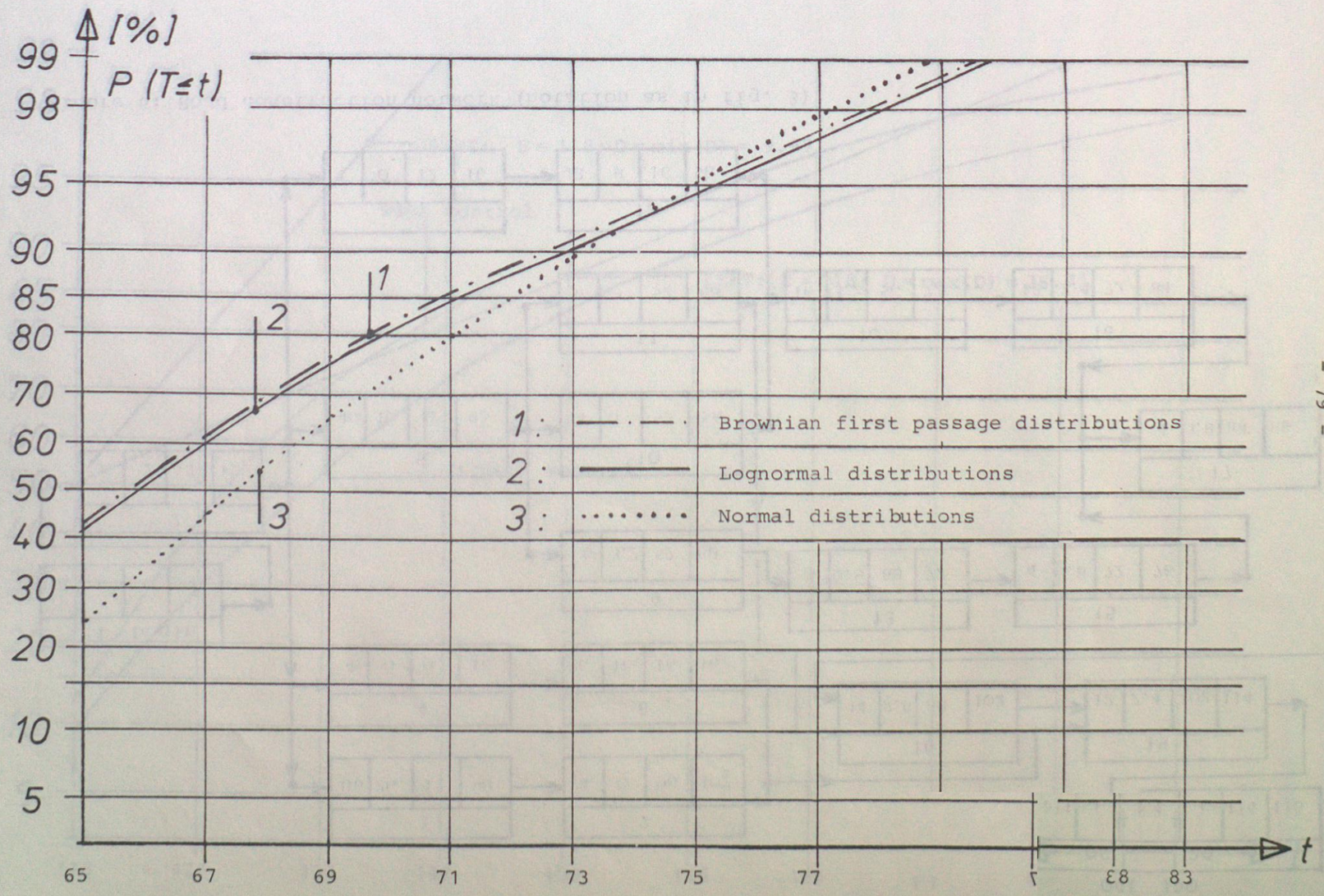


Figure 5



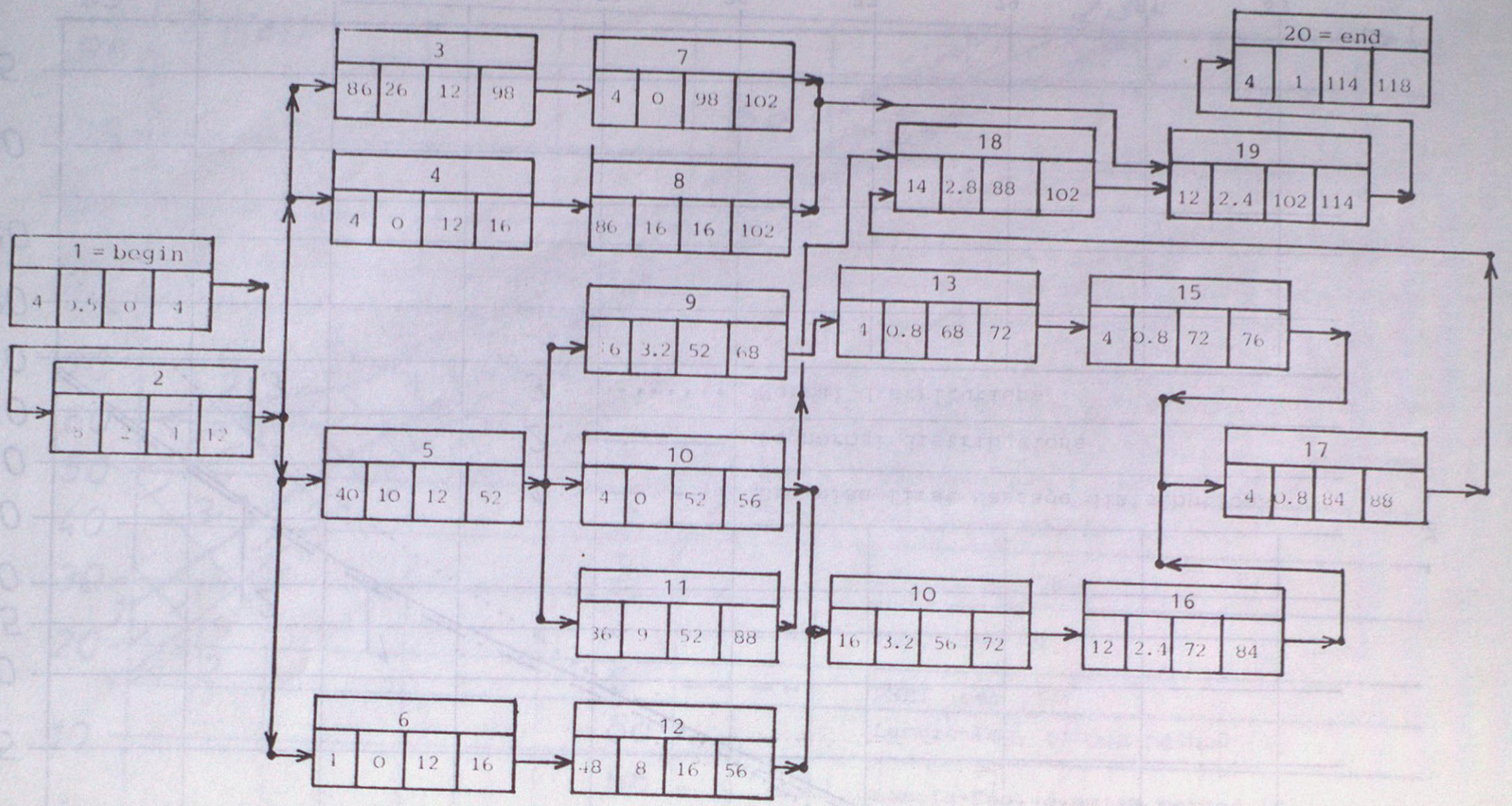
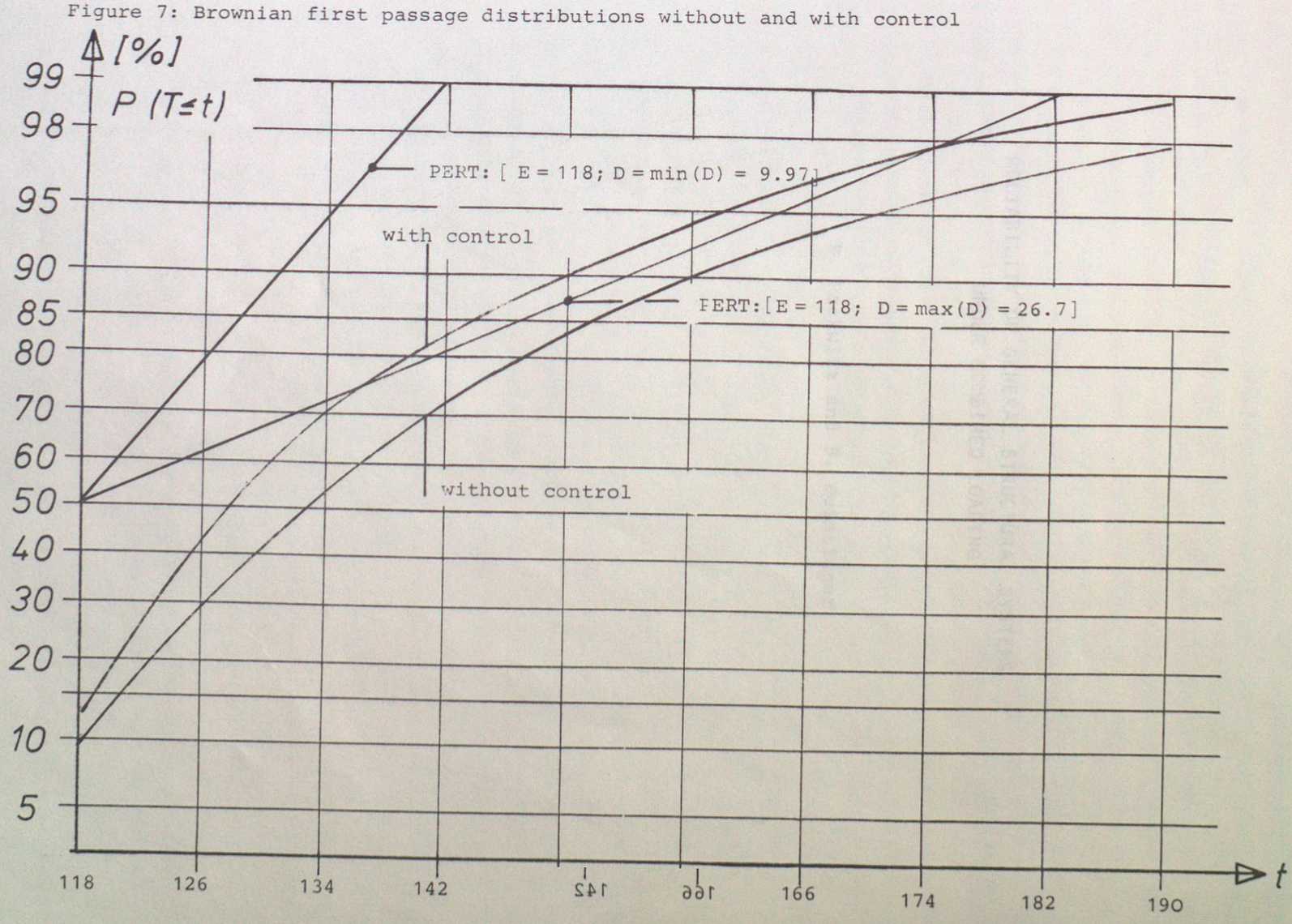


Figure 6: Road construction network (notation as in fig. 3)

Figure 7: Brownian first passage distributions without and with control



RELIABILITY OF GENERAL STRUCTURAL SYSTEMS UNDER COMBINED LOADING

R. Rackwitz and B. Peintinger

Introduction

Traditional design of structural systems generally is based on member by member design assuming unfavourable actions and conservative resistances between which a certain margin of "safety" is kept in order to take account of uncertainties in those asumptions and in the structural model ordinarily applied. The rise of a probabilistic treatment of the uncertainties in the structures area has put out such procedures to a revaluation yielding sometimes surprising results, confirming present practice in other cases, but also exhibiting interesting possibilities for the extrapolation of traditional rules to new areas of building construction. Many deterministic approaches have found their adequate probabilistic counter part. This is particularly so for simple structural component reliability.

Structural reliability or better its complement, the structural risk quantified here as a failure probability per design life should be a small number. It should be related to some structural state being defined as adverse - specifically being related to a given loss. A state of given loss is denoted as a limit state. The usual formulation of structural reliability then is based on the selection of a number of basic uncertainty variables, e.g. actions, dimensions, resistances or model errors, whose realisations belong either to the safe set of data points or to the disjoint set of failure points. These sets are separated by the limit state equation. For example, let $\underline{X} = (X_1,\ldots,X_q)^T$ be the time-invariant vector of random uncertainty variables with joint distribution function $F_{\underline{X}}(\underline{x})$ and define the limit state by $g(\underline{x}) = 0$, while for the safe set it is $S = \{\underline{X} : g(\underline{x}) > 0\}$. Then, the simpliest reliability problem can be formulated as

$$P_f = P[F] = P[g(\underline{X}) \leq 0] = 1 - \int_{\{S\}} dF_{\underline{X}}(\underline{x}) = \int_{\{F\}} dF_{\underline{X}}(\underline{x}) = \int_{\{F\}} f_{\underline{X}}(\underline{x})\, d\underline{x} \quad (1)$$

where $f_{\underline{X}}(\underline{x})$ is the density of $\underline{X}$ if it exists and F denotes the failure event $F = \{\underline{X} : g(\underline{x}) \leq 0\}$. For convenience, we will use

this formulation for component (section, element, ...) failure, only.

Structural systems, on the other hand, usually can fail in a number of modes each of which may be viewed as a component in a series system which, per definition, fails if any one component fails. For example, if a yield hinge model is assumed for structural behaviour each of the possible mechanisms represents a failure mode. Each multiple component mod of failure, in turn, generally has various ways to carry th load, e.g. it can survive a given load although part of its components have "failed". Therefore, each mode may be viewe as a parallel system which fails if none of the possible combinations of components survive. For example, part of th concrete in a stressed zone may have been ruptured while th rest sustains the load. Many other types of systems exist. an example, for the system just described the failure problity can formally be written as:

$$P_f = P[\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} F_{ij}] = P[\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} (g_{ij}(\underline{X}) \leq 0)] \quad (2)$$

a formulation which is similar to those used in classical reliability theory (see, for example, [4]). However, the numerical difficulties in pursuing an exact formulation as given by eqs. (1) and (2) are exorbitant. This is even more so if the basic variables or the failure boundary depend on time in which case one has to determine the probability that the random vector process $\underline{X}(t)$ leaves the safe domain in a certain time.

Subsequently, some concepts of the so-called first-order reliability method will briefly be reviewed because these have proven to be efficient tools in solving problems of the type as given by eq. (1). They will then be generalized in order to determine system reliabilities. Further, the formulation of eq. (2) is extended to include time-variant basic variables. It is believed that it is just this potential of possible generalizations which makes first-order reliability methods most useful in structural reliability. What appears important is that a number

of quite simple solutions can be produced which under certain in each practical case more or less restictive conditions are either exact or give strict narrow bounds from a mathematical point of view. Therefore, the type and nature of approximations and simplification which will be necessary when making the method more rich and powerful remains known at any stage of development and application. On the other hand, it is usually straightforward to simplify these methods in a number of areas such that they even can serve as direct design tools in every day work as exemplified by some recent safety codes.

What is perhaps crucial and which is essentially the case of this report is that since it rests on a random variable representation by standard normal variables dependencies among variables can properly be treated. This is particularly important in system reliability as will shown below.

Nevertheless, in concluding this introduction it is emphasized that perhaps none of the methods to be discussed is exact and, therefore, numerical results are conditional on the selected approximation method. If, however, structural reliability is a number close to one fairly accurate numerical results can be obtained in most cases. When judging the numerical accuracy of approximate reliability methods one has also to keep in mind that the numerical results are conditioned on the set of stochastic models selected for the uncertainties. The selection of stochastic models generally remains rather ambiguous and is prone to modifications whenever new data become available or the technical problem requires more information than the one the original model was capable to convey. In this respect the interpretation of calculated failure probabilities needs care. These, in fact, must not be interpreted as relative frequencies of failure. They are rather operational failure probabilities to be used in and consistent with a decision theoretic framework of design.

2. Estimates and Bounds on Component Failure Probabilities

Essential advantages when solving eq.(1) are gained if the original space of uncertainty variables can be transformed into a space of Gaussian variables, specifically into a space of independent standard normal variates. Then, the probabilistic manipulations can be performed in terms of Gaussian variables which by far have found most interest among statisticians and, fortunately, possess quite a number of useful analytical and/or numerical features. Such transformations have been given in [8,23,29,30,31] at various levels of sophistication but the one outlined in [17] is considered to be the most general. Let:

$$\underline{X} = \underline{T}(\underline{U}) \tag{3}$$

so that the component failure probability becomes

$$P_f = P(g(\underline{X}) \leq 0) = P(g(\underline{T}(\underline{U})) \leq 0) = P(\bar{g}(\underline{U}) \leq 0) \tag{4}$$

where $\underline{U}$ is an independent standard normal vector. For example, if $F_1(x) = P(X_1 \leq x)$ denotes the distribution function of X_1, $F_i(x|X_1,\ldots,X_{i-1}) = P(X_i \leq x|X_1,\ldots,X_{i-1})$, $i = 2,3,\ldots,q$ the sequence of conditional distribution functions of X_i conditioned on $X_1,\ldots,X_{i-1}$ and $\phi(.)$ the standard normal distribution function, it is

$$X_1 = F_1^{-1}[\phi(U_1)] = \min\{x : F_1(x) \geq \phi(U_1)\} \tag{5a}$$

$$X_2 = F_2^{-1}[\phi(U_2)|X_1] = \min\{x : F_2(x|X_1) \geq \phi(U_2)\} \tag{5b}$$

$$\vdots \qquad\qquad \vdots$$

$$X_q = F_q^{-1}[\phi(U_q)|X_1,\ldots,X_{q-1}] = \min\{x : F_q(x|X_1,\ldots,X_q) \geq \phi(U_q)\} \tag{5q}$$

In the following it is assumed that the component reliability problem is already formulated in the normalized space. It will be shown that, in practice, the transformation (3) resp.(5) must only be carried out in some discrete points. Also, eqs.(5) simplify greatly for independent components of the basic variable vector, i.e.: $X_i = F_i^{-1}[\phi(U_i)]$, and specialize to the well-known

transformations for dependent normal-lognormal vectors. In general, the inversion of distribution functions must be carried out numerically, e.g. by application of a Newton-algorithm if the corresponding density is also known. Sometimes, fairly accurate initial solutions can be obtained by using the first few terms of a Cornish-Fisher expansion for the inverse distribution [1].

The basic idea of the simplified reliability methods to be discussed is to approximate the true limit state equation or failure surface (formulated in the u-space) by simpler surfaces so that the probability content of the failure set can easily be computed. For example, tangent hyperplanes, tangent quadratic forms or polyhedra with tangent plane surfaces may be used. If, in particular, the tangent hyperplane approximation is used it is intuitively clear that due to the rotational symmetry of the independent multi-normal density the linearization should be made at a point on $\bar{g}(\underline{u}) = 0$ which is nearest to the coordinate origin. This point may be called the most likely failure point. Any such approximating tangent failure surface cuts off a failure set with largest probability content among all other possible linearizations on $\bar{g}(\underline{u}) = 0$. This point has first been identified to be an outstanding point by Hasofer and Lind [16] who measured structural reliability by its distance to the coordiante origin and which is denoted by the safety index β_{HL}. It can be determined by suitable search algorithms:

$$\beta_{HL} = \min\{\sqrt{\underline{u}^T\underline{u}}\} \quad \text{for } \{\underline{u} : \bar{g}(\underline{u}) = 0\} \tag{6}$$

Note that neither the transformation (3) nor the failure surface needs to have some special properties, e.g. differentiability, in this formulation although both the probability distribution transformation and the search for β_{HL} as well is greatly facilitated in those cases. In particular, if $\bar{g}(\underline{u})$ is differentiable a suitable search algorithm is as follows:

$$\underline{u}^{i+1} = \frac{\dot{\underline{g}}_i}{\|\dot{\underline{g}}^i\|}\left(\underline{u}^i \frac{\dot{\underline{g}}^i}{\|\dot{\underline{g}}^i\|} - \frac{\bar{g}(\underline{u}^i)}{\|\dot{\underline{g}}^i\|}\right)$$

where $\dot{\underline{g}}$ is the vector of partial derivatives to be evaluated numerically and $\|\dot{\underline{g}}\|$ its Euklidian norm. Other similar algorithms have been proposed, for example, in [14,16,17,29,30,32] and yet other alternatives can be constructed on the basis of the well-known methods in mathematical programming. Certainly, the one requiring the least number of calls of the function g usually is preferable.

The starting vector usually can be set to be $\underline{u} = 0$. If the failure surface exhibits several local minima the one with smallest safety index must be found by a suitable choice of the starting point.

The approximation of $\bar{g}(\underline{u}) = 0$ by a plane

$$\bar{g}(\underline{u}) \approx l(\underline{u}) = \underline{u}\,\underline{\alpha}_{HL} + \beta_{HL} = 0 \tag{7}$$

where $\underline{\alpha}_{HL}$ is the gradient of $\bar{g}(\underline{u}) = 0$ in the most likely failure point (Hasofer-Lind point, β_{HL}-point) produces the very simple result

$$P_f \approx \Phi(-\beta_{HL}) \tag{8}$$

In general, the physical models yield sufficiently "flat" failure surfaces so that eq.(8) is accurate enough for most practical applications. The probability distribution transformations, however, may cause substantial deviations from "flat" surfaces. Therefore, in view of the system reliabilit considerations to come it is useful to introduce the notion a "generalized" or "equivalent" safety index defined by Ditlevsen [10] and give it a specific interpretation. Let $a(\underline{u}) = 0$ be any approximating surface whose probability conte can be evaluated without too much effort. An "equivalent" linear failure surface can then be defined as having the same gradient as the linearization in the originally most likely failure point and distance to the origin:

$$\beta_E = -\Phi^{-1}[P(a(\underline{u}) \leq 0)] \tag{9}$$

so that the "equivalent" failure set has the same probability content as the more informative failure set defined by $a(\underline{u}) \leq 0$ (see figure 1). Therefore,

$$\bar{g}(\underline{u}) \approx l_E(\underline{u}) = \underline{u}\ \underline{\alpha}_{HL} + \beta_E = 0 \qquad (10)$$

Suitable substituting failure surfaces $a(\underline{u}) = 0$ can be constructed in a number of ways only few of which are useful in practice for their easy computability. Quadratic forms have found most interest (see e.g. [9,13,15,28]). Also, polyhedrical approximations have been proposed [11]. They are not discussed further, herein. The concept of equivalent failure surfaces will next be used in a much broader perspective.

3. Failure Probabilities of Parallel Systems

The formulations given above are readily extended to structural systems where it is assumed that the safe and failure set of the components are separated by an equivalent hyperplane with distance β (according to eq.(9)) and gradient $\underline{\alpha}$ corresponding to the Hasofer-Lind point. Proceeding formally, define $Z_j = \underline{U}\ \underline{\alpha}_j$ a standard normal state variable. Hence, the component failure probability can also be written as:

$$P_{f,j} = P(\bar{g}_j(\underline{U}) \leq 0) \approx P(l_j(\underline{U}) \leq 0) = P(\underline{U}\ \underline{\alpha}_j + \beta_j \leq 0) = P(Z_j \leq -\beta_j) \qquad (11)$$

The variables Z_j may also be denoted by standardized safety margin.

Failure of parallel systems occurs if all of its m components fail. Therefore,

$$P_{f,p} = P(\bigcap_{j=1}^{m} (Z_j \leq -\beta_j)) = \Phi_m(-\underline{\beta}; \underline{\underline{R}}_Z) = \int_{-\infty}^{-\underline{\beta}} \varphi(z; \underline{\underline{R}}_Z)\, dz \qquad (12)$$

where $\varphi(\underline{z}; \underline{\underline{R}}_Z)$ is the standard multi-normal density and $\underline{\underline{R}}_Z$ the matrix of correlation coefficients with elements

$$\rho_{ij} = \mathrm{Corr}[Z_i, Z_j] = \underline{\alpha}_i^T\ \underline{\alpha}_j \qquad (13)$$

using well-known second moment algebra for the correlation coefficient between two linear functions.

Before further discussing the physical interpretation of eq.(12) we briefly comment on the evaluation of multi-normal probabilities over rectangles as required by eq.(12). Numerical integration simply is prohibitive for larger dimensions than, say, $m > 5$ [25]. A series expansion as proposed by Kendall [20] and generalized in [27] and [33] is of extreme slow convergence. So-called Monte-Carlo integration requires much computing time as does ordinary simulation and poses non-trivial problems with the (infinite) integration limits. However, the integral can be bounded [33] noting a theorem given by Šidak [35] saying that for two normal vectors with correlation coefficients $\{\kappa_{ij}\} \leq \{\rho_{ij}\}$ for $i \neq j$, the following inequality holds (analogously for $\geq$):

$$P_\kappa(\cap Y_j \leq a_j) \leq P_\rho(\cap Y_j \leq a_j) \qquad (14)$$

Now, if a random vector $\underline{Y}$ can be represented

$$Y_i = \lambda_i V + \sqrt{1-\lambda_i^2}\ V_i,\quad i = 1,2,\ldots,m \qquad (15)$$

where V and V_i are uncorrelated standard normal variates and, hence,

$$\kappa_{ij} = \mathrm{Corr}[Y_i, Y_j] = \lambda_i\ \lambda_j \qquad (16)$$

with $|\lambda_i| \leq 1$, the evaluation of multinormal probabilities reduces to

$$\Phi_m(-\underline{\beta}, \underline{\underline{R}}_Y) = \int_{-\infty}^{+\infty} \varphi(t) \prod_{j=1}^{m} \Phi\left(\frac{-\beta_j - t\cdot\lambda_j}{\sqrt{1-\lambda_j^2}}\right) dt \qquad (17)$$

It is important to note that evaluation of eq.(17) only requires one-dimensional integration which can be done numerically without effort. Clearly, any set of constants $\{\lambda_j\}$ producing a correlation matrix where all correlation coefficients are larger than the given correlation coefficients yields an upper bound to eq.(12) according to inequality (14). In the contrary case, one obtains a lower bound. Fortunately, there are some techniques to choose the set of constants in such a way to make the bounds as narrow as possible. The simpliest choice is by setting

$$\lambda_j = [\max\{\rho_{ij}\}]^{1/2} \tag{18a}$$

for the upper bound and

$$\lambda_j = [\min\{\rho_{ij}\}]^{1/2} \tag{18b}$$

for the lower bound provided that $\min\{\rho_{ij}\} \geq 0$ [33]. The same values may be used for the largest/smallest correlation coefficient and subsequent values can be determined for the second, third, ... largest (smallest) correlation coefficients from

$$\lambda_i = \max_{j \leq i-1} \{\frac{\rho_{ij}}{\lambda_j}\} \tag{19a}$$

and

$$\lambda_i = \min_{j \leq i-1} \{\frac{\rho_{ij}}{\lambda_j}\} \tag{19b}$$

Somewhat different choices are proposed in [11]. Rough estimates, on the other hand, can be obtained by selecting "average" λ-values, e.g. from

$$\sum_{i,j} (\lambda_i \lambda_j - \rho_{ij})^2 \rightarrow \min \tag{20}$$

Finally, it is obvious that

$$\phi_m(-\underline{\beta}, \underline{R}_Z) \leq \min_{j=1}^{m} \{\phi(-\beta_j)\} \tag{21a}$$

and

$$\phi_m(-\underline{\beta}, \underline{R}_Z) \geq \prod_{j=1}^{m} \phi(-\beta_j) \quad \text{for } \rho_{ij} \geq 0 \tag{21b}$$

As pointed out in [11] other choices for the λ's are possible provided that the resulting matrices of correlation coefficients

are non-negative definite and fulfill the condition necessary for the validity of eq.(14). Although the bounds obtained in this manner are strict bounds on failure probabilities (within the framework of first-order or generalized first-order reliability), they are not necessarily sufficiently narrow as will be demonstrated later at an example. For arbitrary correlation coefficient matrices one alternatively may use a method given by Hohenbichler [19] which essentially is an application of the transformation technique as described before and the first-order reliability method and which usually yields highly accurate probability estimates.

We return to the interpretation of eq.(12). The most elementary interpretation is a direct analogue to classical parallel systems, i.e. the structure consists of m components each with a failure domain defined by $\bar{g}_j(\underline{u}) \leq 0$. The first component is loaded. If it fails, the second one is loaded. If this fails, the third component is loaded and so on. Hence,

$$P_{f,p} = P(\bigcap_{j=1}^{m} (\bar{g}_j(\underline{U}_j) < 0)) \tag{22}$$

One of the most prominent examples certainly is a nuclear power plant where the containment functions as a standby-component in case of failure of the pressure vessel caused by internal accidents. Further direct applications of eq.(22) will be discussed later on. In general, the definition of a "component" in the reliability sense is much more complex and depends on the physical context. For example, in the structural case failure of a component or a set of components usually increases the loads on the remaining components, modifies the distribution of loads among the components or even changes their strength. Furthermore, a physical component is not necessarily a component in the reliability sense. For example, one of the simpliest redundant structures is given in figure 4 where it is assumed that the applied load distributes itself

equally among all unfailed components. For illustration purposes we assume perfect elastic-brittle behaviour of the components. The deformation modulus is constant for all elements. The elemental strength values are distributed according to $F_X(x) = P(X_i \leq x)$ and independent. Obviously, a system consisting of m elements can carry a load in exactly 2^m-1 different ways, e.g.: no element fails, any one of m elements fails while the other elements survive, any two elements fail and the other survive, ..., any m-1 elements fail and one element survives. Each of these 2^m-1 variations may be interpreted as components in the reliability sense. Due to the large number of possibilities for larger m this definition of a component is generally not useful although inevitable in some cases. Our aim, therefore, is to reduce the number of components in parallel by suitable redefinitions. Specifically, the strength of the same system can also be given as:

$$R_m = R(X_1,\ldots,X_m) = \max_{j=1}^{m}\{(m-j+1)\hat{X}_j\}$$

where $\hat{X}_j$ is the j-th order statistic in a sample $(X_1,\ldots,X_m)$ and whose distribution function can easily be constructed from $F_X(x)$ [34]. An appropriate formulation of system failure probability then is given by:

$$P_{f,p} = P(\max_{j=1}^{m}\{(m-j+1)\hat{X}_j\} \leq x) = P(\max_{j=1}^{m}\{(m-j+1)\hat{X}_j - x\} \leq 0) =$$
$$= P(\max_{j=1}^{m}\{(m-j+1)\,T_j(\underline{U}) - x\} \leq 0) \qquad (23)$$

which is just the formulation chosen in eq.(22). Here, each of the "max-terms" can be interpreted as a "component" in the sense of eq.(22). Note that the equivalent to eq.(21a) corresponds to the number j with smallest failure probability. This idealized system has been investigated firstly by Daniels [7] in some detail. More realistic mechanical assumptions are studied in [18] using the order statistics approach as described before.

More generally, we assume that the force-deformation relationship of a physical component is of the form as shown in figure 2, i.e. with increasing deformation the force reaches a maximum and falls off in a certain manner beyond that point. For the k-th component one has

$$R_k(\delta) = R_k(\delta;\ \underline{\Pi}_k) \qquad (24)$$

in which $\underline{\Pi}_k$ is a random vector of parameters defining the special force-deformation characteristic, e.g. initial slope, maximum resistance and corresponding deformation. Let us assume that structural behaviour is modelled by assuming the location of the necessary number of hinges and the formation of a given mechanism (failure mode) implies failure (compare figure 3). All deformations (rotations) are taken to concentrate at the hinges. The limit state equation then may read:

$$\sum_{(k)} a_k\, M_k(\delta_k(\delta),\ \underline{\Pi}_k) - \sum_{(l)} b_l\, L_l = 0 \qquad (25)$$

where $M_k(\delta_k(\delta))$ is the moment at hinge k as a function of the rotation δ_k which in turn is, by geometrical compatibility, a function of a global deformation parameter δ (see figure 3). The coefficients a_k and b_k are geometrical constants. Note that the assumption of rigid-plastic behaviour of the structure implicitly made before can easily be relaxed at the expense of some more numerical effort. Each of the deformation states may now be taken as a failure mode in a parallel system. The idea of interpreting different deformation states as failure modes in parallel probably has been put forward first in [21]. For the standardized space we formally write:

$$g_\delta(\underline{U};\delta) = \sum_{(k)} a_k\, M_k(\delta_k(\delta), T_{\underline{\Pi}}(\underline{U}_k)) - \sum_{(l)} b_l\, T_{\underline{L}}(\underline{U}_l) = 0 \qquad (26)$$

In discretising the parameter $\delta = \delta_j$, $j = 1,2,\ldots,m$ the mechanism viewed as a parallel system with m components (in the sense of eq.(12)) can be treated in the usual way. Clearly, the accuracy of the probability estimate highly depends on the choice of the parameter δ but also on the type of function in eq.(24). In general, there will be a value of $\delta = \delta^*$ for which the safety index β becomes maximal and for which application of eq.(21a) yields an upper bound. In [34] it has been shown that this bound is close to the exact result for very small and for very

large systems. For medium systems it is generally worthwhile to use the sharper bound provided by eqs.(17) through (20) or to use one of the failure probability estimates mentioned before. The choice of the number m and the spacing $\Delta_i = \delta_{i+1} - \delta_i$ of modes is somewhat arbitrary. Nevertheless any choice-even those which do not include the mode with largest safety index-yields an upper bound.

For later use, it is convenient to define an "equivalent" safety index β_p if such improvements are carried out:

$$\beta_p = - \phi^{-1}[P(\bigcap_{\delta} g_\delta(\underline{U};\delta) \leq 0)] \tag{27}$$

so that the probability of system failure corresponds to the failure probability of one component (in the reliability sense) representing the whole system. Typically, this is the component with largest safety index $\beta_{\delta*}$, the same gradient $\underline{\alpha}_{\delta*}$ but modified safety index $\beta_{\delta*} \rightarrow \beta_p$. Hence, the parallel system is represented by an equivalent linear failure surface

$$l_p(\underline{U}) \equiv \underline{\alpha}_{\delta*}\, \underline{U} + \beta_p = 0 \tag{28}$$

Whether β_p differs substantially from $\beta_{\delta*}$ depends to a large extent on the type of the system under consideration, the number of its "components", but especially on the type of force-deformation relationship eq.(24) and the dependence structure of its parameters. Quite generally and not unexpected, the difference becomes small for force-deformation relationships which tend towards perfect plastic behaviour. On the other hand, considerable differences have been observed for peaked force-deformation relationships and not substantially differing safety indices of the modes.

4. Failure Probabilities of Series Systems

In a similar manner the probability of failure of pure series systems can be computed. A series system fails if any of its components, specifically the weakest, fails. Its failure probability is defined by the probability of the union (in contrast to the intersection with parallel systems) of the failure events, i.e.:

$$P_{f,S} = P(\bigcup_{i=1}^{n} F_i) = P(\bigcup_{i=1}^{n} (Z_i \leq -\beta_i))$$

$$= 1 - P(\bigcap_{i=1}^{n} (Z_i > -\beta_i))$$

$$= 1 - \int_{-\underline{\beta}}^{\infty} \varphi(z;\underline{\underline{R}}_Z)\, dz \tag{29}$$

with the notations as before. Techniques to evaluate eq.(29) have found continuous interest among structrural reliability engineers (see, for example, in chronological order [6,36,27 33,11]. The bounds given by Ditlevsen in [11] are perhaps the most easy to use and sufficiently narrow. (Their general formulation, i.e. for arbitrary failure sets, is used lateron.) Figure 5 demonstrates, however, that the problem of correlation among modes is less significant as compared with the parallel case (see eqs.(21)). This is readily been seen in writing down the bounds resulting from the assumption of fully dependent and independent failure modes.

$$\max_{i=1}^{n} \{\phi(-\beta_i)\} \leq P_{f,S} \leq \sum_{i=1}^{n} \phi(-\beta_i) \leq 1 \tag{30}$$

In fact, for the worst case of equicorrelated modes and equal safety indices for the different modes it is observed that the correlation coefficient must be rather high in order to substantially diminish the failure probability of series systems in comparison to the upper bound given in eq.(30). Excellent bounds are obtained when using the equivalent to eq.(17), i.e.:

$$P_{f,S} = 1 - \int_{-\infty}^{+\infty} \varphi(t) \prod_{i=1}^{n} \phi\left(\frac{+\beta_i - t\cdot\lambda_i}{\sqrt{1-\lambda_i^2}}\right) dt \tag{31}$$

where now, due to the symmetry of $P(\bigcap_i (Z_i > -\beta_i) = 1 - P(\bigcup_i (Z_i \leq \beta_i))$, eq.(19a) gives a lower and eq.(19b) an upper bound. In analogy to parallel systems, one can obtain weaker but simplier bounds in setting $\lambda_i = \sqrt{\rho} = \sqrt{\max\{\rho_{ij}\}}$ for a lower bound and $\lambda_i = \sqrt{\rho} = \sqrt{\min\{\rho_{ij}\}}$ for an upper bound. Accurate estimates can again be obtained by using the algorithm proposed in [19].

The formulation in the second and third line of eq.(29) has important implications for a different type of lower bounds first mentioned by Augusti/Barrata [2]. Observe that the probability of the union of survival events gets larger with increasing number of modes considered. Thus, the consideration of only a limited number of modes always produces a lower bound on the failure probability.

In analogy with the foregoing section one might define an equivalent single component system having the equivalent safety index

$$\beta_S = - \phi^{-1}[P(\bigcup_{i=1}^{n} (Z_i \leq - \beta_i)] \qquad (32)$$

and the same gradient $\underline{\alpha}_S$ as the component with smallest (!) safety index. This formulation may prove useful if a given structural system is composed of series systems in parallel (in a reliability sense).

5. Failure Probabilities of Parallel (Series) Sub-Systems in Series (Parallel)

We are now ready to compute the general problem of eq.(2) in simply combining section 3 and 4. We have

$$P_f = P(\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} F_{ij}) \approx P(\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} (Z_{ij} \leq - \beta_{ij})$$

$$\approx P(\bigcup_{i=1}^{n} (Z_{P,i} \leq - \beta_{P,i})) \qquad (33)$$

where the safety index $\beta_{P,i}$ and the gradient $\underline{\alpha}_{P,i}$ in mode i corresponds to the equivalent values used in eq.(28) and an appropriate method is applied to bound or estimate the failure probability of the series system. For series subsystems in parallel quite an analogous formula holds:

$$P_f = P(\bigcap_{i=1}^{n} \bigcup_{j=1}^{n_i} F_{ij}) \approx P(\bigcap_{i=1}^{n} \bigcup_{j=1}^{n_i} (Z_{ij} \leq -\beta_{ij})$$

$$\approx P(\bigcap_{i=1}^{n} (Z_{S,i} \leq -\beta_{S,i}) \qquad (34)$$

Clearly, the validity of the approximations in eqs.(33) and (34) first rests on the assumption that the failure surfaces of the various modes of failure in parallel and in series are sufficiently flat. Moreover, the subsystems must be characterizeable by a dominant mode and the other "significant" modes should have similar gradient. The "equivalent" failure surfaces may, in fact, only be sufficiently representative if they somehow envelop the individual mode surfaces in the region of interest, i.e. the β_{HL}-points for the series modes should lie in the "equivalent" failure set.

If these conditions are not fulfilled, the problem becomes more involved. However, the upper bound of the parallel-in-series system failure probability in the sense of eq.(30) which then is a reasonable basis for further calculations can be improved at the expense of some more effort. The probability bounds initially derived for simple series systems by Ditlevsen are easily generalized, i.e.:[11]:

$$P_f = P(\bigcup_i \bigcap_j F_{ij}) = \begin{cases} \leq P(\bigcap_j F_{1j}) + \sum_{i=2}^{n} (P(\bigcap_j F_{ij}) - \max_{k=1}^{i-1} \{P(\bigcap_j F_{ij} \cap \bigcap_j F_{kj}\}) \\ \geq P(\bigcap_j F_{1j}) + \sum_{i=2}^{n} \max\{0, P(\bigcap_j F_{ij}) - \sum_{k=1}^{i-1} P(\bigcap_j F_{ij} \cap \bigcap_j F_{kj})\} \end{cases} \qquad (35)$$

where the probabilities $P(\bigcap_j F_{ij} \cap \bigcap_j F_{kj})$, i.e. the probability of the intersection of the failure domains of two parallel systems, are computed in the usual way.

A somewhat weaker lower bound may be obtained by replacing the probability for the intersection of two parallel systems by the smallest intersection probability of any two components of these systems.

$$P_f \geq P(\bigcap_j F_{1j}) + \sum_{i=2}^{n} \max\{0, P(\bigcap_j F_{ij} - \sum_{k=1}^{i-1} \min_{j,l} \{P(F_{ij} \cap F_{kl})\}\} \qquad (36)$$

Appearently, there are no such simple rules for series-in-parallel systems and one is essentially left with the two elementary bounds:

$$P_f = P(\bigcap_i \bigcup_j F_{ij}) = \begin{cases} \leq \min_i \{P(\bigcup_j F_{ij})\} & (37a) \\ \geq \prod_i P(\bigcup_j F_{ij}) & (37b) \end{cases}$$

with which one might compare the result of eq.(34) in order to judge the quality of the approximation. On the other hand it would generally be misleading to use either of the two bounds since only slight dependencies between the various series systems can drastically modify the system failure probability.

Although it would be desireable to have a general, direct calculation method for series systems in parallel of a quality comparable with that of eq.(35) thus is not necessary from a mathematical point of view. In fact, any system the states of which are either intact or not intact, and the same idealization of states is valid for its components, can be represented by series systems in parallel (minimum failure paths representaion) or, equivalently, by parallel systems in series (minimum cut set representation) [4]. The possibility is just the one required for application of eq.(35). The details of representing arbitraty (structural) systems in this form will be given in a separate report. The mathematical problem essentially reduces to implement one of the various algorithms for searching for minimal cut sets available in the literature (see, for example, [4]) with further references). The engineering problem of interpreting components and subsystems in this abstract formulation with the aim to improve the design appears much more difficult.

Therefore, we return to the more direct interpretation of eq.(2).

6. Combined Loading

To make the spectrum of structural reliability problems complete we finally comment briefly on load combination. The probably most flexible formulation for time-variant reliability problems has been given in [37] , i.e.:

$$P_f(T) \leq P_f(0) + \int_0^T \lambda(S,t)\, dt \qquad (38)$$

where $P_f(0)$ is the initial failure probability, T the design life and $\lambda(S,t)$ the mean crossing rate of the random vector load process $\underline{X}(t)$ out of the safe domain S which here is meant

to correspond to the safe domain of a specific failure mode. Frequently, $P_f(0)$ is negligible small and it suffices to consider the stationary case so that the second term in eq.(38) becomes $\lambda(S)T$. Unfortunately, very few exact results are available for the mean crossing rate out of arbitrary safe domains (see [5,24,25,37]. But quite a number of processes can be dealt with if the safe domain is bounded by a hyperplane. Moreover, easily calculated upper and lower bounds can be derived [22,26] in this case. For example, let the failure surface be given by $g(\underline{x}(t)) \equiv X - \sum_{r=1}^{N} X_r(t) = 0$. Then, the mean upcrossing rate of $\sum_{r=1}^{N} X_r(t)$ of a threshold x can be given as

$$\lambda(x) \leq \sum_{r=1}^{N} \int_{-\infty}^{+\infty} \nu_r(y)\, f_r^*(x-y)\, dy \qquad (39)$$

in which $f_r^*(.)$ is the probability density of $\sum_{\substack{j=1 \\ j\neq r}}^{N} X_j$ which must be computed by repeated convolution and $\nu_r(y)$ is the mean upcrossing rate of the r-th load process for level y. Taking now a random threshold eq.(38) reads with $P_f(0) = 0$:

$$P_f(T) \leq T \int_{-\infty}^{+\infty} \Big[\sum_{r=1}^{N} \int_{-\infty}^{+\infty} \nu_r(y)\, f_r^*(x-y)\, dy \Big] f_X(x)\, dx$$

$$= T \sum_{r=1}^{N} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \nu_r(y)\, f_r^*(x-y)\, f_X(x)\, dy\, dx \qquad (40)$$

where the integrals correspond to the formulation given in eq. (1) for a linearly bounded failure set. Therefore, we may also write:

$$P_f(T) \approx \sum_{r=1}^{N} P[g_i(X_1, X_2, \ldots, X_{r-1}, Y_r, X_{r+1}, \ldots, X_N, \underline{Z}) \leq 0] \qquad (41)$$

where $g_i(.)$ is the failure surface of the i-th mode, $X_1, X_2, \ldots, X_{r-1}, X_{r+1}, \ldots, X_N$ represent loads with their random point-in-time (first-order) distribution, Y_r, if properly interpreted [25], a variable having an outcrossing and improper distribution function $F(y) = 1 - \nu_r(y)T$ and $\underline{Z}$ a random vector, collecting the time-invariant variables. Note, that the interpretation of Y_r as a random variable is not mandatory. It is only necessary to introduce $\nu_r(y)T$ as the complimentary

distribution function of a fictitious variable Y_r. The "equivalent" safety index is now defined by

$$\beta_{i,r^*} \approx - \phi^{-1}[\sum_{r=1}^{N} P(g_i(X_1,\ldots,X_{r-1},Y_r,X_{r+1},\ldots,X_N,\underline{Z}) \leq 0)] \quad (42)$$

The "equivalent" gradient corresponds to the term in the sum with smallest "local" safety index. The inequality sign of eq.(38) is changed into an approximation sign in eq.(41) which not only indicates that there can be small errors when replacing the exact convolution integrals of eq. (36) by the approximate convolution of first order reliability methods. It also regards the fact that it is not yet known precisely although conjectured whether the equivalent to the bound eq.(39) is valid for arbitrary safe domains (see [12]). But it is important to mention that the "point crossing" approach outlined before has potential to consider quite general processes with rather complex stochastic dependence strutures. A special case will be dealt within an example later on.

7. Structural System Reliability under Combined Loading

The complete formulation of system failure probability under combined loading, therefore, is

$$\begin{aligned} P_{f,System}(T) &\approx P[\bigcup_{i=1}^{n} F_i(T)] \\ &= P[\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} F_{ij}(T)] \\ &= P[\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} (Z_{ij}(T) \leq - \beta_{ij})] \\ &= P[\bigcup_{i=1}^{n} (Z_{p,r^*;i} \leq - \beta_{p,r^*;i})] \end{aligned}$$

where $\beta_{p,r^*;i}$ is the equivalent safety index with respect to the deformation mode with largest individual safety index and with respect to the combination mode with smallest individual safety index. It is here where a random variable interpretation of Y_r is needed and the whole dependence structure of the random vector $\underline{X}(t)$ but also between $\underline{X}(t)$ and the extremes of the

components of $\underline{X}(t)$ must be known. Since this is to be expected only rarely in applications an alternative formulation is proposed. In particular, we treat each term in eq.(41) as a different "loading case" and, therefore:

$$P_{f,System}(T) \approx \sum_{r=1}^{N} P(\bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} (Z_{ijr} \leq - \beta_{ijr})) \quad (43)$$

ignoring the dependence of outcrossings which might exist in the different loading cases. Also, eq.(43) is consistent with eq.(40) which, in essence, corresponds to Turkstra's rule . Formulation (43) further allows to improve eq.(40) by introducing a more refined description of loadings, for example, by incorporating information on durations of load pulses. An example where eq.(43) can be made sharp within first-order reliability, e.g. by taking account of the dependencies between the failure events corresponding to the "loading" cases will be given in the illustration.

8. Illustration

Example 1: Steel frame subjected to wind and snow load

We consider a simple unsymmetric steel frame loaded by self-weight of the roof, snow load and wind load and which is designed according to present design standards (see figure 6). The time-variant loadings are modelled as sparse filtered Poisson processes with rectangular (enveloping) pulses of the kind proposed by Wen [38,39],i.e. there are three independent processes generating either snow, wind from the left-, or wind from the right-hand side of the structure as shown in figure 6. Further there is one process (weather condition) which generates after a certain delay time, generally different for wind and snow, snow with certain probability and with another probability wind from a given direction. Naturally, wind from both directions is excluded. If snow and wind coincide it is assumed that their load amplitudes are somehow correlated. The three distribution functions if the loads are "on" are normal. Table 1 collects the set of parameters. According to [39] coincidences of any two time-variant loads occur approximately with rate

$$\lambda_{ij} = \lambda_i \lambda_j (\delta_i + \delta_j) \left[1 + \frac{p_i p_j \nu}{\lambda_i \lambda_j} \frac{1}{(\kappa_i + \kappa_j)}\right]$$

while occurrences of one load have intensity

$$\lambda_{ii} = \lambda_i - \left[1 - \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{\lambda_{ij}}{\lambda_i}\right]$$

where the second term avoids "double-counting" and $\lambda_i = \nu_i + p_i \nu$. It is ν_i the occurrence rate of one load process, ν the occurrence rate of the process generating two other processes with probabilities p_i and p_j having mean delay time κ_i and κ_j and mean pulse durations δ_i and δ_j, respectively. Therefore, the "loading cases" shown in table 2 have to be considered.

The structure, on the other hand, is taken to be sufficiently represented by a number of mechanisms as shown in figure 7. The modes 9 to 12 have been ignored in the numerical calculations since their contribution to the total failure probability is negligible. It is assumed that beams behave perfectly rigid between hinges. The general form of the moment rotation curves is shown in figure 8 reflecting, for example, local buckling of the flanges beyond the yield point. Any other moment rotation curve, of course, could have been used. The specific type of moment-rotation curve has been chosen because it is rather flexible to fit various mechanical situations by simple modifications of its parameters. Yield strengths and yield strains are assumed to be jointly normally distributed and positively correlated within one beam but independent between beams. The structure has been designed according to the present German standard which assumes an elastic structural model without second-order effects. The beam shapes and dimensions have been selected such that the admissible stresses are attained at least in one cross-section in loading cases I, III and V.

Figure 9 shows two examples of the safety index β_{HL} versus the deformation parameter δ. Figure 9a exhibits one maximum while figure 9b shows two peaks although this case is of minor interest because the safety indices are relatively large. The first

supremum in figure 9b corresponds to the maximum carrying capacity of hinges 6 and 7; the second supremum develops when hinges 1 and 2 attain their maximum carrying capacity. The upper parts of the figure show some components of the gradient (sensitivity-factors) in the original space of basic variables as functions of the deformation parameter. At the supreme safety index the system obviously can carry the load in two completely different states. This is readily explained by the special type of the moment-rotation relationship where a given moment can correspond to two different rotations. It means also that the deformation modes with supreme safety indices are markedly curved and a simple linearization is inadequate; perhaps less with respect to the mode failure probability as demonstrated below but with respect to its validity for calculating the union of modes in series. In principle, one ought to approximate each parallel mode by at least two tangent hyperplanes and make use of the improvements indicated at the end of section 5 when combining parallel systems in series in each loading case. However, the safety indices differ substantially in each of the series modes and loading cases. Therefore, the lower bound eq.(30) is nearly exact in this example.

Table 4 collects the results for all loading cases. The first column of safety indices represents the equivalent (parallel or series mode) safety indices based on linear approximation of the failure surfaces of the individual modes. The failure probabilities given correspond to this index according to eq.(9). The second column contains the corresponding largest (parallel-mode) safety index. The third column assembles the equivalent safety indices of the parallel modes with largest safety index based on a non-central-hypersphere-approximation with the same mean curvature as the true failure surface. The difference between the Hasofer-Lind index and the equivalent index generally remains small but one can envisage cases in which this can become considerable. All other safety indices reported in tables 3 to 5 are based on Hasofer-Lind indices for the individual parallel modes. The series system failure probabilities have been calculated by using eq.(31). Since certain modes dominate in this

example, no essential improvements are obtained as compared with the simple bounds given by eq.(30). Due to generally large correlations between series modes in a loading case, the exact probability mostly is close to the lower bound. The last two columns of table 4 indicate that the assumption of perfect elastic-plastic behaviour can lead to a significant overestimation of structural reliability.

Table 5 shows the various bounds on the mode/loading case safety indices. It is seen that certain sets of bounds, can, in fact, become rather wide.

Finally, it should be mentioned that changes in the correlation structure of the uncertain parameters defining the moment-rotation curves in the various hinges have non-negligible effects on the total failure probability.

Example 2: Bundle of threads with perfect elastic-brittle behaviour

In a second example the mode interpretation as given in eq.(23) (order statistics interpretation) is used in the system shown in figure 4 with 10 independent elements. Elemental strength and ultimate strain are assumed to be fully correlated and to vary normally with a coefficient of variation of 0.2. The example is especially interesting because one can compare with some exact results. Figure 10 shows the distribution of system strength. The right-hand thick line corresponds to the exact distribution, the left-hand thick line represents Daniels' asymptotic distribution. The dashed line is the exact distribution of the order-statistics mode with largest bearing capacity. For the computational details the reader is referred to [18]. The dotted line corresponds to the same dominating mode but has been computed by the first-order reliability technique (linearisation at the β_{HL}-point) while the thin line has been obtained by using the approximation in [19] to the multinormal integral (12). In this case the improvement of considering all failure modes is substantial but the first-order estimate is in error by approximately half an order of ten. Presumeably,

if the individual modes were computed more accurately, e.g. by applying the equivalent safety index concept as outlined in section 2, one would have obtained almost exact results for the whole system.

Example 3: Event and fault tree calculation

Figure 11 shows an event tree for selected accident sequen[illegible] caused by a pipe bread in a nuclear power plant during an [illegible] quake. In this case the consequences of failure depend str[illegible] on the type and number of protective systems which fail. T[illegible] corresponding probabilities are given in figure 12 in gene[illegible] terms. Due to the common cause the failure and survival of each of the protective systems can be highly dependent. In figure 12 a simplified example of the fault tree for the emergency cooling system is given.

The EC-system essentially is a series system with a few components in parallel. Its failure probability can directly be written down in general terms.

$$P(ECS) = P[(\bigcap_{i=1}^{n_A} A_i) \cup (B_1 \cap B_2) \cup (C_1 \cap C_2 \cap C_3) \cup D \cup (\bigcup_{i=1}^{n_E} E_i) \cup F]$$

$$= \prod_{i=1}^{n_A} P(A_i) + P(B_1) \cdot P(B_2) + P(\bigcap_{i=1}^{3} C_i, D, \bigcup_{i=1}^{n_E} E_i, F))$$

The last term can at least be bounded in applying eq.(35). The failure events C to F are dependent through the common loading but possibly also through other uncertainties shared by two or more components. Failures of the sensors and the mechanical failure of the pumps during operation are assumed to be independent events.

Unfortunately, redundant systems composed of series system cannot be evaluated so easily unless the series systems possess a dominant mode so that the concept of equivalent failure surfaces is applicable. In this case the evaluation of the accident sequence probabilities is a straightforward matter on the basis of section 5. If there are no dominating modes in the different protective series systems in parallel and the only source of stochastic dependence

among the components is the common load it is well known that an analysis can be made by conditional system analysis, i.e. by conditioning on a given value of the load and then integrating over the probabilities with which they occur. Further variables introducing dependencies into the system may be treated in the same manner but numerical effort can soon become intractable. Alternatively, one may use the representation of such complex systems by minimal cut sets as mentioned earlier. Details of this approach must be left to a separate report where it will be shown that the resulting parallel-in-series systems are still sufficiently simple.

Example 4: Parallel systems in series

We study the case of parallel systems in series a little bit further. Consider the system schematically shown in figure 13. It may represent a system resulting from a search for minimal cut sets but here, for illustration purposes, a bridge crossing a river and sitting on 10 supports of 3 piles each. It is loaded by ice-pressure. The functioning of the protective structure does not depend on the magnitude of the ice-pressure. Note that this representation of an uncertain variable may also be used for truly booleans variables. If one or two piles break the ice-pressure is somewhat reduced for the remaining pile(s). Here, we use the second component definition given in section 3. Therefore, if $X_{i,o}$ is the resistance of the protective structure of the i-th support, $X_{i,k}$, $k = 1,2,3$ the strength of the corresponding piles, X an overall modelling error and S the ice-load, we have for any of the supports failing:

$$P_f = P\left(\bigcup_{i=1}^{10} \bigcap_{j=1}^{8} F_{i,j}\right)$$

where

$$\{F_{ij}\} = \begin{cases} \{g_{i,1}(\underline{X}) \le 0\} = \{3X + X_{i,1} + X_{i,2} + X_{i,3} - S \le 0\} \\ \{g_{i,2}(\underline{X}) \le 0\} = \{2X + X_{i,1} + X_{i,2} - \frac{1}{2}S \le 0\} \\ \vdots \\ \{2X + X_{i,1} + X_{i,3} - \frac{1}{2}S \le 0\} \\ \{2X + X_{i,2} + X_{i,3} - \frac{1}{2}S \le 0\} \\ \{X + X_{i,1} - \frac{1}{4}S \le 0\} \\ \{X + X_{i,2} - \frac{1}{4}S \le 0\} \\ \{X + X_{i,3} - \frac{1}{4}S \le 0\} \\ \{g_{i,8}(\underline{X}) \le 0\} = \{X_{i,o} - c \le 0\} \end{cases}$$

With normally distributed basic variables and the parameters as given in figure 13 (the variables $X_{i,k}$ are independent for all i,k) one obtains for $\sigma_X = 0.1$:

$$13.40 \ge 7.26 \ge 6.94 \ge -\phi^{-1}[P_F] \ge 6.94 \ge 6.94 \ge 4.33$$

and for $\sigma_X = 0.2$:

$$10.67 \ge 5.53 \ge 5.11 \ge -\phi^{-1}[P_f] \ge 5.11 \ge 5.11 \ge 2.83$$

where the relevant set of elementary bounds is also added, i.e. according to

$$\max_j\{\prod_i P(F_{ij})\} \le \max_j\{P(\bigcap_j F_{ij})\} \le P_{f,L} \le P_f \le P_{f,U} \le \sum_j \min_i\{P(\bigcap_j F_{ij})\} \le \sum_j \min_i\{P(F_{ij})\}$$

and the bounds $P_{f,L}$ and $P_{f,U}$ are determined from eqs.(35).

It is recognized that a correct calculation of the parallel system is of great importance whereas the sub-systems in series are almost uncorrelated. The bounds eqs.(35) are almost exact in this case.

9. Discussion and Some Open Questions

The aim of the foregoing presentation was to attempt a rather general formulation of structural reliability for given limit states and to demonstrate that convenient numerical solutions are possible within the frame-work of first-order reliability methods. It rests on the presupposition that irrespective of the type and number of loadings, the number of potential failure modes and the degree of redundancy in any of the modes, constant reliability designs are desireable. This is, no doubt, a reasonable objective under certain circumstances. In other cases such designs are suboptimal. The design for a particular loading case or for a special failure mode for a given reliability may require enormous investments while the same reliability level can be achieved with small efforts in other modes. It appears that present practice, in general, is closer to optimal design than to constant reliability design. This, in turn, means that different modes will have considerably differing reliability levels and, hence, there are only very few critical modes in most cases. If this is so the foregoing considerations simplify greatly. The art of the determination of system reliability, then, essentially consists in identifying the critical modes and loading cases.

In the general case, the first step in an analysis is always an appropriate formulation of the physical model for the relevant limit state and the determination of the component safety index (or failure probability) given the stochastic models for the various uncertainties. Both tasks can get quite involved. It is obvious that the sophistication in modelling should not have too much bias to either the mechanical or the stochastic part. Moreover, both types of modelling must be compatible particularly with respect to their use in generalized first-order reliability methods. For example, limit state equation should be differentiable since most search algorithm for the safety index require the existence of, at least, the first derivatives. Since some physical phenomena, in fact, do not share this property, more research work ought to be done to develop less restrictive search algorithms. They should in any

case be made as efficient as possible although experience with those already available is promising [14].

It should be mentioned that the usual mechanical formulations must be modified in some cases. The parallel system studied in example 1 has been formulated in the most simple way. Realistic formulations may require the determination of system behaviour under a set of imposed deformations which simultaneously ensures static equilibrium and geometrical compatibility under realistic assumptions for stiffnesses and second-order effects between hinges.

It is known that in ordinary component reliability the Hasofer-Lind safety index β_{HL} together with the failure probability estimate (see eq.(8)) from tangent linearization in this point (or, by a hyperplane with gradient corresponding to the β_{HL}-point) yields unacceptable inaccurate results only in extreme cases. But small errors in each of the Hasofer-Lind probability can cause larger errors in system failure probability.

From the unfavourable case of example 2 one might conjecture that the chief error is in the mode failure probability and, therefore, the concept of an approximation of a single mode by more realistic failure surfaces generally deserves further attention. If the simpliest first-order estimate is inaccurate polyhedrical approximations undoubtedly should be preferred as an alternative. These may not be as good as quadratic approximations for the single mode but have the potential to carry at least part of the information needed in system reliability. They enable to check with slightly more effort (see second half of section 5) whether the much simpler approach with "equivalent" failure surfaces is not only feasible with respect to the single mode but can also be applied to systems.

The concept of "equivalent" failure surfaces is, as mentioned, only applicable if certain modes in the various sub-systems dominate. Parallel subsystems in series can still be handled appropriately (see, for example, eq.(35)). The numerically more important case of series-subsystems in parallel cannot be treated

satisfactorily as of yet and needs more research work. And, clearly, the same is true for systems at an higher level. But, again, a combination with the many available minimal cut sets algorithms may at least solve any system reliability problem in a mathematical resp. numerical sense. Difficulties can arise when the physical component definition is not so obvious and, therefore, one is left with relatively abstract operations. Furthermore, the numerical effort may become quite considerable and some research is needed to establish suitable criteria to reduce the numerical work in the structural area - probably on similar lines as in classical reliability theory.

It has been shown that one of the key points in general structural system reliability is the computation of the multi-normal integral. Presumeably, the approximate evaluation according to [19] satisfies most practical needs but further attempts should be made for an easier and more accurate computation.

The derivations in this paper have been governed by arguments of mathematical and numerical convenience where in many cases bounding techniques were obligatory. How can these approximations be verified by higher order methods? At present, there are only a few studies which verify to a satisfying degree certain aspects in specific examples either by stochastic simulation or numerical integration. The study of the reliability of systems of the general nature as considered herin might no more be feasible on the basis of "exact" probabilistic methods as mentioned before. Therefore, there is an urgent need to develop methods at the next level of accuracy so that the simplier methods can be checked and, possibly, be improved.

As concerns conclusions for the practical structural analysis and design only little can yet be said which goes beyond what is known already. Firstly, the most critical series mode must be found in each loading case. Caution is in order if the search is done deterministically since the deterministically critical mode does not necessarily coincide with the probabilistically critical one. Unless the system is very large and complex, safety checking of the most critical mode then may be sufficiently representative for the whole system. Secondly, in order to find the parallel mode with largest carrying capacity one may assume mean value (or slightly below mean value) mechanical properties at all physical components except for those which attain their (mean) peak resistance for the same value of imposed deformation. For the latter components the mechanical properties should be selected at the so-called design level, i.e. a level reduced by some appropriate saf[illegible] elements. The last conclusion is based on the behaviour of direction cosines (or sensitivity factors α) with running sed deformation. An example is given in figure 9 where the sequence of α-values is shown for increasing deformation.

Any reasonable combination of components at their respective level, in principle, may be chosen but the search for the one with largest carrying capacity usually deserves some effort. However, it should be emphasized that such recommendations are still tentative and are prone to modifications for specific structures.

10 Summary and Conclusions

The computation of reliability resp. failure probability of general systems under combined loading within the framework of generalized first-order reliability methods is proposed. It is shown that quite general arrangements of components (in parallel and in series) can be dealt with by using suitable techniques for the transformation of non-normal vectors into independent standard normal vectors, by applying the concept of equivalent failure surfaces for conveniently chosen sub-

systems and by estimating multi-normal probability integrals over rectangles or by bounding them by simpler expressions. The reliability under combined loading can easily be evaluated or at least be bounded by the use of the so-called point-crossing method. It appears that in most practical applications significant reductions of the failure modes and loading cases to be considered are possible and, thus, the probabilistic part is greatly simplified. If the concept of equivalent failure surfaces cannot be applied solutions are still possible by representing any arbitrary system by parallel-in-series systems via "minimal cut sets" and then applying certain bounding techniques. Unfortunately, the numerical effort may be quite high and the physical interpretation of the resulting failure sets can become dubious. Nevertheless, the problem of computing reliabilities of some structural systems may be considered as solved in a first-order sense. In order to check the results of generalized first-order system reliability methods higher order methods are most sucessfully derived on the basis of better approximations to the individual mode failure surfaces than by tangent hyperplanes.

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Type of Variable	Distribution type	Mean	Standard Deviation	ν/year	Remarks
X_i	N	240 N/mm^2	10 %	-	$\rho[X_i, Y_i] = \rho[X_i, X_{i+1}] = \rho[Y_i, Y_{i-1}] = 0.9$
Y_i	N	1 %	10 %	-	$\rho[X_i, Y_{i-1}] = 0.9^{2}$ *)
D	LN	321.6 kN	10%	-	
S_{max}	(N)	14.24	75 %	2.5	
$W_{max,l}$	(N)	8.74	50 %	42	Parameters for instantaneous value distribution in "on" states
$W_{max,r}$	(N)	4.37	50 %	42	
S + W	(N)			15	

*) No correlation between beams!

Table 1: Parameters of Basic Variables

Loading Case	Dead Load	Snow Load	Wind Load from the Left	Wind Load from the Right
I	D	$S_{max}(\lambda_S)$	-	-
II	D	-	$W_{max}(\lambda_W)$	-
IIIa	D	$S'_{max}(\lambda_{SW})$	W	-
IIIb	D	S	$W'_{max}(\lambda_{SW})$	-
IV	D	-	-	$W_{max}(\lambda_W)$
Va	D	$S'_{max}(\lambda_{SW})$	-	W
Vb	D	S	-	$W'_{max}(\lambda_{SW})$

Table 2: Loading Cases

$\delta \cdot 10^{-4}$	β_δ*	R						
3	0.85	1.000	0.065	0.070	0.144	0.148	0.167	0.204
4	3.75	0.065	1.000	0.999	0.981	0.870	0.802	0.767
5	3.23	0.070	0.999	1.000	0.987	0.878	0.811	0.775
6	2.82	0.144	0.981	0.987	1.000	0.938	0.888	0.860
7	2.20	0.148	0.870	0.878	0.938	1.000	0.992	0.980
8	1.42	0.167	0.802	0.811	0.888	0.992	1.000	0.996
9	0.66	0.202	0.767	0.775	0.860	0.980	0.996	1.000

Table 3: Correlation Matrix of Parallel Modes 3 - 9 in Series mode 15, Loading Case Va

Load case	Mode	Mechanical behaviour according to figure 8 (γ=0.25) Safety indices β_p	β_{δ^*}	β_{E,δ^*}	Failure probability	Perfectly plastic behaviour (γ=1) β_p	P_f	Number of basic variables
I	1	3.828	3.247	3.160	$6.46 \cdot 20^{-5}$	5.364	$4.07 \cdot 10^{-8}$	8
	2	4.925	3.882	3.828	$4.22 \cdot 10^{-7}$	6.353	$1.06 \cdot 10^{-10}$	8
	3	5.678	4.400	4.361	$6.83 \cdot 10^{-9}$	6.994	$1.34 \cdot 10^{-12}$	8
	4	4.747	3.784	3.726	$1.03 \cdot 10^{-6}$	6.069	$6.45 \cdot 10^{-10}$	8
	U	3.828			$6.46 \cdot 10^{-5}$	5.364	$4.08 \cdot 10^{-10}$	8
II	5	10.887	6.504	-	$1.09 \cdot 10^{-28}$	12.986	$7.64 \cdot 10^{-39}$	9
	6	10.854	9.454	-	$1.05 \cdot 10^{-27}$	19.567	$1.58 \cdot 10^{-85}$	9
	7	10.362	9.548	-	$1.92 \cdot 10^{-25}$	20.476	$1.91 \cdot 10^{-93}$	9
	8	10.268	9.040	-	$5.03 \cdot 10^{-25}$	14.934	$1.03 \cdot 10^{-50}$	9
	13	7.689	7.350	7.330	$7.23 \cdot 10^{-15}$	16.965	$7.90 \cdot 10^{-65}$	11
	14	9.580	7.494	-	$4.89 \cdot 10^{-22}$	10.552	$2.53 \cdot 10^{-26}$	11
	U	7.690			$7.23 \cdot 10^{-15}$	10.560	$2.34 \cdot 10^{-26}$	11
IIIa	13	7.264	6.392	6.359	$1.91 \cdot 10^{-13}$	13.157	$8.07 \cdot 10^{-40}$	11
	14	5.630	5.963	5.959	$1.18 \cdot 10^{-14}$	9.370	$3.67 \cdot 10^{-21}$	11
	U	7.266			$2.03 \cdot 10^{-13}$	9.370	$3.67 \cdot 10^{-21}$	11
IIIb	13	8.011	6.936	6.912	$2.89 \cdot 10^{-24}$	13.194	$4.95 \cdot 10^{-40}$	11
	14	8.438	6.730	6.324	$1.87 \cdot 10^{-16}$	9.200	$1.81 \cdot 10^{-20}$	11
	U	8.012			$1.87 \cdot 10^{-16}$	9.200	$1.81 \cdot 10^{-20}$	11
IV	9-12				~ 0		~ 0	9
	15	7.483	6.841	6.853	$3.37 \cdot 10^{-14}$	13.594	$2.25 \cdot 10^{-42}$	11
	16	8.820	6.813	5.74	$5.74 \cdot 10^{-19}$	14.846	$3.87 \cdot 10^{-50}$	11
	U	7.487			$3.37 \cdot 10^{-14}$	13.594	$2.25 \cdot 10^{-42}$	11
Va	15		3.747	3.699	$2.46 \cdot 10^{-5}$	6.512	$3.71 \cdot 10^{-11}$	11
	16	5.819	4.643	4.593	$2.97 \cdot 10^{-9}$	8.361	$3.14 \cdot 10^{-17}$	11
	U	4.060			$2.46 \cdot 10^{-5}$	6.512	$3.71 \cdot 10^{-11}$	11
Vb	15	6.343	5.400	5.442	$1.13 \cdot 10^{-10}$	8.453	$1.44 \cdot 10^{-17}$	11
	16	7.928	6.142	6.144	$1.12 \cdot 10^{-15}$	8.316	$4.62 \cdot 10^{-17}$	11
	U	6.343			$1.13 \cdot 10^{-10}$	8.287	$5.86 \cdot 10^{-17}$	11
$\Sigma=P_f$		3.748			$8.92 \cdot 10^{-5}$	5.363	$4.08 \cdot 10^{-8}$	11

Table 4: Results for Frame Example

Loading case	Mode	Eq. 21b	Eq. 18b	Eq. 19b	β_H	Eq. 19a	Eq. 18a	
I	1	9.27	7.47	6.18	3.83	3.85	3.25	
	2	13.7o	8.75	6.97	4.93	3.88	3.88	
	3	16.31	11.5o	-	5.68	4.4o	4.4o	
	4	12.86	8.55	6.79	4.75	3.78	3.78	
II	5	∞	18.91	11.81	1o.89	6.5o	6.5o	6.5o
	6	∞	15.16	12.74	1o.85	9.45	9.45	9.45
	7	∞	19.48	14.41	1o.36	9.6o	9.55	9.55
	8	∞	19.38	12.72	1o.27	9.o4	9.o4	9.o4
	13	∞	-	-	7.69	7.36	7.35	7.35
	14	∞	17.92	13.2o	9.58	7.49	7.49	7.49
IIIa	13	2o.o9	15.55	-	7.26	6.51	6.39	6.39
	14	16.55	11.16	9.42	7.63	5.96	5.96	5.96
IIIb	13	∞	13.14	1o.7o	8.o1	6.94	6.94	6.94
	14	2o.3	1o.36	9.65	8.44	7.18	6.73	6.73
IV	9-12	∞	∞	∞	∞	∞	∞	∞
	15	∞	14.23	9.o9	7.49	6.85	6.84	6.84
	16	∞	2o.69	-	8.82	7.oo	6.81	6.81
Va	15	7.69	6.84	6.o4	4.o6	3.79	3.75	3.74
	16	1o.o8	9.6o	8.39	5.82	5.26	4.64	4.64
Vb	15	16.73	7.77	6.85	6.35	5.42	5.4o	5.4o
	16	18.64	9.87	8.72	7.93	6.14	6.14	6.13

Table 5: Safety indices - bounds and best estimate according to[19] for frame example

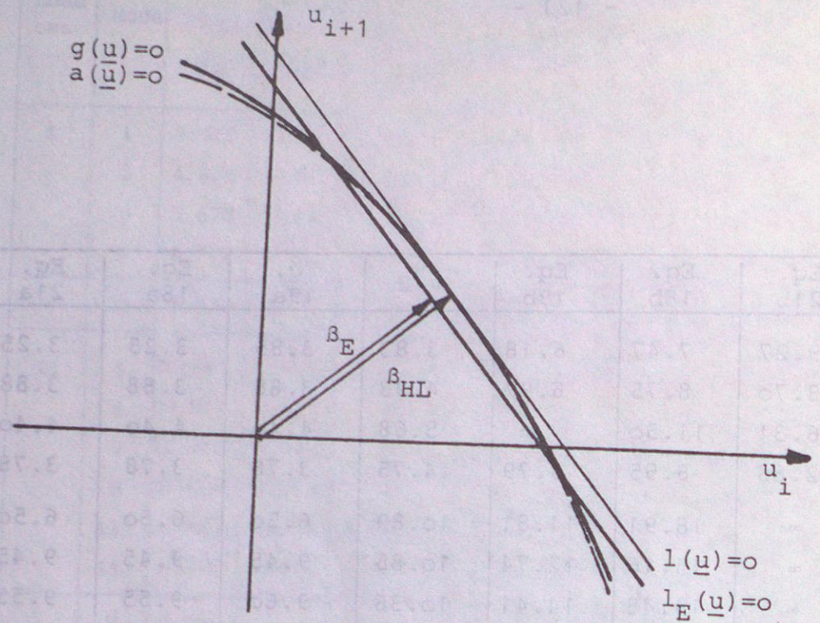


Figure 1: Failure surface $g(\underline{u})=o$, approximating failure surface $a(\underline{u})=o$ and equivalent linear failure surface $l_E(\underline{u})=o$

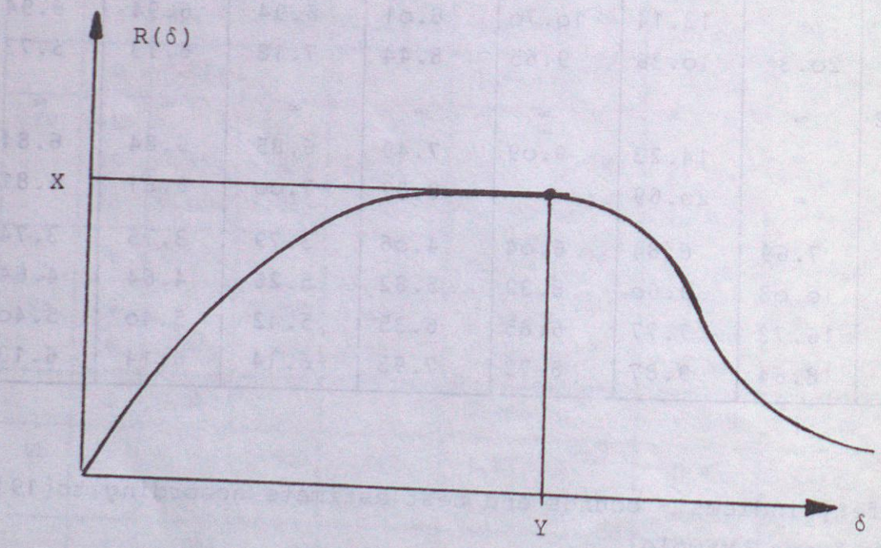


Figure 2: Non-linear force-deformation relationship with two random parameters X and Y

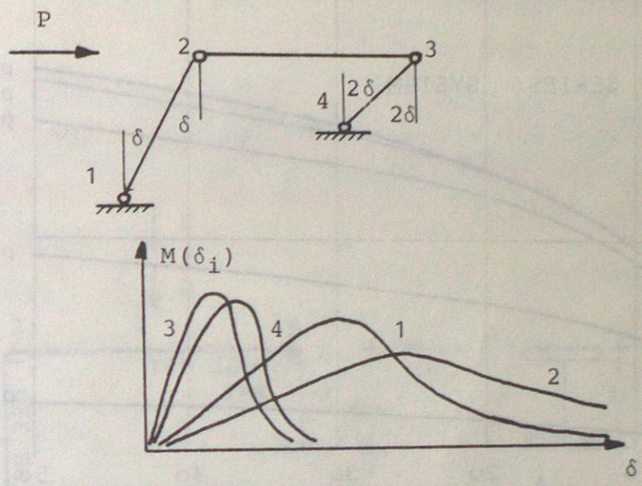


Figure 3: Random moment-rotation curves under imposed leading rotation δ

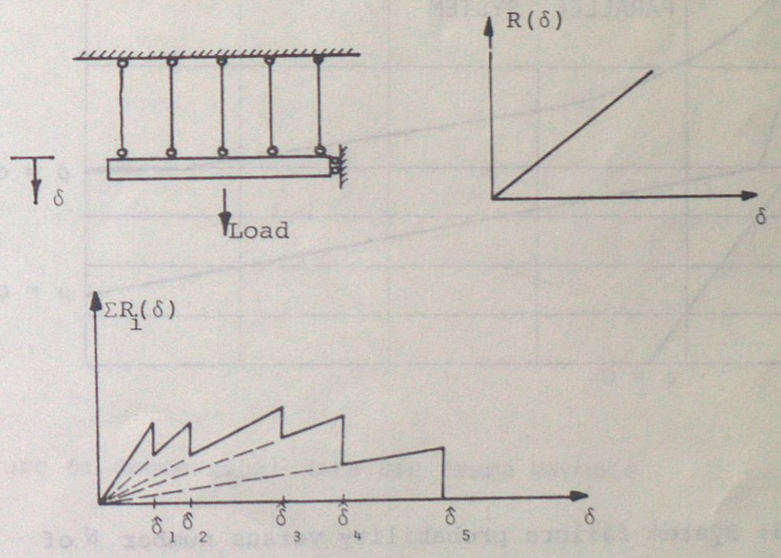


Figure 4: Example of the system resistance behaviour for ideally brittle elements under imposed deformation δ

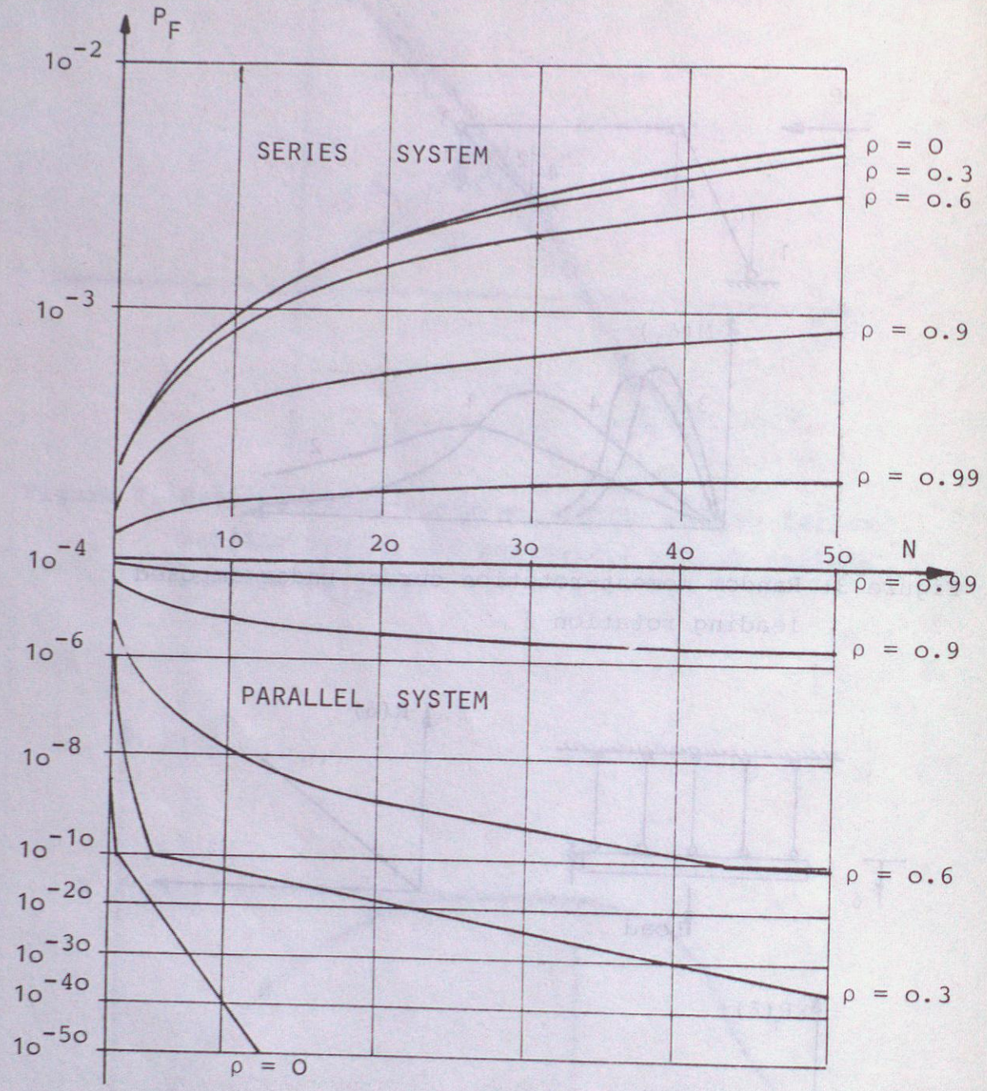


Figure 5: System failure probability versus number N of equicorrelated components with component failure probability 10^{-4}

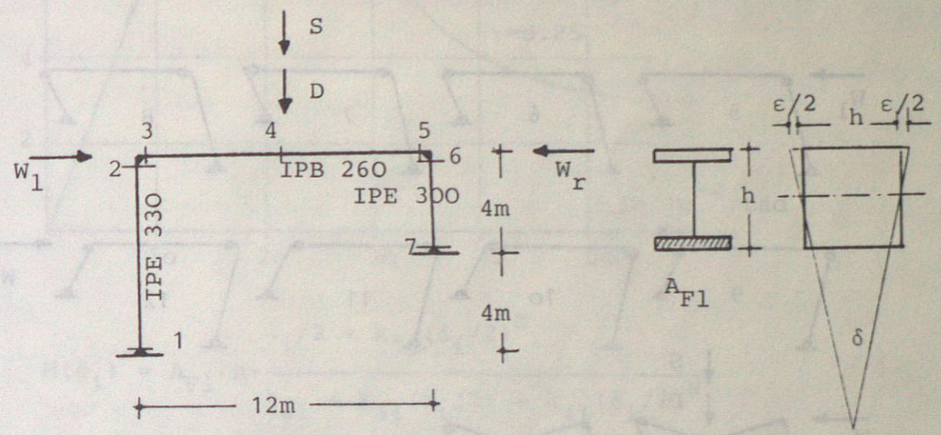


Figure 6: Geometrical data for frame example

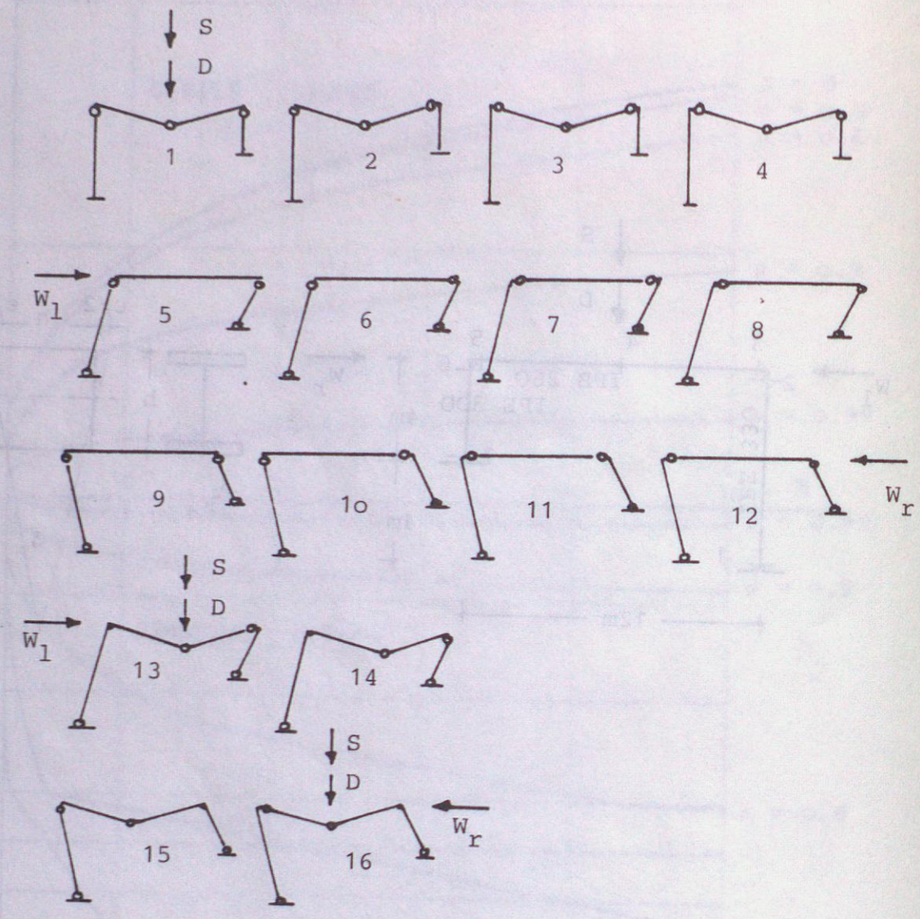
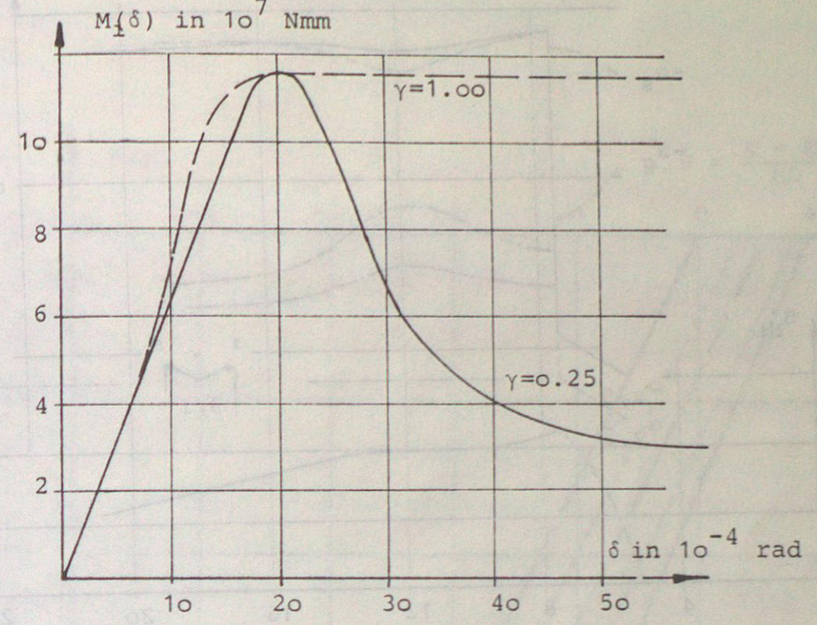


Figure 7: Selected series failure modes



$$M(\delta_i) = A_{Fl} \cdot h \cdot \frac{\delta_i/2 + k_{1i}(\delta_i/2)^N}{k_{2i} + k_{3i}(\delta_i/2) + k_{4i}(\delta_i/2)^N}$$

$k_{1i} = \gamma \cdot X_i \cdot k_{4i}$

$k_{2i} = 1/E = (2.1 \cdot 10^5)^{-1}$

$k_{3i} = 1/X_i - N/(N-1) \cdot k_{2i}/Y_i$

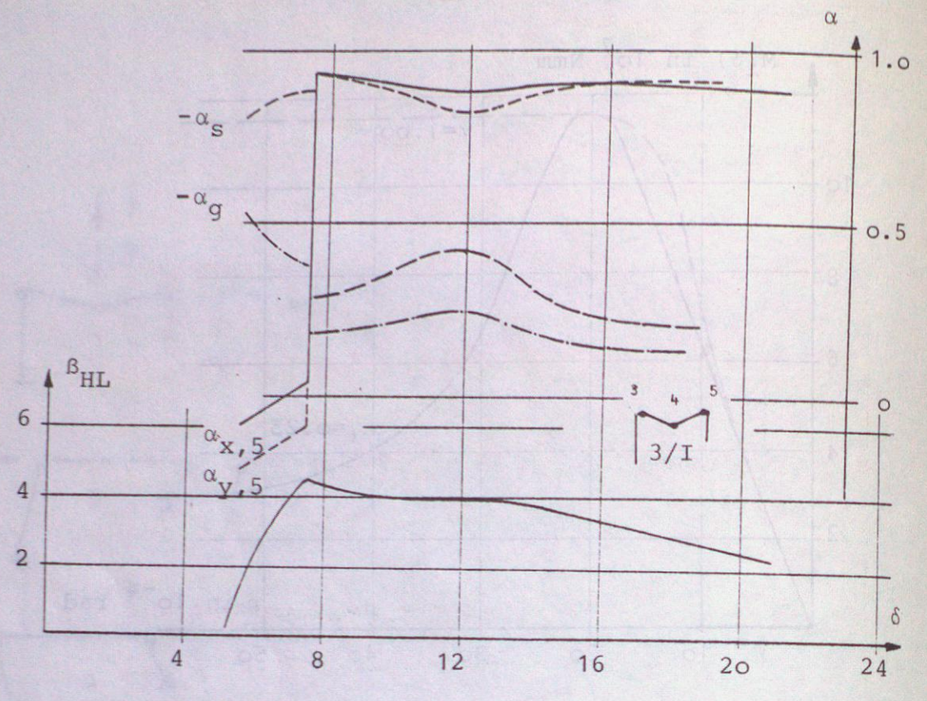
$k_{4i} = 1/((N-1) \cdot k_{2i}^{-1} \cdot Y_i^N \cdot (1-\gamma)$

$N = 6$

$\gamma = 0.25/1.00$

Figure 8: Moment-rotation curve for example 1

a).



b).

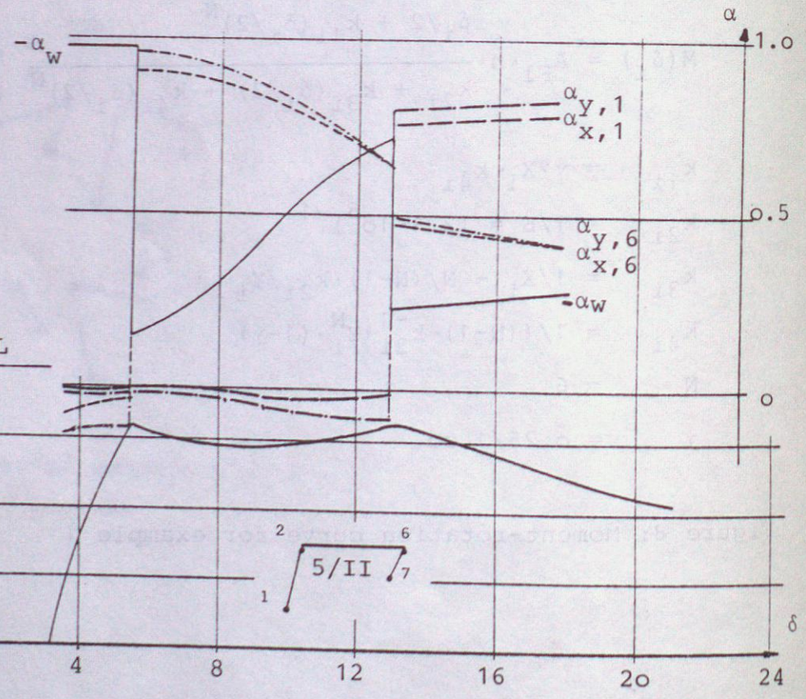


Figure 9: Safety index and some direction cosines versus imposed rotation for frame example

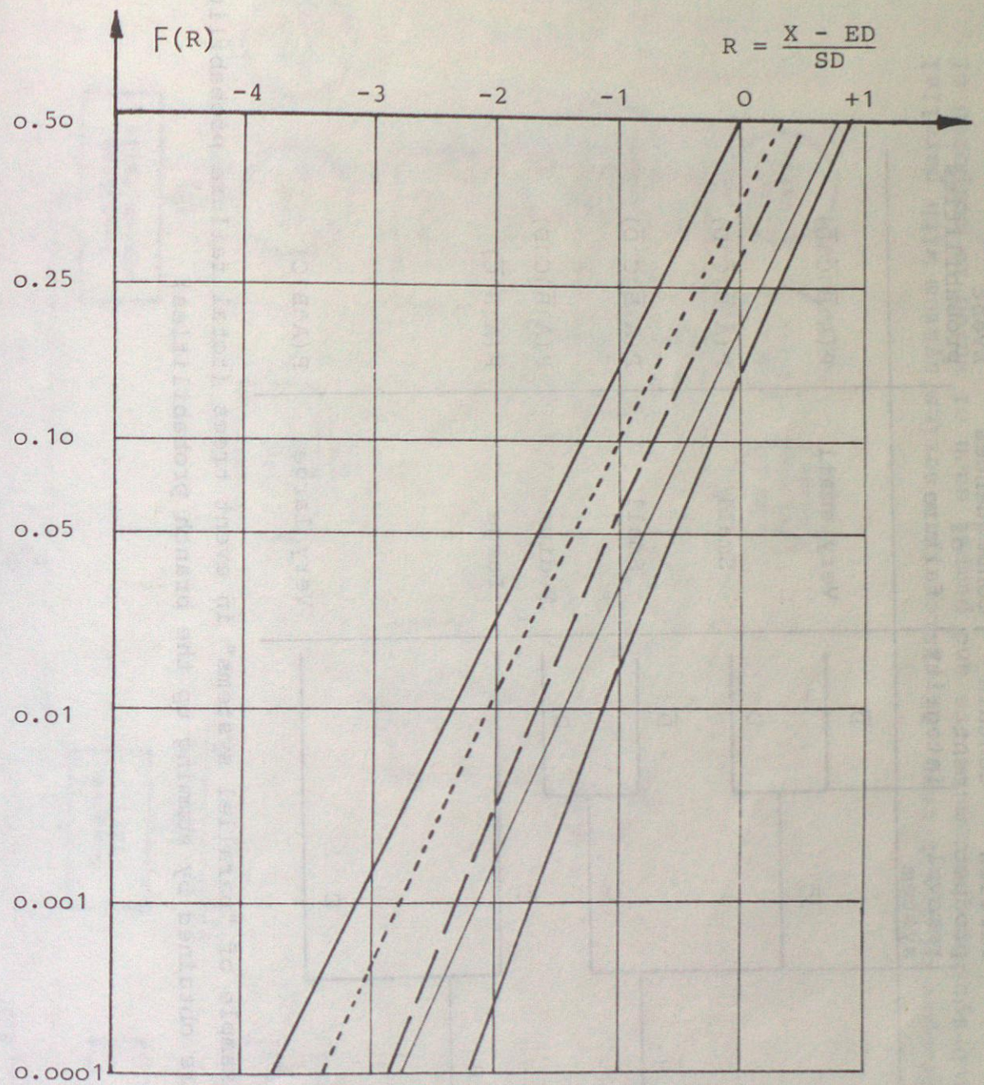


Figure 10: Comparison of various probability bounds for a Daniels-system with exact and asymptotic probabilities. 10 elements (see text)

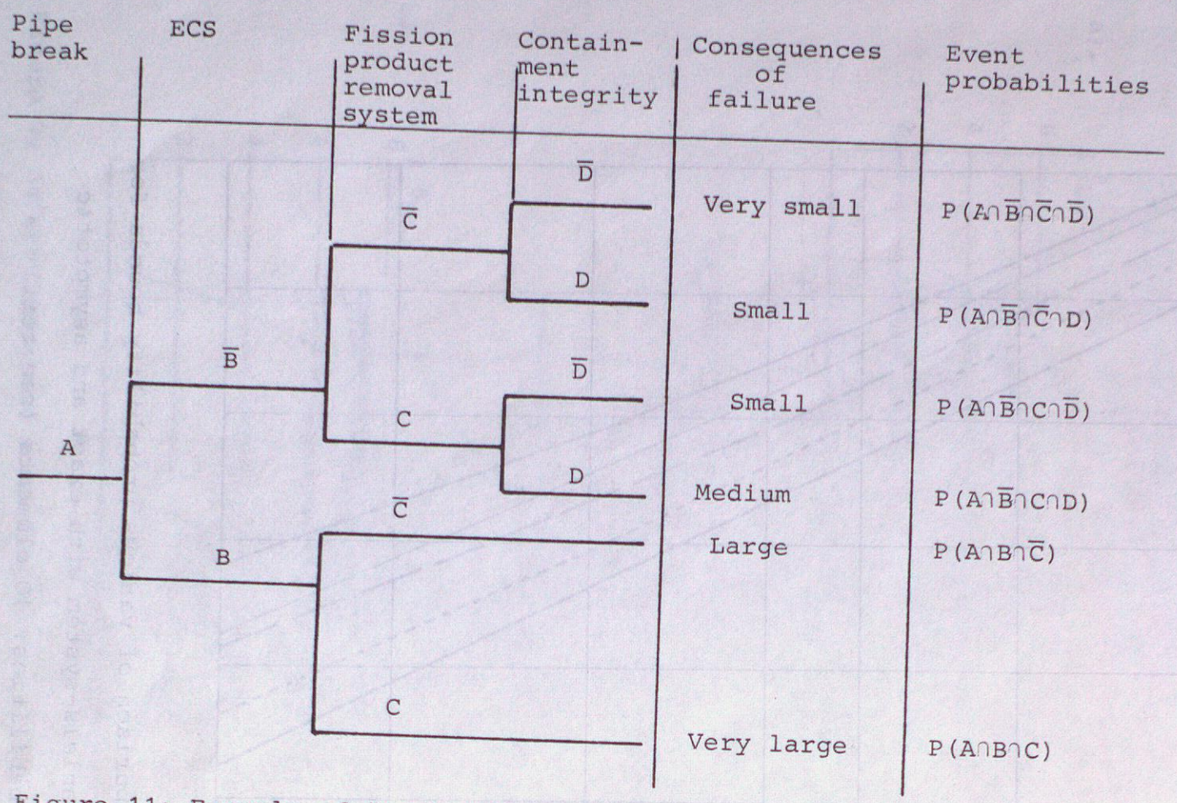


Figure 11: Example of "parallel systems" in event trees (total failure probabilities are obtained by summing up the branch probabilities)

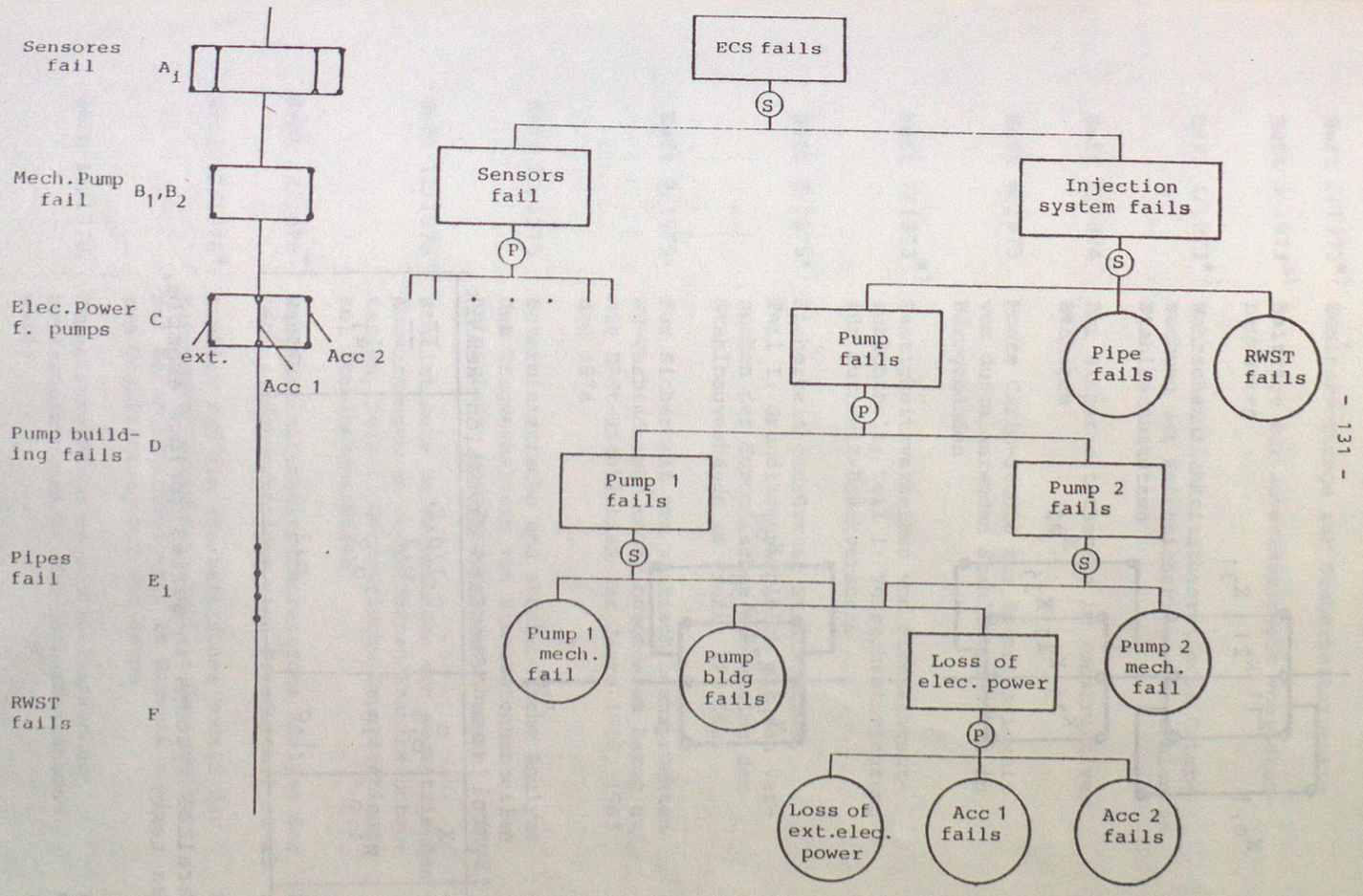
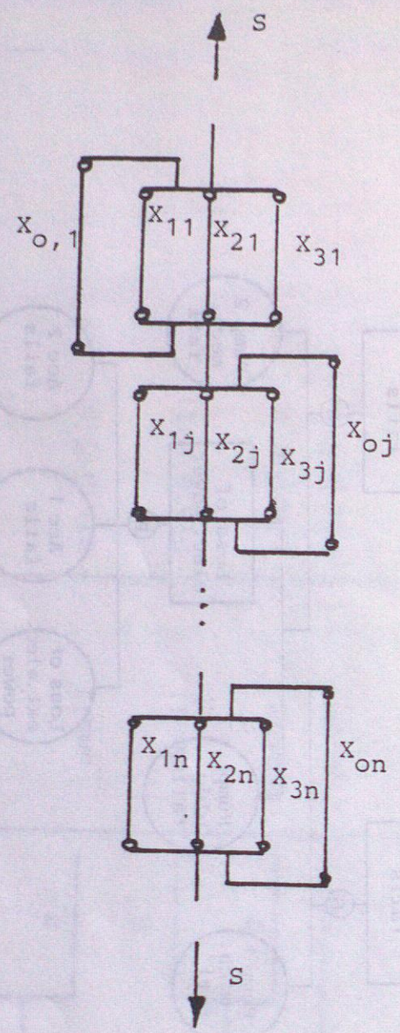


Figure 12: Example of fault tree. EC-system is a series system with parallel sub-systems (accumulators and pumps) each of which is composed of elements in series



Variable	Symbol	mean	Standard deviation	Remark
Modell error	X	o	o.1/o.2,	--
Load		o.5	o.2	--
Protection system	$X_{i,o}$	1.6	o.2	c=1
Piles $X_{i,1}$, $X_{i,2}$, $X_{i,3}$		1.o	o.2	--

Figure 13: Parallel systems in series (bridge example, see text)

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Heft 55/1981	Zwei Anwendungen der Bayesschen Statistik	K. Breitung
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