SONDERFORSCHUNGSBEREICH 96

Rackwitz

LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

BERICHTE
zur
ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

Non-normal vectors, quadratic limit state criteria, systems and extreme loads in first order reliability theory

Zuverlässigkeitstheorie I. Ordnung und nicht-normale Vektoren, quadratische Grenzzustandsflächen, Tragsysteme bzw. außergewöhnliche Einwirkungen

von

B. Fießler, B. Krzykacz, H.-J. Neumann, R. Rackwitz

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NON-NORMAL VECTORS, QUADRATIC LIMIT STATE CRITERIA, SYSTEMS AND EXTREME LOADS IN FIRST ORDER RELIABILITY THEORY

ZUVERLÄSSIGKEITSTHEORIE I. ORDNUNG UND NICHT-NORMALE VEKTOREN, QUADRATISCHE GRENZZUSTANDSFLÄCHEN, TRAGSYSTEME BZW. AUSSERGEWÖHNLICHE EINWIRKUNGEN

von

B. Fießler, B. Krzykacz, H.-J. Neumann, R. Rackwitz

SONDERFORSCHUNGSBEREICH 96 (SFB 96)

Der SFB 96 "Zuverlässigkeit von Bauwerken" ist eine Einrichtung der Technischen Universität München und der Deutschen Forschungsgemeinschaft. Der SFB hat sich die Entwicklung baustoff- und bauartenübergreifender Sicherheitssysteme auf wahrscheinlichkeitstheoretischer Grundlage zum Ziel gesetzt.

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V O R W O R T

Dieses Berichtsheft enthält fünf Beiträge zur Theorie der Berechnung der Versagenswahrscheinlichkeit von Bauwerken. Der Beitrag "Non-Normal Vectors in Structural Reliability" geht auf einen in Heft 14 dieser Berichtsreihe, Anhang B, abgedruckten Vortrag des Erstverfassers zurück und stellt den derzeitigen Stand der Entwicklung der Zuverlässigkeitstheorie 1. Ordnung innerhalb des SFB 96 dar.

Der zweite Beitrag "Quadratic Limit States in Structural Reliability" enthält wesentliche Verallgemeinerungen der in den Heften 21 (Rackwitz, R.: Quadratic Limit State Criteria in Structural Reliability), 22 (Neumann, H.-J.; Fießler, B.; Rackwitz, R.: Die genäherte Berechnung der Versagenswahrscheinlichkeit mit Hilfe rotationssymmetrischer Grenzzustandsflächen 2. Ordnung) und 25 (Neumann, H.-J.: Methoden zur Berechnung der Versagenswahrscheinlichkeit) dieser Berichtsreihe vorgelegten Untersuchungen für eine Zuverlässigkeitstheorie 2. Ordnung. Nach Ansicht der Verfasser können damit die Untersuchungen zur Zuverlässigkeitstheorie 2. Ordnung in gewissem Sinne als abgeschlossen betrachtet werden. Die vorgeschlagenen Rechenmethoden sind als Computer-Programm realisiert und stehen Anwendungen zur Verfügung.

Im dritten und vierten Beitrag wird ein erster Versuch gemacht, nunmehr auch Tragsysteme, die auf verschiedenen Wegen versagen können, mit Hilfe der Zuverlässigkeitstheorie 1. Ordnung zu behandeln. Es wird gezeigt, daß eine entsprechende Formulierung möglich ist. Allerdings ist das dabei auftretende Problem der Ermittlung des Integrals der mehrdimensionalen Normalverteilung noch nicht voll befriedigend gelöst.

Im letzten Beitrag wird schließlich der Sonderfall der nichtlinearen Kombination von extremen Einwirkungen im Rahmen der vorstehend angedeuteten Ansätze studiert.

Alle Beiträge sind als Einzelbeiträge abgefaßt und daher getrennt lesbar.

Die vorgelegten Arbeiten sind im Rahmen der Teilprojekte A 5, C 1 und D 12 des SFB 96 "Zuverlässigkeitstheorie der Bauwerke" durchgeführt worden.

München, April 1978 Die Verfasser

P R E F A C E

This report collects five contributions on the theory for the calculation of failure probabilities of structures. The first paper "Non-Normal Vectors in Structural Reliability" is based on a presentation of the first named author which is printed in these series (report No. 14). The present version illustrates the state of development of first order reliability theories within SFB 96.

The second contribution "Quadratic Limit States in Structural Reliability" contains an essential generalization of towards a second order reliability theory. Earlier studies have been published in report No. 21 (Rackwitz, R.: Quadratic Limit State Criteria in Structural Reliability), report No. 22 (Neumann, H.-J.; Fießler, B.; Rackwitz, R.: Die genäherte Berechnung der Versagenswahrscheinlichkeit mit Hilfe rotationssymmetrischer Grenzzustandsflächen 2. Ordnung) and report No. 25 (Neumann, H.-J.: Methoden zur

Berechnung der Versagenswahrscheinlichkeit). Herewith the major problems in the second order reliability theory can in a certain sense be considered as solved in the authors view. The proposed methods are realized in computer programs and are available for further applications.

The third and fourth contribution are a first attempt to apply the concepts of first order reliability theory to structural systems which can fail in different modes. A suitable formulation is given. However, it requires the evaluation of the multi-dimensional normal distribution which is not yet solved satisfactorily.

The last paper is concerned with non-linear combination of extreme loadings as a special case of application of first order reliability concepts.

No attempt has been made to avoid repetition in the different contributions so that they can be read separatly.

The foregoing studies are part of the projects A 5, C 1 and D 12 of the SFB 96 "Structural Reliability".

Munich, April 1978 The authors

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NON-NORMAL VECTORS IN STRUCTURAL RELIABILITY

by

R. Rackwitz and B. Fießler

Abstract

Informative probabilistic reliability assessments for structural facilities require the prior choice of probabilistic models for the uncertainties, particularly their distributional characteristics. In principle, multi-dimensional numerical integration is necessary for the calculation of reliabilities which is rather tedious as the type of models for the uncertain variables becomes more complex and their number increases. Therefore, suitable approximations are in order. The well-known second moment reliability method as proposed e.g. by Hasofer and Lind is generalized to take account of any arbitrary type of distribution of the uncertainties by means of a discrete first order transformation into a normal distribution. A suitable algorithm is presented for finding the appropriate point of transformation. The method is applied to some extreme cases of limit state functions. The possible error of the approximative method is calculated. It is proved that the inclusion of second order terms for the expansion of the limit state function can yield results which are in error only insignificantly.

Introduction

One of the major problems in the application of probability based design methods to the elaboration of design codes or to direct design of structural facilities is to develope an efficient method for determining the probability of not exceeding a given structural limit state. If $\underline{X}$ is a parameter-invariant vector of n basic uncertain variables such as actions, dimensions and strength of materials with joint distribution function $F(\underline{x})$, then the failure probability

$$P_f = 1 - \int_{\{D\}} dF(\underline{x}) \qquad (1)$$

is the complement to the probability content of the safe domain D which is separated from the unsafe domain by the function describing the limit state. Analytical solutions of (1) exist only for a few special cases concerning the distribution function $F(\underline{x})$ and shapes of D and, therefore, are of minor practical interest. The following study deals with an approximate calculation of the integral (1) for arbitraty distribution functions $F(\underline{x})$ and shapes D of the safe domain.

First Order Reliability Methods

Recently, approximate "first order reliability methods" as proposed by Ditlevsen [2], Hasofer/Lind [7], Paloheimo [10] and Veneziano [11] reduced the fundamental problem of multi-dimensional integration to a numerically much simpler

problem of mathematical programming. Let the vector $\underline{X} \equiv (X_1, X_2 \ldots X_n)^T$ be represented by its mean value vector $M_{\underline{X}} = E[\underline{X}]$ and by its covariance matrix $\Sigma_{\underline{X}} = E[(\underline{X} - E[\underline{X}]) \cdot (\underline{X} - E[\underline{X}])^T]$. Then, there exists an orthogonal transformation such that the components of $\underline{X}$ become uncorrelated yielding a new variable vector

$$\underline{Y} = \underline{\underline{R}}^T \cdot \underline{X} \qquad (2)$$

with the rotation matrix $\underline{\underline{R}}^T$ and the matrix of Eigenvectors $\underline{\underline{R}}$ of Σ_X, the mean value vector $E[Y] = \underline{\underline{R}}^T \cdot E[\underline{X}]$ and diagonal covariance matrix $\Sigma_Y = \underline{\underline{R}}^T \Sigma_{\underline{X}} \underline{\underline{R}}$. If the vector $\underline{Y}$ is standardized by

$$Z_i = \frac{Y_i - E[Y_i]}{D[Y_i]} \qquad i = 1, 2, \ldots, n \qquad (3)$$

with $D[Y_i]$ representing the standard deviation of Y_i, it is $E[Z_i] = 0$ and $D[Z_i] = 1$ and so $\Sigma_{\underline{Z}} = I$ (unit matrix). The so-called safety index ß can be found by minimizing the distance b between the limit state function or failure surface in the formulation $g(\underline{z}) = 0$ and the coordinate origin

$$ß = \min b = \min_{\underline{z} \in \{g(\underline{z}) = 0\}} \{\sqrt{\underline{z}^T \underline{z}}\} \qquad (4)$$

The point representing the smallest distance is denoted by the "checking" or "approximation" point $\underline{z}^*$. Several authors consider ß a convenient reliability measure since further information on the stochastic characteristics of $\underline{X}$ is dispensable or may not be available.

The informativeness of ß about the reliability in terms of a probability statement, however, remains poor and is of the Tchebychev-inequality-type [11]. More precise statements can be made if distributional assumptions on the components of $\underline{X}$ are adopted. For example, assuming the vector $\underline{X}$ being a normal vector and $g(\underline{z}) = 0$ being continuous at the point where eq.(4) is satisfied, the safety index ß produces two useful and simple reliability bounds when approximating the actual, generally non-linear failure surface by either a tangent hyperplane or a supporting hypersphere. For well-behaved convex safe regions the failure probability P_f is, then, bounded by (see Veneziano [11]):

$$1 - \chi_n^2(ß^2) \geq P_f \geq 1 - \phi(ß) \qquad (5)$$

Herein, $\chi_n^2(.)$ denotes the chi-square distribution for n degrees of freedom whereas $\phi(.)$ is the invariate standard normal integral. Essentially, these bounds are related to pure normal uncertainty vectors. In practice, though a lower bound, the right hand side limit frequently yields an accurate estimate of the failure probability P_f.

If the uncertainty vector is log-normal, a simple transformation $\underline{z} \rightarrow \underline{u} : u_i = \ln z_i$ for $i = 1, 2, \ldots, n$ reduces this case to the normal one [3]. Of course, the limit state function now has to be formulated in the new u-space.

However, many uncertain phenomena are only poorly described by either the normal or log-normal model. It is also known that the results in terms of estimates or of bounds to failure probabilities significantly depend on the stochastic model adopted for the basic variable vector. A generalization towards non-normal models would, therefore, considerably increase the applicability of first order reliability methods.

Review of Extensions to Non-Normal Distributions

Paloheimo [10] approximated a non-normal distribution by a normal distribution having the same mean and the same P_f - or $(1-P_f)$-fractile. Setting

$$p = \phi\left(\frac{x_p-\mu}{\sigma'}\right) = F(x_p; \underline{\Theta}) \qquad (6)$$

it is by solution for the new standard deviation σ'

$$\sigma' = \frac{x_p-\mu}{\phi^{-1}(p)} = \frac{F^{-1}(p;\Theta)-\mu}{\phi^{-1}(p)} \qquad (7)$$

yielding the standardized normal variate by the transformation

$$x \to u: \; u = \frac{(x-\mu)\cdot\phi^{-1}(p)}{F^{-1}(p; \underline{\Theta})-\mu} \qquad (8)$$

where $F^{-1}(.)$ is the inverted non-normal marginal distribution with parameter vector $\underline{\Theta}$, μ the mean value and $\phi^{-1}(.)$ the inverted standard normal distribution function, respectively. p is equal either to the target survival probability $1 - P_f$, if the variable is a loading variable or to the failure probability P_f, if the variable is a resisting variable.

Ditlevsen [3] suggested a similar approximation to the distribution of extremes of independent normal variables. Again, the approximation is chosen such, that the new normal distribution fits the non-normal distribution best in the vicinity of the fractiles corresponding to the target failure (survival) probabilities. Alternatively, he proposed to fit the non-normal distribution by a normal distribution having the same values in two different extreme points but, again, with no strong arguments for the choice of these points.

Lind [9] verified the basic idea of applying a continuous mapping which transforms a non-normal distribution into a normal distribution. For example, if the basic uncertainty vector $\underline{X}$ has independent components with different distribution type it is:

$$x \to u: \; u = h(x) = \phi^{-1}\left[F(x; \underline{\Theta})\right] \qquad (9)$$

This idea is, no doubt, implicit in many of the earlier works. The approach is formally appealing. The transformation (9) is, in general, not elementary. However, eq.(9) can easily be applied in computerized analysis where distribution functions and their inverses can be given by suitable series expansions or rational approximations [1, 8]. In each but Lind's approach the aforementioned probability bounds are no longer valid.

Linear Approximation in the "Checking Point" - Independent Uncertainty Vectors

Since the checking method described before employs one single point on the failure surface it suffices to apply eq.(9) in that point, only (see fig. 1). Following Paloheimo's idea, eq.(8) can be improved by taking the value p at the checking point, giving the transformation

$$x \to u: \; u = \frac{(x-\mu)}{(x^*-\mu)} \; \phi^{-1} \; [F(x^*; \underline{\Theta})] \qquad (10)$$

which is linear in x. The checking point must be known. Alternatively, the mean μ might be substituted by any other appropriate central parameter, e.g. the median $\hat{\mu}$ of $F(\underline{x})$.

$$x \to u: \; u = \frac{(x-\hat{\mu})}{(x^*-\hat{\mu})} \; \phi^{-1} \; [F(x^*; \underline{\Theta})] \qquad (10a)$$

It is obvious that these expressions give a correct mapping only with respect to the value of the distribution function at point x^*. A discrete mapping ought to be accurate in the vicinity of the checking point x^*, as well. Hence, it is proposed to linearize the mapping function (9) in the checking point e.g. by taking its Taylor expansion up to the linear term:

$$u = \phi^{-1}(F(x;\cdot)) \approx \phi^{-1}(F(x^*;\cdot)) + (x-x^*) \cdot \left.\frac{\partial \phi^{-1}(F(x;\cdot))}{\partial x}\right|_{x=x^*}$$

$$= \phi^{-1}(F(x^*;\cdot)) + (x-x^*) \cdot \frac{f(x^*;\cdot)}{\varphi(\phi^{-1}(F(x^*;\cdot)))} = \qquad (11)$$

The variable U is then a standard normal variate. In the original space, it has mean

$$\mu' = x^* - \sigma' \; \phi^{-1}(F(x^*;\cdot)) \qquad (12)$$

and standard deviation

$$\sigma' = \frac{\varphi(\phi^{-1}(F(x^*;\cdot)))}{f(x^*;\cdot)} \qquad (13)$$

This linear transformation is equivalent to adjusting a continuous non-normal distribution to a normal distribution having the same cumulative probability and the same probability density in the checking point.

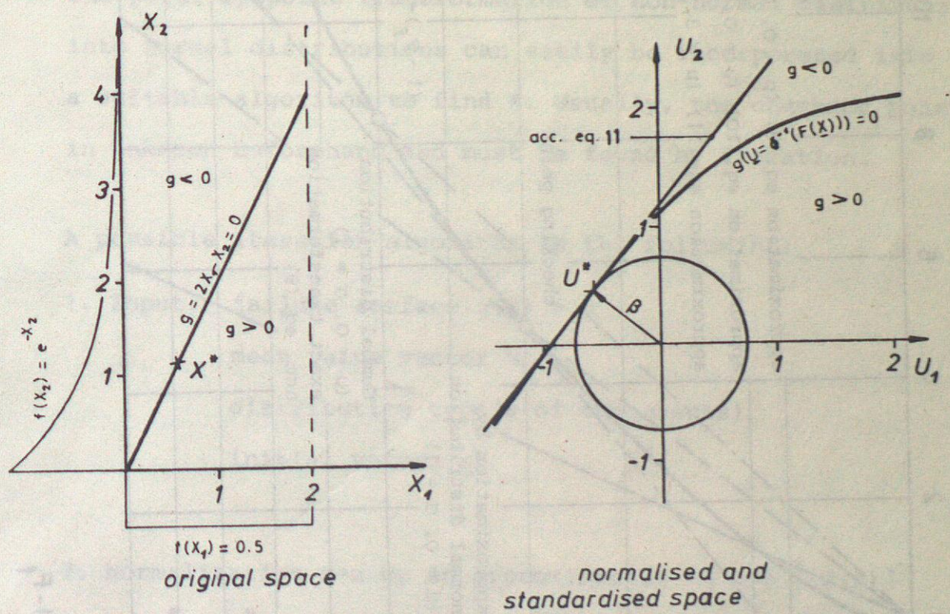

Figure 1

In fig. 2 the different methods as expressed by eqs.(8), (10) and (11) are illustrated for the simple one-dimensional case of an extreme value distribution type I.

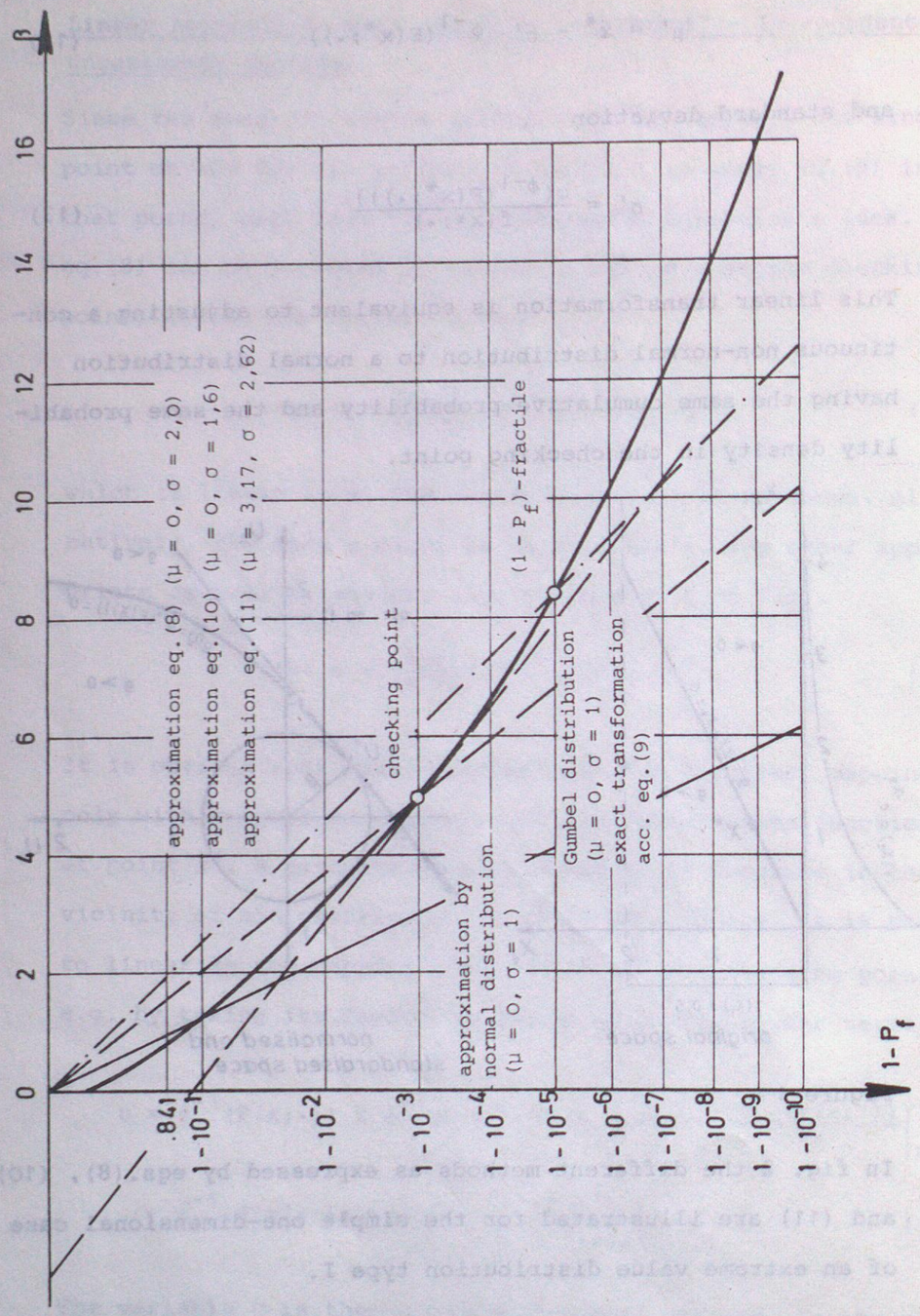

Fig.2: Various approximations of a non-normal distribution by normal distributions with modified distribution parameters

The hyperplane H approximates the failure surface in the point $\underline{u}^*$ obtained by applying the transormation (9) in the same sense as the original linear approximation of $g(\underline{x}) = 0$ (see fig. 1). Therefore, it can be said that the linearized transformation (11) is exact within the first order theory under consideration. It is also obvious that a similar argument holds for the approximating hypersphere S.

A Suitable Iteration Algorithm for finding the safety Index ß

The point by point transformation of non-normal distributions into normal distributions can easily be incorporated into a suitable algorithm to find ß. Usually, the checking point in unknown beforehand and must be found by iteration.

A possible iteration algorithm is the following:

1. Input: failure surface $g(\underline{x}) = 0$
   mean value vector $M_{\underline{X}}$
   distribution type(s of components)
   initial values $\underline{x}^*$

2. Normalization $x \rightarrow u$ by an approximation to $u = \phi^{-1}[F(x)]$ in $\underline{x}^* = \underline{u}^*$ according to eq. (11).

3. Standardization

4. Transformation of $g(\underline{x}) = 0$ into $g(u) = 0$ using the results of 2. and 3.

5. Calculation of direction cosines of search direction by

$$\alpha^{(i+1)} = \frac{\partial g}{\partial \underline{u}}\Big|_{\underline{u}^*} \cdot \left[\sum_{(j)} \left(\frac{\partial g}{\partial \underline{u}}\Big|_{\underline{u}^*}\right)^2 \right]^{-1/2}$$

with $\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \sigma'$ and σ' according to eq.(13) (see appendix B)

6. Calculation of $b^{(i+1)}$ from

$$g(u^*) = 0 \text{ with } u^* = \alpha^{(i+1)} \cdot b^{(i)}$$

7. Inversion of standardization and normalization $u \rightarrow v \rightarrow x$

8. Output: Checking point $\underline{x}^*$, safety index $\beta = b^{(m)}$

If the result at step 8 is unsatisfactory, the sequence of steps has to be repeated from step 2 downwards until convergence is reached. Obviously, the main iterative improvement occurs in the consecutive alteration of the vector of direction cosines $\underline{\alpha}$ for the distance b. Other methods are possible.

To be convergent the algorithm requires local differentiability of the failure surface as well as local continuity and monotony of the original density function. The latter condition is sufficient to yield parameters of the approximating normal distribution which are monotonic functions of the checking point. In practical cases of any dimension with continuous failure surfaces and distribution functions deviating not too far from the normal, excellent results have been obtained (see, for example [3] to [6]).

The search for β becomes numerically more complex if the original failure surface has several local minima or if the original distribution function is discontinuous and/or its

density function is locally non-monotonic, e.g. in case of multimodal distributions. For the latter case, the search for all local minima may turn out to be quite cumbersome. The iteration may even diverge depending on the relative position of a mode and the approximation point. Application of the rougher approximation eq.(10) may help in some cases which also holds if the distribution is discontinuous.

Accuracy of the Method

The accuracy of the results decreases as the number of loading or resisting variables having very skewed and/or limited distributions increase. The convexity properties of the safe domain may be modified.

In order to check the accuracy of the method comparisons can be made with a few results form exact probability theory. To avoid additional influences the following linear limit state surface is chosen

$$\pm C_n \mp \sum_{i=1}^{n} X_i = 0 \qquad (14)$$

in which C_n is a constant derived from the reference case of identically normally distributed variables. Let the constant be defined by $C_n = n \cdot \mu_X \pm \beta \cdot \sigma_X \cdot \sqrt{n}$ and the X_i's be some identically distributed random variables. β is the pre-selected safety index. For rectangular distributions with probability density

$$f(x) = \begin{cases} 1/a & \text{for } 0 \leq x \leq a \\ 0 & \text{elsewhere} \end{cases} \qquad (15)$$

the distribution function of $X = \sum_{i=1}^{n} X_i$ is [8]

$$F(y) = \frac{1}{a^n\, n!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} (y - \nu \cdot a)^n \qquad (16)$$

and, therefore,

$$P_f = 1 - F(C_n)$$

For sums of gamma-distributed variables with probability density function

$$f(x) = \begin{cases} \dfrac{\lambda(\lambda x)^{k-1} \exp[-\lambda x]}{\Gamma(k)} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (17)$$

the type of distribution is retained but with parameters λ and $n \cdot k$ [8]. Remember that $k = 1$ corresponds to the extremely skewed case of an exponential distribution. In Figure 3 some examples are presented for $\beta = 3$.

One recognizes that only in the exceptional cases of the rectangular and the gamma distributions with, say $k < 5$, the approximate method results in significant errors. They increase with the dimension of the basic variable vector. The maximum error associated with each type of distribution function approaches a limit which in using the Central Limit Theorem of probability theory is found as follows. For symmetrical reasons and from eq.(14) the exact approximation point is known to be $x_i^* = C_n/n$ for $i = 1, 2, \ldots, n$. Therefore, the limiting approximate failure probability can be derived from

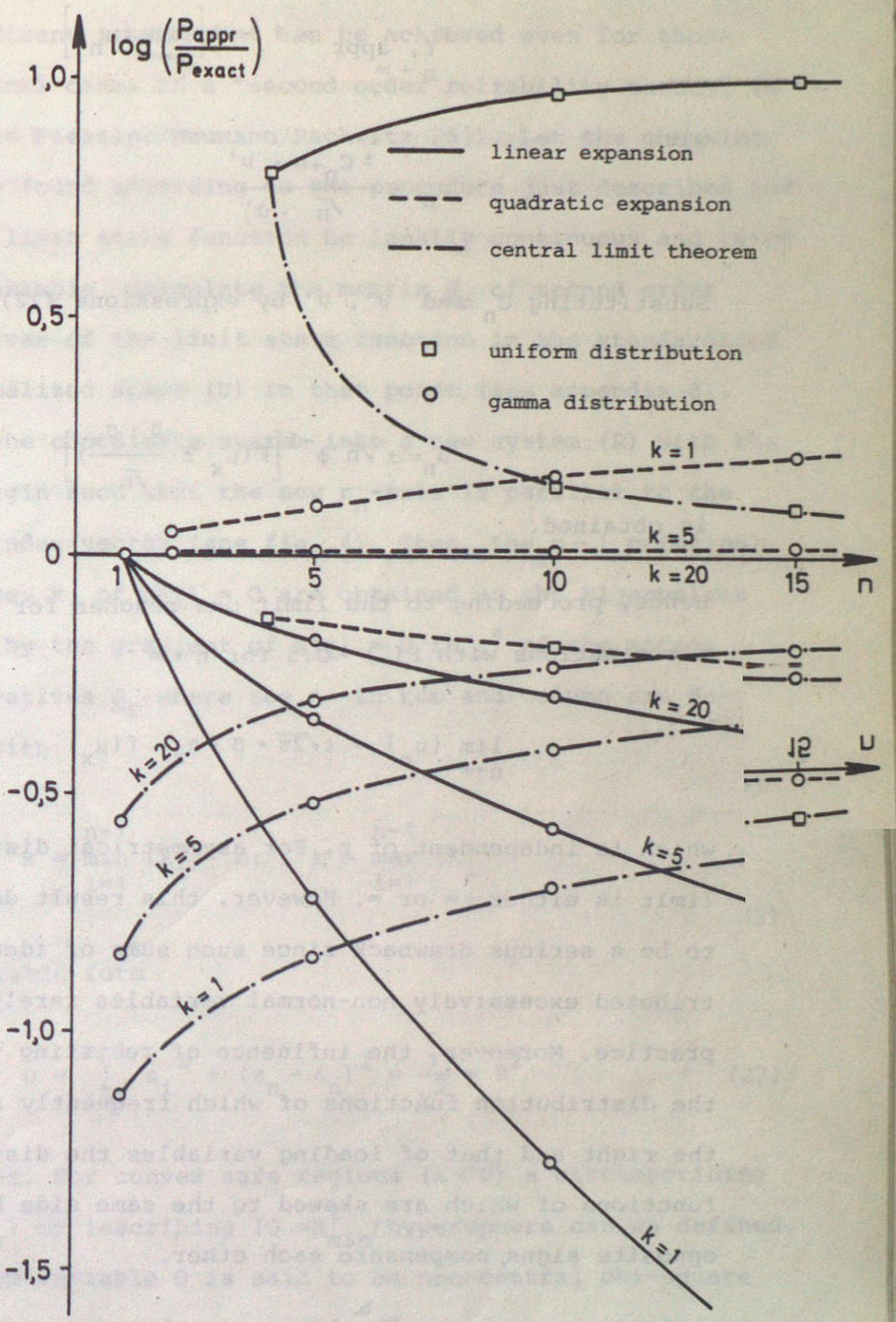

Fig. 3: Approximation errors as a function of dimension n for sums of uniformly and gamma-distributed variables

$$P_{f,\ n\to\infty,\ appr} = 1 - \phi\left[\lim_{n\to\infty}\{u_n\}\right] \tag{18}$$

with

$$u_n = \frac{\pm C_n \mp n \cdot \mu'}{\sqrt{n}\ \cdot \sigma'} \tag{19}$$

Substituting C_n and μ', σ' by expressions (12) and (13)

$$u_n = \pm \sqrt{n}\ \phi^{-1}\left[F(\mu_x \pm \frac{\beta \cdot \sigma_x}{\sqrt{n}})\right] \tag{20}$$

is obtained.

Hence, proceeding to the limit one reaches for symmetrical distributions with $F(.) \to 0{,}5$ for $n \to \infty$

$$\lim_{n\to\infty}\{u_n\} = \pm\sqrt{2\pi} \cdot \beta \cdot \sigma_x\ \ f(\mu_x) \tag{21}$$

which is independent of n. For asymmetrical distributions the limit is either $-\infty$ or ∞. However, this result does not seem to be a serious drawback since such sums of identically distributed excessively non-normal variables rarely occur in practice. Moreover, the influence of resisting variables, the distribution functions of which frequently are skewed to the right and that of loading variables the distribution functions of which are skewed to the same side but have opposite signs, compensate each other.

$\pm\sqrt{2\pi} \cdot \beta \cdot \frac{\alpha^{\frac{1}{\alpha}}}{\sqrt{12}}$

Second Order Expansions

A significant improvement can be achieved even for those exceptional cases if a "second order reliability theory" is used (see Fiessler/Neumann/Rackwitz [5]). Let the checking point be found according to the procedure just described and let the limit state function be locally continuous and twice differentiable. Calculate the matrix $\underline{\underline{G}}_u$ of second order derivatives of the limit state function in the standardized and normalized space (U) in that point (see appendix B). Rotate the coordinate system into a new system (R) with the same origin such that the new r_n-axis is parallel to the safety index vector (see fig. 4). Then, the n - 1 principal curvatures k_i of $g(\underline{r}) = 0$ are obtained as the Eigenvalues divided by the gradient of $g(\underline{r}) = 0$ in $\underline{r}^*$ of the matrix of derivatives $\underline{\underline{G}}_r$ where the n - th row and column are deleted. With

$$k = \min_{i=1}^{n-1}\{k_i\} \quad \text{or} \quad k = \max_{i=1}^{n-1}\{k_i\} \tag{22}$$

the quadratic form

$$Q = \sum_{i=1}^{n-1} s_i^2 + (s_n - \delta_n)^2 = \frac{1}{k^2} = R^2 \tag{23}$$

can be set. For convex safe regions ($k < 0$) a circumscribing ($Q = R^2_{max}$) or inscribing ($Q = R^2_{min}$) hypersphere can be defined. The random variable Q is said to be non-central chi-square distributed with n degrees of freedom and non-centrality parameter $\delta_n^2 = (R - \beta)^2$. Thus, the failure probability is

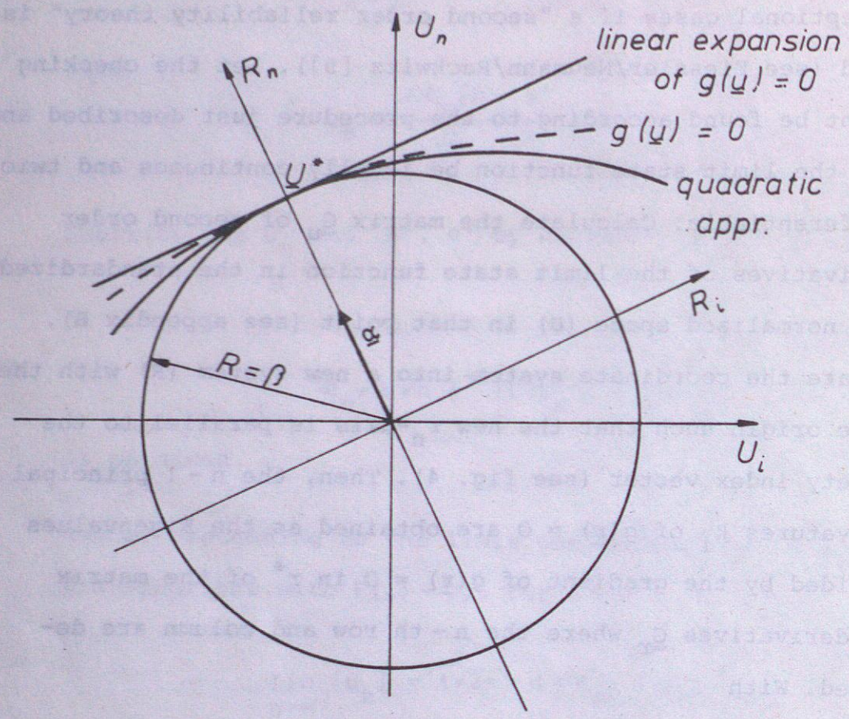

Fig.4: Linear and quadratic approximation of the limit state function

$$P_f = 1 - \chi^2_{n,\delta}(R^2) \qquad (24)$$

Inserting the minimum or maximum radius R eq.(24) yields two bounds to the true failure probability. The non-central chi-square distribution can be evaluated by using expansions as given, for example, in [1] or [8]. When applying eq.(24) to the limit state function (14) one recognizes from fig. 3 that a second order reliability theory is in error only insignificantly. The hypersphere shows excellent agreement with the exact results. It can be proved that more general quadratic forms yield results even still closer to the exact ones (see [5]).

Conclusions

The discrete safety checking methods as proposed by Hasofer/Lind and others can be generalized with respect to arbitrary distributional assumptions for the basic uncertainty vector. In essence, non-normal distributions are approximated in a first order sense by normal distributions in certain checking points. In general, the accuracy of the method is sufficient for engineering purposes. A suitable iteration algorithm is presented to find the appropriate checking point. As an alternative a second order reliability method is proposed which appears to be in error only insignificantly but requires continuity and twice differentiability of the limit state function.

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Appendix B: Transformation of Derivatives

If X is a non-normal variable with the distribution function F(x) and if $\partial g/\partial x$ is the derivative of the limit state function and the transformation

$$u = \phi^{-1} F(x) \tag{B.1}$$

is applied to independent variables X it is from elementary calculus of differentials

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial \phi(u)}{\partial F(x)} = \frac{\partial g}{\partial x} \cdot \frac{\varphi\{\phi^{-1}[F(x)]\}}{f(x)} = \frac{\partial g}{\partial x} \cdot \sigma' \tag{B.2}$$

where σ' is identical with expression (13). Further, the second derivatives become

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial(\partial g/\partial x \cdot \sigma')}{\partial x} \cdot \frac{\partial x}{\partial u} = \left\{\frac{\partial^2 g}{\partial x^2} \cdot \sigma' + \frac{\partial g}{\partial x} \cdot \frac{\partial \sigma'}{\partial x}\right\} \sigma'$$

$$= \left\{\frac{\partial^2 g}{\partial x^2} \sigma' + \frac{\partial g}{\partial x} \cdot \frac{\partial\left(\frac{\varphi\{\phi^{-1}[F(x)]\}}{f(x)}\right)}{\partial x}\right\} \sigma'$$

$$= \left\{\frac{\partial^2 g}{\partial x^2} \cdot \sigma' - \frac{\partial g}{\partial x} \cdot \left[\phi^{-1}[F(x)] + \frac{\frac{\partial f(x)}{\partial x}}{f(x)} \cdot \sigma'\right]\right\} \sigma' \tag{B.3}$$

and

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = \frac{\partial g}{\partial x_i \partial x_j} \cdot \sigma_i' \cdot \sigma_j' \tag{B.4}$$

It is noted that evaluation of (B.3) requires existence of the derivative of the probability density function f(x).

Appendix C: Notations

$f(.)$ = probability density function

$F(.)$ = probability distribution function

$F^{-1}(.)$ = inverse probability distribution function

$g(.)$ = limit state function

μ, $\underline{M}$ = $E[X]$, $E[\underline{X}]$ = mean value (scalar or vector)

P = probability

P_f = failure probability

x, $\underline{X}$ = uncertain quantity (scalar or vector)

β = safety index

σ = $D[X]$ = standard deviation

Σ = covariance matrix

$\varphi(.)$ = standardized normal probability density function

$\phi(.)$ = standardized normal probability distribution function

$\phi^{-1}(.)$ = inverse normal probability distribution function

$\chi^2_n(.)$ = chi-square probability distribution function for n degrees of freedom

QUADRATIC LIMIT STATES IN STRUCTURAL RELIABILITY

by

B. Fießler, H.-J. Neumann, R. Rackwitz

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Abstract

Second moment methods are widely applied in structural reliability. Recently, so-called first order reliability methods have been developed which are capable to produce reliable estimates of the failure probability for arbitrary design situations and distributional assumptions for the uncertainty vector. In essence, non-linear functional relationships or transformations are approximated by linear Taylor expansions so that the simple second moment calculus is retained. Failure probabilities are obtained by evaluating the standard normal integral being the probability content of a circular normal distribution in a domain bounded by a hyperplane.

In this paper second order expansions are studied to approximate the failure surface and some results of the statistical theory of quadratic forms in normal variates are used to calculate improved estimates of the failure probability. Also, close bounds on failure probabilities are derived. Some numerical studies indicate when the application of the "second order reliability method" as proposed herein is in order.

Introduction

The computation of sensitive reliabilities for arbitrary structural problems in principle involves the evaluation of a n-dimensional volume integral. If, for example, $\underline{X}$ is a vector of time and space independent uncertain variables with joint distribution function $F(\underline{x})$, then the probability of failure is one minus the reliability,

$$P_f = 1 - \int_{\{D\}} d\,F(\underline{x}) \qquad (1)$$

with D representing the domain in which the structure operates in safe states. The surface which separates the safe and unsafe domain may be denoted by the limit state or failure surface which is taken to be associated with some given utility loss. Numerical integration over some arbitrary safe domain generally turns out to be rather tedious if feasible at all so that simpler methods have been developed. An effective method has been proposed by Hasofer and Lind [1] and others. In its original form it reports reliability in terms of the safety index ß and makes use only of the first and second statistical moments of the uncertainty vector. It is a discrete point checking method, measuring the minimum distance between the boundary of the safe domain and the mean of the uncertainty vector in terms of standard deviations of the function describing the limit state. Certain extensions to include more information on the stochastic nature of the uncertainties have recently been developed [7, 10].

The method has been applied successfully to a number of engineering problems. In spite of its conceptual simplicity, practical examples usually require the use of computers which, however, is mainly due to the complexity of the mechanical problem to be dealt with. In some cases the results have been checked by numerical integration displaying inaccurate results only for some exceptional design situations. In the following, another simpler method is proposed to examine the accuracy of the "first order reliability methods".

Critics of Present First Order Reliability Approaches

Some objections have been raised against the informativeness of the Hasofer/Lind safety index ß on survival or failure probabilities. Even if the vector of uncertain variables $\underline{X}$ is an independent unit normal vector which always can be achieved through appropriate transformations the safety index does not take proper account of the particular shape of the safe domain (see [7] or [13]. In general, the relationship

$$P_f \geq 1 - \Phi(\beta) \qquad (2)$$

is used which is associated with a tangential linearization of the limit state surface at the checking point. $\Phi(.)$ is the univariate standard normal integral. In this presentation the checking point is simply the point $\underline{x}^*$ on the limit state surface in the formulation $g(\underline{x}) = 0$ which is nearest to the coordinate origin. The inequality (2) is valid for convex safe regions. In practice, $P_f = 1 - \Phi(\beta)$ is frequently used as an estimate for the true failure probability. Another

equally elementary and fairly conservative upper bound has been given by Hasofer [2]

$$P_f \leq 1 - \chi_n^2(\beta^2) \qquad (3)$$

$\chi_n^2(.)$ is the chi-squared-distribution for n degrees of freedom (equals the dimension of the random vector $\underline{X}$). This bound corresponds to a supporting hypersphere which substitutes the true failure surface. It is immediately recognized that in this case P_f depends on n and hence the safety index ß is no more dimension-invariant (see also [13]). Depending on the values of n and ß both bounds are not always sufficiently close so that, in fact, the aforementioned objections apparently are justified; at least as long as the range of application of the simple reliability approach described before is not clearly defined. Several improvements to the method have recently been suggested by various authors.

Ditlevsen [1] developed sharper bounds for the true failure probability by calculating the probability content of inscribing and circumscribing rotational paraboloids. Those involve a convolution of a normal with a chi-squared variable and will be explained in more detail in the sequel. Recently, Neumann/Fiessler/Rackwitz [8] investigated further approximating quadratic forms with rotational symmetry and gave suitable tables. The additional forms investigated are the hypersphere with the same (minimum or maximum) curvature in the checking point leading to the evaluation of the non-central chi-square distribution and the rotational ellipsoid

or hyperboloid both again involving operations with non-central chi-square variables. Simple numerical integration is likewise required in the latter cases. Horne and Price [4] investigated the error in the failure probability given by eq.(2) by studying an approximating hypersphere with radius corresponding to the mean curvature in the checking point.

If the safe domain is of a certain well-behaved shape, at least in the neighbourhood of the checking point, these rotational forms clearly yield sharper bounds for the true failure probability. By taking the mean curvature one also arrives at better estimates for the failure probability than those obtainable by use of eq.(2) or (3). However, the limit state surface now must be continuous and twice differentiable since the second derivatives are used as additional information about the limit state surface. This is a more or less severe complication of such approaches. Also, physical reasoning must be used to choose among the parabolic, elliptical or hyperbolic form.

For convenience, we denote the method involving second derivatives by "second order reliability theory"; in contrast to its well-known "first order" version and exact reliability methods. In the following it is shown that there exist exact non-rotational quadrics whose probability content can be evaluated without undue difficulties in most of the cases.

Thus, their application may be recommended in all exceptional cases and primarily to check the results of less accurate methods. The results on quadratic forms in normal variables given below are not novel from a statistician's point of view but appear to be applied here for the first time to engineering problems in more detail.

General Derivation

Assume that a given limit state surface is twice differentiable in the neighbourhood of the checking point P^* in the standardized and normalized coordinate system (X) of basic uncertain variables. Also, let the vector $\underline{X}$ be an independent and uncorrelated normal vector.

The standardization requires the operation $X_i := (X_i - E[X_i])/D[X_i]$ whereas normalization is achieved by the transformation $X := \phi^{-1}[F(X_i)]$ for all components of $\underline{X}$ with $E[X_i]$ the mean, $D[X_i]$ the standard deviation and $F(X_i)$ the distribution function of X_i. The direction cosines of the location vector of P^* are given by the vector $\underline{\alpha}$ (see fig. 1). Expand the limit state surface $g(\underline{x}) = 0$ into a second order Taylor series about P^*.

$$g(\underline{x}) = g(\underline{x}^*) + \frac{1}{1!}\sum_{i=1}^{n} \frac{\partial g(x)}{\partial x_i}\Big|_{\underline{x}^*} (x_i - x_i^*) + \frac{1}{2!}\left[\sum_{i=1}^{n} \frac{\partial^2 g(x)}{\partial x_i^2}\Big|_{\underline{x}^*} (x_i - x_i^*)^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\partial^2 g(x)}{\partial x_i \partial x_j}\Big|_{\underline{x}^*} (x_i - x_i^*)(x_j - x_j^*)\right] = 0 \qquad (4)$$

or in matrix notation after some rearrangements

$$g(\underline{x}) = (\underline{x}-\underline{x}^*)^T \cdot \underline{\underline{G}}_x \cdot (\underline{x}-\underline{x}^*)+2 \cdot \underline{g}_x^T \cdot (\underline{x}-\underline{x}^*) + 2 \cdot g(\underline{x}^*) = 0 \qquad (5)$$

where $\underline{\underline{G}}_x$ is the matrix of second and mixed derivatives and $\underline{g}_x$ the vector of first order derivatives.

Eq.(5) constitutes a general quadric which easily can be brought into one of the standard forms in a new coordinate system (Z) by certain linear transformations (see appendix B). If the n-m variables occuring only in linear terms are denoted by $\tilde{Z}$, it is:

$$\sum_{i=1}^{n} \lambda_i (z_i-\delta_i)^2 = K_1 \qquad (6)$$

or

$$\sum_{i=1}^{m-1} \lambda_i (z_i-\delta_i)^2 + \sum_{i=m}^{n} k_i \tilde{z}_i = K_2 \qquad (7)$$

in which the λ_i's are the eigenvalues of the matrix $\underline{\underline{G}}$ and the δ_i's are the non-centralities in the Z-coordinate system whereas the constants K_1, K_2 and k_i, respectively, are determined by the transformation conditions (see appendix B). If all λ's are greater than zero the quadric is denoted by a positive definite. The positive definite case of eq.(8) clearly is an ellipsoid with origin at the point $(\delta_1, \delta_2, \ldots, \delta_n)$ and semi-axis $((K_1/\lambda_1)^{1/2}, (K_1/\lambda_2)^{1/2}, \ldots, (K_1/\lambda_n)^{1/2})$. If some of the λ's are zero, cylindrical forms are obtained. The indefinite case is for λ's with different signs. Further detailed classifications are given in appendix B.

Since the Z's are standardized uncorrelated normal variables new stochastic variables W and V can be defined by:

$$W = \sum_{i=1}^{n} \lambda_i (Z_i-\delta_i)^2 \qquad (8)$$

$$V = \sum_{i=1}^{m-1} \lambda_i (Z_i-\delta_i)^2 + \sum_{i=m}^{n} k_i \tilde{Z}_i = K_2 \qquad (9)$$

Obviously, the variable W is a linear combination of non-central chi-squared distributed variables while the variable V additionally contains a linear combination of normally distributed variables. Therefore, the distribution function of W is simply the probability content of a spherical normal distribution over a region defined by eq.(6). A similar interpretation holds for the variables V. It follows from the definition of the safe domain in either of the forms $g(\underline{x}) > 0$ or $g(\underline{z}) > 0$ that the probability of failure can then be estimated by:

$$P_f = P(W > K_1) = 1 - F_W(K_1) \qquad (8a)$$

$$P_f = P(V > K_2) = 1 - F_V(K_2) \qquad (9a)$$

Thus, the probability distribution functions of W and V must be known.

The Distribution of W and V

General

Quadratic forms have received much less attention by statisticans than other problems involving normal variables. The statistical literature on this topic is exhaustively reviewed in Johnson/Kotz [6]. However, only few of the results available are useful in the context of structural reliability. They require the use of a computer which in the light of its necessity in non-trivial engineering applications, anyhow, appears to be no serious obstacle. Since most of the readers may not be familiar with the theoretical results on quadratic forms several formulas are given below.

The Positive Definite Case for W

In analogy to the non-central chi-square distribution which can be expressed as mixtures of central chi-square distribution with weights given by the Poisson density, Ruben [11, 12] showed that in the positive definite case the distribution of W can be given as a mixture of chi-square-probability functions. The distribution function has the expansion

$$F(w;\underline{\lambda},\underline{\delta}) = \sum_{j=0}^{\infty} e'_j \cdot P[\chi^2_{n+2\cdot j} < w/k] \qquad (10)$$

wherein $\underline{\lambda}$ is the vector of coefficients λ_i, $\underline{\delta}$ the vector of deviations from the origin and $\kappa(>0)$ a suitably chosen constant. Here, n is the dimension of the basic variable vector. The e'_j's can be obtained from

$$e'_o = \left[\exp(-\tfrac{1}{2}\sum_{j=1}^{n}\delta_j^2)\right] \prod_{j=1}^{n} (\kappa/\lambda_j)^{1/2} \qquad (11)$$

$$e'_j = \frac{1}{2j}\sum_{k=0}^{j-1} G_{j-k}\cdot e'_k \quad (j \geq 1) \qquad (12)$$

with

$$G_r = \sum_{i=1}^{n}(1-\kappa/\lambda_i)^r + r\kappa\cdot\sum_{i=1}^{n}(\delta_i^2/\lambda_i)\,(1-\kappa/\lambda_i)^{r-1} \qquad (13)$$

and

$$0 \leq \kappa \leq \min_{(i)} \lambda_i \qquad (14)$$

The series (10) is uniformly convergent over any finite interval of w. Ruben also gave the following bound for error in truncating the series (10) at the N-th term (see [6] or [11]).

$$\sum_{j=N}^{\infty} e'_j \cdot P\left[\chi^2_{n+Z_j} < w/\kappa\right]$$

$$\leq e'_o \frac{\Gamma(\frac{1}{2}n+N)}{\Gamma(\frac{1}{2}n)}\cdot\frac{M^N}{N!}\,(1-M)^{-\frac{1}{2}(n+2N)}\cdot P\left[\chi^2_{n+2N} \leq (1-M)\;\; w/\kappa\right] \qquad (15)$$

where $M = \frac{1}{2}\kappa\sum_{i=1}^{n}(\delta_i^2/\lambda_i) + \max_{(i)}|1-\kappa/\lambda_i|$ for $M<1$.

It is, therefore, particularly useful for computerized evaluations.

The Indefinite Case for W

Another quite general and useful expression for the indefinite case as well has been derived by Imhof [5]. The solution of Imhof is based on an inversion of the characteristic function of W. The characteristic function of W is given by (see Johnson/Kotz [6])

$$\varphi(t) = \prod_{j=1}^{n} (1-2 \cdot i \cdot \lambda_j \cdot t)^{-\frac{1}{2}} \quad \exp\{i \cdot \sum_{j=1}^{n} \frac{\delta_j^2 \cdot \lambda_j \cdot t}{1-2 \cdot i \cdot \lambda_j \cdot t}\} \tag{16}$$

and the distribution function of W is obtained from

$$P(W=x) = \bar{F}_H(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} t^{-1} \, J\{e^{-itx} \varphi(t)\} \, dt \tag{17}$$

where J{ } is the imaginary part. It is shown in [5] that eq.(17) can be evaluated by

$$P(W > x) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin\Theta(u)}{u \cdot \rho(u)} \, du \tag{18}$$

with

$$\Theta(u) = \frac{1}{2} \sum_{j=1}^{n} \left[\tan^{-1}(\lambda_j \cdot u) + \delta_j^2 \cdot \lambda_j \cdot u(1+\lambda_j^2 \cdot u^2)^{-1}\right] - \frac{1}{2} x \cdot u \tag{19}$$

$$\rho(u) = \prod_{j=1}^{n} (1 + \lambda_j^2 \cdot u^2)^{1/4} \quad \exp\{\frac{1}{2} \sum_{j=1}^{n} (\delta_j \cdot \lambda_j \cdot u)^2 / (1 + \lambda_j^2 \cdot u^2)\} \tag{20}$$

For $u \to 0$ we have

$$\lim_{u \to 0} \frac{\sin\Theta(u)}{u \cdot \rho(u)} = \frac{1}{2} \sum_{j=1}^{n} \lambda_j (1+\delta_j^2) - \frac{1}{2} x \tag{21}$$

Note that eq.(18) must be evaluated by numerical quadrature.

The truncation error of integration can be bounded from the above by

$$|t_U| = \left|\frac{1}{\pi} \int_U^{\infty} \frac{\sin\Theta(u)}{u \cdot \rho(u)} \, du\right| \leq \left[\pi \cdot k \cdot U^k \prod_{j=1}^{n} |\lambda_j|^{1/2} \quad \exp\{\frac{1}{2} \sum_{j=1}^{n} \delta_j^2 \lambda_j^2 U^2 (1+\lambda_j^2 U^2)^{-1}\}\right]^{-1} \tag{22}$$

with $k = \frac{n}{2}$

From the integrand of formula (18) it is obvious that the integral converges although the sinus operation in the nominator indicates an alterating convergence. The authors agree with a conclusion of Imhof that a simple trapezoidal rule applied to a sufficient number of steps and starting with the value given by eq.(21) is a convenient procedure to keep both the relative quadrature and the truncation error sufficiently small. The same author also obtained the density function

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos\Theta(u)}{\rho \cdot (u)} \, du \tag{23}$$

in which the same expression (19) and (20) are valid. Imhof's formulas hold for both the definite and indefinite case so that it might be used throughout. Finally, it is mentioned that for small failure probabilities the formulas by Press [9] may be preferred.

The General Parabolic Case

From eqs.(8a) or (9a) it is seen that the variable V is the sum of a normal variable Z_n and a variable distributed like W but with fewer degrees of freedom, say m. Therefore, to obtain the distribution of V it is necessary to convolute a W-variable with Z_n. It is

$$F(v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{v} \varphi(z_n) \quad f_W(t-z_n) \; dt \; dz_n$$

$$= \int_{-\infty}^{+\infty} \varphi(z_n) \quad F_W(v-z_n) \; dz_n \qquad (24)$$

in which $\varphi(.)$ is the standard normal density and $F_W(.)$ the distribution according to the foregoing two sections. Note that the non-centrality in W can be omitted in certain cases which further simplifies the expressions in an easily recognized manner. Formula (24) generally must be evaluated by an appropriate method of numerical integration. Since the evaluation of $F_W(.)$ generally already involves numerical integration a normally less time-consuming alternative to eq.(24) is

$$F_P(v) = \int_{-\infty}^{+\infty} \Phi(v-w) \quad f_W(w) \; dw \qquad (25)$$

in which $\Phi(.)$ can be evaluated by one of the well-known rational approximations to the normal integral and $f_W(.)$ is given by eq.(23). In some cases, however, the integral (23) does converge quite slowly which is obvious because u is missing in the denominator of the integral (23) (compare with expression (18)).

Special Forms with Pre-Determined Principal Axis

General

It is possible to use other quadratic approximations to the boundary of the safe domain $g(\underline{x}) \geq 0$ whose probability content can be calculated much easier in some cases and, therefore, shall be studied in some detail. In essence, such forms are set by preselecting the direction of their principal axis. Their curvatures in a nodal point are chosen such to comply with the curvatures of the original limit state surface. But, it should be clear that such forms may give an increasingly worse approximation to the true failure surface as the distance from the nodal point increases.

For example, let the original coordinate system (X) be rotated into a new system (Y) with the same origin such that the point P^* is on the Y_n-axis and has coordinates (0, 0, ... (see fig. 2). Then, calculate the second order and mixed derivatives in P^* and rotate the system (Y) about the Y_n-axis such that the mixed derivatives vanish (see also [1]). The new system has coordinates $z_1, z_2, \ldots, z_n = y_n$. Consequently, an approximating quadric derived from the remaining diagonal $(n-1)$ matrix of second derivatives has the same principal curvatures in the point P^* (for further details see appendix C). Then, either of the forms

$$\sum_{i=1}^{n-1} p_i \, z_i^2 + p_n (z_n - \delta_n)^2 = 1 \qquad (26)$$

or

$$\sum_{i=1}^{n-1} p_i \, z_i^2 - (z_n - \delta_n) = 0 \qquad (27)$$

can be set with the coefficients p_i simply related to the curvatures in point P^* (see appendix C). Analogically to the section for general quadratic forms the left hand part of eqs.(26) and (27) represent a function of central or non-central, linear or squared standard normal variables. Since eqs.(26) and (27) are possible approximations to the true failure surface, the probability of failure is the probability of a unit, uncorrelated normal vector falling outside the domains defined by eqs.(26) or (27). Hence, the formulas of the preceding sections likewise may be used to estimate the failure probability. It is noted that they simplify to a certain extent since only the one non-centrality parameter δ_n is retained. As mentioned before, the choice of either a complete quadratic form or a paraboloid is now somewhat arbitrary. Information on the connectiveness of the safe domain and its convexity properties may be used to select the appropriate one.

Hyperspheres and Rotational Paraboloids

Major computational benefits are obtained when the forms (26) or (27) are simplified towards rotational surfaces with the rotation axis being the Z_n-axis. The nodal curvature may be chosen as the mean of the principal curvatures and so probability calculations may produce reliable estimates for the failure probability, or as the extreme curvatures to yield inscribing or circumscribing surfaces which result in upper and lower bounds for the failure probability.

Although rotational ellipsoids or two-shelled hyperboloids can principally be handled with the material presented in the foregoing most gain is achieved if one concentrates on two elementary forms, the hypersphere and the rotational paraboloid, respectively. The former is obtained by taking the mean, maximum or minimum curvature so that the quadric (26) becomes

$$\sum_{i=1}^{n-1} z_i^2 + [z_n - (R+\beta)]^2 = R^2 \qquad (28)$$

which is a hypersphere with radius R and centre at the point (O, O, ..., O, R+ß). If the Z's are unit uncorrelated normal variables, the left hand expression of (28) is known to be non-central chi-squared distributed with non-centrality parameter $\delta = [R+\beta]^2$ and, thus

$$P_f = 1 - \chi^2_{n,\delta}\,(R^2)$$

Similarly, the rotational paraboloid receives the form

$$\frac{1}{2R}\sum_{i=1}^{n-1} z_i^2 - (z_n - \beta) = 0 \qquad (29)$$

If the Z's are unit, uncorrelated normal variables the first term is chi-squared distributed and the latter is normal. Hence, the probability of failure is the probability of a chi-square variable convoluted with a non-central normal variable being greater than zero. In using expression (24) one arrives after some elementary manipulations at

$$P_f = \int_0^\infty \phi[-(\frac{1}{2R} \cdot t + \beta)] \cdot f_{\chi^2_{n-1}}(t)\, dt \qquad (30)$$

in which $\phi[\ ]$ is the standard normal integral and

$$f_{\chi^2_\nu}(x) = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \cdot x^{(\nu/2-1)} \cdot \exp(-\frac{x}{2}) \qquad (31)$$

is the density of a chi-squared variable with ν degrees of freedom. Though expression (30) must be evaluated by numerical quadrature accurate results generally can be obtained quite easily. Also, some tables are available in [1] and [8].

Discussion and Examples

A general view of the dependance of P_f on the safety index β the curvature in the checking point and the dimension of the basic variable vector can easily be gained when considering simple examples of rotational symmetry. In figure 3 a simple two-dimensional case is demonstrated for $\beta = 3$ showing the plane, spherical and parabolic approximations for various curvatures in the checking point. Figure 4 illustrates the relation between probability of failure calculated for hyperplanes, rotational hyperparaboloids and hyperspheres versus the dimension of the uncertainty vector $\underline{X}$. Figure 5 finally expresses the sensitivity of the failure probability versus the dimension of $\underline{X}$ and the curvature in the checking point for two reliability levels as expressed by $\beta = 3$ and $\beta = 7$.

These results indicate that there may exist significant errors when simply using expression (1). However, from the authors experience only in extreme practical cases the curvature of $g(\underline{x}) = 0$ exceeds values of about 0.05. For a simple example, take a tension bar with known load and random, normally distributed strength and diameter. Figure 6 illustrates the corresponding limit state curves and in the table below, some results for the plane, parabolic and spherical approximation are given. It is noted that the differences are negligible from an engineering point of view. This conclusion also holds for the majority of more complex engineering problems even in higher dimensions as long as the uncertainty vector has distributions not too far from the normal. But if the original uncertainty vector has a distribution function which deviates significantly from the normal originally sufficiently smooth failure surfaces can become distinctively curved in the normalized space. Then, the accuracy of simple estimates according to eqs.(1) or (2) may no more be sufficient. Rackwitz/ Fießler [10] have proved that adopting the foregoing "second order" approach in the transformed space will yield accurate probability estimates in almost all practically relevant cases. The use of quadratic forms appears to be even more crucial in parametric cases, for example for time-dependent reliability problems.

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Appendix B: Quadratic Taylor Expansions

Eq.(5) can be rewritten as follows:

$$g(\underline{x}) = \underline{x}^T \cdot \underline{\underline{G}}_x \underline{x} + 2\underline{x}^T(\underline{g}_x - \underline{\underline{G}}_x \underline{x}^*) + \underline{x}^{*T}(\underline{\underline{G}}_x \underline{x}^* - 2\underline{g}_x) = 0 \qquad (B.1)$$

In using the transformation $\underline{\underline{G}}_x = \underline{\underline{T}}\,\underline{\underline{\Lambda}}\,\underline{\underline{T}}^T$, $\underline{x} = \underline{\underline{T}}\,\underline{z}$ and $\underline{g}_x = \underline{\underline{T}}\,\underline{g}_z$ where $\underline{\underline{T}}$ is the modal matrix of $\underline{\underline{G}}_x$ with $T \cdot T^T = I$ (identity matrix) and $\underline{\underline{\Lambda}} = (\lambda_i)$ the diagonal matrix of the Eigenvalues of $\underline{\underline{G}}_x$, one obtains the principal form where all mixed terms vanish.

$$g(\underline{z}) = \underline{z}^T \cdot \underline{\underline{\Lambda}}\,\underline{z} + 2\underline{z}^T(\underline{g}_z - \underline{\underline{\Lambda}} \cdot \underline{z}^*) + \underline{z}^{*T}(\underline{\underline{\Lambda}}\,\underline{z}^* - 2\underline{g}_z) = 0 \qquad (B.2)$$

If $\underline{\underline{G}}_x$ is regular, that is $\lambda_i \neq 0$ for all $i = 1, 2, \ldots, n$ eq.(B.2) represents a complete quadric. After applying the linear translation $\underline{z} = \underline{z}' + \underline{\delta}$ which implies

$$\underline{\delta} = \underline{z}^* - \underline{\underline{\Lambda}}^{-1} \cdot \underline{g}_z \qquad (B.3)$$

and, of course, $\underline{\underline{\Lambda}}^{-1} = (1/\lambda_i)$, the standard form is constructed:

$$(\underline{z} - \underline{\delta})^T\,\underline{\underline{\Lambda}}\,(\underline{z} - \underline{\delta}) = \underline{g}_z^T\,\underline{\underline{\Lambda}}^{-1}\,\underline{g}_z \qquad (B.4)$$

or

$$\sum_{i=1}^{n} \lambda_i (z_i - \delta_i)^2 = K_1 \qquad (B.5)$$

with $K_1 = \underline{g}_z^T\,\underline{\underline{\Lambda}}^{-1}\,\underline{g}_z$ and which is the same as eq.(6) in the main text.

If G_x is singular, that is some of the λ's equal zero (the case where all λ's equal zero is of no further interest since $g(\underline{x}) = 0$ is linear in each component of $\underline{X}$) and the corresponding elements of $\underline{g}_z$ vanish, too, then cylindrical forms are obtained. Such forms are dealt with as complete quadrics but with a smaller dimension.

One arrives at parabolic forms if for some components of $\underline{Z}$ it is $\lambda_i = 0$ but $g_{z,i} \neq 0$. In general, it is

$$(\underline{z}-\underline{\delta})^T \wedge (\underline{z}-\underline{\delta}) + 2\tilde{\underline{z}}^T \underline{g}_z = K_2 \qquad (B.6)$$

with K_2 similar to K_1 but without the terms where $\lambda_i = 0$ and where $\tilde{\underline{z}}$ is the vector of these components for which the foregoing conditions hold and $\underline{\delta}$ contains only the non-zero components for which the conditions are not valid, specifically all components where $\lambda_i \neq 0$. Finally, the standard form (B.6) can be written in the form of eq.(7)

$$\sum_{i=1}^{m-1} \lambda_i (z_i-\delta_i)^2 + 2 \sum_{i=m}^{n} \tilde{z}_i g_{z_i} = K_2 \qquad (B.7)$$

where K_2 is given by the right hand side of eq.(B.6). For convenience, the components of $\underline{z}$ in eq.(B.7) have been ordered to include the m quadratic terms in the first part and the n-m linear terms in the second part of the left hand side of eq.(B.7).

Appendix C: Principal curvatures in the checking point P^*

Let the checking point P^* with the vector of direction cosines $\underline{\alpha}$ be found by a suitable algorithm in the (X)-space. There always exists an orthogonal rotation matrix $\underline{\underline{T}}$ such that the Y_n-axis of the new coordinate system is parallel to the vector $\underline{\alpha}$.

$$\underline{Y} = \underline{\underline{T}}^T \cdot \underline{X} \qquad (C.1)$$

Clearly, the last column vector of $\underline{\underline{T}}$ is the vector $\underline{\alpha}$. The other columns may be found by one of the well-known orthogonalization procedures. In the new system it is for the derivatives

$$g_{y,i} = \frac{\partial g(\underline{y})}{\partial y}\Big|_{\underline{Y}^*}$$

$$g_{y,i} = 0 \text{ for } i = 1, 2, \ldots, n-1 \qquad (C.2)$$

The matrix $\underline{\underline{G}}_y$ of second and mixed derivatives in P^* is a symmetrical matrix with the n-th column and row deleted. In applying elementary results of differential geometry, for example the two Gaussian fundamental theorems for curves on n-dimensional surfaces, the principal curvatures in P^* are obtained from the roots of the characteristic equation.

$$\det\left(\frac{1}{g_{y,n}} \underline{\underline{G}}_y - \kappa \cdot \underline{\underline{I}}\right) = 0 \qquad (C.3)$$

Then, the radius of curvature in P^* with respect to the i-th principal axis is $R_i = 1/\kappa_i$. In order to establish suitable quadratic forms with the same principal curvatures we set

$$(\underline{z}-\underline{\delta})^T \underline{\underline{P}} (\underline{z}-\underline{\delta}) = 1 \qquad (C.4)$$

where $\underline{\delta} = (0, 0, \ldots, \delta_n)$ and $\underline{\underline{P}}$ is the diagonal matrix of coefficients indicating the length of the semi-axis a_i by the relation $p_i = 1/a_i^2$. The semi-axis a_i depends on the curvatures by $a_i^2 = a_n/\kappa_i$. The requirement that the form (C.4) has the same gradient in P^* as the original form leads to $p_n = g_{y_n}^2/4$ whereas the other elements p_i can be calculated by using the aforementioned relationship between semi-axis and curvatures in the nodal point. It is immediately deducted that

$$p_i = -\frac{\kappa_i \; g_{x_n}}{2} \quad \text{for } i = 1, 2, \ldots, n-1 \qquad (C.5)$$

Of course, the radius is given by $R_i = 1/\kappa_i$. With these expressions eq.(26) can easily be derived. Similarly, the relations between nodal curvatures and coefficients p_i are used in the parabolic case. Here, it is

$$p_i = \kappa_i/2 \qquad (C.6)$$

Hence, also form (27) of the main text may be set with the p's as given by (C.6) and $\delta_n = \beta$.

For rotational forms the curvatures correspond to

$$\kappa = \frac{1}{m} \sum_{i=1}^{m} \kappa_i \quad \text{or} \quad \kappa = \min_{i=1}^{m} \{\kappa_i\} \quad \text{or} \quad \kappa = \max_{i=1}^{m} \{\kappa_i\} \qquad (C.7)$$

whatever type is selected, with m being the number of quadratic terms in eqs.(26) or (27).

Appendix D: Notations

$\underline{a}$, $\underline{a}^T$	column vector, row vector
$\underline{\underline{A}}$, $\underline{\underline{A}}^T$	matrix, transposed matrix
$g(\underline{x})$	state function of $\underline{x}$
X, Z	random variables
$\underline{x}^*$	location vector of checking point P^*
$\underline{\alpha}$	vector of direction cosines of checking point P^*
E(X)	expectation (mean) of X
D(X)	standard deviation of X
P	probability
P_f	probability of failure
β	reliability index
F(.)	probability distribution function
f(.)	probability density function
$\phi(.)$	standard normal probability function
$\phi^{-1}(.)$	inverse standard normal probability function
$\varphi(.)$	standard normal density function
χ_n^2	(central) chi-square distribution with n degrees of freedom
$\chi_{n;\delta}^2$	non-central chi-square distribution with n degrees of freedom and non-centrality parameter δ

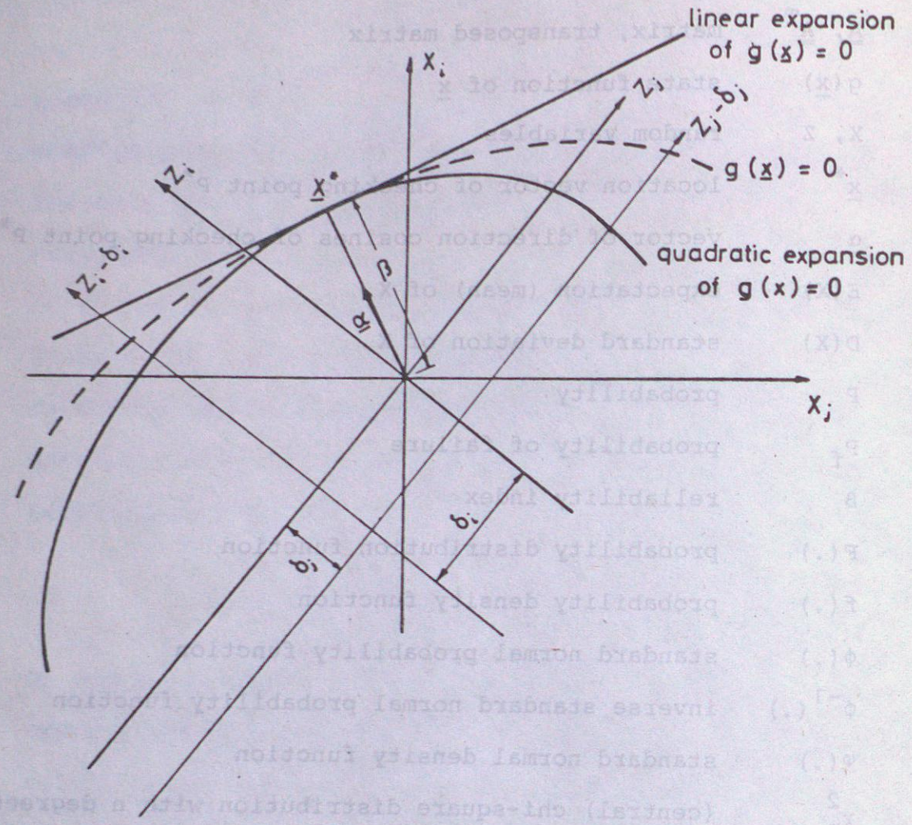


Fig. 1: Linear and quadratic approximations to limit state surface $g(\underline{x}) = 0$

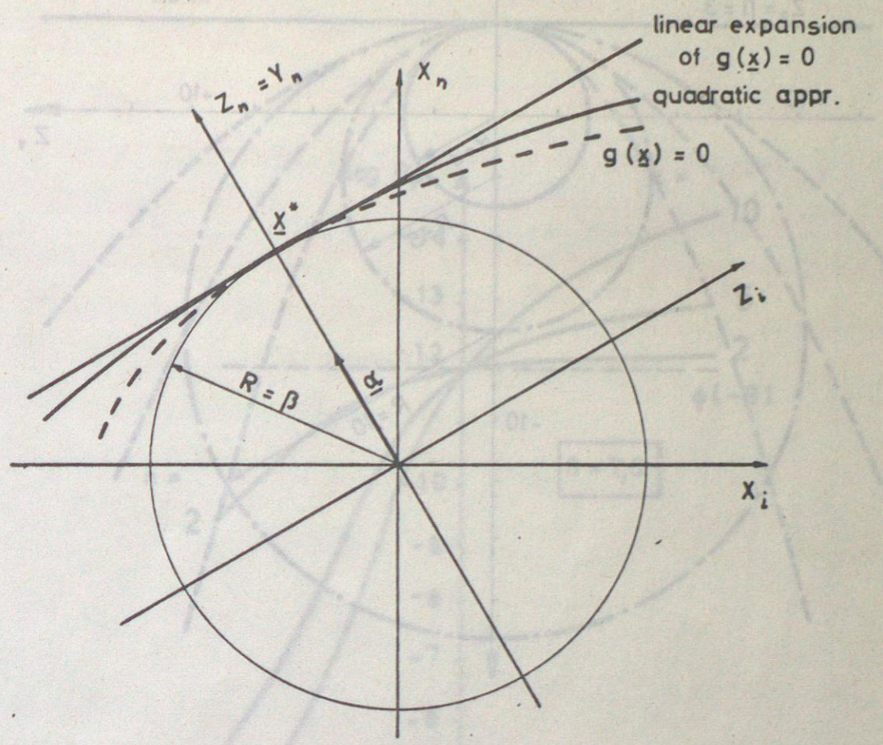


Fig. 2: Derivation of rotational quadratic forms

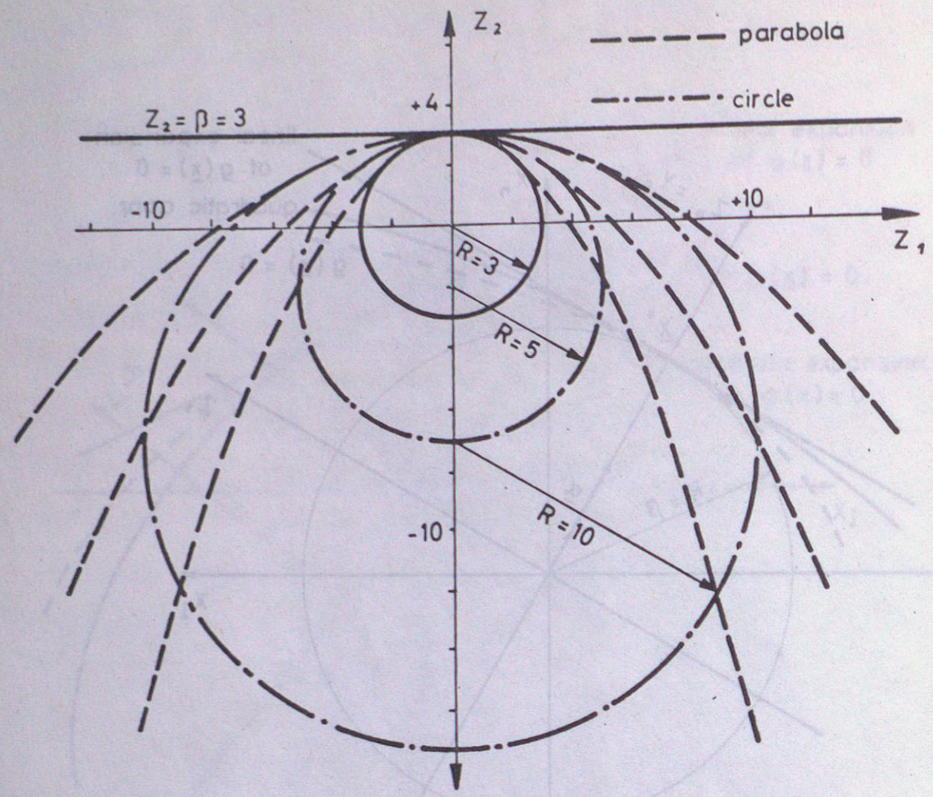


Fig. 3: Plane, spherical and rotational parabolic approximations to $g(\underline{x}) = 0$ for $\beta = 3$ in two dimensions

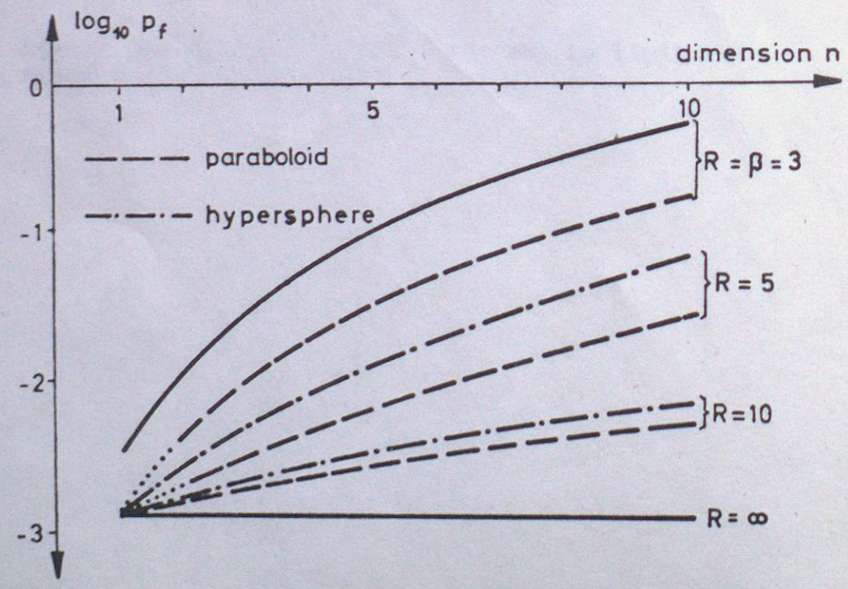


Fig. 4: Failure probability of plane, spherical and parabolic approximations of $g(\underline{x}) = 0$ versus problem dimension ($\beta = 3$)

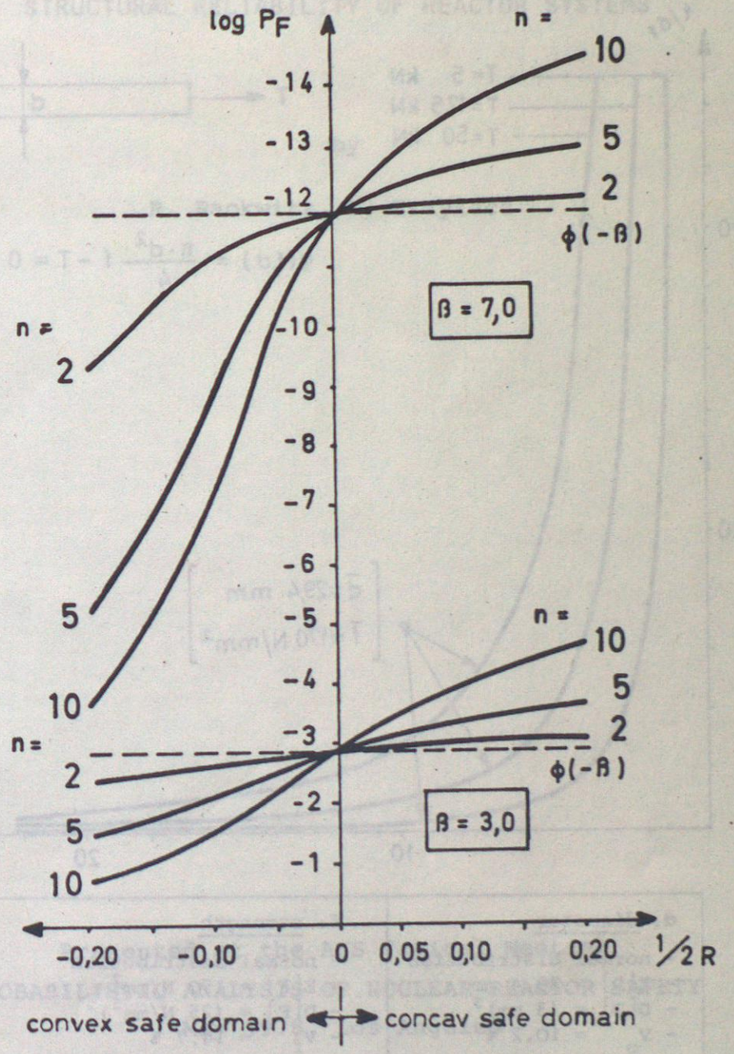
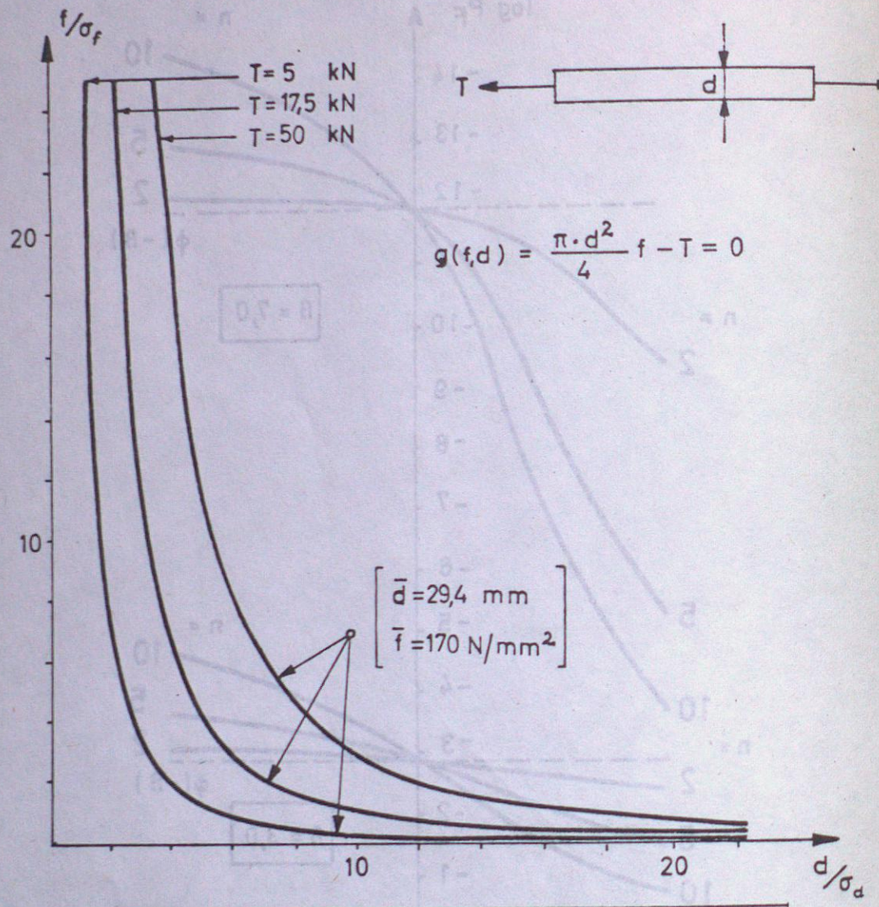


Fig. 5: Failure probability of paraboloids for $\beta = 3$ and $\beta = 7$ versus problem dimension



$$g(f,d) = \frac{\pi \cdot d^2}{4} f - T = 0$$

d: diameter - normal distribution - E[D] = 29,4 mm - D[D] = (3 mm)² - V_o = 10,2 %				f: strength - normal distribution - E[F] = 170 N/mm² - D[F] = (25 N/mm²)² - V_f = 14,7 %		
T	β	$\phi(-\beta)$		$P_{F,(Parabola)}$	$P_F/\phi(-\beta)$	R
5	2,90	1,9	10^{-3}	$2,8 \cdot 10^{-3}$	1,47	8,8
17,5	5,54	1,5	10^{-8}	$4,1 \cdot 10^{-8}$	2,73	6,6
50	6,49	4,3	10^{-11}	$4,4 \cdot 10^{-11}$	1,02	45,6

Fig. 6: Limit state function of tensile bar

STRUCTURAL RELIABILITY OF REACTOR SYSTEMS

by

R. Rackwitz, B. Krzykacz

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ABSTRACT

Reliability analysts working on reactor systems should be concerned with common mode failure. Component or mode failures then are dependent events. The analytical and numerical tools for realistic system reliability assessments in those cases are still not yet fully developed. The well-known lower or upper reliability bounds are only poor guides for optimal design decisions, particularly, if there is a large number of potential failure modes. Therefore, an alternative formulation within the framework of so-called first order reliability methods is presented. It allows the approximate determination of component reliabilities by a local transformation of non-normal basic uncertainty vectors into normal vectors and by local linearization of arbitrary mechanical failure criteria. Component reliabilities then are simply the values of the univariate normal integral with the so-called safety index being the argument. Due to the stochastic dependence of the different modes of failure, the calculation of system reliabilities requires the evaluation of multinormal probabilities, instead. Some numerical methods for their evaluation are given. Finally, the method is applied to Markovian systems the states of which are defined by sets of components or subsystems having either failed or survived. The presentation concentrates on the mathematical concepts and solution techniques rather than on direct practical application.

I. INTRODUCTION

For complex systems with high reliability requirements it is customary to introduce sufficient redundancies so that failure of one or more components does not necessarily cause system failure. "Hot" as well as "standby" redundancies can at least limit the consequences of partial failure and can maintain a certain degree of operability of the system after an adverse event has occured, for example earthquake, fire, missile impact, etc. Thereby, most concern is mandatory if a single cause can produce simultaneous failure of many components and/or subsystems and so can eliminate any degree of redundancy employed in the system. Component failures then are dependent events which makes a probabilistic evaluation of system reliability rather difficult. Further, the system may be exposed to several consecutive "shocks" between which no preventive action or repair is possible and so the system may be destroyed gradually.

In the following an attempt is made to reformulate the problem of system reliability such that use can be made of a probabilistic model for which stochastic dependencies can be handled rather easily. In fact, the only probabilistic model for which this is true appears to be the multidimensional normal vector or Gaussian process. Many reliability problems are independent of parameters such as time or can be reduced accordingly. Hence, an essential part of formulation will be the proper transformation of arbitrary models for the uncertainties into normal vectors. In particular, the formulation makes use of some recent results of "first order reliability theory" [1] which, in essence, replaces multidimensional integration of probabilities by simple algorithms in a transformed space of basic uncertain variables and by the evaluation of the univariate normal integral. It will be seen that systems, instead, require the evaluation of multidimensional normal integrals for which some methods are presented. The formulation to be followed thus further develops several proposals made earlier, e.g. by Moses [2] and Vanmarcke [3]. Finally, the classification of the states of a system which traditionally rests on two basic states, survival and failure, respectively, is generalized towards a more realistic description. The criteria for classification are not purely physical or mathematical but include utility aspects insofar as states are ordered according to their potential losses when entered. This gives rise to consider not only the most unfavourable event, e.g. total failure with large consequences but also to take account of losses due to intermediate partial failure which may govern the design in a number of cases.

II. COMPONENT RELIABILITY

Let the mathematical model describing the "limit state" in which the component j fails to accomplish its intended function be given by

$$Z_j = g_j(\underline{X}) = 0 \qquad (1)$$

where $\underline{X}$ is a vector collecting all uncertain (random) variables, for example the possibly vector actions, strength of materials, geometrical properties and even variables taking account of uncertainties in the model for the limit state function itself. The formulation of the limit state function is such that the random state variable Z_j is greater than zero for any non-failure state and less than zero for failure. Hence, the failure probability can be determined from

$$P_{f,j} = 1 - \int_{\{\underline{x}: g_j(\underline{x}) > 0\}} dF(\underline{x}) \qquad (2)$$

with $F(\underline{x})$ the joint distribution function of $\underline{X}$. In first order reliability methods the evaluation of expression (2) is drastically simplified according to the following steps:

i. Transformation of a non-normal, dependent uncertainty vector into a standardized, uncorrelated normal vector. For

example, if the components of $\underline{X}$ are non-normal but independent it suffices to transform each component separately. From the required identity $F_i(x_i) = \phi(z_i)$ the transformation $x \to z$ becomes

$$x_i = F_i^{-1}[\phi(z_i)] \tag{3}$$

in which $F_i^{-1}(.)$ is the inverse of the marginal distribution function of the i-th component and $\phi(.)$ the univariate standard normal integral. Clearly, eq.(1) can then be rewritten as

$$g_j(..., F_i^{-1}[\phi(Z_i)], ...) = 0 \tag{4}$$

Making these transformations one profitably may use suitable series expansions or rational approximations for the inverse distribution functions if these cannot be obtained in analytical form (see, for example [4]).

Formally, it is easier first to work with the inverse of the transformation (3) and to determine the parameters μ_i' and σ_i' of an approximating normal distribution. This is equivalent to expanding it into a linear Taylor series about a given point x_i^* on the limit state function.

$$z_i = \phi^{-1}[F_i(x_i)] \approx \phi^{-1}[F_i(x_i^*)] + \left.\frac{\partial \phi^{-1}[F_i(x_i^*)]}{\partial x_i}\right|_{x_i^*} (x_i - x_i^*) \tag{5}$$

After some elementary manipulations the "normalized" variable X_i' then can be shown to have mean

$$\mu_i' = x^* - \sigma_i' \cdot \phi^{-1}[F_i(x_i^*)] \tag{6a}$$

and standard deviation

$$\sigma_i' = \frac{f_i(x_i^*)}{\varphi\{\phi^{-1}[F_i(x_i^*)]\}} \tag{6b}$$

Note, that the "approximation point" $\underline{x}^*$ must be known. Its determination will be explained under the next item.

If the uncertainty vector is a dependent non-normal vector a formally similar but usually more difficult procedure can be followed. Assume that the uncertainty vector is a vector the components of which are somehow linearly related so that its covariance matrix Σ_X is informative about the stochastic dependence structure. Such models frequently result from stochastic model building of more complex uncertain phenomena. Rotate the coordinate system such that the uncertainty vector becomes uncorrelated. For X being the original uncertainty vector with joint distribution function $F(\underline{x})$ we obtain with

$$\underline{Y} = \underline{\underline{R}}^T \underline{X} \tag{7}$$

and $\underline{\underline{R}}^T$ the transpose of the matrix of Eigenvectors of Σ_X

$$F_Y(\underline{y}) = F_X[(\underline{\underline{R}}^T)^{-1}\underline{y}] \tag{8}$$

In analogy to the one-dimensional case, we now define $\underline{z}$ by:

$$F_Y(\underline{y}) = \phi_Z(\underline{z}) = \prod_{i=1}^{n} \phi_i(z_i) \tag{9}$$

The last identity can be set since for multidimensional normal distributions no correlation means stochastic independence as well [5]. Therefore, the joint distribution equals the product of the marginal distributions.

Inversion of eq.(9) for the variable z_i yields

$$z_i = \phi^{-1}\left[\frac{F_Y(\underline{y})}{\prod_{\substack{j=1 \\ j\neq i}}^{n} \phi_j(z_j)}\right] \tag{10}$$

which can be expanded in the same manner as expression (5). The z_i's are unknown. Those can be evaluated from a suitable set of equations for the (marginal) densities of being equal, for example

$$\left.\frac{\partial F_Y(\underline{y})}{\partial y_i}\right|_{y^*} = \left.\frac{\partial \phi_Z(\underline{z})}{\partial z_i}\right|_{z^*} = \left.\prod_{\substack{j=1 \\ j\neq i}}^{n} \phi_j(z_j) \cdot \varphi(z_i)\right|_{z^*} \tag{11}$$

Again, rearrangement of the expanded eqs.(10) will yield the parameters of the approximating marginal normal distributions.

It is useful to standardize these variables by

$$Z_i = \frac{X_i' - \mu_i'}{\sigma_i'} \tag{12}$$

so that the limit state function finally can be written in the form

$$g_j(..., Z_i \cdot \sigma_i' + \mu_i', ...) = 0 \tag{13}$$

with the z's being independent standard univariate normal variables.

ii. Approximation of the limit state function by a hyperplane in a point $\underline{z}^*$ nearest to the coordinate origin. This point on g(.) can always be found by a suitable search algorithm (see, for example [6, 7]). The minimum distance is denoted by the safety index β_j

$$\beta_j = \min \{\sqrt{\underline{z}^T \underline{z}} : g_j(\underline{z}) = 0\} \tag{14}$$

Obviously, an approximating hyperplane is then given by

$$\underline{z}\ \underline{\alpha}_j - \beta_j = 0 \tag{15}$$

with $\underline{\alpha}_j$ the vector of direction cosines of the point $\underline{z}^*$ on $g_j(.) = 0$. Note, that the pointwise transformation of non-normal vectors into normal vectors ought to be made at just this point.

iii. Calculation of failure (survival) probabilities by

$$P_f = 1 - P[\underline{Z}\ \underline{\alpha}_j - \beta_j > 0] \tag{16}$$

and since $\underline{Z}$ is an independent normal vector it is

$$P_f = 1 - P[\underline{Z} \cdot \underline{\alpha}_j > \beta_j] = 1 - P[U_j > \beta_j] = 1 - \phi(\beta_j) = \phi(-\beta_j) \tag{17}$$

in which U_j is a standard normal variate.
It has been shown that expression (17) yields rather accurate results. The numerical effort depends mainly on the type of formulation of the limit state surface and on the speed of convergence of the algorithm for the search of the safety index β. Step i. can easily be incorporated into the search algorithm . Frequently, a good approximation can also be obtained if the transformation is made only for a suitably chosen unfavourable fractile of the distribution of $\underline{X}$ [8].

III. RELIABILITY OF SIMPLE SYSTEMS

Consider now as introductory examples the reliability of several elementary systems as there are: series systems, parallel systems and "k out of n"-systems. An ideal series system fails if at least one component fails. An ideal parallel system fails if all components have failed. A "k out of m"-system fails if more than m-k components have failed. According to the foregoing the j-th component fails if $U_j \leq -\beta_j$ and survives if $U_j > -\beta_j$. Therefore, the failure probability of a series system with m components is given by [10]

$$P_f = 1 - P[\bigcap_{j=1}^{m} (U_j > -\beta_j)] \tag{18}$$

For a parallel system we have

$$P_f = P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j)] \tag{19}$$

whereas the probability of failure for a given "k out of m"-system is

$$P_f = 1 - \sum_{l=k}^{m} \sum_{I_l} P\left[\bigcap_{\nu \in I_l} \{U_\nu > -\beta_\nu\} \cap \bigcap_{\nu \in I \setminus I_l} \{U_\nu \leq -\beta_\nu\}\right] \tag{20}$$

where $I = \{1, 2, \ldots, m\}$ and $I_l \subset I$ with $|I_l| = l$. In words, the failure probability is the sum of the probabilities for which k of the U's are greater than the corresponding limits whereas all other U's are smaller than the limits. The summation goes over all combinations of k in a total of m elements.

From eq.(13) it follows that the standard normal variates U_j are correlated since the same components of the uncertainty vector enter different limit state functions. According to the calculus of first and second statistical moments [9] we have for the covariance (correlation coefficient) for any two U's

$$\mathrm{Cov}\,[U_p, U_q] = \underline{\alpha}_p^T\ \underline{\underline{\Sigma}}_Z\ \underline{\alpha}_q \tag{21a}$$

and

$$\rho[U_p, U_q] = \mathrm{Cov}[U_p, U_q] / (\sigma_p \cdot \sigma_q); \tag{21b}$$

It is recognized that the evaluation of either of the eqs.(19) reduces to the determination of the probability content of multinormal densities with correlation coefficient matrix $\underline{\underline{R}}$ with elements $\{\rho_{p,q}\}$ over rectangles. For the three systems mentioned above we have

$$\text{Series:}\quad P_f = 1 - P[\bigcap_{j=1}^{m} (U_j > \beta_j)] = 1 - \int_{-\underline{\beta}}^{\infty} \varphi(\underline{u}; \underline{\underline{R}})\, d\underline{u} \tag{22a}$$

$$\text{Parallel:}\quad P_f = P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j)] = \int_{-\infty}^{-\underline{\beta}} \varphi(\underline{u}; \underline{\underline{R}})\, d\underline{u} \tag{22b}$$

"k out of m":

$$P_f = 1 - \sum_{l=k}^{m} \sum_{I_l} \int_{-\infty}^{-\underline{\beta}(I_l)} \int_{-\infty}^{-\underline{\beta}(I \setminus I_l)} \varphi(\underline{u}, \underline{\underline{R}})\, d\underline{u}_{(I \setminus I_l)}\, d\underline{u}_{(I_l)} \tag{22c}$$

$\varphi(.)$ is the multidimensional normal density.

Of course, the method given before can also be applied to more complex systems. Obviously, the crucial point is the calculation of multi-normal probabilities for which some details are given in the appendix.

IV. MARKOVIAN SYSTEMS

1. General Concepts

For many redundant systems Markovian properties can be assumed for the description of their state behaviour. Then, a number of problems can be resolved in using well-known results of the theory of Markovian chains. Let a system be composed of several elements. The system can realize the states $s_0, s_1, \ldots, s_n$ which from a reliability point of view can be distinguished. Denote by s_0 the initial state where all per-

formance requirements are fully accomplished, s_n by the final failure state and by the s_i's all possible intermediate states, respectively. During consecutive load applications the system may change its state until the system reaches s_n. This state is an absorbing or trapping state. The state behaviour is said to be Markovian if it possesses the following property: Given the system is in a certain state at a given point in time then the future performance of the system depends only on the present state but not on time nor on any previously occupied state. Hence, the state description in conjunction with the transition probabilities p_{ij} completely defines the stochastic behaviour of the system.

For example, let a nuclear reactor be exposed to an earthquake with several aftershocks between which there is no time for repair or to finish the switch-off procedure of the reactor. Let us consider the system of pressure vessel (V) and containment (C). This system has four principal states as measured by the damage caused by the event:

s_0: = entire system intact
s_1: = V leak, C intact
s_2: = C leak, PV intact
s_3: = C leak, PV leak

Hence, there is the following matrix of states and possible transitions.

state left \ state entered	0	1	2	3
0	*	*	*	*
1	-	*	-	*
2	-	-	*	*
3	-	-	-	*

The corresponding matrix of transition probabilities is

state left \ state entered	0	1	2	3
0	p_{00}	p_{01}	p_{02}	p_{03}
1	0	p_{11}	0	p_{13}
2	0	0	p_{22}	p_{23}
3	0	0	0	p_{33}

In general, we have the matrix $\underline{P}$ of transition probabilities $p_{ij} \geq 0$ for i, j = 0, 1, ..., n.

Note, that the sum of the probabilities in any row equals unity. p_{00} is the probability of non-failure of any system component. p_{0n} is the classical system failure probability and p_{0i} the probability of failure of a subsystem. The general term for the transition probabilities can be given as follows. If the state s_i can be described by a set of limit state functions $U_\nu^{(i)}$ the transition probabilities p_{ij} can be represented by

$$p_{ij} = P[\text{some } U_\nu^{(i)} \leq -\beta_\nu, \text{ all other } U_\nu^{(i)} > -\beta_\nu] \tag{23a}$$

More precisely:

$$p_{ij} = P[\bigcap_{\nu \in I_j} \{U_\nu^{(i)} \leq -\beta_\nu^{(i)}\} \cap \bigcap_{\nu \in I_i \setminus I_j} \{U_\nu^{(i)} > -\beta_\nu^{(i)}\}] \tag{23b}$$

where $I_j \subset I_i$ if a transition from s_i to s_j is possible. Its computation can be carried out using the results of the foregoing sections.

In our illustrating example of the pressure vessel-containment system we have, if m_V and m_C are the numbers of critical components (points, elements, sections, failure modes, etc.), in the vessel and the containment causing the same damage

$$p_{00} = P[\bigcap_{i=1}^{m_V} (U_{V,i}^{(0)} > -\beta_{V,i}) \cap \bigcap_{i=1}^{m_C} (U_{C,i}^{(0)} > -\beta_{C,i})]$$

5)

$$p_{01} = P[\bigcap_{i=1}^{m_V} (U_{V,i}^{(0)} \leq -\beta_{V,i}) \cap \bigcap_{i=1}^{m_C} (U_{C,i}^{(0)} > -\beta_{C,i})]$$

$$p_{02} = P[\bigcap_{i=1}^{m_V} (U_{V,i}^{(0)} > -\beta_{V,i}) \cap \bigcap_{i=1}^{m_C} (U_{C,i}^{(0)} \leq -\beta_{C,i})]$$

$$p_{03} = P[\bigcap_{i=1}^{m_V} (U_{V,i}^{(0)} < -\beta_{V,i}) \cap \bigcap_{i=1}^{m_C} (U_{C,i}^{(0)} < -\beta_{C,i})] = 1-(p_{00}+p_{01}+p_{02})$$

$$p_{11} = P[\bigcap_{i=1}^{m_C} (U_{C,i}^{(1)} > -\beta_{C,i})] \qquad p_{13} = 1 - p_{11}$$

$$p_{22} = P[\bigcap_{i=1}^{m_V} (U_{V,i}^{(2)} > -\beta_{V,i})] \qquad p_{23} = 1 - p_{22}$$

$$p_{33} = 1$$

2. Some Applications of the Theory of Markov Chains [11]

Most of the problems in the reliability of Markovian systems can be reduced to simple operations with the matrix $\underline{\underline{P}}$ of transition probabilities. For example, introduce the discrete reliability function of the system

$$R(k) := P(K > k) \tag{24}$$

where the random variable K is the number of load applications until system fails (= entrance of state s_n). It can be shown that

$$R(k) = P(K > k) = 1 - p_{On}^{(k)} \tag{25}$$

$p_{On}^{(k)}$ is the last element in the first row of the matrix of k-step transition probabilities. The latter is given by $\underline{\underline{P}}^k$. Similarly,

$$F(k) = P(K \leq k) = p_{On}^{(k)} \tag{26}$$

is the probability of failure before the k-th load application. Further,

$$p(k) = P(K = k) = p_{On}^{(k)} - p_{On}^{(k-1)} \tag{27}$$

is the probability of failure at the k-th load application.

Let us determine E(K), the average number of load applications until failure. We denote

$$m_0 := E[K \mid \text{initial state} = s_0]$$
$$\vdots$$
$$m_{n-1} := E[K \mid \text{initial state} = s_{n-1}]$$

be shown, that

$$m_i = 1 + \sum_{j=0}^{n-1} p_{ij} \cdot m_j \quad \text{for } i = 0, 1, \ldots, n-1 \tag{28}$$

Under the assumption that the system originally is in state s_0, the mean number of loads until failure is

$$E[K] = m_0 \tag{29}$$

For our example, determination of m_0 yields

$$m_0 = E[K] = \frac{1}{(1-p_{00})} \left(1 + \frac{p_{01}}{p_{13}} + \frac{p_{02}}{p_{23}}\right) \tag{30}$$

Similarly, the variance Var[K] or other moments can be obtained.

There is an immediate and simple extension to a random number of "shocks". If N is this number and K the number of shocks until failure, the survival probability for the system is

$$R(N) = P[K > N] = \sum_{k=1}^{\infty} P[K > k] \cdot P[N = k] \tag{31}$$

For example, if the number N is Poisson distributed with parameter λ, then

$$R(N) = \sum_{k=1}^{\infty} P[K > k] \cdot e^{-\lambda} \cdot \frac{\lambda^{k-1}}{(k-1)!} = 1 - \sum_{k=0}^{\infty} P[K \leq k+1] \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= 1 - e^{-\lambda} \sum_{k=0}^{\infty} p_{On}^{(k+1)} \frac{\lambda^k}{k!} = 1 - e^{-\lambda}\, q_{On} \tag{32}$$

where q_{On} is the (O,n)-th element of the matrix

$$\underline{\underline{Q}} := \underline{\underline{U}}\ \underline{\underline{D}}\ e^{\lambda \underline{\underline{D}}}\ \underline{\underline{U}}^{-1} \tag{33}$$

with

$$\underline{\underline{D}} := \begin{bmatrix} p_{00} & & 0 \\ & \ddots & \\ 0 & & p_{nn} \end{bmatrix}; \quad e^{\lambda \underline{\underline{D}}} := \begin{bmatrix} e^{\lambda p_{00}} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda p_{nn}} \end{bmatrix};$$

and $\underline{\underline{U}}$ the matrix of Eigenvectors of the matrix $\underline{\underline{P}}$ of transition probabilities with Eigenvalues $p_{00}, \ldots, p_{nn}$.

Some other elementary counting processes yield compact results too, which may be applied under given circumstances. If the shock-generating process is a time-dependent Poisson process, the foregoing model has a straightforward application for systems where no repair is possible, for example, for space crafts. The concept of Markovian systems may also be applied if the sequence of shocks is generated by different causes. In this case the Markov chain becomes non-homogenous.

Further theoretical considerations and applications to engineering systems will be given in a separate paper.

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VI. APPENDIX: EVALUATION OF MULTINORMAL PROBABILITIES

Although the evaluation of multinormal probabilities has attracted many statisticians the results are scarce for higher dimensions of the variable vector. An exhaustive review may be found in Johnson/Kotz [12].

Kendall [13] developed integrals of the form (22) into an infinite Mac Laurin series which is given below in a slightly generalized form.

$$\phi(\underline{\alpha}, \underline{\beta}; \underline{R}) = \int_{\underline{\alpha}}^{\underline{\beta}} \varphi(\underline{u}; \underline{R})\, d\underline{n}$$

$$= \sum_{j_{12}=0}^{\infty} \sum_{j_{13}=0}^{\infty} \cdots \sum_{j_{n-1,n}=0}^{\infty} \left[\prod_{\substack{m,l=1 \\ m<1}}^{n} \left(\frac{\rho_{ml}^{j_{ml}}}{j_{ml}!} \right) \cdot \prod_{k=1}^{n} \sqrt{h_k!}\, \{\tau_{h_k}(\beta_k) - \tau_{h_k}(\alpha_k)\} \right] \qquad (A.1)$$

where

$$h_k = \sum_{i=1}^{k-1} j_{ik} + \sum_{i=k+1}^{n} j_{ki} \quad \text{for } k = 1, 2, \ldots, n$$

and $\tau_j(.)$ being the j-th tetrachoric function defined by

$$\tau_j(x) = H_{j-1}(x)\ \exp[-\frac{x^2}{2}]/[(j!)^{1/2} \cdot (2\pi)^{1/2}] \qquad (A.2)$$

with $H_j(x)$ the jth Hermite polynomial

$$H_j(x) = \exp[-\frac{x^2}{2}]\{(-D)^i[\exp -\frac{x^2}{2}]\}$$

$$= j! \sum_{m=0}^{[\frac{j}{2}]} (-1)^m \frac{x^{j-2m}}{2^m \cdot m! \cdot (j-2m)!} \qquad (A.3)$$

It can be seen that only elementary operations are required to determine the various terms of eq.(A.1). Unfortunately, the series converges only slowly for large absolut values of the correlation coefficients. Also, its computation requires considerable time for higher dimension of the $\underline{U}$-vector even in a computerized analysis since n(n-1)/2 different summations have to be carried out. Nevertheless, the method is usually much less time-consuming than direct numerical integration.

However, there are some special cases of the matrix $\underline{R}$ where multinormal probabilities can be obtained by simple one-dimensional numerical quadrature. If the correlation coefficients can be expressed in the form

$$\rho_{i,j} = \lambda_i \cdot \lambda_j \qquad (A.4)$$

then, according to Dunnett and Sobel [14] one can write

$$P[\bigcap_{j=1}^{m} (\alpha_j \le U_j \le \beta_j)]$$

$$= \int_{-\infty}^{+\infty} \varphi(t) \cdot \prod_{j=1}^{m} \{\phi(\frac{\beta_j - \lambda_j \cdot t}{\sqrt{1-\lambda_j^2}}) - \phi(\frac{\alpha_j - \lambda_j \cdot t}{\sqrt{1-\lambda_j^2}})\}\, dt \qquad (A.5)$$

In general, the correlation coefficients cannot be expressed by a product of the λ's as required. Then, a fairly good estimate can be obtained by chosing the λ's such that

$$(\lambda_i \cdot \lambda_j - \rho_{ij})^2 \rightarrow \min \qquad (A.6)$$

provided that it is $|\lambda_j| < 1$ for $j = 1, 2, \ldots, m$. An even better estimate is obtained if the relationship (A.4) is fulfilled exactly for the first two or three absolutely largest correlation coefficients and the criterion (A.6) is applied to the rest. If the requirements on the accuracy are not too strict and if the probabilities to be calculated are sufficiently small the integrals may easily be evaluated by simple Monte-Carlo integration. In fact, this method may be recommended for practical application. Further research on appropriate computation techniques for multinormal probabilities, however, appears necessary.

CLOSE BOUNDS FOR THE RELIABILITY OF STRUCTURAL SYSTEMS

R. Rackwitz

A B S T R A C T

First order reliability methods are applied to two limiting structural systems, the series and the parallel system, respectively. In principle, any distribution and correlation structure can be assumed for the uncertain quantities to be considered for the description of the component behaviour. It is shown that correlation increases reliability for series system and essentially decreases reliability for parallel systems.

Structural systems have successfully been modelled by two limiting idealizations, the series system and the (redundant) parallel system, respectively. For series systems the system is said to fail if at least one component(structural element, section, point in the structure where yielding occurs, etc.) fails or the conditions are met to fail in one of a number of mutually exclusive modes (mechanisms). For parallel systems the system is said to fail if all components have failed or all originally redundant structural configurations or paths to carry the load have been eliminated. Usually, component failures are dependent events since the same uncertain variables are imbedded in the functions describing the passage into failure states of the different components. This fact makes a probabilistic evaluation of system reliability rather difficult. Therefore, practice frequently makes use of several easily assessed and well-known reliability bounds. If there are m different components (failure modes, load paths, etc.) the reliability R (= survival probability) of a series system is bounded by

$$\prod_{j=1}^{m} R_j \leq R_S \leq \min_{j=1}^{m} \{R_j\} \qquad (1)$$

in which the R_j are the individual component reliabilities. The left hand bound is associated with stochastic independence whereas the right hand bound assumes full stochastic dependence between the components [1]. For parallel systems the limits are

$$1 - \max_{j=1}^{m} \{1-R_j\} \leq R_P \leq 1 - \prod_{j=1}^{m} (1-R_j) \qquad (2)$$

Full stochastic dependence is valid for the left hand bound and the right hand bound corresponds to stochastic independence. Note that stochastic dependence increases system reliability for series systems but decreases it for parallel systems. In some applications the upper bound in eq.(1) has been used as a lower bound for parallel systems. Although, the exact reliability frequently is close to the bound corresponding to stochastic independence the bounds have proven to be too wide for systems with a large number of components. Much sharper bounds, however, can be assessed if some results of the so-called "first order reliability methods" as outlined, for example, in [2] are applied.

Let $\underline{X}$ be the vector collecting all uncertain (random) quantities relevant for system performance such as actions, strengths, geometrical properties and other uncertainties, for example in the prediction model of system behaviour. If the prediction model is formulated such that a state variable for the j-th component, $Z_j := g_j(\underline{x})$ is less than zero for failure, its reliability becomes

$$R_j = \int_{\{\underline{x}: g_j(\underline{x}) > 0\}} d\, F_{\underline{X}}(\underline{x}) \qquad (3)$$

with $F_{\underline{X}}(\underline{x})$ the joint distribution of the basic uncertainty vector $\underline{X}$. If the vector $\underline{X}$ is a non-normal dependent vector with given covariance matrix Σ_X, then, there exists an

orthogonal transformation (rotation) (X) → (Y) such that the new vector $\underline{Y}$ has uncorrelated components. Also, there exists a transformation (Y) → (Z) such that the vector $\underline{Z}$ is a standard normal vector with independent components (for further details see [4, 6, 8]). In first order reliability a rather good estimate of the component reliability is achieved by approximating the failure surface $g_j(\underline{z}) = 0$ by a hyperplane in a point on $g_j(\underline{z}) = 0$ nearest to the coordinate origin [6]. For example, if $\underline{z}^*$ is that point having direction cosines $\underline{\alpha}_j$ and which may be found by a suitable search algorithm [8] the integral (3) is replaced by

$$R_j \approx \int_{\{\underline{z}: g_j(\underline{z}) > 0\}} d\, \Phi_{\underline{Z}}(\underline{z}) = P[U_j > -\beta_j] = \phi(\beta_j) \qquad (4)$$

with

$$g_j(\underline{z}) = \underline{z}\, \underline{\alpha}_j - \beta_j = 0 \qquad (5)$$

being the approximating hyperplane, $\Phi_{\underline{Z}}$ the m-dimensional norm distribution function, $\phi(\,)$ the univariate standard normal in

$$\beta_i = \min \{\sqrt{\underline{z}^T \underline{z}} : g_j(\underline{z}) = 0\} \qquad (6)$$

the so-called safety index and $U_j = \underline{Z}\, \underline{\alpha}_j$ a standard normal variate. It can be shown that the reliability estimate obtained by expression (4) is usually in error by only a few percent [4].

For any two components characterized by the same uncertainty vector $\underline{Z}$ but possibly different functions for the failure surface $g(\underline{z}) = 0$ the state variables U_i and U_j are

correlated. According to the calculus for statistical moments [10] the covariance is

$$\mathrm{Cov}[U_i, U_j] = \underline{\alpha}_i^T \, \underline{\underline{\Sigma}}_{\underline{Z}} \, \alpha_j \tag{7}$$

and since the variables Z_i are already standardized the correlation coefficient becomes

$$\rho[U_i, U_j] = \mathrm{Cov}[U_i, U_j] \tag{8}$$

with $\underline{\underline{\Sigma}}_{\underline{Z}} = I$ the covariance matrix of the vector $\underline{Z}$. Hence, the reliability of a series system can be written as

$$R_S = P[\bigcap_{j=1}^{m} (U_j > -\beta_j)]$$

$$= \int_{-\underline{\beta}}^{\infty} \varphi(\underline{u};\, \underline{\underline{R}})\, d\underline{u} \tag{9}$$

and the reliability of a parallel system is given by

$$R_P = 1 - P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j)]$$

$$= 1 - \int_{-\underline{\infty}}^{-\underline{\beta}} \varphi(\underline{u};\, \underline{\underline{R}})\, d\underline{u} \tag{10}$$

with the standard multinormal density

$$\varphi(\underline{u};\, \underline{\underline{R}}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(|\underline{\underline{R}}|)^{1/2}} \exp\left[-\frac{1}{2}\,(\underline{u}^T \, \underline{\underline{R}}^{-1} \, \underline{u})\right] \tag{11}$$

and $\underline{\underline{R}}$ the matrix of correlation coefficients ρ_{ij}.

Unfortunately, the evaluation of integrals of type (9) or (10) is rather tedious if possible at all for arbitrary forms of the correlation matrix (see [5] and [9]). However, there are some useful bounds. Dunnett and Sobel [3] showed that if the correlation coefficients can be expressed by $\rho_{ij} = \lambda_i \, \lambda_j$, then, the following expression holds

$$P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j)] = \int_{-\infty}^{+\infty} \varphi(u_o) \cdot \prod_{j=1}^{m} \phi\left(\frac{(-\beta_j) - \lambda_j \cdot u_o}{\sqrt{1 - \lambda_j^2}}\right) du_o \tag{12}$$

The integral can easily be evaluated by simple numerical quadrature. If, in particular, $\rho_{ij} = \rho > 0$, the integrand simplifies significantly. For negative values of ρ, Steck and Owen [12] gave a useful reduction formula which is not repeated here. Now, note that for symmetrical reasons, it is

$$P[\bigcap_{j=1}^{m} (U_j > -\beta_j)] \equiv P[\bigcap_{j=1}^{m} (U_j \leq \beta_j) \tag{13}$$

Šidak [11] proved the following probability inequality. If the elements of two correlations matrices obey $\{\kappa_{ij}\} \leq \{\rho_{ij}\}$ for all i, j, then

$$P_\kappa[\bigcap_{j=1}^{m} (X_j \leq c_j)] \leq P_\rho[\bigcap_{j=1}^{m} (X_j \leq c_j)] \tag{14}$$

wherein the c_j are some given constants.

Therefore, let $\underline{\lambda}'$ be a vector producing a correlation matrix $\underline{\underline{R}}$ for which $\{\rho_{ij}'\} = \{\lambda_i' \cdot \lambda_j'\} \leq \{\rho_{ij}\}$ for all i, j and $\underline{\lambda}''$

be a vector producing a matrix $\underline{\underline{R}}''$ for which $\{\rho''_{ij}\} = \{\lambda''_i \cdot \lambda''_j\} \geq \{\rho_{ij}\}$ for all i, j. Then, the two new reliability bounds for a series system are in using expressions (9), (13) and (14):

$$P[\bigcap_{j=1}^{m} (U_j \leq \beta_i) \,|\, \underline{\underline{R}}'] \leq R_S \leq P[\bigcap_{j=1}^{m} (U_j \leq \beta_j) \,|\, \underline{\underline{R}}''] \qquad (15)$$

whereas for parallel systems application of eq.(10) and (14) yields

$$1 - P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j) \,|\, \underline{\underline{R}}''] \leq R_P \leq 1 - P[\bigcap_{j=1}^{m} (U_j \leq -\beta_j) \,|\, \underline{\underline{R}}'] \qquad (16)$$

The probabilities P[.] can easily be evaluated by eq.(12). Frequently, it suffices to set $\{\rho_{ij}\} = \rho' = \min \{\rho_{ij}\}$ and $\{\rho''_{ij}\} = \rho'' = \max \{\rho_{ij}\}$. If, further, the limits β_j are equal and absolutely less than 3.5 tables published by Gupta [5] can be used to determine the bounds in eqs. (15) or (16). In fig. 1 to 3 some further results are presented which may help to judge when calculation of the bounds according to eqs. (15) and (16) becomes advisable. In agreement with other authors it is noted that the previous bounds of eqs.(1) and (2) may still be used for moderate m and correlation coefficients less than, say. 0.5 (see for example [13]).

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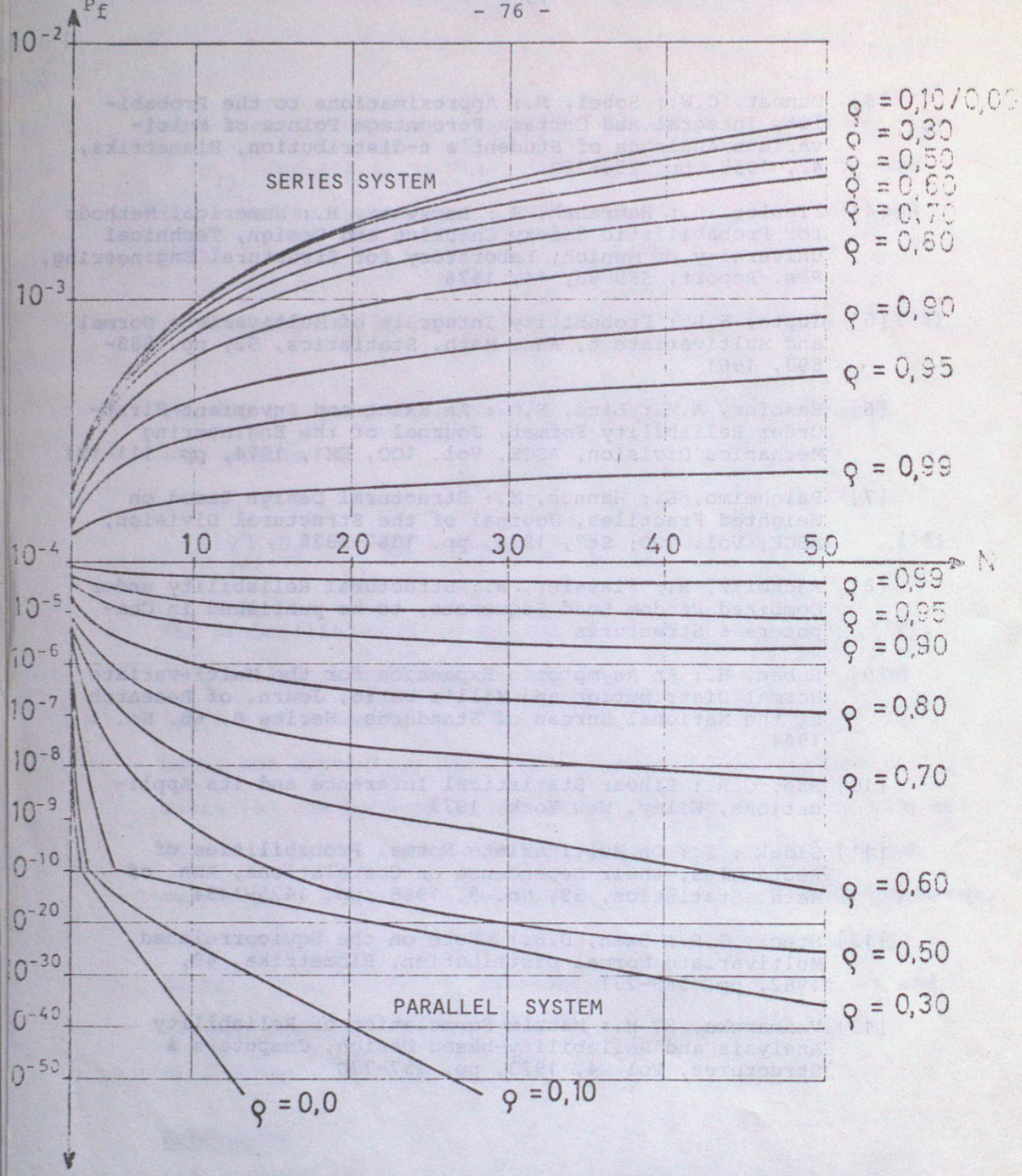

FIG. 1: SYSTEM FAILURE PROBABILITY VERSUS NUMBER N OF EQUICORRELATED COMPONENTS WITH COMPONENT FAILURE PROBABILITY 10^{-4}

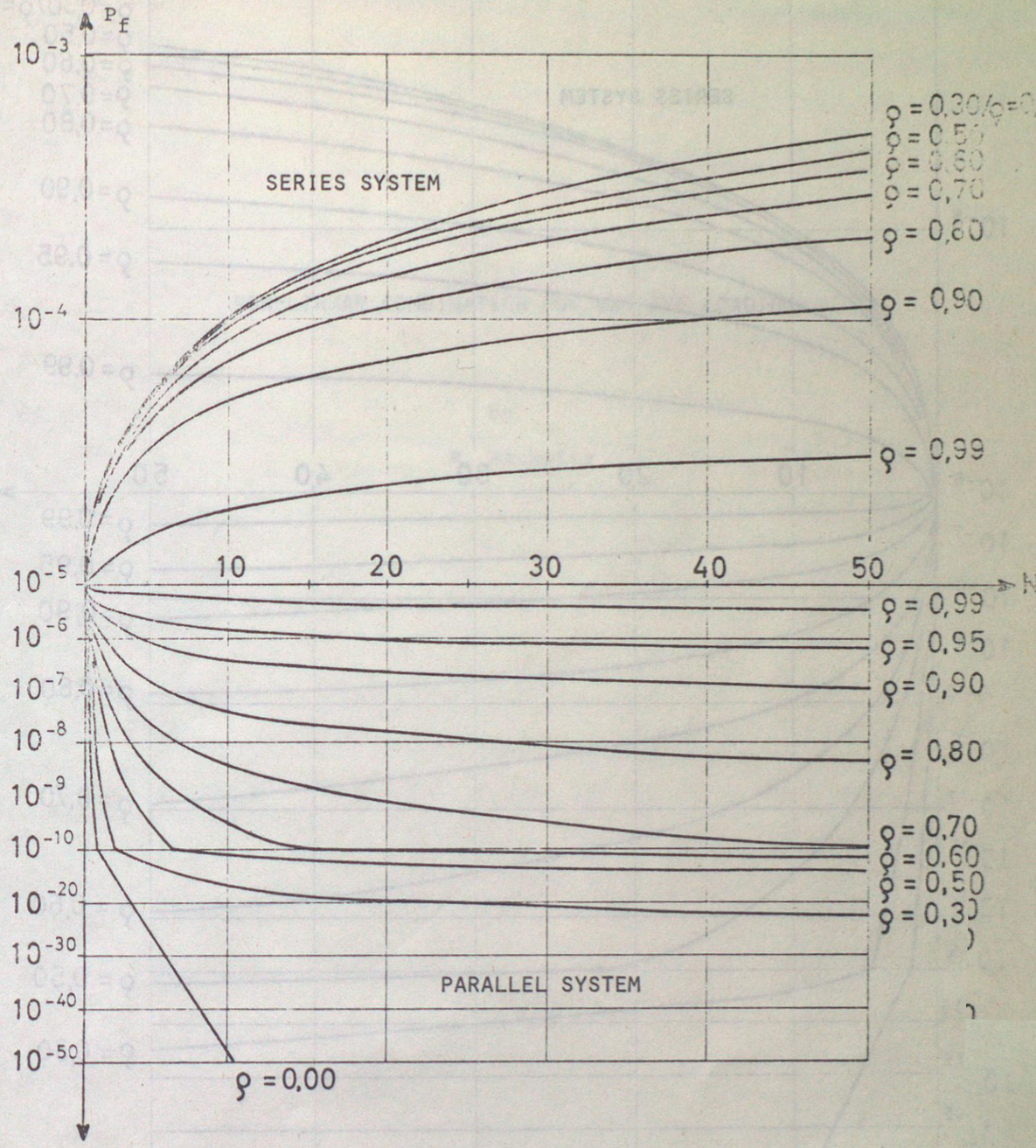

FIG. 2: SYSTEM FAILURE PROBABILITY VERSUS NUMBER N OF EQUICORRELATED COMPONENTS WITH COMPONENT FAILURE PROBABILITY 10^{-5}

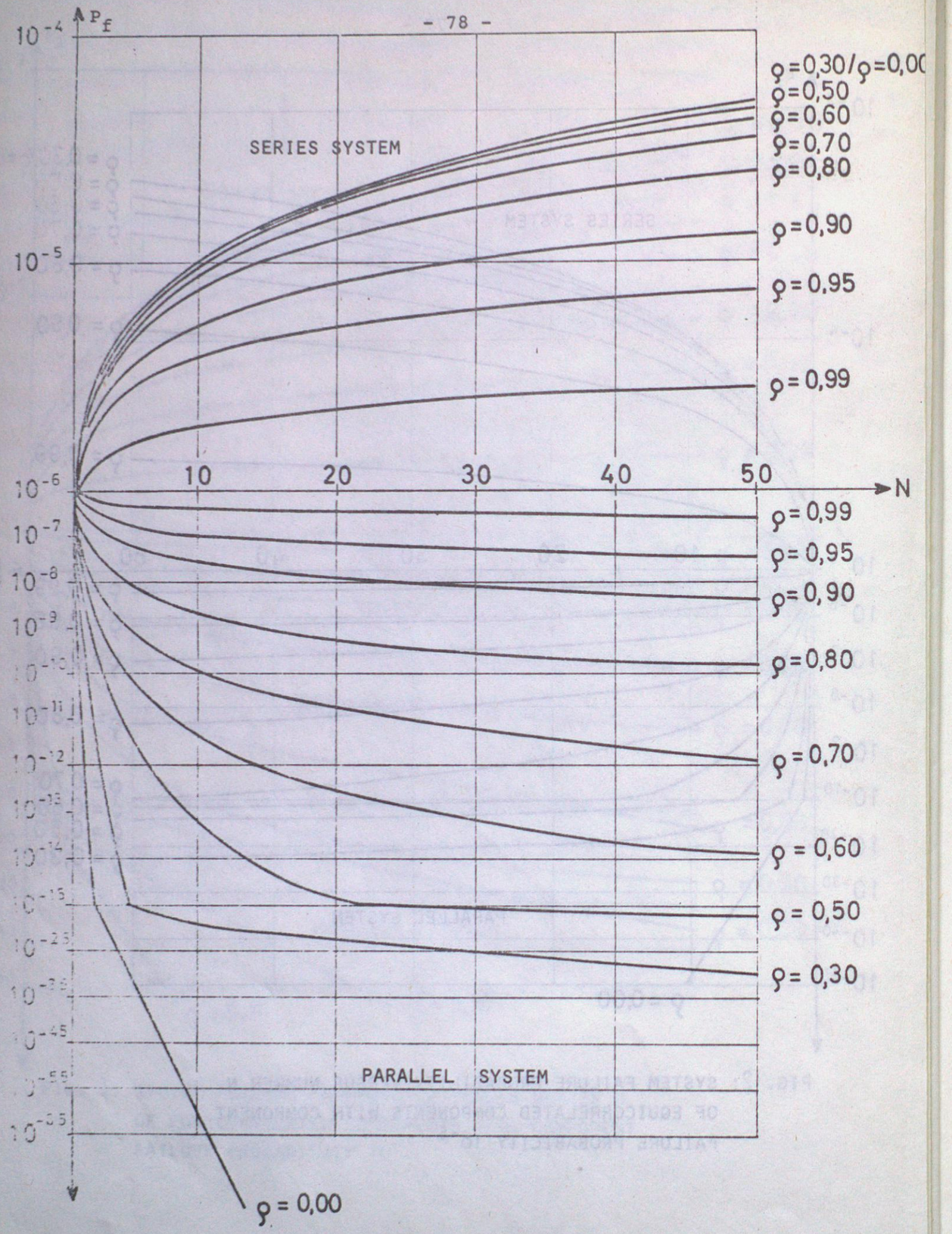

FIG. 3: SYSTEM FAILURE PROBABILITY VERSUS NUMBER N OF EQUICORRELATED COMPONENTS WITH COMPONENT FAILURE PROBABILITY 10^{-6}

NON-LINEAR COMBINATION FOR EXTREME LOADINGS

by

R. Rackwitz

A B S T R A C T

Wen's approximate results on the linear combination of extreme loadings are applied to non-linear combination within the context of first order reliability theory. The failure events are treated as dependent events. By suitable transformations, the reliability can then be obtained by an evaluation of the mulitnormal integral. A simple example illustrates the method.

Introduction

Extreme loading conditions govern to a large extent the desing of structures for unclear power plants and other outstanding structural facilities. Extreme loading events are normally characterized by their short duration but considerable magnitude. Examples are earthquake, tornado, vehicle impact, fire. In some cases the consideration of the simultaneous occurrence of two or more such extreme loadings must be required. Wen [9] showed that if the extreme loading can be modelled by a filtered Poisson process the joint occurrences of two or more actions also form a filtered Poisson process. He derived approximate formulas for the extreme value distribution of the extreme loadings and their sum. However, just under extreme loading conditions structures are unlikely to behave in a simple linear manner. As will be shown this shortcoming of Wen's approach can easily be overcome by use of some concepts of the so-called "first order reliability theory" as presented, for example, in [2] or [4].

Review of Wen's [9] results

Assume that an extreme loading can be represented by a filtered Poisson process

$$X(t) = \sum_{\nu=1}^{N(t)} h_\nu \, (W_\nu, d_\nu, \tau_\nu, t) \qquad (1)$$

in which ν = occurrence number; $N(t)$ = total number of occurrences in the time-interval $(0,t)$, generated by a homogeneous Poisson process with intensity λ_i; h_ν = load amplitude function at each occurrence (here, a rectangular function is assumed); $W_{i,\nu}$ = (maximum) load amplitude in each occurrence with distribution function $F_{W,i}(w)$; d_ν = the random exponentially distributed duration of the load W_ν with mean μ_{ij}; τ_ν = random occurrence time. The interval $(0,t)$ is denoted by the reference period which usually is

taken to correspond to the anticipated time of use of the structure. Since extreme loadings are characterized by $\lambda_i \cdot \mu_i << 1$ and $\mu_i << \tau_i$, the coincidence of more than one impulses of the same loading at any point in time can be neglected and, thus, the maximum in time of X)t) can be given as

$$F_{\max_{[0,t]}}(x) = \sum_{\nu=0}^{\infty} F_W^{\nu}(x) \cdot \{\frac{(\lambda t)^{\nu}}{\nu!} e^{-\lambda t}\} = \exp\{-\lambda t[1-F_W(x)]\} \quad (2)$$

Wen [9] proved for two extreme loadings of the type described by eq.(1) with parameters λ_i, λ_j and μ_i, μ_j, respectively, that the joint occurrence of both loads is asymptotically again a filtered Poisson process with intensity

$$\lambda_{ij} \approx \lambda_i \cdot \lambda_j (\mu_i + \mu_j) \quad (3)$$

Then it is easily derived that the extreme value distribution becomes

$$F_{\max_{[0,t]}}(x) = \exp\left[-t\{\lambda_i(1-F_i(x)) + \lambda_j(1-F_j(x)) + \lambda_{ij}\cdot(1-F_{ij}))\}\right] \quad (4)$$

where $F_{ij}(x)$ is the distribution of $W_i + W_j$. Unfortunately, the necessary convolution of W_1 with W_2 is generally not elementary.

Wen [9] demonstrated the validity of his approach by extensive simulation studies.

Application of First Order Reliability Theory

Let $g(\underline{x}) > 0$ define the domain of safe structural states in the space of all basic uncertain variables. The equality $g(x) = 0$ may be called the limit state surface of the structural problem under consideration. The vector $\underline{X}$ collects the relevant uncertain variables such as loads, strength of

of materials or geometrical parameters having any type of marginal distribution function. If the components of $\underline{X}$ are uncorrelated which always can be achieved by a suitable transformation (see [4]) and are transformed into standard normal variates by using the relation

$$F_j(x_j) = \phi(z_j) \quad (5)$$

the so-called Hasofer-Lind safety index [ ] can be found as the minimum distance between the limit state surface and the coordinate origin by a suitable search algorithmus [4].

$$\beta = \min\{\sqrt{\underline{z}^T \underline{z}} : g(\underline{z}) = 0\} \quad (6)$$

If the limit state surface is linearized in the point P^* nearest to the origin by a hyperplane

$$\underline{z}\,\underline{\alpha} + \beta = 0 \quad (7)$$

where $\underline{\alpha}$ is the vector of direction cosines of P^* it can be shown [4] that a fairly good estimate of the survival probability for one load application can be obtained from

$$P_S = P[g(\underline{X})>0] = P[g(\underline{Z}) > 0] \approx P[\underline{Z}\,\underline{\alpha} > -\beta] = P[U > -\beta] = \phi(\beta) \quad (8)$$

where U is the standard normal variate. For the probability of survival in an interval (0,t) the distribution of amplitudes of an extreme loading has to be replaced by its maximum distribution e.g. according to eq.(2). Consequently, if there are two extreme loadings the probability of survival has to be evaluated from

$$\begin{aligned} P_S(t) &= P[\{g_i(\underline{x}) > 0\} \cap \{g_j(\underline{x}) > 0\} \cap \{g_{ij}(\underline{x}) > 0\}] \\ &= P[\{g_i(\underline{z}) > 0\} \cap \{g_j(\underline{z}) > 0\} \cap \{g_{ij}(\underline{z}) > 0\}] \end{aligned} \quad (9)$$

and by substituting the linearized limit state surface as given by eq.(7)

$$P_S(t) = P[\{\underline{Z}\,\underline{\alpha}_i > -\beta_i\} \cap \{\underline{Z}\,\underline{\alpha}_j > -\beta_j\} \cap \{\underline{Z}\,\underline{\alpha}_{ij} > -\beta_{ij}\}]$$

$$= P[U_i > -\beta_i \cap U_j > -\beta_j \cap U_{ij} > -\beta_{ij}] \qquad (10)$$

Clearly, the survival events are not independent since the same uncertain variables are imbedded in the various limit state surfaces. In fact, they are correlated with correlation coefficient

$$\rho_{ij} = \underline{\alpha}_i \,\underline{\underline{\Sigma}}_Z \cdot \underline{\alpha}_j \qquad (11)$$

where the covariance matrix $\underline{\underline{\Sigma}}_Z = \underline{\underline{I}}$ (unit matrix) if the components of $\underline{Z}$ are made already uncorrelated. It follows that the survival probability of the system under individual and combined extreme loading can be calculated from the three-dimensional normal integral

$$P_S(t) = \int_{-\beta_1}^{\infty} \int_{-\beta_2}^{\infty} \int_{-\beta_{12}}^{\infty} \varphi(u_k, u_2, u_3;\ \rho_{12}, \rho_{23}, \rho_{13})\, du_{12}\, du_2\, du_1 \qquad (12)$$

where $\varphi(.)$ is the trivariate normal density. This integral can be evaluated easily by use of the methods given, for example, by Kendall [3] or Steck [7], or for positive correlation coefficients by simple numerical quadrature of

$$P_S(t) = P[\bigcap_{i=1}^{3} U_i > -\beta_i] = P[\bigcap_{i=1}^{3} U_i \leq \beta_i]$$

$$= \int_{-\infty}^{+\infty} \varphi(t) \cdot \prod_{i=1}^{3} \phi\left[\frac{\beta_i - \lambda_i \cdot t}{\sqrt{1 - \lambda_j^2}}\right] dt \qquad (13)$$

with $\varphi(t)$ the standard univariate normal density and

$$\lambda_1 = \sqrt{\frac{\rho_{12} \cdot \rho_{13}}{\rho_{23}}};\quad \lambda_2 = \sqrt{\frac{\rho_{23} \cdot \rho_{12}}{\rho_{13}}};\quad \lambda_3 = \sqrt{\frac{\rho_{13} \cdot \rho_{23}}{\rho_{12}}}; \qquad (14)$$

FAILURE PROBABILITY TO

Numerical Example

For illustration consider the elementary case in which failure occurs if a random resistance quantity X_1 is exeeded by any of two independent extreme load effects X_2 and X_3 or their combination that is the joint action of two independent loads X_4 and X_5 with the same amplitude distribution as X_2 and X_3, respectively. Therefore, the three limit state surfaces are

$$g_1(x_1, x_2) = x_1 - x_2 = 0$$

$$g_2(x_1, x_3) = x_1 - x_3 = 0$$

$$g_3(x_1, x_4, x_5) = x_1 - (x_4 + x_5) = 0$$

Let X_1 be log-normally distributied with median $\hat{\mu}_1$ and logarithmic standard deviation δ_1. Assume further that the amplitudes of the extreme loadings are exponentially distributed with parameters $\nu_2 = \nu_4$ and $\nu_3 = \nu_5$. The intensities are λ_2, λ_3 and, according to eq.(3), $\lambda_4 = \lambda_5 = \lambda_2 \cdot \lambda_3 (\mu_2 + \mu_3)$ with μ_2, μ_3 the mean durations of loads X_2 and X_3, respectively. In applying the transformation (5) with $F(x) = \phi(\ln(\frac{x}{\hat{\mu}})/\delta) = \phi(z)$ and $F(x) = 1 - e^{-\nu x} = \phi(z)$ the foregoing limit state surfaces can be written as:

$$g_1(z_1, z_2) = g_2(F_1^{-1}\{\phi(z_1)\}, F_2^{-1}\{\phi(z_2)\})$$

$$= \check{\mu}_1 \cdot \exp[z_1 \cdot \delta_1] + \frac{1}{\nu_2} \ln\left[-\frac{\ln \phi(z_2)}{\lambda_2 \cdot t}\right] = 0$$

$$g_2(z_1, z_3) = \check{\mu}_1 \cdot \exp[z_1 \cdot \delta_1] + \frac{1}{\nu_3} \ln\left[-\frac{\ln \phi(z_3)}{\lambda_3 \cdot t}\right] = 0$$

$$g_3(z_1, z_4\, z_5) = \check{\mu}_1 \cdot \exp[z_1 \cdot \delta_1] + \frac{1}{\nu_2} \ln\left[-\frac{\ln \phi(z_4)}{\lambda_4 \cdot t}\right]$$

$$+ \frac{1}{\nu_3} \ln\left[-\frac{\ln \phi(z_5)}{\lambda_5 \cdot t}\right] = 0$$

As an numerical example let $\mu_1 = 10$, $\delta_1 = 0{,}2$, $\nu_2 = \nu_4 = 1$, $\nu_2 = \nu_5 = 0{,}5$, $\lambda_2 = 2$, $\lambda_3 = 0{,}05$, $\mu_2 = 0{,}005$ and $\mu_3 = \mu_2/10$ and, therefore, $\lambda_4 = \lambda_5 = 5{,}5\ \ 10^{-4}$. The direction cosines of the approximating planes and the safety indices are found to be

	direction cosines z_1	z_2	z_3	z_4	z_5	safety index	failure probability
g_1	- 0,579	0,815				2,145	$1{,}596 \cdot 10^{-2}$
g_2	- 0,3561		0,934			1,982	$2{,}374 \cdot 10^{-2}$
g_3	- 0,2098			0,1846	0,9602	3,819	$6{,}704 \cdot 10^{-5}$

and by application of formula (11) the matrix of correlation coefficients becomes

$$\underline{\underline{R}} = \begin{Vmatrix} 1 & 0{,}206 & 0{,}122 \\ 0{,}206 & 1 & 0{,}075 \\ 0{,}122 & 0{,}075 & 1 \end{Vmatrix}$$

Evaluation of the probability of failure according to eq.(13) yields $P_f = 3{,}942 \cdot 10^{-2}$. As expected from the relatively small correlation coefficients its value is close to the lower bound $1 - \Pi\ (1 - P_{f,i}) = 3{,}938 \cdot 10^{-2}$. Extensive parameter studies for the example just described including less unfavourable distribution functions for the extreme loading indicated that unless the coefficient of variation of the resistance is close to or exeeds that one of the loadings the lower probability bound yields an accurate estimate of the more exact result.

Closing Remarks

In the same manner more complex structural problems involving multi-dimensional basic uncertainty vectors with arbitrary marginal distribution functions and distinct non-linear limit state surfaces can be dealt with. In fact, it may be

realized that there is mathematically no distinction between mechanically non-linear failure surfaces and the non-linearity caused by the transformation of non-normal distribution into normal distributions. Inclusion of more than two independent extreme loadings follows the same pattern. The necessary analogues to eqs.(3) and (4) are given in [9]. Even multi-mode failure of structural systems can be treated since it just increases the dimension of the multi-normal integral to be solved [6]. For example, assume that a structure can fail in different modes each of which can be reached by n loading situations, then the failure probability must be calcuated from

$$P_f = 1 - P[\bigcap_{i=1}^{m} \bigcap_{j=1}^{n} g_{ij}(\underline{X}) > 0] \approx 1 - P[\bigcap_{i=1}^{m} \bigcap_{j=1}^{n} (U_{ij} > \beta_{ij})]$$

Hence, the dimension of the integral would be $n \cdot m$.

In general, the calculations may require the use of a computer. Frequently, the correlation coefficients are so small that any correlation may be neglected. Then, the survival probability is as known close to but greater than $\prod_{(i)} \phi(\beta_i)$ and always less than $\max_{(i)} \phi(\beta_i)$

The foregoing concept may even be used for the combination of ordinary loads in a concept as proposed by Turkstra [8] and outlined in more detail by Ditlevsen [1]. A thorough discussion of this concept will be given in a separate paper.

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