NON-NORMAL VECTORS IN STRUCTURAL RELIABILITY

by

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Abstract

Informative probabilistic reliability assessments for structural facilities require the prior choice of probabilistic models for the uncertainties, particularly their distributional characteristics. In principle, multidimensional numerical integration is necessary for the calculation of reliabilities which is rather tedious as the type of models for the uncertain variables becomes more complex and their number increases. Therefore, suitable approximations are in order. The well-known second moment reliability method as proposed e.g. by Hasofer and Lind is generalized to take account of any arbitrary type of distribution of the uncertainties by means of a discrete first order transformation into a normal distribution. A suitable algorithm is presented for finding the appropriate point of transformation. The method is applied to some extreme cases of limit state functions. The possible error of the approximative method is calculated. It is proved that the inclusion of second order terms for the expansion of the limit state function can yield results which are in error only insignificantly.

Introduction

One of the major problems in the application of probability based design methods to the elaboration of design codes or to direct design of structural facilities is to develope an efficient method for determining the probability of not exceeding a given structural limit state. If $\underline{X}$ is a parameter-invariant vector of n basic uncertain variables such as actions, dimensions and strength of materials with joint distribution function $F(\underline{x})$, then the failure probability

$$P_f = 1 - \int_{\{D\}} dF(\underline{x}) \qquad (1)$$

is the complement to the probability content of the safe domain D which is separated from the unsafe domain by the function describing the limit state. Analytical solutions of (1) exist only for a few special cases concerning the distribution function $F(\underline{x})$ and shapes of D and, therefore, are of minor practical interest. The following study deals with an approximate calculation of the integral (1) for arbitraty distribution functions $F(\underline{x})$ and shapes D of the safe domain.

First Order Reliability Methods

Recently, approximate "first order reliability methods" as proposed by Ditlevsen [2], Hasofer/Lind [7], Paloheimo [10] and Veneziano [11] reduced the fundamental problem of multidimensional integration to a numerically much simpler

problem of mathematical programming. Let the vector $\underline{X} \equiv (X_1, X_2 \ldots X_n)^T$ be represented by its mean value vector $M_{\underline{X}} = E[\underline{X}]$ and by its covariance matrix $\Sigma_{\underline{X}} = E[(\underline{X} - E[\underline{X}]) \cdot (\underline{X} - E[\underline{X}])^T]$. Then, there exists an orthogonal transformation such that the components of $\underline{X}$ become uncorrelated yielding a new variable vector

$$\underline{Y} = \underline{\underline{R}}^T \cdot \underline{X} \qquad (2)$$

with the rotation matrix $\underline{\underline{R}}^T$ and the matrix of Eigenvectors $\underline{\underline{R}}$ of Σ_X, the mean value vector $E[Y] = \underline{\underline{R}}^T \cdot E[\underline{X}]$ and diagonal covariance matrix $\Sigma_Y = \underline{\underline{R}}^T \, \Sigma_{\underline{X}} \, \underline{\underline{R}}$. If the vector $\underline{Y}$ is standardized by

$$Z_i = \frac{Y_i - E[Y_i]}{D[Y_i]} \qquad i = 1, 2, \ldots, n \qquad (3)$$

with $D[Y_i]$ representing the standard deviation of Y_i, it $E[Z_i] = 0$ and $D[Z_i] = 1$ and so $\Sigma_{\underline{Z}} = I$ (unit matrix). The so called safety index ß can be found by minimizing the distance b between the limit state function or failure surface in the formulation $g(z) = 0$ and the coordinate origin

$$ß = \min b = \min_{\underline{z} \in \{g(\underline{z}) = 0\}} \{\sqrt{z^T \; z}\} \qquad (4)$$

The point representing the smallest distance is denoted by the "checking" or "approximation" point $\underline{z}^*$. Several authors consider ß a convenient reliability measure since further information on the stochastic characteristics of $\underline{X}$ is dispensable or may not be available.

The informativeness of β about the reliability in terms of a probability statement, however, remains poor and is of the Tchebychev-inequality-type [11]. More precise statements can be made if distributional assumptions on the components of $\underline{X}$ are adopted. For example, assuming the vector $\underline{X}$ being a normal vector and $g(\underline{z}) = 0$ being continuous at the point where eq.(4) is satisfied, the safety index β produces two useful and simple reliability bounds when approximating the actual, generally non-linear failure surface by either a tangent hyperplane or a supporting hypersphere. For well-behaved convex safe regions the failure probability P_f is, then, bounded by (see Veneziano [11]):

$$1 - \chi_n^2 (\beta^2) \geq P_f \geq 1 - \phi(\beta) \qquad (5)$$

Herein, $\chi_n^2(.)$ denotes the chi-square distribution for n degrees of freedom whereas $\phi(.)$ is the invariate standard normal integral. Essentially, these bounds are related to pure normal uncertainty vectors. In practice, though a lower bound, the right hand side limit frequently yields an accurate estimate of the failure probability P_f.

If the uncertainty vector is log-normal, a simple transformation $\underline{z} \rightarrow \underline{u} : u_i = \ln z_i$ for $i = 1, 2, \ldots, n$ reduces this case to the normal one [3]. Of course, the limit state function now has to be formulated in the new u-space.

However, many uncertain phenomena are only poorly described by either the normal or log-normal model. It is also known that the results in terms of estimates or of bounds to failure probabilities significantly depend on the stochastic model adopted for the basic variable vector. A generalization towards non-normal models would, therefore, considerably increase the applicability of first order reliability methods.

Review of Extensions to Non-Normal Distributions

Paloheimo [10] approximated a non-normal distribution by a normal distribution having the same mean and the same P_f - or $(1-P_f)$-fractile. Setting

$$p = \phi\left(\frac{x_p - \mu}{\sigma'}\right) = F(x_p; \underline{\Theta}) \qquad (6)$$

it is by solution for the new standard deviation σ'

$$\sigma' = \frac{x_p - \mu}{\phi^{-1}(p)} = \frac{F^{-1}(p;\Theta) - \mu}{\phi^{-1}(p)} \qquad (7)$$

yielding the standardized normal variate by the transformation

$$x \rightarrow u: \; u = \frac{(x-\mu) \cdot \phi^{-1}(p)}{F^{-1}(p; \underline{\Theta}) - \mu} \qquad (8)$$

where $F^{-1}(.)$ is the inverted non-normal marginal distribution with parameter vector $\underline{\Theta}$, μ the mean value and $\phi^{-1}(.)$ the inverted standard normal distribution function, respectively. p is equal either to the target survival probability $1 - P_f$, if the variable is a loading variable or to the failure probability P_f, if the variable is a resisting variable.

Ditlevsen [3] suggested a similar approximation to the distribution of extremes of independent normal variables. Again, the approximation is chosen such, that the new normal distribution fits the non-normal distribution best in the vicinity of the fractiles corresponding to the target failure (survival) probabilities. Alternatively, he proposed to fit the non-normal distribution by a normal distribution having the same values in two different extreme points but, again, with no strong arguments for the choice of these points.

Lind [9] verified the basic idea of applying a continuous mapping which transforms a non-normal distribution into a normal distribution. For example, if the basic uncertainty vector $\underline{X}$ has independent components with different distribution type it is:

$$x \to u: \; u = h(x) = \phi^{-1} \, [F(x; \underline{\theta})] \qquad (9)$$

This idea is, no doubt, implicit in many of the earlier works. The approach is formally appealing. The transformation (9) is, in general, not elementary. However, eq.(9) can easily be applied in computerized analysis where distribution functions and their inverses can be given by suitable series expansions or rational approximations [1, 8]. In each but Lind's approach the aforementioned probability bounds are no longer valid.

Linear Approximation in the "Checking Point" - Independent Uncertainty Vectors

Since the checking method described before employs one single point on the failure surface it suffices to apply eq.(9) in that point, only (see fig. 1). Following Paloheimo's idea, eq.(8) can be improved by taking the value p at the checking point, giving the transformation

$$x \to u: \; u = \frac{(x-\mu)}{(x^*-\mu)} \, \phi^{-1} \, [F(x^*; \underline{\theta})] \qquad (10)$$

which is linear in x. The checking point must be known. Alternatively, the mean μ might be substituted by any other appropriate central parameter, e.g. the median $\hat{\mu}$ of $F(\underline{x})$.

$$x \to u: \; u = \frac{(x-\hat{\mu})}{(x^*-\hat{\mu})} \, \phi^{-1} \, [F(x^*; \underline{\theta})] \qquad (10a)$$

It is obvious that these expressions give a correct mapping only with respect to the value of the distribution function at point x^*. A discrete mapping ought to be accurate in the vicinity of the checking point x^* as well. Hence, it is proposed to linearize the mapping function (9) in the checking point e.g. by taking its Taylor expansion up to the linear term:

$$u = \phi^{-1}(F(x;\cdot)) \simeq \phi^{-1}(F(x^*;\cdot)) + (x-x^*) \cdot \left. \frac{\partial \phi^{-1}(F(x;\cdot))}{\partial x} \right|_{x=x^*}$$

$$= \phi^{-1}(F(x^*;\cdot)) + (x-x^*) \cdot \frac{f(x^*;\cdot)}{\varphi(\phi^{-1}(F(x^*;\cdot)))} = \qquad (11)$$

The variable U is then a standard normal variate. In the original space, it has mean

$$\mu' = x^* - \sigma' \ \phi^{-1}(F(x^*;\cdot)) \qquad (12)$$

and standard deviation

$$\sigma' = \frac{\varphi(\phi^{-1}(F(x^*;\cdot)))}{f(x^*;\cdot)} \qquad (13)$$

This linear transformation is equivalent to adjusting a continuous non-normal distribution to a normal distribution having the same cumulative probability and the same probability density in the checking point.

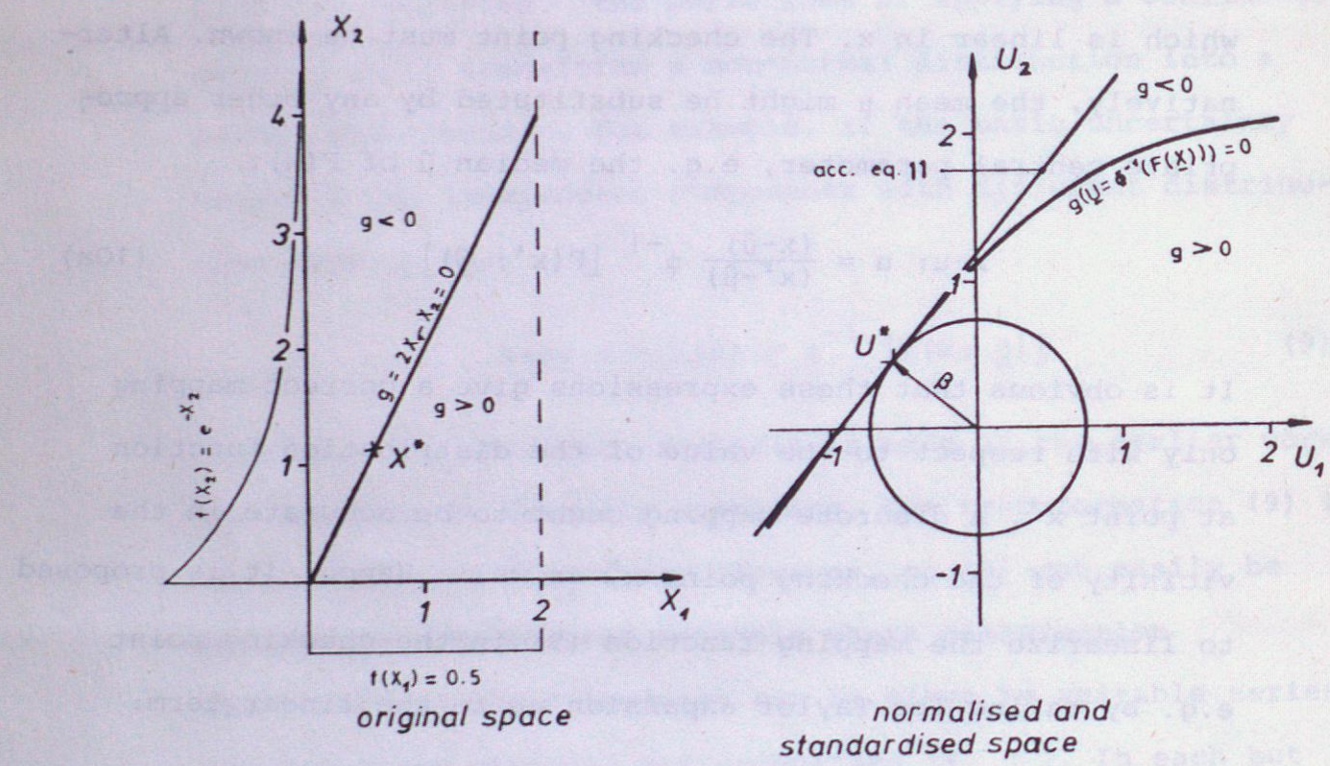

Figure 1

In fig. 2 the different methods as expressed by eqs.(8), (10) and (11) are illustrated for the simple one-dimensional case of an extreme value distribution type I.

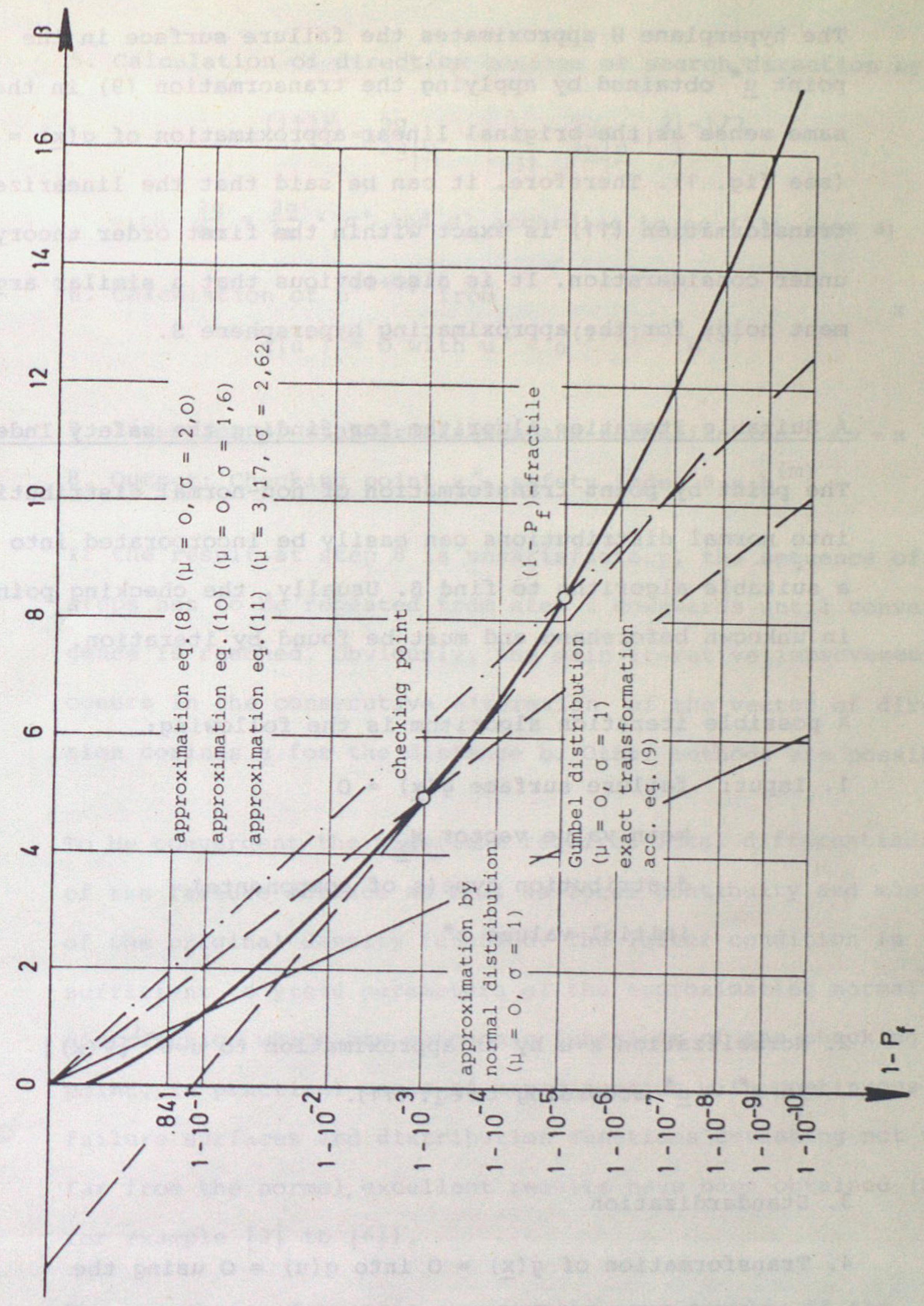

Fig.2: Various approximations of a non-normal distribution by normal distributions with modified distribution parameters

The hyperplane H approximates the failure surface in the point $\underline{u}^*$ obtained by applying the transormation (9) in the same sense as the original linear approximation of $g(\underline{x}) = 0$ (see fig. 1). Therefore, it can be said that the linearized transformation (11) is exact within the first order theory under consideration. It is also obvious that a similar argument holds for the approximating hypersphere S.

A Suitable Iteration Algorithm for finding the safety Index ß

The point by point transformation of non-normal distributions into normal distributions can easily be incorporated into a suitable algorithm to find ß. Usually, the checking point in unknown beforehand and must be found by iteration.

A possible iteration algorithm is the following:

1. Input: failure surface $g(\underline{x}) = 0$
   mean value vector $M_{\underline{X}}$
   distribution type(s of components)
   initial values $\underline{x}^*$

2. Normalization $x \rightarrow u$ by an approximation to $u = \phi^{-1}[F(x)]$ in $\underline{x}^* = \underline{u}^*$ according to eq. (11).

3. Standardization

4. Transformation of $g(\underline{x}) = 0$ into $g(u) = 0$ using the results of 2. and 3.

5. Calculation of direction cosines of search direction by

$$\alpha^{(i+1)} = \frac{\partial g}{\partial \underline{u}}\bigg|_{\underline{u}^*} \cdot \left[\sum_{(j)} \left(\frac{\partial g}{\partial \underline{u}}\bigg|_{\underline{u}^*}\right)^2 \right]^{-1/2}$$

with $\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \sigma'$ and σ' according to eq.(13) (see appendix B

6. Calculation of $b^{(i+1)}$ from

$$g(u^*) = 0 \text{ with } u^* = \alpha^{(i+1)} \cdot b^{(i)}$$

7. Inversion of standardization and normalization $u \rightarrow v \rightarrow x$

8. Output: Checking point $\underline{x}^*$, safety index $ß = b^{(m)}$

If the result at step 8 is unsatisfactory, the sequence of steps has to be repeated from step 2 downwards until convergence is reached. Obviously, the main iterative improvement occurs in the consecutive alteration of the vector of direction cosines $\underline{\alpha}$ for the distance b. Other methods are possible.

To be convergent the algorithm requires local differentiability of the failure surface as well as local continuity and monotony of the original density function. The latter condition is sufficient to yield parameters of the approximating normal distribution which are monotonic functions of the checking point. In practical cases of any dimension with continuous failure surfaces and distribution functions deviating not too far from the normal, excellent results have been obtained (see, for example [3] to [6]).

The search for ß becomes numerically more complex if the original failure surface has several local minima or if the original distribution function is discontinuous and/or its density function is locally non-monotonic, e.g. in case of multimodal distributions. For the latter case, the search for all local minima may turn out to be quite cumbersome. The iteration may even diverge depending on the relative position of a mode and the approximation point. Application of the rougher approximation eq.(10) may help in some cases which also holds if the distribution is discontinuous.

Accuracy of the Method

The accuracy of the results decreases as the number of loading or resisting variables having very skewed and/or limited distributions increase. The convexity properties of the safe domain may be modified.

In order to check the accuracy of the method comparisons can be made with a few results form exact probability theory. To avoid additional influences the following linear limit state surface is chosen

$$\pm C_n \mp \sum_{i=1}^{n} X_i = 0 \qquad (14)$$

in which C_n is a constant derived from the reference case of identically normally distributed variables. Let the constant be defined by $C_n = n \cdot \mu_X \pm \beta \cdot \sigma_X \cdot \sqrt{n}$ and the X_i's be some identically distributed random variables. β is the pre-selected safety index. For rectangular distributions with probability density

$$f(x) = \begin{cases} 1/a & \text{for } 0 \leq x \leq a \\ 0 & \text{elsewhere} \end{cases} \qquad (15)$$

the distribution function of $X = \sum_{i=1}^{n} X_i$ is [8]

$$F(y) = \frac{1}{a^n\, n!} \sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} (y - \nu \cdot a)^n \qquad (16)$$

for $0 \leq y \leq n \cdot a$

and, therefore,

$$P_f = 1 - F(C_n)$$

For sums of gamma-distributed variables with probability density function

$$f(x) = \begin{cases} \dfrac{\lambda(\lambda x)^{k-1} \exp[-\lambda x]}{\Gamma(k)} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (17)$$

the type of distribution is retained but with parameters λ and $n \cdot k$ [8]. Remember that $k = 1$ corresponds to the extremely skewed case of an exponential distribution. In Figure 3 some examples are presented for $\beta = 3$.

One recognizes that only in the exceptional cases of the rectangular and the gamma distributions with, say $k < 5$, the approximate method results in significant errors. They increase with the dimension of the basic variable vector. The maximum error associated with each type of distribution function approaches a limit which in using the Central Li[mit] Theorem of probability theory is found as follows. For symmetrical reasons and from eq.(14) the exact approximat[e] point is known to be $x_i^* = C_n/n$ for $i = 1, 2, \ldots, n$. Therefore, the limiting approximate failure probability can be derived from

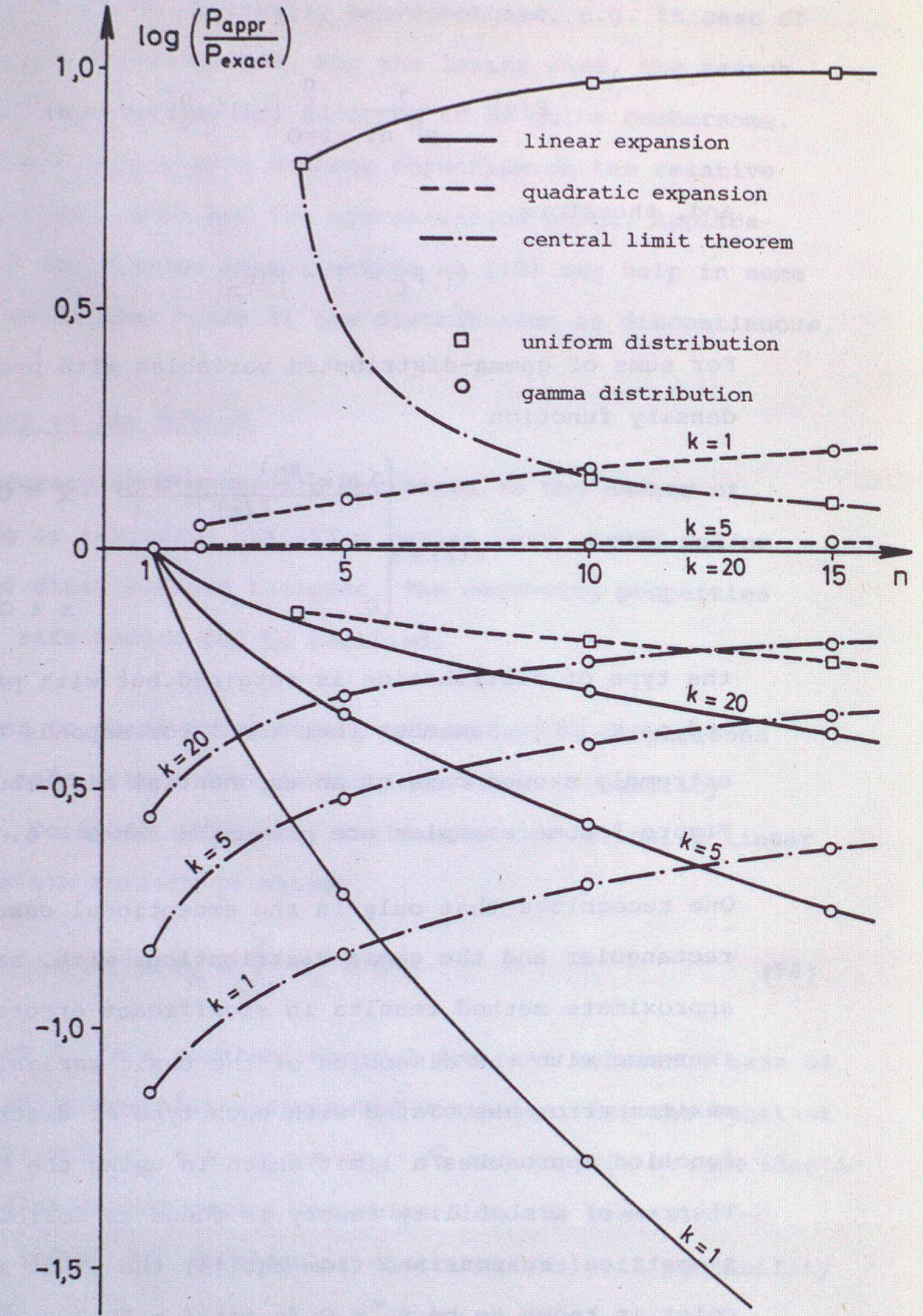

Fig. 3: Approximation errors as a function of dimension n for sums of uniformly and gamma-distributed variables

$$P_{f,\,appr \; (n\to\infty)} = 1 - \phi\left[\lim_{n\to\infty}\{u_n\}\right] \tag{18}$$

with

$$u_n = \frac{\pm C_n + n\cdot\mu'}{\sqrt{n}\cdot\sigma'} \tag{19}$$

Substituting C_n and μ', σ' by expressions (12) and (13)

$$u_n = \pm\sqrt{n}\;\phi^{-1}\left[F\left(\mu_x \pm \frac{\beta\cdot\sigma_x}{\sqrt{n}}\right)\right] \tag{20}$$

is obtained.

Hence, proceeding to the limit one reaches for symmetrical distributions with $F(.) \to 0{,}5$ for $n\to\infty$

$$\lim_{n\to\infty}\{u_n\} = \pm\sqrt{2\pi}\cdot\beta\cdot\sigma_x \quad f(\mu_x) \tag{21}$$

which is independent of n. For asymmetrical distributions the limit is either $-\infty$ or ∞. However, this result does not seem to be a serious drawback since such sums of identically distributed excessively non-normal variables rarely occur in practice. Moreover, the influence of resisting variables, the distribution functions of which frequently are skewed to the right and that of loading variables the distribution functions of which are skewed to the same side but have opposite signs, compensate each other.

Second Order Expansions

A significant improvement can be achieved even for those exceptional cases if a "second order reliability theory" is used (see Fiessler/Neumann/Rackwitz [5]). Let the checking point be found according to the procedure just described and let the limit state function be locally continuous and twice differentiable. Calculate the matrix $\underline{\underline{G}}_u$ of second order derivatives of the limit state function in the standardized and normalized space (U) in that point (see appendix B). Rotate the coordinate system into a new system (R) with the same origin such that the new r_n-axis is parallel to the safety index vector (see fig. 4). Then, the n - 1 principal curvatures k_i of $g(\underline{r}) = 0$ are obtained as the Eigenvalues divided by the gradient of $g(\underline{r}) = 0$ in $\underline{r}^*$ of the matrix of derivatives $\underline{\underline{G}}_r$ where the n - th row and column are deleted. With

$$k = \min_{i=1}^{n-1} \{k_i\} \quad \text{or} \quad k = \max_{i=1}^{n-1} \{k_i\} \tag{22}$$

the quadratic form

$$Q = \sum_{i=1}^{n-1} s_i^2 + (s_n - \delta_n)^2 = \frac{1}{k^2} = R^2 \tag{23}$$

can be set. For convex safe regions (k < 0) a circumscribing ($Q = R^2_{max}$) or inscribing ($Q = R^2_{min}$) hypersphere can be defined. The random variable Q is said to be non-central chi-square distributed with n degrees of freedom and non-centrality parameter $\delta_n^2 = (R - \beta)^2$. Thus, the failure probability is

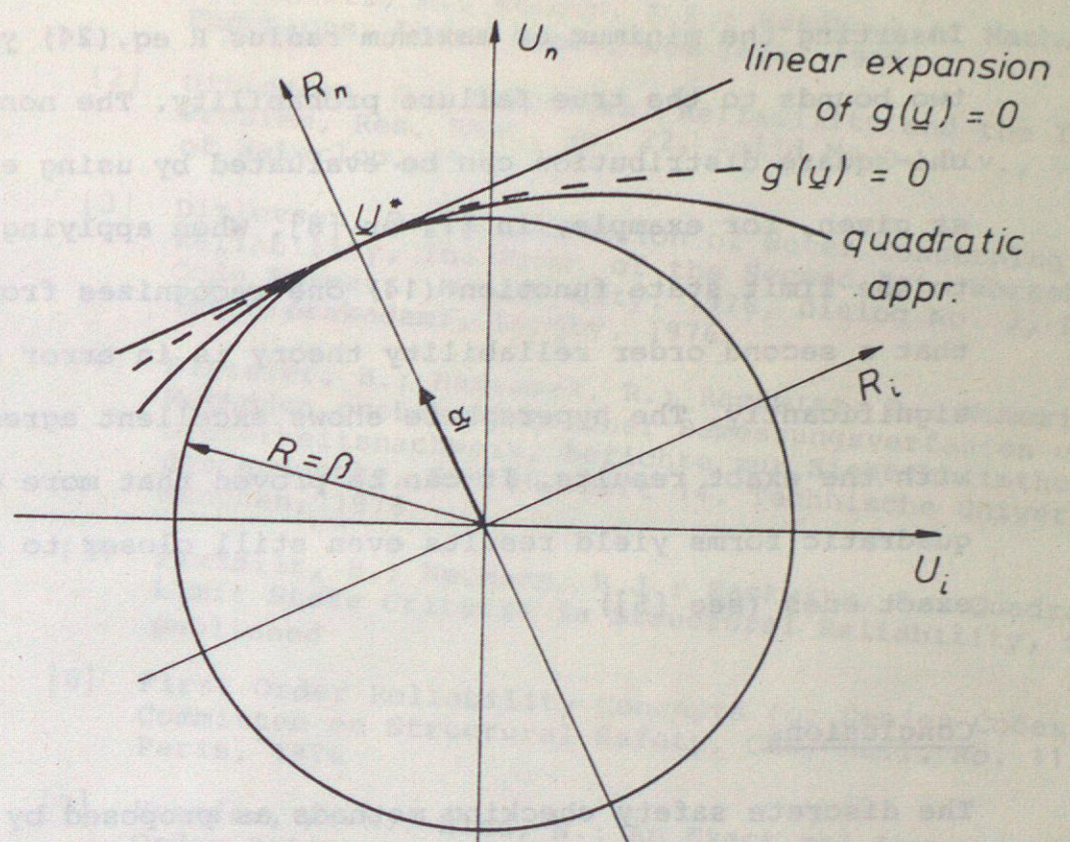

Fig.4: Linear and quadratic approximation of the limit state function

$$P_f = 1 - \chi^2_{n,\delta}(R^2) \tag{24}$$

Inserting the minimum or maximum radius R eq.(24) yields two bounds to the true failure probability. The non-central chi-square distribution can be evaluated by using expansions as given, for example, in [1] or [8]. When applying eq.(24) to the limit state function (14) one recognizes from fig. 3 that a second order reliability theory is in error only insignificantly. The hypersphere shows excellent agreement with the exact results. It can be proved that more general quadratic forms yield results even still closer to the exact ones (see [5]).

Conclusions

The discrete safety checking methods as proposed by Hasofer/Lind and others can be generalized with respect to arbitrary distributional assumptions for the basic uncertainty vector. In essence, non-normal distributions are approximated in a first order sense by normal distributions in certain checking points. In general, the accuracy of the method is sufficient for engineering purposes. A suitable iteration algorithm is presented to find the appropriate checking point. As an alternative a second order reliability method is proposed which appears to be in error only insignificantly but requires continuity and twice differentiability of the limit state function.

Appendix A: References

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Appendix B: Transformation of Derivatives

If X is a non-normal variable with the distribution funct if $\partial g/\partial x$ is the derivative of the limit state function the transformation

$$u = \phi^{-1} F(x)$$

is applied to independent variables X it is from elemen calculus of differentials

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\frac{\partial \phi(u)}{\partial u}}{\frac{\partial F(x)}{\partial x}} = \frac{\partial g}{\partial x} \cdot \frac{\varphi\{\phi^{-1}[F(x)]\}}{f(x)} = \frac{\partial g}{\partial x} \cdot \sigma' \quad (B.2)$$

where σ' is identical with expression (13). Further, the second derivatives become

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial(\partial g/\partial x \cdot \sigma')}{\partial x} \cdot \frac{\partial x}{\partial u} = \left\{\frac{\partial^2 g}{\partial x^2} \cdot \sigma' + \frac{\partial g}{\partial x} \cdot \frac{\partial \sigma'}{\partial x}\right\} \sigma'$$

$$= \left\{\frac{\partial^2 g}{\partial x^2} \sigma' + \frac{\partial g}{\partial x} \cdot \frac{\partial\left(\frac{\varphi\{\phi^{-1}[F(x)]\}}{f(x)}\right)}{\partial x}\right\} \sigma'$$

$$= \left\{\frac{\partial^2 g}{\partial x^2} \cdot \sigma' - \frac{\partial g}{\partial x} \cdot \left[\phi^{-1}[F(x)] + \frac{\frac{\partial f(x)}{\partial x}}{f(x)} \cdot \sigma'\right]\right\} \sigma' \quad (B.3)$$

and

$$\frac{\partial^2 g}{\partial u_i \partial u_j} = \frac{\partial g}{\partial x_i \partial x_j} \cdot \sigma_i' \cdot \sigma_j' \quad (B.4)$$

It is noted that evaluation of (B.3) requires existence of the derivative of the probability density function f(x).