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BASIS CONCEPTS FOR RELIABILITY ANALYSIS OF SYSTEMS[1)]

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1. Introduction – Notions and Definitions

In general,a system is understood as a technical arrangement of clearly identifiable (system–) components whose functioning depends on the proper functioning of all or a subset of its components. A reliability analysis requires precise definitions. It is assumed that the components can attain only two states, i.e. one functioning (safe, working, active,...) and one failure (unsafe, defect, inactive,...) state. This is an idealisation which is not always appropriate but we will maintain it throughout the text. If there is a natural multi–state description of a component or a system we shall assume that this is reduced to a two–state description in a sensible manner. In practice, this step of modelling might be not an easy task. It is, nevertheless, mandatory in pratical system reliability analyses because the division into only two states is crucial for any straightforward quantitative reliability statement. Several attempts have been made to establish concepts for analysing systems with multi–state components (see, for example, Caldarola, 1980; Fardis/Cornell, 1981). It should be clear that system have also multiple states and the definition of safe or failure states requires great care. Such relatively recent extensions of the classical concepts cannot be dealt with herein.

A representation of component performance by only two states is called a Boolean representation but we shall avoid the explicit use of Boolean algebra as far as possible. As a consequence of the Boolean component representation, systems can only be either in the functioning or the failure state. We shall only deal with so–called coherent systems, i.e. systems which remain intact if an additional functioning component is added.

One can distinguish two basic types of systems, the series and the parallel system. Later, we shall add other related types whose separate definition is useful for classsification and calculation purposes. A series system consisting of n components is said to fail if any of its components fail. Classical examples are the chain whose failure is a consequence of the failure of any of its links or a four–wheel car where any flat tire prohibits further use of the car (usually). A parallel system of n components is said to fail if all components fail. As an example, assume that a town is supplied by several electrical life–lines and each one is capable to deliver the required power. Or, in aircraft control, two computers are installed in ideal stand–by redundancy. If the first fails the second takes over and can fulfill all demands but only if this, too, fails no control is possible. Remember, however, that in most technical systems such as structures failure of some components causes higher loads on the remaining components. This problem will require some thought. A system is called redundant if it is intact inspite of the failures of some of its components.

In general, systems are built up by many components in a complex logical arrangement of series and parallel subsystems. Let $F_i = \{X \in V_i\}$ be the failure event of the i–th system component. Denote by F the system failure event. Clearly, for a series system ("or"–connection) we have F as the union of the individual failure events (see figure 1.1)

[1)]This manuscript has been compiled from notes produced for the *ISPRA Course on STRUCTURAL RELIABILITY, Lisbon, Nov. 13–17, 1989* as well as for the *KFA Course on PROBABILISTIC STRUCTURAL MECHANICS, Jülich, May 17, 1990.*

Fig. 1.1: Series system

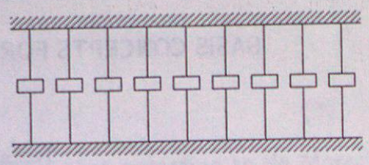


Fig. 1.2: Parallel system

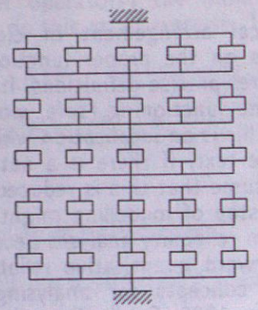


Fig. 1.3: Parallel systems in series

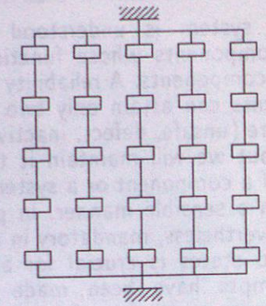


Fig. 1.4: Series systems in parallel

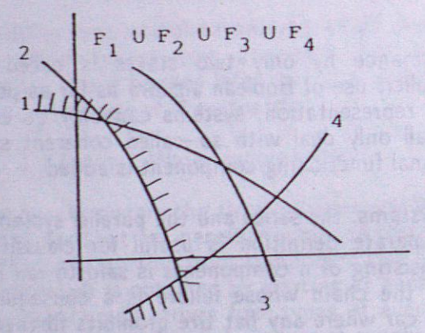


Fig. 1.5: Series system

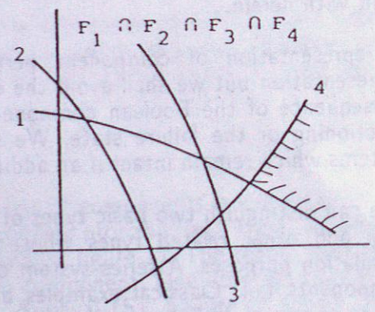


Fig. 1.6: Parallel system

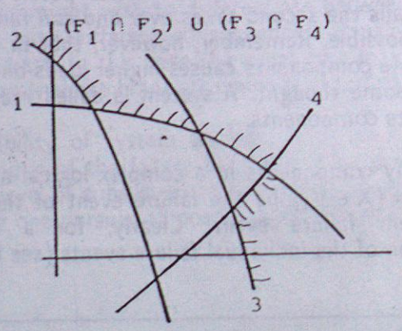


Fig. 1.7: Parallel systems in series

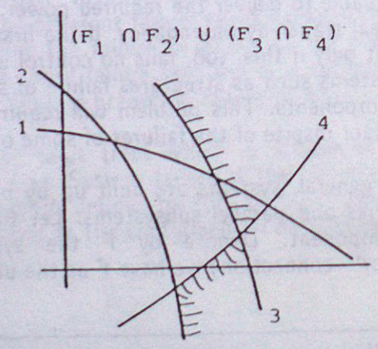


Fig. 1.8: Series systems in parallel

$$F_s = \cup F_i \tag{1.1}$$

while for the parallel system ("and"–connection) F is the intersection of the F_i's (figure 1.2)

$$F_p = \cap F_i \tag{1.2}$$

Correspondingly, for parallel systems in series (unions of intersections) we have (figure 1.3)

$$F = \cup \cap F_{ij} \tag{1.3}$$

whereas series systems in parallel (intersections of unions) are desribed by (figure 1.4)

$$F = \cap \cup F_{st} \tag{1.4}$$

For convenience, the same systems are also presented in figures 1.5 to 1.8 when the F_i's are given by certain domains ion the space of uncertain basic variables $X = (X_1, X_2)$.

Of utmost importance in reliability theory is the fact that any system can be reduced to either of the two forms in making extensive use of the distributive laws of set algebra

$$F_i \cap (F_j \cup F_k) = (F_i \cap F_j) \cup (F_i \cap F_k) \tag{1.5}$$

$$F_i \cup (F_j \cap F_k) = (F_i \cup F_j) \cap (F_i \cup F_k) \tag{1.6}$$

Furthermore, essential reductions are usually possible by applying the so–called absorption rules, i.e. for $F_i \subseteq F_j$ there is

$$F_i \cup F_j = F_j \quad \text{and} \quad F_i \cap F_j = F_i \tag{1.7a}$$

or for $F_i \subset F_j$ and $F_k \subset F_j$ there is

$$(F_i \cup F_k) \subset F_j \text{ and } (F_i \cap F_k) \subset F_j \tag{1.7b}$$

It also follows that the union or intersection of an event with itself is the event. The absorption rules are important when making certain sets "minimal". If, in particular, eq. (1.3) is a minimal set it is denoted by a "minimal cut set". Cut sets are minimal if they contain no other cut set as a genuine subset. Analogously, representation (1.4) is called a "tie set". Such sets are minimal if no tie set contains another tie set as a genuine subset.

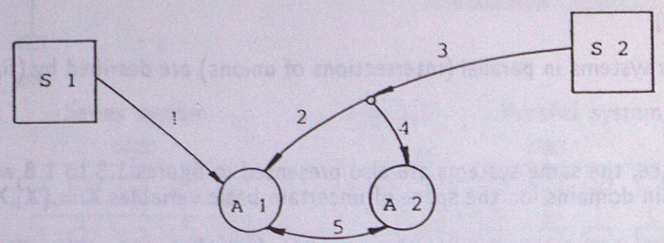
The first step in an analysis of systems is the investigation of the logical structure of the interaction of its components. In a second step which usually is a more formal one the reduction to a minimal form is carried out. Only the third step includes probability evaluations. A forth step may then be added which includes the determination of sensitivity and importance measures of parameters, components and subsystems, respectively. The first step involves classical engineering evaluations and, probably, is the most difficult task. It requires much care and experience to model components and the system realistically and, in a reliability sense, completely. It should be obvious that this modelling phase must not be done without due consideration of the various consecutive steps. It cannot be a subject of these considerations. The second step will be highly formalized. A few hints will be given subsequently and the reader is referred to the vast literature in this area for additional information. The third and to a lesser degree the forth step will be the main subject of these notes.

Illustration 1.1: Water supply system

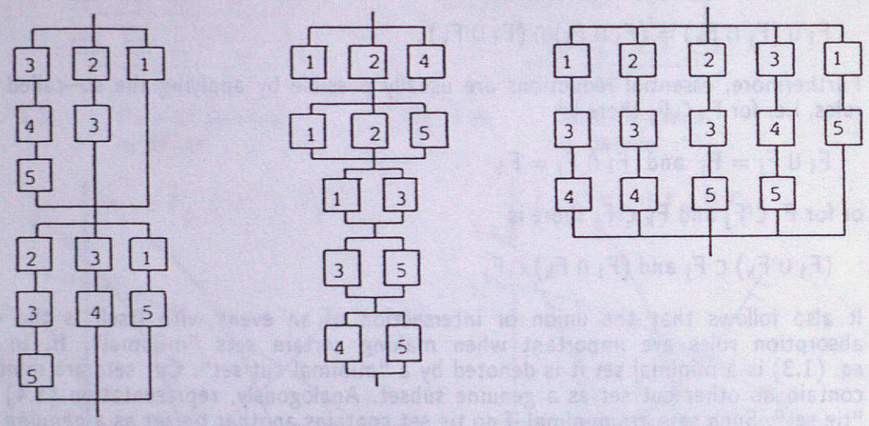
We consider a very simple life–line system as shown in figure 1. Two sources S1 and S2 supply two consumers (town areas) A1 and A2. The arrows indicate the possible direction of flow. The system fails, e.g. at an extraordinary event such as flood, earthquake or war

action, if one of the two consumers is no more supplied. Here, it is easy to write down all possible connections leading to system failure.

$$F = \{[F_1 \cap (F_2 \cup F_3) \cap (F_3 \cup F_4 \cup F_5)] \cup$$
$$\cup [(F_1 \cup F_5) \cap (F_3 \cup F_4) \cap (F_2 \cup F_3 \cup F_5)]\} \qquad (1)$$



System failure is when A1 or A2 are not supplied



System representations

The system can also be represented in terms of a block diagram in which one can easily recognise the logical structure. As in eq. (1) one considers the supply of A1 and A2 separately. For example, A1 is not supplied if line 1 "and" line 2 "or" 3 "and" line 3 "or" 4 "or" 5 are broken.

This system of events in eq. (1) is not yet minimal. One may now apply the laws (1.5) and (1.6) and obtains for the supply of A1

$$F_{A1} = \{(F_1 \cap F_2 \cap F_3) \cup (F_1 \cap F_2 \cap F_4) \cup (F_1 \cap F_2 \cap F_5) \cup$$
$$\cup (F_1 \cap F_3 \cap F_4) \cup (F_1 \cap F_3 \cap F_5)\} \qquad (2)$$

(Carry out all "cuts" in the upper half of the block diagram which make the system fail). Next the absorption laws are applied. First all multiple events in a cut set are deleted except one. Next, multiple cut sets are deleted exept one. Finally, all cut sets which are subsets of other cut sets are deleted. In doing so one arrives at

$$F = \{(F_1 \cap F_3) \cup (F_3 \cap F_5) \cup (F_4 \cap F_5) \cup$$
$$\cup (F_1 \cap F_3 \cap F_4) \cup (F_1 \cap F_3 \cap F_5)\} \qquad (3)$$

Quite analogously, one can produce tie sets. We remember that according to de Morgan's law, it is $A \cap B = \bar{A} \cup \bar{B}$ and $A \cup B = \bar{A} \cap \bar{B}$. Therefore, for the representations (1.3) and (1.4) we have $\cup \cap F_{ij} = \Omega \backslash (\cap \cup \bar{F}_{ij})$ and $\cap \cup F_{ij} = \Omega \backslash (\cup \cap \bar{F}_{ij})$. Hence, having found the minimal cut set for the failure events yields by passing over to the complementary events (by reversing the set operators) the minimal tie set of safe events and vice versa. It is recommended to do this as an exercise. The result is

$$F = \{(F_1 \cup F_5) \cap (F_1 \cup F_3 \cup F_4) \cap (F_2 \cup F_3 \cup F_4) \cap$$
$$\cap (F_2 \cup F_3 \cup F_5) \cap (F_3 \cup F_4 \cup F_5)\} \qquad (4)$$

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2. Formal Logical Analysis of Systems

2.1 EVENT– AND FAILURE TREE ANALYSIS

Complex systems require more formal tools when assessing and reducing the logical structure because a direct analysis can be rather error prone and lengthy.

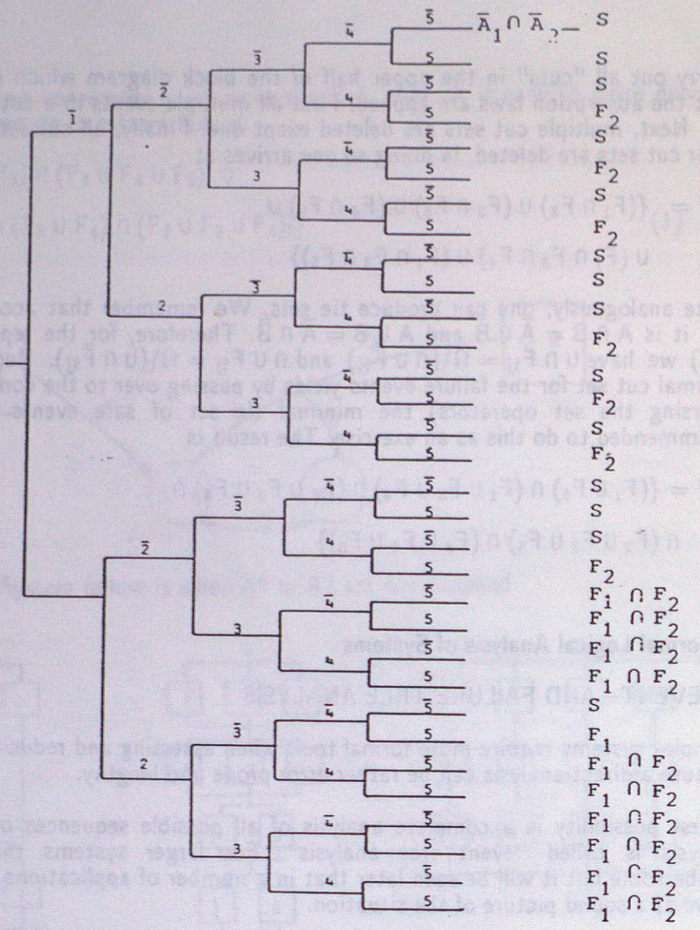
A first possibility is a complete analysis of all possible sequences of events. This type of analysis is called "event tree analysis". For larger systems this can become quite cumbersome but it will be seen later that in a number of applications this is the only way to arrive at a sound picture of the situation.

Illustration 2.1.1 (cont.): Water supply system

In our water supply system the time–sequence of failures of components is irrelevant for the final system states (but not necessarily for the corresponding probabilities). Here, we develop sequences of events starting from component one (compare figure 1). The reader may verify that the same result is obtained by starting at another component. One observes that at the end of each branch one arrives four types of events: Survival (S), Failure of supply of A1 (F_1), Failure of supply of A2 (F_2), Failure of supply of A1 and A2 ($F_1 \cap F_2$).

All survival and failure events constitute an exhaustive, disjoint system of events. In applications one might wish to differentiate between the different failure types because they are associated with different consequences. Usually only the failure branches of the event tree are of interest. In this case the event tree may be called failure tree and it is sufficient to investigate only those branches which lead to failure (failure branches, failure paths).

Another possibility of system analysis is by so–called fault trees which is a backwards analysis technique of the failure branches of an event tree. System failure is the top event. Then, a next level of subsystem and its logical connection is defined by "or" or "and" gates. In this manner one pursues all possibilities until one arrives at the componential basic events.



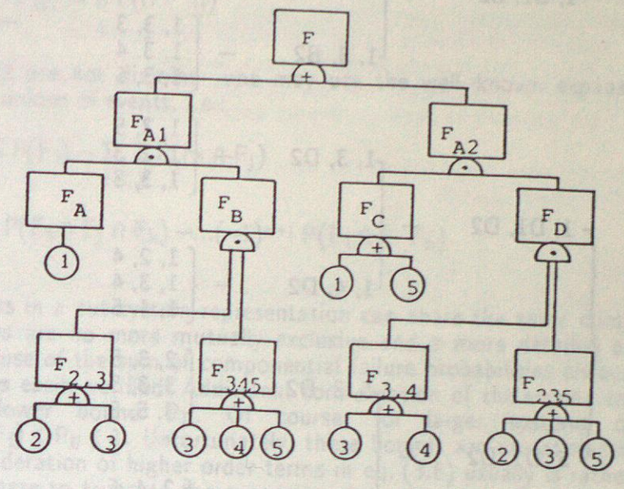
Complete Event Tree

$$
\begin{aligned}
F = \; & \bar{1} \cap \bar{2} \cap \bar{3} \cap 4 \cap 5 \\
& \bar{1} \cap \bar{2} \cap 3 \cap \bar{4} \cap 5 \\
& \bar{1} \cap \bar{2} \cap 3 \cap 4 \cap 5 \\
& \bar{1} \cap 2 \cap \bar{3} \cap 4 \cap 5 \\
& \bar{1} \cap 2 \cap 3 \cap \bar{4} \cap 5 \\
& \bar{1} \cap 2 \cap 3 \cap 4 \cap 5 \\
& 1 \cap \bar{2} \cap \bar{3} \cap 4 \cap 5 \\
& 1 \cap \bar{2} \cap 3 \cap \bar{4} \cap \bar{5} \\
& 1 \cap \bar{2} \cap 3 \cap \bar{4} \cap 5 \\
& 1 \cap \bar{2} \cap 3 \cap 4 \cap \bar{5} \\
& 1 \cap \bar{2} \cap 3 \cap 4 \cap 5 \\
& 1 \cap 2 \cap \bar{3} \cap \bar{4} \cap 5 \\
& 1 \cap 2 \cap \bar{3} \cap 4 \cap \bar{5} \\
& 1 \cap 2 \cap \bar{3} \cap 4 \cap 5 \\
& 1 \cap 2 \cap 3 \cap \bar{4} \cap \bar{5} \\
& 1 \cap 2 \cap 3 \cap \bar{4} \cap 5 \\
& 1 \cap 2 \cap 3 \cap 4 \cap \bar{5} \\
& 1 \cap 2 \cap 3 \cap 4 \cap 5
\end{aligned}
$$

Disjoint Cut Set

Illustration 2.1.2 (cont.): Water supply system

Figure 1 demonstrates the fault tree of our water supply system which in view of the above discussions does not need further explanations.



Fault Tree

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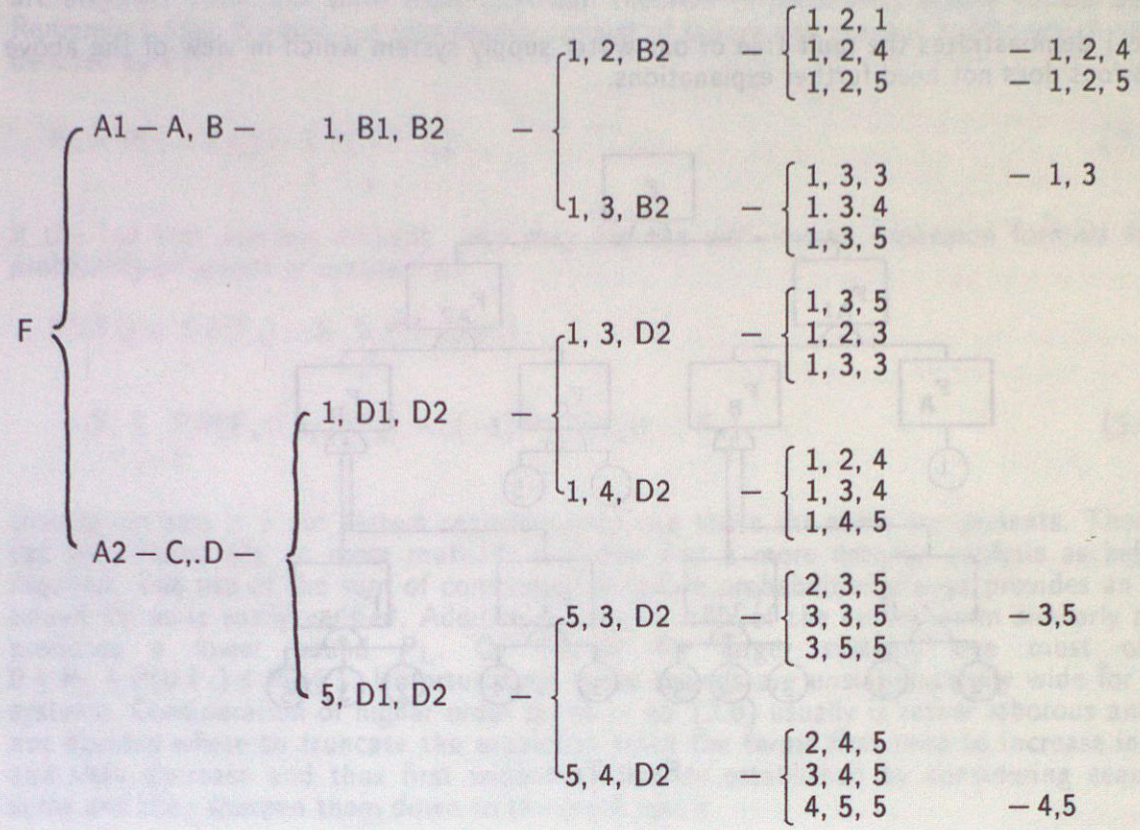
2.2 MINIMAL CUT SETS

Neither the result of the event tree nor of the fault tree has been reduced. Many formal algorithms exist for these reductions but they resemble each other to a large degree. Their differences can frequently only be recognised for very large systems. An account of several methods and some special tasks is given in Yen (1975). Here, we illustrate an algorithm described in Barlow/Proschan (1975) and originally proposed by Fussell (1971).

Illustration 2.2.1: Minimal cut set of water supply system

The algorithm proceeds as follows. Starting with the top event of a fault tree each event of a cut set ("and" connection) is treated as the input into the columns of a row of a matrix. "Or" connections form new rows. Each subsystem in parallel with others consisting of components in series, therefore, produces as many new lines as the subsystem in parallel had components in series. In our example, the development of the full matrix looks as in figure 1.

Herein, the absorptions have been made only at the last step by first deleting all multiple sets except one, and by deleting all those sets containing {1,3}, {3,5} and {4,5} because unions of intersections of larger sets are included in the remaining ones. In practical applications one, of course, might wish to carry out absorptions as early as possible. In a similar manner the dual representation, the minimal tie set, could be derived with some obvious modifications. The production of a minimal tie set of failure events probably is best carried out by starting with the counter–part of a fault tree, the survival tree, from which the minimal cut set of survival events is formed and herefrom, using de Morgan's law, the corresponding tie set. On the other hand tie sets are difficult to handle probabilistically and, therefore, will not be studied furtheron.



Reduction of fault tree to minimal cut set

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3. Elementary probabilistic evaluation of systems – Probability bounds

In this section we compute the failure probability of systems as analysed before under the more or less restrictive assumption that the componential failure events are either independent or fully dependent.

For a series system ("or"–connection) we have for independent events in passing to the complementary events

$$P_{f,S} = P(\bigcup_i F_i) = 1 - P(\bigcap_i \bar{F}_i)$$
$$= 1 - \prod_i P(\bar{F}_i) = 1 - \prod_i (1 - P(F_i)) \tag{3.1}$$

and, analogously, for the parallel system

$$P_{f,P} = P(\bigcap_i F_i) = \prod_i P(F_i) \tag{3.2}$$

In the fully dependent case it is

$$P_{f,S} = P(\bigcup_i F_i) = \max_i \{P(F_i)\} \tag{3.3}$$

$$P_{f,P} = P(\bigcap_i F_i) = \min_i \{P(F_i)\} \tag{3.4}$$

Systems with more complex structure are more difficult to handle. Assume that the cut sets are disjoint. Then, the third Kolmogorovian theorem of probability theory applies directly. Remember, that disjoint cut sets usually consist of failure and survival events which here are denoted by F_i^*.

$$P_f = P(\dot{\cup}_i \cap_j F^*_{ij}) = \sum_i P(\cap_j F^*_{ij}) \tag{3.5}$$

If the cut sets are not disjoint one may use the well–known expansion formula for the probability of unions of events, i.e.:

$$P(\cup_i F_i) = \sum_i P(F_i) - \sum_{i<j} P(F_i \cap F_j)$$

$$+ \sum_{i<j<k} P(F_i \cap F_j \cap F_k) - \ldots (-1)^{n+1} P(F_1 \cap \ldots F_n) \tag{3.6}$$

Several cut sets in a cut system representation can share the same components. Therefore, cut set failures are no more mutually exclusive and a more detailed analysis as before is required. The use of the sum of componential failure probabilities always provides an upper bound P_U as is easily verified. Additional consideration of the second term similarly always produces a lower bound P_L. Of course, for larger systems one must observe $0 \leq P_L \leq P(\cup F_i) \leq P_U \leq 1$. Unfortunately, these bounds are unsiatisfactorily wide for larger systems. Consideration of higher order terms in eq. (3.6) usually is rather laborous and it is not obvious where to truncate the expansion since the terms first tend to increase in value and then decrease and thus first widen the bounds established by considering sequential sums and then sharpen them down to the exact result.

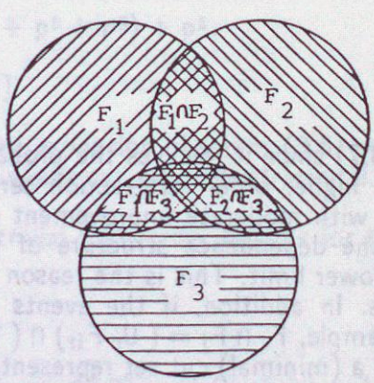

Figure 3.1: Derivation Ditlevsen's bounds

It is possible, however, to derive simple bounds of increasing order and increasing narrowness. The idea can be deduced from figure 3.1. For the first two events we have

$$P(F_1 \cup F_2) = P(F_1) + P(F_2) - P(F_1 \cap F_2)$$

For the third event in a union, an upper bound is obtained if the intersection with the larger probability is substacted, i. e. $P(F_1 \cap F_3)$ or $P(F_2 \cap F_3)$, from the additional term $P(F_3)$. A lower bound is to substract the sum of these probabilities provided that they are not larger than $P(F_3)$. Repeated application of this scheme for more than three events yields:

$$P(\cup_{i=1}^{n} F_i) = \begin{cases} \leq P(F_1) + \sum_{i=2}^{n} \{P(F_i) - \max_{j<i} \{P(F_i \cap F_j)\} \\ \geq P(F_1) + \sum_{i=2}^{n} \max \{0, P(F_i) - \sum_{i=2}^{n} P(F_i \cap F_j)\} \end{cases} \tag{3.7}$$

This elementary result has been derived repeatedly, in one or the other forms, e.g. by Konias (1966), Hunter (1976,1977), Ditlevsen (1979). The narrowness of these bounds depends on the ordering of the events. A different ordering may be necessary for the upper and the lower bound. An algorithm for a best ordering is given by Dawson/Sankoff (1967). Hohenbichler (1980) (see also Hohenbichler/Rackwitz (1983)) generalised these bounds to include more than two–dimensional intersections (but not all higher dimensional intersections). It was found by numerical studies that little is gained by those extensions except for smaller (!) systems. The (two–dimensional) bounds become exact for fully dependent events, are very close to the exact result for independent events, become less satisfactory with increasing number of events but are generally of high quality for small probability events.

Irrespective of the dependence structure of the events, we have, as mentioned shortly after eq. (3.6), the so–called trivial bounds

$$\max_{i=1}^{n} \{P(F_i)\} \leq P(\bigcup_{i=1}^{n} F_i) \leq \sum_{i=1}^{n} P(F_i) \leq 1 \tag{3.8}$$

for the series system and

$$0 \leq P(\bigcap_{i=1}^{n} F_i) \leq \min_{i=1}^{n} \{P(F_i)\} \tag{3.9}$$

for the parallel system being direct consequences of the elementary probability theorem. The bounds (3.8) become obsolete for larger systems and/or larger individual event probabilities. The bound (3.9) generally is of little use. Slightly better is the following obvious relationship

$$0 \leq P(\bigcap_{i=1}^{n} F_i) \leq \min_{j<i=1}^{n} \{P(F_i \cap F_j)\} \tag{3.10}$$

which is of the same nature as eq. (3.7) since it involves the probability of the intersection of any two events. As for eq. (3.7), higher order intersection terms could be included in bounds of the type of eq. (3.10) but with moderate improvement of the upper bound and, without further information about the dependence structure of the componential failure events, no possibility to sharpen the lower limit. This is the reason why we shall not further investigate tie sets of failure events. In addition, if the events F_i and F_j in eq. (3.10) correspond to unions we have, for example, $F_i \cap F_j = (\bigcup_r F_{ir}) \cap (\bigcup_s F_{js})$ whose probability in turn is only easily computed if first a (minimal) cut set representation is found and one of the formula (3.5) to (3.7) are applicable.

Example 3.1: Application to a water supply system (cont.)

The foregoing schemes are now used to compute failure probabilitties for the water supply system introduced in section 2.1. For the complete (disjoint) event tree the calculation of the failure probability is very easy but laborous. With $p_i = P(F_i) = p$, $q_i = 1 - p_i = q$, and independence of the events, it is for those branches which lead to system failure

$$P(F) = q^3p^2 + q^3p^2 + \ldots + p^5 = p^2(3 - 4p^2 + 2p^3) \tag{1}$$

The second exact way to compute the failure probability is by eq. (3.6). We start with the minimal cut set representation in illustration 2.1.1 but any other representation of unions of intersection could be used. Expansion yields:

$$P(F) = P((F_1 \cap F_2 \cap F_4) \cup (F_1 \cap F_2 \cap F_5) \cup (F_1 \cap F_3) \cup (F_3 \cap F_5) \cup (F_4 \cap F_5))$$

$$= P(F_1 \cap F_2 \cap F_4) + P(F_1 \cap F_2 \cap F_5) + P(F_1 \cap F_3) + P(F_3 \cap F_5) + P(F_4 \cap F_5)$$

$$- (P(F_1 \cap F_2 \cap F_4 \cap F_5) + P(F_1 \cap F_2 \cap F_3 \cap F_4) + P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5)$$

$$+ P(F_1 \cap F_2 \cap F_4 \cap F_5) + P(F_1 \cap F_2 \cap F_3 \cap F_5) + P(F_1 \cap F_2 \cap F_3 \cap F_5)$$

$$+ P(F_1 \cap F_2 \cap F_4 \cap F_5)$$

$$+ \ldots + P(F_3 \cap F_4 \cap F_5)$$

$$+ (P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5) +$$

$$+ \ldots + P(F_1 \cap F_3 \cap F_4 \cap F_5)) - (P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5) +$$

$$+ \ldots + P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5)) + P(F_1 \cap F_2 \cap F_3 \cap F_4 \cap F) \qquad (2)$$

Of course, all intersections are reduced, i.e. the first joint intersection $P((F_1 \cap F_2 \cap F_4) \cap (F_1 \cap F_2 \cap F_5))$ to $P(F_1 \cap F_2 \cap F_4 \cap F_5)$ and similarly for all others. A rather lengthy computation gives:

$$P(F) = (p^3 + p^3 + p^2 + p^2 + p^2)$$

$$- (p^4 + p^4 + p^5 + p^4 + p^4 + p^4 + p^4 + p^3 + p^4 + p^3)$$

$$+ (p^5 + p^5 + p^4 + p^5 + p^5 + p^5 + p^4 + p^5 + p^5 + p^4)$$

$$- (p^5 + p^5 + p^5 + p^5 + p^5) + p^5$$

$$= p^2(3 - 4p^2 + 2p^3) \qquad (3)$$

One can observe that the expansion theorem (3.6) requires the evaluation of 31 cut sets which is more than the disjoint cut set of illustration 2.11 in which one counts 18 cut sets.

On the other hand, the trivial first–order bounds for the minimal cut set system according to eq. (3.8) are

$$p^2 \leq P(F) \leq 3p^2 + 2p^3 \qquad (4)$$

The second–order bounds eq. (3.7) read (without ordering and using representation (3) of illustration 2.1.1):

$$p^2 + (p^2 - \max\{0, p^3 + p^4 + p^3\}) + (p^2 - \max\{0, p^4 + p^4\}) + (p^3 - p^4) + p^3$$

$$= 3p^2 - 4p^4 \leq P(F) \leq p^2 + (p^2 - p^3) + (p^2 - p^4) + (p^3 - p^4) + p^3$$

$$= 3p^2 + p^3 - 2p^4 \qquad (5)$$

In order to get some feeling about the numbers we set p=0.1. The exact result then is P(F)=0.02962. The trivial bounds become $0.02 \leq P(F) \leq 0.032$ while the second–order bounds are $0.0296 \leq P(F) \leq 0.0308$. The upper bound for the tie set representation as in eq. (2) of illustration 2.1.1 is 0.19 which demonstrates its inadequacy.

#

4. First–order reliability methods for system analysis (FORM)

4.1 COMPONENTS

Let $U = (U_1,...,U_n)^T$ be an independent standard normal vector and the failure domain be given as $V = \{g(U) \leq 0\} = \{\alpha^T U + \beta \leq 0\}$ with $g(0) > 0$ and $\|\alpha\| = 1$. Then, the exact failure probability is

$$P_f = P(F) = P(U \in V) = \Phi(-\beta) \tag{4.1.1}$$

$u^* = -\beta\,\alpha$ the β–point, α its vector of direction cosines and β the safety index $\beta = +\|u^*\|$.

The standard normal integral $\Phi(c)$ may be determined by one of the expansions given in the literature. Note that we distinguished in eq. (4.1.1) the failure event F and the failure domain V. This is formally correct but at the moment not really necessary. Therefore, we shall use P(V) instead of $P(U \in V)$ for P(F) in the sequel.

Next, we generalize this result for non–linear, differentiable failure surfaces $g(u) = 0$ by expanding it to first–order in the so–called β–point which, for the moment, is defined as the minimal distance of points u on $g(u) = 0$ to the coordinate origin. For β we use the convention

$$\beta = \begin{cases} +\|u^*\| & \text{for } g(0) > 0 \\ -\|u^*\| & \text{for } g(0) \leq 0 \end{cases} \tag{4.1.2}$$

while

$$\|u^*\| = \min\{\|u\|\} \text{ for } \{u:\, g(u) \leq 0\} \tag{4.1.3}$$

Finding u^* is a problem of optimization (minimizing $\|u\|$) under an inequality constraint. The inequality condition is written here only as a reminder that the failure surface $g(u) = 0$ resp. the failure domain $V = \{g(U) \leq 0\}$ is assumed to be non–degenerate, i.e. that the failure set in a sufficiently small neighbourhood of u^* is non–empty and has non–zero probability. Practically, the equality condition is sufficient. The search for u^* will be discussed later. Here, it is further assumed that $g(u^*)$ possesses all first–order derivatives so that a tangent linear approximation of the failure surface is uniquely defined. Intuitively, this expansion point is an obvious one since the multi–normal density drops off with $\exp[-1/2\,\|u\|^2]$ and in the β–point the function $\|u\|$ is not only minimum. Simultaneously, one has $\max\{\varphi(u)\}$ for u^*. As a consequence, u^* is also denoted as the "most likely failure point". In the non–linear case, one, therefore, has

$$P_f \approx \Phi(-\beta) \tag{4.1.4}$$

with, for the moment, unquantifiable error. But it is emphasised already here, that the approximation is widely acceptable. Again, $u^* = -\beta\alpha$ where $\alpha = g(u^*)/\|g(u^*)\|$ the normalized gradient of $g(u^*) = 0$ in u^* and the linear approximation to $g(u) = 0$ is $h(u) = \alpha^T(u - u^*) = \alpha^T u + \beta = 0$.

4.2 UNIONS AND INTERSECTIONS

The failure event (domain) can also be given as a union or intersection of individual (componential) failure domains. Let

$$V = \bigcap_{i=1}^{m} V_i$$

with $V_i = H_i = \{\alpha_i^T U + \beta_i \le 0\} = \{Z_i \le -\beta_i\}$. If, originally, the individual failure domains V_i are bounded by non–linear failure surfaces, we understand that these failure domains have been replaced by linearly bounded half–spaces H_i as described in section 4. The covariance matrix for the vector $\mathbf{Z}$ is given by $\Sigma_Z = \{\alpha_i^T \alpha_j \; ; \; i,j = 1,...,m\}$ which is equal to the correlation coefficient matrix $\mathbf{R}$ because $\mathbf{Z}$ is a zero mean, unit variance vector. The failure probability becomes (Hohenbichler/Rackwitz, 1983):

$$P_f = P(\bigcap_{i=1}^{m} \{Z_i \le -\beta_i\}) = \Phi_m(-\beta \; ; \; \mathbf{R}) \qquad (4.2.1)$$

Φ_m is the multinormal integral. Similarly, for a union of events

$$V = \bigcup_{i=1}^{m} V_i$$

we have

$$P_f = P(\bigcup_{i=1}^{m} \{Z_i \le -\beta_i\}) = 1 - P(\bigcap_{i=1}^{m} \{Z_i > -\beta_i\})$$

$$= 1 - P(\bigcap_{i=1}^{m} \{Z_i \le \beta_i\}) = 1 - \Phi_m(\beta \; ; \; \mathbf{R}) \qquad (4.2.2)$$

The numerical evaluation of the standard multinormal integral $\Phi_m(\mathbf{c} \; ; \; \mathbf{R})$ is essential for eqs. (4.2.1) and (4.2.2) to be of any practical use. Unfortunately, simple general and exact results do not exist. The fact that the approximate evaluation of the general case just on the basis of the first–order reliability methodis possible must be viewed as one of the prominent applications of this approach.

4.3 CUT SET SYSTEMS

As outlined in section 2 more general failure domains (systems) must be given either in terms of a disjoint or (minimal) cut set representation. In the first case the cuts can also contain safe events and no additional difficulties will be met. The system failure probability is simply the sum of all cut probabilities. If, for example, a cut set is given by $\{V_i, V_j\}$ where $V_i = \{g_i(\mathbf{U}) \le 0\}$ and $V_j = \{g_j(\mathbf{U} > 0\}$ it is clear that by multiplying $g_j(\mathbf{U})$ with (-1) one obtains $V_j = \{-g_j(\mathbf{U} \le 0\}$ as required for formulae (4.1) and (4.2). In contrast to our general assumption V_j now is a large probability event but its intersection with small probability events V_i may still yield small joint probabilities. In a first–order context the multiplications of $g_j(\mathbf{U})$ by (-1) results in a sign–change of the original correlation coefficient.

If the system is represented by minimal cut sets one straight forward calculation method is to use the Ditlevsen bounds as derived in section 3 in eq. (3.7). They require the evaluation of the intersection of any two intersections of failure domains, i.e. the probabilities

$$P(V_i \cap V_j) = P(\bigcap_{r=1}^{m_i} V_r \cap \bigcap_{s=1}^{m_j} V_s) = \Phi_{m_i+m_j}(\beta_{i+j}; \mathbf{R}_{i+j}) \qquad (4.3.1)$$

with $\beta_{i+j} = (\beta_1,...,\beta_{m_i},\beta_{m_i+1},...,\beta_{m_i+m_j})^T$ and

$$\mathbf{R}_{I \setminus J} = \{\alpha_p^T \alpha_q \; ; \; p,q = 1,...,m_i, m_i + 1,...,m_i + m_j\}$$

Alternatively, a formally exact result is also obtained when applying the expansion theorem (4.6) but the numerical effort may become great for larger systems. The same is true for the calculation of disjoint cut set probabilities so that, in practice, one might prefer the bounds (4.7). These, in turn, may be weakened in that, after arranging the cut sets according to their (descending) probabilities, the intersection probabilities of any two cut sets are only computed for the k first few dominating sets while the rest of the cut sets is taken into account by either their upper bound or their lower bound. Larger systems can actually require rather detailed considerations as concerns the reasonable numerical effort involving also the valuation of the various approximations. They cannot be presented herein.

Illustration 4.3.1 (cont.): Water supply system

We are ready to apply the above results to the water supply system discussed in section 2. Assume that the componential failure events are now given by $V_i = \{X_i - Y \leq 0\}$ where the X_i represent some "resistance" variables which are assumed to be independent and normally distributed with mean $m_i = m$ and standard deviation $\sigma_i = \sigma$. Y is a normal "loading" variable with mean μ and standard deviation τ. Therefore, the componential failure probabilities are

$$P(F_i) = P(V_i) = P(X_i - Y \leq 0) = P(Z_i \leq -\beta_i) = \Phi(-\beta_i) \qquad (1)$$

It follows that $\beta = \beta_i = (m - \mu)(r\sigma^2 + \tau^2)^{-1/2}$ and $\rho = \rho_{ij} = \mathrm{Cov}[Z_i, Z_j] = \tau^2(\sigma^2 + \tau^2)^{-1}$. Let the parameters be chosen such that $\beta = 3$ and $\rho = 0.5$. The state variables are represented by

$$Z_i = \sqrt{\rho}U + \sqrt{1-\rho}U_i \qquad (2)$$

By conditioning first on the variable $U = u$ one recognizes that the variables Z_i are conditionally independent and the results of section 3 apply. In particular, it can easily be verified that

$$P(F) = \int_{-\infty}^{+\infty} [3p^2(u) - 4p^4(u) + 2p^5(u)]\, \varphi(u)\, du \qquad (3)$$

with $p(u) = \Phi((-\beta - \sqrt{\rho}u)/(1-\rho)^{1/2})$ and the term in square brackets the exact system failure probability if the components were independent. Numerical integration yields $P(F) = 2.31\ 10^{-4}$ with component probability $P(F_i) = 1.35\ 10^{-3}$. Note that $P(F) = 5.47\ 10^{-6}$ would have been obtained if the F_i's were independent. This emphasises the significance of stochastic dependencies among componential failures.

The same system is used to illustrate the material in this section. One determines $P_{f.ik} = P(F_i \cap F_k) = 8.19 \cdot 10^{-5}$ and $P_{f.ijk} = P(F_i \cap F_j \cap F_k) = 1.51 \cdot 10{-}5$. The trivial bounds for the system failure probability eq. (3.8) become

$$\max\{P_{f.k}\} = 8.19\ 10^{-5} \leq P_f \leq 2.76\ 10^{-4} = \sum_{k=1}^{5} P_{f.k} \qquad (4)$$

Ditlevsen's bounds require the probabilities of the intersections of any two cut sets in the system. These probabilities are collected in the following matrix $\underset{=}{P} = \{p_{ij}: i,j = 1,\ldots,5\}$:

$$
\mathbf{P} = \begin{vmatrix}
1.51\cdot10^{-5} & 4.65\cdot10^{-6} & 4.65\cdot10^{-6} & 1.90\cdot10^{-6} & 4.65\cdot10^{-6} \\
 & 1.51\cdot10^{-5} & 4.65\cdot10^{-6} & 4.65\cdot10^{-6} & 4.65\cdot10^{-6} \\
 & & 8.19\cdot10^{-5} & 1.51\cdot10^{-5} & 4.65\cdot10^{-6} \\
 & \text{symm.} & & 8.19\cdot10^{-5} & 1.51\cdot10^{-5} \\
 & & & & 8.19\cdot10^{-5}
\end{vmatrix}
$$

For example, the element $p_{12} = P((F_1 \cap F_2 \cap F_4) \cap (F_1 \cap F_2 \cap F_5)) =$
$= P(F_1 \cap F_2 \cap F_4 \cap F_5) = \Phi_4(\{-3\};\{0.5\}) = 4.65\cdot10^{-6}$. Then, the sharper bounds eq. (3.7) give

$$2.26\ 10^{-4} \leq P_f \leq 2.42\ 10^{-4}$$

with essentially the same numerical values whatever sequence of the five cut sets is considered. These bounds are appreciably narrower than the trivial bounds and, of course, contain the exact result.

#

4.4 PROBABILITY DISTRIBUTION TRANSFORMATIONS

The foregoing results are very special as they only apply to independent standard normal variates. However, if the distribution of the original basic variables are continuous it is always possible to find a probability distribution transformation

$$\mathbf{X} = \mathbf{T}(\mathbf{U}) \tag{4.4.1}$$

such that

$$P_f = P(h(\mathbf{X}) \leq 0) = P(h(\mathbf{T}(\mathbf{U})) \leq 0) = P(g(\mathbf{U}) \leq 0) \tag{4.4.2}$$

where we used the abbreviation $h(\mathbf{T}(\mathbf{U})) = g(\mathbf{U})$. Such transformations are well known from simulation. Remember that if a random number generator is available producing uniformly distributed variables G_i in [0,1], then, we use the identity $P(G_i \leq g) = F_G(g) = F_X(x) = P(X_i \leq x)$ to produce random numbers for the variable X. By solving for X_i we obtain $X_i = F_X^{-1}[G_i]$ as random numbers distributed according to F_X. A similar concept is applied to eq. (4.4.1). Let $\mathbf{X}$ be an independent vector with marginal distribution functions $X_i \sim F_i(x)$. It follows that the identity

$$F_i(x_i) = \Phi(u_i) \tag{4.4.3}$$

holds and, therefore (Rackwitz/Fiessler,1978),:

$$X_i = F_i^{-1}[\Phi(U_i)] \tag{4.4.4a}$$

or

$$U_i = \Phi^{-1}[F_i(X_i)] \tag{4.4.4b}$$

Illustration 4.4.1: Transformation of Gumbel–distributed variables

The Gumbel–distribution is given by

$$F(x) = \exp[-\exp[-\alpha(x-u)]] \tag{1}$$

where α is a dispersion– and u a location parameter. Applying eqs. (3.4.4) we have

$$X = T(U) = u - \ln(-\ln\Phi(U))/\alpha \tag{2}$$

$$U = T^{-1}(X) = \Phi^{-1}[\exp[-\exp[-\alpha(X-u)]]] \tag{3}$$

#

The multidimensional dependent case is more involved. If $\mathbf{X}$ has distribution function $F_{\mathbf{X}}(x) = P(\mathbf{X} \le x) = P\left(\cap_{i=1}^{n}\{X_i \le x_i\}\right)$, then it is always possible to represent this distribution function as a product of conditional distribution functions, i.e.

$$F_{\mathbf{X}}(x) = F_1(x_1)\, F_2(x_2|x_1) \ldots F_n(x_n|x_1,\ldots,x_{n-1}), \text{ where}$$

$$F_i(x_i|x_1,\ldots,x_{i-1}) = \int_{-\infty}^{x_i} \frac{f_i(x_1,\ldots,x_{i-1},s)}{f_{i-1}(x_1,\ldots,x_{i-1})}\, ds$$

and

$$f_j(x_1,\ldots,x_j) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f_{\mathbf{X}}(x_1,\ldots,x_j,s_{j+1},\ldots,s_n)\, ds_{j+1}\ldots ds_n$$

This elementary result is used to construct a transformation which has been proposed by Hohenbichler/Rackwitz (1981) following an idea by Rosenblatt (1952). It will be denoted by the Rosenblatt–transformation in the sequel. We transform sequentially using the identities

$$\Phi(u_1) = F_1(x_1) \tag{4.4.5a}$$
$$\Phi(u_2) = F_2(x_1|x_2) \tag{4.4.5b}$$
$$\vdots$$
$$\Phi(u_n) = F_n(x_n|x_1,\ldots,x_n) \tag{4.4.5n}$$

Hence,

$$\mathbf{X} = \mathbf{T}(\mathbf{U}) = (T_1(U_1), T_2(U_1,U_2),\ldots,T_n(U_1,\ldots,U_n))^T \tag{4.4.6}$$

or

$$X_1 = F_1^{-1}[\Phi(U_1)] \tag{4.4.6a}$$
$$X_2 = F_2^{-1}[\Phi_2)|F_1^{-1}[\Phi(U_1)]] \tag{4.4.6b}$$
$$X_3 = F_3^{-1}[\Phi(U_3)|F_2^{-1}[\Phi(U_2)|F_1^{-1}[\Phi(U_1)]], F_1^{-1}[\Phi(U_1)]] \tag{4.4.6c}$$

and the inverse transformation

$$U = \mathbf{T}^{-1}(\mathbf{X}) = (T_1^{-1}(X_1), T_2^{-1}(X_1,X_2),\ldots,T_{n-1}(X_1,\ldots,X_n))^T \tag{4.4.7}$$

$$U_1 = \Phi^{-1}[F_1(X_1)] \tag{4.4.7a}$$
$$U_2 = \Phi^{-1}[F_2(X_2|X_1)] \tag{4.4.7b}$$
$$U_3 = \Phi^{-1}[F_3(X_3|X_1,X_2)] \tag{4.4.7c}$$

In words: In the first step the first variable is transformed. In the second step all variables conditioned on the first are transformed, and so forth.

Illustration 4.4.2: Exponential distribution with uncertain parameter

Let X have an exponential distribution

$$F_X(x) = 1 - \exp[-\lambda x] \tag{1}$$

but with the parameter $\lambda = 1/E[X]$ unknown. Bayesian statistics tell us that given a sample of size m the posterior distribution of the uncertain parameter Λ is gamma

$$F_\Lambda(\lambda;p,q) = \frac{\gamma(p,\lambda q)}{\Gamma(p)} \tag{2}$$

For the conditional distribution of X we write $F_\chi(x|\lambda)$. Therefore, with $X^T = (\Lambda,X) = (X_1,X_2)$ we have

$$X_1 = \gamma^{-1}(p,\Phi(U_1)\Gamma(p))/q \tag{3}$$

$$X_2 = -\frac{\ln\ \Phi(-u_2)}{\gamma^{-1}(p,\Phi(u_1)\Gamma(p))/q} \tag{4}$$

#

Illustration 4.4.3: Correlated normal variables

Let $\mathbf{X} \sim N_n(\mathbf{m};\Sigma)$. We first standardise $\mathbf{X}$ by applying $Y_i = (X_i - m_i)/\sigma_{ii}$ so that $\mathbf{Y} \sim N_n(\mathbf{0};\mathbf{R})$ with $\rho_{ij} = \sigma_{ij}/(\sigma_{ii}\sigma_{jj})^{1/2}$. We now transform according to

$$\mathbf{Y} = \mathbf{A}\,\mathbf{U} \tag{1}$$

where $A = \{a_{ij}\ ;\ 1 \le i,j \le n\}$ and $a_{ji} = 0$ for $j > i$. The a_{ij}'s are determined from $\mathrm{Var}[Y_i] = \sum_{k=i}^{i} a_{ik}^2 = 1$ and $\mathrm{Cov}[Y_i,Y_j] = \sum_{k=i}^{i} a_{ik}a_{jk} = \rho_{ij}$. Clearly, $a_{11} = \rho_{11} = 1$. One finds with $q_{ii} = 1$.

$$a_{i1} = \rho_{i1}\ ;\ 2 \le i \le n \tag{2a}$$

$$a_{ii} = (\rho_{ii} - \sum_{k=1}^{i-1} a_{ik}^2)^{1/2}\ ;\ 2 \le i \le n \tag{2b}$$

$$a_{ij} = (q_{ij} - \sum_{k=1}^{j-1} a_{jk})/a_{jj}\ ;\ 1 < j < i \le n \tag{2c}$$

Obviously, the Rosenblatt–transformation precisely corresponds to Cholesky's triangularization procedure for symmetric, positive definite matrices. If $\mathbf{Y}$ is only centralized beforehand the ρ_{ij}'s must be replaced by the σ_{ij}'s in which case we denote the transformation matrix by $\mathbf{C}$. Hence, the complete transformation is

$$\mathbf{X} = \mathbf{C}\mathbf{U} + \mathbf{m} \tag{3}$$

#

4.5 COMPUTATION OF THE MULTINORMAL INTEGRAL

In the general case one has to evaluate the multi–normal integral. Unfortunately there are only a few analytical solutions. It is, however, possible to derive very good approximations and an asymptotic formula. In view of its many applications to properties of a multi–normal vector are discussed first. The density of the multi–normal vector $\mathbf{Y} = (Y_1,\ldots,Y_n)^T$ is

$$\varphi_n(\mathbf{y}) = (2\pi \det(\Sigma))^{-1/2} \exp[-\tfrac{1}{2}[(\mathbf{y}-\mathbf{m})^T\Sigma^{-1}(\mathbf{y}-\mathbf{m})]] \tag{4.5.1}$$

and after stadardization by $X_i = (Y_i - \mu_i)/\sigma_i$ such that $\Sigma = \mathbf{D}^T\mathbf{R}\mathbf{D}$ with $\mathbf{D} = \text{diag}\{\sigma_i\}$ the diagonal matrix of the standard deviations and $\mathbf{R}$ the matrix of the correlation coefficients:

$$\varphi_n(\mathbf{x}) = 2\pi^{-u/2}(\det(\mathbf{R}))^{-1/2} \exp[-\tfrac{1}{2}[(\mathbf{x}^T\mathbf{R}^{-1}\mathbf{x})] \tag{4.5.2}$$

If $\mathbf{R} = \mathbf{I}$ ($\mathbf{I}$ = unit matrix), the vector $\mathbf{Z}$ is uncorrelated:

$$\varphi_n(\mathbf{z}) = (2\pi)^{-u/2} \exp[-\tfrac{1}{2}\mathbf{z}^T\mathbf{z}] \tag{4.5.3}$$

This also implies independence of the components of the vector. Let now $\mathbf{Y} = \mathbf{AZ} + \mathbf{m}$. The covariances σ_{ij} are

$$\sigma_{ij} = \text{Cov}[Y_i,Y_j] = \sum_{i=1}^{n} a_{ij}\, a_{ik}\, E[X_i^2] - m_j\, m_k$$

Consequently there is $\mathbf{A} = \mathbf{R}$. The multi–normal distribution function can now be written as:

$$\Phi_n(\mathbf{x};\mathbf{R}) = \int_{-\infty}^{x} \varphi_n(\mathbf{t};\mathbf{R})\, d\mathbf{t} \tag{4.5.4}$$

It is symmetric in the sense that:

$$\Phi_n(\mathbf{x};\mathbf{R}) = 1 - \Phi_n(\mathbf{x};\mathbf{R}) \tag{4.5.5}$$

An important property is that for $\{\rho_{ij}\} \leq \{\kappa_{ij}\}$ (Slepian, 1962; Sidak, 1964)

$$\Phi_n(\mathbf{x};\mathbf{R}) \leq \Phi_n(\mathbf{x};\mathbf{K}) \tag{4.5.6}$$

Unfortunately only the two and three dimensional case have simpler solution (Owen, 1956, Steck, 1958). If the variables can be represented by

$$X_i = \lambda_i\, Y_0 + \sqrt{(1-\lambda_i^2)}\, Y_i \tag{4.5.7}$$

where $Y_0,Y_1,\ldots,Y_n$ are independent standard normal variables and, therefore, $\rho_{ij} = \kappa_i\kappa_j$, it is (Dunnet/Sobel, 1955):

$$\Phi_n(\mathbf{x};\mathbf{R}) = \int_{-\infty}^{+\infty} \varphi(y_0) \prod_{i=1}^{n} \Phi\left(\frac{x_i - \lambda_i y_0}{\sqrt{(1-\lambda_i^2)}}\right) dy_0 \tag{4.5.8}$$

For the special case of equicorrelation we have $\sqrt{\rho} = \lambda_i = \lambda_j \geq 0$. On the basis of eq. (4.5.8) bounds can be constructed which, however, are not always sufficiently narrow.

The two dimensional case is needed more frequently. It can be computed as a special case of eq. (4.5.8) or by numerical integration according to

$$\Phi_2(x,y;\rho) = \Phi(x)\,\Phi(y) + \int_0^{\rho} \varphi_2(x,y;t)\,dt \tag{4.5.9}$$

with

$$\varphi_2(x,y;\rho) = \frac{1}{2\pi\,(1-\rho^2)^{1/2}} \exp[-\frac{1}{2}\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}]$$

For the general case the following scheme has been proposed (Hohenbichler/ Rackwitz, 1985). Obviously there is:

$$\Phi_m(\mathbf{c};\mathbf{R}) = P(\bigcap_{i=1}^{n}\{Z_i \le c_i\}) = P(Z_1 \le c_1)\,P(\bigcap_{i=2}^{n}\{Z_i \le c_i\}\,|\,\{Z_1 \le c_i\}) \tag{4.5.10}$$

The Z_i's have the Rosenblatt–transformation

$$Z_i = \sum_{i=1}^{n} a_{ij}\,U_j \tag{4.5.11}$$

with $a_{11} = 1$ and $a_{i1} = \rho_{i1}$. The condition in the second term of eq. (4.5.11) can be removed by observing that it affects only the first variable. The distribution function of a new conditional (truncated) variable $\bar{U}_1$ is for $\bar{U}_1 \le c_1$

$$F_{\bar{U}_1|c_1}(\bar{u}_1) = P(U_1 \le \bar{u}_1 | U_1 \le c_1) = \frac{P(\{U_1 \le \bar{u}_1\} \cap \{U_1 \le c_1\})}{P(U_1 \le c_1)} = \frac{\Phi(\bar{u}_1)}{\Phi(c_1)} \tag{4.5.12}$$

Using $F_{\bar{U}_1|c_1}(\bar{u}_1) = \Phi(u_1)$ with U_1 a new auxiliary standard normal variable

$$\bar{U}_1 = \Phi^{-1}[\Phi(U_1)\Phi(c_1)] \tag{4.5.13}$$

in eq. (4.5.11), one obtains:

$$\begin{aligned}
P(\bigcap_{i=2}^{m}\{Z_i \le c_i | Z_1 \le c_1\} &= P(\bigcap_{i=2}^{m}\{\alpha_{i1}\bar{U}_1 + \sum_{j=2}^{i}\alpha_{ij}U_j \le c_i\}) = \\
&= P(\bigcap_{i=2}^{m}\{\alpha_{i1}\,\Phi^{-1}[\Phi(U_1)\,\Phi(c_1)] + \sum_{j=2}^{i}\alpha_{ij}U_j \le c_i\} \\
&= P(\bigcap_{i=2}^{m}\{g_i(\mathbf{U}) \le 0\} = P(\bigcap_{i=2}^{m}\{\boldsymbol{\alpha}_i^{(2)T}\mathbf{U} \le c_i^{(2)}\} \\
&= \Phi_{m-1}(\mathbf{c}^{(2)};\mathbf{R}^{(2)})
\end{aligned} \tag{4.5.14}$$

such that eq. (4.5.10) can be written as

$$\Phi_m(\mathbf{c};\mathbf{R}) = \Phi(c_1)\,\Phi_{m-1}(\mathbf{c}^{(2)};\mathbf{R}^{(2)}) \tag{4.5.15}$$

Hence the dimension of the multinormal integral has been diminished by one. In line 2 of eq. (4.5.14) one recognizes that only the first variable enters non–linearly. The functions $g_i(\mathbf{u})$ in the third line can be linearized in their respective β–points. Repeated application of this scheme leads to the approximation

$$\Phi_m(c;R) \approx \Phi(c_1)\,\Phi(c_2^{(2)})\ldots\Phi(c_m^{(m)}) \qquad (4.5.16)$$

Several improvements are possible which cannot be discussed here. One special asymptotic result due to Ruben (1964) is given for its simplicity. If all c_i's are negative and the solution of

$$\gamma = R^{-1}c \qquad (4.5.17)$$

results in a vector γ with positive elements than, for $\|c\| \to \infty$,

$$\Phi(-c;R) \sim \varphi(c;R)\,(\det(R))^{1/2}\,(\prod_{i=1}^{n}\gamma_i)^{-1} \qquad (4.5.18)$$

The condition of negative c_i's but positive γ_i's restricts the domain of application to a certain extent, however.

Illustration 4.1: Chain with n links (Ditlevsen, 1982)

Consider a chain with n links whose resistances X_i are independently normally distributed with mean $m = m_i$ and standard deviation $\sigma = \sigma_i$ and which is loaded by a normally distributed load Y with mean μ and standard deviation τ. Failure is for

$$F = \bigcup_{i=1}^{n}\{Z_i \le 0\} = \bigcup_{i=1}^{n}\{X_i - Y \le 0\} \qquad (1)$$

One determines $\beta = (m-\mu)/(\sigma^2+\tau^2)^{1/2}$ and $\rho = \rho_{ij} = \mathrm{Corr}[Z_i,Z_j] = \sigma^2/(\sigma^2+\tau^2)$. Therefore, using the simple correlation structure of the Z_i's and the exact formula for the multinormal integral with equicorrelation one determines

$$P_f = 1 - \Phi_n(\beta\,;\rho) = 1 - \int_{-x}^{-\infty}\varphi(u)\Phi^n\left(\frac{\beta - u\lambda}{\sqrt{1-\lambda^2}}\right)dv \qquad (2)$$

with $\lambda = +\sqrt{\rho}$ and $U_i = (Z_i - (m-\mu))/(\sigma^2+\tau^2)^{1/2}$.

Alternatively, eq.(3.7) can be used. It receives the form

$$\Phi(-\beta_1 + \sum_{i=2}^{n}\max\{0\,,\,\Phi(-\beta_i) - \sum_{j=1}^{i=1}\Phi(-\beta_i\,,-\beta_j\,;\rho_{ij}\} \le P_f$$

$$\le \Phi(-\beta_1) + \sum_{i=2}^{n}\Phi(-\beta_i) - \max_{j=1}^{i-1}\{\Phi(-\beta_i\,,-\beta_j\,;\rho_{ij})\} \qquad (3)$$

where Φ_2 is the two–dimensional normal integral. Ditlevsen (1982) gave the following bounds to Φ_2. For $\rho_{ij} \ge 0$

$$\max\{\Phi(-\beta_i)\Phi(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}})\,,\,\Phi(-\beta_j)\Phi(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}})\} \qquad (4)$$

is a lower bound and the sum of the two terms in brackets an upper bound. For $\rho_{ij} < 0$ the "max" operation has to be changed into "min" to produce an upper bound while the lower bound is zero in this case. For $\rho = \rho_{ij} \ge 0$ and $\beta = \beta_i$ we have

$$\Phi(-\beta)\Phi(-\beta\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}) \le \Phi(-\beta,-\beta;\rho) \le 2\Phi(-\beta)\Phi(-\beta\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}) \quad (5)$$

Application to eq. (3) produces

$$\Phi(-\beta)\max_{i=2}^{n}\{i[1-(i-1)\Phi(-\beta\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}})]\} \le P_f \le \Phi(-\beta)[(n-(n-1)\Phi(-\beta\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}})] \quad (6)$$

The lower bound becomes largest for:

$$i = \text{int}\{(1/\Phi(-\beta\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}) + 1)/2\}$$

#

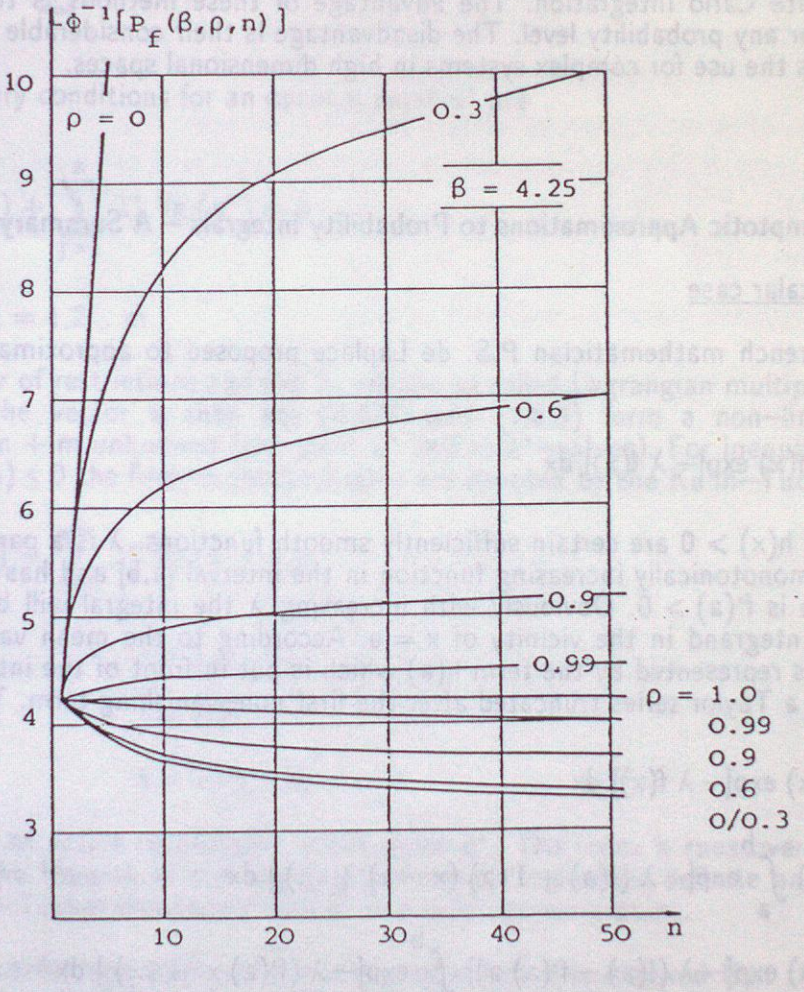

Figure 4.5.1: Failure probability versus number of components and componental correlation coefficients in series and parallel systems

In figure 4.5.1 the influence of the number of components and their correlation in a system is demonstrated. The componental safety index is $\beta = 4.75$. The components are assumed to be equicorrelated. The upper part of the figure presents the results for parallel systems and the lower part for series systems. One can recognize that for series systems neither the number of components in the system nor their correlation is very important unless the

system becomes very large. The contrary is the case for parallel systems. Their failure probability decreases dramatically with the number of components especially for small correlation coefficients. We conclude that the study of parallel systems requires some effort because correlations or more generally dependencies between the components must be taken into account properly.

4.6 Search Algorithms

The mathematical basis for the search for the β–point given only one failure domain (restriction) or a cut set of failure domains (several restrictions) is the existence of an optimum point in the admissible domain. In the first order context, discussed before, the optimum point corresponds to the maximum density of the standard normal vector in the failure domain. Due to the rotational symmetry of the standard space the point is also the point in V or on its boundary closest to the origin. The existence of such a point is defined by Lagrange's theorem. The Lagrangian function is defined as:

$$L(\mathbf{u}) = f(\mathbf{u}) + \sum_{j=1}^{m} \lambda_j \, g_j(u) \tag{4.6.1}$$

and the necessary conditions for an optimal point $\mathbf{u}^x$ are

$$\nabla L(\mathbf{u}^x) = \nabla f(\mathbf{u}^x) + \sum_{j=1}^{m} \lambda_j^x \, \nabla g_j(\mathbf{u}^x) = 0 \tag{4.6.2}$$

$$g_j(\mathbf{u}^x) = 0 \text{ for } j = 1,2....m \tag{4.6.3}$$

m is the number of restrictions and the λ_j are the so called Lagrangian multipliers. If n is the dimension of the vector $\mathbf{u}$ then eqs. (4.6.2) and (4.6.3) form a non–linear system of equations with n + m unknowns (the point $\mathbf{u}^x$ and m λ^x–values). For inequality restrictions in the form $g_j(\mathbf{u}) \leq 0$ the first–order conditions are denoted by the Kuhn–Tucker conditions:

$$\nabla L(\mathbf{u}^x) = \nabla f(\mathbf{u}^x) + \sum_{j=1}^{t} \lambda_j^x \, \nabla g_j(\mathbf{u}^x) = 0 \tag{4.6.4a}$$

$$g_j(\mathbf{u}^x) = 0 \qquad j = 1,2....t \tag{4.6.4b}$$

$$\lambda_j^x \geq 0 \qquad j = 1,2....t \tag{4.6.4c}$$

$$g_k(\mathbf{u}^x) \leq 0 \qquad k = t+1,...m \tag{4.6.4d}$$

t is the number of active restrictions in the point $\mathbf{u}^x$. The index k runs over all non–active restrictions. If the Hessian of the Lagrangian function is positive definite and the point $\mathbf{u}^x$ fulfills the Kuhn–Tucker conditions then $\mathbf{u}^x$ is a local optimum point.

On this basis suitable algorithms can be designed where it is required that they are globally convergent when started from an arbitrary initial point. Furthermore convergence should be sufficiently fast (see Gill et al., 1981; Hock/Schittkowski, 1983 and Arora, 1989). In the following we can only outline the main features of a suitable algorithm.

The following function has to be minimized

$$\mathbf{u}^x = \min\{f(\mathbf{u})\} = \min\{\| \mathbf{u} \|^2\} \tag{4.6.5}$$

given the constraints:

$g_j(u) \leq 0$ für j=1,2.....m (4.6.6)

The Lagrangian function with linearised constraints becomes:

$$L(u,\lambda) = \|u^o\|^2 + 2\, u^{oT}\, \Delta u + \Delta u^T\, \Delta u + \sum_{j=1}^{m} \lambda_j \{g_j(u^o) + \nabla g_j^{oT}\, \Delta u\} \tag{4.6.7}$$

and the Kuhn–Tucker conditions are:

$$\nabla L(u,\lambda) = 2\, u^o + 2\, \Delta u + \sum_{j=1}^{t} \lambda_j\, \nabla g_j^o = 0 \tag{4.6.8a}$$

$$g_j(u) = g_j(u^o) + \nabla g_j^{oT}\, \Delta u = 0 \qquad j=1,2,..t \tag{4.6.8b}$$

Let G be the matrix of the gradients of the (active) constraints and Γ a vector with the values of the constraint functions

$$G = [\nabla g_1^o, ... \nabla g_j^o ..... \nabla g_t^o]_{nxt} \tag{4.6.9}$$

Then the system of equations can be written in matrix form as

$$\begin{bmatrix} 2\,I & G \\ G^T & \emptyset \end{bmatrix} \begin{bmatrix} \Delta u \\ \lambda \end{bmatrix} = \begin{bmatrix} -2\, u^o \\ -\Gamma \end{bmatrix} \tag{4.6.10}$$

Solution of this system yields the following iteration scheme:

$$u^{k+1} = G_k\, (G_k^T\, G_k)^{-1}\, (G_k^T\, u^k - \Gamma_k) \tag{4.6.11}$$

The matrices G are given by:

$$G_k = A_k\, N_k \tag{4.6.12}$$

with

$$A_k = [a_1^k, ... a_j^k, ... a_t^k] \tag{4.6.13}$$

$$a_j^k = \frac{1}{\|\nabla g_j^k\|}\, \nabla g_j^k \tag{4.7.14}$$

and N_k = diagonal matrix with the norms of $\|\nabla g_j^k\|$. With these notations the algorithm can be written as

$$u^{k+1} = A_k\, \Sigma_k^{-1}\, (A_k^T\, u^k - N_k^{-1}\, \Gamma_k) \tag{4.6.15}$$

with $\Sigma_k = A_k^T\, A_k$ the covariance matrix of the linearised (active) constraint functions in the point u^k.

Specialization of this scheme to only one constraint yields the algorithm already given by Hasofer/Lind (1974) and Rackwitz/Fießler (1978). This algorithm is not yet surely convergent. It can easily be made convergent by introducing either a deceleration scheme or a suitable step length procedure (see, for example, Abdo/Rackwitz, 1990). The convergence rate can be made especially high if for this step length procedure at least approximate information about the curvature properties of the Lagrangian function are used. This, however, is suitable only if the problem dimension is not too high.

5. Summary and Discussion

The purpose of this note was to develop some of the basic notions and mathematics for the analysis of systems. It was shown that complex systems can be reduced to a minimal cut set representation. Probability evaluations are only straight forward for either independent or fully dependent componental failure events. Otherwise serious computational difficulties arise. The easiest way to handle dependent failure events in complex systems is by use of certain concepts of first order reliability (FORM). Those concepts require a probability distribution transformation, an algorithm for the most important region in the so called standard space and some non–trivial evaluations of the multi normal integral. These results have been improved in two ways. On the one hand a second order reliability method (SORM) has been developed which could be shown to be asymptotically ($P_f \to 0$ or $P_f \to 1$) exact (see Breitung, 1974; Hohenbichler et al., 1987; Breitung/Hohenbichler, 1989). A summary of the relevant results is given in the appendix. Very recently it has even been proposed to apply those asymptotic concepts in the so called original space. Numerical studies have shown that the corrections to FORM by SORM usually are insignificant in practical applications so that the additional effort for SORM is not really necessary. The other very recent route of improvements is based on importance sampling methods for which quite a number of studies are already available. The idea is to locate important regions for numerical Monte Carlo integration. The advantage of these methods is that they furnish exact results for any probability level. The disadvantage is their considerable numerical effort which prohibits the use for complex systems in high dimensional spaces.

Appendix: Asymptotic Approximations to Probability Integrals – A Summary of Results

Introductory scalar case

In 1820 the French mathematician P.S. de Laplace proposed to approximate the following integral

$$I(\lambda) = \int_a^b h(x) \exp[-\lambda f(x)]\, dx \qquad (A.1)$$

where $f(x)$ and $h(x) > 0$ are certain sufficiently smooth functions. λ is a parameter. Assume that $f(x)$ is a monotonically increasing function in the interval [a,b] and has a minimum in a and thus there is $f'(a) > 0$. Obviously with increasing λ the integral will be dominated by values of the integrand in the vicinity of $x = a$. According to the mean value theorem the function $h(x)$ is represented by the term $h(a)$ which is put in front of the integral and $f(x)$ is developed into a Taylor series truncated after the first non–vanishing term. Then

$$\begin{aligned} I(\lambda) &= \int_a^b h(x) \exp[-\lambda f(x)]\, dx \\ &\approx h(a) \int_a^b \exp[-\lambda (f(a) + f'(a)(x-a) + \ldots)]\, dx \\ &\approx h(a) \exp[-\lambda (f(a) - f'(a)\, a)] \int_a^b \exp[-\lambda (f'(a)\, x + \ldots)]\, dx \\ &\approx h(a) \exp[-\lambda (f(a) - f'(a)\, a)]\, (-\lambda f'(a))^{-1} \{\exp[-\lambda f'(a)\, b] - \exp[-\lambda f'(a)\, a]\} \end{aligned} \qquad (A.2)$$

For $a = 0$ it is always possible to choose for arbitrary $b < \infty$ a λ such that

$$I(\lambda) \approx h(0)\ \exp[-\lambda f(0)]\ |\lambda f'(0)|^{-1} \qquad (A.3)$$

In the second case it is assumed that f(x) has a minimum in [a,b] at x^*. Without loss of generality we assume that $x^* = 0$ and $h(0) > 0$. Again f(x) is developed into a Taylor series truncated after the first non–vanishing term. With $f'(0) = 0$ and $f''(0) > 0$ as well as $a = -\epsilon_1$ and $b = \epsilon_2$ there is

$$I(\lambda) = \int_a^b h(x) \exp[-\lambda f(x)]\, dx = h(0) \int_{-\epsilon_1}^{+\epsilon_2} \exp[-\lambda (f(0) + \tfrac{1}{2} f''(0)\, x^2 + \ldots)]\, dx$$

$$\approx h(0) \exp[-\lambda f(0)] \int_{-\epsilon_1}^{+\epsilon_2} \exp[-\tfrac{1}{2} \lambda f''(0)\, x^2]\, dx \qquad \text{(A.4)}$$

With the substitution

$$\xi = \sqrt{(\tfrac{\lambda}{2} f''(0))}\, x$$

one obtains

$$I(\lambda) \approx h(0) \exp[-\lambda f(0)] \sqrt{(2/(\lambda f''(0)))} \int_{-\sqrt{(\lambda f''(0)\epsilon_1/2)}}^{+\sqrt{(\lambda f''(0)\epsilon_2/2)}} \exp[-\xi^2]\, d\xi$$

Even if ϵ_1 and ϵ_2 are small it is always possible to choose a λ such that the integration limits can be set at $\pm\infty$ without to much error

$$I(\lambda) \approx h(0) \exp[-\lambda f(0)] \sqrt{(2/(\lambda f''(0)))} \int_{-\infty}^{+\infty} \exp[-\xi^2]\, d\xi$$

$$\approx h(0) \exp[-\lambda f(0)] \sqrt{((2\pi)/(\lambda f''(0)))} \qquad \text{(A.5)}$$

These approximations can be shown to be asymptotically exact, i.e. for $\lambda \to \infty$ (Copson, 1965). For illustration, the integrand in eq. (A.5) is shown in fig. 1 indicating that the dominating contribution to the integral comes in fact from the vicinity of the minimum (critical) point for large λ.

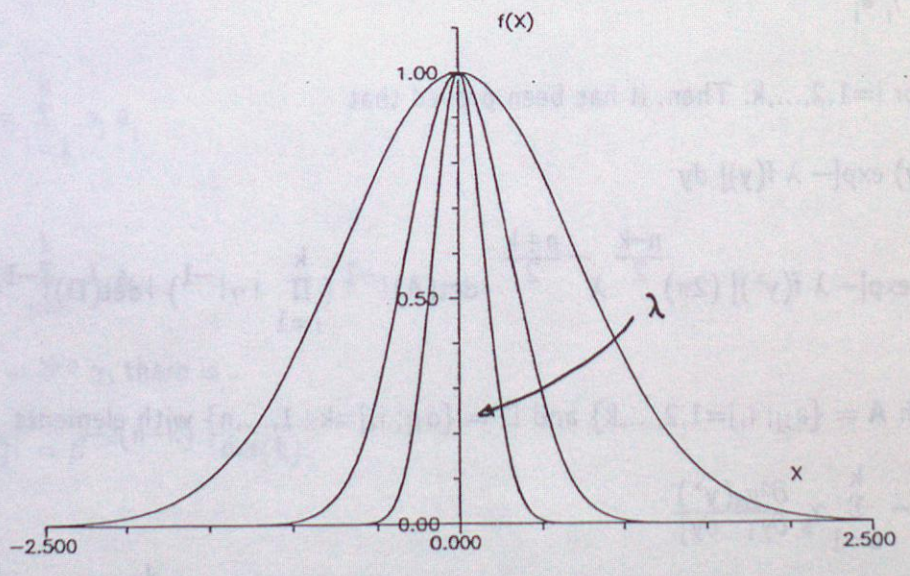

Figure A1: Integrand in eq. (A.5) for increasing λ

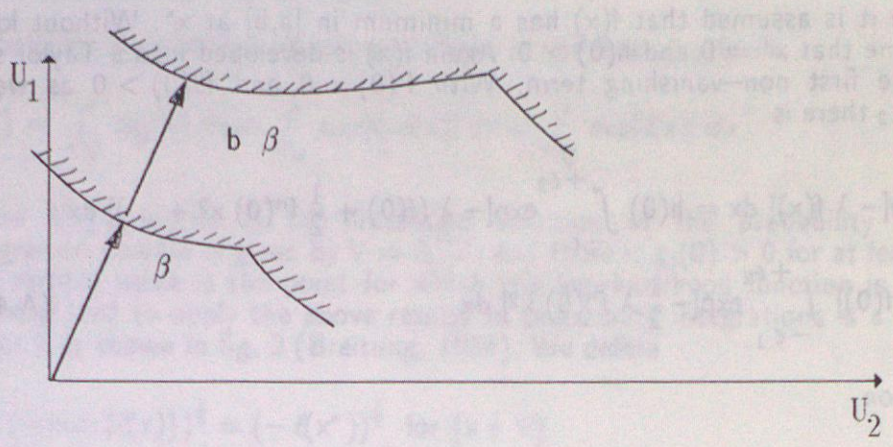

Figure A2: Scaling of integration domain

General Case

Not until very recently these results have been generalized to multivariate integrals of various forms (Bleistein, 1975). A fairly general result for the integral

$$I(\lambda) = \int_D h(\mathbf{y}) \exp[-\lambda f(\mathbf{y})]\, d\mathbf{y} \tag{A.6}$$

für $\lambda \to \infty$ where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, and D a simply connected domain containing the origin has been given in (Breitung/Hohenbichler, 1989). Herein $f(\mathbf{y})$ is at least twice differentiable and has a minimum at $\mathbf{y} = \mathbf{y}^* \neq \mathbf{0}$. $h(\mathbf{y})$ is a slowly varying function and there is $h(\mathbf{0}) \neq 0$. D is given by $D = \cap_{i=1=1}^{k} D_i$ with $D_i = \{\mathbf{y}: g_i(\mathbf{y}) \leq 0\}$ and $k \in \{1,2,\ldots,n\}$. $f(\mathbf{y})$ as well as the functions $g_i(\mathbf{y})$ are at least twice differentiable in $\mathbf{y}^*$ and the function $h(\mathbf{y})$ is continuous in $\mathbf{y}^*$ and $h(\mathbf{y}^*) \neq 0$. In $\mathbf{y}^*$ there is $g_i(\mathbf{y}^*) = 0$ for i=1,2,...,k. The gradients $\mathbf{a}_i = \nabla g_i(\mathbf{y}^*)$ (i=1,2,...,k) are linearly independent. This implies that $a_{ij} = 0$ for i=1,2,...,k and j=k+1,...,n which always can be achieved by a suitable orthogonal transformation. It also means that $\partial f(\mathbf{y}^*)/\partial y_i = 0$ for i=k+1,...,n and the gradient can be represented as

$$\nabla f(\mathbf{y}^*) = \sum_{i=1}^{k} \gamma_i \mathbf{a}_i$$

with $\gamma_i < 0$ for i=1,2,...,k. Then, it has been proved that

$$I(\lambda) = \int_D h(\mathbf{y}) \exp[-\lambda f(\mathbf{y})]\, d\mathbf{y}$$
$$\sim h(\mathbf{y}^*) \exp[-\lambda f(\mathbf{y}^*)]\, (2\pi)^{\frac{n-k}{2}}\, \lambda^{-\frac{n+k}{2}}\, |\det(\mathbf{A})|^{-1} \left(\prod_{i=1}^{k} |\gamma|^{-1}\right) |\det(\mathbf{D})|^{-1/2} \tag{A.7}$$

for $\lambda \to \infty$ with $\mathbf{A} = \{a_{ij};\ i,j=1,2,\ldots,k\}$ and $D = \{d_{ij};\ i,j=k+1,\ldots,n\}$ with elements

$$d_{ij} = \frac{\partial^2 f(\mathbf{y}^*)}{\partial y_i\, \partial y_j} - \sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(\mathbf{y}^*)}{\partial y_i\, \partial y_j}$$

and $\det(\mathbf{D}) = 1$ for $k = n$.

Application to probability integrals

Probability integrals can always be written in the following form

$$P(V) = \int_V \psi_X(x)\, dx = \int_V \exp[\ln\psi(x)]\, dx = \int_V \exp[\ell(x)]\, dx \qquad (A.8)$$

where $\ell(x) = \ln\psi(x)$ is the likelihood function of the probability density $\psi(x)$. The integration domain is given by $V = \cap_{i=1}^{n} V_i$ and there is $g_i(0) > 0$ for at least one $i \in \{1,...,n\}$. The critical point is the point for which the log–likelihood function is maximal in V. The essential idea to apply the above results to probability integrations is a central scaling by a factor b as shown in fig. 2 (Breitung, 1984). We define

$$\beta = (-\max\{\ell(x)\})^{\frac{1}{2}} = (-\ell(x^*))^{\frac{1}{2}} \quad \text{for } \{x \in V\} \qquad (A.9)$$

and

$$f(x) = \beta^{-2}\, \ell(x) \qquad (A.10)$$

In x^* it is $f(x^*) = 1$. Also we assume $\ell(x^*) < 0$. We consider the integral

$$P(b) = \int_V \exp[-b^2 f(x)]\, dx \qquad (A.11)$$

and apply eq. (7) with $h(x) = 1$ and $\lambda = b^2$

$$P(b) \sim (2\pi)^{\frac{n-k}{2}}\, b^{-n-k} \exp[-b^2]\, |\det(A)|^{-1} \left(\prod_{i=1}^{k} |\gamma_i|^{-1}\right) |\det(D)|^{-1/2} \qquad (A.12)$$

In noting

$$\nabla f(x^*) = \beta^{-2}\, \nabla\ell(x^*)$$

$$\nabla\nabla f(x^*) = \beta^{-2}\, \nabla\nabla\ell(x^*)$$

and by

$$\nabla f(x^*) = \sum_{i=1}^{k} \gamma_i\, a_i$$

and

$$\nabla\ell(x^*) = \sum_{i=1}^{k} \delta_i\, a_i$$

with $\delta_i = \beta^{-2}\, \gamma_i$ there is

$$|\det(D)| = \beta^{-2(n-k)}\, |\det(L)|$$

where

$$L = \left\{ \frac{\partial^2\ell(x^*)}{\partial x_i\, \partial x_j} - \sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(x^*)}{\partial x_i\, \partial x_j} \; ; \; i,j = k+1, \ldots, n \right\}$$

Hence

$$P(b) \sim (2\pi)^{\frac{n-k}{2}} \frac{\beta^{n-k}}{b^{n+k}} \beta^{2k} \exp[-b^2] \, |\det(\mathbf{A})|^{-1} \left(\prod_{i=1}^{k} |\gamma_i|^{-1} \right) |\det(\mathbf{L})|^{-1/2} \tag{A.13}$$

and with $b = \beta$ (β is already large)

$$P(V) \approx (2\pi)^{\frac{n-k}{2}} \exp[-\beta^2] \, |\det(\mathbf{A})|^{-1} \left(\prod_{i=1}^{k} |\gamma_i|^{-1} \right) |\det(\mathbf{L})|^{-1/2} \tag{A.14}$$

In particular, for $k = 1$ we have

$$P(V) \approx (2\pi)^{\frac{n-1}{2}} \exp[-\beta^2] \, |\gamma_1|^{-1} \, |\det(\mathbf{L})|^{-1/2} \tag{A.15}$$

with $\gamma_1 = \|\nabla \ell(\mathbf{x}^*)\| / \|\nabla g_1(\mathbf{x}^*)\|$. For $k = n$ we have

$$P(V) \approx \exp[-\beta^2] \, |\det(\mathbf{A})|^{-1} \left(\prod_{i=1}^{n} |\gamma_i|^{-1} \right) \tag{A.16}$$

The special case of multinormal integrals yields nicely compact results. Let $\mathbf{Y}$ be a standard normal vector with probability density $\varphi(\mathbf{y}) = (2\pi)^{-n/2} \exp[-\|\mathbf{u}\|^2/2]$. The scaling of the integral

$$P(V) = \int_V \varphi(\mathbf{y}) \, d\mathbf{y} \tag{A.17}$$

yields upon substitution with $\mathbf{u} = \mathbf{y}\, b^{-1}$

$$P(bV) = \int_{bV} \varphi(\mathbf{y}) \, d\mathbf{y} = b^n \int_V \varphi(b\mathbf{u}) \, d\mathbf{u} = (2\pi)^{-n/2} \, b^n \int_V \exp[-b^2 \|\mathbf{u}\|^2/2] \, d\mathbf{u} \tag{A.18}$$

where it is of course $h(\mathbf{u}) = 1$, $f(\mathbf{u}) = \|\mathbf{u}\|^2/2$ and $\lambda = b^2$. The critical point $\mathbf{u}^*$ has distance $\beta = \|\mathbf{u}^*\|$ to the origin.

Application of eq. (A.7) is again straightforward. Let the integration domain be given by $V = \{\cap_{i=1}^{n} V_i\}$ with $V_i = \{g_i(\mathbf{u}) \le 0\}$. In the critical point $\mathbf{u}^*$ it is $g_i(\mathbf{u}) = \mathbf{a}_i^T (\mathbf{u} - \mathbf{u}^*) = 0$ for $i = 1,2,\ldots,n$. $\mathbf{u}^*$ can be represented as $\mathbf{u}^* = \Sigma_{i=1}^{n} \gamma_i \, \mathbf{a}_i$ with $\mathbf{a}_i = \nabla g_i(\mathbf{u}^*)$. Then

$$P(V) \approx (\det(\mathbf{R}))^{-1/2} \prod_{i=1}^{n} \frac{\varphi_1(u_i^*)}{(-\gamma_i)} \tag{A.19}$$

This formula is a asymptotic approximation for the multinormal integral in noting that $\beta_i = -\mathbf{a}_i^T \mathbf{u}^*$ and $\mathbf{R} = \{\mathbf{a}_i^T \mathbf{a}_j\}$ and therefore $P(V) \sim \Phi_n(-\beta; \mathbf{R})$ (Ruben, 1964). For $k = 1$ with $g_1(\mathbf{0}) \ge 0$ and $V_1 = \{g_1(\mathbf{u} \le 0\}$ one obtains (Breitung, 1984)

$$P(V) \approx \|\mathbf{u}^*\|^{-1} \, |\det(\mathbf{D})|^{-1/2} \, \varphi_1(\mathbf{u}^*) \approx \Phi(-\beta) \, |\det(\mathbf{D})|^{-1/2} \tag{A.20}$$

with

$$\mathbf{D} = \{\delta_{ij} - \frac{\|\mathbf{u}^x\|}{\|\nabla g(\mathbf{u}^x)\|} \frac{\partial^2 g_1(\mathbf{u}^x)}{\partial u_i \, \partial u_j};\ i,j = 2,\ldots,n\}$$

In the second equation use was made of $\Phi(-x) \sim \varphi(x)/x$ and $\beta = \|\mathbf{u}^x\|$. Further there is now $g_i(0) > 0$ for at least one $i \in \{1, 2,\ldots,m\}$ and for the functions there is $g_i(\mathbf{u}^x)$ for $i = 1, 2,\ldots,k$ it is $g_i(\mathbf{u}^x) = 0$, where $k \leq m$. For $i = k+1,\ldots,m$ there is $g_i(\mathbf{u}^x) > 0$. Then, (Hohenbichler, 1984)

$$P(V) \approx |\det(\mathbf{D})|^{-1/2}\, P\Big(\bigcap_{i=1}^{k} V_i\Big) \qquad (A.21)$$

with

$$\mathbf{D} = \{\delta_{ij} - \sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(\mathbf{u}^x)}{\partial u_i \, \partial u_j};\ i,j=k+1,\ldots,n\}$$

and for the cut set probability the result in eq. (A.19) with $n = k$ or other suitable computation schemes for the normal integral (Hohenbichler/Gollwitzer, 1987, Gollwitzer/Rackwitz, 1986).

The terms with $\det(\mathbf{D})$ or $\det(\mathbf{L})$ are second order corrections which, in general, are small compared to the leading remaining first order term. The two types of results are mathematically absolutely equivalent. They reduce probability integration to a problem of nonlinear programming (the search for the critical point) and some simple algebra.

Discussion

A first question when implementing this elegant theory is the accuracy question. Although the probabilities computed by these formulae have asymptotically vanishing error doubts have repeatedly been put forward about the accuracy in the non–asymptotic case and that the theory as presented before does not provide means to quantify the error or at least error bounds. Theoretically, the results given before are only the first term of an asymptotic expansion. Consideration of higher order terms would give indications about the error although still in an asymptotic setting. Only for the case $k = 1$ two more terms have been computed so far (see Tvedt, 1983) and there is little hope that additional terms can easily be computed for the other cases. However, the last five years have seen a very fruitful development of special methods of importance sampling which can quantify the error of first or second order probability estimates with some additional numerical effort or even to arrive at arbitrarily exact results. It is not possible to review these results here and we refer the reader to some references (Hohenbichler/Rackwitz, 1986, Gollwitzer/Rackwitz, 1986, Fujita/Rackwitz, 1988). Practical applications have shown that usually there is no need to worry about accuracy even with the first order results.

A second question is whether the formulation in the original space or in the standard space should be chosen. Historically much more and positive experience has been gained with the standard space formulation but a decision should involve several aspects. In the standard space formulation all variables are suitably scaled, the objective function for the search algorithm is very simple and the formulae (A.19) to (A.21) are easy to apply. The simplicity of the objective function has even be used to design a special efficient and robust search algorithm. But a probability distribution transformation must be performed (see, for example, Hohenbichler/Rackwitz, 1981) which can be somewhat complicated. Also, the probability distribution transformation can distort the original failure domain. However, this generally is not a valid argument as it can make its boundary more or less curved depending on the application at hand. In the original space formulation the lack of scaling can introduce larger round–off errors and instabilities in the search algorithm and the objective function is more complicated. While in the standard space the first and second derivatives of the objective function are analytic, they must be determined numerically in the original space.

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