Reprinted from Materials and Member Behavior
Structures '87 Proceedings/ST Div./
ASCE/Orlando, Florida/Aug. 17-20, 1987

Reliability-oriented Design of Fatigue Experiments

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It is well known in the reliability analysis for structural components subjected to cyclic random load that apart from the usually model parameters uncertainties are most important and significant for the final structural design. The purpose of this study is to discuss several problem in this area, and to develop a reliability-oriented design philosophy of fatigue experiments. A brief account is given of a residual strength model for the probabilistic prediction of structural safety when a component is subjected to cyclic random load. The formulation enables by determining the sensitivity of structural safety against the elements of the test plan to design tests in an optimal way.

Introduction

In practice most reliability analyses for structural components subject to fatigue are performed by using the phenomenological damage accumulation law of Palmgren-Miner together with the information contained in empirical S-N-curves. The "model uncertainty", however, is fairly large in this approach and has to be accounted for appropriately. More physically based approaches are under study in various areas among which fracture mechanics for the prediction of crack initiation, crack growth and crack instability have found most attention. Yet, except under special conditions, the latter approaches have not lead to much better reliability estimates. The reason primarily is the lack of knowledge of the specific values of the parameters. Also, for certain materials there is little justification to use approaches connected with the growth of cracks at all. Real structural components also involve certain additional phenomena such as the presence of (random) initial stresses, spatial redistribution of "far field" stresses or forces during damage accumulation or load history effects which are extremely difficult to consider. Sometimes, the fatigue lives of structural components can differ from those predicted by theories and/or from the usual small size test specimens by several orders of magnitude. As a consequence, practice tends to rely on (full scale) experiments under representative loading spectra or

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at least a combination of theoretical predictions and experiments as far as possible. But the number of those tests is always limited. This implies that there can be significant statistical uncertainties which can affect reliability predictions. It is somewhat surprising that this aspect has attracted very little attention so far. The purpose of this study, therefore, is to highlight several important problems in this area.

Although the proposed approach is rather general and not even limited to fatigue deterioration it is developed for a special deterioration model. In the following a brief review is given of a residual strength model proposed repeatedly in several contexts but, probably most thoroughly studied in a series of papers [Yang/Liu, 1977] on fatigue of composite structures. It, in fact, can be shown that this model is also suitable for many other materials and types of structural components. Furthermore, it allows to bridge the gap between theoretical predictions and experiments because statistical aspects can be incorporated in a rather straight-forward manner. Using this model an appropriate reliability formulation is presented. Given this formulation, it is then shown how to plan experiments optimally.

Yang's Deterioration Model

For easy reference Yang's deterioration model [Yang/Liu, 1977] is briefly reviewed. Assume that the residual strength R(n) of a material is a monotonically decreasing function of n, the number of fatigue cycles. A reasonable starting point for a suitable model of strength degradation is that the decrement of R(n) at the n-th cycle is inversely proportional to some power C - 1 of the residual strength R(n) itself but proportional to some function of maximum cycle stress S:

$$dR(n)/dn = - f(S)/(C\, R^{C-1}(n)) \qquad (1)$$

Integration from n = 0 to n = n yields:

$$R^C(n) = R^C(0) - f(S)n \qquad (2)$$

If the initial strength is random so is R(n). For example, the variability of the initial strength can be modelled by a two-parameter Weibull distribution:

$$F_{R(0)}(x) = 1 - \exp[- (x/B)^A] \qquad (3)$$

At failure in a constant stress experiment it is R(n) = S and n = N. Hence, from eq. (2):

$$N = [R^C(0) - S^C]/f(S) \qquad (4)$$

with which knowing the distribution of R(0) the distribution of fatigue lives can be determined.

$$F_N(n) = P(N \le n) = P((R^C(0) - S^C)/f(S) \le n)$$

$$= P(R(0) \le (nf(S + S^C)^{1/C})$$

$$= 1 - \exp\left(- \left[\frac{n + [S^C/f(S)]}{B^C/f(S)}\right]^{A/C} \right)$$

$$\approx 1 - \exp\left(- \left[\frac{n}{B^C/f(S)}\right]^{A/C} \right) \qquad (5)$$

Note that $B^C/f(S)$ is something like a characteristic fatigue life. Suppose that S-N-curves in the form

$$K\, S^D\, N = 1 \quad \text{or} \quad N = (K\, S^D)^{-1} \qquad (6)$$

are available. Then, an obvious choice for the yet unknown scale parameter in eq. (5) is:

$$B^C/f(S) = (K\, S^D)^{-1} \qquad (7)$$

Eq. (2) becomes:

$$R^C(n) = R^C(0) - B^C\, K\, S^D\, n \qquad (8)$$

while the final version of eq. (5) is obtained by inserting in eq. (7):

$$F_N(n) = 1 - \exp\left(- [n\, K\, S^D]^{A/C} \right) \qquad (9)$$

This also leads by combining eq. (3) and (8) to the distribution of residual strength:

$$F_{R(n)}(x) = P(R(n) \le x)$$

$$= P(R^C(n) \le x^C)$$

$$= P(R^C(0) - K\, B^C\, S^D\, n \le x^C)$$

$$= P(R(0) \le (x^C + K\, B^C\, S^D\, n)^{1/C})$$

$$= 1 - \exp(- [(x^C + K\, B^C\, S^D\, n)/B^C]^{A/C}) \quad (10)$$

Many further details and discussions may be found in the studies [Yang/Liu, 1977]. At least for certain types of composite structures and types of fatigue cases those references also provide an excellent experimental verification of the model. If tests under so-called spectrum loading would have been performed, relation (6) should be modified to

$$K\, E[S^D] N = M \qquad \text{or } N = M/(K\, E[S^D]) \quad (11)$$

according to Palmgren-Miner's damage accumulation hypothesis. Then, eq. (8) becomes:

$$R^C(n) = R^C(O) - B^C\, K\, E[S^D]\, n/\, M \quad (12)$$

Correspondingly, eq. (10) now reads:

$$F_{R(n)}(x) = 1 - \exp(- [(x^C + K\, B^C\, E[S^D]\, n/M)/B^C]^{A/C}) \quad (13)$$

Herein, the parameter M ought to be interpreted as a model uncertainty variable. It takes account not only of the randomness in the sequences of applied loads and, possibly, associated retardation effects in damage accumulation. It must also be interpreted as an adjustment factor to fit the model to experimental results. For the validity of eq. (11) it is required that N is large so that the variability of Palmgren-Miner's damage indicator vanishes.

Reliability Formulation

Under stationary and ergodic loading there is for high-reliable components

$$P_f(t) \sim 1 - \exp[- E[N(t)]] \quad (14)$$

where

$$E[N(t)] = \int_O^t \nu_X(\tau) d\tau \quad (15)$$

and

$$\nu_X(\tau) = \lim_{\Delta \to O} \frac{1}{\Delta} P((S(\tau) \le X(\tau) \cap (S(\tau + \Delta) > X(\tau + \Delta))) \quad (16)$$

the upcrossing rate above the threshold $X(\tau)$. If, in particular and without loss of generality, $S(\tau)$ is a Gaussian process with zero mean and unit standard deviation and the threshold is given as in eq. (12) it is [Lindley, 1976]

$$\nu(R(\tau) = \omega_O \exp\left[- \frac{1}{2} R^2(\tau)\right] \left[\varphi \frac{\dot{R}(\tau)}{\omega_O} - \frac{\dot{R}(\tau)}{\omega_O} \Phi(- \frac{\dot{R}(\tau)}{\omega_O})\right] \quad (17)$$

with

$$R(\tau) = [R^C(O) - B^C\, K\, E[S^D]\, \tau/M]^{1/C} \quad (18)$$

and $\dot{R}(\tau)$ its time derivative. ω_O is the zero crossing frequency of $S(\tau)$. Usually, the integration in eq. (15) has to be performed numerically. A simple approximation based on the method of Laplace is given in Guers and Rackwitz [Guers/Rackwitz, 1986].

In eq. (18) the parameters R(O), C, B, K, D and M, however, must be assumed to be uncertain so that the total failure probability is:

$$P_f(t) \sim 1 - \int \exp[- E[N(t) | \underline{Q} = \underline{q}]]\, dF_{\underline{Q}}(\underline{q}) \quad (19)$$

$\underline{Q}$ collects the (non-ergodic) uncertain parameters. An evaluation of eq. (19) by numerical integration can become rather laborous and, therefore, it is proposed to perform it approximately by FORM or SORM [Rackwitz, 1985]. In order to formulate problem (14) such that these methods can be used we introduce the equality:

$$P(U_T \le u_t) = P(T \le t) = P_f(t) \quad (20)$$

Solving for T with $P_f(t)$ in eq. (14) and $\varkappa(t) = E[N(t) | \underline{Q}]$ gives:

$$T = \varkappa^{-1}[- \ln \Phi(-U_T) | \underline{Q}] \quad (21)$$

Therefore, we can write

$$P_f(t) = P(T - t \le O)$$

$$= P(\varkappa^{-1}[- \ln \Phi(-U_T) | \underline{Q}] - t \le O) \quad (22)$$

which is the required formulation. The inversion of $\varkappa^{-1}$ must again be performed numerically, e.g. by Newton's algorithm which here reads:

$$T^{k+1} = T^k - \frac{\int_0^{T^k} \nu(\tau|\underline{Q})d\tau + \ln \Phi(U_T)}{\nu(T^k|\underline{Q})} \tag{23}$$

In accordance with the FORM-methodology we now have to represent R(O) in eq. (18) by its Rosenblatt-transformation which is [Hohenbichler/Rackwitz, 1981]:

$$R(O) = B[- \ln \Phi(- U_{R(O)})]^{1/A} \tag{24}$$

If the parameters A, B, C, D, K and M were known the probability computation could now be carried out in the usual way. If this is the case one probably prefers to work directly with eq. (14). Only if higher dimensional probability integrations are necessary application of FORM/SORM might be useful. However, those parameters usually are unknown beforehand and little confidence is given in theoretical predictions.

Statistical Considerations

The problem, then, is the planning and evaluation of the tests for the estimation of the various parameters and, furthermore, the quantification of their statistical variability. Obviously, one can perform four types of tests. Type I are static strength tests to determine the parameters A, B in eq. (3). Type-II-tests are the usual tests for S-N-curves. Pairs of measurements (N_i, S_i) can be used for the estimation of the parameters in eq. (5). If in those tests no failure is observed and/or if the number of cycles in the tests are limited, the residual strength as in eq. (10) is determined in type-III-tests. Finally, type-IV-tests may be performed under spectrum loading especially for the estimation of the parameter M. Again, if the number of cycles in a text is limited and/or no failure is observed one still can determine the residual strength.

It is proposed to use those tests jointly for a maximum likelihood estimation. Other alternatives as estimation by moments or quantiles appear to be discriminated by purely statistical reasons.

The densities of the Weibull-like distributions eq. (3), (9), (10) and (13) are all easily determined but rather lengthy. They are not given herein. The likelihood functions are

$$L(\underline{q}|\underline{\underline{x}}) = \prod_{i=1}^{n_j} f(\underline{x}_i|\underline{q}), \quad j = I,II,III,IV \tag{25}$$

where $\underline{\underline{x}}$ is the observation matrix (note that in type II-IV-tests one has pairs or even triples of observations per experiment, denoted by $\underline{x}_i$ and $\underline{q}$ the unknown parameter vector, here with dimension 6). The maximum likelihood estimations are obtained from the system of (non-linear) equations

$$\frac{\partial \ln L(\underline{q}|\underline{\underline{x}})}{\partial q_k} = 0, \quad k = 1,\ldots,6 \tag{26}$$

yielding the estimation vector $\hat{\underline{q}} = (\hat{a},\hat{b},\hat{c},\hat{d},\hat{k},\hat{m})^T$. Eq. (26) must be solved numerically by one of the methods suitable for this task, e.g. a Newton-Kantorovich-Raphson algorithm. Since the sample sizes are limited we have to consider also statistical uncertainties. If the sample size is sufficiently large, one can make use of the result that maximum likelihood estimators are asymptotically normal with mean value vector $E[\underline{Q}] = \hat{\underline{q}}$ and covariance matrix $\underline{\underline{\Sigma}}_Q$ whose elements are defined by:

$$\underline{\underline{\Sigma}}_Q^{-1} = \left(- \frac{\partial^2 \ln L(\underline{\underline{x}}|\hat{\underline{q}})}{\partial q_r \partial q_s}; \; r,s = 1,\ldots,6 \right) \tag{27}$$

For any normal vector $\underline{Q}$ a Rosenblatt-transformation $\underline{Q} = \underline{T}(\underline{U}_Q)$ exists which if inserted into eq. (21) produces the final reliability formulation amenable to FORM/SORM. Since, by definition, none of our parameters can be non-positive it is proposed to assume $\underline{Q}$ jointly log-normal with the same means and covariances, instead. This additional assumption does not only avoid certain numerical problems but is believed to approximate the parameter distribution more realistically for smaller sample sizes.

The approach so far assumes that any prior information on the parameters is dominated by the actual data. A rigorous application of Bayes' theorem for the incorporation of prior information is actually avoided. In an earlier attempt the rigorous formulation has been pursued which unfortunately lead to serious numerical problems and considerable computational effort. Instead, a simpler alternative for practical applications is proposed. Prior information can be quantified in terms of a prior point estimate for the parameters associated with an equivalent sample size expressing the "degree of belief" in those estimates.

Reliability-based Planning of Experiments

The considerations so far permit to design an adaptive scheme for the decisions on types and corresponding sample sizes of experiments. We assume that a reliability constraint in terms of a failure probability $P_{f,0}$ or a generalized safety index $\beta_0 = -\Phi^{-1}[P_{f,0}]$ is given. In order to meet the reliability requirement we need to introduce a design parameter δ, e.g. a cross-section modulus, which can be used to modify the stresses according to $S = \delta L$ where L is the load process. Initially, only the prior estimates together with the equivalent sample sizes are available. Note that especially the covariance matrix of $\underset{\sim}{Q}$ depends on the equivalent sample sizes. A prior reliability analysis now is performed such that the reliability requirement is fulfilled by varying δ together with an appropriate (pre-posterior) increase of the sample sizes in the various types of experiments. This is most easily done by treating δ and the n_j as auxiliary independent Gaussian variables with vanishing standard deviation and means equal to their (deterministic) values. The α-values, i.e. the gradient of the limit state function in eq. (24) at the β-point, normalized appropriately, precisely indicates how to distribute a given budget for a first set of different tests. The results of those tests are then used to update the parameter estimates according to previous section. A second reliability analysis formally identical to the first one yields a second set of tests most likely with a modified δ. With this δ the second set of experiments is performed. A third reliability analysis, which usually will be the last one on the basis of the newly updated parameters will yield a new design parameter δ meeting the reliability constraint and the total budget for testing as well.

For brevity of presentation, we have omitted any detail both in the estimator updating procedure and the reliability analysis which are of a rather technical nature though not at all trivial.

A more interesting general problem is the assignment of the budget for the tests to be carried out in each step relative to a constraint on the total budget and, secondly, to build up a "stopping rule", i.e. a rule which terminates testing if the gain of information from further experiments becomes uninterestingly small. A principal question in optimisation of those steps is whether the related optimisation tasks are always convex. There is certain numerical evidence that this is so only with respect to some special parameters. These and related questions are still under study.

Numerical Investigations

For the numerical verification of the study, some experimental data of graphite/epoxy laminates [Yang/Liu, 1977] were applied. Firstly, mean value vector $E[\underset{\sim}{Q}] = \hat{\underset{\sim}{q}}$ and covariance matrix $\underline{\underline{\Sigma}}_Q$ were calculated by the maximum likelihood estimation with the jointly likelihood function eq. (25) and its likelihood equation eq. (26). The sample sizes of test are

$$n_I = 12, \quad n_{II} = 20, \quad n_{III} = 25, \quad n_{IV} = 8$$

The data of type I, II, III tests are from Yang [Yang/Liu, 1977] and the data of type IV test are generated empirically where the condition of spectrum loading is assumed to be narrow-band Gaussian process with $\mu(t) = 0$, standard deviation $\sigma(t) = E[R(0)]/8.67 = 60.26$ MPa (8.70 ksi) and $\omega_0 = 2\pi$. Thus,

$$E[S^D] = (2\sqrt{2}\,\sigma)^D\,\Gamma(1 + D/2)$$

The calculation results of mean value vector $E[\underset{\sim}{Q}] = \hat{\underset{\sim}{q}}$ was

$$\hat{a} = 7.0380,\ \hat{b} = 68.2332,\ \hat{c} = 4.4468,\ \hat{d} = 17.6564,\ \hat{m} = 2.4171$$

and variance covariance matrix $\underline{\underline{\Sigma}}_Q$ was

$$\begin{bmatrix} .8417D{+}00 & .8942D{+}00 & .2228D{+}00 & .5882D{-}02 & -.6510D{+}00 \\ .8942D{+}00 & .3937D{+}01 & -.9824D{-}01 & .1315D{-}01 & -.1306D{+}01 \\ .2228D{+}00 & -.9824D{-}01 & .3639D{+}00 & .4794D{-}03 & .1233D{+}00 \\ .5882D{-}02 & .1315D{-}01 & .4794D{-}03 & .1223D{-}02 & .5179D{-}02 \\ -.6510D{+}00 & -.1306D{+}01 & .1233D{+}00 & .5179D{-}02 & .6436D{+}01 \end{bmatrix}$$

where estimation parameter vector is 5 dimensional vector $\hat{\underset{\sim}{q}} = (\hat{a},\hat{b},\hat{c},\hat{d},\hat{m})^T$. Since parameter K is extremely small, it is assumed that the value of K is a constant value ($K = 1.8285 \times 10^{-36}$) because of numerical reason.

Secondly, in order to evaluate the parameter sensitivity, when it is assumed the only 1 specimen was added to the samples of each type tests corresponding to Tabel 1, each mean value vector $E[\underset{\sim}{Q}]$ and covariance matrix $\underline{\underline{\Sigma}}_Q$ were estimated respectively again to the cases of Table 1.

TABLE 1.-Combination of Case Study

	n_I	n_{II}	n_{III}	n_{IV}
Basic case	12	20	25	8
Case I	13	20	25	8
Case II	12	21	25	8
Case III	12	20	26	8
Case IV	12	20	25	9

The value of median point of the probability distribution for each test is selected as a reasonable way how to determine the additional sample data, that is,

Type-I-test: $$x^* = \hat{b}(-\ln 0.5)^{1/\hat{a}}, \qquad (28)$$

Type-II-test: $$n^* = \frac{1}{K S_0^d}(-\ln 0.5)^{\hat{c}/\hat{a}}, \quad \text{(under specified } S_0) \qquad (29)$$

Type-III-test: $$x^* = \{\hat{b}^{\hat{c}}(-\ln 0.5)^{\hat{c}/\hat{a}} - K\,\hat{b}^{\hat{c}}\,S_o^{\hat{d}}\,N_0\}^{1/\hat{c}}, \quad \text{(under specified } S_0,\ N_0) \qquad (30)$$

Type-IV-test:
$$x^* = \{\hat{b}^{\hat{c}}(-\ln 0.5)^{\hat{c}/\hat{a}} - K\,\hat{b}^{\hat{c}}(2\sqrt{2}\,\sigma_0)^{\hat{d}}\,\Gamma(1+\hat{d}/2)N_0/\hat{m}\}^{1/\hat{c}}, \quad \text{(under specified } \sigma_0,\ N_0) \qquad (31)$$

In these cases, the results of maximum likelihood estimation were

Case I: $E[\underline{Q}] = [7.0890,\ 68.1880,\ 4.4696,\ 17.6560,\ 2.3871]^T$

$$\underline{\underline{\Sigma}}_{\underline{Q}} = \begin{bmatrix} .8250D{+}00 & .8522D{+}00 & .2194D{+}00 & .5758D{-}02 & -.6140D{+}00 \\ .8522D{+}00 & .3729D{+}01 & -.9560D{-}01 & .1263D{-}01 & -.1206D{+}01 \\ .2194D{+}00 & -.9560D{-}01 & .3643D{+}00 & .4354D{-}03 & .1217D{+}00 \\ .5758D{-}02 & .1263D{-}01 & .4354D{-}03 & .1221D{-}02 & .5345D{-}02 \\ -.6140D{+}00 & -.1206D{+}01 & .1217D{+}00 & .5345D{-}02 & .6117D{+}01 \end{bmatrix}$$

Case II: $E[\underline{Q}] = [7.1025,\ 68.3689,\ 4.3853,\ 17.6599,\ 2.2826]^T$

$$\underline{\underline{\Sigma}}_{\underline{Q}} = \begin{bmatrix} .8452D{+}00 & .8678D{+}00 & .2320D{+}00 & .5161D{-}02 & -.5347D{+}00 \\ .8678D{+}00 & .3834D{+}01 & -.7159D{-}01 & .1135D{-}01 & -.1044D{+}01 \\ .2320D{+}00 & -.7159D{-}01 & .3427D{+}00 & .9466D{-}03 & .7627D{-}01 \\ .5161D{-}02 & .1135D{-}01 & .9466D{-}03 & .1100D{-}02 & .5524D{-}02 \\ -.5347D{+}00 & -.1044D{+}01 & .7627D{-}01 & .5524D{-}02 & .4807D{+}01 \end{bmatrix}$$

Case III: $E[\underline{Q}] = [7.1532,\ 68.3958,\ 4.4500,\ 17.6552,\ 2.1467]^T$

$$\underline{\underline{\Sigma}}_{\underline{Q}} = \begin{bmatrix} .8308D{+}00 & .8304D{+}00 & .2222D{+}00 & .6038D{-}02 & -.4334D{+}00 \\ .8304D{+}00 & .3735D{+}01 & -.1020D{+}00 & .1321D{-}01 & -.8414D{+}00 \\ .2222D{+}00 & -.1020D{+}00 & .3559D{+}00 & .4433D{-}03 & .7605D{-}01 \\ .6038D{-}02 & .1321D{-}01 & .4433D{-}03 & .1195D{-}02 & .5108D{-}02 \\ -.4334D{+}00 & -.8414D{+}00 & .7605D{-}01 & .5108D{-}02 & .3749D{+}01 \end{bmatrix}$$

Case IV: $E[\underline{Q}] = [7.0896,\ 68.1815,\ 4.4628,\ 17.6568,\ 2.3761]^T$

$$\underline{\underline{\Sigma}}_{\underline{Q}} = \begin{bmatrix} .8245D{+}00 & .8537D{+}00 & .2200D{+}00 & .5623D{-}02 & -.5913D{+}00 \\ .8537D{+}00 & .3731D{+}01 & -.9247D{-}01 & .1236D{-}01 & -.1169D{+}01 \\ .2200D{+}00 & -.9247D{-}01 & .3614D{+}00 & .5497D{-}03 & .1146D{+}00 \\ .5623D{-}02 & .1236D{-}01 & .5497D{-}03 & .1208D{-}02 & .5453D{-}02 \\ -.5913D{+}00 & -.1169D{+}01 & .1146D{+}00 & .5453D{-}02 & .5772D{+}01 \end{bmatrix}$$

Thirdly, for each normal vectors $\underline{Q}$ a Rosenblatt-transformation $\underline{Q} = \underline{T}(\underline{U}_Q)$ exists which if inserted into eq. (21) produces the final reliability formulation amenable to FORM/SORM. It is assumed that $\underline{Q}$ is jointly log-normal with the same means and covariances, since none of parameters can be non-positive. This additional assumption does not only avoid certain numerical problems but is believed to approximate the paramter distribution more realistically for smaller sample sizes as these cases.

The Tabel 2 presents the results obtained by FORM/SORM method with the threshold function eq. (18) and time inversion algorithm eq. (23) through the insertion of Rosenblatt-transformations $\underline{Q} = \underline{T}(\underline{U}_Q)$ and eq. (24) into eq. (21). The time inversion technique in eq. (23) has been carried out by requiring a relative precision which is compatible with the precision in the FORM/SORM algorithm, where the standard deviation of load process $\sigma(t)$ and parameter K is treated as independent constants as:

$$\sigma(t) = E[R(0)]/8.67 = 60.26 \text{ MPa } (= 8.70 \text{ ksi}),$$
$$K = 1.8285 \times 10^{-36}$$

Finally, we can easily evaluate the sensitivity of β with respect to sample size n_j (j = I,II,III,IV) from the results of Table 2. Now, the sensitivity of β is defined by

$$\gamma_j = \frac{\partial \beta}{\partial n_j} \quad (j = \mathrm{I,II,III,IV}) \tag{32}$$

which is represented as follows when the additional sample size is equal to Δn_j

$$\gamma_j = \frac{\Delta \beta}{\Delta n_j} = \frac{\beta(n_j + \Delta n_j) - \beta(n_j)}{\Delta n_j} \tag{33}$$

Thus, we can formulate the simple decision rule on the allocation of sample size for the types of materials test as

$$n_j^{*} = n^{0} \cdot \gamma_j^{*} \tag{34}$$

TABLE 2.-Reliability Calculation for Componental Fatigue of Each Case

SERVICE TIME t		$10.^5$	$10.^4$	$10.^3$	$10.^2$	10.	1.
	BETA1	1.392	1.995	2.346	2.723	3.165	3.666
BASIC CASE	BETA2	1.330	1.974	2.362	2.602	3.131	3.689
	BETA3	1.344	1.962	2.330	2.724	3.166	3.712
	BETA1	1.398	2.006	2.359	2.738	3.181	3.682
CASE I	BETA2	1.336	1.984	2.375	2.614	3.146	3.704
	BETA3	1.347	1.972	2.343	2.739	3.182	3.727
	BETA1	1.410	2.015	2.367	2.745	3.187	3.687
CASE II	BETA2	1.326	1.995	2.384	2.615	3.152	3.709
	BETA3	1.350	1.983	2.351	2.741	3.188	3.732
	BETA1	1.417	2.027	2.382	2.761	3.204	3.703
CASE III	BETA2	1.344	2.009	2.401	2.735	3.166	3.725
	BETA3	1.377	1.996	2.366	2.775	3.206	3.748
	BETA1	1.404	2.006	2.359	2.738	3.181	3.682
CASE IV	BETA2	1.342	1.985	2.375	2.614	3.146	3.704
	BETA3	1.346	1.873	2.343	2.739	3.182	3.727

(BETA1: FORM, BETA2: SORM, BETA3: SIMULATION)

where n_j^{*} is preferable sample size for the type-j-test, n^0 is the total sample size, and γ_j^{*} is sensitivity ratio given by

$$\gamma_j^{*} = \frac{\|\gamma_j\|}{\Sigma \|\gamma_j\|} \tag{35}$$

Table 3 shows the sensitivity γ_j which were calculated from the result of Table 2, where $\Delta n_j = 1$. With the value of sensitivity in Table 3, the sensitivity ratio γ_j^{*} and recommended allocation of sample size are given in Table 4 and 5, where $n^0 = 100$, which are corresponding to the value of service time and reliability indices.

From the information of Table 5, one can easily make a decision for the allocation of sample size by some kinds of criterion in line with his design target. For example, under the service time criterion

TABLE 3.-Sensitivity γ_j for Service Time and Reliability Indices

SERVICE TIME t		$10.^5$	$10.^4$	$10.^3$	$10.^2$	10.	1.
	BETA1	.006	.011	.013	.015	.016	.016
γ_{I}	BETA2	.006	.010	.013	.012	.015	.015
	BETA3	.003	.010	.013	.015	.016	.015
	BETA1	.018	.020	.021	.022	.022	.021
γ_{II}	BETA2	-.004	.021	.022	.013	.021	.020
	BETA3	.006	.021	.021	.017	.022	.020
	BETA1	.025	.032	.036	.038	.039	.037
γ_{III}	BETA2	.014	.035	.039	.133	.035	.036
	BETA3	.033	.034	.036	.051	.040	.036
	BETA1	.012	.011	.013	.015	.016	.016
γ_{IV}	BETA2	.012	.011	.013	.012	.015	.015
	BETA3	.002	.011	.013	.015	.016	.015

TABLE 4.-Sensitivity Ratio γ_j^* for Service Time and Reliability Indices

SERVICE TIME t		$10.^5$	$10.^4$	$10.^3$	$10.^2$	10.	1.
γ_I^*	BETA1	.098	.149	.157	.167	.172	.178
	BETA2	.167	.130	.149	.071	.174	.174
	BETA3	.068	.132	.157	.153	.170	.174
γ_{II}^*	BETA1	.295	.270	.253	.244	.237	.233
	BETA2	.111	.273	.253	.076	.244	.233
	BETA3	.136	.276	.253	.173	.234	.233
γ_{III}^*	BETA1	.410	.432	.434	.422	.419	.411
	BETA2	.389	.455	.448	.782	.407	.419
	BETA3	.750	.447	.434	.520	.426	.419
γ_{IV}^*	BETA1	.197	.149	.157	.167	.172	.178
	BETA2	.333	.143	.149	.071	.174	.174
	BETA3	.045	.145	.157	.153	.170	.174

TABLE 5.-Preferable Allocation of Sample Size

SERVICE TIME t		$10.^5$	$10.^4$	$10.^3$	$10.^2$	10.	1.
n_I^*	BETA1	10	15	16	17	17	18
	BETA2	17	13	15	7	17	17
	BETA3	7	13	16	15	17	17
n_{II}^*	BETA1	30	27	25	24	24	23
	BETA2	11	27	25	8	24	23
	BETA3	14	28	25	17	23	23
n_{III}^*	BETA1	41	43	43	42	42	41
	BETA2	39	45	45	78	41	42
	BETA3	75	45	43	52	43	42
n_{IV}^*	BETA1	20	15	16	17	17	18
	BETA2	33	14	15	7	17	17
	BETA3	5	14	16	15	17	17

at $t = 10.^5$:

$$n_I^* = 11, \quad n_{II}^* = 18, \quad n_{III}^* = 52, \quad n_{IV}^* = 19$$

at $t = 10.^4$:

$$n_I^* = 14, \quad n_{II}^* = 27, \quad n_{III}^* = 44, \quad n_{IV}^* = 15$$

at $t = 10.^3$:

$$n_I^* = 15, \quad n_{II}^* = 25, \quad n_{III}^* = 44, \quad n_{IV}^* = 15$$

at $t = 10.^2$:

$$n_I^* = 13, \quad n_{II}^* = 16, \quad n_{III}^* = 57, \quad n_{IV}^* = 13$$

at $t = 10.$:

$$n_I^* = 17, \quad n_{II}^* = 24, \quad n_{III}^* = 42, \quad n_{IV}^* = 17$$

at $t = 1.$:

$$n_I^* = 18 \quad n_{II}^* = 23 \quad n_{III}^* = 42 \quad n_{IV}^* = 18$$

On the other hand, the test plan to design materials test will be decided when the optimal rule is based on the averaged criterion as

$$n_I^* = 15 \quad n_{II}^* = 22 \quad n_{III}^* = 47 \quad n_{IV}^* = 16$$

Conclusions

The fatigue deterioration model by Yang used herein is by no means superior to most physically based models. However, it is simple and captures the most significant aspects. Most important is its capability to be easily reformulated such that reliabilities can be evaluated conveniently by FORM/SORM even if there are substantial multi-dimensional parameter uncertainties. This allows to plan experiments such that those parameter uncertainties can be reduced in an optimal manner together with a selection of suitable design parameters. Essential is that there is a relatively simple procedure for the Bayesian updating of prior or preliminary information on unknown parameters in the model. It has been found that asymptotic concepts in Bayesian analysis are very helpful. In particular, the well-known result that maximum likelihood estimators are asymptotically normally distributed with easily evaluated means and covariances is used. Prior information, if available, is to be quantified in terms of point estimates for the parameters associated with an "equivalent" sample size expressing the "degree of belief" in those point estimates prior to testing.

The presented concept to design experiments on a reliability basis should apply quite generally and even in structural fatigue it is not restricted to Yang's fatigue model studied in more detail herein. For example, approaches based on crack growth laws together with stress intensity instability criteria can be cast into a similar formulation. Yang's model is by its very nature empirical and, probably, has in its presented form explanatory character only to a limited degree. Although several points need further study it is believed that the suggested approach to the reliability of fatigue-deteriorating structural components will be soon ready for practical application.

Appendix I. - References

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