Asymptotic crossing rate of Gaussian vector processes into intersections of failure domains

M. Hohenbichler and R. Rackwitz

Institut fuer Bauingenieurwesen III, Technical University of Munich, Munich, FRG

By use of asymptotic analysis, the asymptotic rate of exits of Gaussian vector processes with continuously differentiable sample paths into intersections of failure domains with piecewise twice differentiable boundaries is derived. After some convenient orthogonal transformations, the result only involves local properties of the failure surface at the so-called Beta-point and the cross-correlation matrix between the process and its time-derivative.

INTRODUCTION

The failure probability of highly reliable structural systems under stationary time-variant loading can well be approximated or bounded by $P_F(t) \leq P_F(0) + \nu_F T$ where $P_F(0)$ is the initial failure probability, ν_F the outcrossing rate and $[0, t]$ a given time-interval[1]. In fact, this type of bounds for the failure probability were developed even for nonstationary loading as well as failure conditions and utilized in applications as early as the 1960's[2,3,4]. If the loading can be modelled by a Gaussian vector process, only a few special, exact results for ν_F are known (see, for example, Refs 5 and 6). A complete set of results only exists for failure domains bounded by hyperplanes (see Refs 5, 6 and 7). Nonlinear failure surfaces as, for example, those arising from v. Mises yield criterion exhibit serious problems. However, application of certain methods of asymptotic analysis recently yielded results of quite general nature and computational ease not only for the initial failure probability[9,10] but also for the outcrossing rate[11,12]. These results are asymptotic in the sense that the relative error in the failure probability becomes negligible when these probabilities are small. The results available so far are for simple failure domains with at least twice differentiable failure surfaces and for unions thereof[11,13]. In the following the asymptotic crossing rate of Gaussian vector processes into intersections of failure domains is given supplementing the tools for the treatment of redundant systems.

RESULTS

Let $\underline{U} = \underline{U}(t) = (U_1(t), \ldots, U_n(t))^T$, $n \geq 2$, be a stationary normal process with continuously differentiable sample paths[14], whose autocorrelation functions $r_i(t)$ of $U_i(t)$ are twice differentiable at $t = 0$. The time derivative of $\underline{U}(t)$ is denoted by $\underline{\dot{U}} = \underline{\dot{U}}(t) = (\underline{\dot{U}}_1(t), \ldots, \underline{\dot{U}}(t))^T$. Without loss of generality it is assumed that, for each fixed but arbitrary value of t, the variables $U_i = U(t)$ are stochastically independent with zero mean and unit variance. The correlation matrix of $\underline{\dot{U}}$ is $\underline{\underline{\ddot{R}}} = E[\underline{\dot{U}}\underline{\dot{U}}] = (E[\dot{U}_i \dot{U}_j]$: $1 \leq i, j \leq n)$. The cross-correlation matrix of $\underline{U}$ and $\underline{\dot{U}}$ is $\underline{\underline{\dot{R}}} = E[\dot{U}U^T] = (E[\dot{U}_i U_j]$: $1 \leq i, j \leq n)$. Let further $k \geq 2$ and $g_1 = g_1(\underline{u}), \ldots, g_k = g_k(\underline{u})$ be piecewise continuously differentiable functions such that for the probability density $\varphi(\underline{u})$ of $\underline{U}$ the surface integrals

(A1) $$S_i = \int_{\{g_i = 0\}} \|\underline{u}\| \varphi(\underline{u}) \, ds(\underline{u}) < \infty$$

over the failure surfaces $\partial F_i = \{g_i = 0\}$ exist. $\|\underline{u}\| = (\underline{u}^T \underline{u})^{1/2}$ is the Euclidean norm of $\underline{u}$ and $\varphi_n(\underline{u}) = \varphi_n(\underline{u}; \underline{\underline{I}})$ is the multinormal density function with correlation matrix $\underline{\underline{K}} = \underline{\underline{I}}$ ($\underline{\underline{I}}$ = unit matrix). Assumption (A1) holds in almost all engineering applications.

The outcrossing rate ν_F of the process $\underline{U}(t)$ from a 'safe domain' $\mathbb{R}^n \backslash F$ into the 'failure domain' of structural states defined by

$$F := \bigcap_{i=1}^{k} \{g_i \leq 0\} \tag{1}$$

is given by the generalized Rice formula[13]:

$$\nu_F = \int_{\partial F} E[\{-\underline{\alpha}(\underline{u})^T \underline{\dot{U}}\}^+ | \underline{U} = u] \varphi(\underline{u}) \, ds(\underline{u}) \tag{2}$$

Here, $\alpha(\underline{u})$ is the outwards directed unit normal vector at a point $\underline{u}$ on the surface ∂F of F, $E[.|.]$ is the conditional mean. The notation $\{x\}^+ = \max\{0, x\}$ is used.

It is assumed that:

(A2) the failure domain F has a unique 'Beta-point' $\underline{u}^*$, i.e., a point $\underline{u}^*$ in F (or its boundary) with minimal distance to the coordinate origin. Also, the origin is not contained in F which implies that $\underline{u}^* \in \partial F$.

(A3) In a small environment of $\underline{u}^*$, the functions g_i $(1 \leq i \leq k)$ are twice continuously differentiable, and it is $g_i(\underline{u}^*) = 0$ for $1 \leq i \leq k$.

(A4) The gradients $\underline{a}_i := \text{grad}\, g_i(\underline{u}^*)$ $(1 \leq i \leq k)$ are linearly independent, and it is $\|a_i\| = 1$ for $1 \leq i \leq k$ which can always be achieved by premultiplying g_i by $1/\|\underline{a}_i\|$.

According to Lagrange's theorem $\underline{u}^*$ is always a linear combination

$$\underline{u}^* = \sum_{i=1}^{k} \gamma_i \underline{a}_i \qquad \text{with } \gamma_i \leq 0 \text{ for } 1 \leq i \leq k$$

of the $\underline{a}_i$'s, where, due to (A4), the γ_i's are uniquely

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determined. In addition, we require here that:

(A5) $\underline{u}^* = \sum_{i=1}^{k} \gamma_i \underline{a}_i$ with $\gamma_i < 0$ for $1 \leq i \leq k$.

Assumptions (A2) to (A5) parallel very much those in Ref. 10.

Next, assume that:

(A6) $\underline{a}_i^T(\dot{\underline{\underline{R}}}\,\underline{u}^*) = \underline{a}_i^T \dot{\underline{\underline{R}}}\,\underline{u}^* \neq 0$ for $1 \leq i \leq k$.

As a simple consequence of (A5) and (A6) it is

$$\underline{a}_i^T(\dot{\underline{\underline{R}}}\,\underline{u}^*) < 0 \quad \text{for at least one } i \quad (1 \leq i \leq k), \tag{3}$$

since due to $\underline{u}^T \underline{\underline{R}}\,\underline{u} = 0$ ($\underline{\underline{R}}$ is skew-symmetric)

$$0 = (\underline{u}^*)^T \underline{\underline{R}}\,\underline{u}^* = \sum_{i=1}^{k} \gamma_i (\underline{a}_i^T \dot{\underline{\underline{R}}}\,\underline{u}^*).$$

Without loss of generality, it is further assumed that the last n–k components of the vectors $\underline{a}_i = (a_{i1}, \ldots, a_{in})$ are zero:

(A7) $a_{ij} = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$.

This can always be achieved by a suitable orthogonal transformation. It implies $u_{k+1}^* = \cdots = u_n^* = 0$. Finally, define by I the (nonempty) set of all indices i $(1 \leq i \leq k)$ for which it is,

$$a_i^T(\dot{\underline{\underline{R}}}\,\underline{u}^*) < 0.$$

Under these assumptions the asymptotic behaviour of the outcrossing rate

$$\nu_F(b) := b^n \int_{\partial F} E\left[\left\{-\underline{a}(\underline{u})^T \frac{1}{b}\dot{\underline{U}}\right\}^+ \middle| \frac{1}{b}\underline{U} = \underline{u}\right] \varphi(b\underline{u})\, \mathrm{ds}(\underline{u}) \tag{4}$$

of the process $(1/b)\underline{U}(t)$ (with derivative $(1/b)\dot{\underline{U}}(t)$) from $\mathbb{R}^n \backslash F$ into F, with $b \to \infty$ a central scaling factor as in Refs 9, 10, 11, 12 can be derived. The complete derivations and proofs are given in Ref. 12.

By classical regression theory the variable $-\underline{a}^T\dot{\underline{U}}$ conditioned on $\underline{U} = \underline{u}$ is normally distributed with mean $E[-\underline{a}^T\dot{\underline{U}}|\underline{u}] = -\underline{a}^T\dot{\underline{\underline{R}}}\,\underline{u}$ and variance $\mathrm{Var}[-\underline{a}^T\dot{\underline{U}}|\underline{u}] = \underline{a}^T\underline{\underline{S}}\,\underline{a}$, the latter being independent of $\underline{u}$, where the matrix $\underline{\underline{S}} = \ddot{\underline{\underline{R}}} - \dot{\underline{\underline{R}}}\,\dot{\underline{\underline{R}}}^T$. According to Ref. 5 the conditional mean in equation (4) is:

$$\begin{aligned}\psi(\underline{a}, b, \underline{u}) &:= E\left[\left\{-\underline{a}\frac{1}{b}\dot{\underline{U}}\right\}^+ \middle| \frac{1}{b}\underline{U} = \underline{u}\right] \\ &= -\underline{a}^T\dot{\underline{\underline{R}}}\,\underline{u}\,\phi\left(b\frac{-\underline{a}^T\dot{\underline{\underline{R}}}\,\underline{u}}{(\underline{a}^T\underline{\underline{S}}\,\underline{a})^{1/2}}\right) \\ &\quad + \frac{1}{b}(\underline{a}^T\underline{\underline{S}}\,\underline{a})^{1/2}\varphi\left(b\frac{-\underline{a}^T\dot{\underline{\underline{R}}}\,\underline{u}}{(\underline{a}^T\underline{\underline{S}}\,\underline{a})^{1/2}}\right).\end{aligned} \tag{5}$$

With $\partial F_j := \partial F \cap \{g_j = 0\}$, the outcrossing rate $\nu_F(b)$ becomes

$$\begin{aligned}\nu_F(b) &= b^n \int_{\partial F} \psi(\underline{\alpha}(\underline{u}), b, \underline{u})\varphi(b\underline{u})\, \mathrm{ds}(\underline{u}) \\ &\sim b^n \sum_{j=1}^{k}\left[\psi(\underline{a}_j, b, \underline{u}^*)\int_{\partial F_j} \varphi(b\underline{u})\, \mathrm{ds}(\underline{u})\right] \\ &\sim \frac{b}{\sqrt{d}} \sum_{j=1}^{k} [\psi(\underline{a}_j, b, \underline{u}^*)\varphi(b\beta_j)\phi_{k-1}(b\underline{c}_j; \underline{\underline{K}}_j)] \\ &\sim \frac{b}{\sqrt{b}} \sum_{i\in I} [(-\underline{a}_i^T\dot{\underline{\underline{R}}}\,\underline{u}^*)\varphi(b\beta_i)\phi_{k-1}(b\underline{c}_i; \underline{\underline{K}}_i)].\end{aligned} \tag{6}$$

where

$$d := \begin{cases} 1 & \text{for} \quad k = n \\ \det(\underline{\underline{I}} - \underline{\underline{D}}) & \text{for} \quad 2 \leq k < n \end{cases}$$

$\underline{\underline{D}} = (d_{ij}: k+1 \leq i, j \leq n)$ with $d_{ij} = \sum_{s=1}^{k} \gamma_s \dfrac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j}$

$\underline{\underline{I}} = (\delta_{ij}: k+1 \leq i, j \leq n)$ (δ_{ij} = Kronecker's delta)

and for $i \in \{1, \ldots, k\}$

$\beta_i := \underline{a}_i^T\underline{u}^*$

$c_i := (c_{is}: 1 \leq s \leq k, s \neq i) \in \mathbb{R}^{k-1}$

with $c_{is} = \beta_s - ((\underline{u}^*)^T\underline{a}_i)(\underline{a}_s^T\underline{a}_i)$,

$\underline{\underline{K}}_i := (k_{ist}: 1 \leq s, t \leq k, s \neq i, t \neq i) \in \mathbb{R}^{k-1,k-1}$

with $k_{ist} := \underline{a}_s^T\underline{a}_t - (a_s^T\underline{a}_i)(\underline{a}_t^T a_i)$.

In Ref. 10 it is shown that $d \geq 0$. Here, it is additionally assumed that $d > 0$. In the second line of equation (6) the disjointness of the crossing events for different surfaces ∂F_j is used together with the fact that $\psi(\underline{a}_j, b, \underline{u})$ is a slowly varying function in a small environment of $\underline{u}^*$. Due to (A6) and equation (3) the function equation (5) approaches $-\underline{a}_j^T\dot{\underline{\underline{R}}}\,\underline{u}^*$ for $b \to \infty$ which is independent of b. The result for the surface integral in line three of equation (6) may also be derived by reasoning analogously to Ref. 5 or Ref. 7, i.e., by computing two volume integrals $P(\cap\{a_j(\underline{U}, \underline{u}^*) \leq 0\})$ and $P(\cap\{a_j(\underline{U}, \underline{u}^* + \underline{a}_j\Delta) \leq 0\})$ and letting $\Delta \to 0$. Here, the $a_j(\underline{U}, \underline{x})$'s are the asymptotic linear approximations of the failure surfaces at $\underline{u}^*$. The fourth line of equation (6) is a consequence of the fact that the function equation (5) approaches zero for $b \to \infty$ for all summation terms for which $-\underline{a}_j^T\dot{\underline{\underline{R}}}\,\underline{u}^*$ is negative.

This yields for sufficiently large β_i's an asymptotic approximation for ν_F by setting $b = 1$.

$$\nu(F) \approx \frac{1}{\sqrt{d}} \sum_{i\in I} [(-\underline{a}_i^T\dot{\underline{\underline{R}}}\,\underline{u}^*)\varphi(\beta_i)\phi_{k-1}(\underline{c}_i; \underline{\underline{K}}_i)] \tag{7}$$

Since

$$\phi_{k-1}(\underline{c}_i; \underline{\underline{K}}_i) = P\left[\bigcap_{\substack{j=1 \\ j\neq i}}^{k} \{\underline{\alpha}_j^T(\underline{U} - \underline{v}^*) \leq 0\}\right],$$

where

$$\underline{\alpha}_j = \alpha_{(i)j} = \underline{a}_j - (\underline{a}_i^T\underline{a}_j)^T\underline{a}_i$$

$$\underline{v}^* = \underline{v}_{(i)}^* = \underline{u}^* - (\underline{a}_i^T\underline{u}^*)^T\underline{a}_i$$

are the projections of $\underline{a}_j$ and $\underline{u}^*$ onto the plane orthogonal to $\underline{a}_i$ and since further due to assumption (A5) there is

$$\underline{v}^* = \sum_{\substack{j=1 \\ j\neq i}}^{k} \gamma_j \underline{\alpha}_j,$$

Ruben's asymptotic formula for the multidimensional normal integral can be written as:[10]

$$\phi_{k-1}(\underline{c}_i;\underline{\underline{K}}) = (\det(\underline{\underline{K}}_i))^{1/2}\left[\sqrt{2\pi}\sum_{s=1}^{k}\varphi(v_s^*)\right]\prod_{\substack{j=1\\j\neq i}}^{k}(-\gamma_j)^{-1}$$

Furthermore, there is

$$\beta_i^2+\sum_{s=1}^{k}(v_s^*)^2=(\underline{a}_i^T\underline{u}^*)^2+(\underline{v}^*)^T\underline{v}^*=(\underline{u}^*)^T\underline{u}^*$$

or

$$\varphi(\beta_i)\prod_{s=1}^{k}\varphi(v_s^*)=\frac{1}{\sqrt{2\pi}}\prod_{s=1}^{k}\varphi(u_s^*),$$

and, consequently:

$$\varphi(\beta_i\phi_{k-1}(\underline{c}_i;\underline{\underline{K}}_i) = (\det(\underline{\underline{K}}_i))^{1/2}\prod_{s=1}^{k}\varphi(u_s^*)\prod_{\substack{j=1\\j\neq i}}^{k}(-\gamma_j)^{-1}.$$

Together with equation (7) this yields finally:

$$v(F)\approx\frac{1}{\sqrt{d}}\prod_{s=1}^{k}\varphi(u_s^*)\sum_{i\in I}\times\left[(-\underline{a}_i^T\dot{\underline{\underline{R}}}\underline{u}^*)(\det(\underline{\underline{K}}_i))^{1/2}\prod_{\substack{j=1\\j\neq i}}^{k}(-\gamma_i)^{-1}\right]. \quad (8)$$

It can furthermore be shown that if the second part of assumption (A2) is not fulfilled, i.e., there is $g_i(\underline{v}^*)>0$ for some indices i in equation (1), then, these failure domains can be neglected asymptotically in equation (1).

In applications, the first-order version obtained by setting $d=1$ might also provide sufficient numerical accuracy. Then, an important result which presumably holds quite generally is that asymptotically the dependence of the expectation term in equation (2) can be reduced to a dependence on the Beta-point and the surface integration in equation (2) can be carried out for linear approximations of the failure surfaces in the Beta-point. The Beta-point must be found by a suitable optimization procedure[5]. It should be emphasized that the specific reasoning presented before does not apply to simple surfaces ($k=1$) as treated in Ref. 11 since $\underline{a}_i^T(\dot{\underline{\underline{R}}}\underline{u}^*)=0$ in this case.

Potential applications of these results are, for example, for redundant structural systems with brittle components and where, at a single 'load wave' more than one such components can fail.

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