Multi-Failure Mode Systems under Time-Variant Loading

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SUMMARY

The time-dependent failure probability of multicomponent systems with time-invariant system properties under stationary random loading is estimated in terms of the unconditional rate of failure resp. outcrossings. The system can have operational as well as structural components. Gaussian shock, rectangular wave and continuous processes are dealt with.

INTRODUCTION

The following upper bound and approximation (for small failure probabilities) for the failure probability of general, non-repairable, stationary systems is well-known:

$$P_f(t) \le P_f(0)+E[N(t)] = P_f(0)+\nu t \qquad (1)$$

N(t) is the counting process of failure events in [0,t], ν is the so-called unconditional outcrossing (failure) rate which can be given as

$$\nu = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\{S(t)=1\}\cap\{S(t+\Delta)=0\}) \qquad (2)$$

if S(t) is the system's state function possessing a value of 1 if it is in a safe state and zero otherwise. If the failure event can be expressed as the exit of a vector process $\underline{X}(t)$ into the failure domain V one may also define

$$\nu = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\{\underline{X}(t)\in \bar{V}\}\cap\{\underline{X}(t+\Delta)\in V\}) \qquad (3)$$

and as for the failure process in eq.(2) the process of exits

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an orderly point process. In this case, S(t) is unity if $\underline{X}(t) \in \bar{V}$ and zero otherwise. However, systems usually consist of a number of "components" for which various combinations of componential failure events form the different modes in which a system can fail. The system failure event can always be expressed as a (minimal) cut set of componential failure events, i.e.

$$F = \bigcup_i \bigcap_j F_{ij} \qquad (4)$$

Consequently, there are three basic problems in system reliability that is the calculation of

i. the componential failure probabilities (rates)
ii. the failure probability (rates) for intersections of componential failure events
iii. the failure probabilities (rates) for unions of individual or intersections of componential failure events.

In the following, some results are given if the system consists of non-structural components and of structural components subjected to time-varying loads. Structural properties and states are described in terms of a random vector process Y(t) of input quantities (loads, stiffnesses, resistances) some components of which can be time-invariant yielding the componential output quantities $\underline{X}(t)$ (load effects, stresses, etc.) Even if the input processes were independent the output processes are not because they are related by (for linear structural behaviour):

$$\underline{X}(t) = \underline{X}(0) + \underline{\underline{H}}\ \underline{Y}(t) \qquad (5)$$

or, more generally, for linear dynamic systems by:

$$\underline{X}(t) = \underline{X}(0) + \int_0^t \underline{\underline{h}}(t-\tau)\ \underline{Y}(\tau)\,d\tau \qquad (6)$$

Even if it is sometimes possible and advisable to work in the input space for formulation (5) this is generally impossible for formulation (6). Primary interest will therefore lie

in a proper quantification of the dependencies thus introduced. It will further be assumed that $\underline{Y}(t)$ or, slightly less restrictive, $\underline{X}(t)$ are zero mean, unit variance Gaussian processes (with respect to their amplitude distribution) and, in this paper, failure events are independent of the state of the system. Componential failure will be determined either by a simple renewal point process (figure I a), so-called marked shocks (figure I b) or marked renewal processes, rectangular wave renewal processes where the mark now is a rectangular wave (figure I c) or Gaussian processes with an existing second derivative of their autocorrelation function (figure I d). In figure 1 renewals are denoted by crosses, failure events as circles. In the case of a multivariate renewal process, it is assumed that it is marginally and crosswise orderly, i.e. the probability of more than one renewal in one or more components of the process in a small time interval is of negligible order.

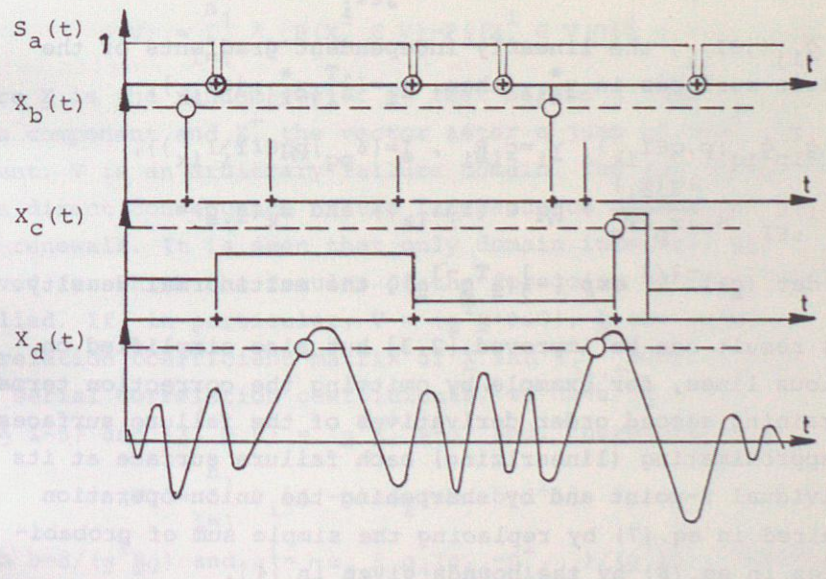

Figure 1: Types of Renewal-Failure-Outcrossing Events

ZERO-TIME FAILURE PROBABILITY

By definition, components of type a. are in a functioning state at t=O and, as an assumption, for a type b. component no shock is present at that time. It follows that

$$P_f(0) = P(\underline{X}(0) \in \cup \cap V_{ij}) \tag{7}$$

where $\underline{X}(0)$ includes only processes of type c. and d. Solution of eq. (7) is straight-forward if one uses the well-known techniques of the first-order reliability methods [1]. These can be made asymptotically exact yielding [2]

$$P_f(0) \sim \sum_{i \in I} \varphi_{j \in I_i}(\underline{c}_i; \underline{\underline{R}}_i) \prod_{j \in I_i} (-\gamma_{ij})^{-1} [\det(\underline{\underline{I}} - \underline{\underline{B}}_i)]^{-1/2} \tag{8}$$

where I_i is the subset of components in the i-th cut set for which $V_{ij} = \{g_{ij}(\underline{x}) \le 0\}$ but $g_{ij}(\underline{x}) = 0$ at least for one j. If $\underline{x}^*_{ij}$ is the joint β-point, i.e. the point for which

$$\beta^*_{ij} = \min\{||\underline{x}||\} \text{ for } \{\underline{x}: \bigcap_{j \in I_i} g_{ij}(\underline{x}) \le 0\} \tag{9}$$

and $\dot{\underline{g}}_{ij}(j \in I_{ik})$ the linearly independent gradients of the failure surfaces in $\underline{x}^*_{ij}$, then, $\underline{c}_i = \{\dot{\underline{g}}^T_{ij} \underline{x}^*_{ij}; j \in I_{ik}\}$,

$\underline{\underline{R}}_i = \{\dot{\underline{g}}^T_{ip} \dot{\underline{g}}_{iq}; p,q \in I_{ik}\}$, $\gamma_i = \underline{c}_i \underline{\underline{R}}_i^{-1}$, $\underline{\underline{I}} = \{\delta_{pq}; pq \in (I_i \setminus I_{ik})\}$,

$\underline{\underline{B}}_i = \{\sum_{s \in I_{ik}} \gamma_s \frac{\partial g_s(\underline{x}_i)}{\partial x_p \partial x_q}; pq \in (I_i \setminus I_{ik})\}$ and $\varphi_k(\underline{a}; \underline{\underline{c}}) =$

$(2\pi \det(\underline{\underline{c}}))^{-1/2} \exp[-\frac{1}{2}(\underline{a}^T \underline{\underline{c}}^{-1} \underline{a})]$ the multinormal density.

This result can be improved [2,3] but also simplified on various lines, for example, by omitting the correction terms containing second order derivatives of the failure surfaces, by approximating (linearizing) each failure surface at its individual β-point and by sharpening the union operation required in eq.(7) by replacing the simple sum of probabilities in eq.(8) by the bounds given in [4].

STATIONARY OUTCROSSING RATES

Despite the suggestions made in [5] to use the so-called point of maximum density of outcrossings as an approximation point of an arbitrary failure surface, some simulation results and, stronger, a still limited set of asymptotic results as, for example, given in [6,7] let us presume that it is generally acceptable to use the β-point found in the foregoing section as a convenient approximation point for the failure surface. As for the time-invariant case, the inclusion of quadratic terms might be necessary in the sense of asymptotically correct results. Yet, linear approximations in the individual or joint β-points may be used for first-order estimates and which are the subject of the following.

Let the linearizations of the failure surfaces be found, i.e. $V_{rs} = \{g_{rs}(\underline{x}) \le 0\} \approx \{\underline{\alpha}^T_{rs} \underline{x} + \beta_{rs} \le 0\}$. Assume further the first n_1 components of $\underline{X}(t)$ of type c. and the renewals occurring independently with rate λ_i. Overall orderliness of the renewals implies that [5]

$$\nu(V) = \sum_{i=1}^{n_1} \lambda_i [P(\underline{X}^i_+ \in V) - P((\underline{X}^i_+ \in V) \cap (\underline{X} \in V))] \tag{10}$$

where $\underline{X}$ is the random vector at rest before a jump of the i-th component and $\underline{X}^+_i$ the vector after a jump of that component. V is an arbitrary failure domain. The sum in eq.(10) is a direct consequence of the independence assumption for the renewals. It is seen that only domain integrals are involved for which the results of the foregoing section can be applied. If, in particular, $V = \{\underline{\alpha}^T \underline{x} + \beta \le 0\}$, $\underline{\underline{R}}$ the cross-correlation coefficient matrix of $\underline{X}$ and $\rho^i_{ji,+} = \mathrm{Corr}[X_j, X^i_+]$ the serial correlation coefficients, we have $\{\underline{X} \in V\} = \{\underline{\alpha}^T \underline{X} \le -\beta\}$ and $\{\underline{X}^i_+ \in V\} = \{\underline{\alpha}^T \underline{X}^i_+ \le -\beta\}$ and, therefore [8]:

$$\nu(V) = \sum_{i=1}^{n_1} \lambda_i [\phi(-b) - \phi_2(-b, -b; \rho^i)] \tag{11}$$

with $b = \beta/(\underline{\alpha}^T \underline{\underline{R}} \underline{\alpha})$ and $\rho^i = 1 - \alpha_i \sum_{j=1}^{n} \alpha_j (\rho_{ij} - \rho^i_{ij,+})/(\underline{\alpha}^T \underline{\underline{R}} \underline{\alpha})$. Furthermore, a parallel system experiences an outcrossing if before the jump the vector $\underline{X}$ is in at least one safe componential domain but then jumps into the joint failure domain. It is easily shown that V in eq.(10) must be replaced by $\cap V_r$ with $V_r=\{\underline{\alpha}_r^T\underline{x} + \beta_r \le 0\}$ and, hence

$$\nu(\cap V_r) = \sum_{i=1}^{n} \lambda_i \, P[(\underline{X}_+^i \in \bigcap_r V_r) \cap (\underline{X} \in (\Omega \setminus \bigcap_r V_r))]$$

$$= \sum_{i=1}^{n} \lambda_i [\phi_m(-\underline{c};\underline{\underline{R}}_c) - \phi_{2m}(-\underline{d},-\underline{d};\underline{\underline{R}}_d^i)] \quad \text{in which}$$

$$\underline{c} = \{\beta_r/(\underline{\alpha}_r^T\underline{\underline{R}}\underline{\alpha}_r)^{1/2}; r=1,\ldots,m\}, \quad \underline{d} = [\underline{c},\underline{c}]^T$$

$$\underline{\underline{R}}_c = \{(\underline{\alpha}_r^T\underline{\underline{R}}\underline{\alpha}_s)/((\underline{\alpha}_r^T\underline{\underline{R}}\underline{\alpha}_r)^{1/2}(\underline{\alpha}_s^T\underline{\underline{R}}\underline{\alpha}_s)^{1/2}); r,s=1,\ldots m\}$$

$$\underline{\underline{R}}_d^i = \begin{bmatrix} \underline{\underline{R}}_c & \underline{\underline{R}}_e^i \\ \underline{\underline{R}}_e^i & \underline{\underline{R}}_c \end{bmatrix}$$

$$\underline{\underline{R}}_e^i = \{[\underline{\alpha}_r^T\underline{\underline{R}}\underline{\alpha}_t - \alpha_{t,i}\sum_{j=1}^{n}\alpha_{rj}(\rho_{ij}-\rho_{ij,+}^i)]/((\underline{\alpha}_r^T\underline{\underline{R}}\underline{\alpha}_r)^{1/2}(\underline{\alpha}_t^T\underline{\underline{R}}\underline{\alpha}_t)^{1/2})\} \tag{12}$$

A non-redundant (series) system fails if a jumping component of $\underline{X}$ enters any of the system component failure domains but, before the jump was in the joint safe domain of all system components. More generally, a cut set system fails if any of its parallel systems experiences an outcrossing of $\underline{X}$ but the jumping i-th component of $\underline{X}$ remains in the safe domain of at least one system component in the other parallel systems. We have

$$v(V) = \sum_{k=1}^{K}\sum_{i=1}^{n} \lambda_i \, P((\underline{X}_+^i \in \bigcap_{q\in M_k} V_q) \cap \bigcap_{j=1}^{K} \bigcup_{p\in M_j} (\underline{X} \in \bar{V}_p))$$

$$= \sum_{k=1}^{K}\sum_{i=1}^{n} \lambda_i [P(\underline{X}_+^i \in \bigcap_{q\in M_k} V_q) -$$

$$P(\bigcup_{j=1}^{K} (\underline{X}_+^i \in \bigcap_{q\in M_k} V_q) \cap (\underline{X} \in \bigcap_{r\in M_j, M_k} V_r))] \tag{13}$$

The second term is a correction term and can be bounded by the use of the results in [4].

It is now very easy to add processes of type b. According to the independence assumption for the renewals the same formulae apply with $\underline{X}$ as before but $\underline{X}_i^+$ the vector of rectangular wave processes augmented by the i-th shock component. Remember, shock components have zero amplitude except during the jump.

Furthermore, system components of type a. can be added. The probabilities in square brackets in eq.(10) must be set as unity. The componential failure probability is bounded from the above by $\lambda_i t$ but more accurate values can be used if their distribution of the time to failure is known. In this case, system components can also be assumed repairable with componential failure probabilities replaced by the corresponding unavailabilities [9].

The consideration of processes of type d. is more involved. The multivariate generalisation of Rice's formula reads [10]

$$\nu(V) = \int_{\partial V} E_+[-\underline{\alpha}^T(\underline{x})\dot{\underline{X}} \mid \underline{X}=\underline{x}] \; \varphi(\underline{x})\,ds(\underline{x}) \tag{14}$$

where $\underline{\alpha}(\underline{x})$ is the unit normal of the failure surface $g(\underline{x})=0$ at $\underline{x}$, ∂V the surface of V and $ds(\underline{x})$ the surface element in $\underline{x}$. Without loss of generality, one can assume $\underline{X}(t)$ an independent vector. Only if the expectation in eq.(14) is independent of $\underline{x}$, (implies independence of $\underline{X}$ and $\dot{\underline{X}}$), we have for $V = \{\underline{\alpha}^T\underline{x} + \beta \le 0\}$ and H the failure half-space

$$\nu(V) = E_+[-\underline{\alpha}^T\dot{\underline{X}}] \int_{\partial H} \varphi(\underline{x})\,ds(\underline{x}) = E_+[\dot{X}_N]\varphi(\beta) \tag{15}$$

Otherwise the surface integral in eq.(14) appears intractable analytically as well as numerically in spaces of higher dimension.

However, as pointed out in [11], eq.(15) is an upper bound with $E_+[-\dot{X}_N] = \frac{1}{\sqrt{2\pi}} D[\underline{\alpha}^T\dot{\underline{X}}]$ even for dependent vectors $\dot{\underline{X}}$ and $\underline{X}$. Moreover, it can be shown that under some not too restrictive conditions and increasing β the integral (14) is dominated by the second factor in the integrand and here only by the vicinity of the β-point. Therefore, eq.(15) can be modified to

$$\nu(V) = E_+[\dot{X}_N | \underline{x}^*]\varphi(\beta) \tag{16}$$

with

$$E_+[\dot{X}_N | \underline{x}^*] = \mu\phi(\frac{\mu}{\sigma})+\sigma\varphi(\frac{\mu}{\sigma}) \tag{17}$$

$$\mu = E[-\underline{\alpha}^T\dot{\underline{X}} | \underline{x}^*] = -\underline{\alpha}^T\underline{\underline{\Sigma}}_{\dot{X}X}\underline{x}^* \tag{18}$$

$$\sigma^2 = \mathrm{Var}[-\underline{\alpha}^T\dot{\underline{X}} | \underline{x}^*] = \underline{\alpha}^T(\underline{\underline{\Sigma}}_{\dot{X}\dot{X}}-\underline{\underline{\Sigma}}^T_{X\dot{X}}\underline{\underline{\Sigma}}_{X\dot{X}})\underline{\alpha} \tag{19}$$

In generalisation of the interpretation for Rice's formula, i.e. that the outcrossing rate is the probability (per small unit time) that the velocity $\dot{X}_N$ is such to move $\underline{X}$ into the failure domain given that $\underline{X}$ is on the failure surface times the probability that it is on that surface, the outcrossing rate for a parallel system is the probability of $\underline{X}$ being in all failure domains except one where $\underline{X}$ is on its failure surface times the probability that $\underline{X}$ moves into the respective failure domain. Obviously, the various outcrossing events are disjoint. Let $\underline{x}^*$ now be the joint β-point of $V=\cap V_i$. It can then be shown that, in first order approximation,

$$\nu(\cap V_i) = \sum_{i=1}^{n_2} E_+[-\underline{\alpha}_i^T\dot{\underline{X}} | \underline{x}^*]\varphi(\underline{\alpha}_i^T\underline{x}^*)\phi_{n_2-1}(\underline{c}_i;\underline{\underline{R}}_i) \tag{20}$$

with $E_+[\ .\ ]$ according to eq.(17), $\underline{\alpha}_i$ the gradients of the failure surfaces in $\underline{x}^*$ (can, alternatively and approximately, be evaluated in the individual β-points)with safety indices $\beta_i=\underline{\alpha}_i^T\underline{x}^*$, $\underline{c}_i=\{\beta_s-(\underline{\alpha}_i^T\underline{x}^*)(\underline{\alpha}_s^T\underline{\alpha}_i);s\neq i\}$ and $\underline{\underline{R}}_i=\{\underline{\alpha}_s^T\underline{\alpha}_t-(\underline{\alpha}_s^T\underline{\alpha}_i)(\underline{\alpha}_t^T\underline{\alpha}_i);$ $s,t\neq i\}$.

The formulation can be extended to (minimal) cut set systems. Eqs. (14),(15) or (20) can be interpreted as follows.The outcrossing rate is the probability of moves of $\underline{X}$ into the failure domain (conditioned on $\underline{X}=\underline{x}^*$) in first approximation times the probability that $\underline{X}$ is on the failure surface. This last probability is non-zero because the set $\underline{X} \in G$ with $G=\{g(\underline{X})=0\}$ is non-empty except for some easily detected degenerate cases. Therefore, the following obvious short hand

notation of eq.(16) is used.

$$\nu(V) = P(\underline{X}\in G)E_+[\dot{X}_N | \underline{x}^*] \tag{21}$$

In analogy to the jump process case a (minimal) cut set system fails if $\underline{X}$ is in all failure domains of a cut except one where it is on the failure surface and moves into the last failure domain provided that it remains in at least one safe domain in the other cuts.

Formally, one has:

$$\nu(V) = \sum_{k=1}^{K}\sum_{q\in M_k} P((\underline{X}\in G_q)\cap\bigcap_{\substack{i\in M_k\\ i\neq q}}(\underline{X}\in V_i)\cap\bigcap_{\substack{j=1\\ j\neq k}}\bigcup_{\substack{p\in M_j\\ p\neq q}}(\underline{X}\in\overline{V}_p))E_+[\dot{X}_{N,q} | \underline{x}_k^*] \tag{22}$$

The probabilities in eq.(22) can be expanded.

$$P((\underline{X}\in G_q)\cap\bigcap_{\substack{i\in M_k\\ i\neq q}}(\underline{X}\in V_i)\cap\bigcap_{\substack{j=1\\ j\neq k}}\bigcup_{\substack{p\in M_j\\ p\neq q}}(\underline{X}\in\overline{V}_p)) =$$

$$P((\underline{X}\in G_q)\cap\bigcap_{\substack{i\in M_k\\ i\neq q}}(\underline{X}\in V_i)) -$$

$$P(\bigcup_{\substack{j=1\\ j\neq k}}^{K}\bigcap_{\substack{r\in M_j, M_k\\ r\neq q}}(\underline{X}\in V_r)\cap(\underline{X}\in G_q)) \tag{23}$$

The computation of the last correction term can be quite involved and may be simplified in practical applications.

It is now straight forward but rather lengthy to formulate the outcrossing rate (unconditional failure rate) if the failure phenomenon (type a) is present as well as the load effect processes of types b. to d. For a simple component one has

$$\nu(V) = \sum_{i=1}^{n_a}\nu_i + \sum_{i=1}^{n_b}\lambda_{b,i}P(\underline{X}_{b+c+i}\in\overline{V})+$$

$$+\sum_{i=1}^{n_c} \lambda_{c,i} P(\{\underline{X}_{b+c} \in \nabla\} \cap \{\underline{X}^{i}_{b+c,+} \in V\})$$

$$+P(\underline{X}_{d+c} \in G) E_{+}[\dot{\underline{X}}_{N,d} | \underline{x}^{*}_{d+c}] \qquad (24)$$

with obvious reference to the uncertainty space by the indices. It is clear that time-invariant components of dimension e simply increase the uncertainty space by e dimensions. The analogous formulae for systems are not given here.

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