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OUTCROSSING RATES OF GAUSSIAN VECTOR PROCESSES FOR CUT SETS OF COMPONENTIAL FAILURE DOMAINS

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ABSTRACT

The outcrossing rate for not necessarily stationary, differentiable Gaussian vector processes into intersections, unions or unions of intersections of deterministically time-varying linearly bounded failure domains is given on the basis of the generalized Rice-formula. The expectation term in this formula is calculated exactly or closely bounded. The remaining surface integral is expressed in terms of the multi-normal integral which can be determined numerically. The formulae can also be used as approximations for smooth non-linear failure boundaries which is justified by reference to some recent results of asymptotic analysis.

KEYWORDS

Gaussian processes; outcrossing rates; time-variant failure domains; minimal cut sets; first-order reliability

1. INTRODUCTION

The exact analysis of the reliability of structures under time-variant multidimensional load processes with or without dynamic effects requires the determination of the distribution function of the time T to first failure. The failure probability is $P_f(t)=P(T\le t), t\ge 0$. Unfortunately, no general solution exists but the following well-known upper bound has been established [1]

$$P_f(t)\le P_f(0)+\int_0^t \nu(\tau)d\tau \qquad (1)$$

where $P_f(0)$ is the initial failure probability and $\nu(\tau)$ the rate of exits of a mean-square differentiable vector process $\underline{X}(\tau)=(X_1(\tau),X_2(\tau),\ldots,X_n(\tau))^T$ into the failure domain $F(\tau)$ with boundary $\partial F(\tau)$. Rather generally, $F(\tau)$ is given by $F(\tau)=\cup\cap F_{rs}(\tau)$ where $F_{rs}(\tau)=\{\underline{x}:g_{rs}(\underline{x},\tau)<0\}$ and $\partial F_{rs}(\tau)=\{\underline{x}:g_{rs}(\underline{x},\tau)=0\}$ a deterministic function of time τ. Problems of this kind, for example, are met in the analysis of multicomponent structures subjected to random excitation with or without load-effect induced variation of $F_{rs}(\tau)$ with time (for example, due to damage accumulation).

An exact evaluation of eq.(1) for arbitrary boundaries $\partial F_{rs}(\tau)$ and processes $\underline{X}(\tau)$ still appears hopeless. However, $P_f(0)$ as well as $\nu(\tau)$ can be approximated or bounded rather efficiently if $\underline{X}(\tau)$ is a Gaussian process whose autocorrelation functions are twice differentiable at $\tau=0$ and $F(\tau)$ consists of combinations of intersections and unions of half spaces and the time-derivative of $g(\underline{x},\tau)$ exists on $\partial F(\tau)$. Without loss of generality the required first and second moments of the standardized process can, for any arbitrary but fixed τ, be given as (reference to τ now being omitted),

$$E[\underline{X}] = E[\dot{\underline{X}}] = \underline{0}$$

$$\underline{\underline{R}} = E[\underline{X}\ \underline{X}^T] = \underline{\underline{I}}$$

$$\dot{\underline{\underline{R}}} = E[\dot{\underline{X}}\ \underline{X}^T]$$

$$\ddot{\underline{\underline{R}}} = E[\dot{\underline{X}}\ \dot{\underline{X}}^T]$$

and the individual failure domains are represented by

$$F_i(\tau) = \{\underline{x}:g_i(\underline{x},\tau)<0\}=\{\underline{x}:-\underline{\alpha}_i^T(\tau)\underline{x}+\beta_i(\tau)<0\} \qquad (2)$$

with $-\underline{\alpha}_i(\|\underline{\alpha}_i\|=1)$ the vector of direction cosines of $g_i(\underline{x},\tau)$ on $\partial F_i(\tau)$ and $\beta_i(\tau)=\min\{\|\underline{x}\| \,|\, g_i(\underline{x},\tau)\leq 0\}$ the so-called safety-index. The supposed existence of the time derivative of $g_i(\underline{x},\tau)$ here implies that the time-derivatives of $\underline{\alpha}_i(\tau)$ and $\beta_i(\tau)$ exist. In the following some results are derived utilizing the fact that multinormal integrals can now be calculated with moderate effort and high numerical accuracy [2]. The same method can be applied to determine $P_f(0)$.

2. OUTCROSSING RATES FOR INTERSECTIONS AND UNIONS

Let $F=\bigcap_{i=1}^{k} F_i$ with $1 \leq k \leq n$ and the $\underline{\alpha}_i$'s, $i=1,2,\ldots,k$ be linearly independent. Also, $|\beta_i(\tau)|>0$ for all i meaning that the origin is not contained in any $F_i(\tau)$. Application of the generalized Rice-formula [3] yields in noting that the normal vectors $\underline{\alpha}_i$ on each subsurface $\partial F_i(\tau)=\partial F(\tau) \cap \{\underline{x}:\underline{\alpha}_i^T(\tau)\underline{x}+\beta_i(\tau)=0\}$ do not depend on $\underline{Y}=\underline{x}$

$$\nu = \sum_{i=1}^{k} \int_{\partial F_i} E[\{\underline{\alpha}_i^T\underline{\dot{X}}-(\dot{\beta}_i-\sum_{j=1}^{n}\dot{\alpha}_{ij})\}^+|\underline{X}=\underline{y}]\varphi_n(\underline{y})ds(\underline{y}) \tag{3}$$

where $\{a\}^+=\max\{0,a\}$, $\dot{\beta}_i-\sum_{j=1}^{n}\dot{\alpha}_{ij}$ the change of $F_i(\tau)$ during a small time-interval, $\varphi_n(\underline{y})$ the density of $\underline{X}$ and $ds(\underline{y})$ denotes surface integration over ∂F_i. Using some standard results of normal distribution theory [4] the conditional distribution of $\underline{\alpha}_i^T\underline{\dot{X}}$ given $\underline{X}=\underline{y}$ is univariate normal with mean $m_i(\underline{y})=\underline{\alpha}_i^T\underline{\underline{\dot{R}}}\underline{y}$ and variance $\sigma_i^2=\underline{\alpha}_i^T(\underline{\underline{\ddot{R}}}-\underline{\underline{\dot{R}}}\,\underline{\underline{\dot{R}}}^T)\underline{\alpha}_i$. Performance of the expectation operation in eq.(3) results in

$$\Psi(\dot{a}_i,m_i(\underline{y}),\sigma_i) = E[\{\underline{\alpha}_i^T\underline{\dot{X}}-(\dot{\beta}_i-\sum_{i=1}^{n}\dot{\alpha}_{ij})\}^+|\underline{X}=\underline{y}]$$

$$= \sigma_i\varphi(\frac{\dot{a}_i-m_i(\underline{y})}{\sigma_i})-((\dot{a}_i-m_i(\underline{y}))\phi(-\frac{\dot{a}_i-m_i(\underline{y})}{\sigma_i}) \tag{4}$$

with $\dot{a}_i=\dot{\beta}_i-\sum_{j=1}^{n}\dot{\alpha}_{ij}$ and $\varphi(.)$ and $\phi(.)$ the standard normal density and distribution function, respectively. Eq.(3) is rewritten with eq.(4) as

$$\nu = \sum_{i=1}^{k} \int_{\partial F_i} \Psi(\dot{a}_i,m_i(\underline{y}),\sigma_i)\varphi_n(\underline{y})ds(\underline{y}) \tag{5}$$

where it is noted that the function Ψ also depends on $\underline{y}$. No general solution appears possible for this integral.

A special case has already been pointed out in [1] when $\underline{X}$ and $\underline{\dot{X}}$ are independent implying that Ψ does not depend on $\underline{y}$ and, therefore, $m_i(\underline{y})=0$ and $\sigma_i^2=\kappa_i^2=\underline{\alpha}_i^T\underline{\underline{\ddot{R}}}\underline{\alpha}_i$. Also, if $\underline{X}$ is stationary one can make use of the fact that the scalar X and its derivate $\dot{X}$ are uncorrelated. Since the conditional distribution of $\underline{\alpha}_i^T\underline{\dot{X}}$ given

$\underline{X}=\underline{y}$ is the same as the conditional distribution of $\underline{\alpha}_i^T\underline{\dot{X}}$ given $\underline{\alpha}_i^T\underline{\underline{\dot{R}}}\underline{X}=\underline{\alpha}_i^T\underline{\underline{\dot{R}}}\underline{y}$ a suitable coordinate rotation will always establish the independence of $\underline{\alpha}_i^T\underline{\dot{X}}$ and $\underline{\alpha}_i^T\underline{\underline{\dot{R}}}\underline{X}$. Then, the expectation term in eq. (3) can be separated as before and the variance of $\underline{\alpha}_i^T\underline{\dot{X}}$ becomes $\kappa_i^2=\underline{\alpha}_i^T\underline{\underline{\ddot{R}}}\underline{\alpha}_i$ [5]. It follows that

$$\nu = \sum_{i=1}^{k} \Psi(\dot{a}_i,0,\kappa_i) \int_{\partial F_i} \varphi_n(\underline{y})\,ds(\underline{y}) \qquad (6)$$

For $\dot{a}_i=0$, it is simply $\Psi(0,0,\kappa_i)=\kappa_i/\sqrt{2\pi}$. The remaining surface integrals can be solved by generalizing an argument in [5] and [6] (compare also figure 1):

$$\Gamma_\cap^i = \int_{\partial F_i} \varphi_n(\underline{y})\,ds(\underline{y}) = \varphi(\beta_i)\,\phi_{k-1}(\underline{b}_i;\underline{\underline{B}}_i) \qquad (7)$$

with $\underline{b}_i = (\beta_s-\beta_i\underline{\alpha}_s^T\underline{\alpha}_i;\ 1\le s\le k;\ s\neq i)$

$\underline{\underline{B}}_i = (\underline{\alpha}_s^T\underline{\alpha}_t-(\underline{\alpha}_s^T\underline{\alpha}_i)(\underline{\alpha}_t^T\underline{\alpha}_i);\ 1\le s,t\le k;\ s,t\neq i)$

and $\phi_0 = 1$

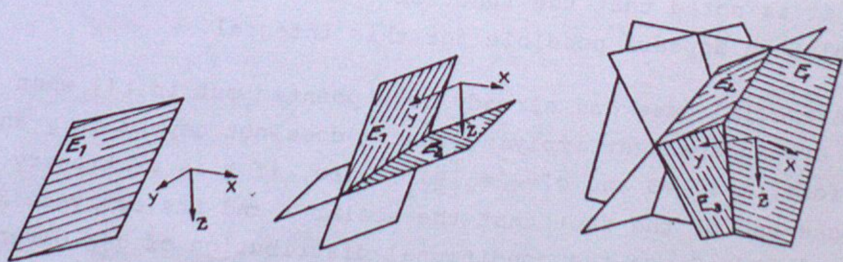
The outcrossing rate for a union of failure domains can also be given following a completely analogous derivation:

$$\nu = \sum_{i=1}^{k} \Psi(\dot{a}_i,0,\kappa_i)\,\Gamma_\cup^i \qquad (8)$$

in which

$$\Gamma_\cup^i = \varphi(\beta_i)\,[1-\phi_{k-1}(\underline{b}_i;\underline{\underline{B}}_i)] \qquad (9)$$

with the same notation as before but now $\phi_0=0$.



k=1 k=2 k=3

Figure 1: Intersecting failure domains

In the general case, an upper bound can be derived by substituting $\Psi(\dot{a}_i, m_i(\underline{y}), \sigma_i)$ by its maximum value for $\underline{y} \in \partial F_i$. Since it can be shown that $\Psi(\dot{a}_i, m_i(\underline{y}), \sigma_i)$ is monotonically increasing with $m_i(\underline{y})$, it suffices to solve the following problem of linear programming:

$$m_i(\underline{y}_i') = \max\{\alpha_i^T \dot{\underline{R}} \underline{y}\} \text{ for } \begin{cases} \{\underline{y}: g_i(\underline{y})=0 \cap \bigcap_{i \neq j} g_j(\underline{y}) \leq 0\} \\ \\ \{\underline{y}: g_i(\underline{y})=0 \cap \bigcap_{i \neq j} g_j(\underline{y}) > 0\} \end{cases} \tag{10}$$

The upper set of constraints is to be used for intersections while the lower set of constraints is for unions of failure domains. One obtains

$$\nu \leq \sum_{i=1}^{k} \Psi(\dot{a}_i, m_i(y_i'), \sigma_i)\Gamma_q^i \tag{11}$$

where q equals either $\cap$ or $\cup$ whichever case is of interest. It should be noted that in contrast to [5], the function Ψ here is bounded or approximated instead of the surface integral Γ in (7).

3. OUTCROSSINGS INTO (MINIMAL) CUT SETS

We now discuss the case of exits into unions of cut sets

$$F = \bigcup_{i=1}^{m} \bigcap_{j(i)=j_1}^{n_i} F_{ij(i)} = \bigcup_{i=1}^{m} C_i \tag{12}$$

with $j(i) \in \{j_1, \ldots, n_i\} \subset \{1, \ldots, k\}$ indicating those failure domains which occur in the i-th cut, $i=1,\ldots,m$.

As pointed out in [6] an explicit expression for the crossing rate can be obtained by using the method of exclusion and inclusion, but the resulting formula is quite cumbersome. For practical purposes suitable bounding techniques appear appropriate. Observing that ν_F can be considered as a finite positive set function the bounds in [7] apply as well to crossing rates:

$$\max_{i=1}^{m}\{\nu_{C_i}\} \leq \nu_F^{\,l} \leq \nu_F \leq \nu_F^{\,u} \leq \sum_{i=1}^{m} \nu_{C_i} \tag{13}$$

with

$$\nu_F^{\,l} = \nu_{C_1} + \sum_{i=2}^{m} \{\nu_{C_i} - \sum_{j<i} \nu_{C_i \cap C_j}\}^+$$

and

$$\nu_F^{\,u} = \nu_{C_1} + \sum_{i=2}^{m} (\nu_{C_i} - \max_{j<i}\{\nu_{C_i \cap C_j}\}).$$

The componential crossing rates ν_{C_i} and $\nu_{C_i \cap C_j}$ can be calculated by equations (6) or (11) with eq.(7).

4. THE NON-REGULAR CASE

Some degenerate cases can occur by weakening the assumptions preceeding eq.(3). Recall that under the assumption of linear independence of the $\underline{\alpha}_i$'s, k=1 corresponds to a simple halfspace while k=n corresponds to an unrestricted cone in $\mathbb{R}^n$. Let $k \leq n$ and assume the vectors $\underline{\alpha}_i$, i=1,...,k are linearly dependent. Then the hyperplanes ∂F_i do not intersect in an unique manner. A number of cases is possible. Figures 2.a-b show two possible arrangements for k=n=3 each resulting in a redundancy of some of the failure domains. In these cases the redundent components have to be omitted and then the crossing rates are calculated as above.

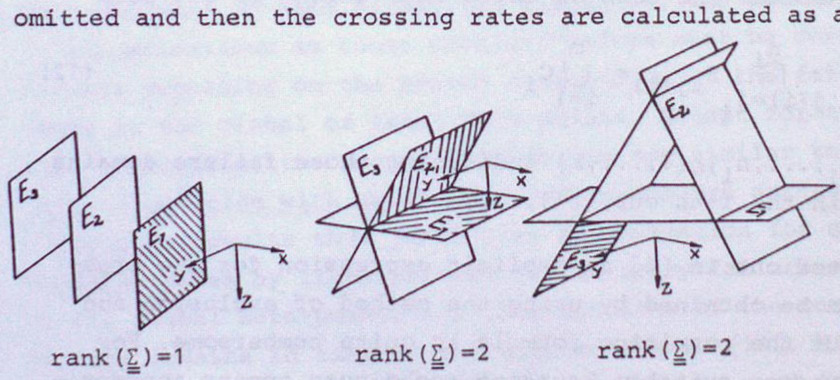


Figure 2: Linearly dependent failure surfaces

Another possibility is shown in figure 2.c where the first failure component whose boundary is given by the plane E_1 is only of interest in the case of a union failure domain. In order to calculate the surface integrals the equations of the straight lines in which the planes intersect must be known. This example also gives an idea for the other degenerate case, k>n, i.e. the $\underline{\alpha}_i$'s always being dependent. The calculation of the surface integrals now requires the knowledge of the cuts of the individual surfaces as the solution of a system of linear equations with defective rank. In both cases the linear dependence of the $\underline{\alpha}_i$'s results in a degenerate behaviour of the probability distribution of the vector $\underline{Y}(t)=(\underline{\alpha}_1^T\underline{X}(t),\ldots,\underline{\alpha}_k^T\underline{X}(t))^T$, as can easily be seen by inspection of the covariance matrix $\underline{\underline{\Sigma}}=\mathrm{Cov}(\underline{Y}(t),\underline{Y}(t))=((\underline{\alpha}_i^T\underline{\alpha}_j);$ $i,j=1,\ldots,k)$ which is not of full rank k, i.e. rank $(\underline{\underline{\Sigma}})<k$. The non-regular case must be solved by individual inspection. No simple general strategy appears available.

5. OUTCROSSING RATES FOR GENERAL NON-LINEAR FAILURE BOUNDARIES

If the componental failure surfaces are given in terms of arbitrary smooth functions g_i, exact results for the outcrossing rates are rarely available. Recently, asymptotically exact approximations have been derived in [8] for k=1 and in [9] for $2\leq k\leq n$ and intersection failure domains. In these papers it is shown that the major contribution to the crossing rate is asymptotically given by the outcrossings near the (local or global) Beta-points. (See [10] for precise definitions of local and global Beta-points.) To gain asymptotic exactness, the estimates based on linearisations as those obtained before must be corrected by factors depending on the second derivatives of the failure surfaces at the global or local Beta-points. Except for the curvature correction, the asymptotic results are similar to the result (6) and coincide with eq.(6) for linear failure surfaces. The asymptotic results thus justify an approximation for general outcrossing rates by linearisation of arbitrary failure surfaces at the global Beta-points for intersections of the contributing failure domains in line of the discussions in [10]. With some loss of accuracy local Beta-points may even be used for intersections. Equally important is the result in [9] that the function Ψ as defined in eq.(4) asymptotically depends only on the global Beta-point $\underline{y}^*$ for which $\beta_r=\min\{\|\underline{y}\| \mid \bigcap_s g_{rs}(\underline{y})\leq 0\}$ justifying the approach leading to inequality (11). Therefore, the relation (11) becomes asymptotically correct if $\underline{y}'$ is replaced by

$\underline{y}^*$. Furthermore, the asymptotic outcrossing rate of unions of failure domains was proved to be just a limited sum of the crossing rates of the individual domains (see [8] and [11]) which might simplify the considerations in section 3. The material given in sections 2 and 3 is considered as the time-variant equivalent to the first-order reliability concepts for time-invariant problems as outlined in [2] and [9] with the exception that the rigorous treatment of non-gaussian processes is far more complex if possible at all.

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