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RELIABILITY FORMULATIONS FOR TIME-VARIANT NON-DETERIORATING AND DETERIORATING STRUCTURAL SYSTEMS

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1. Introduction

With few exceptions structural reliability methods so far concentrated on so-called first-passage failures, i.e. failure upon completion of a structure or failure under individual or combined stationary loading and random but essentially non-deteriorating system properties. Also, the computation of the reliability of structural systems now appears possible - at least in the time-invariant case and in a few simplier cases of time-dependent reliability. Only recently, strength degradation problems began to deserve the interest structural failure statistics suggest. The reliability methodology as available as of yet for these problems appears still much less developed than for first-passage failures - not to mention that degrading structures were and will be inspected, maintained or repaired. In fact, the systems and deterioration aspects bring up a whole set of new questions which are common and partially solved in classical reliability theory but appear to be not even touched in structural reliability approaches.
Rather than asking for the probability of structural system failure in a given reference time one now might ask in addition for the best inspection strategy, the number of maintenance and/or repair actions, and the portion

of time a structural facility cannot be used as designed due to these operations. Even if one restricts the considerations to the special frequently not unrealistic case where the facility is simply put out of function for the times of maintenance or repairs, i.e. neither the system nor any of its components can fail or deteriorate during these times, the the computations become much more involved than in the time-invariant case.

In the following, setting out from some well-known results the formulation of the time-dependent failure probability of systems with not load-induced smoothly varying componental failure domains is given. Some preliminary comments on deteriorating systems and maintenance and repair problems are presented in the final section.

2. Non-Deteriorating Cut Set Systems

If the joint distribution of times to first failure in each component of a structural system were known, the system failure probability would be given by

$$P_f(t) = P(\bigcup_i \bigcap_j F_{ij}) \qquad (1)$$

with $F_{ij} = \{T_{ij} \leq t\}$, the componental failure event and $\cup \cap F_{ij}$ the minimal (disjoint) cut set of failure events. The componental time to failure distribution, however, is hard to assess even for simple components. It can be bounded from the above by

$$F(t) = P_f(t) \leq P_f(0) + \int_0^t \nu(\tau)d\tau \qquad (2)$$

where

$$\nu(\tau) = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\underline{X}(\tau)\in S \cap \underline{X}(\tau+\Delta)\in F) \qquad (3)$$

is the unconditional outcrossing rate (intensity) of the vector load process $\underline{X}(\tau)$ crossing the boundary of the safe domain S towards the exterior failure domain F. For simplicity, the boundary of S is assumed to be time-invariant. From its derivation it is known that the inequality (2) derives from the fact that the actual number of outcrossings is replaced by the mean number of exits in the same interval. In addition, the condition that the process $\underline{X}(\tau)$ must start in the safe domain before an exit can occur is conservatively neglected and, hence, making the outcrossing rate slightly different from the unconditional failure rate (renewal, failure intensity) defined in classical reliability. This assumes the validity of the last condition just mentioned. The fact, that eq. (2) does not assume any particular form or structure of the safe resp. failure event, let one conclude its validity for arbitrary failure events of complex systems in the space of $\underline{X}(\tau)$, too. As known, eq. (2) has found a number of numerical or analytical solutions one of which, the so-called point-crossing formula, written in a slightly generalized form, is

$$\begin{aligned}\nu(\tau) &= \lim_{\Delta\to 0} \frac{1}{\Delta} P(\underline{X}(\tau)\in S\cap\underline{X}(\tau+\Delta)\in F)\\ &= \lim_{\Delta\to 0} \frac{1}{\Delta} P(\bigcup_{i=1}^{n} (\underline{X}(\tau)\in S)\cap(\underline{X}(\tau)+\Delta X_i(\tau+\Delta))\in F))\\ &\leq \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{i=1}^{n} P((\underline{X}(\tau)\in S)\cap(\underline{X}(\tau)+\Delta X_i(\tau+\Delta))\in F) \qquad (4)\end{aligned}$$

where $\underline{X}(\tau) = (X_1(\tau),\ldots,X_n(\tau))^T$ and $\Delta X_i(\tau+\Delta)$ the change of the i-th component of $\underline{X}(\tau)$ during $[\tau,\tau+\Delta]$. One recognizes that the event that $\underline{X}(\tau)$ leaves the safe domain in the small (unit) time interval is first decomposed into the union of events where an exit can occur due to an outcrossing of any one component of $\underline{X}(\tau)$ in $[\tau,\tau+\Delta]$ whose probability clearly is bounded from the above by the sum of the individual probabilities. As usual, the possibility of more than one outcrossing event in $[\tau,\tau+\Delta]$ is neglected, i.e. is considered of zero order. The step leading to the last line of eq. (4) is, in fact, crucial for any computationally amenable result in the analysis of many time-dependent systems, because the same concept is now applied to the components of a cut set (parallel system which fails if all components experienced an outcrossing of their safe domains). Hence, if the number of components in the k-th cut set of a system is m_k and the possibility of one or more outcrossings in the interval $[t,t+\Delta]$ in the same component is neglected, the k-th cut set fails if any of the components experiences an outcrossing while the other components are already in a failure state at time τ. Therefore,

$$\nu_k = \lim_{\Delta\to 0} \frac{1}{\Delta} P(\bigcup_{j=1}^{m_k} \{(\underline{X}(\tau)\in S_j)\cap(\underline{X}(\tau+\Delta)\in F_j) \bigcap_{\substack{r=1\\ r\neq j}}^{m_k} (\underline{X}(\tau)\in F_r)\})$$

$$\leq \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{j=1}^{m_k} P((\underline{X}(\tau)\in S_j)\cap(\underline{X}(\tau+\Delta)\in F_j) \bigcap_{\substack{r=1\\ r\neq j}}^{m_k} (\underline{X}(\tau)\in F_r))$$

$$\leq \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{j=1}^{m_k} \sum_{i=1}^{n} P((\underline{X}_i(\tau)+\Delta X_i(\tau+\Delta)\in F_j)\cap(\underline{X}_i(\tau)\in S_j) \bigcap_{\substack{r=1\\ r\neq j}}^{m_k} (\underline{X}(\tau)\in F_r)) \quad (5)$$

Note that cut set failure can only occur if the process has been in $S_j \bigcap_{\substack{r=1\\ r\neq j}}^{m_k} F_r$ at time τ which generally is a small probability event and then an outcrossing occurs due to a change during time Δ in any of the components of $\underline{X}$ in any of the system components. Eq. (5) can be evaluated at least numerically once the second order distribution $F(\underline{x}_1,t_1;\underline{x}_2;t_2)=P(\underline{X}(t)\leq\underline{x}_1;t=t_1\cap\underline{X}(t)\leq\underline{x}_2;t=t_2)$ or even simplier $F(\underline{x}_1, t_1; \underline{x}_1+\Delta x_{i2}, t_2)=P(\underline{X}(t)\leq\underline{x}_1, t=t_1;\underline{X}(t)+X_i\leq\underline{x}_2, t=t_2)$ can be given. Then, the first-order reliability techniques can be applied. The crossing probabilities in eq. (5) may further be bounded from the above by making use of $P(\cap A_i) \leq \min_i P(A_i)$ where the events A_i refer to time τ. In particular, one may delete the event $\{\underline{X}(\tau)\in S_j\}$ and, ultimately, retain only the event for which $P(\underline{X}(\tau)\in F_r)$ is smallest so that

$$\nu_k \leq \lim_{\Delta\to 0} \frac{1}{\Delta} P(\bigcup_{j=1}^{m_k} \{(\underline{X}(\tau)\in F_{r*})\cap(\underline{X}(\tau+\Delta)\in F_j)\}) \quad (5a)$$

with r^* denoting the event with smallest probability. The larger the dependence among the different events the better this bound might become. Under the same circumstances the probabilities in eq. (5a) should be close to $\max_j \{P((\underline{X}(\tau)\in F_{r*})\cap(\underline{X}(\tau+\Delta)\in F_j))\}$, a lower bound to eq. (5a) with respect to the union operation.

The unconditional failure rate of the whole system is the probability per unit time that none of the total of K cut sets exists at time τ and one or more cut sets fail in the interval $\tau,\tau+\Delta$.

$$\nu_F(\tau) = \lim_{\Delta\to 0} \frac{1}{\Delta} P((\bigcap_{k=1}^{K} \bar{C}_k)\cap \bigcup_{k=1}^{K} C_k^*)$$

$$= \lim_{\Delta\to 0} \frac{1}{\Delta} [P(\bigcup_{k=1}^{K} C_k^*) - P((\bigcup_{k=1}^{K} C_k)\cap \bigcup_{k=1}^{K} C_k^*)]$$

$$\leq \lim_{\Delta\to 0} \frac{1}{\Delta} P(\bigcup_{k=1}^{K} C_k^*) \leq \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{k=1}^{K} P(C_k^*) \quad (6)$$

$\bigcap_{k=1}^{K} \bar{C}_k$ is the event that at time τ no cut set is in a failure state. C_k^* is the event that the k-th cut set experiences failure in $[\tau,\tau+\Delta]$ and is given explicitly in eq. (5). The second line makes use of the identity $P(A\cap B) = P(B) - P(\bar{A}\cap B)$. The first inequality sign results from neglecting the second term in the second line which simply is the probability that at time τ one or more cut sets are already in a failure state. Its probability clearly is always smaller than the first and should be relatively small for high-reliable systems. The second inequality is the well-known upper bound for unions of events. Inserting now the upper bound (5) yields

$$\nu_F(\tau) \le \lim_{\Delta\to 0} \frac{1}{\Delta} \sum_{k=1}^{K} \sum_{j=1}^{m_k} \sum_{i=1}^{n} P((\underline{X}(\tau)+\Delta X_i(\tau+\Delta)\in F_{jk})\cap$$

$$\cap(\underline{X}(\tau)\in S_{jk}) \bigcap_{\substack{r=1\\ r\neq j}}^{m_k} (\underline{X}(\tau)\in F_{rk})) \qquad (7)$$

In principle, the event in the first line of eq. (6), especially $\bar{C}_k$, could have been written out. The formation of a cut set representation, however, now amounts to an enormous number of cut sets of considerable length whose numerical computation might be possible only for relatively small systems As an alternative the bound (7) is proposed for a numerical analysis which at the most involves $K\cdot m_k\cdot n$ (m_k+1)-dimensional intersections. Practically, one might concentrate only on the dominant cut-sets which then either yield more or less reliable estimates or upper bounds to eq. (7). In passing it is also noted that lower bounds for $\nu_F(\tau)$ appear difficult to assess. Finally, it should be observed that the developments given so far are very much in accordance with the formulations in classical reliability where the computations facilitate greatly if the componental failure events can be assumed to be independent (see, for example, Henley/Kumamoto, 1981).

3. Some Preliminary Formulations for Elementary Deteriorating Systems

The same type of formulation does not appear to be useful for deteriorating systems, i.e. systems whose properties depend on the load history and especially for those where the load on the components depends on the number and the type of components which have already failed. With few exceptions deterioration failures (fatigue, corrosion, abrasion, etc.) are brittle failures. Hence, series systems fail if any of its components reaches a critical deterioration state. Parallel systems require all components to fail but if a component fails its shared load must be redistributed among the remaining components. This generally introduces a rather complex dependence structure of failure events and makes numerically feasible formulations so difficult.

A rather general deterioration model for the damage state X(t) is the following (Arone, 1981)

$$\frac{dX_i(t)}{dt} = h(X_i(t))g_i(Z_i(t)) \qquad (8)$$

which upon integration yields

$$G(X_i(t)) - G(X_i(t_o)) \sim N(E[\int_{t_o}^{t} g_i(Z_i(t))dt], D[\int_{t_o}^{t} g_i(Z_i(t))dt]) \qquad (9)$$

if $Z(t)$ resp. $g(Z(t))$ are load processes whose auto-dependence is such that the central limit theorem holds and where $X_i(t_o)$ is the damage at time t_o. Let, for simplicity and without substantial loss of generality, $g_i(Z(t)) = l_i Z(t)$ be positive white noise with mean $l_i m$ and standard deviation $l_i\sigma$ and $h(X_i(t)) = C_i$ a random component property. Then,

$$P(\max_{[t_o,t]} \{X_i(\tau)\} \le x) = P(X_i(t) \le x - X_i(t_o))$$

$$\approx \phi\left(\frac{x-[C_i l_i\, m(t-t_o)-X_i(t_o)]}{C_i l_i \sigma \sqrt{t-t_o}}\right) \qquad (10)$$

is the probability distribution of the damage at time t and with a_i the critical damage threshold and $C_i=1$,

$$P(T_i \le t) \approx \phi\left(\frac{a_i-[l_i m(t-t_o)-X_i(t_o)]}{l_i \sigma \sqrt{t-t_o}}\right) \quad ; \ t \ge t_o \qquad (11)$$

is the probability distribution of the time to first failure of the i-th system component. In rewriting the failure event as

$$F_i(t) = \{a_i - (X_i(t) - X_i(t_o)) \le 0\} \qquad (12)$$

and transforming it into the independent normal space

$$F_i(t) = \{F^{-1}_{a_i}[\phi(U_{i1})] - (F^{-1}_{C_i}[\phi(U_{i2})] l_i (m(t-t_o)$$

$$+ U_o \sigma \sqrt{t-t_o}) + F^{-1}_x[\phi(U_{i3}), t_o]) \le 0\} \qquad (13)$$

with $F_j^{-1}[\ ]$ the inverse distribution function of the j-th independent

random variable, shows that first-order reliability methods can be applied to evaluate eq. (11) for arbitrary distributional assumptions for the initial damage $X_i(t_o)$, the material property C_i and the threshold a_i. It follows further that the failure probability of a series system as given by

$$F_S(t) = \cup\ F_i(t) \qquad (14)$$

can be calculated in the usual way (Hohenbichler/Rackwitz, 1983).

Parallel systems appear to be more difficult to compute. Assume a certain sequence of component failures and consider the i-th component failure. Clearly, eq. (13) holds where l_i now is the load distribution factor for this system state. The initial damage is the random damage $X_i(t_o)$ accumulated in the i-th component at $t_o=T_{i-1}$, the random failure time of the (i-1)-th component. But the distribution of both quantities conditioned on the failure time of the (i-2)-th component is known. Therefore, the failure event of the n-th component can be given in terms of the states and failure times of the previously failed components. And if there are N exhaustive and mutually exclusive component failure sequences the system failure probability simply is the sum over all sequence probabilities.

In practice, only a few dominating sequences may have to be evaluated and even then each sequence may be truncated after a few component failures since load redistribution shortens the survival time of the remaining components considerably. Numerical studies for equal load-sharing systems indicate that at least for smaller systems the study of only the first two or three component failures is sufficient. It should be noted that since eq. (11) resp. (13) is nothing else than the first passage time distribution of failure times at time t of the parallel system, the (numerical) derivative with respect to t is the unconditional failure rate and, thus, can be interpreted as the unconditional cut set failure rate. Therefore, the considerations of section 2 become valid implying a suitable procedure to follow for systems at a higher level in the system hierarchy. If there are also non-deteriorating components in the system, however, the outcrossing approach might become intractable. Then, the time-to-failure approach may still work with the time-to-failure distribution bounded from the above by eq. (2), i.e. by inverting eq. (2) with respect to T_j

$$F_j(t) = \{T_j - t \le 0\}$$

$$\le \{\frac{1}{\nu_j}\,[\phi(U_j) - P_{f,j}(0)] - t \le 0\} \qquad (15)$$

with ν_j and $P_{f,j}(0)$ evaluated as given in section 2. Via conditioning of ν_j and $P_{f,j}(0)$ on variables common to the various components it may still be possible to handle dependencies among various components.

4. Summary

Present formulations of structural reliability cover component reliability under rather general circumstances but system reliability only for time-invariant systems. The purpose of this paper was to extend the formulations to time-variant non-deteriorating and, at a still modest level, to deteriorating systems. It focuses at formulations and solution procedures in the context of the first-order reliability methodology which, in fact, appears to be the key to handle various dependencies which might exist between systems components. Solutions, even numerical ones, are urgently needed in order to get hold of the important questions of maintenance and

repair. However, much further research work needs to be done in this area.

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