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RESISTANCE OF LARGE BRITTLE PARALLEL SYSTEMS

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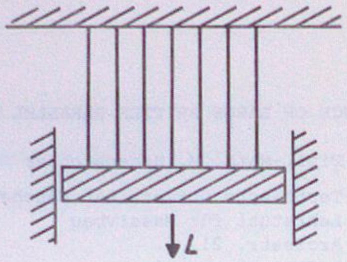
One of the main tasks in the development of reliability based design codes is to establish a suitable set of distribution functions for the uncertain resistance of structural elements and to relate the parameters of those distributions to the characteristic properties of the elements like material constants or size. Those relations should be described in a simple but still sufficient form. Confirmation through statistical data is unavoidable, but beforehand some insight should be gained by studying simple resistance models which at least help to classify the typical stochastic performance of structural elements.

An alternative between the simple extremes "weakest link system" and "ductile parallel system", whose resistances are the minimum and the sum of component resistances, is the "brittle parallel system" with equal load sharing described in the figure. Let n be the number of components and L the total load on the system.



Initially, the load per element is L/n; if some component resistances are smaller, these components break and the load is redistributed equally among the surviving ones. At a final stage either a certain number k $(0 \le k \le n-1)$ of components is broken and the resistance of each surviving component is at least $L/(n-k)$, or the system breaks down. Equivalently, the system survives if, and only if, there exists a value $x \ge 0$ (imagined as load per surviving element) such that the number $G_n(x)$ of those components, whose resistance is not smaller than x, multiplied with x is not smaller than the total load L:

(1) Survival $\Leftrightarrow$ $x\,G(x) \ge L$ for some $x \ge 0$.

Or, denoting $F_n(x)$ the fraction of components with resistance smaller than x or $nF_n(x)$ their absolute number, the system survives if, and only if

(2) $$\frac{1}{n} R_n = \max_{x \ge 0} x(1-F_n(x)) \ge \frac{L}{n}$$

Obviously, R_n is the system resistance. Daniels [4] modelled the component resistances X_i as independent, non-negative random variables and found an exact recursion formula for the failure probability, but here more important the asymptote

(3) $$\lim_{n\to\infty} P\left(\frac{R_n - ne}{\sqrt{n}d} < r\right) = \Phi(r)$$

where Φ is the standard normal integral and

(4) $$e = \max_{x \ge 0} x(1-F(x)) = x_o(1-F(x_o)) = E[x_o(1-F_n(x_o))]$$

$$nd^2 = \mathrm{var}[nx_o(1-F_n(x_o))],\quad d^2 = x_o^2 F(x_o)(1-F(x_o)),$$

provided that the maximum is unique and $\lim_{x\to\infty} x(1-F(x)) = 0$. Here, $F(x) = P(X_i < x) = E[F_n(x)]$ is the distribution function of X_i and equals the mean fraction of component resistances smaller than x. With formulation (2) which is not contained in Daniels' paper, Sen and Bhattacharyya [9] found a connection between the convergence of the empirical process $H_n = (\sqrt{n}(F_n(x) - F(x)) : x \ge 0)$ and the convergence of R_n. More specifically, they prooved that weak convergence of the empirical process $H_n = (H_n(x) : x \ge 0)$ in distribution to a normal process $G = (G(x) : x \ge 0)$

(5) $$H_n = \sqrt{n}\,[F_n - F] \xrightarrow[d]{} G \quad \text{(relative to the Skorokhod-topology)}$$

implies

(6) $$\frac{R_n - ne}{\sqrt{n}\,x_o} \xrightarrow[d]{} G(x_o),$$ which is a normal variable with zero mean.

(e and x_o are defined as in eq. (4)). In other words, R_n is then asymptotically normal with mean ne and variance $nx_o^2[\mathrm{var}\,G(x_o)]$. The latter method extends Daniels' result about independent sequences $X_1, X_2, \ldots$ to the case of "weakly dependent" sequences.

Extension of Daniels' Result to weakly dependent components

In view of eqs. (5) and (6) it suffices to assure weak convergence of the empirical process $H_n = \sqrt{n}(F_n - F)$ to a normal process, relative to the Skorokhod-topology. Apart from stationarity, the usual conditions therefore are socalled "mixing-conditions". For theorems of that type see, for example, [3] and [10].

A theorem about convergence of the empirical process H_n requiring only the more general "functions of mixing-condition", is given in [2]. All these conditions mean "weak" or "asymptotic" independence of the X_i's, i.e. that for large $|i-j|$ the stochastic dependence (correlation) between X_i and X_j is small. In the papers known to the author, the formula for the variance of $G(x)$, which enters in eq. (6), is

$$(7)\quad \mathrm{var}[G(x)] = F(x)(1-F(x)) + 2\sum_{i=1}^{\infty} [P(X_1<x,\ X_{1+i}<x)-F^2(x)]$$

Extension of Daniels' Result to continuous systems

One of the problems in modelling e.g. the resistance of a concrete cross-section by a Daniels' system, is to find a proper discretisation of the originally continuous area. It seems more convenient instead to investigate continuous Daniels-systems. Let therefore $(X_t: t\geq 0)$ be a stationary, separable stochastic process with continuous parameter t; X_t (≥ 0) is the local resistance at location t, i.e. the resistance of a small section dt around t is approximately $X_t dt$. In analogy with the discrete case, the resistance of a continuous brittle parallel system of length T with equal load sharing ("continuous Daniels system") is

$$(8)\quad \frac{1}{T} R_T = \max_{x\geq 0} x(1-F_T(x)).$$

Here $F_T(x)$ is that fraction of the interval $0\leq t\leq T$ where the local resistance X_t is smaller than x.

Also, the results given above read analogously in the continuous case, except possibly eq. (7) which becomes

$$(7^*)\quad \mathrm{var}\,[G(x)] = 2\int_0^{\infty} [P(X_0<x,\ X_t<x) - F^2(x)]\, dt.$$

Again to ascertain the asymptotic normal distribution of R_T, weak convergence of the empirical process $H_T = \sqrt{T}(F_T-F)$ to a normal process G must be prooved. This is usually simple under "mixing conditions", where the proofs are similar to Billingsley's one and are easily rewritten for continuous processes. However, there are only few continuous processes known to be mixing the most important of which are markovian, while on the other hand a transcription of Berkes' and Philipp's theorem, requiring only the "functions of mixing"-condition, is not as obvious. For that reason the author himself prooved weak convergence of H_T in the latter case relative to a topology weaker than Skorokhod's, but which is on the other hand still strong enough to imply the distributional convergence of the system resistance R_T. The resulting formulas are the same as given above. For some more details, see the appendix A.

Before continuing with further extensions, note the accordance of the asymptotic law, eqs. (6) and (3) with the central limit theorem applied to the variables $B_i = x_o \cdot 1_{[x_o,\infty)}(X_i)$ ($i\in\mathbb{N}$) which vanish for $X_i<x_o$ and equal x_o for $X_i\geq x_o$. This means that the system resitance R_n is asymptotically the sum $B_1 + \dots + B_n$. In other words, for very large systems all components with resistance smaller than x_o can be neglected, while the rest should be stressed at most with the load x_o per element, any higher load would probably lead to a collaps of the whole system.

Convergence of system resistance per size unit

Since convergence in eq. (3) or (6) is extremely slow, while improvements exist only for independent components [1] and since it is very difficult to ascertain the required "mixing" [6] or even the "functions of mixing"-conditions, other propositions about convergence of the relative system resistance per number of elements $\frac{1}{n} R_n$ or system size $\frac{1}{T} R_T$ resp., are useful which require much less restrictive assumptions.

Let $(Y_i: i\in\mathbb{N})$ be a stationary normal sequence with autocorrelation function $\rho(i)$ that satisfies

$$(9)\quad \sup_{n\in\mathbb{N}} [n^{-\delta} \sum_{i=1}^{n} |\rho(i)|] < \infty \qquad \text{for some } 0 \le \delta < 1,$$

and let f be a monotonly increasing, non-negative function, such that for the distribution function F of the resistance variables $X_i = f(Y_i)$, eq. (4) has a unique solution x_o and the moment

$$(10)\quad E\,[X_i^{\frac{3}{1-\delta}}] < \infty$$

exists. Then the relative system resistance $\frac{1}{n} R_n$ for a "Daniels-system" with component resistances X_i converges in probability:

$$(11)\quad \frac{1}{n} R_n \xrightarrow[P]{} e = x_o\,(1-F(x_o)).$$

Typical examples are the lognormal sequence (with f=exp, $X_i = \exp(Y_i)$) and the truncated normal sequence (with $X_i = \mu - \sigma\,\Phi^{-1}\,[\Phi(\frac{\mu}{\sigma})\Phi(\frac{\mu - Y_i}{\sigma})]$), who satisfy eq. (10) for each $\delta < 1$. The respective conditions for the continuous model $(X_t: t\ge 0) = (f(Y_t): t \ge 0)$ are similar but with

$$(12)\quad E\,[(\sup_{0\le t\le \tau} X_t)^{\frac{3}{1-\delta}}] < \infty \qquad \text{for some } \tau > 0$$

instead of eq. (10). It is however somewhat awkward to verify eq. (12) directly, so that the following sufficient condition is useful: if

$$(13a)\quad \lim_{t \searrow 0} t^{-\alpha_1}\,(1-\rho(t)) < \infty \qquad \text{for some } \alpha_1 > 0$$

and if there exists a function $\hat{f} \ge 0$ and an $\alpha_2 > 0$ such that

$$(13b)\quad |f(y_1) - f(y_2)| \le (\hat{f}(y_1) + \hat{f}(y_2)) \cdot |y_1 - y_2|^{\alpha_2} \quad \text{for all } y_1, y_2,$$

and if also

$$(13c)\quad E\,[X_t^{\frac{3}{1-\delta}}] < \infty$$

and

$$(13d)\quad E\,[\hat{F}(Y_t)^{\gamma}] < \infty \qquad (\text{with } \gamma = 4 + \max\{\frac{2}{\alpha_1\,\alpha_2}, \frac{6}{1-\delta}\}),$$

then eq. (12) holds. Again lognormal and truncated normal processes satisfy the conditions (14 b, c, d,). A sketch of the proofs is given in appendix B.

Strong Correlation between Components

The results of the foregoing section allows also to study the behaviour of large systems with strongly dependent components. Often such strong dependencies occur due to a random parameter which acts uniformly in all components, while given the respective value of that parameter, the components are only weakly correlated. An equivalent model is the following: Let Q be the common random parameter and $(Y_i : i \in \mathbb{N})$ a sequence of random variables describing those properties which vary from one component to the other. The component resistance $X_i = f(Y_i, Q)$ is a function of Y_i and Q. Assume further that for each fixed value q of Q the relative resistance of a system converges in probability (see the foregoing section)

$$(14)\quad \frac{1}{n} R_n = \frac{1}{n} R_n(q) \xrightarrow[P]{} e = e(q)$$

where $F_q(x) = P[X_i(q) < x] = P[f(Y_i, q) < x]$ and

$$(15)\quad e(q) = \max_{x \ge 0} x\,(1 - F_q(x)).$$

Integrating out the uncertainty of q, one obtains

$$(16)\quad \frac{1}{n} R_n \xrightarrow[d]{} e(Q),$$

i.e. the relative resistance of a large system is distributed like e(Q). For instance, equicorrelated lognormal component resistances imply asymptotically a lognormal distribution of $\frac{1}{n} R_n$ (see [5]).

Remark: Phoenix and Taylor [7] kept the assumption of stochastic independence and generalized the mechanical model introducing uniform strain but random stress-strain relationships. They also obtained an asymptotic normal distribution of R_n.

Appendix A

Through a simple transformation is suffices to prove convergence of the empirical process H_n or H_T resp., for uniformly distributed variables $0 \leq X_i \leq 1$ or $0 \leq X_t \leq 1$. The essential point is to introduce a metric on $D[0,1]$ weaker than Skorokhod's one but sharing all of its relevant properties and which is still strong enough for a conclusion like Sen's and Bhattacharyya's. Let $x=x(t)$ and $y=y(t)$ be functions in $D[0,1]$. With these, compact sets $\bar{x}$ and $\bar{y}$ in $[0,1] \times \mathbb{R}$ are associated:

$$\bar{x} = \{(t,z): 0 \leq t \leq 1,\ \min\{x^-(t),\, x(t)\} \leq z \leq \max\{x^-(t),\, x(t)\}\}$$

where $x^-(t) = \lim_{\tau \nearrow t} x(\tau)$ for $t > 0$ and $x^-(0) = x(0)$. Then, $d(x,y)$ is defined as the Hausdorff-distance between $\bar{x}$ and $\bar{y}$. Denoting with H_n and $\tilde{H}_n$ the empirical processes associated with $(X_i: 1 \leq i \leq n)$ and $(\tilde{X}_i: 1 \leq i \leq n)$, there is

$$d(H_n, \tilde{H}_n) \leq 2\,\varepsilon\,\sqrt{n}$$

whenever

$$0 < \tilde{X}_i,\ X_i \leq 1 \qquad \text{for all } 1 \leq i \leq n \qquad \text{and}$$

$$|X_i - \tilde{X}_i| \leq \varepsilon \qquad \text{for all } 1 \leq i \leq n.$$

This property allows to generalize Yoshihara's proof [10] to the case "functions of mixing variables". The exact formulation is:

Let $(Y_i: i \in \mathbb{Z})$ and $(Y_i^{(n)}: i \in \mathbb{Z})$ $(n \in \mathbb{N})$ be stationary sequences of random variables which are uniformly distributed on $[0,1]$. $K \geq 0$, $0 < \tau \leq 3$, $a \geq 1$ and γ are constants such that the mixing function $\varphi^{(n)}$ of the φ-mixing sequence $(Y_i^{(n)}: i \in \mathbb{Z})$ satisfies

$$\varphi^{(n)}(i) \leq K \cdot (i-2n)^{-2(1+\tau)} \qquad \text{for } i > 2n$$

and

$$E[\,|Y_i - Y_i^{(n)}|^{a}\,] \leq K\, n^{-\gamma}$$

$$\gamma > \max\{a(1+a),\ 2(1+a),\ (2+a)\tfrac{3+\tau}{1+\tau}\}.$$

Then, the empirical process H_n of $(Y_i: i \in \mathbb{N})$ converges weakly in distribution (relative to the d-metric) to a normal process whose covariance function is the usual one.

The respective conditions for the continuous parameter process are quite parallel, but also

$$(*)\quad \begin{aligned} &E[\,|Y(t_1) - Y(t_2)|^{a_1}] \leq K \cdot |t_1 - t_2|^{b} \\ &E[\,|Y^{(n)}(t_1) - Y^{(n)}(t_2)|^{a_1}] \leq K \cdot |t_1 - t_2|^{b} \end{aligned}$$

for some $a_1 \geq 1$, $b > 1$ and all $|t_1 - t_2| \leq 1$ is required, and the last term $(2+a)\frac{3+\tau}{1+\tau}$ in the inequality for γ must be replaced by

$$\frac{3+\tau}{1+\tau}\left(\frac{2b}{b-1} + a \max\left\{\frac{b}{b-1}, \frac{a_1}{b}\right\}\right).$$

Also, the proof of the continuous version is parallel to the discrete one, if you note that $(*)$ relates the approximation error $Z^{(n)} = Y^{(n)} - Y$ within the intervals $[k, k+1]$ to its value at the endpoints:

$$P(\{\max_{0 \leq s \leq \frac{1}{2}} |Z^{(n)}(k+s) - Z^{(n)}(k)| > \varepsilon\}) \leq K\, \varepsilon^{-aa_1 \lambda(1-\lambda)}\, n^{-\gamma(1-\lambda)}$$

for some constant K and each $\frac{1}{b} < \lambda < 1$, and similar for the second half of the interval.

Appendix B

Following [9], the moment condition $E[X_i^\gamma] < \infty$ for $\gamma > 1$ (compare eq. (10) together with

$$(*) \qquad \forall \eta>0\ \exists r\geq 0\ \exists n_o \in N\ \forall n\geq n_o:\ P[\max_{x\geq 0} n^{\frac{1}{\gamma}} |F_n(x)-F(x)| \geq r] \leq \eta$$

implies

$$P[\max_{|x-x_o|\leq\varepsilon} x(1-F_n(x)) = \max_{x\geq 0} x(1-F_n(x))] \xrightarrow[n\to\infty]{} 1 \quad \text{for each } \varepsilon>0.$$

The condition (*) induces also the convergence

$$|\max_{|x-x_o|\leq\varepsilon} x(1-F_n(x)) - x_o(1-F(x_o))| \leq (x_o+\in) \max_{x\geq 0}|F_n(x)-F(x)| \xrightarrow[P]{} 0 .$$

Remembering that $\frac{1}{n} R_n = \max_{x\geq 0} x(1-F_n(x))$, eq. (11) follows if only (*) is verified, which itself can be deduced from

$$E[| F_n(x)-F(x)|^b] \leq K\cdot n^{-\alpha} \quad \text{for all } x\geq 0 \text{ and some } \alpha > \frac{1+b}{\gamma}.$$

For normal sequences $(X_i: i\in N)$ an upper bound of $E[|F_n(x)-F(x)|^2]$ can be given through the correlations of the X_i (see [8]).

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