Indirect Model Reference Adaptive Control of Piecewise Affine Systems with Concurrent Learning

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Abstract: In this paper, we propose a concurrent learning-based indirect model reference adaptive control approach for multivariable piecewise affine systems as an enhancement of our previous work. The main advantage of the concurrent learning-based approach is that the linear independence condition of the recorded data suffices for the convergence of the estimated system parameters. The classical persistent excitation assumption of the input signal is not required. Moreover, it is proved that the closed-loop system is stable even when the system enters the sliding mode. The numerical example shows that the concurrent learning-based approach exhibits better tracking performance and achieves parameter convergence when compared with our previously proposed approach.

Keywords: piecewise affine systems, adaptive control, concurrent learning, parameter estimation, hybrid systems.

1. INTRODUCTION

Most of the engineering systems are characterized by nonlinearity and hybrid (continuous states mixed with discontinuous events) phenomenon, which complicates the analysis of such systems. There already exist various approaches and methods in the control community to analyze and control linear systems. This motivates the concept to model the hybrid systems or approximate the nonlinear systems with a set of linear systems, such that the existing approaches for linear systems can also be applied to nonlinear and hybrid systems.

Piecewise affine (PWA) systems, as a kind of switched systems, are proposed to realize this concept because of its universal approximation capability. The state space in PWA systems is partitioned into several convex regions by hyperplanes. Each region is viewed as a linear subsystem, which can be obtained by linearizing the nonlinear system around certain operation point. If the trajectory of state goes through different regions, the dynamics of the PWA system switches among the corresponding subsystems.

The major challenges we face when we apply the control theory into the real world are model uncertainties and disturbances from the environment. With large uncertainties and disturbances, a single pre-tuned controller might not be able to stabilize the closed-loop system. By using adaptive control, the control gains are adapted to eliminate the negative influence of the disturbances and uncertainties. For the adaptive identification of PWA systems, both hyperplanes and subsystem parameters are to be estimated. In Kersting and Buss (2015), the hyperplane estimation is explored by using the total least square approach, while subsystem parameter identification is studied in Kersting and Buss (2017a).

In the literature, there exist various adaptive control approaches for switched systems. The work of Sang and Tao mainly focuses on the model reference adaptive control (MRAC) of piecewise linear (PWL) systems. Sang and Tao (2012a) and Sang and Tao (2012b) represent the results of state tracking and output tracking, respectively.

Di Bernardo et al. develops various MRAC approaches for PWA systems in control canonical form. di Bernardo et al. (2013) provides a passivity-based MRAC approach whereas di Bernardo et al. (2016) extends the previous results to the case where the sliding mode can not be neglected. Moreover, this method has its counterpart for the discrete-time case and is presented in Bernardo et al. (2013).

All of the above-mentioned MRAC methods are direct MRAC approaches in the sense that the controller gains are updated based on tracking error without estimating the system parameters. Considering the case, where the identification of system parameters is a part of the control objective, the indirect MRAC can be applied. In indirect MRAC, the control gains are updated based on the estimated system parameters. In Kersting and Buss (2017b), an indirect MRAC for multivariable PWA systems is proposed. The asymptotic convergence of tracking error is proved by using a common Lyapunov function. Under the persistently exciting (PE) condition, all the estimated subsystem parameters are proved to converge to the real values.

The PE assumption requires that the input signals should contain different frequencies. This causes oscillations and vibrations in the real engineering systems, which might be harmful to the plants. A recently proposed approach, concurrent learning Chowdhary et al. (2013), replaces the restrictive PE condition with some mild assumption on the linear independence of the recorded data.

This work enhances the indirect MRAC approach presented in Kersting and Buss (2017b) by integrating concurrent learning. Without requiring the PE condition, the proposed approach guarantees the convergence of the subsystem parameters to their real values. Besides, the controller gains are converged to the nominal values. Moreover, the closed-loop system is proved to be stable when the system enters the sliding mode. Compared to the previous approach, the concurrent learning-based approach exhibits better tracking performance and guarantees parameter convergence without PE input signals.

The rest of this paper is structured as follows. Section 2 defines the PWA systems and introduces the MRAC approach. The concurrent learning-based indirect MRAC is displayed in Section 3. The stability proof and convergence analysis are also provided. The validated through a numerical simulation in Section 4. The conclusion and discussion are followed in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

In this section, the definition of PWA systems and the model reference adaptive control approach are revisited.

A piecewise affine system with $s \in \mathbb{N}$ subsystems takes the form

$$\dot{x} = A_i x + B_i u + f_i, \quad i = 1, 2, \cdots, s \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state of the PWA system and control input. The system matrix $A_i \in \mathbb{R}^{n\times n}$, the input matrix $B_i \in \mathbb{R}^{n\times p}$ and the affine term $f_i \in \mathbb{R}^n$ characterize the dynamics of i-th subsystem.

The state space of a PWA system is partitioned into s convex regions $\Omega_i, i = 1, \cdots, s$. The boundaries of the convex regions can be described by a set of hyperplanes in the state space, which are analytically expressed by a set of inequalities

$$\Omega_i = \left\{ x \in \mathbb{R}^n | H_i \begin{bmatrix} x \\ 1 \end{bmatrix} \preceq 0 \right\} \tag{2}$$

where a hyperplane is expressed by one row of H_i. The operator $\preceq$ represents $<$ or $\leq$ in the element-wise. In the context of PWA systems, the region partitions do not exhibit overlap, i.e., $\Omega_i \cap \Omega_j = \emptyset, i \neq j$. The following indicator function can be utilized to determine the current activated subsystem

$$\chi_i(t) = \begin{cases} 1, & \text{if } x^T(t) \in \Omega_i \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Since there is no overlap among the regions, we have $\sum_{i=1}^{s} \chi_i = 1$ and $\chi_i(t)\chi_j(t) = 0$ if $i \neq j$.

Here we make an usual assumption in the study of the adaptive control of PWA systems (see di Bernardo et al. (2013), di Bernardo et al. (2016), Bernardo et al. (2013)) that the partitions of the state space is known, which means that the indicator function $\chi_i(t)$ is also known. In the case, where the PWA system is used to approximate an uncertain nonlinear system, the region partitions can be determined by the designer. Based on the partitions, one can proceed with controller design as we will present in this paper.

The goal of the MRAC is to enforce the trajectory of the controlled system to track the trajectory generated by a reference model. Consider a PWA reference system

$$\dot{x}_m = A_m x_m + B_m r + f_m, \tag{4}$$

where $A_m = \sum_{i=1}^{s} A_{mi}\chi_i$, $B_m = \sum_{i=1}^{s} B_{mi}\chi_i$, $f_m = \sum_{i=1}^{s} f_{mi}\chi_i$ are the parameters of the reference system. $x_m \in \mathbb{R}^n$ and $r \in \mathbb{R}^p$ denote the state of the reference system and reference input.

For simplicity and without loss of generality, we assume that the space partition of the reference system is the same as the one of the controlled PWA system. If more flexibility is desired in the practice, each convex region of the controlled PWA system can be further divided into several regions with the same subsystem parameters, while different subsystem parameters can be designed for these regions of the reference system.

Assume each subsystem of the reference system is stable and thus there exists symmetric and positive definite matrix $P_i \in \mathbb{R}^{n\times n}$ for given symmetric and positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$ such that

$$A_{mi}^T P_i + P_i A_{mi} = -Q_i. \tag{5}$$

In this work, a common quadratic Lyapunov function (CQLF) is assumed to exist, i.e. $P_i = P$. In general, the existence of the common P matrix requires the matrix set A_{mi} to fulfill some conditions as shown in Shorten et al. (2003). However, the reference system is given by the user and can be designed to fulfill the conditions to have CQLF. Therefore, this assumption is not restrictive.

The concept of indirect MRAC is to update the control gains based on the estimated parameters. For switching systems, the parameters of each subsystem are estimated separately and each subsystem has its corresponding control gains. If i-th subsystem is activated, the parameters of this subsystem are estimated and the corresponding control gains are adapted by using the estimated i-th subsystem parameters. Therefore, the control law takes the form

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t), \tag{6}$$

with

$$K_x = \sum_{i=1}^{s} \chi_i K_{xi}, \quad K_r = \sum_{i=1}^{s} \chi_i K_{ri}, \quad K_f = \sum_{i=1}^{s} \chi_i K_{fi}. \tag{7}$$

where $K_{xi} \in \mathbb{R}^{p\times n}$, $K_{ri} \in \mathbb{R}^{p\times r}$ and $K_{fi} \in \mathbb{R}^p$ denote the control gains for i-th subsystem.

The problem we would like to solve in this work is formulated as follows: Given a reference system and a PWA system with known state space partitions and unknown subsystem parameters A_i, B_i and f_i, design an adaptive control law $u(t)$ such that the state $x(t)$ of the PWA system tracks the state $x_m(t)$ of the reference system and the system parameters converge to their real values without PE input signals.

3. MAIN RESULTS

In this section, we present the concurrent learning-based indirect MRAC approach for PWA systems. The control and adaptation laws are derived and the stability proof is provided.

Inserting (6) into the PWA system equation (1) yields

$$\dot{x} = \sum_{i=1}^{s} \chi_i((A_i + B_i K_{xi})x + B_i K_{ri} r + (B_i K_{fi} + f_i)) \quad (8)$$

Letting the closed-loop system (8) equal to the reference system (4), we obtain the *matching equations*

$$\begin{aligned} A_{mi} &= A_i + B_i K_{xi}^*, \\ B_{mi} &= B_i K_{ri}^*, \\ f_{mi} &= f_i + B_i K_{fi}^*, \end{aligned} \quad (9)$$

where $K_{xi}^*, K_{ri}^*, K_{fi}^*$ are called *nominal control gains*. They are obtained by solving *matching equations* if all the subsystem parameters A_i, B_i, f_i are known and B_i has full column rank for $i = 1, \cdots, s$

$$\begin{aligned} K_{xi}^* &= B_i^\dagger (A_{mi} - A_i), \\ K_{ri}^* &= B_i^\dagger B_{mi}, \\ K_{fi}^* &= B_i^\dagger (f_{mi} - f_i), \end{aligned} \quad (10)$$

with $(\cdot)^\dagger$ denoting the Moore-Penrose pseudoinverse.

The classical indirect adaptive control approach updates the control gains by replacing the system parameters in (10) with the estimated parameters. This, however, may introduce singularity by calculating $B_i^\dagger(t)$. To avoid this problem, we follow our previous work Kersting and Buss (2017b) to apply the dynamic gain adjustment approach. Define the closed-loop estimation errors as

$$\begin{aligned} \varepsilon_{Ai} &= \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \\ \varepsilon_{Bi} &= \hat{B}_i K_{ri} - B_{mi}, \\ \varepsilon_{fi} &= \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}, \end{aligned} \quad (11)$$

where $\hat{A}_i$, $\hat{B}_i$ and $\hat{f}_i$ denote the estimated system parameters of i-th mode. Based on the closed-loop estimation errors, the control gains adaptation obeys

$$\begin{aligned} \dot{K}_{xi} &= -S_i^T B_{mi}^T \varepsilon_{Ai}, \\ \dot{K}_{ri} &= -S_i^T B_{mi}^T \varepsilon_{Bi}, \\ \dot{K}_{fi} &= -S_i^T B_{mi}^T \varepsilon_{fi}. \end{aligned} \quad (12)$$

We assume that the matrices $S_i \in \mathbb{R}^{p \times p}$ exist and are known such that $K_{ri}^* S_i$ are symmetric and positive definite. This is a common assumption in the adaptive control, see Tao (2014). It remains to determine the adaptation law for parameter estimation. Our proposed concurrent learning-based approach combines the current data and recorded history data for the estimation

$$\dot{\hat{A}}_i = \dot{\hat{A}}_i^C + \dot{\hat{A}}_i^R, \quad \dot{\hat{B}}_i = \dot{\hat{B}}_i^C + \dot{\hat{B}}_i^R, \quad \dot{\hat{f}}_i = \dot{\hat{f}}_i^C + \dot{\hat{f}}_i^R. \quad (13)$$

with the superscript C representing the parameter adaptation with the current data, while the superscript R means the adaptation with the recorded data.

Suppose $\hat{x}$ denoting the predicted states, we define

$$\dot{\hat{x}} = A_m \hat{x} + \sum_{i=1}^{s} ((\hat{A}_i - A_{mi})x + \hat{B}_i u + \hat{f}_i)\chi_i. \quad (14)$$

The parameter update law based on the current data and closed-loop estimation errors takes the form

$$\begin{aligned} \dot{\hat{A}}_i^C &= -\chi_i P_i \tilde{x} x^T - \varepsilon_{Ai}, \\ \dot{\hat{B}}_i^C &= -\chi_i P_i \tilde{x} u^T - \varepsilon_{Ai} K_{xi}^T - \varepsilon_{Bi} K_{ri}^T - \varepsilon_{fi} K_{fi}^T, \\ \dot{\hat{f}}_i^C &= -\chi_i P_i \tilde{x} - \varepsilon_{fi}, \end{aligned} \quad (15)$$

where $\tilde{x} = \hat{x} - x$ denotes the prediction error of system states.

The idea of concurrent learning is to use the history data concurrently to update the estimated parameters. Suppose $x_{i_j}, u_{i_j}, \dot{x}_{i_j}$ to represent the j-th recorded state, input and derivative of state of i-th subsystem, we define ε_{i_j} as

$$\varepsilon_{i_j}(t) = \hat{A}_i(t) x_{i_j} + \hat{B}_i(t) u_{i_j} + \hat{f}_i(t) - \dot{x}_{i_j} \quad (16)$$

and insert (1) to replace $\dot{x}_{i_j}$ leading to

$$\begin{aligned} \varepsilon_{i_j}(t) &= (\hat{A}_i(t) - A_i) x_{i_j} + (\hat{B}_i(t) - B_i) u_{i_j} + (\hat{f}_i(t) - f_i) \\ &= \tilde{A}_i(t) x_{i_j} + \tilde{B}_i(t) u_{i_j} + \tilde{f}_i(t). \end{aligned} \quad (17)$$

with $\tilde{A}_i = \hat{A}_i - A_i, \tilde{B}_i = \hat{B}_i - B_i$ and $\tilde{f}_i = \hat{f}_i - f_i$. We propose the following update law based on the recorded data

$$\begin{aligned} \dot{\hat{A}}_i^R &= -\chi_i \gamma \sum_{j=1}^{q} \varepsilon_{i_j} x_{i_j}^T, \\ \dot{\hat{B}}_i^R &= -\chi_i \gamma \sum_{j=1}^{q} \varepsilon_{i_j} u_{i_j}^T, \\ \dot{\hat{f}}_i^R &= -\chi_i \gamma \sum_{j=1}^{q} \varepsilon_{i_j}, \end{aligned} \quad (18)$$

where q denotes the number of recorded data and $q \geq n + p + 1$ holds, γ is a positive scaling factor.

Compared with the approach proposed in our previous paper Kersting and Buss (2017b), the concurrent learning-based method proposed here supplements the additional adaptation terms (18), which depend on the recorded data. Now we proceed to explore, how the modified adaptive law affects the control and parameter convergence. The state tracking and parameter identification performance are summarized in the following theorem.

Theorem 1. Consider a reference system (4) with CQLF. The PWA system (1) with known state space partitions and unknown subsystem parameters is controlled by (6) with the adaptation laws (12), (11) and (13). Let the recorded data stacks $Z_i \in \mathbb{R}^{(n+p+1)\times q}$ contain $n+p+1$ linearly independent vectors $z_{i_j} = [x_{i_j}^T, u_{i_j}^T, 1]^T$. If the input matrices B_i have full column rank, the pairs (A_{mi}, B_{mi}) are controllable, then the state of the PWA system asymptotically tracks the reference state x_m. Furthermore, the estimated parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ converge to their true values and the control gains K_{xi}, K_{ri}, K_{fi} converge to the nominal gains as $t \to \infty$.

Proof. Consider the following candidate Lyapunov function

$$V = \frac{1}{2}\tilde{x}^T P\tilde{x} + \frac{1}{2}\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\tilde{A}_i) + \mathrm{tr}(\tilde{B}_i^T\tilde{B}_i) + \tilde{f}_i^T\tilde{f}_i + \mathrm{tr}(\tilde{K}_{xi}^T M_{si}\tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si}\tilde{K}_{fi}), \tag{19}$$

with $M_{si} = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p\times p}$. Taking the time derivative of V yields

$$\begin{aligned}\dot{V} =& \underbrace{\frac{1}{2}(\tilde{x}^T P\dot{\tilde{x}} + \dot{\tilde{x}}^T P\tilde{x})}_{\dot{V}_1} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i^C) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i^C) + \tilde{f}_i^T\dot{\tilde{f}}_i^C)}_{\dot{V}_{2a}} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_{si}\dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_{si}\dot{\tilde{K}}_{fi})}_{\dot{V}_{2b}} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i^R) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i^R) + \tilde{f}_i^T\dot{\tilde{f}}_i^R)}_{\dot{V}_3}\end{aligned} \tag{20}$$

Inserting the parameter update law based on current data (15), indirect update law of control gains (12) and closed-loop estimation error (11) into $\dot{V}_1, \dot{V}_{2a}$ and $\dot{V}_{2b}$ yields

$$\begin{aligned}\dot{V}_1 + \dot{V}_{2a} + \dot{V}_{2b} =& -\tilde{x}^T(\frac{1}{2}\sum_{i=1}^{s}Q_{mi}\chi_i)\tilde{x} \\ &- \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \mathrm{tr}(\varepsilon_{fi}^T\varepsilon_{fi}))\end{aligned} \tag{21}$$

Detailed derivations of this step can be found in Kersting and Buss (2017b). Substituting the $\dot{\tilde{A}}_i^R$, $\dot{\tilde{B}}_i^R$ and $\dot{\tilde{f}}_i^R$ in $\dot{V}_3$ with (18) gives

$$\begin{aligned}\dot{V}_3 = -\sum_{i=1}^{s}\chi_i\gamma(&\underbrace{\mathrm{tr}(\tilde{A}_i^T\sum_{j=1}^{q}\varepsilon_{i_j}x_{i_j}^T)}_{\dot{V}_{3ai}} + \underbrace{\mathrm{tr}(\tilde{B}_i^T\sum_{j=1}^{q}\varepsilon_{i_j}u_{i_j}^T)}_{\dot{V}_{3bi}} \\ &+ \underbrace{\mathrm{tr}(\tilde{f}_i^T\sum_{j=1}^{q}\varepsilon_{i_j})}_{\dot{V}_{3fi}})\end{aligned} \tag{22}$$

Inserting (17) into $\dot{V}_{3ai}$ yields

$$\begin{aligned}\dot{V}_{3ai} &= \gamma\mathrm{tr}(\tilde{A}_i^T(\tilde{A}_i\sum_{j=1}^{q}x_{i_j}x_{i_j}^T + \tilde{B}_i\sum_{j=1}^{q}u_{i_j}x_{i_j}^T + \tilde{f}_i\sum_{j=1}^{q}x_{i_j}^T)) \\ &= \gamma\mathrm{tr}(\tilde{A}_i^T\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}\underbrace{\begin{bmatrix}\sum_j x_{i_j}x_{i_j}^T \\ \sum_j u_{i_j}x_{i_j}^T \\ \sum_j x_{i_j}^T\end{bmatrix}}_{\xi_{1i}})\end{aligned} \tag{23}$$

Using the property of trace $\mathrm{tr}(X^TY) = \mathrm{vec}(X)^T\mathrm{vec}(Y)$, (23) can be further transformed as

$$\dot{V}_{3ai} = \gamma\mathrm{vec}(\tilde{A}_i)^T\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}\xi_{1i}) \tag{24}$$

Recalling the compatibility of vectorization with Kronecker product $\mathrm{vec}(ABC) = (C^T\otimes A)\mathrm{vec}(B)$, it follows

$$\begin{aligned}\dot{V}_{3ai} &= \gamma\mathrm{vec}(\tilde{A}_i)^T\mathrm{vec}(I_n\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}\xi_{1i}) \\ &= \gamma\mathrm{vec}(\tilde{A}_i)^T(\xi_{1i}^T\otimes I_n)\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix})\end{aligned} \tag{25}$$

Similarly, we can obtain

$$\dot{V}_{3bi} = \gamma\mathrm{vec}(\tilde{B}_i)^T(\xi_{2i}^T\otimes I_n)\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}) \tag{26}$$

and

$$\dot{V}_{3fi} = \gamma\mathrm{vec}(\tilde{f}_i)^T(\xi_{3i}^T\otimes I_n)\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}) \tag{27}$$

with

$$\xi_{2i} = \begin{bmatrix}\sum_j x_{i_j}u_{i_j}^T \\ \sum_j u_{i_j}u_{i_j}^T \\ \sum_j u_{i_j}\end{bmatrix}, \quad \xi_{3i} = \begin{bmatrix}\sum_j x_{i_j} \\ \sum_j u_{i_j} \\ \sum_j 1\end{bmatrix}. \tag{28}$$

Summing up $\dot{V}_{3ai}$, $\dot{V}_{3bi}$ and $\dot{V}_{3fi}$ yields

$$\begin{aligned}&\dot{V}_{3ai} + \dot{V}_{3bi} + \dot{V}_{3fi} \\ =&\gamma[\mathrm{vec}(\tilde{A}_i)^T\mathrm{vec}(\tilde{B}_i)^T\mathrm{vec}(\tilde{f}_i)^T]\begin{bmatrix}\xi_{1i}^T\otimes I_n \\ \xi_{2i}^T\otimes I_n \\ \xi_{3i}^T\otimes I_n\end{bmatrix}\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix})) \\ =&\gamma\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix})^T\begin{bmatrix}\xi_{1i}^T\otimes I_n \\ \xi_{2i}^T\otimes I_n \\ \xi_{3i}^T\otimes I_n\end{bmatrix}\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix})\end{aligned} \tag{29}$$

Note that

$$\begin{aligned}\begin{bmatrix}\xi_{1i}^T\otimes I_n \\ \xi_{2i}^T\otimes I_n \\ \xi_{3i}^T\otimes I_n\end{bmatrix} &= \sum_j\begin{bmatrix}x_{i_j}x_{i_j}^T & x_{i_j}u_{i_j}^T & x_{i_j} \\ u_{i_j}x_{i_j}^T & u_{i_j}u_{i_j}^T & u_{i_j} \\ x_{i_j}^T & u_{i_j} & 1\end{bmatrix}\otimes I_n \\ &= (\sum_j\begin{bmatrix}x_{i_j} \\ u_{i_j} \\ 1\end{bmatrix}\begin{bmatrix}x_{i_j} & u_{i_j} & 1\end{bmatrix})^T\otimes I_n\end{aligned} \tag{30}$$

Therefore, we obtain

$$\dot{V}_{3ai} + \dot{V}_{3bi} + \dot{V}_{3fi} = \gamma\tilde{\theta}_i^T\Xi_i\tilde{\theta}_i \tag{31}$$

with

$$\Xi_i = (\sum_j\begin{bmatrix}x_{i_j} \\ u_{i_j} \\ 1\end{bmatrix}\begin{bmatrix}x_{i_j} & u_{i_j} & 1\end{bmatrix})^T\otimes I_n, \quad \tilde{\theta}_i = \mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}) \tag{32}$$

So the derivative of the candidate Lyapunov function becomes

$$\begin{aligned}\dot{V} =& -\tilde{x}^T(\frac{1}{2}\sum_{i=1}^{s}Q_{mi}\chi_i)\tilde{x} - \gamma\sum_{i=1}^{s}\chi_i\tilde{\theta}_i^T\Xi_i\tilde{\theta}_i \\ &- \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \mathrm{tr}(\varepsilon_{fi}^T\varepsilon_{fi})).\end{aligned} \tag{33}$$

The linear independence of the $n+p+1$ vectors z_{i_j} implies the full rank of the data stack Z_i, from which it follows that $Z_iZ_i^T$ is positive definite. Since the identity matrix I_n is positive definite, the Kronecker product $\Xi_i = Z_iZ_i^T\otimes I_n$ is also positive definite, which together with the positive definiteness of Q_{mi} implies the negative semidefiniteness of $\dot{V}$.

Now we consider the case, where the closed-loop system exhibits sliding mode behavior. $\dot{V}$ along the sliding mode solution needs to be analyzed. According to the Filippov concept Filippov (2013), we evaluate the convex combinations of the vector fields around the sliding surface. This can be done by substituting $\chi_i \in \{0,1\}$ with $\bar{\chi}_i \in [0,1]$ in the expression of $\dot{V}$. Hence, we have

$$-\tilde{x}^T(\frac{1}{2}\sum_{i=1}^{s} Q_{mi}\bar{\chi}_i)\tilde{x} \leq 0 \tag{34}$$

and

$$-\gamma\sum_{i=1}^{s}\bar{\chi}_i\tilde{\theta}_i^T\Xi_i\tilde{\theta}_i \leq 0, \tag{35}$$

which leads to the negative semidefiniteness of $\dot{V}$ even when the system enters sliding mode. This indicates the boundedness of state prediction error $\tilde{x}$, estimated subsystem parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ (and equivalently $\tilde{\theta}_i \in \mathcal{L}_\infty$) and control gains K_{xi}, K_{ri}, K_{fi}. This further implies $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty$. Moreover, from $\dot{V}$ it follows $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_2$ and $\tilde{\theta}_i \in \mathcal{L}_2$.

From the boundedness of $\tilde{x}$ and $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$, it follows $\tilde{x}, \varepsilon_A, \varepsilon_B, \varepsilon_f \to 0$ as $t \to \infty$, $x, u \in \mathcal{L}_\infty$ and $\lim_{t\to\infty}(x - x_m) = 0$.

Furthermore, the recorded data $x_{i_j}, u_{i_j} \in \mathcal{L}_\infty$ due to the boundedness of x, u. This together with $\tilde{\theta}_i \in \mathcal{L}_\infty$ results in $\epsilon_{i_j} \in \mathcal{L}_\infty$. Considering (18) we have $\dot{\hat{A}}_i^R, \dot{\hat{B}}_i^R, \dot{\hat{f}}_i^R \in \mathcal{L}_\infty$. From (15) we can obtain $\dot{\hat{A}}_i^C, \dot{\hat{B}}_i^C, \dot{\hat{f}}_i^C \in \mathcal{L}_\infty$. Therefore, let $\hat{\theta}_i = \mathrm{vec}([\hat{A}_i\hat{B}_i\hat{f}_i])$ and it yields $\dot{\hat{\theta}}_i, \dot{\tilde{\theta}}_i \in \mathcal{L}_\infty$, which together with $\tilde{\theta}_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\tilde{\theta}_i \to 0$ as $t \to \infty$. Hence, $\hat{A}_i \to A_i, \hat{B}_i \to B_i$ and $\hat{f}_i \to f_i$ as $t \to \infty$.

Finally, we study the convergence of the controller gains. Considering the full column rank assumption of B_i and taking the convergence of $\hat{A}_i, \hat{B}_i, \hat{f}_i, \varepsilon_{A_i}, \varepsilon_{Bi}$ and ε_{fi} into (11), we can conclude that $K_{xi} \to K_{xi}^*, K_{ri} \to K_{ri}^*$ and $K_{fi} \to K_{fi}^*$ as $t \to \infty$.

Remark One condition to guarantee the convergence of the control and subsystem parameters is the linear independence of the sampled data vectors $\{z_{i_j}\}$. Here we use the singular value maximizing data recording algorithm proposed in Chowdhary and Johnson (2011) to maximize the singular value of the data stack Z_i and obtain rich information. By doing so the condition of linear independence can be fulfilled faster.

4. NUMERICAL VALIDATION

In this section, the proposed concurrent learning-based MRAC approach is validated through a numerical example.

We take the mass-spring-damper system from Kersting and Buss (2017b) to validate the proposed algorithm. The system is shown in the Fig. 1, where $m_1 = 5\,\mathrm{kg}, m_2 = 1\,\mathrm{kg}$ denote the masses, $d = 1\,\mathrm{N\,s/m}$ is the damping factor, p_1, p_2 represent the displacement of the two spring, F_1, F_2 are the forces operated on the masses, respectively. The left mass is connected with the static wall by a spring with static spring constant $c_0 = 1\,\mathrm{N/m}$ whereas the two masses are connected with the right spring, which has a stiffness with a PWA characteristics

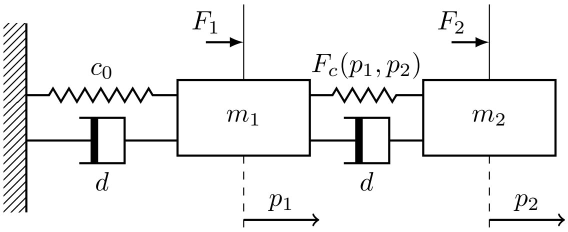

Fig. 1. The mass-spring-damper system

$$F_c(p_1, p_2) = \begin{cases} 10\,\mathrm{N/m}, & \text{if } |p_2 - p_1| \leq 1\,\mathrm{m} \\ 1\,\mathrm{N/m}, & \text{if } p_2 - p_1 > 1\,\mathrm{m} \\ 100\,\mathrm{N/m}, & \text{if } p_2 - p_1 < -1\,\mathrm{m} \end{cases} \tag{36}$$

Defining the state vector $x = [p_1, \dot{p}_1, p_2, \dot{p}_2]^T$ and the input vector $u = [F_1, F_2]^T$, the system dynamics can be written as a PWA system in the state space form as (1). The reference system is chosen as

$$\dot{x}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -25 & -10 \end{bmatrix} x_m + \begin{bmatrix} 0 & 0 \\ 25 & 0 \\ 0 & 0 \\ 0 & 25 \end{bmatrix} r, \tag{37}$$

which exhibits a decoupling motion of the two masses. The control goal of our approach is to enforce dynamics of the controlled PWA system to track the trajectory of the reference system.

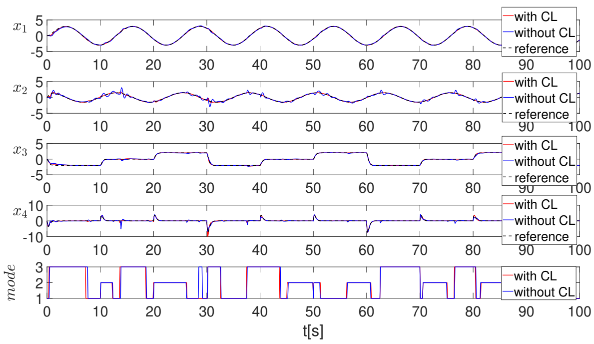

Fig. 2. State tracking performance of indirect MRAC with and without concurrent learning

We use the reference signal $r = [r_1, r_2]^T$, where $r_1 = 3\sin(0.5t)$ and r_2 is a periodic rectangular wave switching among the values $\{-2, 0, 2\}$ with time interval $T = 30\,\mathrm{s}$. The scaling factor γ is specified to be 20. The same common P matrix is utilized as in Kersting and Buss (2017b). Besides, the singular value maximizing data recording algorithm is utilized to manage the data for concurrent learning.

Fig.2 shows the state tracking performance of the indirect MRAC approach with and without concurrent learning. 'CL' in the legends stands for 'concurrent learning'. The black dashed lines depict the states of the reference model. The red lines and blue lines show the states of the controlled PWA system with and without concurrent learning, respectively. No significant difference between the performance of the two approaches is observed for the positions(x_1 and x_3). However, we can see that the red trajectories of the velocity components(x_2 and x_4)

exhibit fewer peaks compared to the corresponding blue lines. Hence, using concurrent learning improves the state tracking performance of the controlled system.

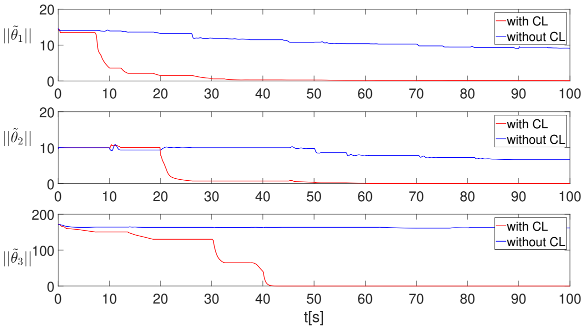

Fig. 3. Parameter convergence of indirect MRAC with and without concurrent learning

In Fig.3, the norm of the parameter estimation errors $\tilde{\theta}_i$ by using algorithms with and without concurrent learning are displayed in red and blue lines, respectively. By using concurrent learning, $||\tilde{\theta}_i||$ converges to zero for $\forall i \in \{1, 2, 3\}$. Compared to the concurrent learning-based approach, the approach from our previous work exhibits unsatisfactory convergence performance of the parameter estimation errors.

Fig.4 displays the convergence of the controller gains of subsystem 2 (the controller gains for other subsystems are similar and thus not shown because of clarity) by applying concurrent learning-based MRAC approach. The dashed lines represent the nominal gains and the solid lines stand for the adaptation gains. The elements in the gain matrices are distinguished by different colors. We can see that the controller gains converge to their nominal values, which validates the conclusion of Theorem 1.

5. CONCLUSION

In this paper, we propose a concurrent learning-based indirect MRAC approach for multivariable PWA systems. With the proposed approach, the controlled PWA system tracks the trajectory of the reference system asymptotically. With the common Lyapunov function, the closed-loop system is stable under arbitrary switching and in sliding mode. Furthermore, if the recorded data of concurrent learning is linearly independent, the system parameters converge to their real values and the control gains converge to the nominal gains. The simulation shows an improvement of the state tracking and parameter estimation performance when compared with our previous work. Future work will be focused on relaxing the assumption of known region partitions of the PWA systems.

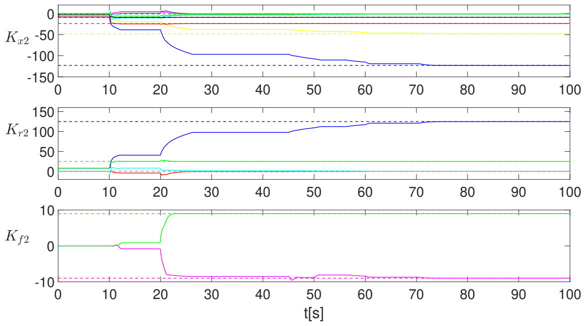

Fig. 4. Convergence of controller gains with concurrent learning-based indirect MRAC

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