Polar Codes: Basics and Recent Advances

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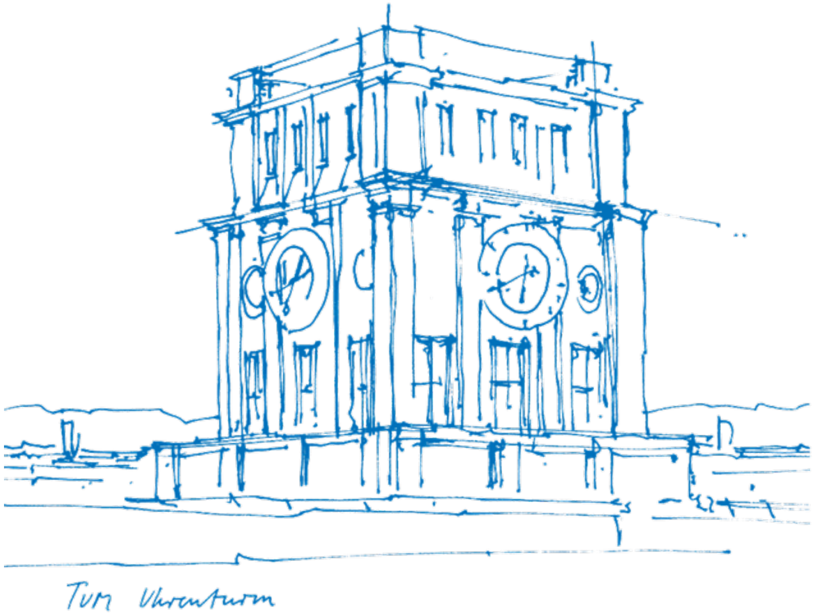
Technical University of Munich (TUM)
German Aerospace Center (DLR)

INCOMING School on Massive IoT, Distributed and Decentralized Information Processing
November 5, 2020



Polar Codes

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 55, NO. 7, JULY 2009 3051

Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

Erdal Arıkan, *Senior Member, IEEE*

***Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity $I(W)$ of any given binary-input discrete memoryless channel (B-DMC) W. The symmetric capacity is the highest rate achievable subject to using the input letters of the channel with equal probability. Channel polarization refers to the fact that it is pos-**

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We write $W : \mathcal{X} \rightarrow \mathcal{Y}$ to denote a generic B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probabilities $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. The input alphabet $\mathcal{X}$ will always be $\{0, 1\}$, the output alphabet and the transition probabilities may

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- They are capacity-achieving on binary memoryless symmetric (BMS) channels with low encoding/decoding complexity [Arı09].
- But successive cancellation (SC) decoding performs poorly for small blocks.

Successive List Cancellation Decoding

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List Decoding of Polar Codes

Ido Tal, *Member, IEEE* and Alexander Vardy, *Fellow, IEEE*

***Abstract*—We describe a successive-cancellation list decoder for polar codes, which is a generalization of the classic successive-cancellation decoder of Arıkan. In the proposed list decoder, L decoding paths are considered concurrently at each decoding stage, where L is an integer parameter. At the end of the decoding process, the most likely among the L paths is selected as the single codeword at the decoder output. Simulations show that the resulting performance is very close to that of maximum-likelihood decoding, even for moderate values of L. Alternatively, if a genie is allowed to pick the transmitted codeword from the list, the results are comparable with the performance of current state-of-the-art LDPC codes. We show that such a genie can be easily implemented using simple CRC precoding. The specific list-decoding algorithm that achieves this performance doubles the number of decoding paths for each information bit, and then uses a pruning procedure to discard all but the L most likely paths. However, straightforward implementation of this**

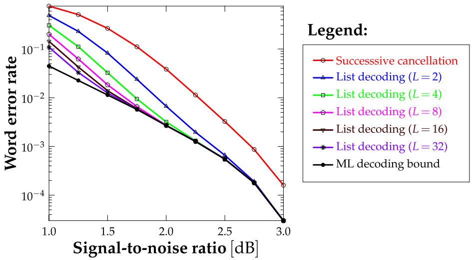

Fig. 1. List-decoding performance for a polar code of length $n = 2048$ and rate $R = 0.5$ on the BPSK-modulated Gaussian channel. The code was constructed using the methods of [15], with optimization for $E_b/N_0 = 2$ dB.

- SC list (SCL) decoding with CRC and large list-size performs very well and matches maximum-likelihood (ML) [TV15].

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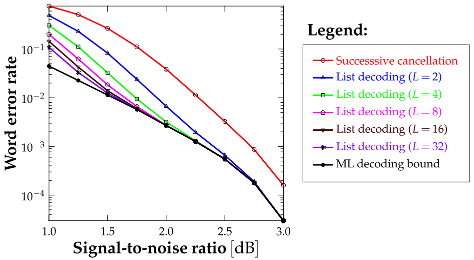

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- SC list (SCL) decoding with CRC and large list-size performs very well and matches maximum-likelihood (ML) [TV15].
- It can also be used to decode other codes (e.g., Reed–Muller codes).

Polar Codes with Dynamic Frozen Bits

254 IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, VOL. 34, NO. 2, FEBRUARY 2016

Polar Subcodes

Peter Trifonov, *Member, IEEE*, and Vera Miloslavskaya, *Member, IEEE*

***Abstract*—An extension of polar codes is proposed, which allows some of the frozen symbols, called dynamic frozen symbols, to be data-dependent. A construction of polar codes with dynamic frozen symbols, being subcodes of extended BCH codes, is proposed. The proposed codes have higher minimum distance than classical polar codes, but still can be efficiently decoded using the successive cancellation algorithm and its extensions. The codes with Arikan, extended BCH and Reed-Solomon kernel are considered. The proposed codes are shown to outperform LDPC and turbo codes, as well as polar codes with CRC.**

RM codes, and are therefore likely to provide better finite length performance. However, there are still no efficient MAP decoding algorithms for these codes.

It was suggested in [17] to construct subcodes of RM codes, which can be efficiently decoded by a recursive list decoding algorithm. In this paper we generalize this approach, and propose a code construction "in between" polar codes and EBCH codes. The proposed codes can be efficiently decoded using the techniques developed in the area of polar coding, but provide

- Later, polar codes were extended with the concept of dynamic frozen bits, which enabled state-of-art designs.

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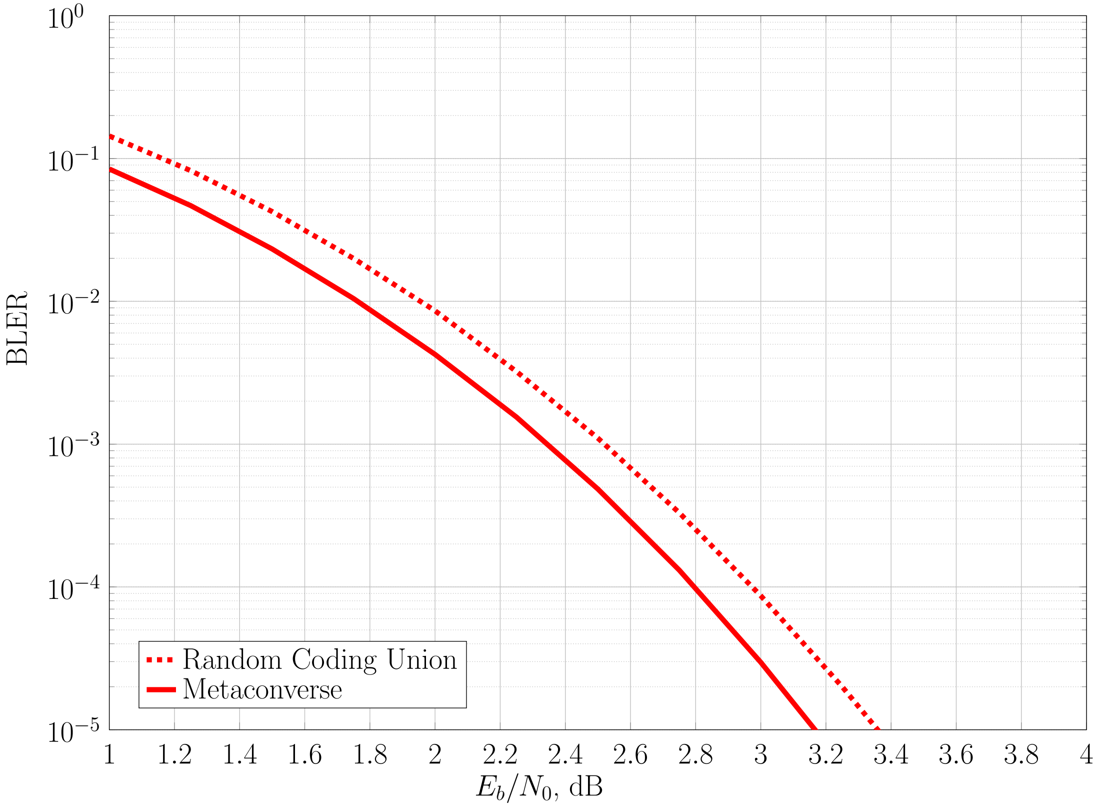
- Later, polar codes were extended with the concept of dynamic frozen bits, which enabled state-of-art designs.
- It is also shown that any code can be decoded using SCL decoding, but some require very large complexity for a good performance.

$n = 128, k = 64$



BLER
E_b/N_0, dB
Random Coding Union
Metaconverse
10^0
10^{-1}
10^{-2}
10^{-3}
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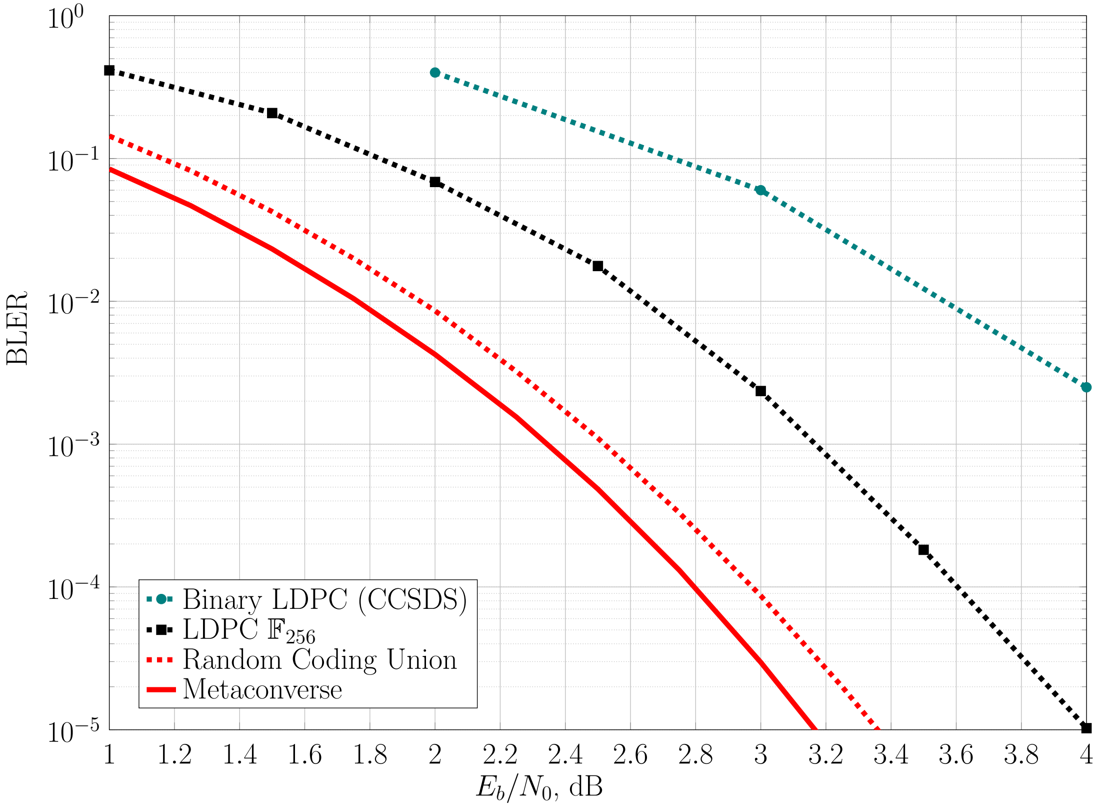

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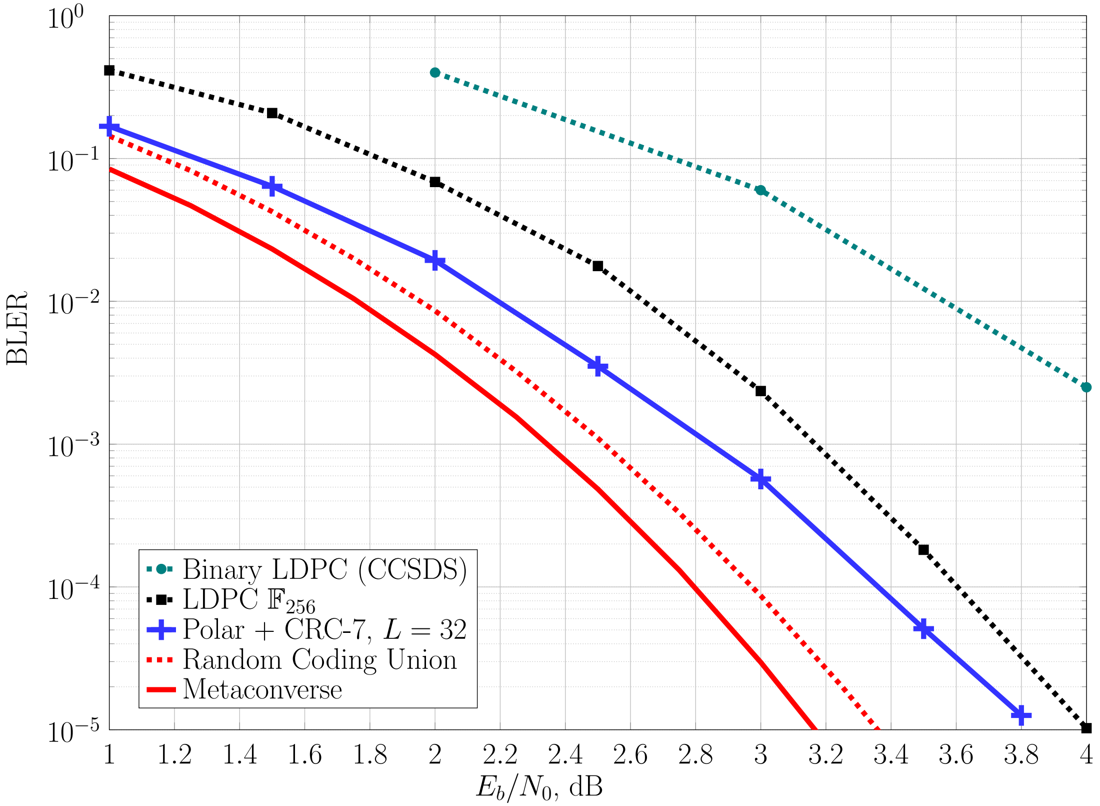

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Polar + CRC-7, $L = 32$
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Metaconverse

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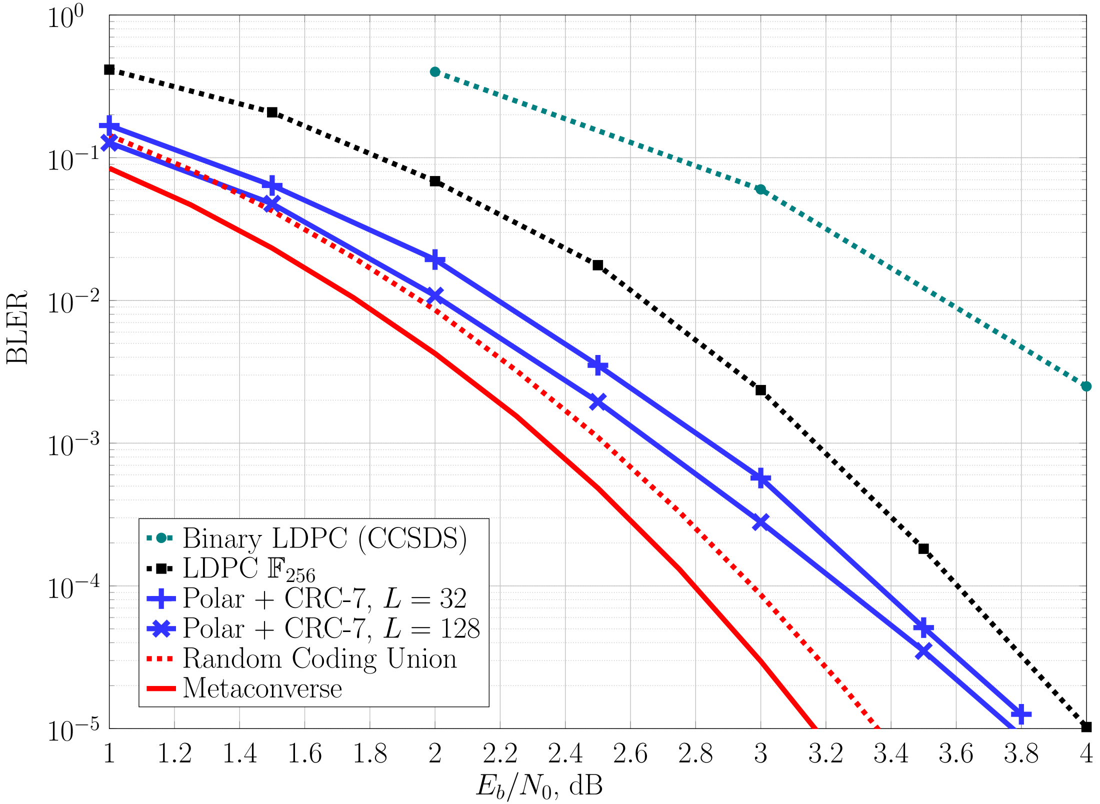

Most of the curves can be obtained on pretty-good-codes.org. For the rest, send an e-mail.

Outline

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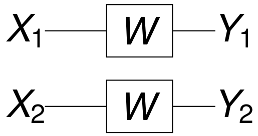
- The technique is lossless in terms of mutual information (required to achieve the capacity).
- The technique is of low complexity (there exists an encoder-decoder pair, realizing the technique with $\mathcal{O}(N \log N)$ complexity, where N is the block length).

Example: Binary Erasure Channel

Given two independent copies of a BEC(ϵ) $W : \{0, 1\} \to \{0, 1, ?\}$, i.e.,

$$Y = \begin{cases} X & \text{w.p. } 1 - \epsilon \\ ? & \text{w.p. } \epsilon \end{cases}$$



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we set

$$X_1 = U_1 \oplus U_2$$
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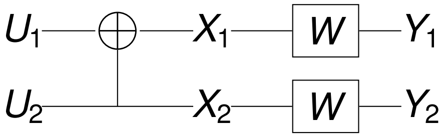

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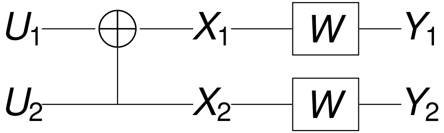

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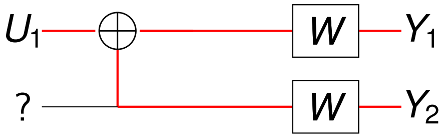

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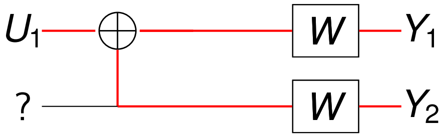

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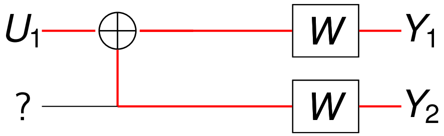

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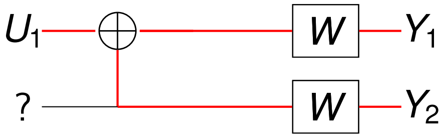

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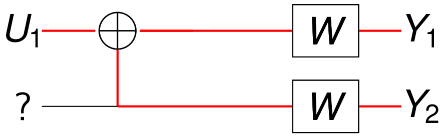

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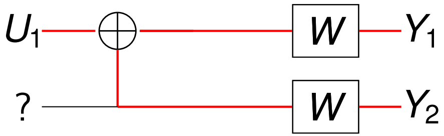

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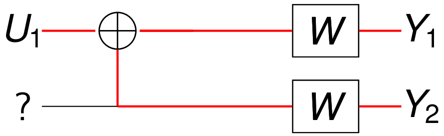

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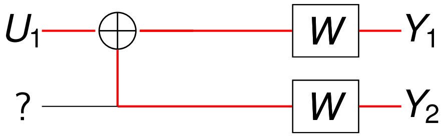

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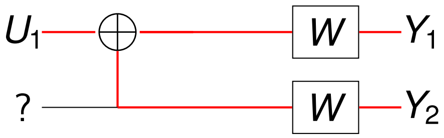

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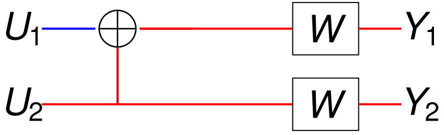

we set

$$X_1 = U_1 \oplus U_2$$
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$$X_1^2 = U_1^2 \mathbf{G}_2$$

$$\mathbf{G}_2 \triangleq \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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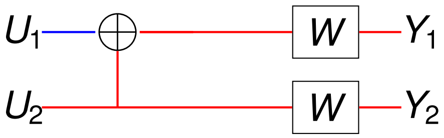

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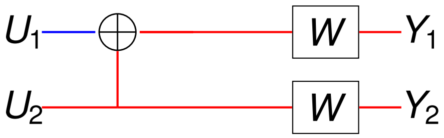

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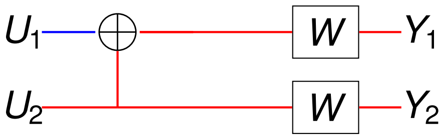

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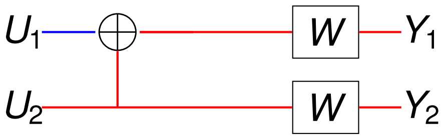

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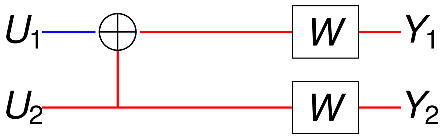

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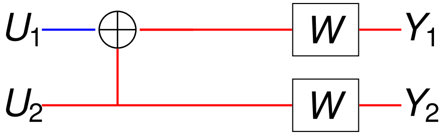

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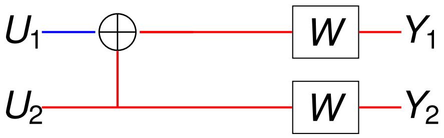

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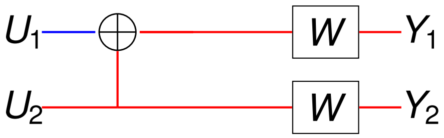

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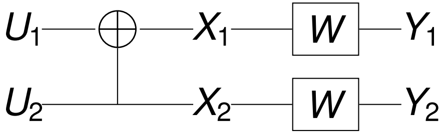

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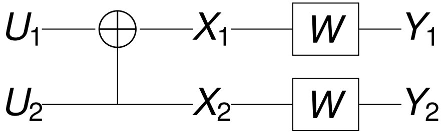

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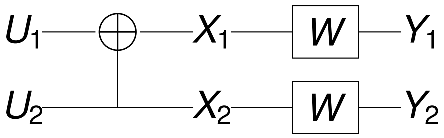

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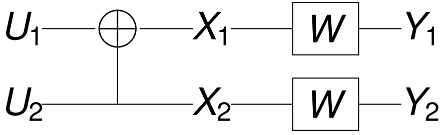

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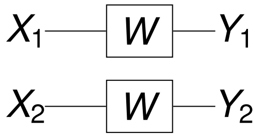
Hence, we have

$$2\epsilon - \epsilon^2 \geq H(X_1|Y_1) = \epsilon \geq \epsilon^2 \quad \text{with equality if and only if } \epsilon \in \{0, 1\}$$

A Basic Transform: General BMS Channels

Given two independent copies of a BMS channel $W : \{0, 1\} \to \mathcal{Y}$,



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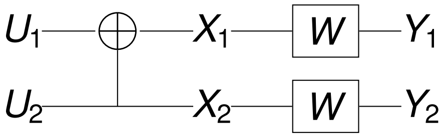

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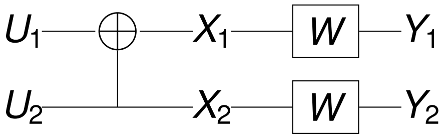

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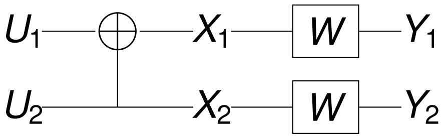

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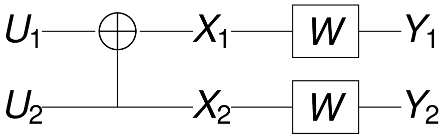

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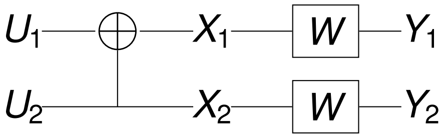

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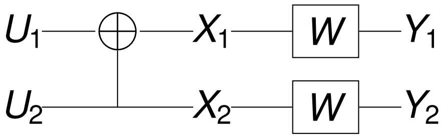

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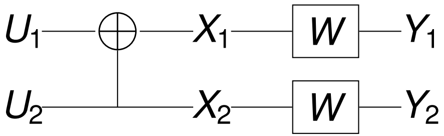

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Combining these, we conclude $H(U_2|Y_1^2 U_1) \leq H(W) \leq H(U_1|Y_1^2)$.

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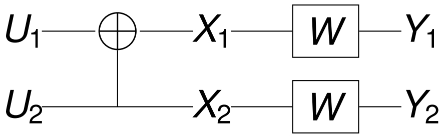

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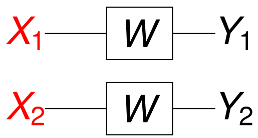
$$H(U_2|Y_1 Y_2 U_1) \leq H(U_2|Y_2) = H(X_2|Y_2) = H(W)$$

Combining these, we conclude $H(U_2|Y_1^2 U_1) \leq H(W) \leq H(U_1|Y_1^2)$. Indeed, the polarization is strict [Arı09], i.e., if $H(W) \notin \{0,1\}$, then

$$H(U_2|Y_1^2 U_1) < H(W) < H(U_1|Y_1^2)$$

Polarized Synthetic Channels

Given two independent copies of $W : \{0, 1\} \to \mathcal{Y}$ with a capacity of $C(W)$, we obtain two synthetic channels:



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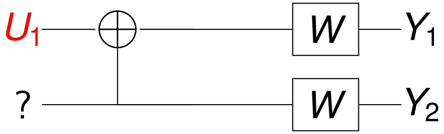

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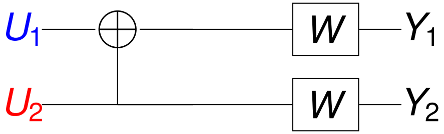

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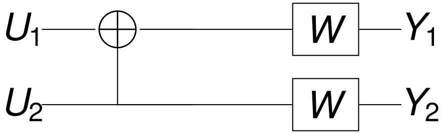

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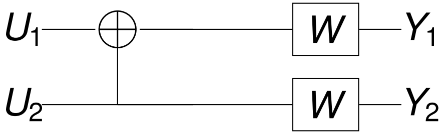

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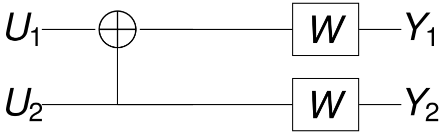

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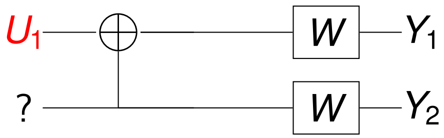
- Then, transmit at a rate $C(W_2^{(2)})$, where the decoder uses $(Y_1^2, \hat{U}_1)$ to output $\hat{U}_2$.

Genie-Aided vs. Real Successive Decoder

- The channel $W_2^{(1)}$ has the input U_1 and output Y_1^2 ✓



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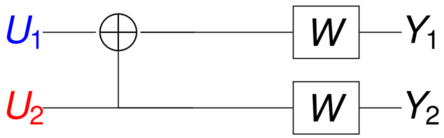

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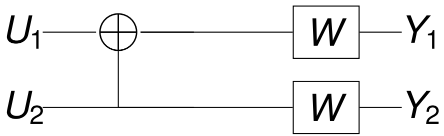

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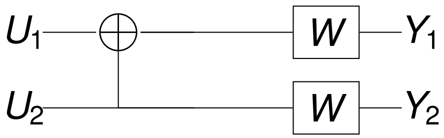

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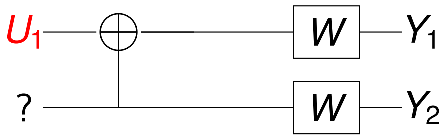

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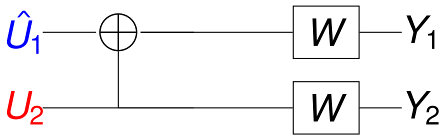

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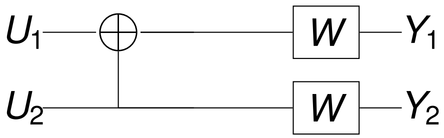

$$\{\hat{U}_1^2 \neq U_1^2\} = \{\tilde{U}_1^2 \neq U_1^2\}$$

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The real decoder makes an error IF AND ONLY IF the genie-aided decoder makes an error!

Polar Transform

We can apply the basic transform recursively to the independent copies of (W), $(W_2^{(1)}, W_2^{(2)})$, $(W_4^{(1)}, W_4^{(2)}, W_4^{(3)}, W_4^{(4)})$, etc., as many times as needed.

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Definition

The Kronecker product of two matrices $\mathbf{X}$ and $\mathbf{Y}$ is

$$\mathbf{X} \otimes \mathbf{Y} \triangleq \begin{bmatrix} x_{1,1}\mathbf{Y} & x_{1,2}\mathbf{Y} & \dots \\ x_{2,1}\mathbf{Y} & x_{2,2}\mathbf{Y} & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, a Kronecker power of a matrix is written as $\mathbf{X}^{\otimes n} = \mathbf{X}^{\otimes(n-1)} \otimes \mathbf{X}$, $\mathbf{X}^{\otimes 0} \triangleq 1$.

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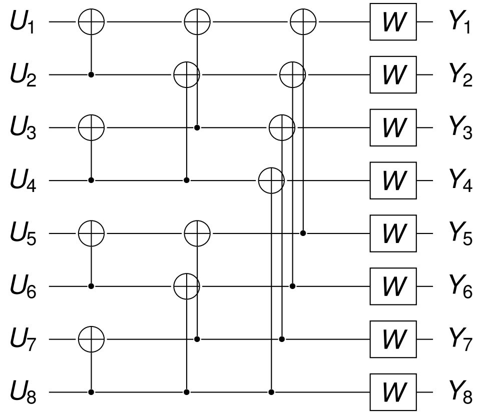
Example

Recall the matrix representing the basic transform $\mathbf{G}_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then, we write

$$\mathbf{G}_2^{\otimes 2} = \mathbf{G}_2 \otimes \mathbf{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

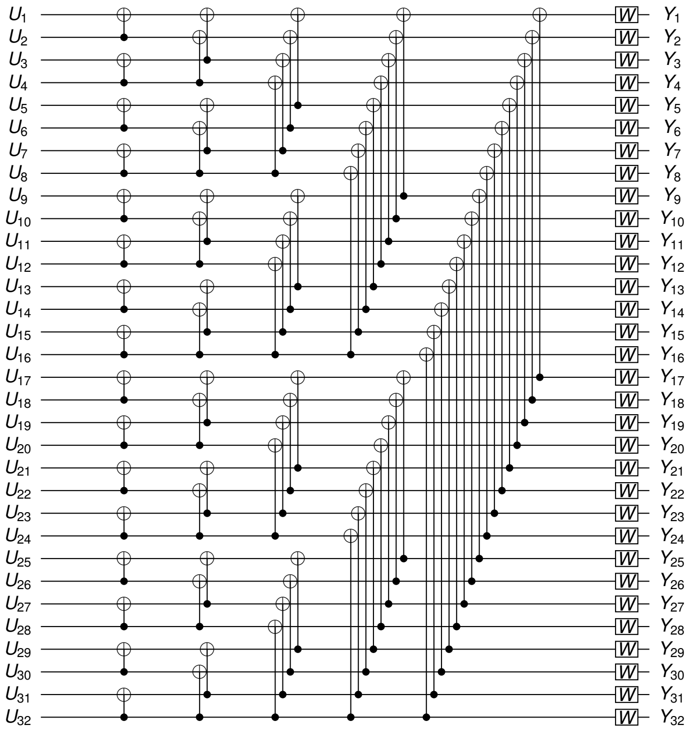
Polar Transform (N=8)

$$U_1^8 \mathbf{G}_2^{\otimes \log_2 8} = X_1^8$$



Polar Transform (N=32)

$$U_1^{32}\mathbf{G}_2^{\otimes \log_2 32} = X_1^{32}$$



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For any fixed $\delta > 0$, the fraction of the mediocre channels vanishes as $N \to \infty$, i.e., we have

$$\lim_{N\to\infty} \frac{1}{N} \left| \{ i \in \{1, \ldots, N\} : \delta < H(W_N^{(i)}) < 1 - \delta \} \right| = 0.$$

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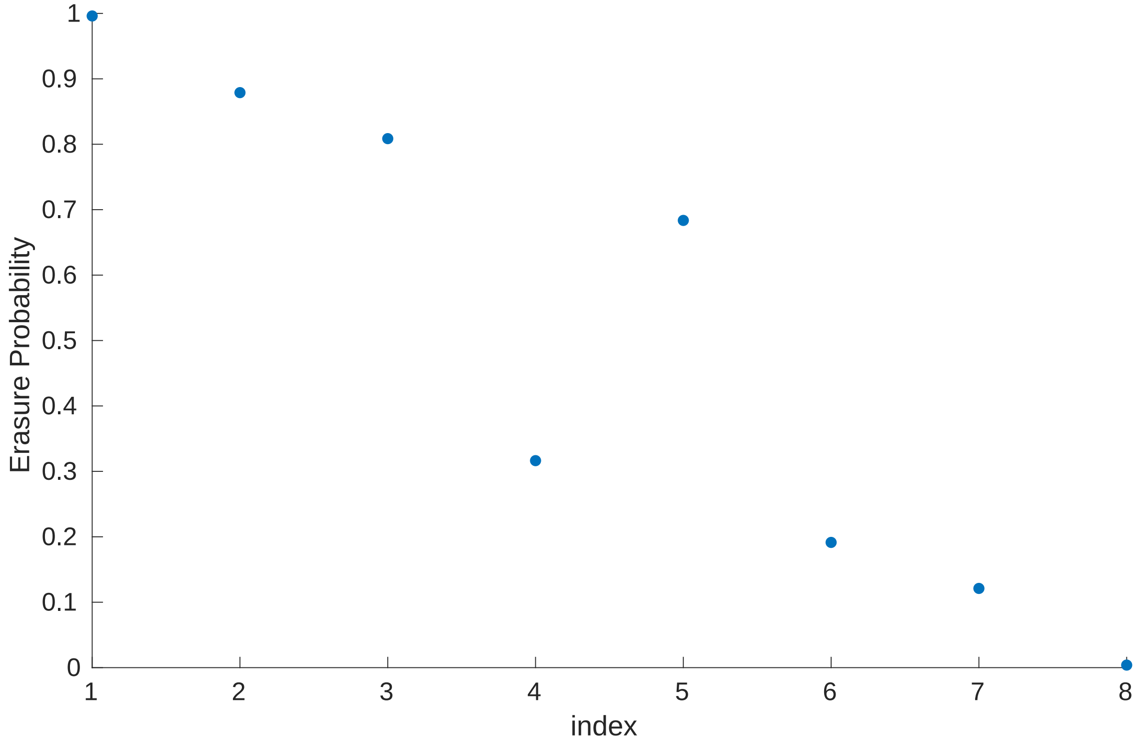
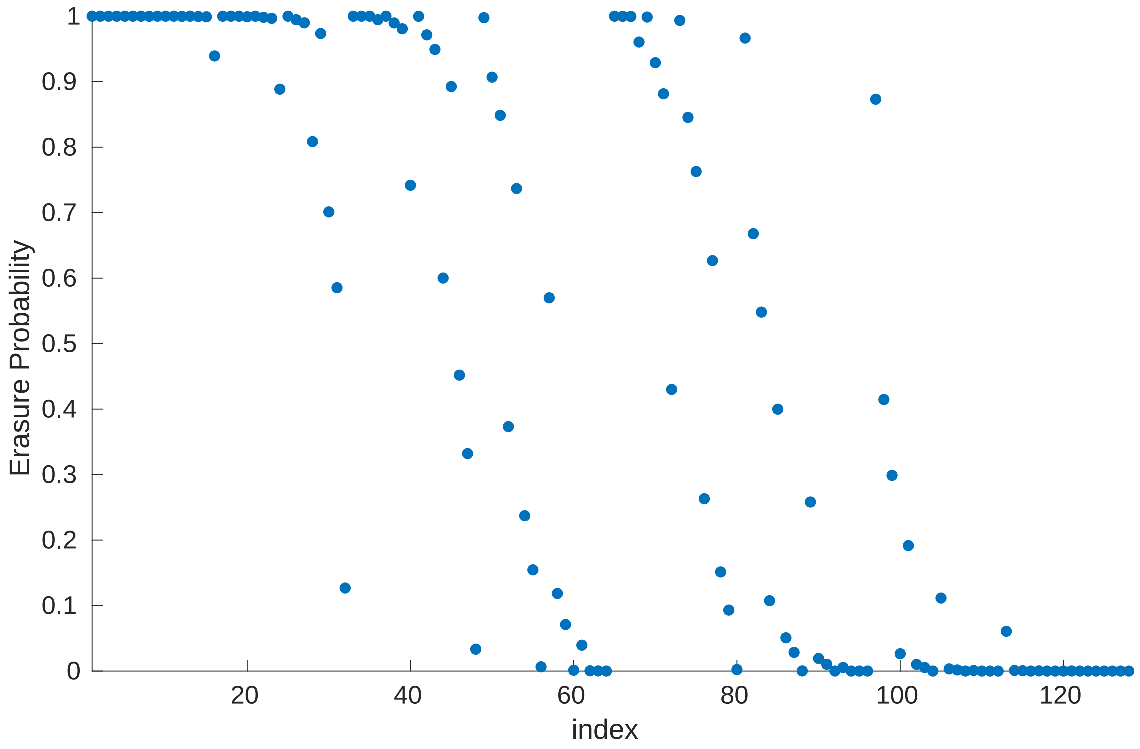
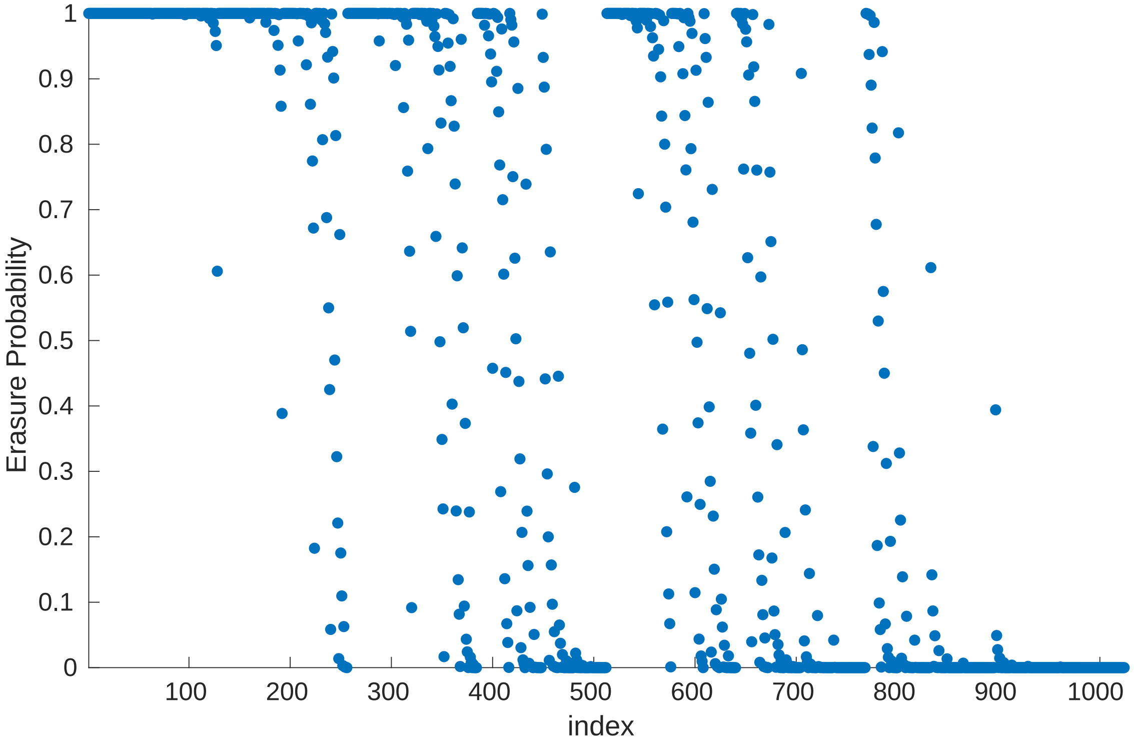
Channel Polarization - Numerical ($N = 2^3$, BEC(0.5))



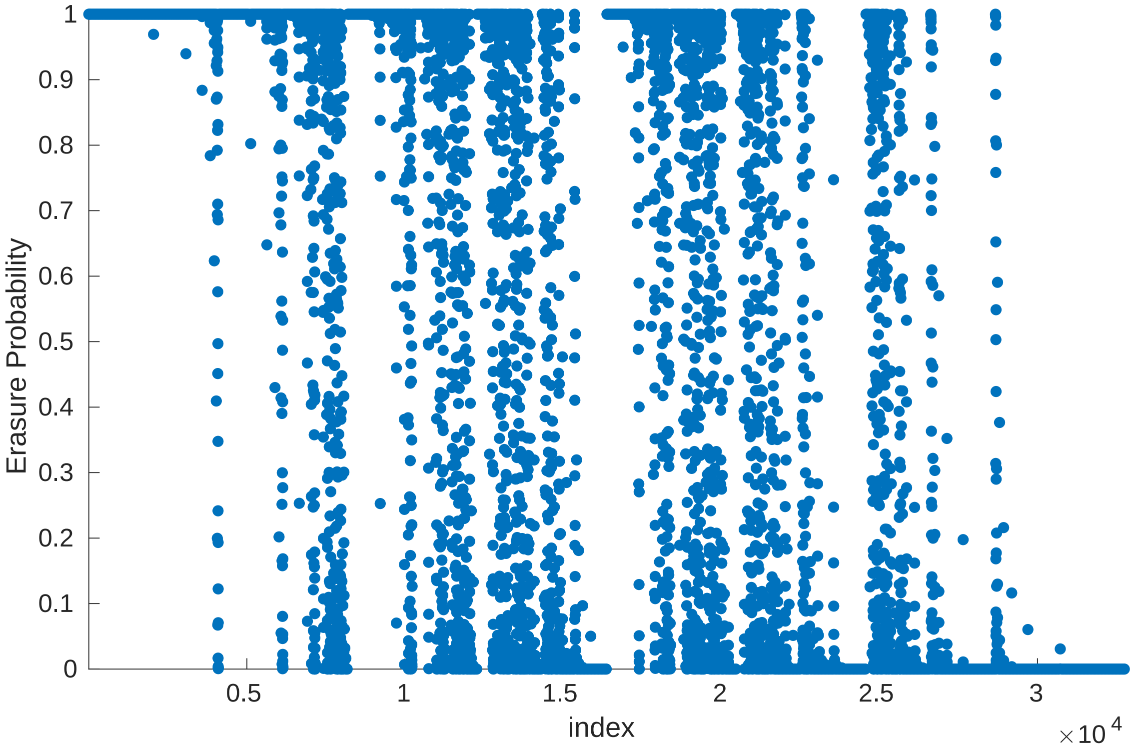
Channel Polarization - Numerical ($N = 2^7$, BEC(0.5))



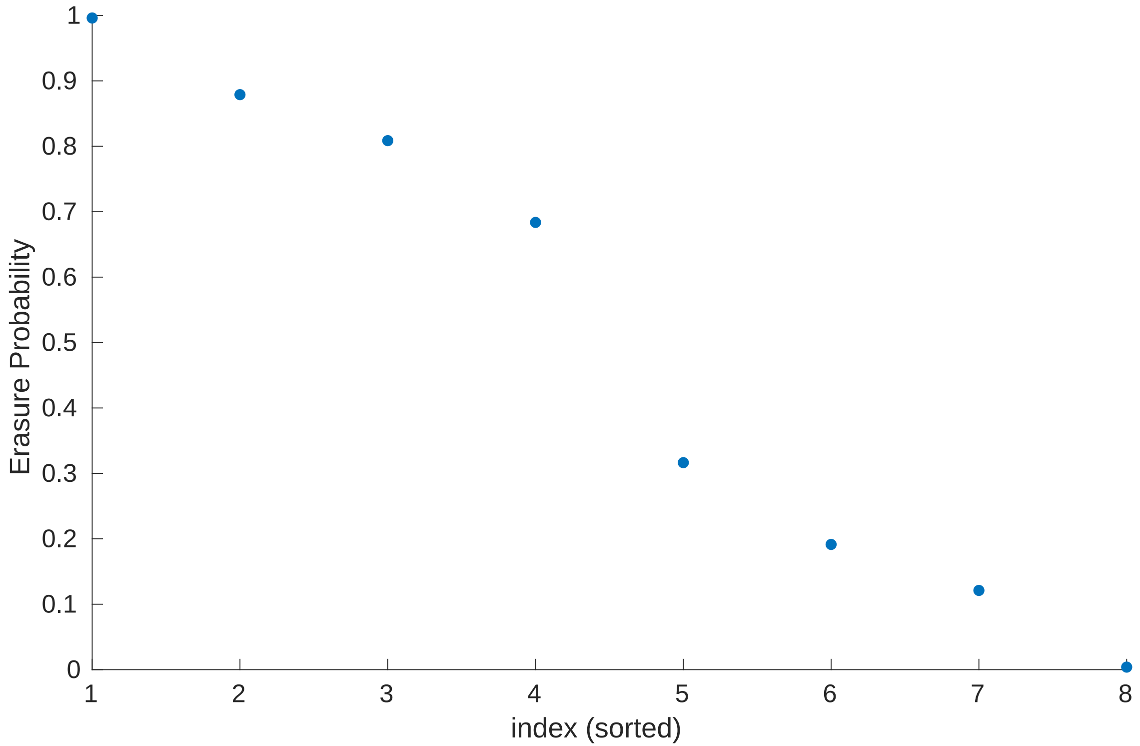
Channel Polarization - Numerical ($N = 2^{10}$, BEC(0.5))



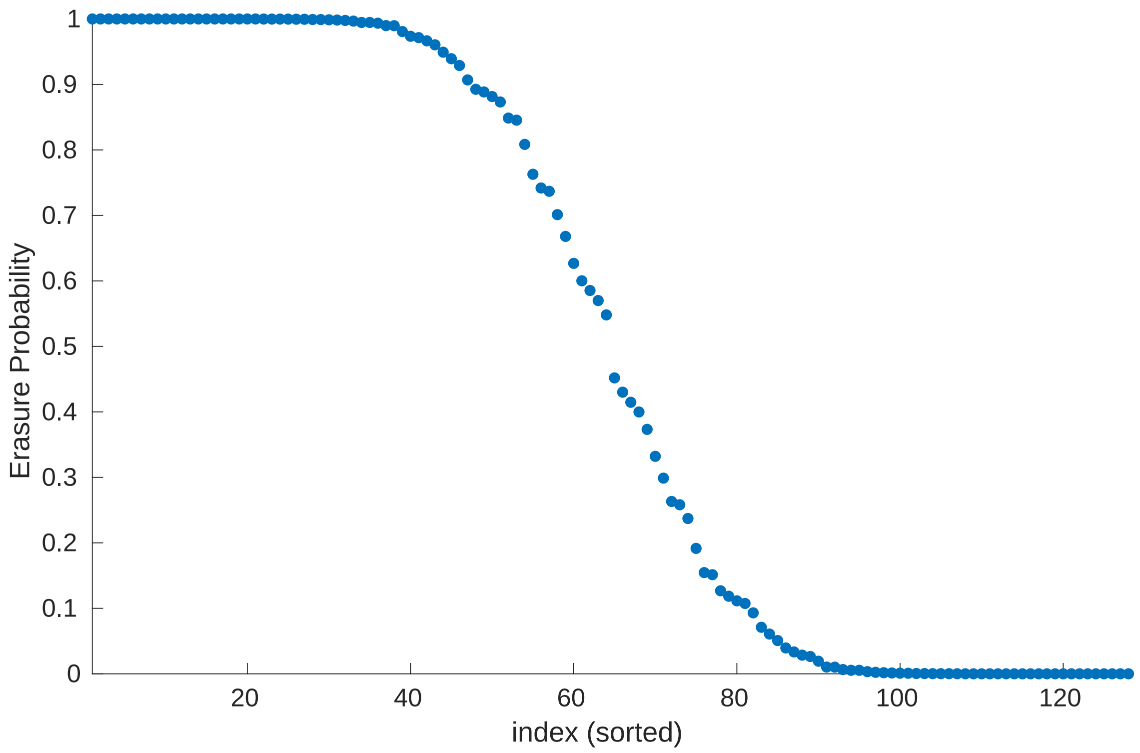
Channel Polarization - Numerical ($N = 2^{15}$, BEC(0.5))



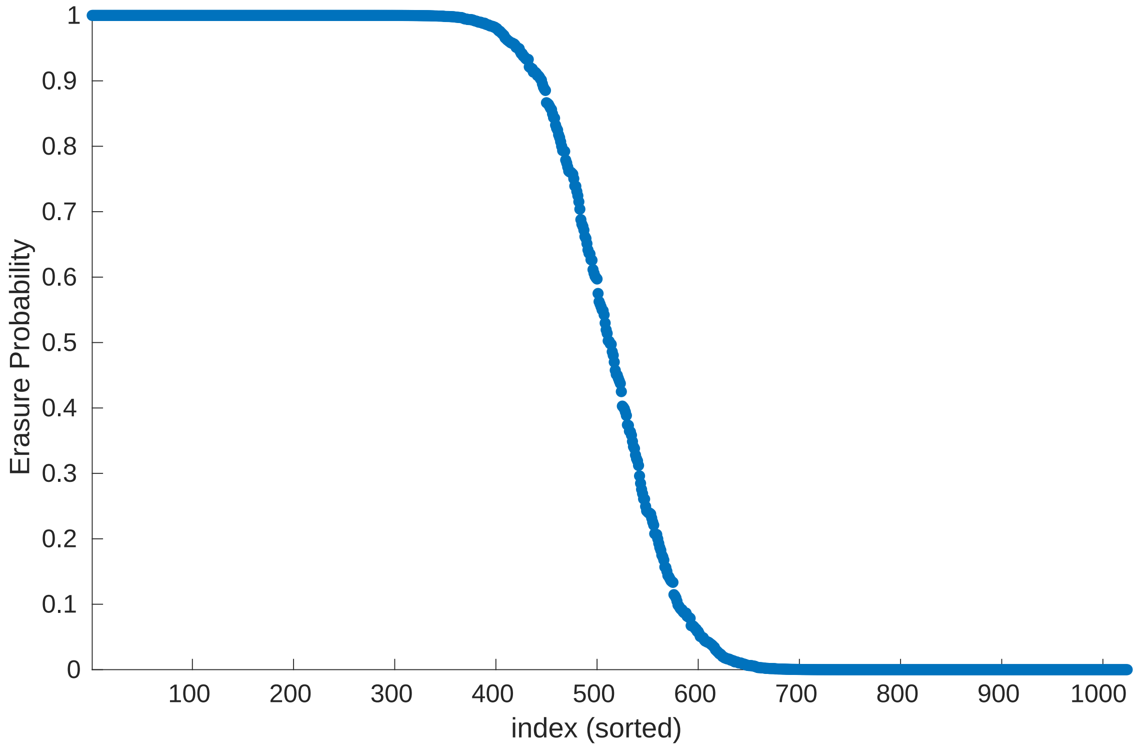
Channel Polarization - Numerical (Sorted, $N = 2^3$, BEC(0.5))



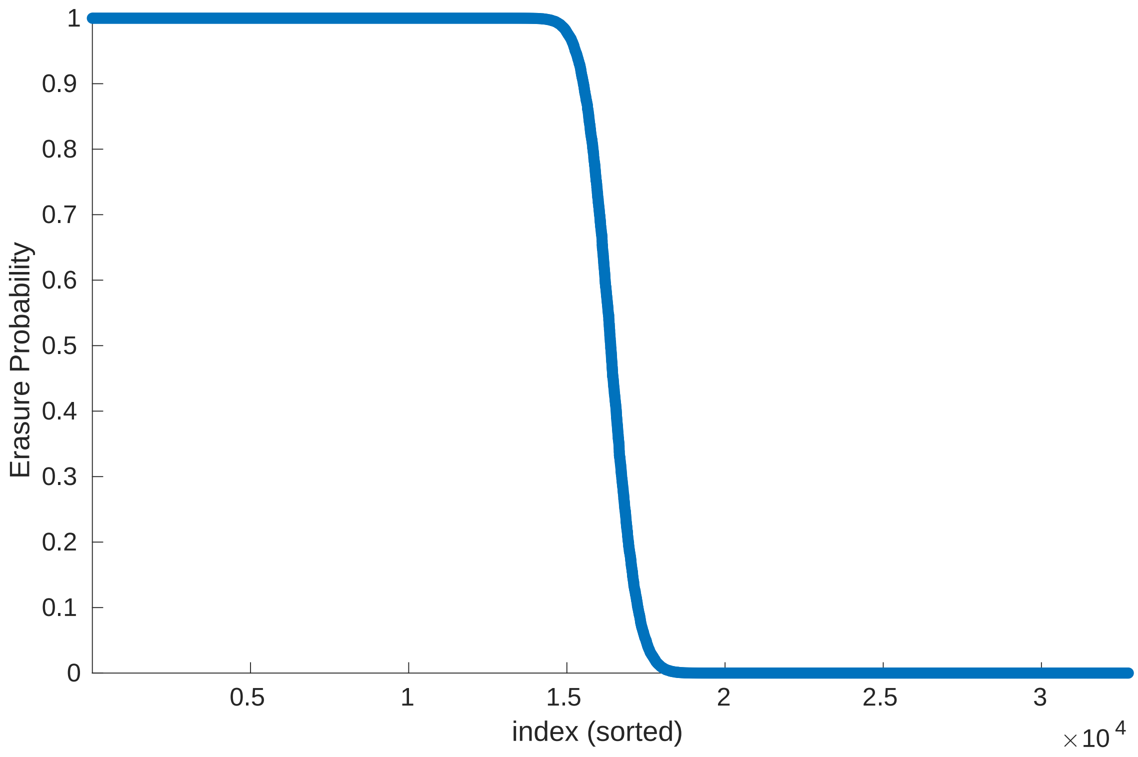
Channel Polarization - Numerical (Sorted, $N = 2^7$, BEC(0.5))



Channel Polarization - Numerical (Sorted, $N = 2^{10}$, BEC(0.5))



Channel Polarization - Numerical (Sorted, $N = 2^{15}$, BEC(0.5))



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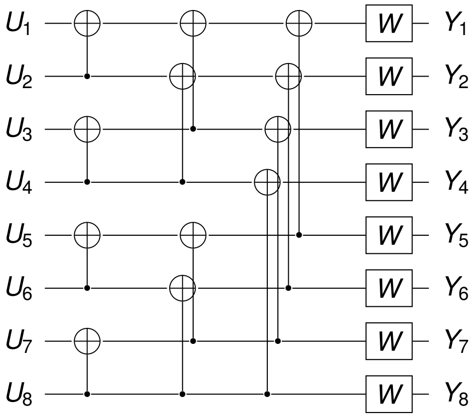
We want to design an (N, k) code, where $N = 2^n$ with $n \geq 1$.



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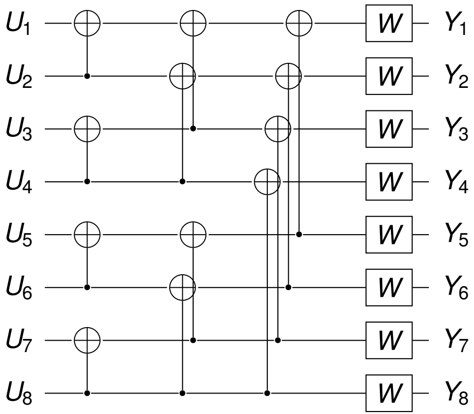

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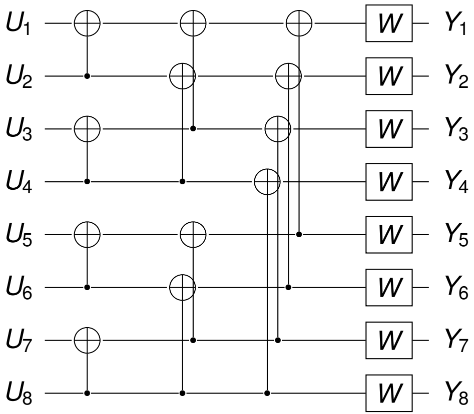

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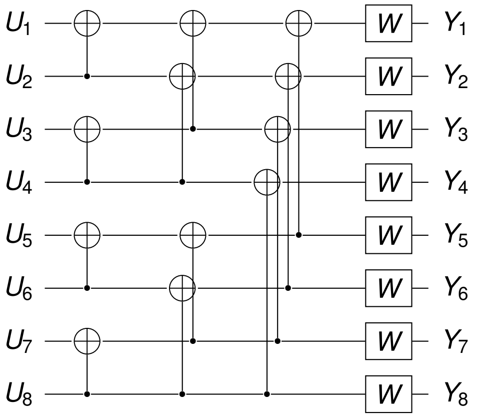

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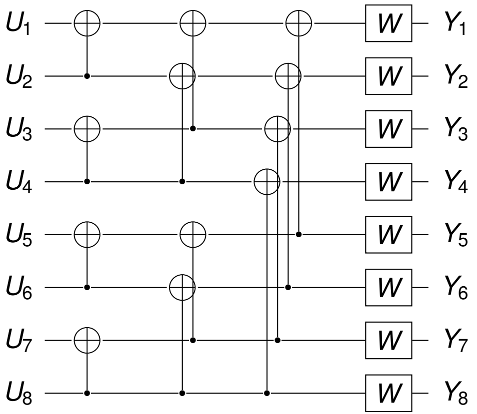
The polar rule minimizes a tight upper bound on the error probability under SC decoding while the RM rule maximizes the minimum Hamming distance.



A Historical Remark

Rekursive Codes mit der Plotkin-Konstruktion und ihre Decodierung

Vom Fachbereich
Elektrotechnik und Informationstechnik
der Technischen Universität Darmstadt
zur Erlangung des Grades
Doktor-Ingenieur genehmigte

Dissertation

von

Dipl.-Ing. Norbert Stolte
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- Observation: Reed-Muller (RM) codes perform poorly under low-complexity SC decoding.

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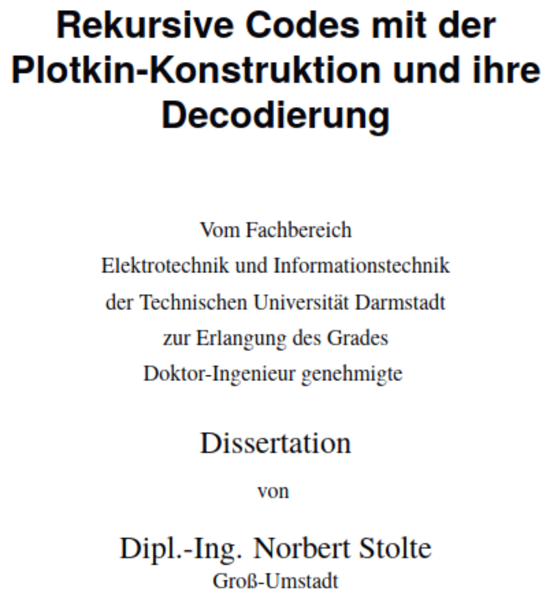

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- They were shown to outperform RM codes under SC decoding.

Generator Matrix

After defining a set $\mathcal{A}$, the generator matrix of the code is obtained by removing the rows in $\mathcal{F} \triangleq \{1, \ldots, N\} \setminus \mathcal{A}$ (frozen set) from $\mathbf{G}_2^{\otimes n}$:

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$$\mathbf{G}_2^{\otimes 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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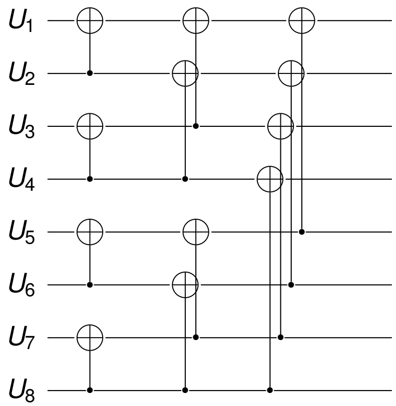
$X_1^8 = V_1^4 \mathbf{G}$ for random information bits V_1^4

Encoding

Let V_1^k denote the random information bits to be encoded:



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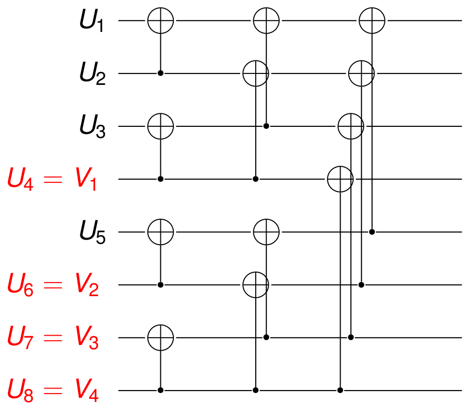
Let V_1^k denote the random information bits to be encoded:

1. For a given set $\mathcal{A}$, map V_1^k onto $U_{\mathcal{A}}$.



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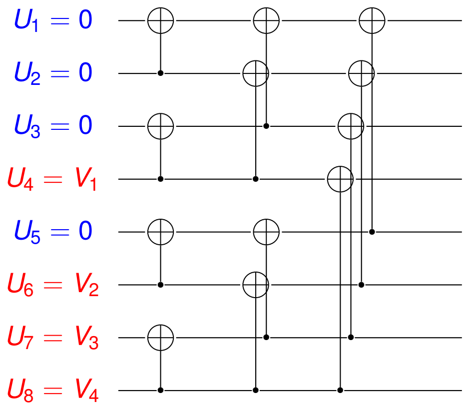
2. Set the remaining elements of U_1^N to 0 (frozen bits), i.e., $U_{\mathcal{F}} = 0^{n-k}$.



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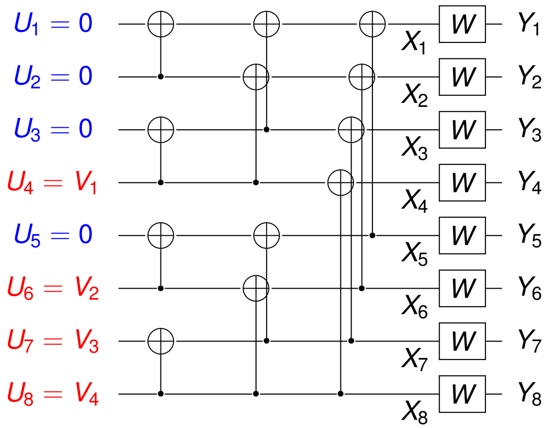

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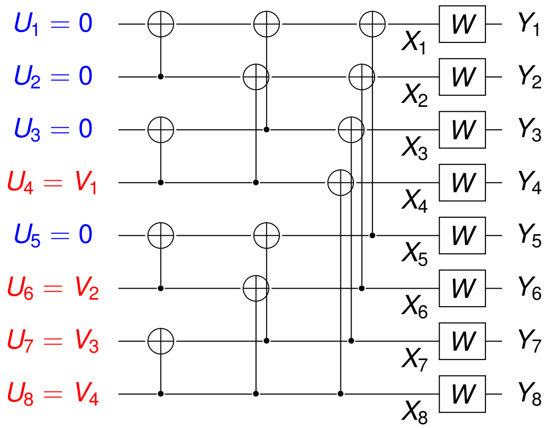
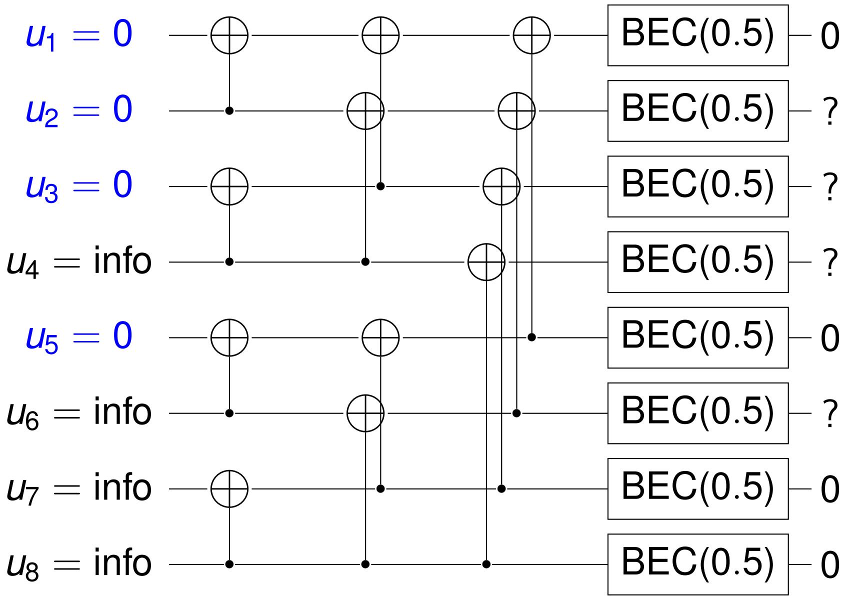
4. This can be done with a complexity of $\mathcal{O}(N \log N)$ instead of $\mathcal{O}(N^2)$.



SC Decoding: BEC Example



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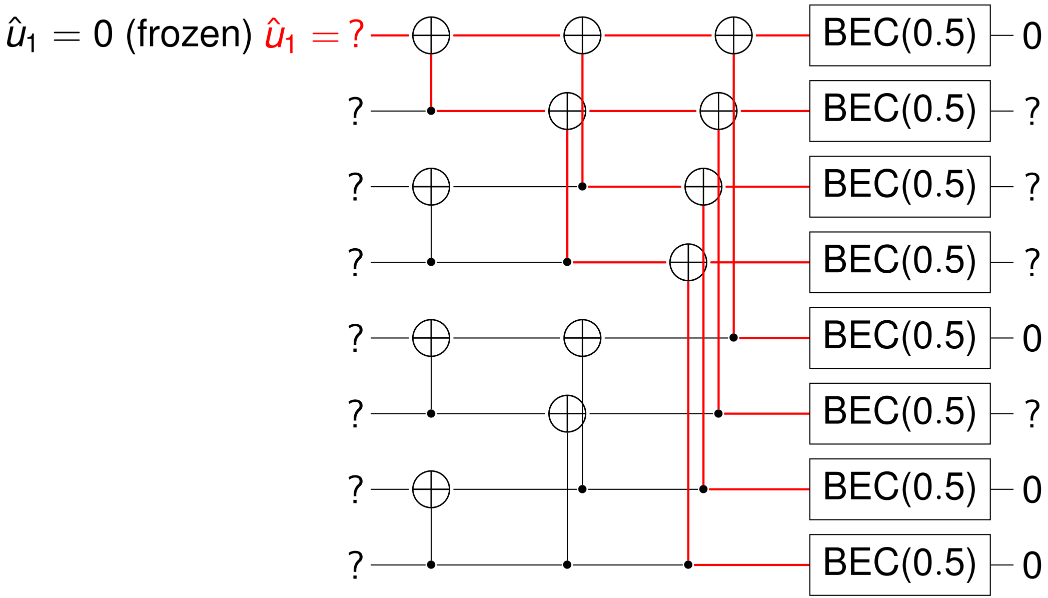

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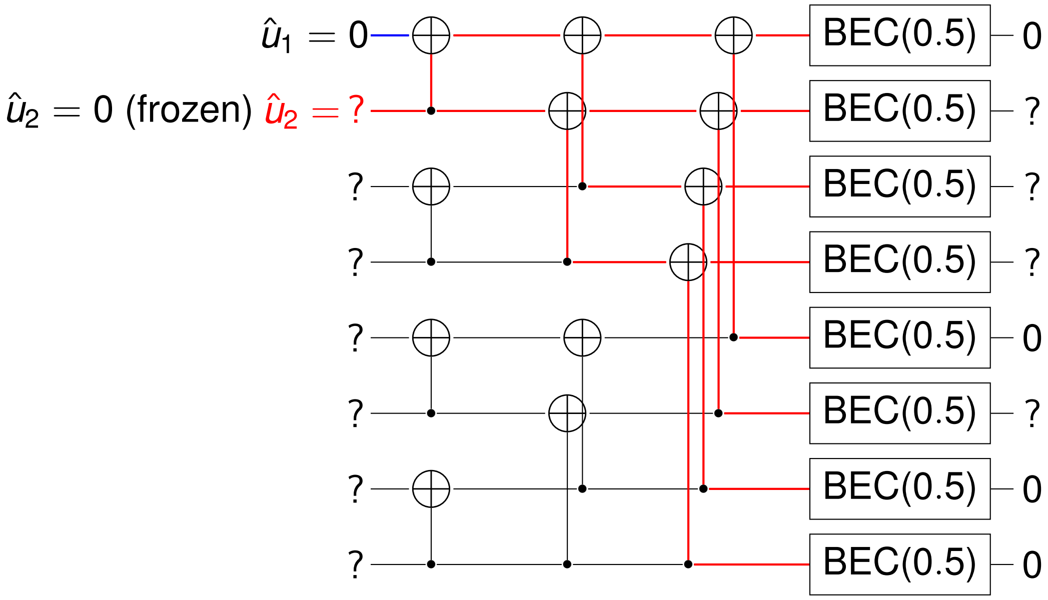

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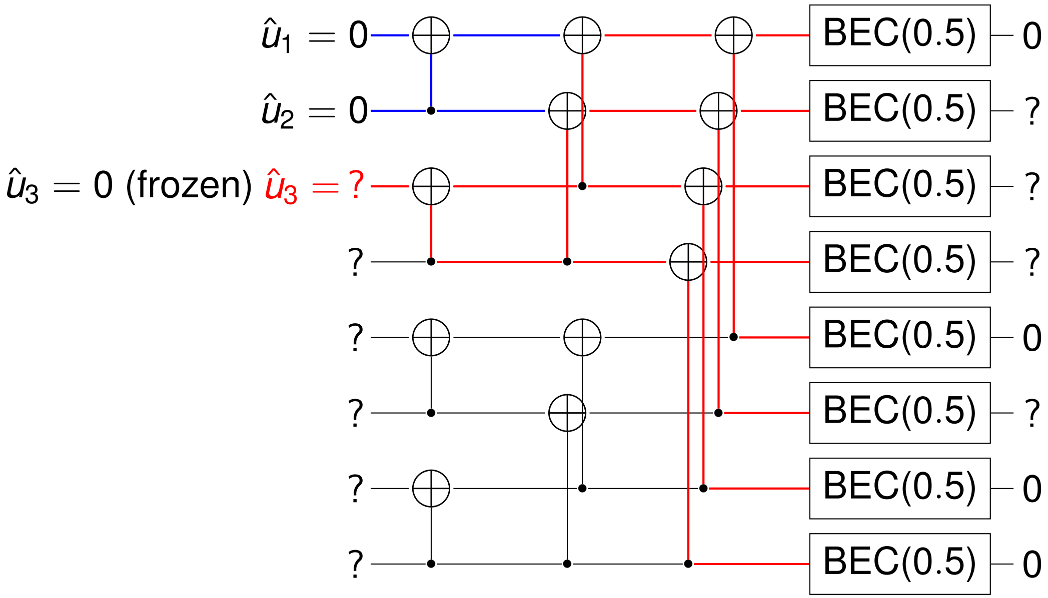

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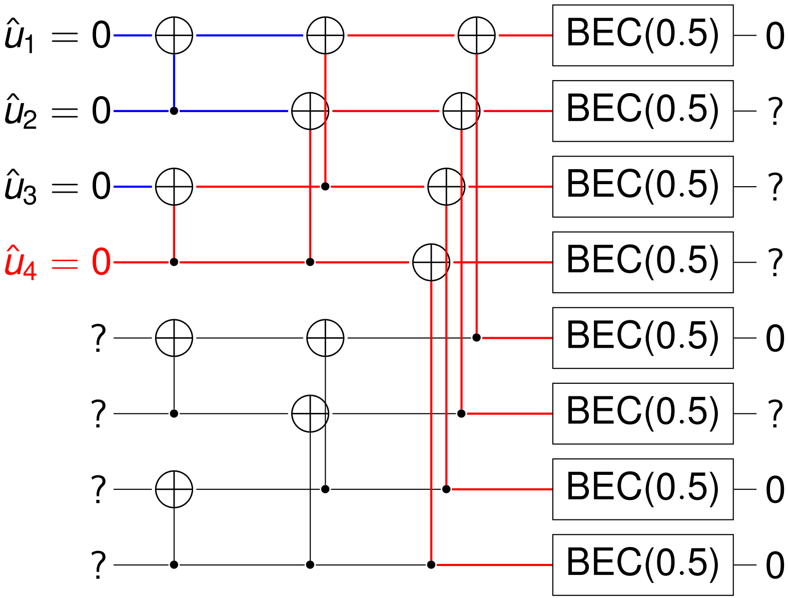

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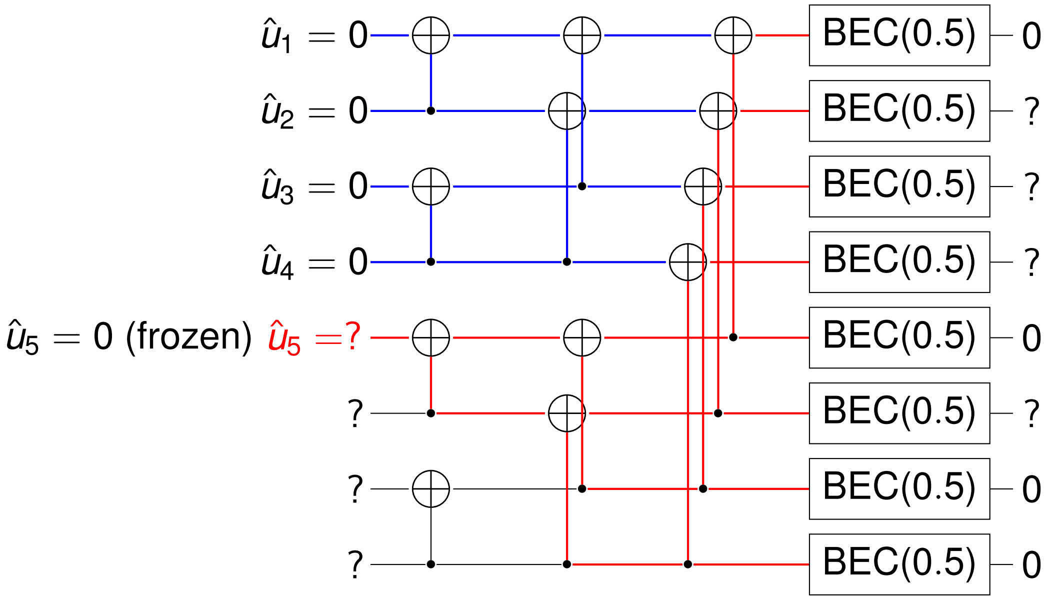

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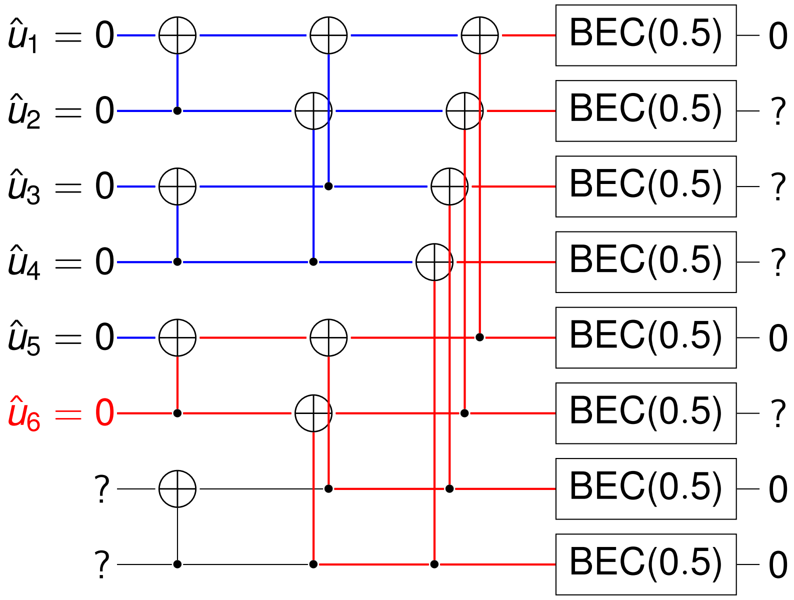

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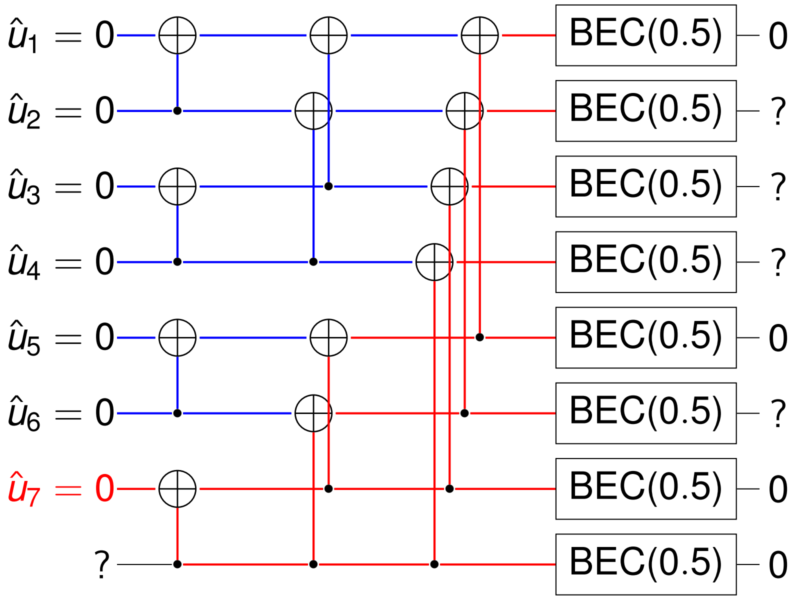

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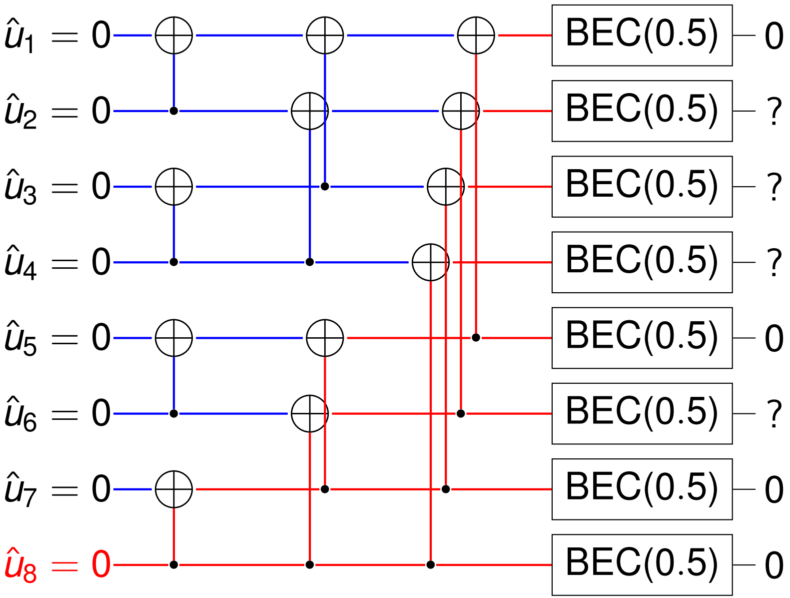

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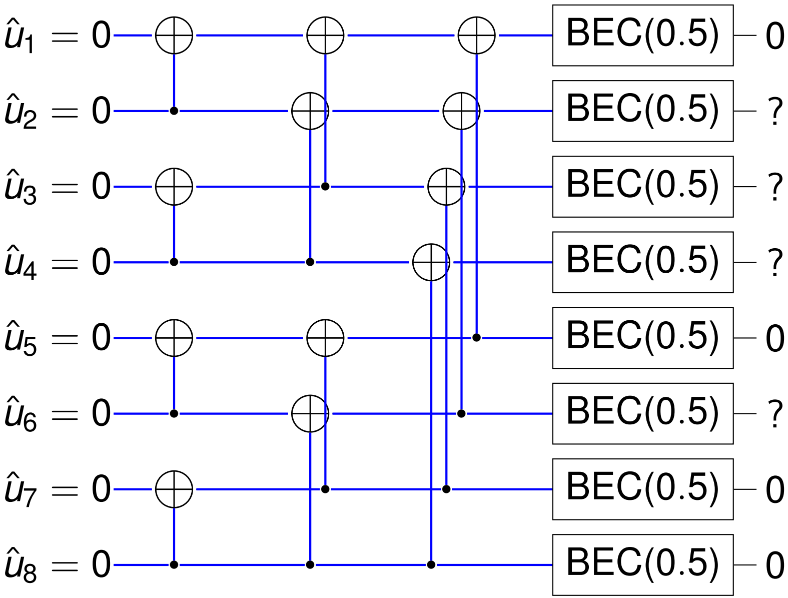
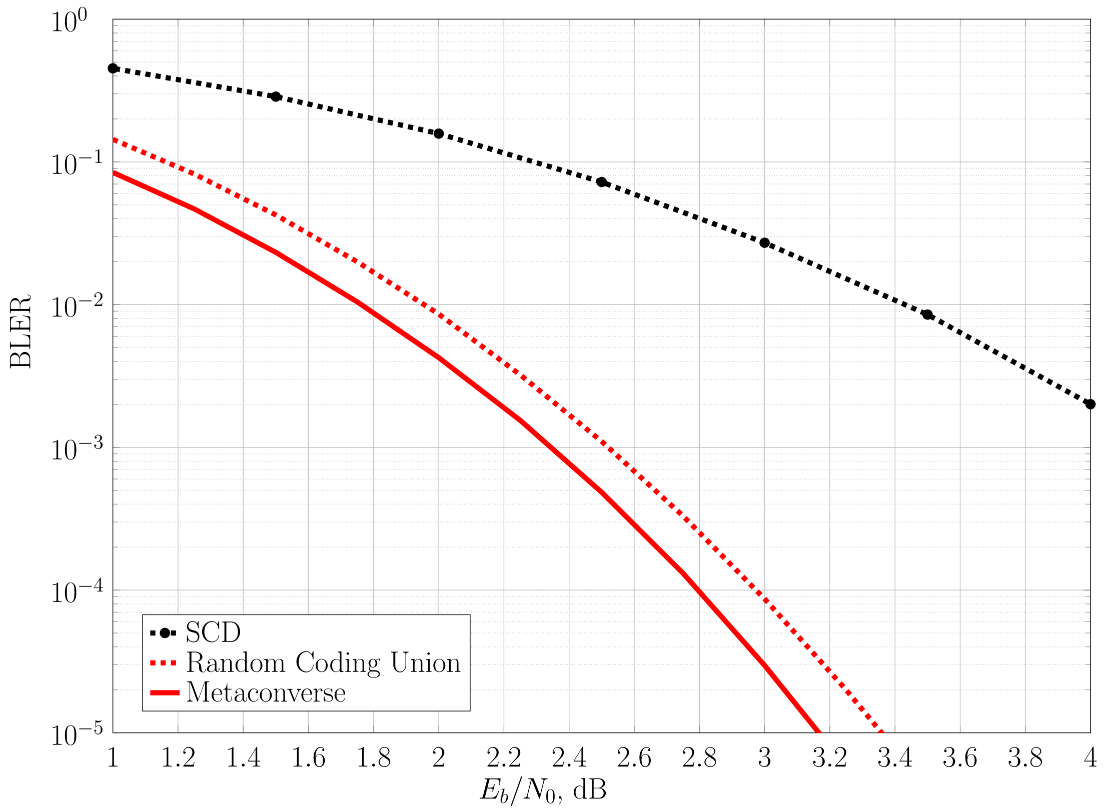


$n = 128, k = 64$



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E_b/N_0, dB
SCD
Random Coding Union
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Successive Cancellation List Decoding

Key idea: Each time a decision is needed on $\hat{u}_i$, both options, i.e., $\hat{u}_i = 0$ and $\hat{u}_i = 1$, are stored. This doubles the number of partial input sequences (paths) at each decoding stage.

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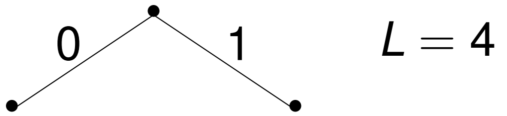

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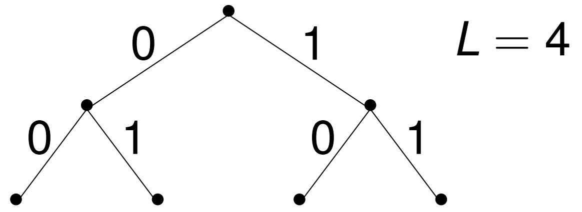

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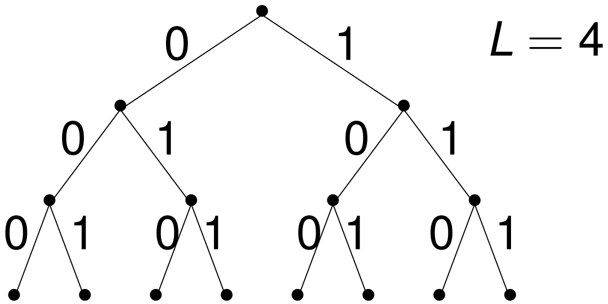

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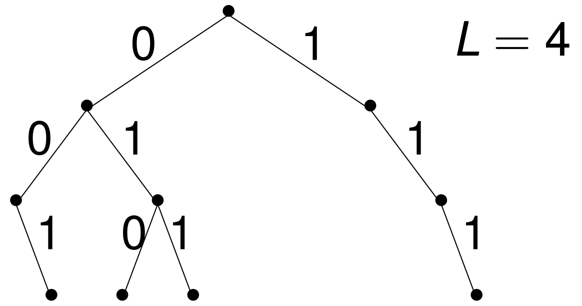

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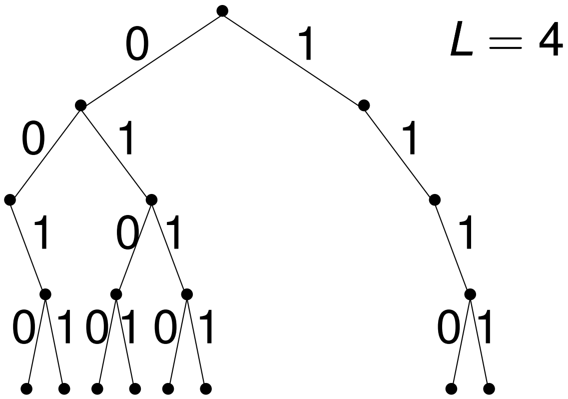

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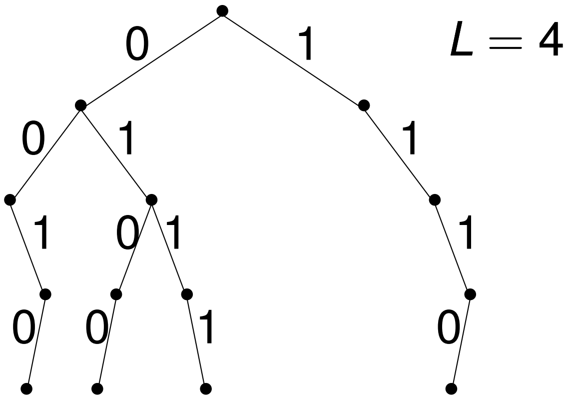

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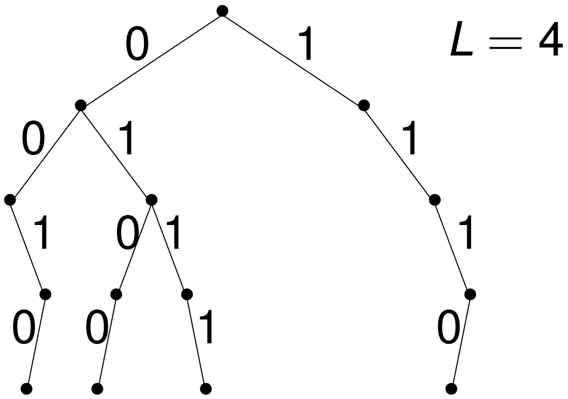

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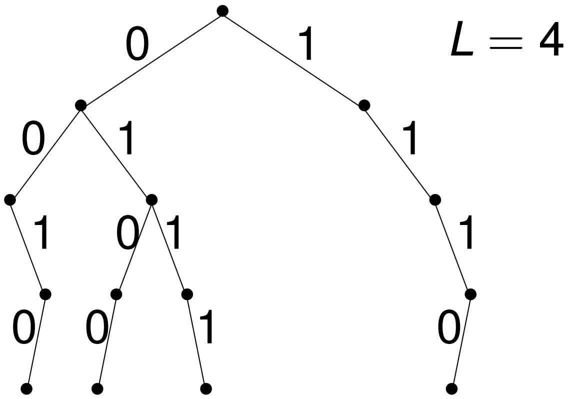

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- The decoder has been applied to RM codes previously (see, e.g., [Sto02, DS06]).

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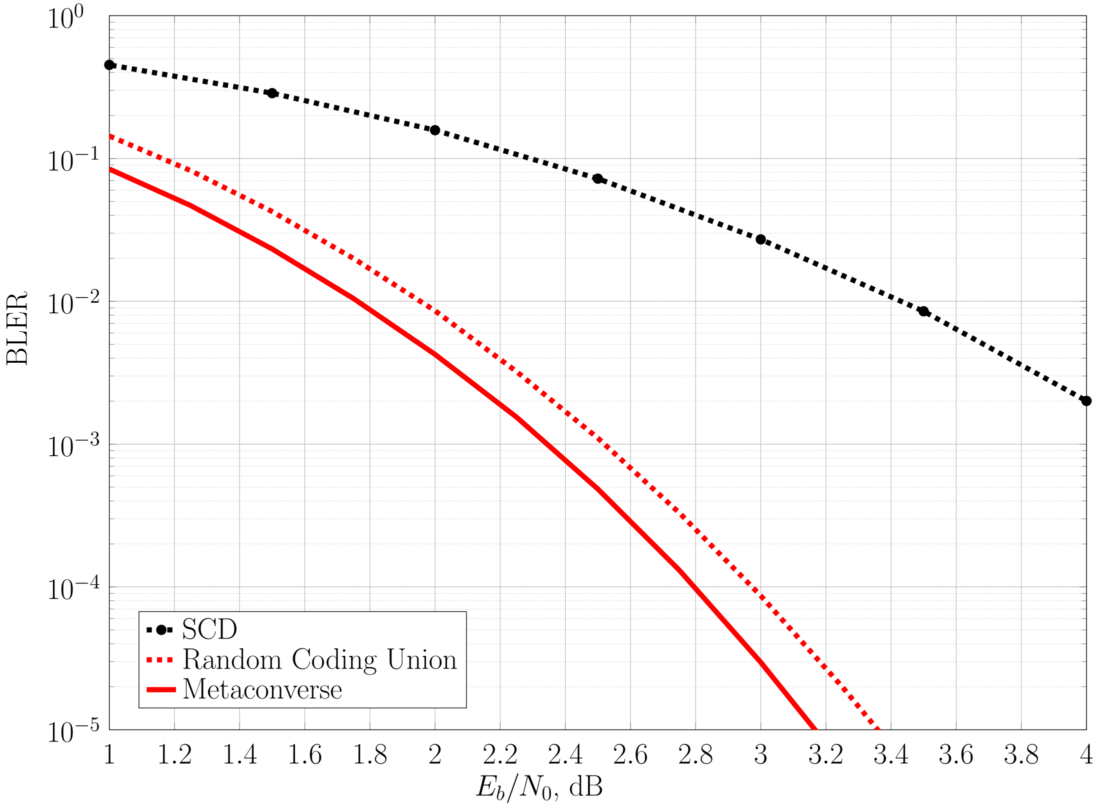

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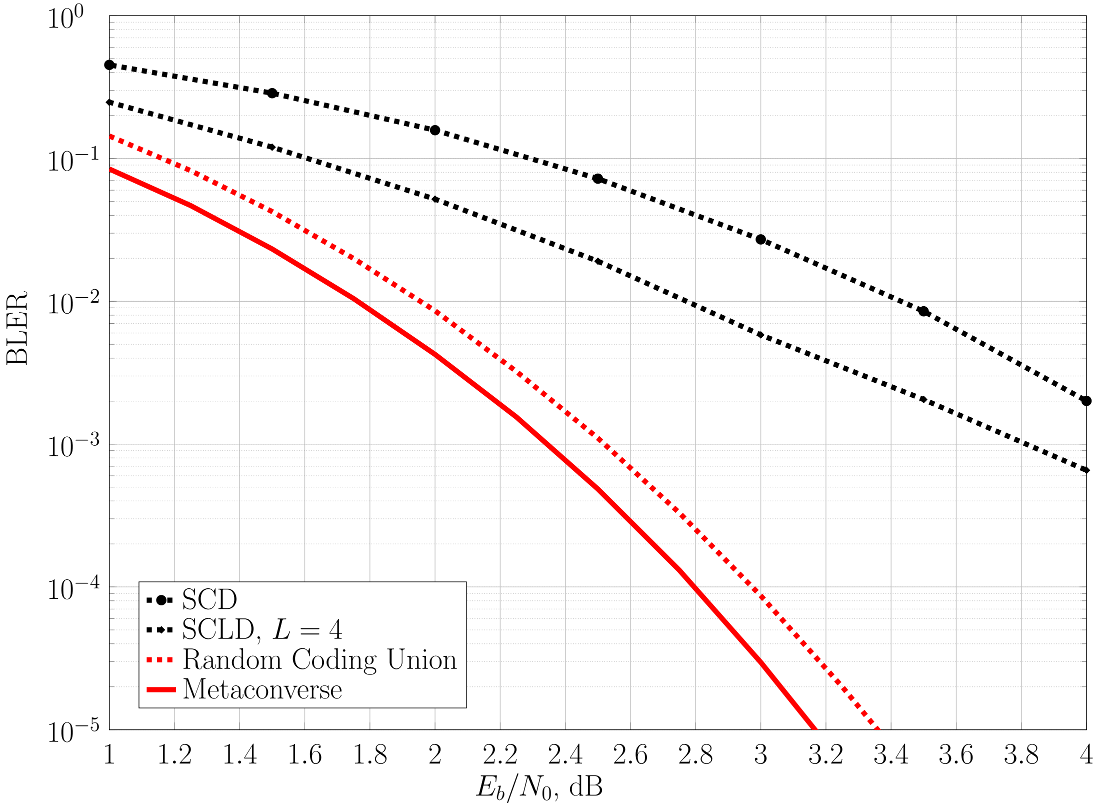

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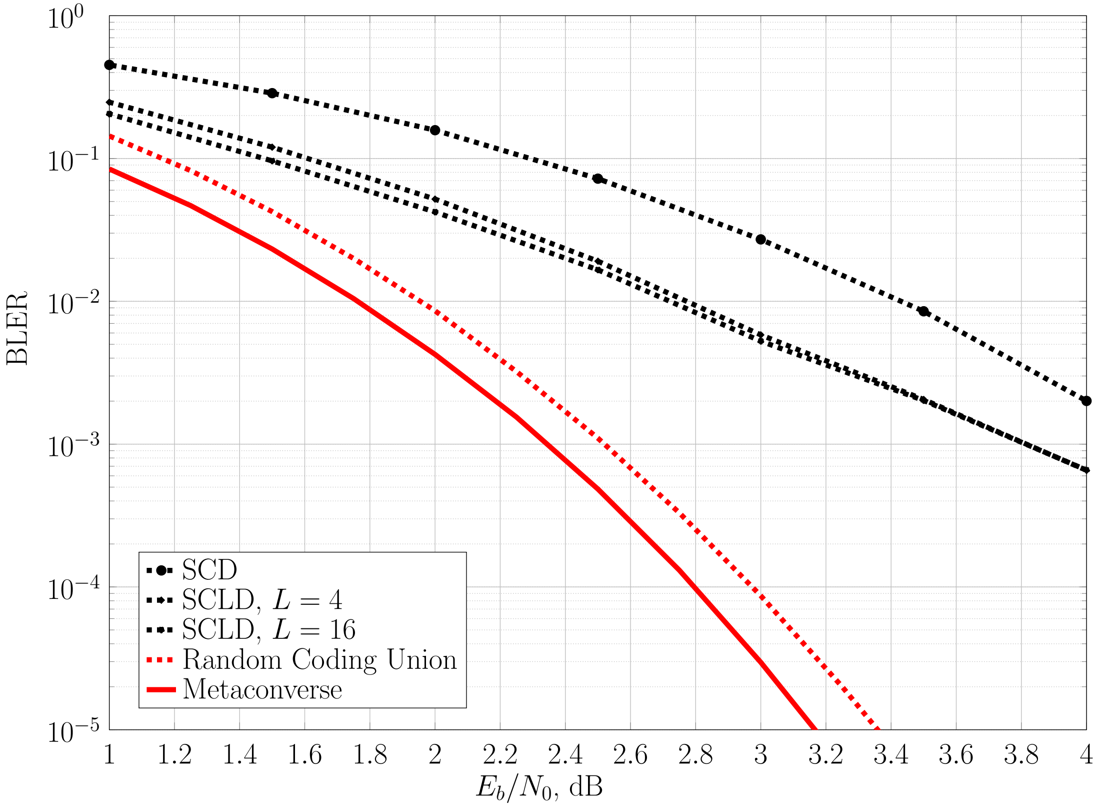
$N = 128, k = 64$



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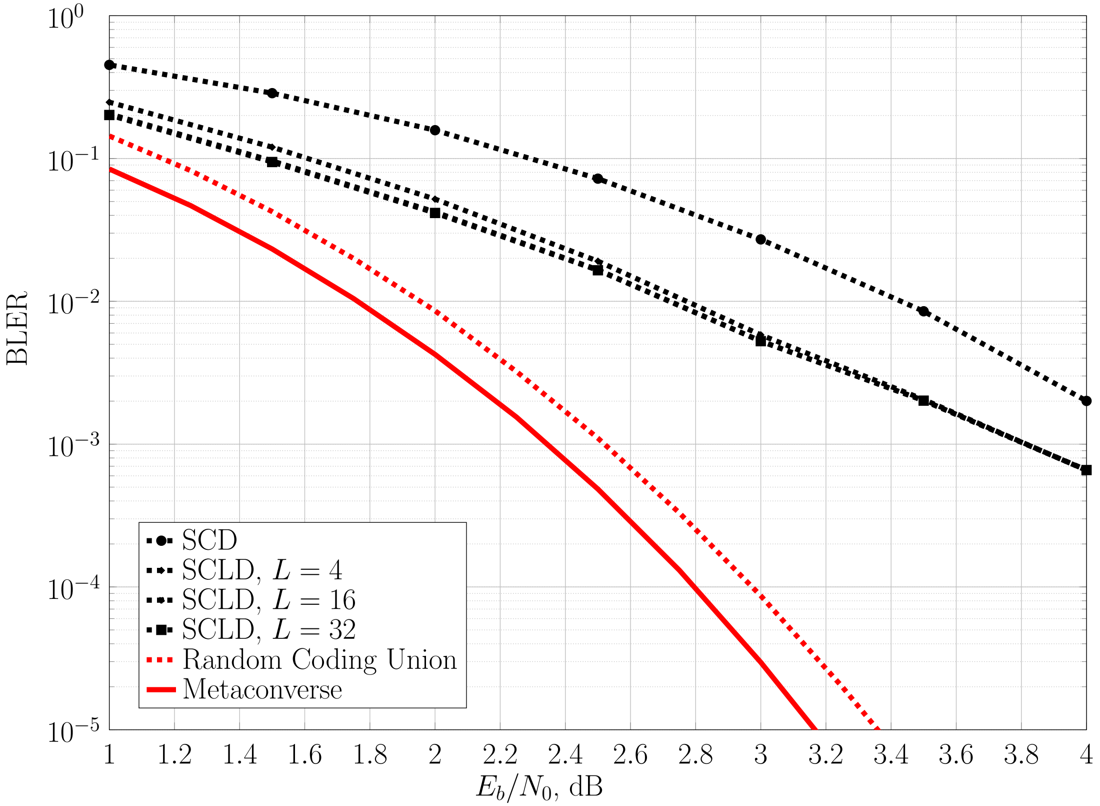
E_b/N_0, dB
SCD
SCLD, $L = 4$
SCLD, $L = 16$
Random Coding Union
Metaconverse

$N = 128, k = 64$



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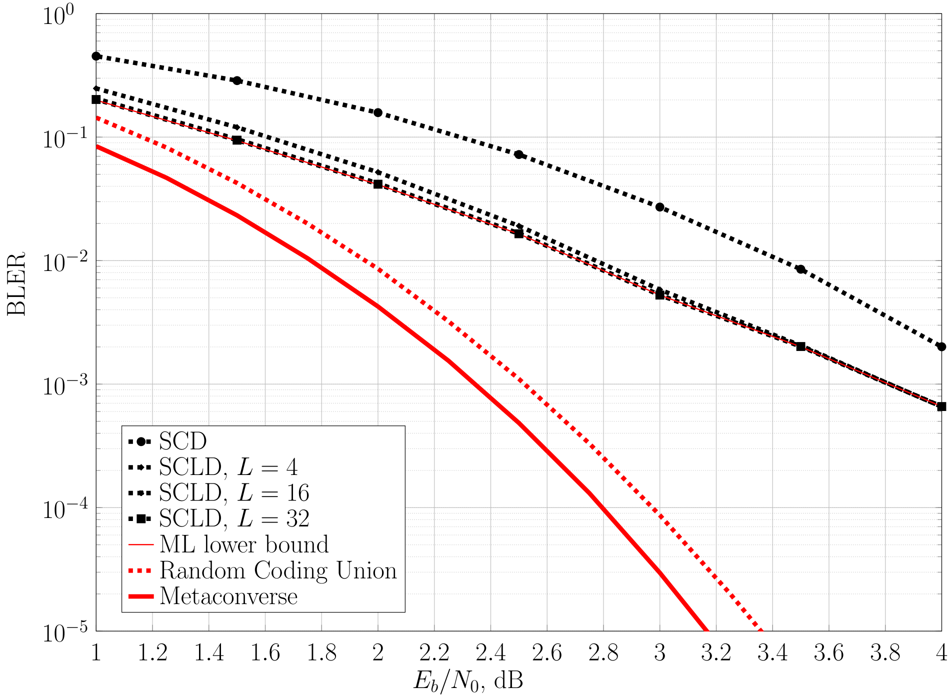
3.6
3.8
4
E_b/N_0, dB
SCD
SCLD, $L = 4$
SCLD, $L = 16$
SCLD, $L = 32$
Random Coding Union
Metaconverse

$N = 128, k = 64$



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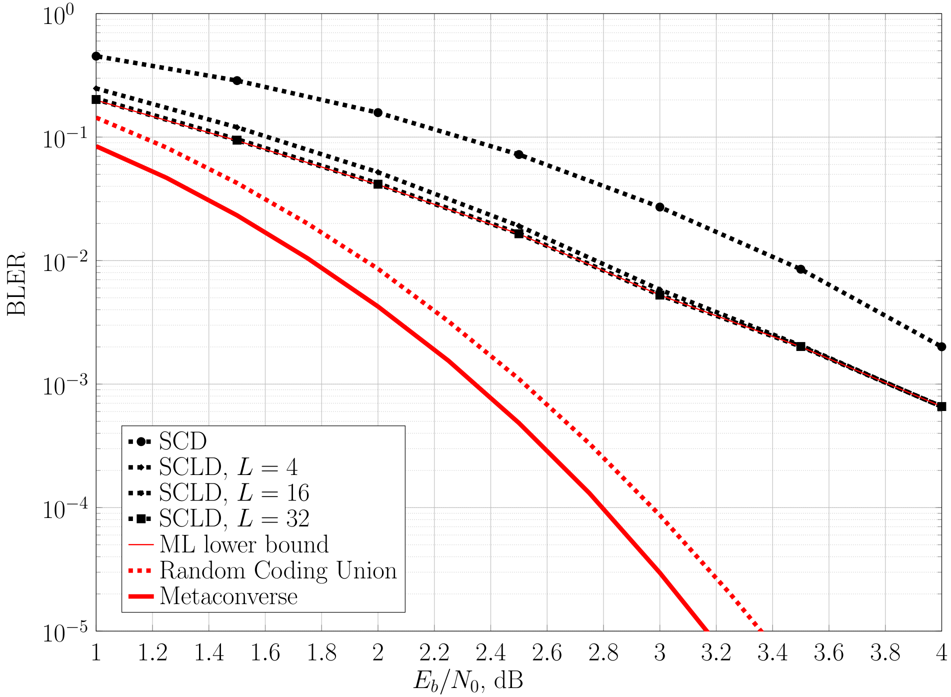

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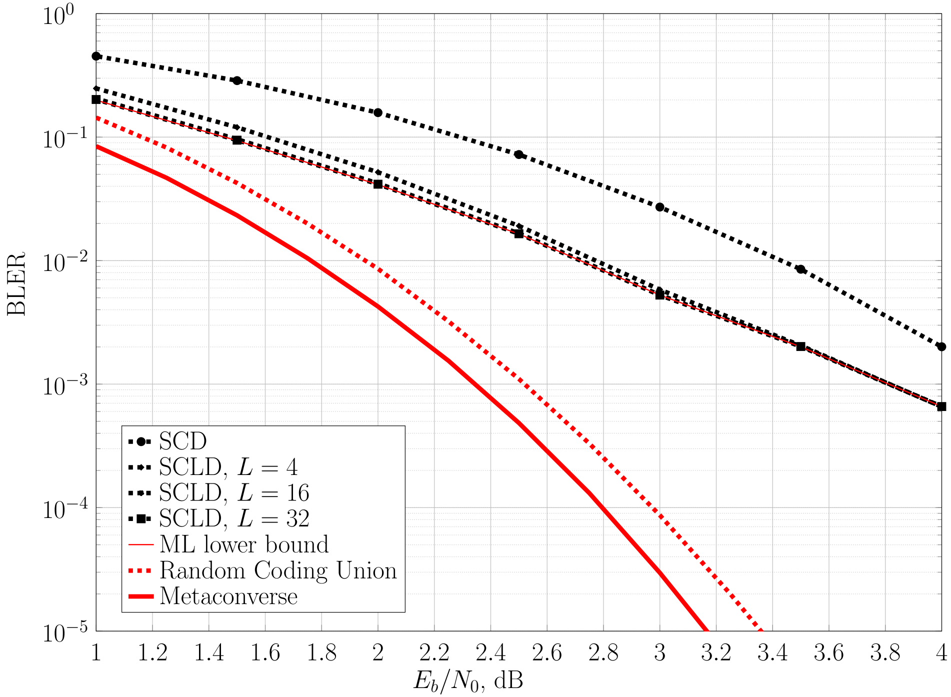

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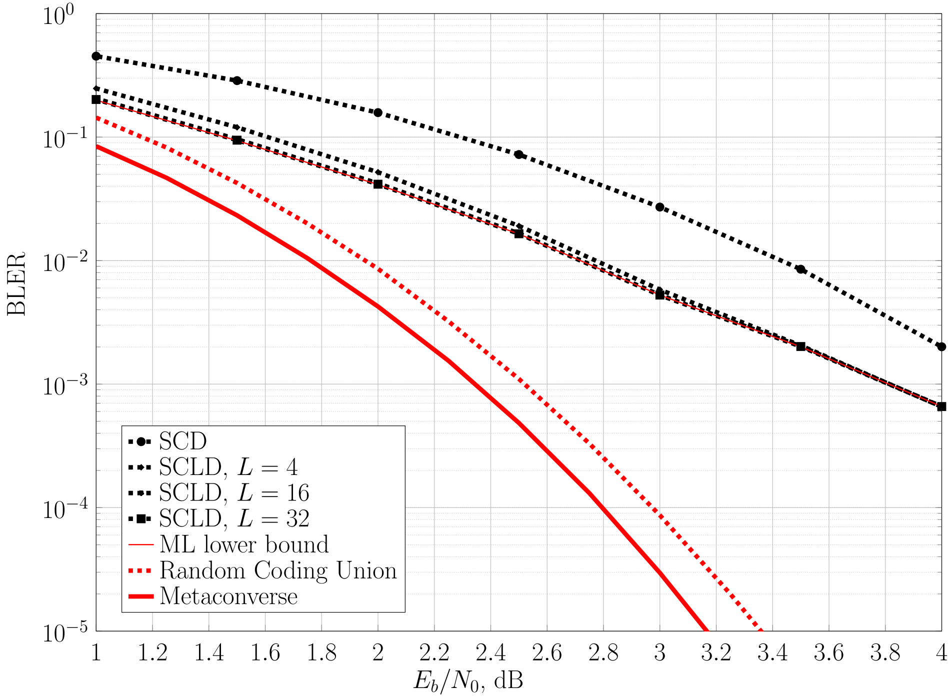

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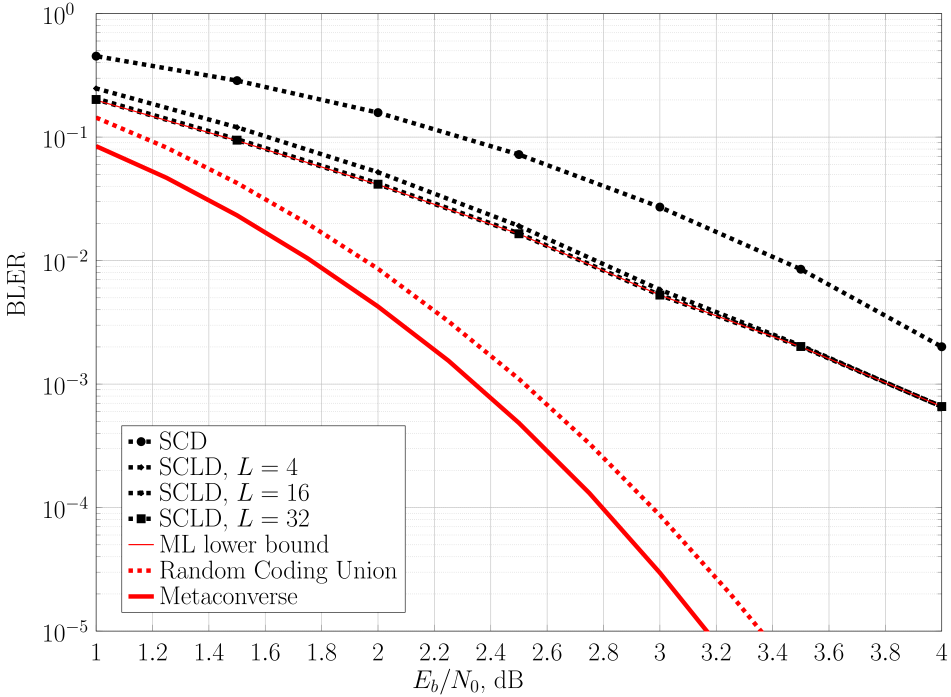

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At the receiver:

- SCL decoding (inner code), followed by syndrome check with outer code: pick the most probably codeword on the list fulfilling the CRC.

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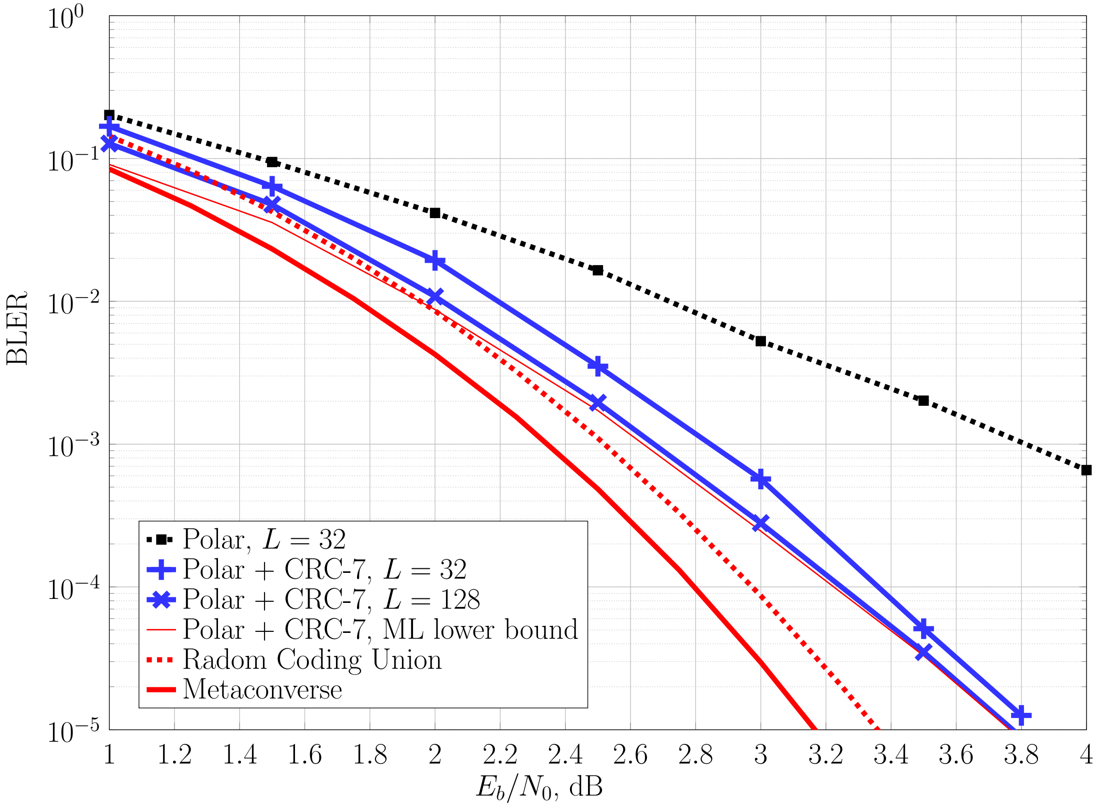

Polar, $L = 32$
Polar + CRC-7, $L = 32$
Polar + CRC-7, $L = 128$
Polar + CRC-7, ML lower bound
Radom Coding Union
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BLER
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10^{-1}
10^{-2}
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10^{-4}
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1
1.2
1.4
1.6
1.8
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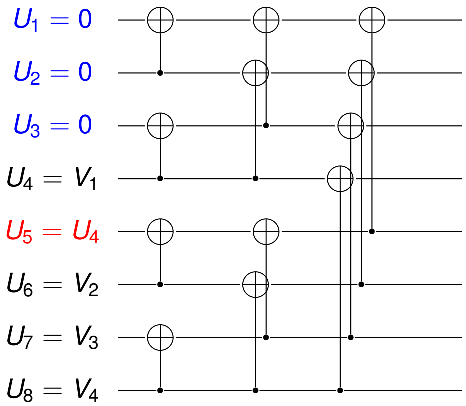
4

Dynamic Frozen Bits

- The value of a frozen bit can also be set to a linear combination of previous information bits (rather than a fixed 0 or 1 value) [TM16]



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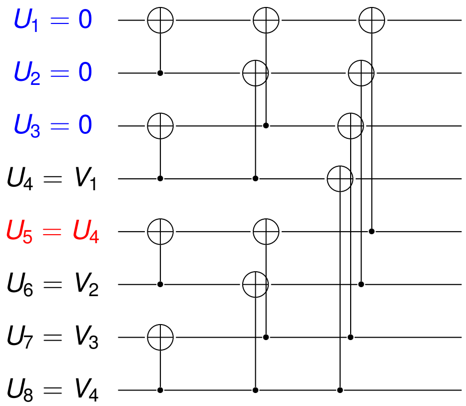

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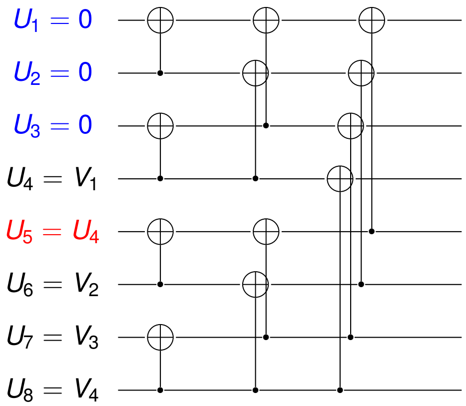

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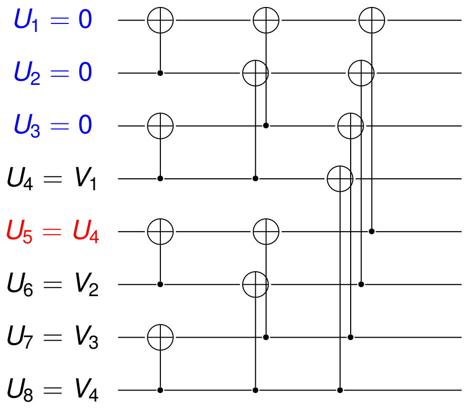
- Any binary linear block code can be represented as a polar code with dynamic frozen bits!



Outline

1 Overview of Polar Codes

2 Recent Advances in Polar Codes

- Binary Erasure Channel

3 Conclusions

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Based on joint works with Henry D. Pfister [CP20, CP21]

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Note: this ignores frozen bits and will be modified soon!

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- For the first m input bits, the information/frozen sets are denoted as

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$$\sum_{i \in \mathcal{A}^{(m)}} H\left(U_i \middle| Y_1^N, U_1^{i-1}\right) - \sum_{i \in \mathcal{F}^{(m)}} \left(1 - H\left(U_i \middle| Y_1^N, U_1^{i-1}\right)\right) \leq \bar{D}_m \leq \sum_{i \in \mathcal{A}^{(m)}} H\left(U_i \middle| Y_1^N, U_1^{i-1}\right)$$

Bounding the List Size

Theorem

Upon observing y_1^N when u_1^N is sent, we define the set (for $\alpha \in (0,1]$) $\mathcal{S}_\alpha^{(m)}\left(u_1^m, y_1^N\right) \triangleq \left\{\tilde{u}_1^m : \mathbb{P}\left(\tilde{u}_{\mathcal{A}^{(m)}} | y_1^N, \tilde{u}_{\mathcal{F}^{(m)}}\right) \geq \alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} | y_1^N, u_{\mathcal{F}^{(m)}}\right)\right\}$. *Then,*

$$\mathrm{E}\left[\log_2 \left|\mathcal{S}_\alpha^{(m)}\right|\right] \leq \bar{D}_m + \log_2 \frac{1}{\alpha} = H\left(U_{\mathcal{A}^{(m)}} \middle| Y_1^N, U_{\mathcal{F}^{(m)}}\right) + \log_2 \frac{1}{\alpha}$$

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Proof.

$$\log_2 \left|\mathcal{S}_\alpha^{(m)}\right| = \log_2 \sum_{\tilde{u}_1^m} \mathbb{1}\left\{\mathbb{P}\left(\tilde{u}_{\mathcal{A}^{(m)}} | y_1^N, \tilde{u}_{\mathcal{F}^{(m)}}\right) \geq \underbrace{\alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} \middle| y_1^N, u_{\mathcal{F}^{(m)}}\right)}_{q}\right\}$$
$$\leq \log_2 1/\left(\alpha \mathbb{P}\left(u_{\mathcal{A}^{(m)}} \middle| y_1^N, u_{\mathcal{F}^{(m)}}\right)\right)$$

Valid for all u_1^N and y_1^N; thus, we take expectation over all u_1^m and y_1^N □

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- For an SCL decoder with max list size L_m during the m-th decoding step,
  - the decoder needs $L_m \geq |\mathcal{S}_1^{(m)}|$ for the true u_1^m to stay on the list
  - Choosing $\alpha < 1$ (say 0.94) captures near misses and matches entropy better.

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  - A first-order code design criterion can be seen as $\log_2 L_m \geq d_m$.
  - Based on this, a small code improvement will be introduced.

Dynamic Reed-Muller Codes

- d-RM code ensemble [CNP20]:
  - Let $\mathcal{A}$ be the information indices of an RM code.
  - u_i is an information bit if $i \in \mathcal{A}$.
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- Closely related to polarization-adjusted convolutional (PAC) codes [Arı19].
- PAC and (random instances of) d-RM code perform very similar under SCL decoding with the same list sizes.

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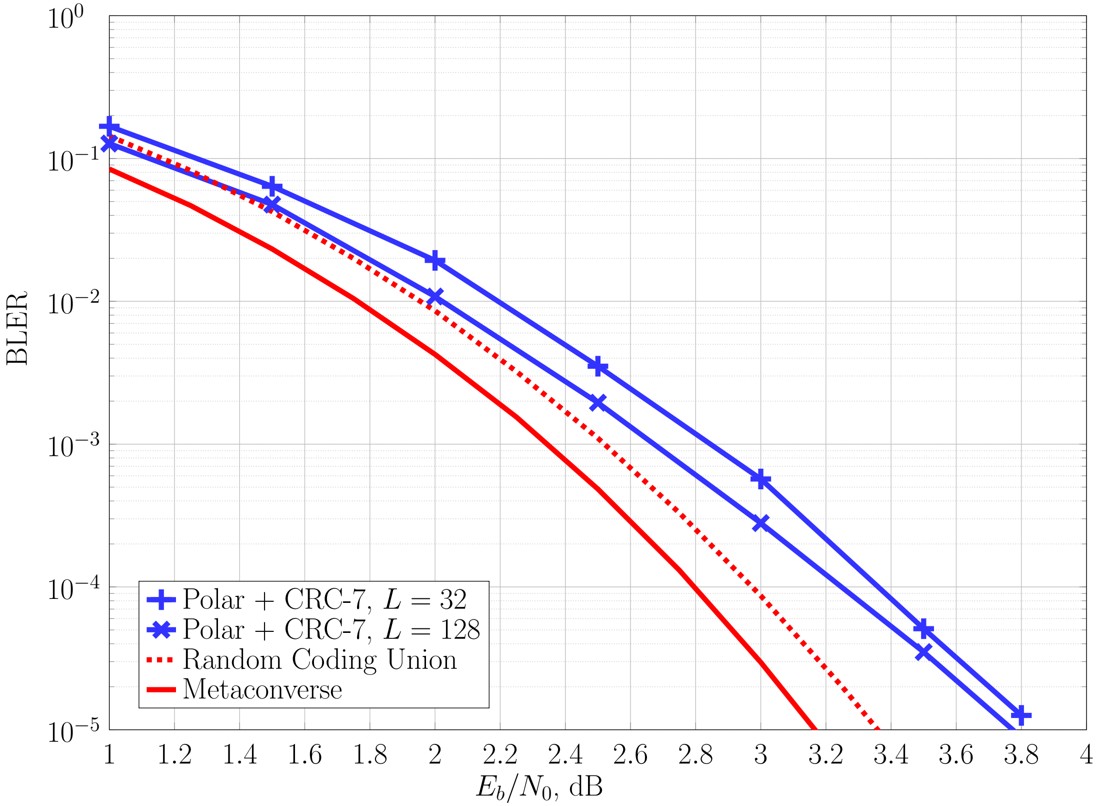

10^{0}
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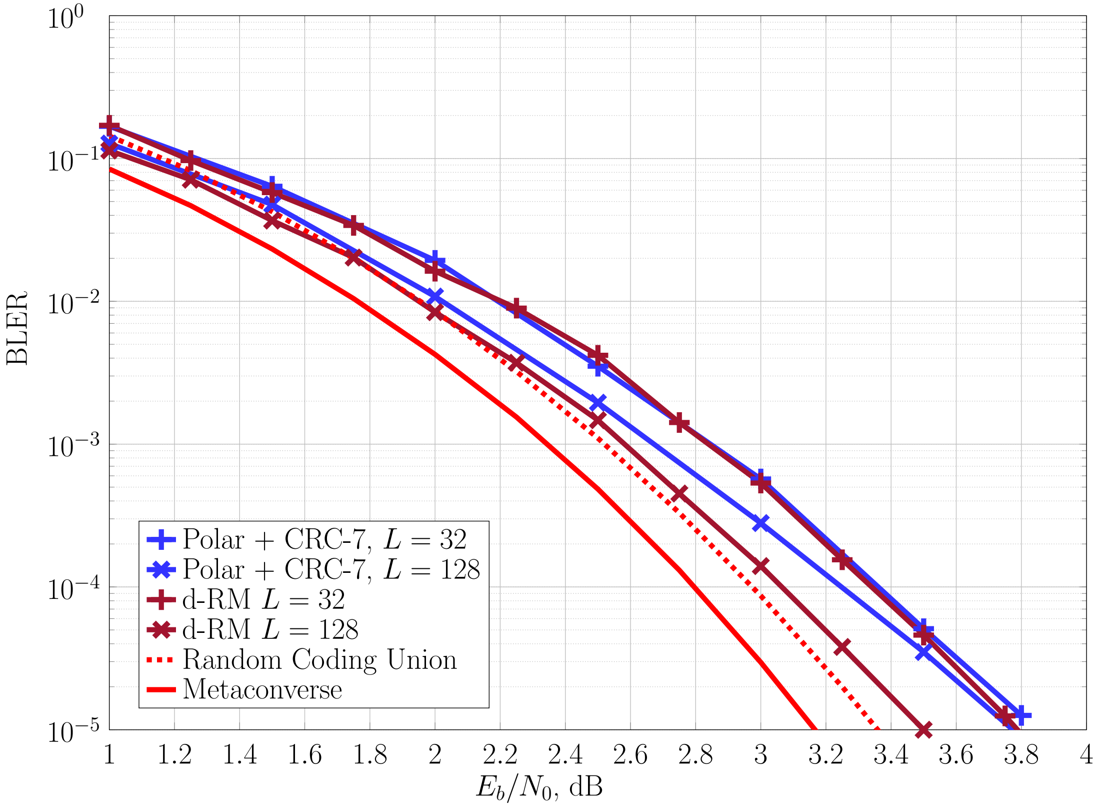

10^0
10^{-1}
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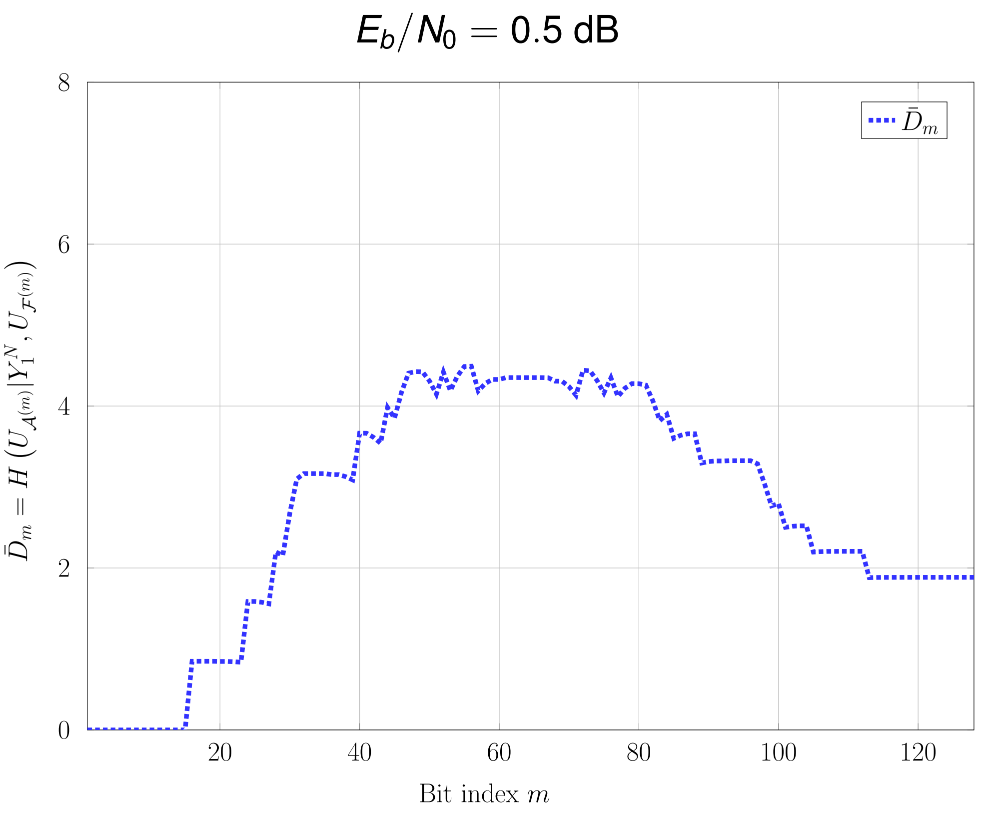
$(128, 64)$ d-RM Code over the AWGN Channel



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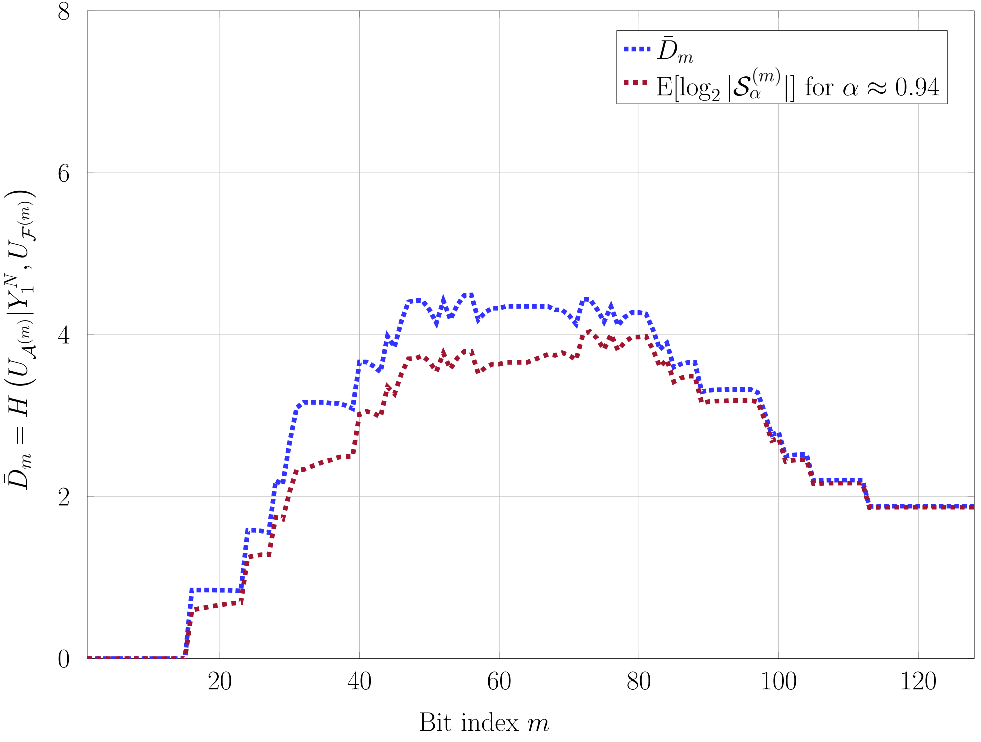

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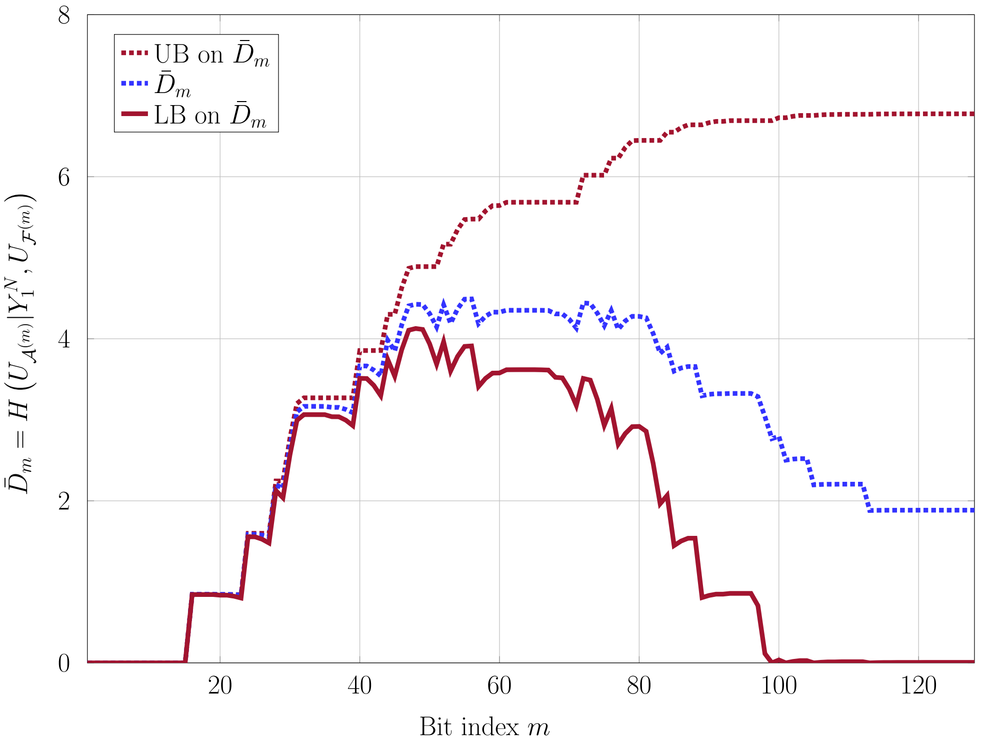
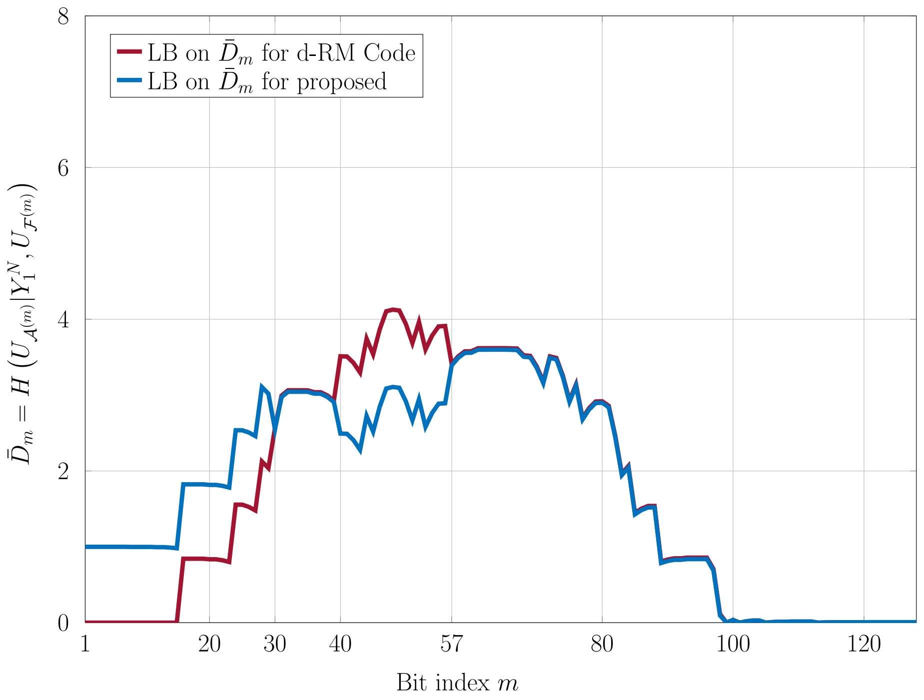


$(128, 64)$ Proposed vs d-RM Code over the AWGN Channel

Proposed Code

- $u_{\{30,40\}}$ dynamic frozen bits
- $u_{\{1,57\}}$ info. bits

$E_b/N_0 = 0.5$ dB



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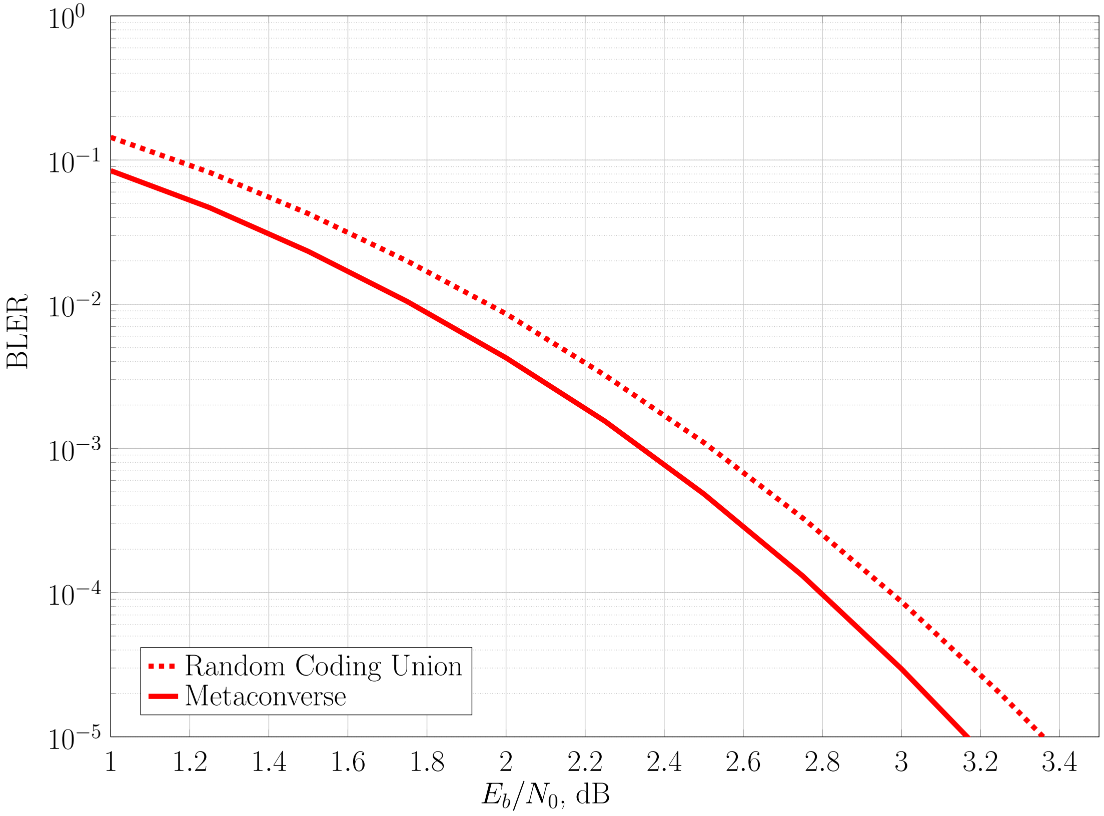

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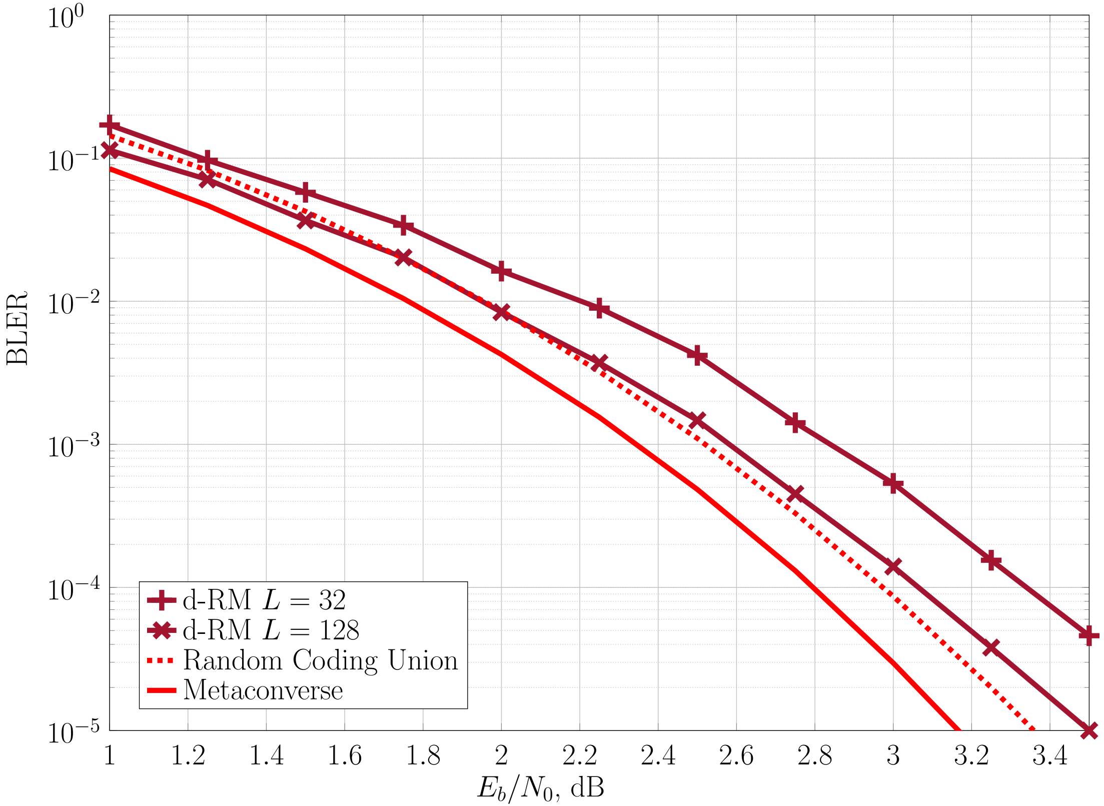

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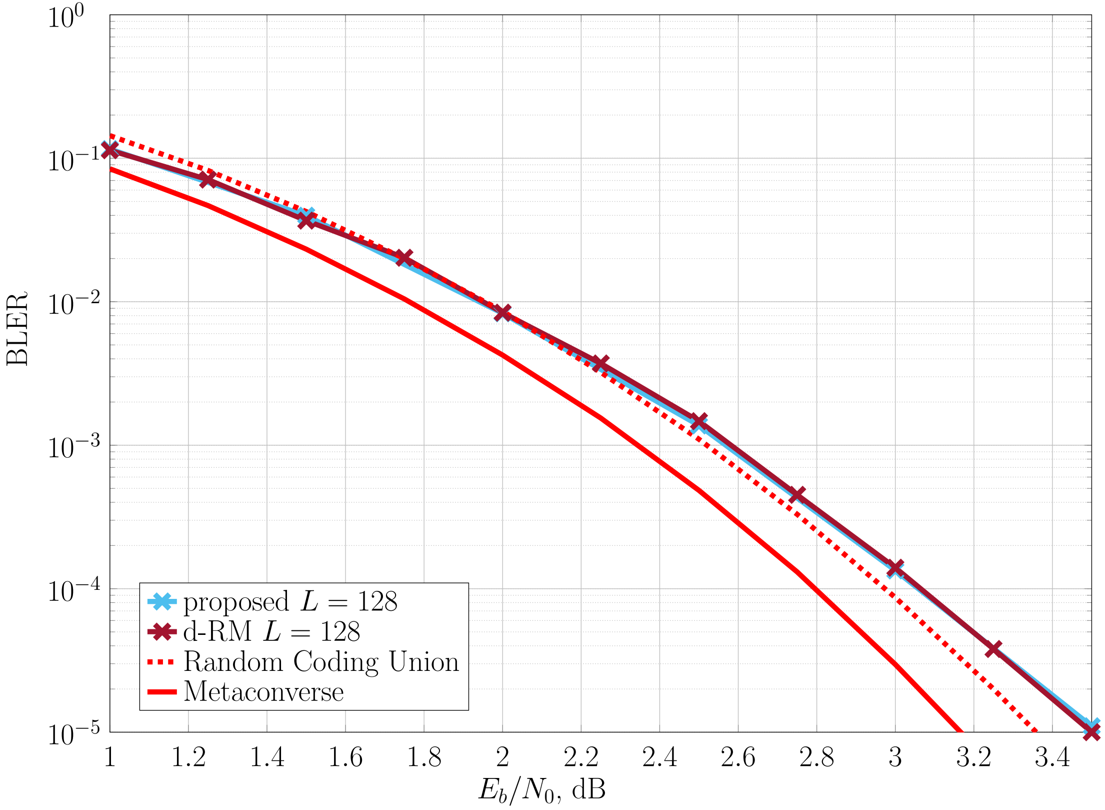

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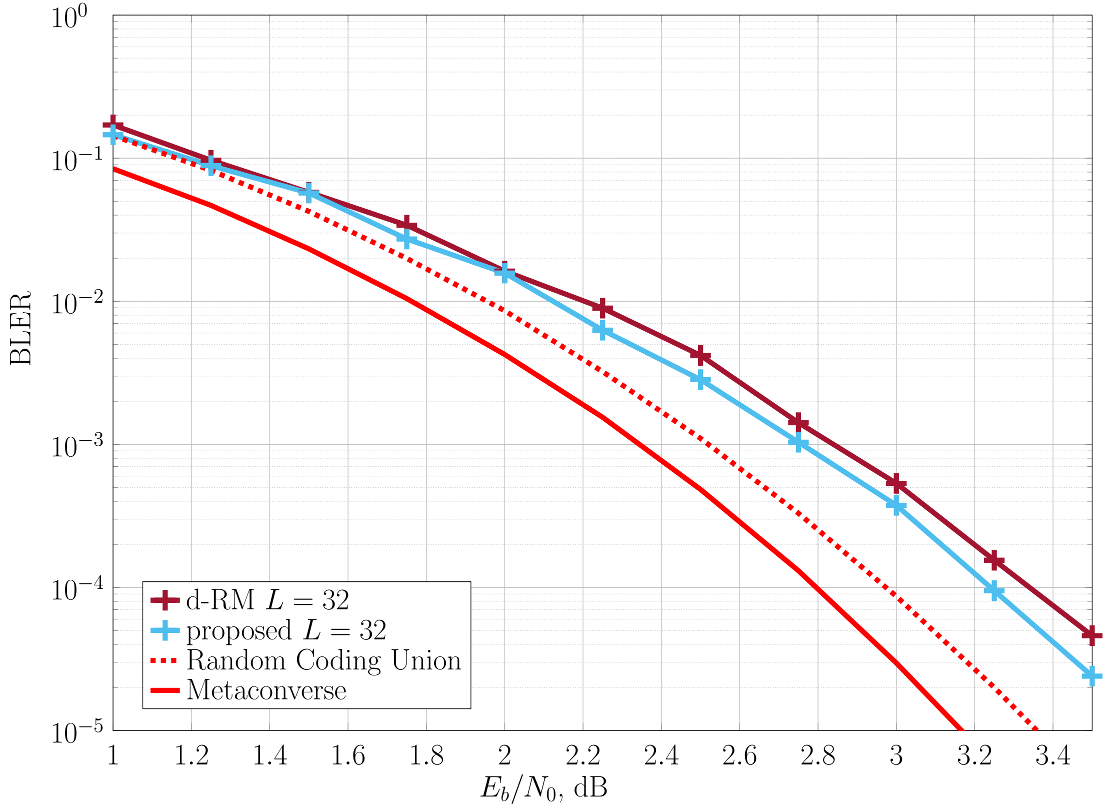

Recent Related Works

- Among many others, there are some recent works to be checked:
  - Works by E. Viterbo and his group: [RV19, RBV20]
  - A paper by A. Vardy and his group: [YFV20]
  - A paper by S. ten Brink and his group: [GEE$^+$20]

Outline

Successive Cancellation Inactivation Decoding

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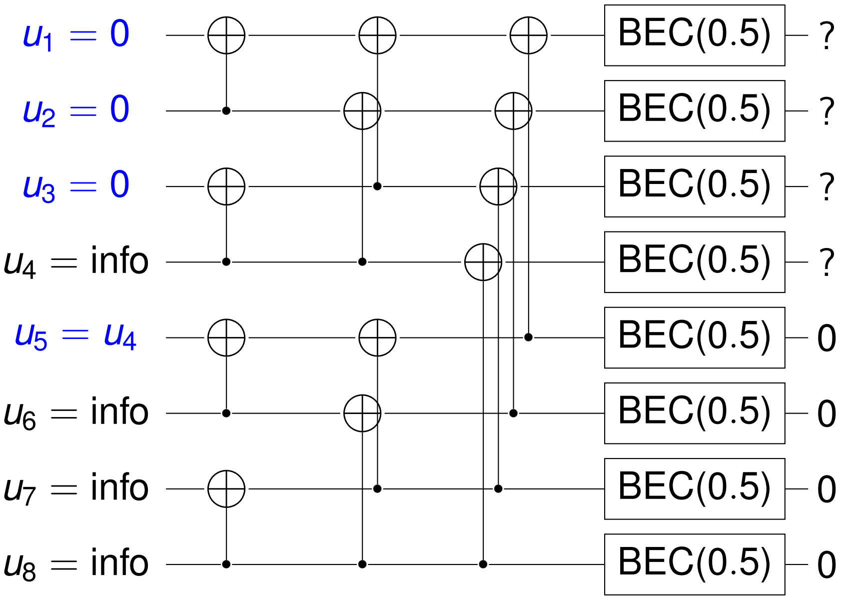
- It continues decoding using SC decoding for the BEC, where the message values are allowed to be functions of all inactivated variables.
- Previously inactivated bits may be resolved using linear equations derived from decoding frozen bits.

Example: SC Inactivation Decoding



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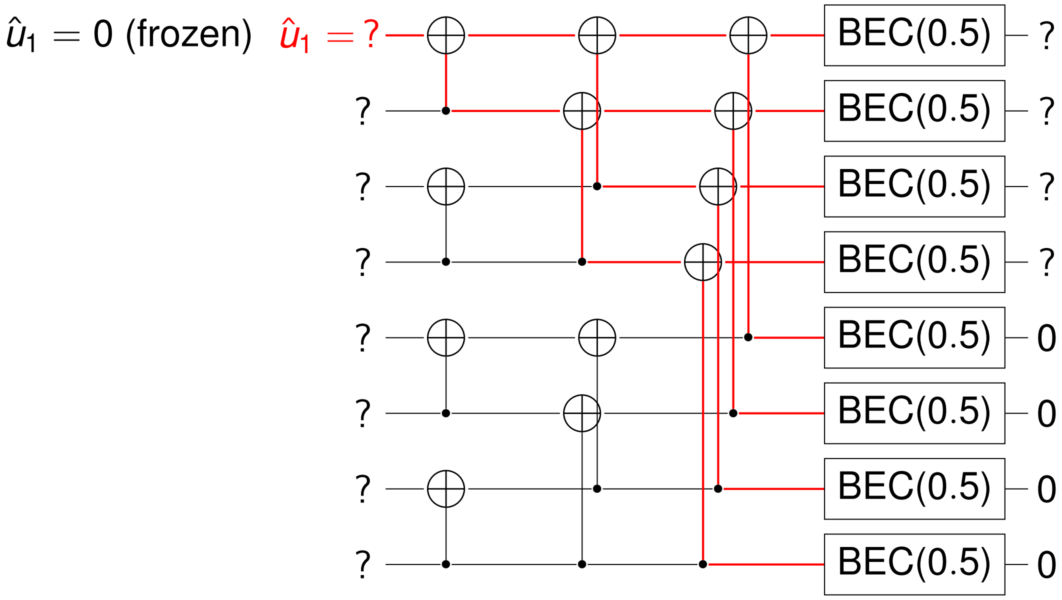

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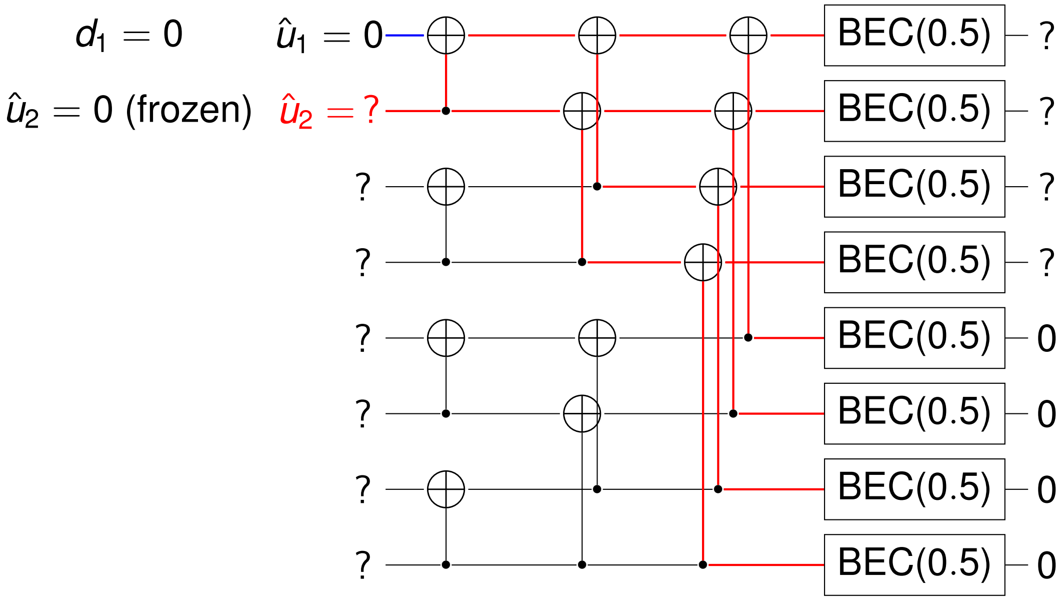

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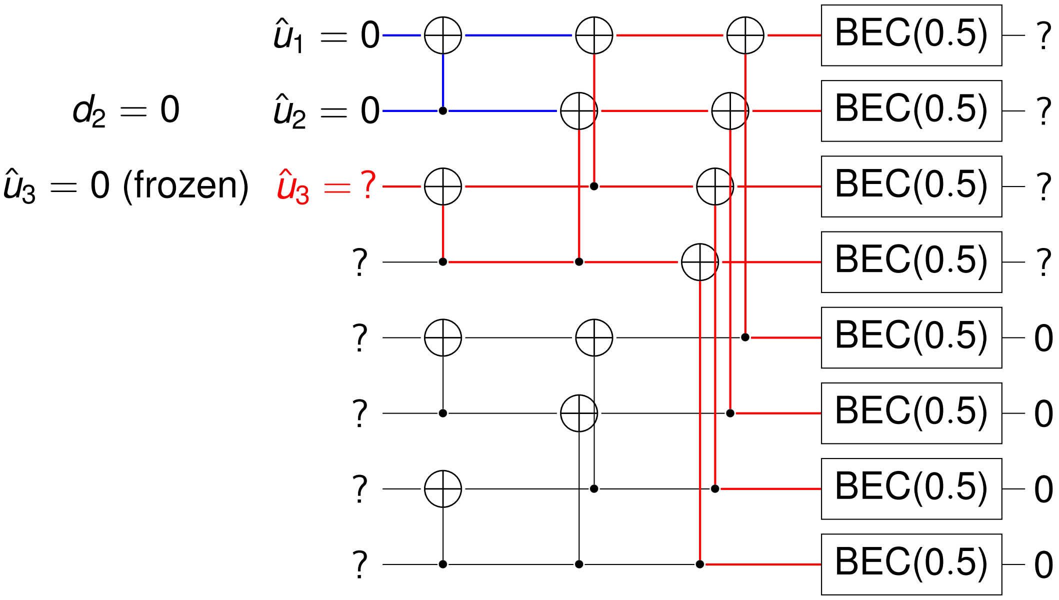

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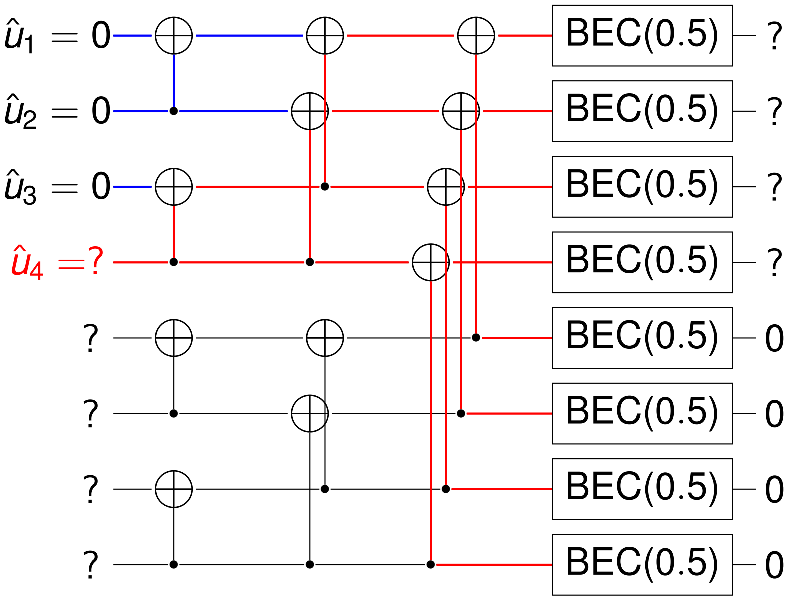

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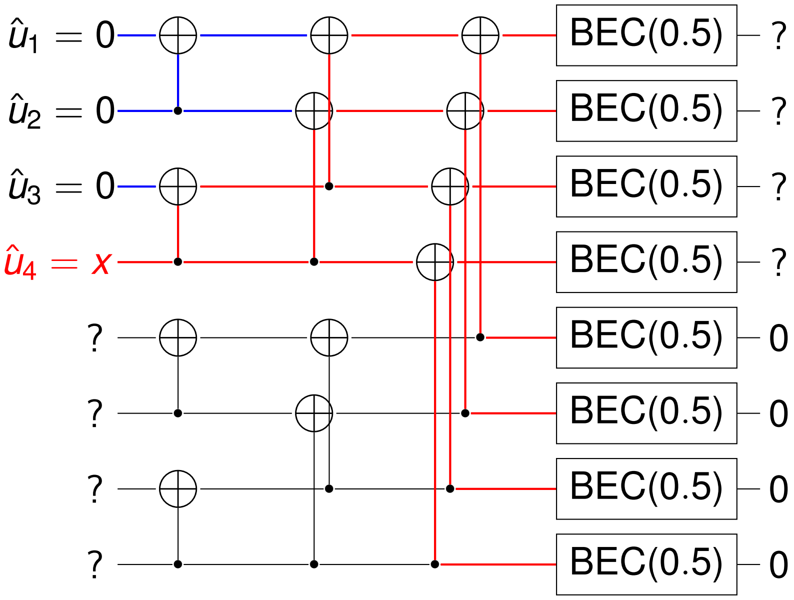
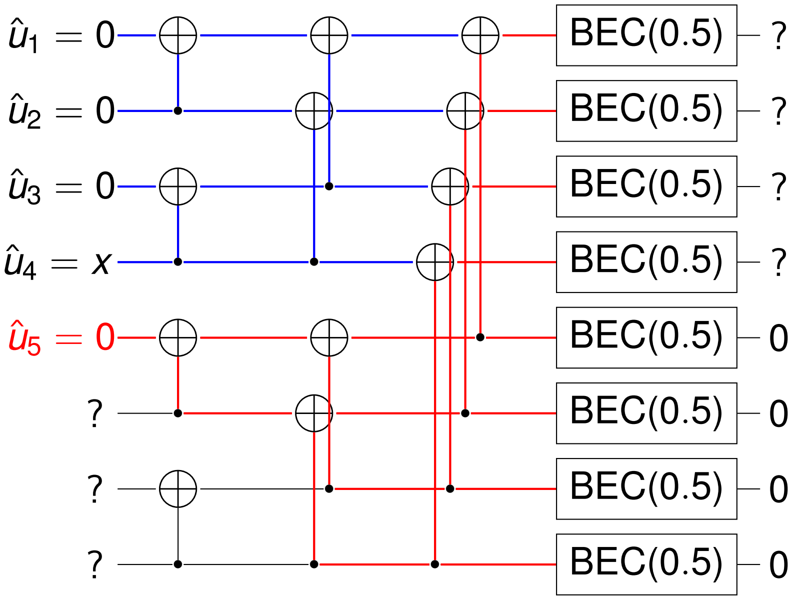


Example: SC Inactivation Decoding

$d_4 = 1$

$\hat{u}_5 = \hat{u}_4$ (frozen)



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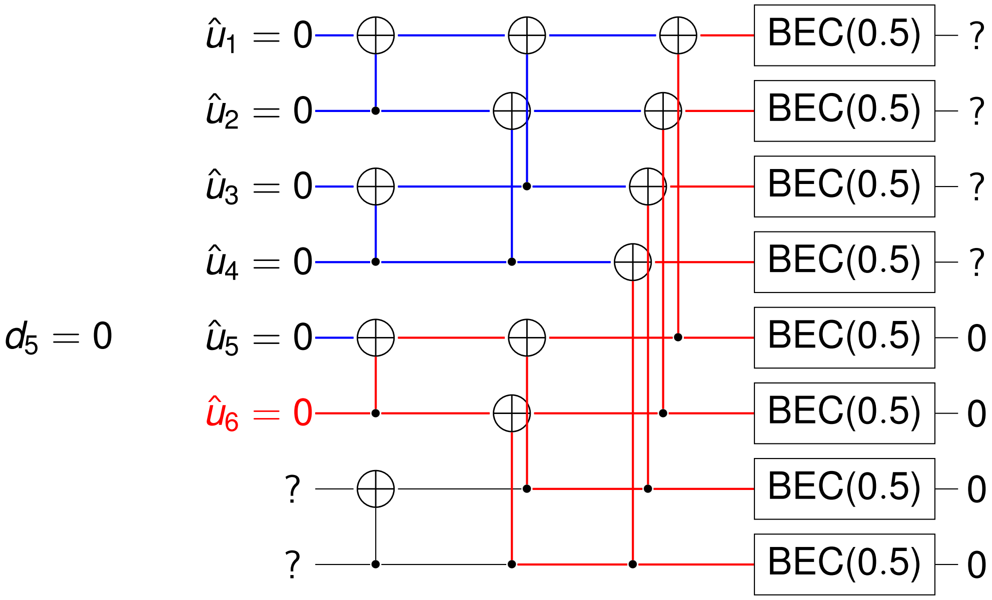

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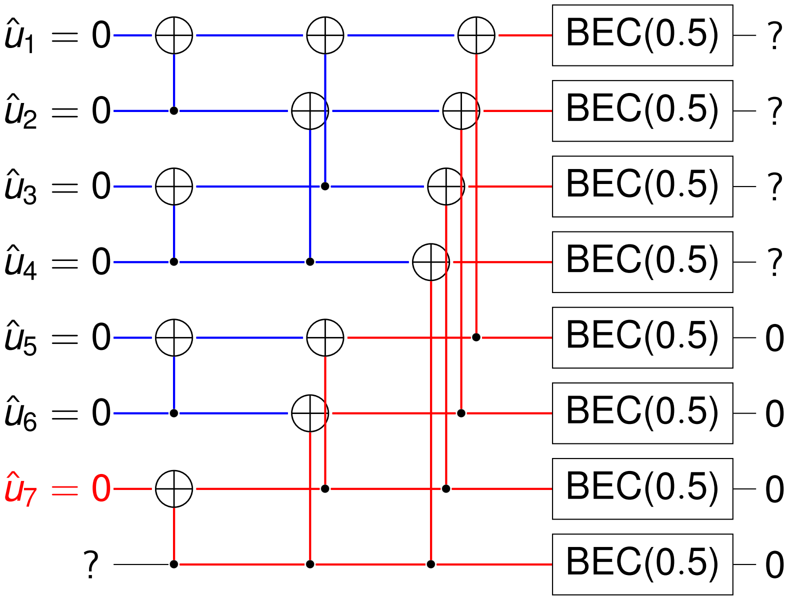
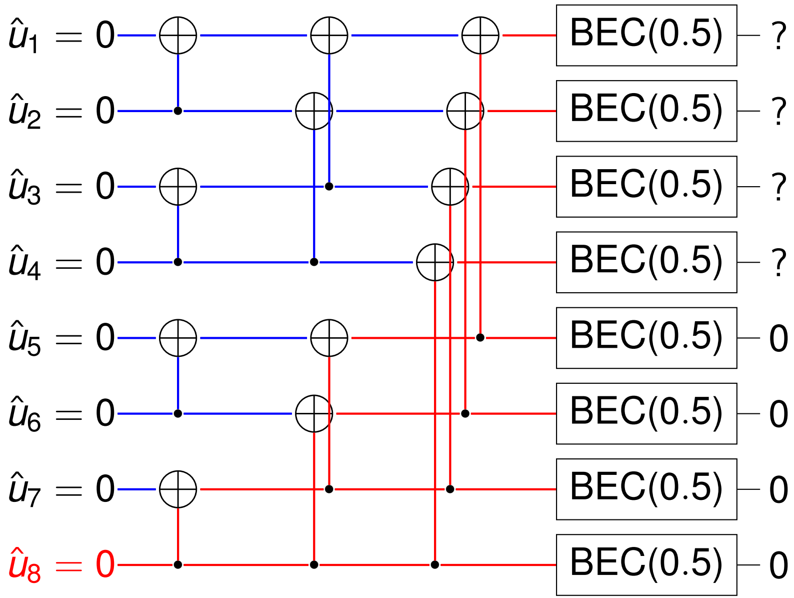
$d_6 = 0$



Example: SC Inactivation Decoding

$d_7 = 0$



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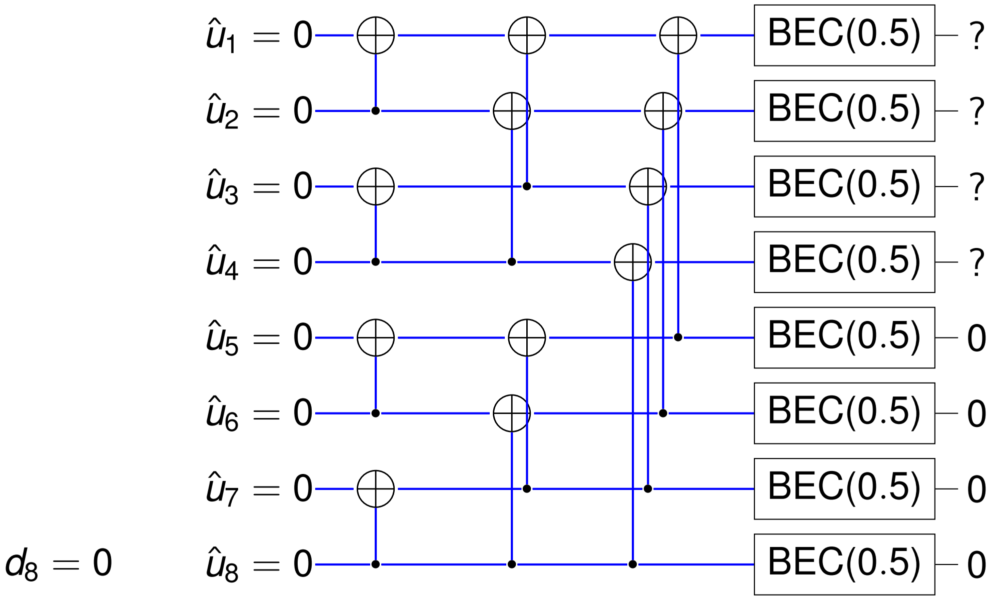
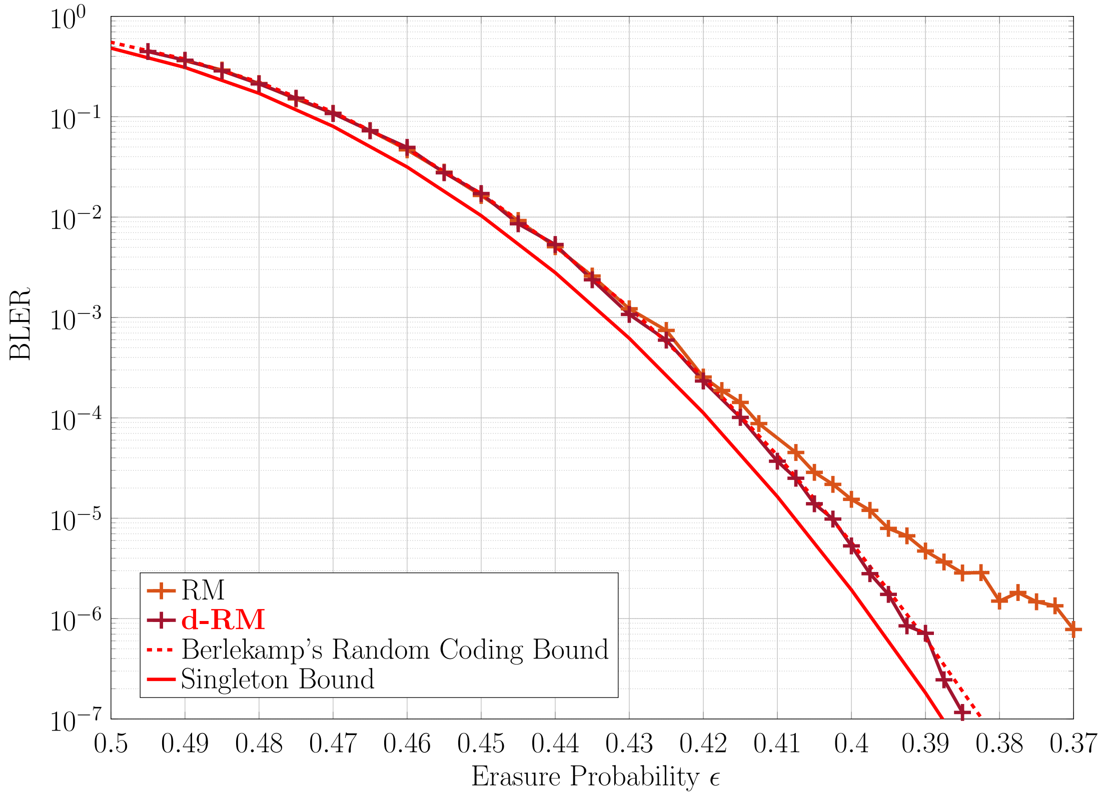


(512, 256) Codes over the BEC



The Subspace Dimension

- For a fixed y_1^N, the subspace dimension is

$$d_m(y_1^N) = H\left(U_{\mathcal{A}^{(m)}} \middle| Y_1^N = y_1^N, U_{\mathcal{F}^{(m)}}\right)$$

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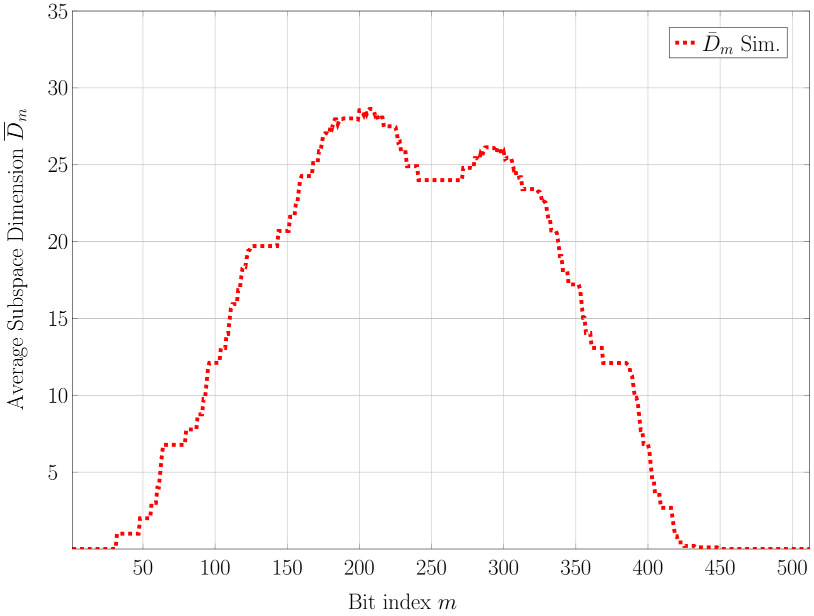

Evolution of the Subspace Dimension

- If U_m is an information bit, then
  - If decoder outputs an erasure, then $d_m(y_1^N) = d_{m-1}(y_1^N) + 1$
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Averaged over all y_1^N, the erasure probabilities are obtained via density evolution.
Must approximate consolidation probabilities.

The Markov Chain Approximation

- The random sequence $D_1, \ldots, D_N$ can be approximated by an inhomogeneous Markov chain with transition probabilities $P_{i,j}^{(m)} \approx \mathbb{P}\left(D_m = j \,|\, D_{m-1} = i\right)$ where

$$P_{i,j}^{(m)} = \begin{cases} \epsilon_N^{(m)} & \text{if } m \in \mathcal{A},\ j = i+1 \\ 1 - \epsilon_N^{(m)} & \text{if } m \in \mathcal{A},\ j = i \\ \epsilon_N^{(m)} + \left(1 - \epsilon_N^{(m)}\right) 2^{-D_{m-1}} & \text{if } m \in \mathcal{F},\ j = i \\ \left(1 - \epsilon_N^{(m)}\right)\left(1 - 2^{-D_{m-1}}\right) & \text{if } m \in \mathcal{F},\ j = i-1 \end{cases}$$

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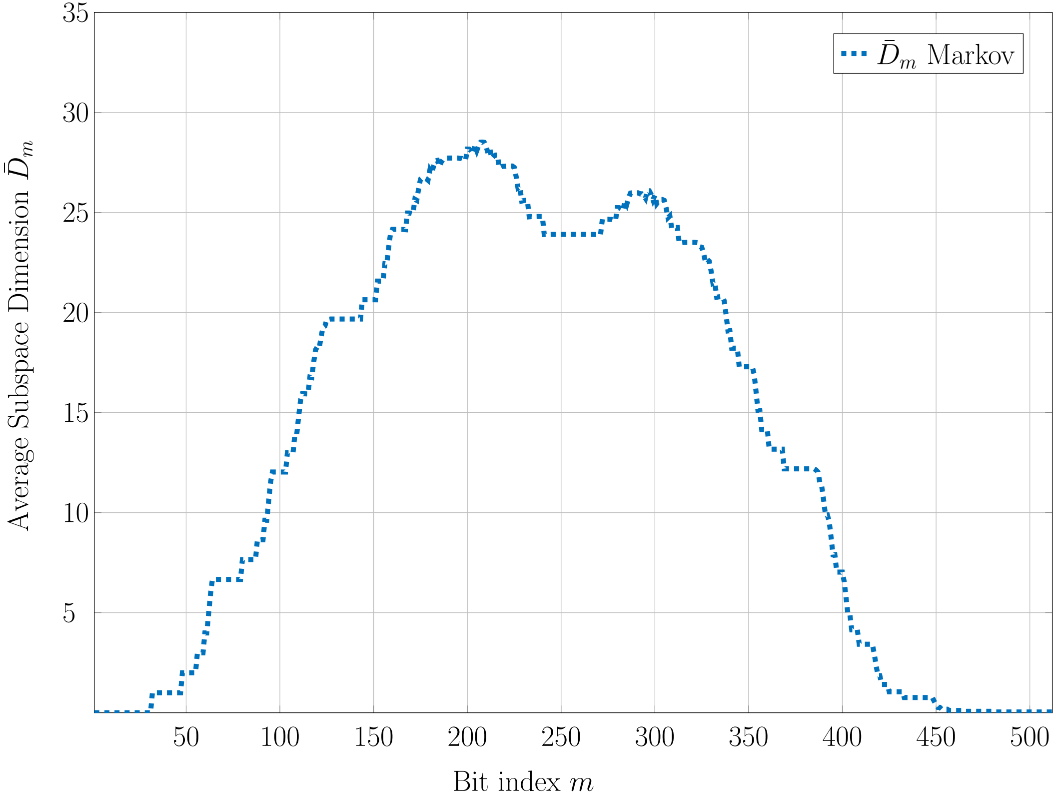
(512, 256) d-RM Code

A fixed-weight BEC with exactly $\texttt{round}(512 \times 0.48) = 246$ erasures



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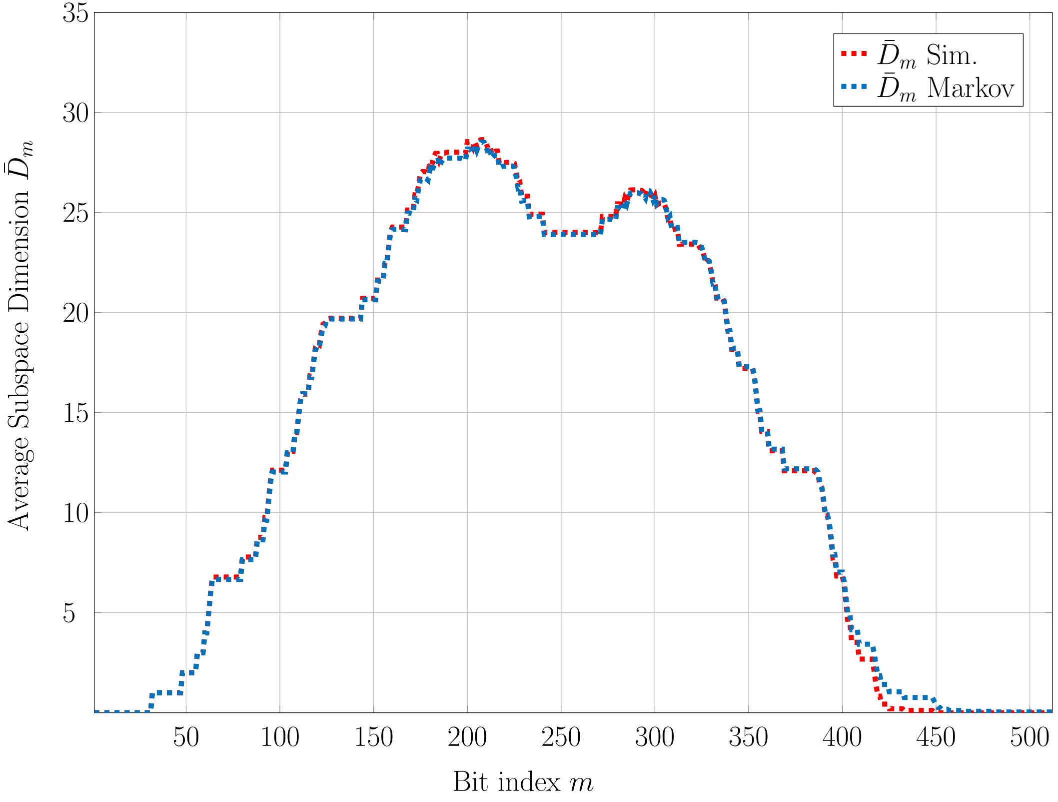

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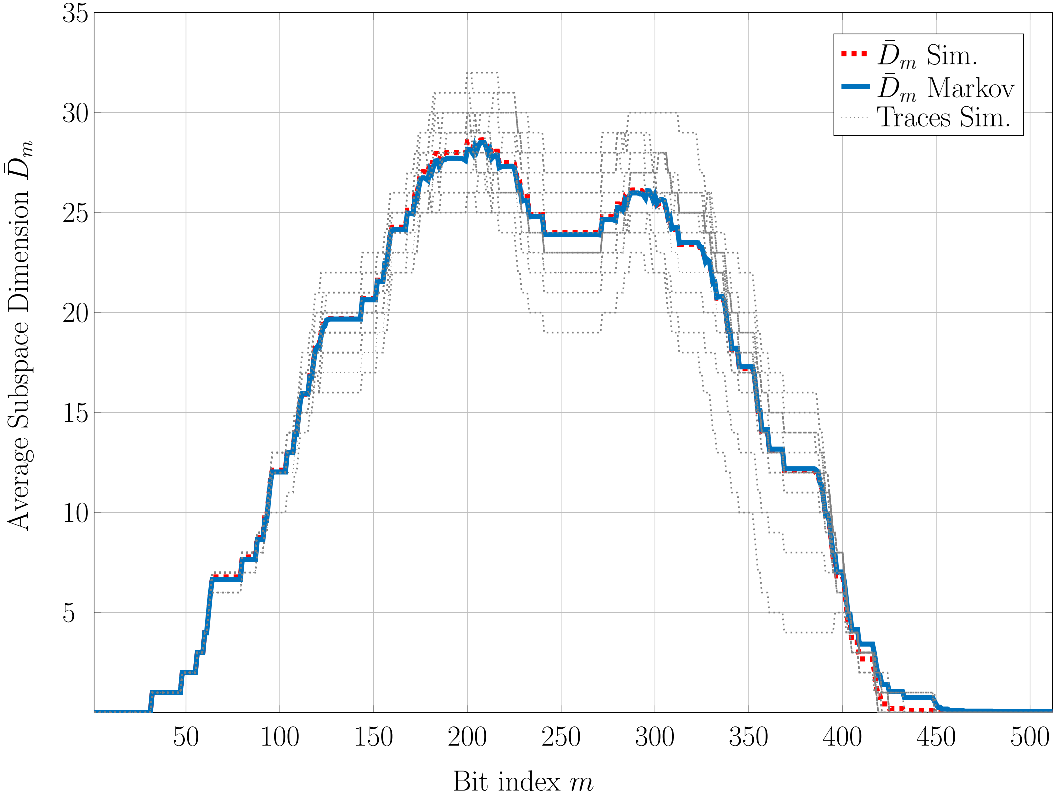

Concentration of the Subspace Dimension

Theorem

The subspace dimension D_m for a particular random realization Y_1^N concentrates around the mean $\bar{D}_m$ for sufficiently large block lengths [CP21], i.e., for any $\beta > 0$, we have

$$\mathbb{P}\left\{\frac{1}{N}|D_m - \bar{D}_m| > \beta\right\} \leq 2\exp\left(-\frac{\beta^2}{2}N\right). \qquad (1)$$

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Proof.

Key observation: at any decoding stage, the subspace dimension satisfies Lipschitz-1 condition: For all $i \in [N]$ and all values y_1^N and $\tilde{y}_i$, we have

$$|d_m(y_1^N) - d_m(y_1^{i-1}, \tilde{y}_i, y_{i+1}^N)| \leq 1.$$

Then, use Azuma-Hoeffding inequality by forming a Doob's Martingale. □

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- We use the theorem above to give bounds on the average complexity of ML decoding of a given code implemented via SCI decoding.
- Extension to general BMS channels is possible (the case of continuous output channels should be tackled with more care).

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- Outlook and Future Work
  - Apply this technique to design longer codes with good SCL performance

Thanks

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