Abstract—In this work, we derive conditions under which compositional abstractions of networks of stochastic hybrid systems can be constructed using the interconnection topology and joint dissipativity-type properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a stochastic hybrid system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process. Moreover, we derive conditions for the construction of abstractions for a class of stochastic hybrid systems involving nonlinearities satisfying an incremental quadratic inequality. In this work, unlike existing results, the stochastic noises and jumps in the concrete subsystem and its abstraction need not to be the same. We provide examples with numerical simulations to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning for interconnected stochastic hybrid systems.

Index Terms—Compositional Abstraction, Jump Diffusions, Dissipativity, Interconnected Systems

I. INTRODUCTION

Abstraction based controller synthesis is becoming a promising approach to design controllers for enforcing complex specifications over large interconnected control systems in a reliable and cost effective way. In this approach, one synthesizes a controller to enforce the specifications over the abstraction instead of the original (concrete) system, and refines the controller (using a so-called interface map) to that of the concrete system. Since the error between the output of the concrete system and that of its abstraction is quantified, one can ensure that the concrete system also satisfies the specifications (within a priori known error bounds).

Constructing abstractions for a complex system when viewed monolithically is a challenging task in itself. One approach to deal with this is to leverage the fact that many large-scale complex systems can be regarded as interconnected systems consisting of smaller subsystems. This motivates a compositional approach for the construction of the abstractions wherein abstractions of the concrete systems can be constructed by using the abstractions of smaller subsystems. Recently, there have been several results on the compositional construction of (in)finite abstractions of deterministic control systems including [1], [2], [3], and of a class of stochastic hybrid systems [4]. These results employ a small-gain type condition for the compositional construction of abstractions. However, as shown in [5], this type of condition is a function of the size of the network and can be violated as the number of subsystems grows. Recently in [6], a compositional framework for the construction of infinite abstractions of networks of control systems has been proposed using dissipativity theory. In this result a notion of storage function is proposed which describes joint dissipativity properties of control systems and their abstractions. This notion is used to derive compositional conditions under which a network of abstractions approximate a network of the concrete subsystems. Those conditions can be independent of the number or gains of the subsystems under some properties for the interconnection topologies.

In this work, we extend this approach to a class of stochastic hybrid systems, namely, jump-diffusions. Stochastic hybrid systems are a general class of systems consisting of continuous and discrete dynamics subject to probabilistic noise and events. In jump-diffusions, the continuous dynamics are modelled by stochastic differential equations and switches are modelled as Poisson processes. We introduce a notion of so-called stochastic storage functions describing joint dissipativity properties of stochastic hybrid subsystems and their abstractions. Given a network of stochastic hybrid subsystems and the stochastic storage functions between subsystems and their abstractions, we derive conditions based on the interconnection topology, guaranteeing that a network of abstractions quantitatively approximate the network of concrete subsystems. For a class of stochastic hybrid subsystems and using the incremental quadratic inequality for the nonlinearity, we derive a set of matrix (in)equalities facilitating the construction of their abstractions together with the corresponding stochastic storage functions. We illustrate the effectiveness of the proposed results in two examples in which compositionality conditions are satisfied independent of the number or gains of the subsystems.

A. Related work

Compositional abstraction for (deterministic) interconnected control systems using dissipativity was introduced in [6]. In a preliminary version of this paper, which appeared in [7], this
technique was extended to a class of stochastic hybrid systems. In both works, the joint dissipativity properties are defined with respect to a static map whose input is the (internal) inputs and outputs of the subsystems and their abstractions. In contrast to this, in this paper we employ a dynamic map based on a similar notion introduced in [8]. This allows for a broader class of (stochastic) hybrid subsystems for which one can find (stochastic) storage functions between them and their abstractions (cf. the second case study). Furthermore, in this work we derive constructive conditions for computing abstractions for a class of stochastic hybrid systems by considering nonlinearities which are more general than the ones considered in [6] and [7].

Compositional abstractions for jump-diffusions are also introduced in [4]. However, in [4] it is assumed that the stochastic noises in a subsystem and its abstraction are the same. This assumption is not realistic in practice, as it requires access to the realization of the noises in the original subsystem in order to refine the constructed controllers for the abstractions to the original subsystems. On the other hand, in this paper concrete subsystems and their abstractions do not share the same stochastic noises. In addition, the results in [4] use small-gain type conditions for the main compositionality result whereas the proposed approach here uses dissipativity-type conditions which can potentially provide scale-free results under some properties over the interconnection topologies. Although the results in [4] derive conditions for constructing abstractions for just linear jump-diffusions, here we provide constructive conditions for a class of nonlinear jump-diffusions.

II. STOCHASTIC HYBRID SYSTEMS

A. Notation

The sets of non-negative integer and real numbers are denoted by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Those symbols are noted with subscripts to restrict them in the usual way, e.g. \( \mathbb{R}_{>0} \) denotes the positive real numbers. The symbol \( \mathbb{R}^{n \times m} \) denotes the vector space of real matrices with \( n \) rows and \( m \) columns. The symbols \( I_n, I_m, 0_{n \times m} \) denote the vector with all its elements to be one, the zero vector, identity and zero matrices in \( \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{n \times m} \), respectively. For \( a, b \in \mathbb{R} \) with \( a \leq b \), the closed, open, and half-open intervals in \( \mathbb{R} \) are denoted by \([a, b], ]a, b], [a, b], \text{ and } ]a, b] \), respectively. For \( a, b \in \mathbb{N} \) and \( a \leq b \), we use \([a; b], ]a; b], [a; b], \text{ and } ]a; b] \) to denote the corresponding intervals in \( \mathbb{N} \). Given \( N \in \mathbb{N}_{>1} \), vectors \( x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}_{>1} \) and \( i \in [1; N] \), we use \( x = [x_1; \ldots; x_N] \) to denote the concatenated vector in \( \mathbb{R}^n \) with \( n = \sum_{i=1}^N n_i \). Similarly, we use \( X = [X_1; \ldots; X_N] \) to denote the matrix in \( \mathbb{R}^{n \times m} \) with \( n = \sum_{i=1}^N n_i \). Given \( N \in \mathbb{N}_{>1} \), matrices \( X_i \in \mathbb{R}^{n_i \times m_i}, n_i \in \mathbb{N}_{>1} \), and \( i \in [1; N] \). Given a vector \( x \in \mathbb{R}^n \), we denote by \( \|x\| \) the Euclidean norm of \( x \). Given a matrix \( M = [m_{ij}] \in \mathbb{R}^{n \times m} \), we denote by \( \|M\| \) the induced 2 norm of \( M \), and the trace of \( M \) by \( \text{Tr}(M) \), where \( \text{Tr}(P) = \sum_{i=1}^n p_{ii} \) for any \( P = \{p_{ij}\} \in \mathbb{R}^{n \times n} \). Given matrices \( M_1, \ldots, M_n \), the notation \( \text{diag}(M_1, \ldots, M_n) \) represents a block diagonal matrix with diagonal matrix entries \( M_1, \ldots, M_n \). Given a symmetric matrix \( A \), \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote the minimum and maximum eigenvalues of \( A \), respectively. Given a function \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \), the (essential) supremum of \( f \) is denoted by \( \|f\|_\infty := (\text{ess}\sup\{|f(t)|, t \geq 0\}) \). Measurability throughout this paper refers to Borel measurability. A continuous function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \gamma(0) = 0 \); \( \gamma \) is said to belong to class \( \mathcal{K}_{\infty} \) if \( \gamma \in \mathcal{K} \) and \( \gamma(r) \to \infty \) as \( r \to \infty \). A continuous function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is said to belong to class \( \mathcal{KL} \) if, for each fixed \( t \), the map \( \beta(t, \cdot) \) belongs to class \( \mathcal{K} \) with respect to \( r \), and for each fixed nonzero \( r \), the map \( \beta(t, r) \) is decreasing with respect to \( t \) and \( \beta(t, r) \to 0 \) as \( t \to \infty \). Given a matrix \( B \), we use the usual symbols \( \mathsf{im} B \) and \( \ker B \) to denote the image and kernel of \( B \), respectively.

B. Stochastic Hybrid Systems

Let \((\Omega, \mathcal{F}, \mathbb{P})\) denote a probability space endowed with a filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions of completeness and right continuity. The expected value of a measurable function \( g(X) \), where \( X \) is a random variable defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), is defined by the Lebesgue integral \( \mathbb{E}[g(X)] := \int_\Omega g(X(\omega))\,d\mathbb{P}(\omega) \), where \( \omega \in \Omega \). Let \((W_t)_{t \geq 0} \) be a d-dimensional F-Brownian motion and \((P_s)_{s \geq 0} \) be an r-dimensional F-Poisson process. We assume that the Poisson process and Brownian motion are independent of each other. The Poisson process \( P_s = [P^1_s, \ldots, P^r_s] \) models r kinds of events whose occurrences are assumed to be independent of each other.

Definition II.1. The class of stochastic hybrid systems studied in this paper is a tuple

\[
\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, U, W, f, \sigma, \rho, \mathbb{Q}^1, \mathbb{Q}^2, h_1, h_2),
\]

where

- \( \mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathbb{Q}^1 \), and \( \mathbb{Q}^2 \) are the state, external input, internal input, external output, and internal output spaces, respectively;
- \( U \) and \( W \) are subsets of sets of all \( \mathbb{F} \)-progressively measurable processes taking values in \( \mathbb{R}^m \) and \( \mathbb{R}^p \), respectively;
- \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n \) is the drift term which is globally Lipschitz continuous; there exist Lipschitz constants \( L_x, L_u, L_w \in \mathbb{R} \) such that \( \|f(x, u, w) - f(x', u', w')\| \leq L_x \|x - x'\| + L_u \|u - u'\| + L_w \|w - w'\| \) for all \( x, x' \in \mathbb{R}^n \), all \( u, u' \in \mathbb{R}^m \), and all \( w, w' \in \mathbb{R}^p \);
- \( \sigma : \mathbb{R}^n \to \mathbb{R}^{n \times b} \) is the diffusion term which is globally Lipschitz continuous with the Lipschitz constant \( L_\sigma \);
- \( \rho : \mathbb{R}^n \to \mathbb{R}^{n \times q} \) is the reset term which is globally Lipschitz continuous with the Lipschitz constant \( L_\rho \);
- \( h_1 : \mathbb{R}^n \to \mathbb{Q}^1 \) is the external output map;
- \( h_2 : \mathbb{R}^n \to \mathbb{Q}^2 \) is the internal output map.

A stochastic hybrid system \( \Sigma \) satisfies

\[
\frac{d\xi(t)}{dt} = f(\xi(t), (v(t), \omega(t)))dt + \sigma(\xi(t))dW_t + \rho(\xi(t))dP_t,
\]

\[
\xi(0) = \xi_0,
\]

\[
\xi(t) = h_1(\xi(t)),
\]

\[
\xi(t) = h_2(\xi(t)),
\]

almost surely (P-a.s.) for any \( v \in U \) and any \( \omega \in W \), where stochastic process \( \xi : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) is called a solution process of \( \Sigma \), stochastic process \( \xi_1 : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{Q}^1 \)
is called an external output trajectory of Σ, and stochastic process ζ2 : Ω × R_≥0 → R^{q_2} is called an internal output trajectory of Σ. We also write ξ_{a,v,w}(t) to denote the value of the solution process at time t ∈ R_≥0 under input trajectories v and w from initial condition ξ_{a,v,w}(0) = a P-a.s., where a is a random variable that is F_0-measurable. We denote by ζ_{1,v,w} and ζ_{2,v,w} the external and internal output trajectories corresponding to solution process ξ_{a,v,w}. Here, we assume that the Poisson processes P_i^a, for any i ∈ [1;r], have the rates λ_i.

We emphasize that the postulated assumptions on f, σ, and ρ ensure existence, uniqueness, and strong Markov property of the solution process [9], [10].

Remark II.2. If the stochastic hybrid system Σ does not have internal inputs and outputs, the system defined in Definition II.1 reduces to Σ = (R^n, R^m, U, f, σ, ρ, R^n, h), where f : R^n × R^m → R^n. Correspondingly, equation (1) describing the evolution of solution processes reduces to:

\[ Σ : \left\{ \begin{array}{l} \dot{ξ}(t) = f(ξ(t), v(t))dt + σ(ξ(t))dW_t + ρ(ξ(t))dP_t, \\ ζ(ξ(t)) = h(ξ(t)). \end{array} \right. \]

(2)

We use the notion of stochastic hybrid system as in (2) later to refer to interconnected systems.

Remark II.3. In our description of stochastic hybrid subsystems in Definition II.1, we distinguish between external and internal inputs and outputs (as illustrated in Figure 1). We use internal inputs and outputs to define the interconnection between subsystems, whereas the external ones are those which are available after the interconnection and can be used to control the interconnected system (defined later in Definition IV.1).

In the next section, we introduce two notions which we use to formally relate a stochastic hybrid system and its abstraction. The first notion, namely stochastic storage functions, relates a stochastic hybrid system introduced in Definition II.1 and its abstraction. The second notion, namely stochastic simulation functions, relates a stochastic hybrid system without internal inputs and outputs (see Remark II.2) and its abstraction.

III. STOCHASTIC STORAGE FUNCTION

In this section, we introduce a notion of so-called stochastic storage functions, adapted from the notion of storage functions from dissipativity theory [11]. Before introducing the notion of stochastic storage functions, we introduce a linear control system which is given by:

\[ \xi_0(t) = A_0 \xi_0(t) + B_0 w(t) \]

(3)

\[ \zeta_0(t) = C_0 \xi_0(t) + D_0 w(t), \]

where \( A_0 \in \mathbb{R}^{l_0 \times l_0}, B_0 \in \mathbb{R}^{l_0 \times m_0}, C_0 \in \mathbb{R}^{q_0 \times l_0}, \) and \( D_0 \in \mathbb{R}^{q_0 \times m_0}, \) where \( B_0, \) and \( D_0 \) have the conformal partitions

\[ B_0 = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \quad D_0 = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \]

(4)

respectively. These conformal partitions will be used later in the paper. We use the tuple \( \Sigma_0 = (A_0, B_0, C_0, D_0) \) to represent such a linear control system. Now we define the infinitesimal generator of a stochastic process which will be used later to define a notion of stochastic storage functions.

Definition III.1. Consider two stochastic hybrid systems \( \Sigma = (R^n, R^m, U, W, f, σ, ρ, R^n, h_1, h_2) \) and \( \hat{Σ} = (R^n, R^m, \hat{U}, \hat{W}, f, σ, ρ, R^n, R^{q_2}, \hat{h}_1, \hat{h}_2) \) with solution processes ξ and \( \hat{ξ} \), respectively. Consider a linear control system \( Σ_0 = (A_0, B_0, C_0, D_0) \) satisfying (3) with state trajectory ξ_0. Consider a twice continuously differentiable function \( V : \mathbb{R}^n × \mathbb{R}^n × \mathbb{R}^n \rightarrow \mathbb{R}_≥0 \). The infinitesimal generator of the stochastic process \( Σ = [\xi; \hat{ξ}; ξ_0] \), denoted by \( L \), acting on function \( V \) is defined as [9]:

\[ LV(x, \hat{x}, \theta) := \begin{bmatrix} \partial_x V & \partial_{\hat{x}} V & \partial_{\theta} V \end{bmatrix} \begin{bmatrix} f(x, u, w) \\ f(\hat{x}, \hat{u}, \hat{w}) \\ A_0 \theta + B_0 w_0 \end{bmatrix} + \frac{1}{2} \text{Tr} \left( \sigma(x)σ^T(x)\partial_x V \right) + \frac{1}{2} \text{Tr} \left( \hat{σ}(x)\hat{σ}^T(\hat{x})\partial_{\hat{x}} V \right) \]

\[ + \sum_{j=1}^{r} \lambda_j(V(x + \rho(x)e_j, \hat{x}) - V(x, \hat{x})) \]

\[ + \sum_{j=1}^{r} \lambda_j(V(\hat{x}, \hat{x} + \rho(\hat{x})e_j) - V(\hat{x}, \hat{x})), \]

where \( e_j \) denotes an r-dimensional vector with 1 on the j-th entry and 0 elsewhere.

Now we have all the ingredients to introduce a notion of stochastic storage functions.

Definition III.2. Consider two stochastic hybrid systems \( \Sigma = (R^n, R^m, U, W, f, σ, ρ, R^n, R^{q_2}, h_1, h_2) \) and \( \hat{Σ} = (R^n, R^m, \hat{U}, \hat{W}, f, σ, ρ, R^n, R^{q_2}, \hat{h}_1, \hat{h}_2) \) with the same external output space dimension and let \( \Sigma_0 = (A_0, B_0, C_0, D_0) \) be a linear control system as in (3). A twice continuously differentiable function \( V : \mathbb{R}^n × \mathbb{R}^n × \mathbb{R}^n \rightarrow \mathbb{R}_≥0 \) is called a stochastic storage function from \( \Sigma \) to \( \Sigma \), with respect to \( Σ_0 \), in the k-th moment (SStf-M_k), where \( k ≥ 1 \), if it has polynomial growth rate and there exist convex functions \( α, η ∈ K_{∞} \), concave function \( ψ_{ext} ∈ K_{∞} \cup \{0\} \), some constant \( c ∈ R_≥0 \), some matrices \( W, W, \) and \( H, \) and some symmetric matrix \( X \) of appropriate dimension such that

\[ D_2^T XD_2 ≤ 0, \]

(5)

where \( D_2 \) is given in (4), and \( ∀ x ∈ R^n, ∀ \hat{x} ∈ R^n, \) and \( ∀ θ ∈ R^{l_0} \) one has

\[ α(||h_1(x) - \hat{h}_1(\hat{x})||_k^k) ≤ V(x, \hat{x}, \theta), \]

(6)

and \( ∀ \hat{u} ∈ R^m, \) \( ∃ u ∈ R^m, \) such that \( ∀ \hat{w} ∈ R^p ∀ w ∈ R^p, \) one obtains

\[ LV(x, \hat{x}, \theta) ≤ -η(V(x, \hat{x}, \theta)) + \psi_{ext}(||\hat{w}||_k) \]

\[ + z^T Xz + c, \]

(7)
where $z = C_θθ + D_θu_θ$ and

$$u_θ = \begin{bmatrix} W_w - \hat{W}_w & \hat{h}_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$  

We use notation $\hat{Σ} \preceq Σ$ if there exists an SSF-$M_k$ $V$ from $\hat{Σ}$ to $Σ$. The stochastic hybrid system $Σ$ (possibly with $\hat{n} < n$) is called an abstraction of $Σ$.

**Remark III.3.** If $C_θ$ is the zero matrix, and $D_θ$ is the identity matrix, then the quadratic term in (7) reduces to the one in [6], [7], with

$$z = \begin{bmatrix} W_w - \hat{W}_w & \hat{h}_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$  

**Remark III.4.** Condition (5) has also appeared in various forms in the literatures as a necessary condition for deriving asymptotic stability from dissipativity properties of a system. See for example [8].

Now, we recall a slightly adapted version of the notion of stochastic simulation function introduced in [4]. This notion is appropriate for relating interconnected systems without internal inputs and outputs.

**Definition III.5.** Let $Σ = (R^n, R^m, U, f, σ, ρ, R^q, h)$ and $\hat{Σ} = (R^{\hat{n}}, R^{\hat{m}}, \hat{U}, \hat{f}, \hat{σ}, \hat{ρ}, R^{\hat{q}}, \hat{h})$ be two stochastic hybrid systems. A twice continuously differentiable function $V : R^n \times R^m \times R^l \rightarrow R_{≥0}$ is called a stochastic simulation function from $Σ$ to $\hat{Σ}$ in the $k$-th moment (SSF-$M_k$), where $k ≥ 1$, if there exist convex functions $α, η ∈ C_∞$, concave function $ψ_{ext} ∈ C_∞ \cup \{0\}$, and some constant $c ∈ R_{≥0}$, such that $∀x ∈ R^n$, $∀\hat{x} ∈ R^{\hat{n}}$, and $∀θ ∈ R^l$, one has

$$α(∥h(x) - \hat{h}(\hat{x})∥) ≤ V(x, \hat{x}, θ),$$  

and $∀u ∈ R^m$ $∃u ∈ R^m$ such that

$$LV(x, \hat{x}, θ) ≤ -η(V(x, \hat{x}, θ)) + ψ_{ext}(∥\hat{u}∥^k) + c.$$  

We say that a stochastic hybrid system $Σ$ is approximately simulated by a stochastic hybrid system $Σ$, denoted by $Σ ≤ AS Σ$, if there exists an SSF-$M_k$ function $V$ from $Σ$ to $Σ$. We call $Σ$ (possibly with lower dimension $\hat{n} < n$) an abstraction of $Σ$.

The next theorem shows the importance of the existence of an SSF-$M_k$ by quantifying the error between the output behaviors of $Σ$ and the ones of its abstractions $\hat{Σ}$.

**Theorem III.6.** Let $Σ = (R^n, R^m, U, f, σ, ρ, R^q, h)$ and $\hat{Σ} = (R^{\hat{n}}, R^{\hat{m}}, \hat{U}, \hat{f}, \hat{σ}, \hat{ρ}, R^{\hat{q}}, \hat{h})$ be two stochastic hybrid systems. Suppose $V$ is an SSF-$M_k$ from $Σ$ to $\hat{Σ}$. Then, there exists a $KL$ function $β$, a $K_∞$ function $γ_{ext}$, and some constant $c' ∈ R_{≥0}$ such that for any $v ∈ U$, any random variable $a$ and $a$ that are $F_0$-measurable, and any $θ_0 ∈ R^l$, there exists $v ∈ U$ such that the following inequality holds for any $t ∈ R_{≥0}$:

$$E[∥z_{ext}(t) - \hat{ζ}_{ext}(t)∥^k] ≤ β(E[V(a, a, θ_0), t]) + γ_{ext}(E[∥\hat{v}∥^k_{ext}]) + c'.$$  

**Proof.** The proof is similar to the one of Theorem 3.5 in [4] and is omitted here due to lack of space.  

In the next section we first provide a definition of interconnected stochastic hybrid systems. We then provide conditions under which we can construct abstractions of interconnected stochastic hybrid systems in a compositional way.

IV. INTERCONNECTED STOCHASTIC HYBRID SYSTEMS

Next definition provides a notion of interconnection for stochastic hybrid subsystems investigated in this paper.

**Definition IV.1.** Consider $N ∈ N_{≥1}$ stochastic hybrid systems $Σ_1, \ldots, Σ_N$, follow by $n = \sum_{i=1}^N n_i$, $m = \sum_{i=1}^N m_i$, $q = \sum_{i=1}^N q_i$, and the functions

$$f(x, u) := [f_1(x_1, u_1); \ldots; f_N(x_N, u_N)],$$  

$$σ(x) := [σ_1(x_1); \ldots; σ_N(x_N)],$$  

$$ρ(x) := [ρ_1(x_1); \ldots; ρ_N(x_N)],$$  

$$h(x) := [h_1(x_1); \ldots; h_N(x_N)],$$  

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$ and with internal variables constrained by

$$[w_1; \ldots; w_N] = M[h_1(x_1); \ldots; h_N(x_N)].$$  

Assume we are given $N$ stochastic hybrid subsystems $Σ_i = (R^{n_i}, R^{m_i}, R^p, U_i, f_i, σ_i, ρ_i, R^{q_i}, R^{r_i}, h_i, h_{ext})$ together with their corresponding abstractions $\hat{Σ}_i = (R^{\hat{n}_i}, R^{\hat{m}_i}, \hat{U}_i, \hat{f}_i, \hat{σ}_i, \hat{ρ}_i, R^{\hat{q}_i}, R^{\hat{r}_i}, \hat{h}_i, \hat{h}_{ext})$ and with SSF-$M_k$ $V_i$ from $Σ_i$ to $\hat{Σ}_i$. We use $σ_i, ρ_i, ψ_{ext}$, $A_{θ_i}, B_{θ_i}, C_{θ_i}, D_{θ_i}, H_i, W_i, W_i$, and $X_i$ to denote the corresponding functions, matrices, and their corresponding conformal block partitions appearing in Definition III.2.

The next theorem provides a compositional approach on the construction of abstractions of networks of stochastic hybrid systems.

**Theorem IV.2.** Consider an interconnected system $Σ = (Σ_1, \ldots, Σ_N)$ induced by $N ∈ N_{≥1}$ stochastic hybrid subsystems $Σ_i$ and the interconnection matrix $M$. Suppose each
subsystem $\Sigma_i$ admits an abstraction $\bar{\Sigma}_i$, with the corresponding SStF-$M_k$, $V_i$ with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i})$, $i \in [1; N]$. Suppose there exists $\mu_i > 0$, $i \in [1; N]$, symmetric matrix $Q \succeq 0$, and matrix $M$ of appropriate dimension such that the matrix (in)equalities (11) and (12) are satisfied, where $\hat{q} = \sum_{i=1}^{N} q_{2i}$, and

$$W = \text{diag}(W_1, \ldots, W_N), \quad \hat{W} = \text{diag}(\hat{W}_1, \ldots, \hat{W}_N),$$

$$H = \text{diag}(H_1, \ldots, H_N),$$

$$A_D = \text{diag}(A_{\theta_1}, \ldots, A_{\theta_N}), \quad B_D = \text{diag}(B_{\theta_1}, \ldots, B_{\theta_N}),$$

$$C_D = \text{diag}(C_{\theta_1}, \ldots, C_{\theta_N}), \quad D_D = \text{diag}(D_{\theta_1}, \ldots, D_{\theta_N}),$$

and $S$ is the following permutation matrix:

$$S = \begin{bmatrix}
I_{r_W} & 0_{r_W} & \cdots & 0_{r_W} & 0_{r_H} & 0_{r_H} & \cdots & 0_{r_H} \\
0_{r_W} & I_{r_W} & \cdots & 0_{r_W} & 0_{r_H} & 0_{r_H} & \cdots & 0_{r_H} \\
0_{r_W} & I_{r_W} & \cdots & 0_{r_W} & 0_{r_H} & 0_{r_H} & \cdots & 0_{r_H} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0_{r_W} & I_{r_W} & \cdots & 0_{r_W} & 0_{r_H} & 0_{r_H} & \cdots & 0_{r_H} \\
0_{r_W} & I_{r_W} & \cdots & 0_{r_W} & 0_{r_H} & 0_{r_H} & \cdots & 0_{r_H} \\
\end{bmatrix}_{1 \times \{N\}},$$

where, for each $i \in [1; N]$, $r_W(i)$ and $r_H(i)$ denote the number of rows in $W_i$ and $H_i$, respectively. Then

$$V(x, \hat{x}, \hat{\theta}) := \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T \hat{Q} \theta,$$

where $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{N \times l_0}$, $l_0 = \sum_{i=1}^{N} l_i$, is an SSF-$M_k$ from the interconnected system $\Sigma := (\Sigma_1, \ldots, \Sigma_N)$, with the coupling matrix $M$ to $\Sigma$.

**Proof.** The proof is inspired by that of Theorem 4.2 in [6]. First we show that the inequality (8) holds for some convex $K_\infty$ function $\alpha$. As also argued in the proof of Theorem 4.2 in [4], for any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, any $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^n$, and any $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{N \times l_0}$, one gets:

$$\|h(x) - \hat{h}(\hat{x})\| \leq N^{\max(\hat{x}, 1)} \sum_{i=1}^{N} \|h_{\theta_i}(x_i) - \hat{h}_{\theta_i}(\hat{x}_i)\| \leq N^{\max(\hat{x}, 1)} \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i, \theta_i)) \leq \hat{\alpha}(V(x, \hat{x}, \theta),$$

for any $k \geq 1$, where $\alpha$ is a $K_\infty$ function defined as

$$\hat{\alpha}(s) := \max_{\mu \in \mathbb{R}^m} \left\{ \mu^T s \right\} \quad \text{s.t.} \quad \mu^T \hat{s} = s,$$

where $\hat{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$ and $\mu = [\mu_1; \ldots; \mu_N]$. The function $\hat{\alpha}$ is a concave function as argued in [4]. By defining the convex function $1$ $\alpha(s) = \alpha^{-1}(s)$, $s \in \mathbb{R}_0^+$, one obtains

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^k) \leq V(x, \hat{x}, \theta),$$

satisfying inequality (8). Now we prove the inequality (9). Consider any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^n$, and $\hat{u} = [\hat{u}_1; \ldots; \hat{u}_N] \in \mathbb{R}^m$. For any $i \in [1; N]$, there exists $u_i \in \mathbb{R}^{m_1}$, consequently, a vector $u = [u_1; \ldots; u_N] \in \mathbb{R}^m$, satisfying (7) for each pair of subsystems $\Sigma_i$ and $\Sigma_i$ with the internal inputs given by $[u_1; \ldots; u_N] = [M_1(h_{2i}(x_1)); \ldots; h_{2i}(x_N)]$ and $[\hat{u}_1; \ldots; \hat{u}_N] = [M_1(h_{2i}(\hat{x}_1)); \ldots; h_{2i}(\hat{x}_N)]$, respectively. The dynamics of $\Sigma_{\theta_i}, i \in [1; N]$, can be lumped together into a single auxiliary system as the following:

$$\dot{\theta}(t) = A_D \theta(t) + B_D S \left( W_{NWN} - \hat{W}_{NWN} \right) h_{2i}(x_1) - H_i h_{2i}(\hat{x}_i) \vdots \vdots \vdots$$

$$\quad = A_D \theta(t) + B_D S \left[ \begin{bmatrix} W_{NWN} - \hat{W}_{NWN} \\ h_{2i}(x_1) - H_i h_{2i}(\hat{x}_i) \\ \vdots \\ h_{2i}(x_N) - H_i h_{2i}(\hat{x}_N) \end{bmatrix} \right],$$

$$z(t) = C_D \theta(t) + D_D S \left[ \begin{bmatrix} W_{NWN} - \hat{W}_{NWN} \\ h_{2i}(x_1) - H_i h_{2i}(\hat{x}_i) \\ \vdots \\ h_{2i}(x_N) - H_i h_{2i}(\hat{x}_N) \end{bmatrix} \right],$$

where $z = [z_1; \ldots; z_N]$. We now consider the infinitesimal generator of the function $V$, and employ the previous auxiliary system and conditions (11) and (12) to derive the chain of inequalities given in (13), where $\alpha'(s) = \sum_{i=1}^{N} \mu_i c_i$.

$$\Theta(x, \theta) := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \\ h_{2i}(x_1) - H_i h_{2i}(\hat{x}_i) \\ \vdots \\ h_{2i}(x_N) - H_i h_{2i}(\hat{x}_N) \end{bmatrix},$$

and the functions $\eta \in K_\infty$ and $\psi_{\text{ext}} \in K_\infty \cup \{0\}$ are defined as

$$\eta(s) := \min_{s \geq 0} \left\{ \sum_{i=1}^{N} \mu_i \eta_i(s_i) \right\} \quad \text{s.t.} \quad \mu^T \hat{s} = s,$$

$$\psi_{\text{ext}}(s) := \max_{s \geq 0} \left\{ \sum_{i=1}^{N} \mu_i \psi_{\text{ext}}(s_i) \right\} \quad \text{s.t.} \quad \|\hat{s}\| \leq s.$$

It remains to show that $\eta$ is a convex function and $\psi_{\text{ext}}$ is a concave one. Let us recall that by assumption functions $\eta_i, \forall i \in [1; N]$, are convex functions. Thus the function $\eta$ above defines a perturbation function which is a convex one; see [12] for further details. Again, by assumption $\psi_{\text{ext}}, \forall i \in [1; N]$, are concave functions. By similar reasoning, we conclude that $\psi_{\text{ext}}$ is a concave function. Hence, we conclude $V$ is an SSF-$M_k$ function from $\Sigma$ to $\Sigma$. \qed
\[
\begin{bmatrix}
A_D^T \dot{Q} + \dot{Q} A_D & \dot{Q} B_D S \\
W M & T
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
T
+ \begin{bmatrix}
C_D & D_D S
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
T
\begin{bmatrix}
\mu_1 X_1 & & \\
& \ddots & \\
& & \mu_N X_N
\end{bmatrix}
\begin{bmatrix}
C_D & D_D S
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
\leq 0,
\]
(11)

\[WMH = \dot{W}M , \]  
(12)

\[
\mathcal{L} V(x, \dot{x}, \theta) = \sum_{i=1}^{N} \mu_i \mathcal{L} V_i(x_i, \dot{x}_i, \theta_i) + \theta^T Q \theta + \dot{\theta}^T Q \dot{\theta} \leq \sum_{i=1}^{N} \mu_i \left(-\eta_i(V_i(x_i, \dot{x}_i, \theta_i)) + \psi_{\text{ext}}(\|u_i\|^k) + z_i^T X_i z_i + c_i\right) + \theta^T Q \dot{\theta} + \dot{\theta}^T Q \theta
\]

\[
= \sum_{i=1}^{N} \mu_i \eta_i(V_i(x_i, \dot{x}_i, \theta_i)) + \sum_{i=1}^{N} \mu_i \psi_{\text{ext}}(\|u_i\|^k) + \Theta(x, \theta)^T \begin{bmatrix}
A_D^T \dot{Q} + \dot{Q} A_D & \dot{Q} B_D S \\
W M & T
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
T
+ \Theta(x, \theta)^T \begin{bmatrix}
C_D & D_D S
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
T
\begin{bmatrix}
\mu_1 X_1 & & \\
& \ddots & \\
& & \mu_N X_N
\end{bmatrix}
\begin{bmatrix}
C_D & D_D S
\end{bmatrix}
\begin{bmatrix}
W M \\
I_q
\end{bmatrix}
\Theta(x, \theta) + \epsilon',
\]
(13)

**Remark IV.3.** If \(C_{\theta_i} = 0\) is the zero matrix and \(D_{\theta_i}\) is the identity matrix (i.e. \(\Sigma_{\theta_i}\) is a static map), \(\forall i \in [1; N]\), then matrix inequality (11) reduces to matrix inequality (15) in [7, Theorem 7] (which is a stochastic counterpart of matrix inequality (IV.1) in [6, Theorem 4.2]).

**Remark IV.4.** The matrix inequality (11) is linear with respect to the decision variables \(\dot{Q}\) and \(\mu = [\mu_1, \ldots, \mu_N]\), and matrix equality (12) is linear with respect to the decision variable \(\dot{M}\), which can be solved by using readily available software packages such as [13].

In the next section, we consider a specific class of stochastic hybrid systems \(\Sigma\), and a specific candidate SSrF-M2 function \(V\). We derive conditions facilitating the construction of \(\Sigma\) as an abstraction of \(\Sigma\) and such that \(\dot{V}\) is an SSrF-M2 from \(\Sigma\) to \(\Sigma\).

**V. A CLASS OF STOCHASTIC HYBRID SYSTEMS**

We consider a specific class of stochastic hybrid systems with the drift, diffusion, reset, and output functions given by

\[
d\xi(t) = (A\xi(t) + Bu(t) + E\varphi(t, F\xi) + D\omega(t))dt + GdW_t + \sum_{i=1}^{r} R_i dP^i_t, \\
\zeta_1(t) = C_1\xi(t), \\
\zeta_2(t) = C_2\xi(t),
\]
(14)

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{n \times p}, E \in \mathbb{R}^{n \times k}, F \in \mathbb{R}^{k \times n}, G \in \mathbb{R}^{n \times 1}, R_i \in \mathbb{R}^{n}, \forall i \in [1; r], C_1 \in \mathbb{R}^{q \times n}, \) and \(C_2 \in \mathbb{R}^{q \times n}\). The vector \(R_i\) and scalar \(\lambda_i > 0\) (rate of the Poisson process), \(\forall i \in [1; r]\), parametrize the jumps associated with events \(i\). The time-varying non-linearity is the one considered in [14], which satisfies an incremental quadratic constraint holds for all \(t \in \mathbb{R}_{\geq 0}\), and \(k_1, k_2 \in \mathbb{R}^k\):

\[
\begin{bmatrix}
k_2 - k_1 \\
\varphi(t, k_2) - \varphi(t, k_1)
\end{bmatrix}
^T
\begin{bmatrix}
k_2 - k_1 \\
\varphi(t, k_2) - \varphi(t, k_1)
\end{bmatrix}
\geq 0.
\]

To facilitate subsequent analysis, we write matrix \(\dot{M}\) in the following conformal partitioned form

\[
\dot{M} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{12}^T & M_{22}
\end{bmatrix},
\]

We use the tuple

\[
\Sigma = (A, B, C_1, C_2, D, E, F, G, R, \varphi, \lambda),
\]
where \(R = \{R_1, \ldots, R_r\}\) and \(\lambda = \{\lambda_1, \ldots, \lambda_r\}\), to refer to the class of system of the form (14).
A. Stochastic storage function

Here, we consider a candidate SSfM-M2 of the form

$$V(x, \dot{x}, \theta) = (x - P\dot{x})^T \tilde{M}(x - P\dot{x}) + \theta^T A\theta,$$  \hspace{1cm} (15)

where \( P, \tilde{M} = \tilde{M}^T \succ 0 \), and \( A = A^T \succ 0 \) are matrices of appropriate dimensions. In order to show that \( V(x, \dot{x}, \theta) \) in (15) is an SSfM-M2 from an abstraction \( \Sigma \) to the concrete system \( \Sigma \), with respect to \( \Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta) \), where \( B_\theta = [B_1 \ B_2] \) and \( D_\theta = [D_1 \ D_2] \), we require the following assumptions on the concrete system \( \Sigma \) and on \( \Sigma_\theta \).

**Assumption V.1.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F, G, R, \varphi, \lambda) \). There exist matrices \( \tilde{M} \succ 0, K, X, L_1, Z, W, \Lambda, A_\theta, C_\theta, B_\theta := [B_1 \ B_2], D_\theta := [D_1 D_2] \), and positive constants \( \kappa \) and \( \tilde{\kappa} \) such that

$$D_1^T X D_2 \preceq 0,$$

and the (in)equalities given in (16) and (17) hold, where

$$\Delta = (A + BK)^T \tilde{M} + \tilde{M} (A + BK).$$

An equivalent geometric characterization of (16) is given by the following lemma.

**Lemma V.2.** Given \( D \) and \( Z \), the condition (16) is satisfied for some matrix \( W \) if and only if

$$\text{im } D \subseteq \text{im } Z.$$  \hspace{1cm} (18)

**Remark V.3.** Remark that when the non-linearity in (14) reduces to the one described in [6, Section V] and \( \Sigma_\theta \) is a static map, matrix inequality (17) reduces to (V.5) in [6, Theorem 5.5]. Note also that in the absence of the non-linearity in (14), matrix inequality (17) is feasible if the pair \( (A, B) \) is stabilizable and \( A_\theta \) is Hurwitz.

Now, we provide one of the main results of this section showing under which conditions \( V \) in (15) is an SSfM-M2.

**Theorem V.4.** Let \( \Sigma = (A, B, C_1, C_2, D, E, F, G, R, \varphi, \lambda) \) and \( \tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}_1, \tilde{C}_2, \tilde{D}, \tilde{E}, \tilde{F}, \tilde{G}, \tilde{R}, \tilde{\varphi}, \tilde{\lambda}) \) with the same external output dimension. Suppose Assumption V.1 holds and there exist matrices \( P, Q, H, W \) and \( L_2 \) of appropriate dimensions such that:

$$AP = P\dot{A} - BQ,$$  \hspace{1cm} (19a)

$$C_1 P = \dot{C}_1,$$  \hspace{1cm} (19b)

$$C_2 P = H\dot{C}_2,$$  \hspace{1cm} (19c)

$$FP = \dot{F},$$  \hspace{1cm} (19d)

$$E = PE + B(L_2 - L_1),$$  \hspace{1cm} (19e)

$$PD = ZW.$$  \hspace{1cm} (19f)

Then, function \( V \) defined in (15) is an SSfM-M2 from \( \tilde{\Sigma} \) to \( \Sigma \), with respect to \( \Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta) \).

**Proof.** We note that from (19b), \( \forall x \in \mathbb{R}^n \) and \( \forall \dot{x} \in \mathbb{R}^n \), we have \( \|C_1 x - C_1 \dot{x}\|^2 = (x - P\dot{x})^T C_1^T C_1 (x - P\dot{x}) \). It can be readily verified that \( \frac{\lambda_{\text{min}}(M)}{\lambda_{\text{max}}(C_1^T C_1)} \|C_1 x - C_1 \dot{x}\|^2 \leq V(x, \dot{x}, \theta) \) for all \( \theta \in \mathbb{R}^l \), implying that inequality (6) holds with \( \alpha(r) = \frac{\lambda_{\text{min}}(M)}{\lambda_{\text{max}}(C_1^T C_1)} r \) for any \( r \in \mathbb{R}_{\geq 0} \), which is a convex function.

We proceed to prove inequality (7). By the definition of \( V \), one has

$$\partial_x V = (x - P\dot{x})^T \tilde{M}, \partial_{xx} V = -2(x - P\dot{x})^T \tilde{M} P,$$

and

$$\partial_{x\dot{x}} V = 2\tilde{M}, \partial_{x\dot{x}x} V = 2P^T \tilde{M} P.$$

Following the definition of \( \mathcal{L} \), for any \( x \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n, \theta \in \mathbb{R}^l \), one obtains:

$$\mathcal{L}V(x, \dot{x}, \theta) = 2(x - P\dot{x})^T \tilde{M} (Ax + E\varphi(Fx) + Bu + Du - 2(x - P\dot{x})^T \tilde{M} P(A\dot{x} + E\varphi(F\dot{x}) + B\ddot{u} + D\ddot{w}) + G^T \tilde{M} G$$

$$+ G^T P^T \tilde{M} P \tilde{G} + 2(x - P\dot{x})^T \tilde{M} \sum_{i=1}^r \lambda_i R_i + \sum_{i=1}^r \lambda_i R_i^T \tilde{M} R_i$$

$$- 2(x - P\dot{x})^T \tilde{M} \sum_{i=1}^r \lambda_i R_i$$

$$+ 2\theta^T A \left( A_\theta \theta + [B_1 \ B_2] \begin{bmatrix} W w - W \ddot{w} \ C_2 x - H \tilde{C}_2 \ddot{x} \end{bmatrix} \right).$$

Given any \( x \in \mathbb{R}^n, \dot{x} \in \mathbb{R}^n, \) and \( \theta \in \mathbb{R}^n \), we use the following interface function to choose \( u \in \mathbb{R}^n \):

$$u = K(x - P\dot{x}) + \tilde{R}u + L_1 \varphi(t, Fx) - L_2 \varphi(t, \dot{F}x),$$

where \( L_2, Q, \) and \( \tilde{R} \) are matrices of appropriate dimension. Using the interface function in (20), and the conditions (16), (19a), (19d), and (19f), one obtains:

$$\mathcal{L}V(x, \dot{x}, \theta) = 2(x - P\dot{x})^T \tilde{M} \left( A(x - P\dot{x}) + BK(x - P\dot{x}) \right.$$  

$$+ Z W w - Z W \ddot{w} + (B\tilde{R} - P\tilde{B})\ddot{u} + (B L_1 + E)\dot{\varphi} \right)$$

$$+ G^T \tilde{M} G + G^T P^T \tilde{M} \tilde{P} \tilde{G} + \sum_{i=1}^r \lambda_i R_i^T \tilde{M} R_i + \sum_{i=1}^r \lambda_i R_i^T \tilde{M} R_i$$

$$+ 2(x - P\dot{x})^T \tilde{M} \left( \sum_{i=1}^r \lambda_i R_i - \sum_{i=1}^r \lambda_i P R_i \right) + 2\theta^T A \Lambda \theta$$

$$+ 2\theta^T \Lambda B \left( W w - W \ddot{w} \right) + 2 \theta^T \Lambda B_2 \left( C_2 x - H \tilde{C}_2 \ddot{x} \right),$$

where \( \dot{\varphi} = \varphi(t, Fx) - \varphi(t, \dot{F}x) \). Using Young’s inequality, Cauchy-Schwarz inequality, (17), and (19c), one obtains the upper bound for \( \mathcal{L}V(x, \dot{x}, \theta) \) as given in (21), where \( \pi, \pi' \in \mathbb{R}_{>0} \) satisfy \( \pi + \pi' < \kappa, \kappa = \min\{\kappa - \pi, \pi', \tilde{\kappa}\} \), and

$$c = G^T \tilde{M} G + G^T P^T \tilde{M} \tilde{P} \tilde{G} + \sum_{i=1}^r \lambda_i R_i^T \tilde{M} R_i + \sum_{i=1}^r \lambda_i R_i^T \tilde{M} R_i$$

$$+ 2c' \left( \sum_{i=1}^r \lambda_i R_i - \sum_{i=1}^r \lambda_i P R_i \right) \leq 0.$$

Using the upper bound (21), the inequality (7) is satisfied, implying that \( V \) is an SSfM-M2 from \( \Sigma \) to \( \Sigma \), with respect to \( \Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta) \), with the convex function \( \eta(s) = \kappa s \), concave function \( \psi_{\text{ext}}(s) = \sqrt{M(B\tilde{R} - P\tilde{B})^2} \), \( \forall s \in \mathbb{R}_{>0} \), matrix \( X \), and \( c = \kappa + c' \).

\( \square \)
Remark V.5. Note that matrix $\hat{R}$ is a free design parameter in the interface function. As explained in [6] and [15], one can choose $\hat{R}$ to minimize the function $\psi_{\text{ext}}$ for $V$ and, hence, lower the upper bound on the error between the output behaviors of $\Sigma$ and $\hat{\Sigma}$. The choice of $\hat{R}$ minimizing $\psi_{\text{ext}}$ is given by

$$
\hat{R} = (B^T \hat{M} B)^{-1} B^T \hat{M} \hat{P} \hat{B}.
$$

Remark V.6. The constant $\varepsilon$, can be also minimized, thereby lowering the upper bound on the error between the output behaviors of $\Sigma$ and $\hat{\Sigma}$. One can choose $\hat{G}$ to be the zero matrix and choose $\hat{\lambda}$ and $\hat{R}$ to solve the following optimization problem:

$$
\arg \min_{\hat{R}, \hat{\lambda} > 0} \sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i^T \hat{P}^T \hat{M} \hat{P} \hat{R}_i
$$

$$
-\sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i^T \hat{M} \hat{P} (\sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i)
$$

$$
\left(\frac{2}{\pi^r} \sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i^T \hat{M} \hat{P} (\sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i) \right) + \left(\frac{2}{\pi^r} \sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i^T \hat{M} \hat{P} (\sum_{i=1}^{r} \hat{\lambda}_i \hat{R}_i) \right),
$$

where $\hat{\lambda} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_r\}$ and $\hat{R} = \{\hat{R}_1, \ldots, \hat{R}_r\}$. This optimization problem is, in general, a non-convex one.
and choose $x = P\hat{x}$ and $\theta = 0$ in (7). One has:

$$0 \leq (C_2 P\hat{x} - H\hat{C}_2 \hat{x})^T D^T_2 X D_2 (C_2 P\hat{x} - H\hat{C}_2 \hat{x}),$$

for all $\hat{x} \in \mathbb{R}^n$. Since $D^T_2 X D_2 \preceq 0$, and $D^T_2 X D_2 \neq 0$ by assumption, one obtains $C_2 P - H\hat{C}_2 = 0$, which implies (19c).

Consider the input signals $\hat{v} \equiv 0, \omega \equiv 0, \omega \equiv 0$. It can be easily seen that the subspace $\{x, \hat{x}, \theta : x = P\hat{x}, \theta = 0\} \subseteq \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\delta$ is invariant [16], which implies that when $\xi(0) = P\hat{\xi}(0)$ and $\theta(0) = 0$, one has:

$$\xi(t) = P\hat{\xi}(t), \quad \theta(t) = 0, \quad d\xi(t) = Pd\hat{\xi}(t),$$

for all $t \in \mathbb{R}_{\geq 0}$, from which we derive that

$$(AP\hat{\xi}(t) + BQ\hat{\xi}(t) + BL_1 \varphi(t, F\hat{\xi}(t)) - BL_2 \varphi(t, \hat{F}\xi(t)) + E\varphi(t, FP\hat{\xi}(t)))dt = (PA\hat{\xi}(t) + PE\varphi(t, \hat{F}\xi(t)))dt,$$

for all $t \in \mathbb{R}_{\geq 0}$, thus implying (19a), (19d), and (19e).

**B. Geometric interpretation of different conditions**

In this section, we provide geometric conditions on matrices appearing on the definition of $\hat{\Sigma}$, of stochastic storage function and its corresponding interface function. The geometric conditions facilitate the construction of the abstraction. First, we recall the following result from [15], providing necessary and sufficient conditions for the existence of $\hat{A}$ and $Q$ satisfying (19a).

**Lemma V.9.** Consider matrices $A$, $B$, and $P$. There exist matrices $\hat{A}$ and $Q$ satisfying (19a) if and only if

$$\text{im } AP \subseteq \text{im } P + \text{im } B.$$  \hfill (24)

Similarly, we provide necessary and sufficient conditions for the existence of $\hat{C}_2$ and $\hat{E}$, $L_2$ satisfying (19c) and (19e), respectively.

**Lemma V.10.** Given $P$ and $C_2$, there exists matrix $\hat{C}_2$ satisfying (19c) if and only if

$$\text{im } C_2 P \subseteq \text{im } H$$  \hfill (25)

for some matrix $H$ of appropriate dimension.

**Lemma V.11.** Given $P$, $B$, and $L_1$, there exist matrices $\hat{E}$ and $L_2$ satisfying (19e) if and only if

$$\text{im } E \subseteq \text{im } B + \text{im } P.$$  \hfill (26)

Lemmas V.9, V.10, and V.11 provide sufficient and necessary conditions on $P$ and $H$, resulting in the construction of matrices $\hat{A}$, $\hat{C}_2$, and $\hat{E}$ and matrices $Q$ and $L_2$ appearing in the interface function (20). The next lemma provides a sufficient and necessary condition on the existence of $\hat{D}$ satisfying (19f).

**Lemma V.12.** Given $Z$, there exists matrix $\hat{D}$ satisfying (19f) if and only if

$$\text{im } Z\hat{W} \subseteq \text{im } P,$$  \hfill (27)

for some matrix $\hat{W}$ of appropriate dimension.

Although condition (27) is readily satisfied by choosing $\hat{W} = 0$, one should preferably aim at finding a nonzero $\hat{W}$ with the highest possible rank to facilitate later the satisfaction of compositionality condition (12).

**Table I:** Construction of $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{R}, \varphi, \hat{\lambda})$ together with the corresponding stochastic storage function $\hat{V}$ in (15), with $\hat{\Sigma}_0 = (A_0, B_0, C_0, D_0)$, and interface function in (20) for a given $\Sigma = (A, B, C_1, C_2, D, E, F, G, R, \varphi, \lambda)$.

\begin{itemize}
  \item 1) Choose matrix $Z$ such that (18) is satisfied;
  \item 2) Choose $W$ such that $D = ZW$;
  \item 3) Choose matrices $\hat{M}, \hat{k}, \hat{L}_1, \hat{X}, A_0, C_0, B_0 = [B_1 B_2], D_0 = [D_1 D_2]$, and constants $\hat{k}, \hat{\lambda}$ such that (17) is satisfied (see Remark V.13);
  \item 4) Determine matrix $P$ of lowest rank with $r_0 = P \in [24, 25, 26, \text{and } (27) \text{(see Remark V.14)}$;
  \item 5) Choose $\hat{A}$ and $Q$ according to (19a);
  \item 6) Choose $L_2$ and $E$ according to (19c);
  \item 7) Compute $\hat{E} = FP$;
  \item 8) Compute $\hat{C}_1 = C_1 P$;
  \item 9) Choose $\hat{G} = 0$. Choose $\hat{R} = \{\hat{R}_1, \ldots, \hat{R}_t\}$ and $\lambda = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_t\}$ according to (23);
  \item 10) Choose $\hat{C}_2$ satisfying $H\hat{C}_2 = C_2 P$ for some $H$;
  \item 11) Choose $\hat{D}$ satisfying $PD = \hat{Z}W$ for some $W$ with the highest possible rank;
  \item 12) Choose $\hat{B}$ freely (e.g. $\hat{B} = I_2$ making $\hat{\Sigma}$ fully actuated);
  \item 13) Compute $\hat{R}$, appearing in (20), according to (22);
\end{itemize}

**C. Construction of abstraction**

We summarize the construction of abstraction $\hat{\Sigma}$, stochastic storage function $\hat{V}$ in (15), and its corresponding interface function in (20) in Table I.

**Remark V.13.** One way to solve the matrix inequality (17) is as follows: First, we select arbitrary $C_0$ and $D_0 = [D_1 D_2]$, and solve the following bilinear matrix inequality for $\hat{\kappa}$, $\hat{X}$, $\hat{M}$, and $\hat{L}_1$:

$$\begin{bmatrix}
\Delta & \hat{M} Z \\
Z^T \hat{M} & 0
\end{bmatrix} \geq \begin{bmatrix}
\hat{M}(B_1 + E) & 0 \\
0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
\hat{L}_1 + E^T \hat{M} & 0 \\
0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
\hat{M} & 0 \\
0 & 0
\end{bmatrix}$$

$$\begin{bmatrix}
C^T_0 D^T_2 X D_2 C_2 - F^T M_1 F & C^T_0 D^T_2 X D_1 - F M_{12} \\
D^T_1 X D_2 C_2 - M_{12} F & D^T_1 X D_1 - M_{22}
\end{bmatrix}$$

We then solve the following bilinear matrix equation for $\hat{\kappa}$, $\lambda$, and $A_0$:

$$A^T_0 \Lambda + \Lambda A_0 = C^T_0 X C_0 - \hat{\kappa} \Lambda.$$  \hfill (28)

Finally, we solve the following linear equations for $B_0 = [B_1 B_2]$:

$$\begin{bmatrix}
\Lambda B_1 & 0 \\
0 & \Lambda B_2
\end{bmatrix}$$

$$\begin{bmatrix}
C^T_0 X D_1 \\
C^T_0 X D_2
\end{bmatrix}$$

**Remark V.14.** One way to satisfy the geometric conditions (24)-(27) is to start with a scalar abstraction (i.e. $\tilde{n} = 1$) and pick $P$ to be an arbitrary column vector, and check if (24)-(27) hold. If not, then increase the state-space dimension of the abstraction by one (i.e. $\tilde{n} = 2$), add a linearly independent column vector to $P$, and check again if (24)-(27) hold. Repeat this process until (24)-(27) are satisfied. Note that in the worst-case scenario, this process will terminate when $\tilde{n} = n$ (i.e. the state-space dimension of the concrete subsystem and abstraction are equal).
In the next section, we provide two examples for compositional construction of abstractions of a network of stochastic hybrid systems using the technique presented in the paper. First, in a physically motivated example, we construct a compositional abstraction of a network of resistor-capacitor (R-C) circuits affected by stochastic noise. In the second example, we illustrate the advantage of using a linear control system $\Sigma_\theta$ over just a static map (which was used in [6], [7]) to conclude the joint dissipativity property of a concrete subsystem and its abstraction.

VI. EXAMPLES

A. Network of RC Circuits

Consider an interconnection of $n$ first order R-C circuits. The $i$-th R-C circuit has a dynamic given by:

$$ \dot{v}_{i,c} = \left( -\frac{1}{R_i C_i} v_{i,c} + \frac{1}{R_i C_i} v_{i,s} + \frac{1}{C_i} \dot{w}_i \right) dt + \zeta dW_i $$

where $\zeta \in \mathbb{R}_{>0}$, $\tau \in \mathbb{R}_{>0}$, $i \in [1:n]$, $v_{i,s} \in \mathbb{R}$ represents the input source voltage (external input), $v_{i,c} \in \mathbb{R}$ is the voltage across capacitor, $C_i$ is the capacitance, $R_i$ is the resistance, and $\dot{w}_i \in \mathbb{R}$ is the total current inflow from other R-C circuits in the network. The continuous noise and jump terms represent the thermal noise (also known as Johnson-Nyquist noise) and the so-called Shot noise [17], respectively. Assume the rate of the Poisson process $P_i$ is $\lambda$. For illustration purposes, in this example we fix $R_i = 1$ Ohm, and $C_i = 1$ Farad $\forall i \in [1:N]$. We consider the above interconnected system as an interconnection of $N$ concrete subsystems $\Sigma_i$, $i \in [1:N]$, wherein each subsystem $\Sigma_i$ is formed by clustering $n_i$ R-C circuits ($n_i \leq n$). We also add a non-linearity belonging to the class of nonlinearities presented in this paper. Each subsystem, $\Sigma_i = (A_i, B_i, C_i, D_i, L_i, N_i, \bar{I}_i, v_{i,s}, \bar{w}_i, \Phi, \lambda)$, generates a scalar external output:

$$ \begin{align*}
\Sigma_i : & \\
\dot{\xi}_i &= (A_i \xi_i + B_i u_i + D_i w_i + \bar{I}_i \Phi(\bar{I}_i^T \xi_i))dt \\
\xi_{i1} &= C_i \xi_i, \\
\zeta_{i1} &= \zeta_{i2} = \xi_i, \\
\end{align*} $$

where $\xi_i = L_i v_i$, $v = [v_{e_1}; \ldots; v_{e_n}]$, $L_i := [e_{i1}; \ldots; e_{in_i}]$, $e_{ij} \in \mathbb{R}^{1 \times n}$ is a row vector whose $k$-th element is defined as $e_{ij}^{(k)} = 1$ if $k$-th R-C circuit is part of the $i$-th cluster and 0 otherwise.

$A_i, B_i, D_i \in \mathbb{R}^{n \times n}$ are readily obtained from (28), $C_i \in \mathbb{R}^{1 \times n}$, $u_i = L_i v_i$, $w_i = [v_{i1}; \ldots; v_{in_i}]$, $w_i = L_i \bar{w}$, $\bar{w} = [\bar{w}_1; \ldots; \bar{w}_n]$, and $\Phi : \mathbb{R} \to \mathbb{R}$ is defined as $\phi(x) = \sin(x)$.

The interconnection topology in this example is given by:

$$ M = \begin{bmatrix}
    n-1 & -1 & \ldots & \ldots & -1 \\
    -1 & n-1 & -1 & \ldots & -1 \\
    -1 & -1 & n-1 & \ldots & -1 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    -1 & \ldots & \ldots & -1 & n-1
  \end{bmatrix} $$

The interconnection topology represents a fully-connected interconnection topology. We aggregate each $\Sigma_i$ into a scalar deterministic abstraction $\hat{\Sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_{i1}, 1, 1, 1, 0, 0, \varphi, 0)$ given by the following dynamics:

$$ \begin{align*}
\hat{\xi}_i &= (\hat{A}_i \hat{\xi}_i + \hat{B}_i \hat{u}_i + \hat{w}_i + \varphi(\hat{\xi}_i))dt \\
\hat{\zeta}_{i1} &= \hat{C}_{i1} \hat{\xi}_i, \\
\hat{\zeta}_{i2} &= \hat{\xi}_i,
\end{align*} $$

where $\hat{A}_i$ satisfies $A_i \bar{I}_i = \bar{I}_i A_i \hat{B}_i$ is chosen arbitrarily (in this example we choose $\bar{B}_i = 1$), $\hat{C}_{i1} = C_i \bar{I}_i$. The function $V_i(x_i, \dot{x}_i) = (x_i - \bar{I}_i \dot{x}_i)^T (x_i - \bar{I}_i \dot{x}_i)$ (i.e. $\bar{M}_i = I_{n_i}, \bar{P}_i = \bar{I}_i n_i, \Lambda_i = 0$) is a SSF-M2 function from $\hat{\Sigma}_i$ to $\Sigma_i$, with the following parameters:

$$ K_i = -\chi I_n, Z_i = I_{n_i}, W_i = I_{n_i}, X_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\
\bar{I}_i & 0_{n_i} \end{bmatrix}, \\
\hat{\eta}_i = 2\chi - 2\lambda \tau - \omega^2 - \lambda \tau^2, Q_i = 0_{n_i}, H_i = \bar{W}_i = \bar{I}_i, \\
L_{i1} = -\bar{I}_i, \bar{A}_{i0} = 0, B_{i0} = 0, C_{i0} = 0, D_{i0} = 0_{2n_i}, \bar{r}_0 = 0,$$

where $\chi > \lambda \tau + \omega^2 + \lambda \tau^2$, and with $\alpha_i(r) = \frac{1}{\lambda_{max}(C_i C_i^T)^{1/2}} r$, $\eta_i(r) = (2\chi - 2\lambda \tau - \omega^2 - \lambda \tau^2 r) \psi(\omega \tau r) = 0$, $\forall r \in \mathbb{R}_{>0}$, and $c_i = \tau^2 + \omega^2$. Inputs $u_i \in \mathbb{R}^{n_i}$ is given via the interface function in (20) (i.e. $\bar{R}_i = \bar{I}_i, L_{i2} = \bar{I}_i$)

$$ u_i = -\chi (x_i - \bar{I}_i \dot{x}_i) + \bar{I}_i \hat{u}_i = \bar{I}_i \varphi(\bar{I}_i^T x_i) + \bar{I}_i \varphi(\hat{x}_i).$$

By selecting $\mu_1 = \ldots = \mu_N = 1$, the function $V(x, \dot{x}, \theta) = \Sigma_{i=1}^N \mu_i V_i(x_i, \dot{x}_i, \theta_i)$ is an SSF-M2 function from $\hat{\Sigma}$ to $\Sigma$, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ with a coupling matrix $\hat{M}$, satisfying condition (12) as the following:

$$ M \text{diag}(\bar{I}_n, \ldots, \bar{I}_N) = \text{diag}(\bar{I}_n, \ldots, \bar{I}_N) \hat{M}. $$

A matrix $\hat{M}$ exists satisfying (30) if there exist $N$ equitable partitions of the graph described by the Laplacian matrix $L = -M$, which is always true here because $L$ represents a fully connected graphs, as explained in [18]. It can be easily seen that condition (11) reduces to $[ -L^T \ 0 \ 0 \ 0 \ I_n ] [ -L \ 0 \ I_n ] = -(L + L^T) \preceq 0$, which always holds since $L = L^T \succeq 0$, which is always true for Laplacian matrices of undirected graphs [18].

1) Controller synthesis: Now, we synthesize a controller for the abstract interconnected system $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ to enforce a specification, and then refine the designed controller to the one for the concrete interconnected system. We fix $n = 9, N = 3, \tau = 0.2, \omega = 0.4, \lambda = 1, \chi = 10$ and

$$ C_{i1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, C_{i2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, C_{i3} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. $$

We synthesize a controller using toolbox SCOTS [19] to enforce the following linear temporal logic specification [20] over the outputs of $\hat{\Sigma}$:

$$ \Psi = \Box S \land \left( \bigwedge_{i=1}^3 \Box (-O_i) \right) \land \Box \Diamond T_1 \land \Box \Diamond T_2, $$
which can be interpreted as follows: the output trajectory of the closed loop system evolves inside the set $\mathcal{S}$, avoids regions $O_i$, $i \in [1; 5]$, indicated with blue boxes in Figure 3, and visits $T_i$, $i \in [1; 2]$ infinitely often, indicated with red boxes in Figure 3. We use (29) to generate the corresponding input enforcing this specification over the original system $\Sigma$.

**B. Example 2**

In this part, we provide compositional abstractions of a network of subsystems wherein the joint dissipativity property of each concrete subsystem and its abstraction is only concluded with respect to a linear control system $\Sigma_{\theta_i}$ rather than a static map. Consider an interconnection of $N$ second order subsystems $\Sigma_i$, where each $\Sigma_i$ is given by

$$
\Sigma_i: \begin{cases} 
\dot{\xi}_i(t) = (A_i \xi_i(t) + B_i v_i(t) + D_i \omega_i(t)) dt, \\
\xi_0(t) = C_i \xi_i(t), \quad \eta_i(t) = \xi_i(t),
\end{cases}
$$

where

$$
A_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ -I_{n_i} & -0.5I_{n_i} \end{bmatrix}, \\
B_i = \begin{bmatrix} 0_{n_i} \\ I_{n_i} \end{bmatrix}, \\
C_i = \begin{bmatrix} 0_{n_i} \\ e_{n_i} \end{bmatrix},
$$

vector $e_{n_i}$ represents a column vector whose first element is 1 and remaining elements are zero. For the sake of simulation we choose $N = 3, n_i = 10, \forall i \in [1; N]$. We consider the following abstract system $\hat{\Sigma}_i$:

$$
\hat{\Sigma}_i: \begin{cases} 
\dot{\hat{\xi}}_i(t) = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \hat{\xi}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{v}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{\omega}_i(t) \end{cases} dt,
$$

$$
\hat{\xi}_0(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\xi}_i(t), \\
\hat{\xi}_0(t) = \hat{\xi}_i(t).
$$

We restrict $K_i$ for each $i \in [1; N]$ appearing in (20) such that the last $n_i$ columns are identically zero. This restriction can appear in practice when for example only some state variables are available to be measured. With this restriction on the structure of $K_i$, one cannot find a storage function with $C_i \theta_i = 0$ in this example. Using the guidelines shown in Table I and the solver package Yalmip [13], it can be shown that the function

$$
V_i(x_i, \hat{x}_i, \theta_i) = (x_i - P \hat{x}_i)^T M_i (x_i - P \hat{x}_i) + \theta_i^T A_i \theta_i
$$

is an SSStF-M$_2$ from $\hat{\Sigma}_i$ to $\Sigma_i$, with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i}), \forall i \in [1; N]$, with the following parameters

$$
\begin{align*}
M_i &= \begin{bmatrix} 2I_{n_i} & I_{n_i} \\ I_{n_i} & I_{n_i} \end{bmatrix}, \\
P_i &= \begin{bmatrix} I_{n_i} & 0_{n_i} \\ 0_{n_i} & I_{n_i} \end{bmatrix}, \\
K_i &= \begin{bmatrix} -0.5I_{n_i} & 0_{n_i} \end{bmatrix}, \\
\kappa_i &= 0.1, W_i = I_{n_i}, Q_i = 0, H_i = \tilde{W}_i = \tilde{I}_{n_i}, L_i = 0, \Lambda = I_{2n_i},
\end{align*}
$$

$$
\begin{align*}
A_{\theta_i} &= -5I_{2n_i}, B_{\theta_i} = \begin{bmatrix} 0_{n_i} \\ 0_{n_i} \end{bmatrix}, \\
D_{\theta_i} &= \begin{bmatrix} 0_{n_i} & I_{n_i} \\ 0_{n_i} & I_{n_i} \end{bmatrix}, \\
X_i &= \begin{bmatrix} 9.47785I_{n_i} & -7.4055I_{n_i} \\ -7.4055I_{n_i} & 1.6526I_{n_i} \end{bmatrix}, \quad \hat{\kappa}_i = 1,
\end{align*}
$$

with $\alpha_i(r) = \lambda_{\min}(M_i) r$, $\eta_i(r) = 0.1r$, $\psi_{\text{ext}} = 0, \forall r \in \mathbb{R}_{\geq 0}$, and $c_i = 0$. Functions $u_i \in \mathbb{R}^{n_i}$ are given via the interface function:

$$
u_i = -K_i(x_i - P_i \hat{x}_i) + \tilde{I}_{n_i} \hat{u}_i,$$

(i.e., $\hat{R}_i = \tilde{I}_{n_i}, L_{2i} = 0$). With the interconnection matrix $M$ given by

$$
M = \begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 1 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & 1 & -2 \end{bmatrix},
$$

and by selecting $\mu_1 = \cdots = \mu_N = 1$, it can be verified that the function $V = \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T \theta$, where $\theta = [\theta_1; \ldots; \theta_N]$, is an SSF-M$_2$ from $\hat{\Sigma}$ to $\Sigma$, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \hat{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ with the coupling matrix $\hat{M}$ given by

$$
\hat{M} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},
$$

satisfying conditions (11) and (12). In the simulation, the input signal to the abstract system is chosen arbitrarily as $\tilde{v}(t) = [\sin(t); 0.1 e^{-t}; -t]$. Figure 4 shows the evolution of the absolute value of the error between the output trajectories of the concrete interconnected system and its abstraction. One can readily verify that the error is always bounded by the computed error bound in Theorem III.6.

**VII. CONCLUSION**

In this work, using tools from stochastic calculus and dissipativity theory, we derived conditions under which abstractions of interconnected stochastic hybrid systems can be constructed compositionally. In future work, we will look at deriving constructive conditions which facilitate the construction of abstractions for classes of non-linear stochastic hybrid systems broader than the one considered in this paper, together with the corresponding stochastic storage functions and interface maps.
Fig. 4: The evolution of $\|\zeta(t) - \hat{\zeta}(t)\|^2$, where $\zeta(t) = [\zeta_{11}(t), \ldots, \zeta_{1N}(t)]$, and $\hat{\zeta}(t) = [\hat{\zeta}_{11}(t), \ldots, \hat{\zeta}_{1N}(t)]$, and the theoretical upper bound obtained for this example according to (10).

REFERENCES


