From Dissipativity Theory to Compositional Abstractions of Interconnected Stochastic Hybrid Systems

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***Abstract*—In this work, we derive conditions under which compositional abstractions of networks of stochastic hybrid systems can be constructed using the interconnection topology and joint dissipativity-type properties of subsystems and their abstractions. In the proposed framework, the abstraction, itself a stochastic hybrid system (possibly with a lower dimension), can be used as a substitute of the original system in the controller design process. Moreover, we derive conditions for the construction of abstractions for a class of stochastic hybrid systems involving nonlinearities satisfying an incremental quadratic inequality. In this work, unlike existing results, the stochastic noises and jumps in the concrete subsystem and its abstraction need not to be the same. We provide examples with numerical simulations to illustrate the effectiveness of the proposed dissipativity-type compositional reasoning for interconnected stochastic hybrid systems.**

***Index Terms*—Compositional Abstraction, Jump Diffusions, Dissipativity, Interconnected Systems**

I. INTRODUCTION

Abstraction based controller synthesis is becoming a promising approach to design controllers for enforcing complex specifications over large interconnected control systems in a reliable and cost effective way. In this approach, one synthesizes a controller to enforce the complex specifications over the abstraction instead of the original (concrete) system, and refines the controller (using a so-called *interface map*) to that of the concrete system. Since the error between the output of the concrete system and that of its abstraction is quantified, one can ensure that the concrete system also satisfies the specifications (within a priori known error bounds).

Constructing abstractions for a complex system when viewed monolithically is a challenging task in itself. One approach to deal with this is to leverage the fact that many large-scale complex systems can be regarded as interconnected systems consisting of smaller *subsystems*. This motivates a *compositional* approach for the construction of the abstractions wherein abstractions of the concrete systems can be constructed by using the abstractions of smaller subsystems.

Recently, there have been several results on the compositional construction of (in)finite abstractions of deterministic control systems including [1], [2], [3], and of a class of stochastic hybrid systems [4]. These results employ a small-gain type condition for the compositional construction of abstractions. However, as shown in [5], this type of condition is a function of the size of the network and can be violated as the number of subsystems grows. Recently in [6], a compositional framework for the construction of infinite abstractions of networks of control systems has been proposed using dissipativity theory. In this result a notion of storage function is proposed which describes joint dissipativity properties of control systems and their abstractions. This notion is used to derive compositional conditions under which a network of abstractions approximate a network of the concrete subsystems. Those conditions can be independent of the number or gains of the subsystems under some properties for the interconnection topologies.

In this work, we extend this approach to a class of stochastic hybrid systems, namely, jump-diffusions. Stochastic hybrid systems are a general class of systems consisting of continuous and discrete dynamics subject to probabilistic noise and events. In jump-diffusions, the continuous dynamics are modelled by stochastic differential equations and switches are modelled as Poisson processes. We introduce a notion of so-called stochastic storage functions describing joint dissipativity properties of stochastic hybrid subsystems and their abstractions. Given a network of stochastic hybrid subsystems and the stochastic storage functions between subsystems and their abstractions, we derive conditions based on the interconnection topology, guaranteeing that a network of abstractions quantitatively approximate the network of concrete subsystems. For a class of stochastic hybrid subsystems and using the incremental quadratic inequality for the nonlinearity, we derive a set of matrix (in)equalities facilitating the construction of their abstractions together with the corresponding stochastic storage functions. We illustrate the effectiveness of the proposed results in two examples in which compositionality conditions are satisfied independent of the number or gains of the subsystems.

A. Related work

Compositional abstraction for (deterministic) interconnected control systems using dissipativity was introduced in [6]. In a preliminary version of this paper, which appeared in [7], this

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technique was extended to a class of stochastic hybrid systems. In both works, the joint dissipativity properties are defined with respect to a static map whose input is the (internal) inputs and outputs of the subsystems and their abstractions. In contrast to this, in this paper we employ a dynamic map based on a similar notion introduced in [8]. This allows for a broader class of (stochastic hybrid) subsystems for which one can find (stochastic) storage functions between them and their abstractions (cf. the second case study). Furthermore, in this work we derive constructive conditions for computing abstractions for a class of stochastic hybrid systems by considering nonlinearities which are more general than the ones considered in [6] and [7].

Compositional abstractions for jump-diffusions are also introduced in [4]. However, in [4] it is assumed that the stochastic noises in a subsystem and its abstraction are the same. This assumption is not realistic in practice, as it requires access to the realization of the noises in the original subsystem in order to refine the constructed controllers for the abstractions to the original subsystems. On the other hand, in this paper concrete subsystems and their abstractions do not share the same stochastic noises. In addition, the results in [4] use small-gain type conditions for the main compostionality result whereas the proposed approach here uses dissipativity-type conditions which can potentially provide scale-free results under some properties over the interconnection topologies. Although the results in [4] derive conditions for constructing abstractions for just linear jump-diffusions, here we provide constructive conditions for a class of nonlinear jump-diffusions.

II. Stochastic Hybrid Systems

A. Notation

The sets of non-negative integer and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. Those symbols are footnoted with subscripts to restrict them in the usual way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. The symbol $\mathbb{R}^{n\times m}$ denotes the vector space of real matrices with n rows and m columns. The symbols $\vec{1}_n, \vec{0}_n, I_n, 0_{n\times m}$ denote the vector with all its elements to be one, the zero vector, identity and zero matrices in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n\times n}, \mathbb{R}^{n\times m}$, respectively. For $a, b \in \mathbb{R}$ with $a \leq b$, the closed, open, and half-open intervals in $\mathbb{R}$ are denoted by $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b]$, $]a; b[$, $[a; b[$, and $]a; b]$ to denote the corresponding intervals in $\mathbb{N}$. Given $N \in \mathbb{N}_{\geq 1}$, vectors $x_i \in \mathbb{R}^{n_i}, n_i \in \mathbb{N}_{\geq 1}$ and $i \in [1; N]$, we use $x = [x_1; \ldots; x_N]$ to denote the concatenated vector in $\mathbb{R}^n$ with $n = \sum_{i=1}^N n_i$. Similarly, we use $X = [X_1; \ldots; X_N]$ to denote the matrix in $\mathbb{R}^{n\times m}$ with $n = \sum_{i=1}^N n_i$, given $N \in \mathbb{N}_{\geq 1}$, matrices $X_i \in \mathbb{R}^{n_i\times m}, n_i \in \mathbb{N}_{\geq 1}$, and $i \in [1; N]$. Given a vector $x \in \mathbb{R}^n$, we denote by $\|x\|$ the Euclidean norm of x. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the induced 2 norm of M, and the trace of M by $\mathsf{Tr}(M)$, where $\mathsf{Tr}(P) = \sum_{i=1}^n p_{ii}$ for any $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$.Given matrices $M_1, \ldots, M_n$, the notation $\mathsf{diag}(M_1, \ldots, M_n)$ represents a block diagonal matrix with diagonal matrix entries $M_1, \ldots, M_n$. Given a symmetric matrix A, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of A, respectively. Given a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty := (\text{ess})\text{sup}\{\|f(t)\|,\ t \geq 0\}$. Measurability throughout this paper refers to Borel measurability. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the map $\beta(r, t)$ belongs to class $\mathcal{K}$ with respect to r, and for each fixed nonzero r, the map $\beta(r, t)$ is decreasing with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$. Given a matrix B, we use the usual symbols im B and $\ker\ B$ to denote the image and kernel of B, respectively.

B. Stochastic hybrid systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right continuity. The expected value of a measurable function $g(X)$, where X is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is defined by the Lebesgue integral $\mathbb{E}[g(X)] := \int_\Omega g(X(\omega))\mathsf{d}\mathbb{P}(\omega)$, where $\omega \in \Omega$. Let $(W_s)_{s\geq 0}$ be a b-dimensional $\mathbb{F}$-Brownian motion and $(P_s)_{s\geq 0}$ be an r-dimensional $\mathbb{F}$-Poisson process. We assume that the Poisson process and Brownian motion are independent of each other. The Poisson process $P_s = [P_s^1; \ldots; P_s^{\mathsf{r}}]$ models r kinds of events whose occurrences are assumed to be independent of each other.

Definition II.1. *The class of stochastic hybrid systems studied in this paper is a tuple*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2),$$

where

- $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$, $\mathbb{R}^{q_1}$, *and* $\mathbb{R}^{q_2}$ *are the state, external input, internal input, external output, and internal output spaces, respectively;*
- $\mathcal{U}$ *and* $\mathcal{W}$ *are subsets of sets of all* $\mathbb{F}$*-progressively measurable processes taking values in* $\mathbb{R}^m$ *and* $\mathbb{R}^p$*, respectively;*
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ *is the drift term which is globally Lipschitz continuous: there exist Lipschitz constants* $L_x, L_u, L_w \in \mathbb{R}_{\geq 0}$ *such that* $\|f(x, u, w) - f(x', u', w')\| \leq L_x\|x - x'\| + L_u\|u - u'\| + L_w\|w - w'\|$ *for all* $x, x' \in \mathbb{R}^n$*, all* $u, u' \in \mathbb{R}^m$*, and all* $w, w' \in \mathbb{R}^p$*;*
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times \mathsf{b}}$ *is the diffusion term which is globally Lipschitz continuous with the Lipschitz constant* L_σ*;*
- $\rho : \mathbb{R}^n \to \mathbb{R}^{n\times \mathsf{r}}$ *is the reset term which is globally Lipschitz continuous with the Lipschitz constant* L_ρ*;*
- $h_1 : \mathbb{R}^n \to \mathbb{R}^{q_1}$ *is the external output map;*
- $h_2 : \mathbb{R}^n \to \mathbb{R}^{q_2}$ *is the internal output map.*

A stochastic hybrid system Σ satisfies

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \upsilon(t), \omega(t))\mathsf{d}t + \sigma(\xi(t))\mathsf{d}W_t + \rho(\xi(t))\mathsf{d}P_t, \\ \zeta_1(t) = h_1(\xi(t)), \\ \zeta_2(t) = h_2(\xi(t)), \end{cases} \tag{1}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) for any $\upsilon \in \mathcal{U}$ and any $\omega \in \mathcal{W}$, where stochastic process $\xi : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is called a *solution process* of Σ, stochastic process $\zeta_1 : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{q_1}$

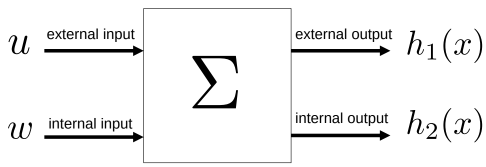

Fig. 1: Stochastic hybrid system defined in Definition II.1.

is called an external output trajectory of Σ, and stochastic process $\zeta_2 : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{q_2}$ is called an internal output trajectory of Σ. We also write $\xi_{a\upsilon\omega}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_{\geq 0}$ under input trajectories υ and ω from initial condition $\xi_{a\upsilon\omega}(0) = a$ $\mathbb{P}$-a.s., where a is a random variable that is $\mathcal{F}_0$-measurable. We denote by $\zeta_{1_{a\upsilon\omega}}$ and $\zeta_{2_{a\upsilon\omega}}$ the external and internal output trajectories corresponding to solution process $\xi_{a\upsilon\omega}$. Here, we assume that the Poisson processes P_s^i, for any $i \in [1; \mathsf{r}]$, have the rates λ_i. We emphasize that the postulated assumptions on f, σ, and ρ ensure existence, uniqueness, and strong Markov property of the solution process [9], [10].

Remark II.2. *If the stochastic hybrid system Σ does not have internal inputs and outputs, the system defined in Definition II.1 reduces to $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h)$, where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$. Correspondingly, equation* (1) *describing the evolution of solution processes reduces to:*

$$\Sigma : \begin{cases} \mathsf{d}\xi(t) = f(\xi(t), \upsilon(t))\mathsf{d}t + \sigma(\xi(t))\mathsf{d}W_t + \rho(\xi(t))\mathsf{d}P_t, \\ \zeta(t) = h(\xi(t)). \end{cases} \tag{2}$$

We use the notion of stochastic hybrid system as in (2) later to refer to interconnected systems.

Remark II.3. *In our description of stochastic hybrid subsystems in Definition II.1, we distinguish between external and internal inputs and outputs (as illustrated in Figure 1). We use internal inputs and outputs to define the interconnection between subsystems, whereas the external ones are those which are available after the interconnection and can be used to control the interconnected system (defined later in Definition IV.1).*

In the next section, we introduce two notions which we use to formally relate a stochastic hybrid system and its abstraction. The first notion, namely stochastic storage functions, relates a stochastic hybrid system introduced in Definition II.1 and its abstraction. The second notion, namely stochastic simulation functions, relates a stochastic hybrid system without internal inputs and outputs (see Remark II.2) and its abstraction.

III. Stochastic Storage Function

In this section, we introduce a notion of so-called stochastic storage functions, adapted from the notion of storage functions from dissipativity theory [11]. Before introducing the notion of stochastic storage functions, we introduce a linear control system which is given by:

$$\begin{aligned} \dot{\xi}_\theta(t) &= A_\theta \xi_\theta(t) + B_\theta \upsilon_\theta(t) \\ \zeta_\theta(t) &= C_\theta \xi_\theta(t) + D_\theta \upsilon_\theta(t), \end{aligned} \tag{3}$$

where $A_\theta \in \mathbb{R}^{l_\theta \times l_\theta}, B_\theta \in \mathbb{R}^{l_\theta \times m_\theta}, C_\theta \in \mathbb{R}^{q_\theta \times l_\theta}$, and $D_\theta \in \mathbb{R}^{q_\theta \times m_\theta}$, where B_θ, and D_θ have the conformal partitions

$$B_\theta = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ D_\theta = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \tag{4}$$

respectively. These conformal partitions will be used later in the paper. We use the tuple $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ to represent such a linear control system. Now we define the infinitesimal generator of a stochastic process which will be used later to define a notion of stochastic storage functions.

Definition III.1. *Consider two stochastic hybrid systems $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$ and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$ with solution processes ξ and $\hat{\xi}$, respectively. Consider a linear control system $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ satisfying* (3) *with state trajectory ξ_θ. Consider a twice continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \rightarrow \mathbb{R}_{\geq 0}$. The infinitesimal generator of the stochastic process $\Xi = [\xi; \hat{\xi}; \xi_\theta]$, denoted by $\mathcal{L}$, acting on function V is defined as [9]:*

$$\begin{aligned} \mathcal{L}V(x, \hat{x}, \theta) &:= \begin{bmatrix} \partial_x V & \partial_{\hat{x}} V & \partial_\theta V \end{bmatrix} \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \\ A_\theta \theta + B_\theta u_\theta \end{bmatrix} \\ &+ \frac{1}{2}\mathsf{Tr}\left(\sigma(x)\sigma^T(x)\partial_{x,x}V\right) + \frac{1}{2}\mathsf{Tr}\left(\hat{\sigma}(\hat{x})\hat{\sigma}^T(\hat{x})\partial_{\hat{x},\hat{x}}V\right) \\ &+ \sum_{j=1}^{\mathsf{r}} \lambda_j (V(x + \rho(x)\mathsf{e}_j^{\mathsf{r}}, \hat{x}) - V(x, \hat{x})) \\ &+ \sum_{j=1}^{\hat{\mathsf{r}}} \hat{\lambda}_j (V(x, \hat{x} + \hat{\rho}(\hat{x})\mathsf{e}_j^{\hat{\mathsf{r}}}) - V(x, \hat{x})), \end{aligned}$$

where $\mathsf{e}_j^{\mathsf{r}}$ denotes an r-dimensional vector with 1 on the j-th entry and 0 elsewhere.

Now we have all the ingredients to introduce a notion of stochastic storage functions.

Definition III.2. *Consider two stochastic hybrid systems $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, \rho, \mathbb{R}^{q_1}, \mathbb{R}^{q_2}, h_1, h_2)$ and $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^{q_1}, \mathbb{R}^{\hat{q}_2}, \hat{h}_1, \hat{h}_2)$ with the same external output space dimension and let $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$ be a linear control system as in* (3)*. A twice continuously differentiable function $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \rightarrow \mathbb{R}_{\geq 0}$ is called a stochastic storage function from $\hat{\Sigma}$ to Σ, with respect to Σ_θ, in the k-th moment (SStF-M$_k$), where $k \geq 1$, if it has polynomial growth rate and there exist convex functions $\alpha, \eta \in \mathcal{K}_\infty$, concave function $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$, some constant $\mathsf{c} \in \mathbb{R}_{\geq 0}$, some matrices $W, \hat{W}$, and H, and some symmetric matrix X of appropriate dimension such that*

$$D_2^T X D_2 \preceq 0, \tag{5}$$

where D_2 is given in (4)*, and $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, and $\forall \theta \in \mathbb{R}^{l_\theta}$ one has*

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^k) \leq V(x, \hat{x}, \theta), \tag{6}$$

and $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$ $\exists u \in \mathbb{R}^m$, such that $\forall \hat{w} \in \mathbb{R}^{\hat{p}}$ $\forall w \in \mathbb{R}^p$, one obtains

$$\begin{aligned} \mathcal{L}V(x, \hat{x}, \theta) \leq &-\eta(V(x, \hat{x}, \theta)) + \psi_{\mathsf{ext}}(\|\hat{u}\|^k) \\ &+ z^T X z + \mathsf{c}, \end{aligned} \tag{7}$$

where $z = C_\theta \theta + D_\theta u_\theta$ *and*

$$u_\theta = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$

We use notation $\hat{\Sigma} \preceq \Sigma$ if there exists an SStF-M$_k$ V from $\hat{\Sigma}$ to Σ. The stochastic hybrid system $\hat{\Sigma}$ (possibly with $\hat{n} < n$) is called an abstraction of Σ.

Remark III.3. *If* C_θ *is the zero matrix, and* D_θ *is the identity matrix, then the quadratic term in* (7) *reduces to the one in [6], [7], with*

$$z = \begin{bmatrix} Ww - \hat{W}\hat{w} \\ h_2(x) - H\hat{h}_2(\hat{x}) \end{bmatrix}.$$

Remark III.4. *Condition* (5) *has also appeared in various forms in the literatures as a necessary condition for deriving asymptotic stability from dissipativity properties of a system. See for example [8].*

Now, we recall a slightly adapted version of the notion of stochastic simulation function introduced in [4]. This notion is appropriate for relating interconnected systems without internal inputs and outputs.

Definition III.5. *Let* $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h)$ *and* $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h})$ *be two stochastic hybrid systems. A twice continuously differentiable function* $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta} \to \mathbb{R}_{\geq 0}$ *is called a stochastic simulation function from* $\hat{\Sigma}$ *to* Σ *in the* k-th *moment (SSF-M$_k$), where* $k \geq 1$, *if there exist convex functions* $\alpha, \eta \in \mathcal{K}_\infty$, *concave function* $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$, *and some constant* $\mathsf{c} \in \mathbb{R}_{\geq 0}$, *such that* $\forall x \in \mathbb{R}^n$, $\forall \hat{x} \in \mathbb{R}^{\hat{n}}$, *and* $\forall \theta \in \mathbb{R}^{l_\theta}$, *one has*

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|^k) \leq V(x, \hat{x}, \theta), \tag{8}$$

and $\forall \hat{u} \in \mathbb{R}^{\hat{m}}$ $\exists u \in \mathbb{R}^m$ *such that*

$$\mathcal{L}V(x, \hat{x}, \theta) \leq -\eta(V(x, \hat{x}, \theta)) + \psi_{\mathsf{ext}}(\|\hat{u}\|^k) + \mathsf{c}. \tag{9}$$

We say that a stochastic hybrid system $\hat{\Sigma}$ is approximately simulated by a stochastic hybrid system Σ, denoted by $\hat{\Sigma} \preceq_{AS} \Sigma$, if there exists an SSF-M$_k$ function V from $\hat{\Sigma}$ to Σ. We call $\hat{\Sigma}$ (possibly with lower dimension $\hat{n} < n$) an abstraction of Σ. The next theorem shows the important of the existence of an SSF-M$_k$ by quantifying the error between the output behaviors of Σ and the ones of its abstractions $\hat{\Sigma}$.

Theorem III.6. *Let* $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h)$ *and* $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \hat{\mathcal{U}}, \hat{f}, \hat{\sigma}, \hat{\rho}, \mathbb{R}^q, \hat{h})$ *be two stochastic hybrid systems. Suppose* V *is an SSF-M$_k$ from* $\hat{\Sigma}$ *to* Σ. *Then, there exists a* $\mathcal{KL}$ *function* β, *a* $\mathcal{K}_\infty$ *function* γ_{ext}, *and some constant* $\mathsf{c}' \in \mathbb{R}_{\geq 0}$ *such that for any* $\hat{\upsilon} \in \hat{\mathcal{U}}$, *any random variable* a *and* $\hat{a}$ *that are* $\mathcal{F}_0$*-measurable, and any* $\theta_0 \in \mathbb{R}^{l_\theta}$, *there exists* $\upsilon \in \mathcal{U}$ *such that the following inequality holds for any* $t \in \mathbb{R}_{\geq 0}$*:*

$$\begin{aligned} \mathbb{E}[\|\zeta_{a\upsilon}(t) - \hat{\zeta}_{\hat{a}\hat{\upsilon}}(t)\|^k] \leq \beta(\mathbb{E}[V(a, \hat{a}, \theta_0)], t) \\ + \gamma_{\mathsf{ext}}(\mathbb{E}[\|\hat{\upsilon}\|_\infty^k]) + \mathsf{c}'. \end{aligned} \tag{10}$$

Proof. The proof is similar to the one of Theorem 3.5 in [4] and is omitted here due to lack of space. □

In the next section we first provide a definition of interconnected stochastic hybrid systems. We then provide conditions under which we can construct abstractions of interconnected stochastic hybrid systems in a compositional way.

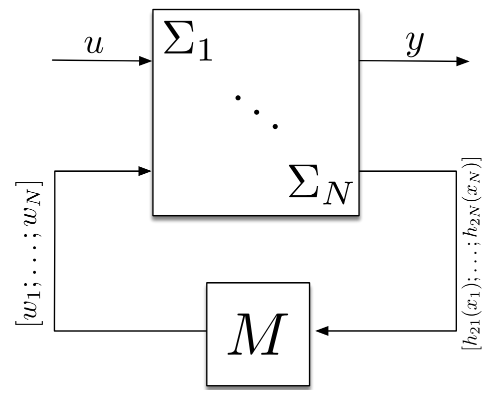

Fig. 2: Interconnected stochastic hybrid system defined in Definition IV.1

IV. Interconnected Stochastic Hybrid Systems

Next definition provides a notion of interconnection for stochastic hybrid subsystems investigated in this paper.

Definition IV.1. *Consider* $N \in \mathbb{N}_{\geq 1}$ *stochastic hybrid subsystems*

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i}),$$

where $i \in [1; N]$, *and a matrix* M *(the interconnection matrix) of an appropriate dimension defining the coupling of these subsystems. The interconnected stochastic hybrid system*

$$\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, \rho, \mathbb{R}^q, h),$$

denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, *follows by* $n = \sum_{i=1}^N n_i, m = \sum_{i=1}^N m_i, q = \sum_{i=1}^N q_{1i}$, *and the functions*

$$\begin{aligned} f(x, u) &\coloneqq [f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)], \\ \sigma(x) &\coloneqq [\sigma_1(x_1); \ldots; \sigma_N(x_N)], \\ \rho(x) &\coloneqq [\rho_1(x_1); \ldots; \rho_N(x_N)], \\ h(x) &\coloneqq [h_{11}(x_1); \ldots; h_{1N}(x_N)], \end{aligned}$$

where $u = [u_1; \ldots; u_N]$, $x = [x_1; \ldots; x_N]$ *and with internal variables constrained by*

$$[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].$$

Assume we are given N stochastic hybrid subsystems $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, \rho_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{q_{2i}}, h_{1i}, h_{2i})$ together with their corresponding abstractions $\hat{\Sigma}_i = (\mathbb{R}^{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{\hat{p}_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \hat{\sigma}_i, \hat{\rho}_i, \mathbb{R}^{q_{1i}}, \mathbb{R}^{\hat{q}_{2i}}, \hat{h}_{1i}, \hat{h}_{2i})$ and with SStF-M$_k$ V_i from $\hat{\Sigma}_i$ to Σ_i. We use α_i, η_i, $\psi_{i\text{ext}}$, A_{θ_i}, B_{θ_i}, C_{θ_i}, D_{θ_i}, H_i, W_i, $\hat{W}_i$, and X_i to denote the corresponding functions, matrices, and their corresponding conformal block partitions appearing in Definition III.2. The next theorem provides a compositional approach on the construction of abstractions of networks of stochastic hybrid systems.

Theorem IV.2. *Consider an interconnected system* $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ *induced by* $N \in \mathbb{N}_{\geq 1}$ *stochastic hybrid subsystems* Σ_i *and the interconnection matrix* M. *Suppose each* *subsystem Σ_i admits an abstraction $\hat{\Sigma}_i$ with the corresponding SStF-M$_k$ V_i with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i})$, $i \in [1;N]$. Suppose there exists $\mu_i > 0$, $i \in [1;N]$, symmetric matrix $\tilde{Q} \succeq 0$, and matrix $\hat{M}$ of appropriate dimension such that the matrix (in)equalities* (11) *and* (12) *are satisfied, where $\tilde{q} = \sum_{i=1}^{N} q_{2i}$, and*

$$W = \mathsf{diag}(W_1, \ldots, W_N), \hat{W} = \mathsf{diag}(\hat{W}_1, \ldots, \hat{W}_N),$$
$$H = \mathsf{diag}(H_1, \ldots, H_N),$$

$$A_D = \mathsf{diag}(A_{\theta_1}, \ldots, A_{\theta_N}), B_D = \mathsf{diag}(B_{\theta_1}, \ldots, B_{\theta_N}),$$
$$C_D = \mathsf{diag}(C_{\theta_1}, \ldots, C_{\theta_N}), D_D = \mathsf{diag}(D_{\theta_1}, \ldots, D_{\theta_N}),$$

and S is the following permutation matrix:

$$S = \begin{bmatrix} I_{r_{W_1}} & 0_{r_{W_2}} & \ldots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \ldots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \ldots & 0_{r_{W_N}} & I_{r_{H_1}} & 0_{r_{H_2}} & \ldots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & I_{r_{W_2}} & \ldots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \ldots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \ldots & 0_{r_{W_N}} & 0_{r_{H_1}} & I_{r_{H_2}} & \ldots & 0_{r_{H_N}} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \ldots & I_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \ldots & 0_{r_{H_N}} \\ 0_{r_{W_1}} & 0_{r_{W_2}} & \ldots & 0_{r_{W_N}} & 0_{r_{H_1}} & 0_{r_{H_2}} & \ldots & I_{r_{H_N}} \end{bmatrix},$$

where, for each $i \in [1;N]$, r_{W_i} and r_{H_i} denote the number of rows in W_i and H_i, respectively. Then

$$V(x, \hat{x}, \theta) := \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T \tilde{Q} \theta,$$

where $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{l_\theta}, l_\theta = \sum_{i=1}^{N} l_{\theta_i}$, is an SSF-M$_k$ from the interconnected system $\hat{\Sigma} := \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$, with the coupling matrix $\hat{M}$, to Σ.

Proof. The proof is inspired by that of Theorem 4.2 in [6]. First we show that the inequality (8) holds for some convex $\mathcal{K}_\infty$ function α. As also argued in the proof of Theorem 4.2 in [4], for any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n$, any $\hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and any $\theta := [\theta_1; \ldots; \theta_N] \in \mathbb{R}^{l_\theta}$, one gets:

$$\begin{aligned} \|h(x) - \hat{h}(\hat{x})\|^k &\le N^{\max\{\frac{k}{2},1\}-1} \sum_{i=1}^{N} \|h_{1i}(x_i) - \hat{h}_{1i}(\hat{x}_i)\|^k \\ &\le N^{\max\{\frac{k}{2},1\}-1} \sum_{i=1}^{N} \alpha_i^{-1}(V_i(x_i, \hat{x}_i, \theta_i)) \\ &\le \underline{\alpha}(V(x, \hat{x}, \theta)), \end{aligned}$$

for any $k \ge 1$, where $\underline{\alpha}$ is a $\mathcal{K}_\infty$ function defined as

$$\underline{\alpha}(s) := \begin{cases} \max\limits_{\vec{s} \ge 0} & N^{\max\{\frac{k}{2},1\}-1} \sum\limits_{i=1}^{N} \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu^T \vec{s} = s, \end{cases}$$

where $\vec{s} = [s_1; \ldots; s_N] \in \mathbb{R}^N$ and $\mu = [\mu_1; \ldots; \mu_N]$. The function $\underline{\alpha}$ is a concave function as argued in [4]. By defining the convex function[1] $\alpha(s) = \underline{\alpha}^{-1}(s), \forall s \in \mathbb{R}_{\ge 0}$, one obtains

$$\alpha(\|h_1(x) - \hat{h}_1(\hat{x})\|^k) \le V(x, \hat{x}, \theta),$$

satisfying inequality (8). Now we prove the inequality (9). Consider any $x = [x_1; \ldots; x_N] \in \mathbb{R}^n, \hat{x} = [\hat{x}_1; \ldots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$, and $\hat{u} = [\hat{u}_1; \ldots; \hat{u}_N] \in \mathbb{R}^{\hat{m}}$. For any $i \in [1;N]$, there exists $u_i \in \mathbb{R}^{m_i}$, consequently, a vector $u = [u_1; \ldots; u_N] \in \mathbb{R}^m$, satisfying (7) for each pair of subsystems Σ_i and $\hat{\Sigma}_i$ with the internal inputs given by $[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)]$ and $[\hat{w}_1; \ldots; \hat{w}_N] = \hat{M}[\hat{h}_{21}(\hat{x}_1); \ldots; \hat{h}_{2N}(\hat{x}_N)]$, respectively. The dynamics of Σ_{θ_i}, $i \in [1;N]$, can be lumped together into a single auxiliary system as the following:

$$\begin{aligned} \dot{\theta}(t) &= A_D\theta(t) + B_D S \begin{bmatrix} W_1 w_1 - \hat{W}_1 \hat{w}_1 \\ \vdots \\ W_N w_N - \hat{W}_N \hat{w}_N \\ h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \\ &= A_D\theta(t) + B_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \begin{bmatrix} h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} z(t) &= C_D\theta(t) + D_D S \begin{bmatrix} W_1 w_1 - \hat{W}_1 \hat{w}_1 \\ \vdots \\ W_N w_N - \hat{W}_N \hat{w}_N \\ h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N) \end{bmatrix} \\ &= C_D\theta(t) + D_D S \begin{bmatrix} WM \\ I_{\tilde{q}} \end{bmatrix} \begin{bmatrix} h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N) \end{bmatrix}, \end{aligned}$$

where $z = [z_1; \ldots; z_N]$. We now consider the infinitesimal generator of the function V, and employ the previous auxiliary system and conditions (11) and (12) to derive the chain of inequalities given in (13), where $\mathsf{c}' = \sum_{i=1}^{N} \mu_i \mathsf{c}_i$,

$$\Theta(x, \theta) := \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_N \\ h_{21}(x_1) - H_1 \hat{h}_{21}(\hat{x}_1) \\ \vdots \\ h_{2N}(x_N) - H_N \hat{h}_{2N}(\hat{x}_N) \end{bmatrix},$$

and the functions $\eta \in \mathcal{K}_\infty$ and $\psi_{\mathsf{ext}} \in \mathcal{K}_\infty \cup \{0\}$ are defined as

$$\eta(s) := \begin{cases} \min\limits_{\vec{s} \ge 0} & \sum_{i=1}^{N} \mu_i \eta_i(s_i) \\ \text{s.t.} & \mu^T \vec{s} = s, \end{cases}$$

$$\psi_{\mathsf{ext}}(s) := \begin{cases} \max\limits_{\vec{s} \ge 0} & \sum_{i=1}^{N} \mu_i \psi_{i\mathsf{ext}}(s_i) \\ \text{s.t.} & \|\vec{s}\| \le s. \end{cases}$$

It remains to show that η is a convex function and ψ_{ext} is a concave one. Let us recall that by assumption functions η_i, $\forall i \in [1;N]$, are convex functions. Thus the function η above defines a *perturbation function* which is a convex one; see [12] for further details. Again, by assumption $\psi_{i\mathsf{ext}}$, $\forall i \in [1;N]$, are concave functions. By similar reasoning, we conclude that ψ_{ext} is a concave function. Hence, we conclude V is an SSF-M$_k$ function from $\hat{\Sigma}$ to Σ. $\square$

[1] The inverse of a strictly increasing concave (resp. convex) function is a strictly increasing convex (resp. concave) function.

$$\begin{bmatrix} A_D^T\tilde{Q}+\tilde{Q}A_D & \tilde{Q}B_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix} \\ \begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}^T S^TB_D^T\tilde{Q} & 0 \end{bmatrix} + \begin{bmatrix} C_D & D_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}\end{bmatrix}^T \begin{bmatrix}\mu_1X_1 & & \\ & \ddots & \\ & & \mu_NX_N\end{bmatrix}\begin{bmatrix} C_D & D_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}\end{bmatrix} \preceq 0, \tag{11}$$

$$WMH=\hat{W}\hat{M}, \tag{12}$$

$$\begin{aligned}
\mathcal{L}V(x,\hat{x},\theta) &= \sum_{i=1}^N \mu_i\mathcal{L}V_i(x_i,\hat{x}_i,\theta_i)+\dot{\theta}^T\tilde{Q}\theta+\theta^T\tilde{Q}\dot{\theta} \le \sum_{i=1}^N \mu_i\Big(-\eta_i(V_i(x_i,\hat{x}_i,\theta_i))+\psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k)+z_i^TX_iz_i+\mathsf{c}_i\Big)+\dot{\theta}^T\tilde{Q}\theta+\theta^T\tilde{Q}\dot{\theta}\\
&= -\sum_{i=1}^N \mu_i\eta_i(V_i(x_i,\hat{x}_i,\theta_i))+\sum_{i=1}^N\mu_i\psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k)+\begin{bmatrix}z_1\\ \vdots\\ z_N\end{bmatrix}^T\begin{bmatrix}\mu_1X_1 & & \\ & \ddots & \\ & & \mu_NX_N\end{bmatrix}\underbrace{\begin{bmatrix}z_1\\ \vdots\\ z_N\end{bmatrix}}_{z}\\
&\quad+\Theta(x,\theta)^T\begin{bmatrix} A_D^T\tilde{Q}+\tilde{Q}A_D & \tilde{Q}B_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix} \\ \begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}^T S^TB_D^T\tilde{Q} & 0 \end{bmatrix}\Theta(x,\theta)+\mathsf{c}'\\
&= -\sum_{i=1}^N \mu_i\eta_i(V_i(x_i,\hat{x}_i,\theta_i))+\sum_{i=1}^N\mu_i\psi_{i\mathsf{ext}}(\|\hat{u}_i\|^k)+\Theta(x,\theta)^T\begin{bmatrix} A_D^T\tilde{Q}+\tilde{Q}A_D & \tilde{Q}B_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix} \\ \begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}^T S^TB_D^T\tilde{Q} & 0 \end{bmatrix}\Theta(x,\theta)\\
&\quad+\Theta(x,\theta)^T\begin{bmatrix} C_D & D_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}\end{bmatrix}^T \begin{bmatrix}\mu_1X_1 & & \\ & \ddots & \\ & & \mu_NX_N\end{bmatrix}\begin{bmatrix} C_D & D_DS\begin{bmatrix}WM\\ I_{\tilde{q}}\end{bmatrix}\end{bmatrix}\Theta(x,\theta)+\mathsf{c}'\\
&\le -\eta(V(x,\hat{x},\theta))+\psi_{\mathsf{ext}}(\|\hat{u}\|^k)+\mathsf{c}',
\end{aligned} \tag{13}$$

Remark IV.3. *If C_{θ_i} is the zero matrix and D_{θ_i} is the identity matrix (i.e. Σ_{θ_i} is a static map), $\forall i\in[1;N]$, then matrix inequality* (11) *reduces to matrix inequality (15) in [7, Theorem 7] (which is a stochastic counterpart of matrix inequality (IV.1) in [6, Theorem 4.2]).*

Remark IV.4. *The matrix inequality* (11) *is linear with respect to the decision variables $\tilde{Q}$ and $\mu=[\mu_1;\ldots;\mu_N]$, and matrix equality* (12) *is linear with respect to the decision variable $\hat{M}$, which can be solved by using readily available software packages such as [13].*

In the next section, we consider a specific class of stochastic hybrid systems Σ, and a specific candidate SStF-M$_2$ function V. We derive conditions facilitating the construction of $\hat{\Sigma}$ as an abstraction of Σ and such that V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ.

V. A Class of Stochastic Hybrid Systems

We consider a specific class of stochastic hybrid systems with the drift, diffusion, reset, and output functions given by

$$\begin{aligned}
\mathsf{d}\xi(t) &= (A\xi(t)+B\upsilon(t)+E\varphi(t,F\xi)\\
&\quad+D\omega(t))\mathsf{d}t+G\mathsf{d}W_t+\sum_{i=1}^{\mathsf{r}}R_i\mathsf{d}P_t^i,\\
\zeta_1(t) &= C_1\xi(t),\\
\zeta_2(t) &= C_2\xi(t),
\end{aligned} \tag{14}$$

where $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, D\in\mathbb{R}^{n\times p}, E\in\mathbb{R}^{n\times l_k}, F\in\mathbb{R}^{l_k\times n}, G\in\mathbb{R}^{n\times 1}, R_i\in\mathbb{R}^n, \forall i\in[1;\mathsf{r}], C_1\in\mathbb{R}^{q_1\times n}$, and $C_2\in\mathbb{R}^{q_2\times n}$. The vector R_i and scalar $\lambda_i>0$ (rate of the Poisson process), $\forall i\in[1;\mathsf{r}]$, parametrize the jumps associated with events i. The time-varying non-linearity is the one considered in [14], which satisfies an incremental quadratic inequality: for all $\tilde{M}\in\mathcal{M}$, where $\mathcal{M}$ is the set of symmetric matrices referred to as incremental multiplier matrices, the following incremental quadratic constraint holds for all $t\in\mathbb{R}_{\ge 0}$, and $k_1,k_2\in\mathbb{R}^{l_k}$:

$$\begin{bmatrix}k_2-k_1\\ \varphi(t,k_2)-\varphi(t,k_1)\end{bmatrix}^T\tilde{M}\begin{bmatrix}k_2-k_1\\ \varphi(t,k_2)-\varphi(t,k_1)\end{bmatrix}\ge 0.$$

To facilitate subsequent analysis, we write matrix $\tilde{M}$ in the following conformal partitioned form

$$\tilde{M}=\begin{bmatrix}M_{11} & M_{12}\\ M_{12}^T & M_{22}\end{bmatrix}.$$

We use the tuple

$$\Sigma=(A,B,C_1,C_2,D,E,F,G,\mathsf{R},\varphi,\lambda),$$

where $\mathsf{R}=\{R_1,\ldots,R_{\mathsf{r}}\}$ and $\lambda=\{\lambda_1,\ldots,\lambda_{\mathsf{r}}\}$, to refer to the class of system of the form (14). We now consider a specific candidate function and derive conditions under which it is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ.

A. Stochastic storage function

Here, we consider a candidate SStF-M$_2$ of the form

$$V(x,\hat{x},\theta)=(x-P\hat{x})^T\widehat{M}(x-P\hat{x})+\theta^T\Lambda\theta, \quad (15)$$

where P, $\widehat{M}^T=\widehat{M}\succ 0$, and $\Lambda=\Lambda^T\succ 0$ are matrices of appropriate dimensions. In order to show that $V(x,\hat{x},\theta)$ in (15) is an SStF-M$_2$ from an abstraction $\hat{\Sigma}$ to the concrete system Σ, with respect to $\Sigma_\theta=(A_\theta,B_\theta,C_\theta,D_\theta)$, where $B_\theta=\begin{bmatrix}B_1 & B_2\end{bmatrix}$ and $D_\theta=\begin{bmatrix}D_1 & D_2\end{bmatrix}$, we require the following assumptions on the concrete system Σ and on Σ_θ.

Assumption V.1. *Let $\Sigma=(A,B,C_1,C_2,D,E,F,G,\mathsf{R},\varphi,\lambda)$. There exist matrices $\widehat{M}\succ 0$, K, X, L_1, Z, W, Λ, A_θ, C_θ, $B_\theta:=[B_1\ B_2]$, $D_\theta:=[D_1\ D_2]$, and positive constants $\hat{\kappa}$ and $\bar{\kappa}$, such that*

$$D_2^TXD_2\preceq 0,$$

and the (in)equalities given in (16) *and* (17) *hold, where*

$$\Delta=(A+BK)^T\widehat{M}+\widehat{M}(A+BK).$$

An equivalent geometric characterization of (16) is given by the following lemma.

Lemma V.2. *Given D and Z, the condition* (16) *is satisfied for some matrix W if and only if*

$$\text{im } D\subseteq\text{im } Z. \quad (18)$$

Remark V.3. *Remark that when the non-linearity in* (14) *reduces to the one described in [6, Section V] and Σ_θ is a static map, matrix inequality* (17) *reduces to (V.5) in [6, Theorem 5.5]. Note also that in the absence of the non-linearity in* (14)*, matrix inequality* (17) *is feasible if the pair (A,B) is stabilizable and A_θ is Hurwitz.*

Now, we provide one of the main results of this section showing under which conditions V in (15) is an SStF-M$_2$.

Theorem V.4. *Let $\Sigma=(A,B,C_1,C_2,D,E,F,G,\mathsf{R},\varphi,\lambda)$, and $\hat{\Sigma}=(\hat{A},\hat{B},\hat{C}_1,\hat{C}_2,\hat{D},\hat{E},\hat{F},\hat{G},\hat{\mathsf{R}},\varphi,\hat{\lambda})$ with the same external output dimension. Suppose Assumption V.1 holds and there exist matrices P, Q, H, $\hat{W}$ and L_2 of appropriate dimensions such that:*

$$AP=P\hat{A}-BQ \quad (19a)$$
$$C_1P=\hat{C}_1 \quad (19b)$$
$$C_2P=H\hat{C}_2 \quad (19c)$$
$$FP=\hat{F} \quad (19d)$$
$$E=P\hat{E}+B(L_2-L_1) \quad (19e)$$
$$P\hat{D}=Z\hat{W}. \quad (19f)$$

Then, function V defined in (15) *is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta=(A_\theta,B_\theta,C_\theta,D_\theta)$.*

Proof. We note that from (19b), $\forall x\in\mathbb{R}^n$ and $\forall\hat{x}\in\mathbb{R}^{\hat{n}}$, we have $\|C_1x-\hat{C}_1\hat{x}\|^2=(x-P\hat{x})^TC_1^TC_1(x-P\hat{x})$. It can be readily verified that $\frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^TC_1)}\|C_1x-\hat{C}_1\hat{x}\|^2\leq V(x,\hat{x},\theta)$ for all $\theta\in\mathbb{R}^{l_\theta}$, implying that inequality (6) holds with $\alpha(r)=\frac{\lambda_{\min}(\widehat{M})}{\lambda_{\max}(C_1^TC_1)}r$ for any $r\in\mathbb{R}_{\geq 0}$, which is a convex function.

We proceed to prove inequality (7). By the definition of V, one has

$$\partial_xV=2(x-P\hat{x})^T\widehat{M},\partial_{\hat{x}}V=-2(x-P\hat{x})^T\widehat{M}P,$$
$$\partial_{x,x}V=2\widehat{M},\partial_{\hat{x},\hat{x}}V=2P^T\widehat{M}P.$$

Following the definition of $\mathcal{L}$, for any $x\in\mathbb{R}^n,\hat{x}\in\mathbb{R}^{\hat{n}},\theta\in\mathbb{R}^{l_\theta}$, one obtains:

$$\begin{aligned}\mathcal{L}V(x,\hat{x},\theta)&=2(x-P\hat{x})^T\widehat{M}(Ax+E\varphi(Fx)+Bu+Dw)\\&-2(x-P\hat{x})^T\widehat{M}P(\hat{A}\hat{x}+\hat{E}\varphi(\hat{F}\hat{x})+\hat{B}\hat{u}+\hat{D}\hat{w})+G^T\widehat{M}G\\&+\hat{G}^TP^T\widehat{M}P\hat{G}+2(x-P\hat{x})^T\widehat{M}\sum_{i=1}^{\mathsf{r}}\lambda_iR_i+\sum_{i=1}^{\mathsf{r}}\lambda_iR_i^T\widehat{M}R_i\\&-2(x-P\hat{x})^T\widehat{M}\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}+\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^TP^T\widehat{M}P\hat{R}_i\\&+2\theta^T\Lambda\Big(A_\theta\theta+\begin{bmatrix}B_1 & B_2\end{bmatrix}\begin{bmatrix}Ww-\hat{W}\hat{w}\\C_2x-H\hat{C}_2\hat{x}\end{bmatrix}\Big).\end{aligned}$$

Given any $x\in\mathbb{R}^n,\hat{x}\in\mathbb{R}^{\hat{n}}$, and $\hat{u}\in\mathbb{R}^{\hat{m}}$, we use the following *interface* function to choose $u\in\mathbb{R}^m$:

$$u=K(x-P\hat{x})+Q\hat{x}+\tilde{R}\hat{u}+L_1\varphi(t,Fx)-L_2\varphi(t,\hat{F}\hat{x}), \quad (20)$$

where L_2, Q, and $\tilde{R}$ are matrices of appropriate dimension. Using the interface function in (20), and the conditions (16), (19a), (19d), (19e), and (19f), one obtains:

$$\begin{aligned}\mathcal{L}V(x,\hat{x},\theta)&=2(x-P\hat{x})^T\widehat{M}\Big(A(x-P\hat{x})+BK(x-P\hat{x})\\&+ZWw-Z\hat{W}\hat{w}+(B\tilde{R}-P\hat{B})\hat{u}+(BL_1+E)\delta\varphi\Big)\\&+G^T\widehat{M}G+\hat{G}^TP^T\widehat{M}P\hat{G}+\sum_{i=1}^{\mathsf{r}}\lambda_iR_i^T\widehat{M}R_i+\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^TP\widehat{M}P\hat{R}_i\\&+2(x-P\hat{x})^T\widehat{M}(\sum_{i=1}^{\mathsf{r}}\lambda_iR_i-\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}_i)+2\theta^T\Lambda A_\theta\theta\\&+2\theta^T\Lambda B_1(Ww-\hat{W}\hat{w})+2\theta^T\Lambda B_2(C_2x-H\hat{C}_2\hat{x}),\end{aligned}$$

where $\delta\varphi=\varphi(t,Fx)-\varphi(t,\hat{F}\hat{x})$. Using Young's inequality, Cauchy-Schwarz inequality, (17), and (19c), one obtains the upper bound for $\mathcal{L}V(x,\hat{x},\theta)$ as given in (21), where $\pi,\pi'\in\mathbb{R}_{>0}$ satisfy $\pi+\pi'<\hat{\kappa}$, $\tilde{\kappa}=\min\{\hat{\kappa}-\pi-\pi',\bar{\kappa}\}$, and

$$\tilde{\mathsf{c}}=G^T\widehat{M}G+\hat{G}^TP^T\widehat{M}P\hat{G}+\sum_{i=1}^{\mathsf{r}}\lambda_iR_i^T\widehat{M}R_i+\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^TP^T\widehat{M}P\hat{R}_i,$$

$$\mathsf{c}'=\frac{\|\sqrt{\widehat{M}}\left(\sum_{i=1}^{\mathsf{r}}\lambda_iR_i-\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_iP\hat{R}_i\right)\|^2}{\pi'}.$$

Here, we have used the fact that for any $x\in\mathbb{R}^n$ and any $\hat{x}\in\mathbb{R}^{\hat{n}}$, one has [14],

$$\begin{bmatrix}x-P\hat{x}\\\delta\varphi\end{bmatrix}^T\begin{bmatrix}F & 0_{l_k}\\0_{l_k} & I_{l_k}\end{bmatrix}^T\tilde{M}\begin{bmatrix}F & 0_{l_k}\\0_{l_k} & I_{l_k}\end{bmatrix}\begin{bmatrix}x-P\hat{x}\\\delta\varphi\end{bmatrix}\geq 0.$$

Using the upper bound (21), the inequality (7) is satisfied, implying that V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta=(A_\theta,B_\theta,C_\theta,D_\theta)$, with the convex function $\eta(s)=\tilde{\kappa}s$, concave function $\psi_{\mathsf{ext}}(s)=\frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}s,\forall s\in\mathbb{R}_{\geq 0}$, matrix X, and $\mathsf{c}=\tilde{\mathsf{c}}+\mathsf{c}'$. □

$$D = ZW, \tag{16}$$

$$\begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1+E) & C_2^T B_2^T \Lambda \\ Z^T\widehat{M} & 0 & 0 & B_1^T\Lambda \\ (BL_1+E)^T\widehat{M} & 0 & 0 & 0 \\ \Lambda B_2 C_2 & \Lambda B_1 & 0 & {A_\theta}^T\Lambda + \Lambda A_\theta \end{bmatrix} \preceq \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^T D_2^T X D_2 C_2 - F^T M_{11} F & C_2^T D_2^T X D_1 & -F^T M_{12} & C_2^T D_2^T X C_\theta \\ D_1^T X D_2 C_2 & D_1^T X D_1 & 0 & D_1^T X C_\theta \\ -M_{12}^T F & 0 & -M_{22} & 0 \\ C_\theta^T X D_2 C_2 & C_\theta^T X D_1 & 0 & {C_\theta}^T X C_\theta - \bar{\kappa}\Lambda \end{bmatrix}, \tag{17}$$

$$\begin{aligned}
\mathcal{L}V(x,\hat{x},\theta) &= \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix}^T \begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1+E) & C_2^T B_2^T \Lambda \\ Z^T\widehat{M} & 0 & 0 & B_1^T\Lambda \\ (BL_1+E)^T\widehat{M} & 0 & 0 & 0 \\ \Lambda B_2 C_2 & \Lambda B_1 & 0 & {A_\theta}^T\Lambda + \Lambda A_\theta \end{bmatrix} \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix} \\
&\quad + 2(x-P\hat{x})^T\widehat{M}(B\tilde{R}-P\hat{B})\hat{u} + 2(x-P\hat{x})^T\widehat{M}\left(\sum_{i=1}^{\mathsf{r}}\lambda_i R_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i P\hat{R}_i\right) + \tilde{\mathsf{c}} \\
&\le \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix}^T \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^T D_2^T X D_2 C_2 - F^T M_{11} F & C_2^T D_2^T X D_1 & -F^T M_{12} & C_2^T D_2^T X C_\theta \\ D_1^T X D_2 C_2 & D_1^T X D_1 & 0 & D_1^T X C_\theta \\ -M_{12}^T F & 0 & -M_{22} & 0 \\ C_\theta^T X D_2 C_2 & C_\theta^T X D_1 & 0 & {C_\theta}^T X C_\theta - \bar{\kappa}\Lambda \end{bmatrix} \begin{bmatrix} x-P\hat{x} \\ Ww-\hat{W}\hat{w} \\ \delta\varphi \\ \theta \end{bmatrix} \\
&\quad + 2(x-P\hat{x})^T\widehat{M}(B\tilde{R}-P\hat{B})\hat{u} + 2(x-P\hat{x})^T\widehat{M}\left(\sum_{i=1}^{\mathsf{r}}\lambda_i R_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i P\hat{R}_i\right) + \tilde{\mathsf{c}} \\
&\le -(\hat{\kappa}-\pi-\pi')(x-P\hat{x})^T\widehat{M}(x-P\hat{x}) + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 - \bar{\kappa}\theta^T\Lambda\theta \\
&\quad - 2\begin{bmatrix} x-P\hat{x} \\ \delta\varphi \end{bmatrix}^T \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix}^T \tilde{M} \begin{bmatrix} F & 0_{l_k} \\ 0_{l_k} & I_{l_k} \end{bmatrix} \begin{bmatrix} x-P\hat{x} \\ \delta\varphi \end{bmatrix} \\
&\quad + \left(C_\theta\theta + \begin{bmatrix} D_1 & D_2 \end{bmatrix}^T \begin{bmatrix} Ww-\hat{W}\hat{w} \\ C_2x - H\hat{C}_2\hat{x} \end{bmatrix}\right)^T X \left(C_\theta\theta + \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} Ww-\hat{W}\hat{w} \\ C_2x - H\hat{C}_2\hat{x} \end{bmatrix}\right) + \tilde{\mathsf{c}} + \frac{\|\sqrt{\widehat{M}}\left(\sum_{i=1}^{\mathsf{r}}\lambda_i R_i - \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i P\hat{R}_i\right)\|^2}{\pi'} \\
&\le -(\hat{\kappa}-\pi-\pi')(x-P\hat{x})^T\widehat{M}(x-P\hat{x}) - \bar{\kappa}\theta^T\Lambda\theta + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 + z^T X z + \tilde{\mathsf{c}} + \mathsf{c}' \\
&\le -\tilde{\kappa}V(x,\hat{x},\theta) + \frac{\|\sqrt{\widehat{M}}(B\tilde{R}-P\hat{B})\|^2}{\pi}\|\hat{u}\|^2 + z^T X z + \tilde{\mathsf{c}} + \mathsf{c}'
\end{aligned} \tag{21}$$

Remark V.5. *Note that matrix $\tilde{R}$ is a free design parameter in the interface function. As explained in [6] and [15], one can choose $\tilde{R}$ to minimize the function ψ_{ext} for V and, hence, lower the upper bound on the error between the output behaviors of Σ and $\hat{\Sigma}$. The choice of $\tilde{R}$ minimizing ψ_{ext} is given by*

$$\tilde{R} = (B^T\widehat{M}B)^{-1}B^T\widehat{M}P\hat{B}. \tag{22}$$

Remark V.6. *The constant c, can be also minimized, thereby lowering the upper bound on the error between the output behaviours of Σ and $\hat{\Sigma}$. One can choose $\hat{G}$ to be the zero matrix and choose $\hat{\lambda}$ and $\hat{\mathsf{R}}$ to solve the following optimization problem:*

$$\begin{aligned}
&\mathit{arg}\min_{\hat{\mathsf{R}},\hat{\lambda}>0} \quad \sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^T P^T\widehat{M}P\hat{R}_i \\
&\quad - \frac{2(\sum_{i=0}^{\mathsf{r}}\lambda_i R_i^T)\widehat{M}P(\sum_{i=0}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i)}{\pi'} + \frac{(\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i^T)P^T\widehat{M}P(\sum_{i=1}^{\hat{\mathsf{r}}}\hat{\lambda}_i\hat{R}_i)}{\pi'},
\end{aligned} \tag{23}$$

where $\hat{\lambda} = \{\hat{\lambda}_1,\ldots,\hat{\lambda}_{\hat{\mathsf{r}}}\}$ and $\hat{\mathsf{R}} = \{\hat{R}_1,\ldots,\hat{R}_{\hat{\mathsf{r}}}\}$. This optimization problem is, in general, a non-convex one.

Remark V.7. *The matrix inequality* (17) *is bi-linear in $\widehat{M}, K, L_1, Z$, and linear in X and Λ if we fix $\hat{\kappa}$, $\bar{\kappa}$, and the matrices $A_\theta, B_\theta, C_\theta$, and D_θ.*

In the following theorem we show that conditions (19a), (19b), (19c), (19d), and (19e) are not only sufficient, but also necessary for (15) to be an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, provided that the interface function is as in (20) for some matrices $K, Q, \tilde{R}, L_1$, and L_2, of appropriate dimensions.

Theorem V.8. *Let $\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda)$ and $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{\mathsf{R}}, \varphi, \hat{\lambda})$ with the same external output space dimension. Assume that $G = \hat{G} = 0$, and $R_i = \hat{R}_i = 0$ $\forall i \in [1;\hat{\mathsf{r}}]$, where 0 represents the zero matrices of appropriate dimensions. Suppose that V, defined in* (15)*, is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, with respect to $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$, with the interface function given in* (20). *Then equations* (19a), (19b), (19c), (19d), *and* (19e) *hold.*

Proof. Since V is an SStF-M$_2$ from $\hat{\Sigma}$ to Σ, there exists a $\mathcal{K}_\infty$ function α such that $\|C_1x - \hat{C}_1\hat{x}\|^2 \le \alpha^{-1}(V(x,\hat{x},\theta))$ for any $x \in \mathbb{R}^n$, any $\hat{x} \in \mathbb{R}^{\hat{n}}$, and any $\theta \in \mathbb{R}^{l_\theta}$. From (15), it follows that $\|C_1P\hat{x} - \hat{C}_1\hat{x}\|^2 \le \alpha^{-1}(V(P\hat{x},\hat{x},0)) = 0$ holds for all $\hat{x} \in \mathbb{R}^{\hat{n}}$ which implies (19b). Let us assume that $D_2^T X D_2 \neq 0$. To prove (19c), we consider the inputs $w \equiv 0, \hat{w} \equiv 0, \hat{u} \equiv 0,$

and choose $x = P\hat{x}$ and $\theta = 0$ in (7). One has:

$$0 \le (C_2 P\hat{x} - H\hat{C}_2\hat{x})^T D_2^T X D_2 (C_2 P\hat{x} - H\hat{C}_2\hat{x}),$$

for all $\hat{x} \in \mathbb{R}^{\hat{n}}$. Since $D_2^T X D_2 \preceq 0$, and $D_2^T X D_2 \neq 0$ by assumption, one obtains $C_2 P - H\hat{C}_2 = 0$, which implies (19c). Consider the input signals $\hat{\upsilon} \equiv 0, \omega \equiv 0, \hat{\omega} \equiv 0$. It can be easily seen that the subspace $\{(x, \hat{x}, \theta) : x = P\hat{x}, \theta = 0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \times \mathbb{R}^{l_\theta}$ is invariant [16], which implies that when $\xi(0) = P\hat{\xi}(0)$ and $\xi_\theta(0) = 0$, one has:

$$\xi(t) = P\hat{\xi}(t), \quad \xi_\theta(t) = 0, \quad \mathsf{d}\xi(t) = P\mathsf{d}\hat{\xi}(t),$$

for all $t \in \mathbb{R}_{\ge 0}$, from which we derive that

$$\begin{aligned}&(AP\hat{\xi}(t) + BQ\hat{\xi}(t) + BL_1\varphi(t, F\xi(t))\\ &- BL_2\varphi(t, \hat{F}\hat{\xi}(t)) + E\varphi(t, FP\hat{\xi}(t)))\mathsf{d}t\\ &= (P\hat{A}\hat{\xi}(t) + P\hat{E}\varphi(t, \hat{F}\hat{\xi}(t)))\mathsf{d}t,\end{aligned}$$

for all $t \in \mathbb{R}_{\ge 0}$, thus implying (19a), (19d), and (19e). □

B. Geometric interpretation of different conditions

In this section, we provide geometric conditions on matrices appearing on the definition of $\hat{\Sigma}$, of stochastic storage function and its corresponding interface function. The geometric conditions facilitate the construction of the abstraction. First, we recall the following result from [15], providing necessary and sufficient conditions for the existence of $\hat{A}$ and Q satisfying (19a).

Lemma V.9. *Consider matrices A, B, and P. There exist matrices $\hat{A}$ and Q satisfying* (19a) *if and only if*

$$\text{im } AP \subseteq \text{im } P + \text{im } B. \tag{24}$$

Similarly, we provide necessary and sufficient conditions for the existence of $\hat{C}_2$ and $\hat{E}$, L_2 satisfying (19c) and (19e), respectively.

Lemma V.10. *Given P and C_2, there exists matrix $\hat{C}_2$ satisfying* (19c) *if and only if*

$$\text{im } C_2 P \subseteq \text{im } H \tag{25}$$

for some matrix H of appropriate dimension.

Lemma V.11. *Given P, B, and L_1, there exist matrices $\hat{E}$ and L_2 satisfying* (19e) *if and only if*

$$\text{im } E \subseteq \text{im } B + \text{im } P. \tag{26}$$

Lemmas V.9, V.10, and V.11 provide sufficient and necessary conditions on P and H, resulting in the construction of matrices $\hat{A}$, $\hat{C}_2$, and $\hat{E}$ and matrices Q and L_2 appearing in the interface function (20). The next lemma provides a sufficient and necessary condition on the existence of $\hat{D}$ satisfying (19f).

Lemma V.12. *Given Z, there exists matrix $\hat{D}$ satisfying* (19f) *if and only if*

$$\text{im } Z\hat{W} \subseteq \text{im } P, \tag{27}$$

for some matrix $\hat{W}$ of appropriate dimension.

Although condition (27) is readily satisfied by choosing $\hat{W} = 0$, one should preferably aim at finding a nonzero $\hat{W}$ with the highest possible rank to facilitate later the satisfaction of compositionality condition (12).

TABLE I: Construction of $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}_1, \hat{C}_2, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{\mathsf{R}}, \varphi, \hat{\lambda})$ together with the corresponding stochastic storage function V in (15), with $\Sigma_\theta = (A_\theta, B_\theta, C_\theta, D_\theta)$, and interface function in (20) for a given $\Sigma = (A, B, C_1, C_2, D, E, F, G, \mathsf{R}, \varphi, \lambda)$.

1) Choose matrix Z such that (18) is satisfied;
2) Choose W such that $D = ZW$;
3) Choose matrices $\widehat{M}, K, L_1, \Lambda, X, A_\theta, C_\theta, B_\theta = [B_1\ B_2], D_\theta = [D_1\ D_2]$, and constants $\hat{\kappa}$, $\bar{\kappa}$ such that (17) is satisfied (see Remark V.13);
4) Determine matrix P of lowest rank with $\ker P = 0$ that satisfies (24), (25), (26), and (27) (see Remark V.14);
5) Choose $\hat{A}$ and Q according to (19a);
6) Choose L_2 and $\hat{E}$ according to (19e);
7) Compute $\hat{F} = FP$;
8) Compute $\hat{C}_1 = C_1 P$;
9) Choose $\hat{G} = 0$. Choose $\hat{\mathsf{R}} = \{\hat{R}_1, \ldots, \hat{R}_{\hat{\mathsf{r}}}\}$ and $\hat{\lambda} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{\hat{\mathsf{r}}}\}$ according to (23);
10) Choose $\hat{C}_2$ satisfying $H\hat{C}_2 = C_2 P$ for some H;
11) Choose $\hat{D}$ satisfying $P\hat{D} = Z\hat{W}$ for some $\hat{W}$ with the highest possible rank;
12) Choose $\hat{B}$ freely (e.g. $\hat{B} = I_{\hat{n}}$ making $\hat{\Sigma}$ fully actuated);
13) Compute $\tilde{R}$, appearing in (20), according to (22);

C. Construction of abstraction

We summarize the construction of abstraction $\hat{\Sigma}$, stochastic storage function V in (15), and its corresponding interface function in (20) in Table I.

Remark V.13. *One way to solve the matrix inequality* (17) *is as follows: First, we select arbitrary C_θ and $D_\theta = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$, and solve the following bilinear matrix inequality for $\hat{\kappa}$, X, $\widehat{M}$, and L_1:*

$$\begin{bmatrix} \Delta & \widehat{M}Z & \widehat{M}(BL_1 + E) \\ Z^T\widehat{M} & 0 & 0 \\ (BL_1 + E)^T\widehat{M} & 0 & 0 \end{bmatrix}$$
$$\preceq \begin{bmatrix} -\hat{\kappa}\widehat{M} + C_2^T D_2^T X D_2 C_2 - F^T M_{11} F & C_2^T D_2^T X D_1 & -F M_{12} \\ D_1^T X D_2 C_2 & D_1^T X D_1 & 0 \\ -M_{12}^T F & 0 & -M_{22} \end{bmatrix}.$$

We then solve the following bilinear matrix equation for $\bar{\kappa}$, Λ, and A_θ:

$$A_\theta^T \Lambda + \Lambda A_\theta = C_\theta^T X C_\theta - \bar{\kappa}\Lambda.$$

Finally, we solve the following linear equations for $B_\theta = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$:

$$\Lambda B_1 = C_\theta^T X D_1^T,$$

$$\Lambda B_2 = C_\theta^T X D_2.$$

Remark V.14. *One way to satisfy the geometric conditions* (24)-(27) *is to start with a scalar abstraction (i.e. $\hat{n} = 1$) and pick P to be an arbitrary column vector, and check if* (24)-(27) *hold. If not, then increase the state-space dimension of the abstraction by one (i.e. $\hat{n} = 2$), add a linearly independent column vector to P, and check again if* (24)-(27) *hold. Repeat this process until* (24)-(27) *are satisfied. Note that in the worst-case scenario, this process will terminate when $\hat{n} = n$ (i.e. the state-space dimension of the concrete subsystem and abstraction are equal).*

In the next section, we provide two examples for compositional construction of abstractions of a network of stochastic hybrid systems using the technique presented in the paper. First, in a physically motivated example, we construct a compositional abstraction of a network of resistor-capacitor (R-C) circuits affected by stochastic noise. In the second example, we illustrate the advantage of using a linear control system Σ_θ over just a static map (which was used in [6], [7]) to conclude the joint dissipativity property of a concrete subsystem and its abstraction.

VI. Examples

A. Network of RC Circuits

Consider an interconnection of n first order R-C circuits. The i-th R-C circuit has a dynamic given by:

$$\mathsf{d}v_{c_i} = \left(-\frac{1}{R_iC_i}v_{c_i} + \frac{1}{R_iC_i}v_{s_i} + \frac{1}{C_i}\tilde{w}_i\right)\mathsf{d}t + \varpi \mathsf{d}W_t + \tau \mathsf{d}P_t, \tag{28}$$

where $\varpi \in \mathbb{R}_{>0}, \tau \in \mathbb{R}_{>0}$, $i \in [1;n]$, $v_{s_i} \in \mathbb{R}$ represents the input source voltage (external input), $v_{c_i} \in \mathbb{R}$ is the voltage across capacitor, C_i is the capacitance, R_i is the resistance, and $\tilde{w}_i \in \mathbb{R}$ is the total current inflow from other R-C circuits in the network. The continuous noise and jump terms represent the thermal noise (also known as Johnson-Nyquist noise) and the so-called Shot noise [17], respectively. Assume the rate of the Poisson process P_t is λ. For illustration purposes, in this example we fix $R_i = 1$ Ohm, and $C_i = 1$ Farad $\forall i \in [1;N]$. We consider the above interconnected system as an interconnection of N concrete subsystems Σ_i, $i \in [1;N]$, wherein each subsystem Σ_i is formed by clustering n_i R-C circuits ($n_i \leq n$). We also add a non-linearity belonging to the class of nonlinearities presented in this paper. Each subsystem, $\Sigma_i = (A_i, B_i, C_{1i}, I_{ni}, D_i, \vec{1}_{ni}, \vec{1}_{ni}^T, \varpi\vec{1}_{ni}, \tau\vec{1}_{ni}, \varphi, \lambda)$, generates a scalar external output:

$$\Sigma_i : \begin{cases} \mathsf{d}\xi_i = (A_i\xi_i + B_iu_i + D_iw_i + \vec{1}_{ni}\varphi(\vec{1}_{ni}^T\xi_i))\mathsf{d}t \\ \qquad + \varpi\vec{1}_{ni}\mathsf{d}W_t + \tau\vec{1}_{ni}\mathsf{d}P_t, \\ \zeta_{1i} = C_{1i}\xi_i, \\ \zeta_{2i} = \xi_i, \end{cases}$$

where $\xi_i = \mathsf{L}_i v$, $v = [v_{c_1}; \ldots; v_{c_n}]$, $\mathsf{L}_i := [e_{i1}; \ldots; e_{in_i}]$, $e_{ij} \in \mathbb{R}^{1\times n}$ is a row vector whose k-th element is defined as

$$e_{ij}^{(k)} = \begin{cases} 1 \text{ if } k\text{-th R-C circuit is part of the } i\text{-th cluster} \\ 0 \text{ otherwise}, \end{cases}$$

$A_i, B_i, D_i \in \mathbb{R}^{n_i\times n_i}$ are readily obtained from (28), $C_{1i} \in \mathbb{R}^{1\times n_i}$, $u_i = \mathsf{L}_i v_s$, $v_s = [v_{s_1}; \ldots; v_{s_n}]$, $w_i = \mathsf{L}_i\tilde{w}$, $\tilde{w} = [\tilde{w}_1; \ldots; \tilde{w}_n]$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\varphi(x) = \sin(x).$$

The interconnection topology in this example is given by

$$M = -\begin{bmatrix} n-1 & -1 & \ldots & \ldots & -1 \\ -1 & n-1 & -1 & \ldots & -1 \\ -1 & -1 & n-1 & \ldots & -1 \\ \vdots & & \ddots & \ddots & \vdots \\ -1 & \ldots & \ldots & -1 & n-1 \end{bmatrix}.$$

The interconnection topology represents a fully-connected interconnection topology. We aggregate each Σ_i into a scalar deterministic abstraction $\hat{\Sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_{1i}, 1, 1, 1, 1, 0, 0, \varphi, 0)$ given by the following dynamics

$$\hat{\Sigma}_i : \begin{cases} \mathsf{d}\hat{\xi}_i = (\hat{A}_i\hat{\xi}_i + \hat{B}_i\hat{u}_i + \hat{w}_i + \varphi(\hat{\xi}_i))\mathsf{d}t \\ \hat{\zeta}_{1i} = \hat{C}_{1i}\hat{\xi}_i, \\ \hat{\zeta}_{2i} = \hat{\xi}_i, \end{cases}$$

where $\hat{A}_i$ satisfies $A_i\vec{1}_{n_i} = \vec{1}_{n_i}\hat{A}_i$, $\hat{B}_i$ is chosen arbitrarily (in this example we choose $\hat{B}_i$ = 1), $\hat{C}_{1i} = C_{1i}\vec{1}_{n_i}$. The function $V_i(x_i, \hat{x}_i) = (x_i - \vec{1}_{n_i}\hat{x}_i)^T(x_i - \vec{1}_{n_i}\hat{x}_i)$ (i.e. $\hat{M}_i = I_{n_i}, P_i = \vec{1}_{n_i}, \Lambda_i = 0$) is a SStF-M$_2$ function from $\hat{\Sigma}_i$ to Σ_i, with the following parameters

$$K_i = -\chi I_{n_i}, Z_i = I_{n_i}, W_i = I_{n_i}, X_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ I_{n_i} & 0_{n_i} \end{bmatrix},$$
$$\hat{\kappa}_i = 2\chi - 2\lambda\tau - \varpi^2 - \lambda\tau^2, Q_i = 0_{n_i}, H_i = \hat{W}_i = \vec{1}_{n_i},$$
$$L_{1i} = -\vec{1}_{n_i}, A_{\theta_i} = 0, B_{\theta_i} = 0, C_{\theta_i} = 0, D_{\theta_i} = I_{2n_i}, \bar{\kappa} = 0,$$

where $\chi > \lambda\tau + \frac{\varpi^2}{2} + \frac{\lambda\tau^2}{2}$, and with $\alpha_i(r) = \frac{1}{\lambda_{\max}(C_{1i}^T C_{1i})}r$, $\eta_i(r) = (2\chi - 2\lambda\tau - \varpi^2 - \lambda\tau^2)r, \psi_{i\mathsf{ext}}(r) = 0$, $\forall r \in \mathbb{R}_{\geq 0}$, and $\mathsf{c}_i = \tau^2 + \varpi^2$. Inputs $u_i \in \mathbb{R}^{n_i}$ is given via the interface function in (20) as (i.e. $\tilde{R}_i = \vec{1}_{n_i}, L_{2i} = \vec{1}_{n_i}$)

$$u_i = -\chi(x_i - \vec{1}_{n_i}\hat{x}_i) + \vec{1}_{n_i}\hat{u}_i - \vec{1}_{n_i}\varphi(\vec{1}_{n_i}^T x_i) + \vec{1}_{n_i}\varphi(\hat{x}_i). \tag{29}$$

By selecting $\mu_1 = \ldots = \mu_N = 1$, the function $V(x, \hat{x}, \theta) = \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i, \theta_i)$ is an SSF-M$_2$ function from $\hat{\Sigma}$ to Σ, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ with a coupling matrix $\hat{M}$, satisfying condition (12) as the following

$$M\mathsf{diag}(\vec{1}_{n_1}, \ldots, \vec{1}_{n_N}) = \mathsf{diag}(\vec{1}_{n_1}, \ldots, \vec{1}_{n_N})\hat{M}. \tag{30}$$

A matrix $\hat{M}$ exists satisfying (30) if there exist N *equitable partitions* of the graph described by the Laplacian matrix $L = -M$, which is always true here because L represents a fully connected graphs, as explained in [18].

It can be easily seen that condition (11) reduces to

$$\begin{bmatrix} -L \\ I_n \end{bmatrix}^T \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} -L \\ I_n \end{bmatrix} = -(L + L^T) \preceq 0,$$

which always holds since $L = L^T \succeq 0$, which is always true for Laplacian matrices of undirected graphs [18].

1) Controller synthesis: Now, we synthesize a controller for the abstract interconnected system $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N)$ to enforce a specification, and then refine the designed controller to the one for the concrete interconnected system. We fix n = 9, N = 3, $\tau = 0.2, \varpi = 0.4, \lambda = 1, \chi = 10$ and

$$C_{11} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, C_{12} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, C_{13} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

We synthesize a controller using toolbox `SCOTS` [19] to enforce the following linear temporal logic specification [20] over the outputs of $\hat{\Sigma}$:

$$\Psi = \Box S \wedge \left(\bigwedge_{i=1}^{5} \Box(\neg O_i)\right) \wedge \Box\Diamond T_1 \wedge \Box\Diamond T_2,$$

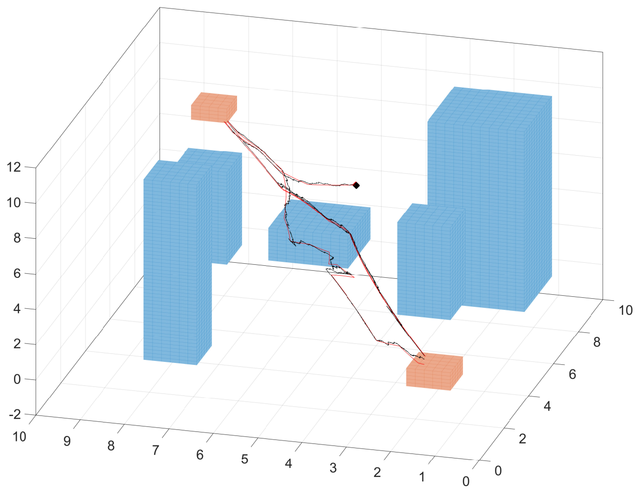

Fig. 3: The figure shows the output trajectories of the abstract (red) and one realization of the concrete (black) interconnected systems. The initial point of the trajectories is represented by the diamond.

which can be interpreted as follows: the output trajectory of the closed loop system evolves inside the set S, avoids regions $O_i, i \in [1;5]$, indicated with blue boxes in Figure 3, and visits $T_i, i \in [1;2]$ infinitely often, indicated with red boxes in Figure 3. We use (29) to generate the corresponding input enforcing this specification over the original system Σ.

B. Example 2

In this part, we provide compositional abstractions of a network of subsystems wherein the joint dissipativity property of each concrete subsystem and its abstraction is only concluded with respect to a linear control system Σ_θ rather than a static map. Consider an interconnection of N second order subsystems Σ_i, where each Σ_i is given by

$$\Sigma_i : \begin{cases} \mathsf{d}\xi_i(t) = (A_i\xi_i(t) + B_i\upsilon_i(t) + D_i\omega_i(t))\mathsf{d}t, \\ \zeta_{1i}(t) = C_{1i}\xi_i(t), \\ \zeta_{2i}(t) = \xi_i(t), \end{cases}$$

where

$$A_i = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ -I_{n_i} & -0.5I_{n_i} \end{bmatrix}, B_i = D_i = \begin{bmatrix} 0_{n_i} \\ I_{n_i} \end{bmatrix}, C_{1i} = \begin{bmatrix} 0_{n_i} \\ e_{n_i} \end{bmatrix}^T,$$

vector e_{n_i} represents a column vector whose first element is 1 and remaining elements are zero. For the sake of simulation we choose $N = 3$, $n_i = 10$, $\forall i \in [1;N]$. We consider the following abstract system $\hat{\Sigma}_i$,

$$\hat{\Sigma}_i : \begin{cases} \mathsf{d}\hat{\xi}_i(t) = \left(\begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}\hat{\xi}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\hat{\upsilon}_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\hat{\omega}_i(t)\right)\mathsf{d}t, \\ \hat{\zeta}_{1i}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}\hat{\xi}_i(t), \\ \hat{\zeta}_{2i}(t) = \hat{\xi}_i(t). \end{cases}$$

We restrict K_i for each $i \in [1;N]$ appearing in (20) such that the last n_i columns are identically zero. This restriction can appear in practice when for example only some state variables are available to be measured. With this restriction on the structure of K_i, one cannot find a storage function with $C_{\theta_i} = 0$ in this example. Using the guidelines shown in Table I and the solver package `Yalmip` [13], it can be shown that the function

$$V_i(x_i, \hat{x}_i, \theta_i) = (x_i - P\hat{x}_i)^T\widehat{M}(x_i - P\hat{x}_i) + \theta_i^T\Lambda\theta_i$$

is an SStF-M$_2$ from $\hat{\Sigma}_i$ to Σ_i, with respect to $\Sigma_{\theta_i} = (A_{\theta_i}, B_{\theta_i}, C_{\theta_i}, D_{\theta_i})$, $\forall i \in [1;N]$, with the following parameters

$$\widehat{M}_i = \begin{bmatrix} 2I_{n_i} & I_{n_i} \\ I_{n_i} & I_{n_i} \end{bmatrix}, P_i = \begin{bmatrix} \vec{1}_{n_i} & \vec{0}_{n_i} \\ \vec{0}_{n_i} & \vec{1}_{n_i} \end{bmatrix}, K_i = \begin{bmatrix} -0.5I_{n_i} & 0_{n_i} \end{bmatrix},$$

$$\hat{\kappa}_i = 0.1, W_i = I_{n_i}, Q_i = 0, H_i = \hat{W}_i = \vec{1}_{n_i}, L_{1i} = 0, \Lambda = I_{2n_i},$$

$$A_{\theta_i} = -5I_{2n_i}, B_{\theta_i} = \begin{bmatrix} 0_{n_i} & 0.207I_{n_i} \\ 0_{n_i} & -0.573I_{n_i} \end{bmatrix}, C_{\theta_i} = 0.1I_{2n_i},$$

$$D_{\theta_i} = \begin{bmatrix} 0_{n_i} & I_{n_i} \\ 0_{n_i} & I_{n_i} \end{bmatrix}, X_i = \begin{bmatrix} 9.47785I_{n_i} & -7.4055I_{n_i} \\ -7.4055I_{n_i} & 1.6526I_{n_i} \end{bmatrix}, \bar{\kappa}_i = 1,$$

with $\alpha_i(r) = \frac{\lambda_{\min}(\widehat{M}_i)}{\lambda_{\max}(C_{1i}^T C_{1i})}r$, $\eta_i(r) = 0.1r$, $\psi_{i\text{ext}} = 0, \forall r \in \mathbb{R}_{\geq 0}$, and $c_i = 0$. Functions $u_i \in \mathbb{R}^{n_i}$ are given via the interface function:

$$u_i = -K_i(x_i - P_i\hat{x}_i) + \vec{1}_{n_i}\hat{u}_i,$$

(i.e. $\tilde{R}_i = \vec{1}_{n_i}, L_{2i} = 0$). With the interconnection matrix M given by

$$M = \begin{bmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & \\ 1 & 0 & 0 & \dots & 1 & -2 \end{bmatrix}.$$

and by selecting $\mu_1 = \dots = \mu_N = 1$, it can be verified that the function $V = \sum_{i=1}^{N} \mu_i V_i(x_i, \hat{x}_i, \theta_i) + \theta^T\theta$, where $\theta = [\theta_i; \dots; \theta_N]$, is an SSF-M$_2$ from $\hat{\Sigma}$ to Σ, where $\hat{\Sigma}$ is the interconnection of the abstract subsystems $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \dots, \hat{\Sigma}_N)$ with the coupling matrix $\hat{M}$ given by

$$\hat{M} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix},$$

satisfying conditions (11) and (12). In the simulation, the input signal to the abstract system is chosen arbitrarily as $\hat{\upsilon}(t) = [\sin(t); 0.1\mathrm{e}^{-t}; -t]$. Figure 4 shows the evolution of the absolute value of the error between the output trajectories of the concrete interconnected system and its abstraction. One can readily verify that the error is always bounded by the computed error bound in Theorem III.6.

VII. Conclusion

In this work, using tools from stochastic calculus and dissipativity theory, we derived conditions under which abstractions of interconnected stochastic hybrid systems can be constructed compositionally. In future work, we will look at deriving constructive conditions which facilitate the construction of abstractions for classes of non-linear stochastic hybrid systems broader than the one considered in this paper, together with the corresponding stochastic storage functions and interface maps.

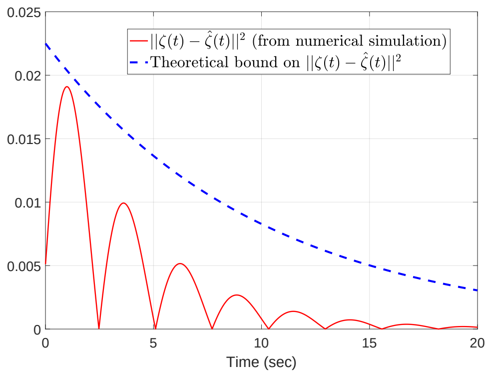

Fig. 4: The evolution of $\|\zeta(t) - \hat{\zeta}(t)\|^2$, where $\zeta(t) = [\zeta_{11}(t); \ldots; \zeta_{1N}(t)]$, and $\hat{\zeta}(t) = [\hat{\zeta}_{11}(t); \ldots; \hat{\zeta}_{1N}(t)]$, and the theoretical upper bound obtained for this example according to (10).

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