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German Bernhart*, Jan-Frederik Mai, and Matthias Scherer

On the construction of low-parametric families of min-stable multivariate exponential distributions in large dimensions

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Abstract: Min-stable multivariate exponential (MSMVE) distributions constitute an important family of distributions, among others due to their relation to extreme-value distributions. Being true multivariate exponential models, they also represent a natural choice when modeling default times in credit portfolios. Despite being well-studied on an abstract level, the number of known parametric families is small. Furthermore, for most families only implicit stochastic representations are known. The present paper develops new parametric families of MSMVE distributions in arbitrary dimensions. Furthermore, a convenient stochastic representation is stated for such models, which is helpful with regard to sampling strategies.

Keywords: MSMVE distributions; Bernstein functions; IDT-frailty copulas; IDT processes; extreme-value copulas

MSC: 60G70

1 Motivation

Min-stable multivariate exponential (MSMVE) distributions constitute an important class of multivariate distributions, in particular for real-world applications. This is mainly due to two reasons: First of all, starting from an MSMVE distribution, one can easily derive corresponding extreme-value distributions by transforming the one-dimensional margins appropriately. Extreme-value distributions in turn can be very useful for modeling particular real-world use cases. Among others, they have been applied in environmental sciences, e.g. when modeling the occurrence of floods at different places or maximal ozone levels at different monitoring stations, see [15, 18], and in Finance, e.g. when modeling the dependence of extreme asset returns, see [25, 33]. The second reason applies to applications in risk-management, respectively modeling of credit portfolios. A natural, widespread, and robust way to model the default times of single components of such a portfolio (e.g. bonds or loans) is based on the exponential distribution. For a portfolio manager, it is daily business to express the default risk associated with a single credit-risky asset in terms of an exponential rate parameter, which is often called the “credit spread”. However, given exponentially distributed univariate marginals, it is not obvious how to model the corresponding joint distribution of multiple credit-risky assets. In practical portfolio credit-risk management, it is quite popular to link the univariate probability distributions to a joint distribution by the use of a copula, see, e.g., [6, 40]. However, it is the authors’ conviction that “true” multivariate exponential concepts should be used because such concepts naturally fit the intuitive

***Corresponding Author: German Bernhart:** Technische Universität München, Parkring 11, 85748 Garching-Hochbrück, Germany, E-mail: german.bernhart@tum.de

Jan-Frederik Mai: XAIA Investment GmbH, Sonnenstraße 19, 80331 München, Germany, E-mail: jan-frederik.mai@xaia.com

Matthias Scherer: Technische Universität München, Parkring 11, 85748 Garching-Hochbrück, Germany, E-mail: scherer@tum.de

motivation of modeling lifetimes. Such “true” multivariate exponential distributions have been defined in the statistical literature, see, e.g., [12]. In [5], one can find further support for the application of these dependence concepts. In the past, some of the criticism surrounding copula modeling in the context of portfolio credit risk can be attributed to the lack of such a natural link between the copula model and the intuition of lifetime modeling.

Though being a very useful class of distributions and being very well studied on an abstract level, (see, e.g., [20, Chapter 6]; [24, 35]), the number of known parametric MSMVE families, in particular in large dimensions, is rather small. Furthermore, the number of parametric models for which concrete stochastic representations are known is even smaller. For most families, only the spectral representation of [7], or an implicit stochastic representation as a limit distribution can be stated. However, concrete stochastic representations are helpful for practical applications as they allow for a more intuitive understanding and can serve as a starting point for developing efficient simulation algorithms.

Motivated by the above considerations, there is some recent work aiming at the construction of new and flexible parametric MSMVE families in large dimensions, see, e.g., [1, 10, 13, 41]. In the same spirit, the aim of the present paper is to develop new parametric models that have a convenient stochastic representation. We will develop two similar classes of MSMVE distributions, giving rise to a huge quantity of parametric MSMVE models in arbitrary dimensions (see Theorems 1 and 2). The underlying stochastic model is given by a frailty construction, i.e. the components can be defined as first-passage times of a stochastic process across independent trigger levels. Similar to [19], the aim is to introduce new classes of models, while an application of these in practice requires a further, very detailed investigation of particular parametric families, which lies outside the scope of this work.

Our findings have at least three important implications:

- (i) They give rise to a huge class of parametric stable tail dependence functions in arbitrary dimensions.
- (ii) The underlying stochastic model can be used for efficient simulations, in particular in large dimensions, to construct non-exchangeable extensions (see Lemma 1), and to investigate statistical properties of the associated MSMVE distribution.
- (iii) Based on the specific stochastic model, which can be interpreted as a so-called frailty construction, one further advantage is observed: When considering large homogeneous credit portfolios, Glivenko–Cantelli type approximations for the portfolio loss process as presented in [28] are available. Since the default times are conditionally independent and identically distributed in such models, for a large portfolio, the portfolio loss can be approximated by the conditional default probability. The famous Vasicek formula is derived using the same idea, see [42]. These approximations allow, e.g., for semi-analytic evaluation of collateralized debt obligations or similar non-linear derivatives depending on the portfolio loss.

The remainder of the paper is organized as follows. In Section 2, the necessary mathematical concepts are introduced, including a brief introduction to MSMVE distributions and their connection to so-called IDT subordinators. In Section 3, a specific family of IDT subordinators and the related family of MSMVE distributions is investigated. A similar example is sketched in Section 4. A note on simulation can be found in Section 5. Section 6 concludes.

2 Mathematical prerequisites and notation

Section 2.1 recalls some background on MSMVE laws, Section 2.2 introduces IDT subordinators and the so-called IDT-frailty model, and Section 2.3 provides an explicit construction of IDT subordinators, which is of prime relevance for our results.

2.1 MSMVE distributions

A random vector (X_1, \dots, X_d) with support $[0, \infty)^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to have a min-stable multivariate exponential (MSMVE) distribution if each minimum over arbitrary subsets of scaled components, i.e. $\min\{c_1 X_{i_1}, \dots, c_k X_{i_k}\}$ for all $1 \leq k \leq d$, $1 \leq i_1 < \dots < i_k \leq d$, $c_1, \dots, c_k > 0$, is (univariate) exponential. Relying on the min-stability as a characterizing property reflects the philosophy of [12] to lift the concepts and properties of the one-dimensional exponential law to higher dimensions. Furthermore, these distributions can be seen as possible distributions of asymptotic component-wise minima, as stated in [32], or as limiting extreme-value distributions with (negative) exponential marginals, as stated in [8]. It is well-known, see, e.g., [20, Theorem 6.2, p. 174], that the survival function \bar{F} of an MSMVE distribution can be written as

$$\bar{F}(x_1, \dots, x_d) := \mathbb{P}(X_1 > x_1, \dots, X_d > x_d) = \exp(-\ell(x_1, \dots, x_d)),$$

$x_i \geq 0$, $i = 1, \dots, d$, where $\ell : \mathbb{R}_+^d \rightarrow \mathbb{R}_+$ is homogeneous of order 1, i.e. $\ell(t\mathbf{x}) = t\ell(\mathbf{x})$ for all $t > 0$, $\mathbf{x} \in \mathbb{R}_+^d$. Additionally assuming all marginal laws to be unit exponentials, such a function ℓ is called a stable tail dependence function and $\ell(\mathbf{e}_i) = 1$ is fulfilled for all unit vectors \mathbf{e}_i , $1 \leq i \leq d$. The previously mentioned link between MSMVE distributions and extreme-value distributions can also be stated using the concept of copulas (for an introduction, see, e.g., the textbooks [20] and [31]): C is the copula of an extreme-value distribution, a so-called extreme-value copula (see [14]), if and only if

$$C(u_1, \dots, u_d) = \exp(-\ell(-\log(u_1), \dots, -\log(u_d))), \quad u_1, \dots, u_d \in (0, 1],$$

with ℓ a stable tail dependence function. Furthermore, the survival copula of an MSMVE vector (X_1, \dots, X_d) is exactly of this kind.

One of the major findings in multivariate extreme-value theory, at least known since [9] and many times re-discovered and re-formulated since then, is a one-to-one relationship between d -dimensional MSMVEs and certain measures on a subspace of \mathbb{R}_+^d , which is somehow comparable with the one-to-one relationship between infinitely divisible distributions and their associated Lévy measures. A quite recent, purely analytical derivation of this result can be retrieved from [36]. Additionally, the latter reference shows that a d -variate function $\bar{F} : [0, \infty)^d \rightarrow [0, 1]$ is an MSMVE survival function with unit exponential marginals if and only if the function

$$\ell(x_1, \dots, x_d) := -\log(\bar{F}(x_1, \dots, x_d))$$

is homogeneous of order 1, fully d -max-decreasing, and $\ell(\mathbf{e}_i) = 1$, $1 \leq i \leq d$, so these are necessary and sufficient conditions for $\ell : \mathbb{R}_+^d \rightarrow \mathbb{R}_+$ to be a stable tail dependence function. This refines a result of [16]. The homogeneity property is a reformulation of the extreme-value property of the underlying extreme-value copula in terms of the function ℓ . The fully d -max-decreasingness property is essentially a reformulation of the d -increasingness property of the associated MSMVE's distribution function after transformation to survival functions and an application of the log-transform. However, it is not easy to investigate this property analytically. Another characterization of stable tail dependence functions in terms of so-called "max-zonoids" is given in [30].

In the well-studied bivariate case, ℓ is characterized by the so-called Pickands dependence function $A : [0, 1] \rightarrow [1/2, 1]$, which is defined by $A(t) := \ell(t, 1-t)$. Sufficient conditions for a function to be a bivariate Pickands dependence function are as follows, see [14, Theorem 2.3]: A is a bivariate Pickands dependence function if and only if A is convex and $\max\{t, 1-t\} \leq A(t) \leq 1$, for all $t \in [0, 1]$. Concerning measures of dependence, like concordance measures and tail dependence coefficients, it is well-known for a bivariate MSMVE vector (X_1, X_2) that these measures can be computed easily from A (see, e.g., [14, 17]), e.g.

$$\rho = 12 \int_0^1 \frac{1}{(1+A(t))^2} dt - 3, \quad (\text{Spearman's } \rho),$$

$$\lambda_L = 2(1 - A(1/2)), \quad (\text{lower tail-dependence coefficient}).$$

Note that $\rho \geq 0$ for all MSMVE distributions. Assuming the dependence structure, respectively function A , to be of a given parametric form, maximum likelihood, maximum pseudo-likelihood or rank-based moment estimators are available, see the literature on estimation of (bivariate extreme-value) copulas, e.g., [21, Chapter 5].

Our approach is dimension-free in the sense that instead of random vectors (X_1, \dots, X_d) we consider MSMVE sequences $\{X_k\}_{k \in \mathbb{N}}$. These are sequences of random variables such that the min-stability property holds for all finite subsets of \mathbb{N} , i.e. $\min_{i \in I} \{c_i X_i\}$ is exponentially distributed for all finite $I \subset \mathbb{N}$ and $c_i > 0$, for all $i \in I$.

2.2 The IDT-frailty model

The starting point of our considerations is a result by [27] which shows that for an exchangeable MSMVE sequence $\{X_k\}_{k \in \mathbb{N}}$ there exists a stochastic representation as a frailty model

$$X_k := \inf\{t > 0 : H_t > E_k\}, \quad k \in \mathbb{N}, \quad (1)$$

with an iid sequence $\{E_k\}_{k \in \mathbb{N}}$ of unit exponential random variables and $\{H_t\}_{t \geq 0}$ a so-called strong IDT subordinator, which is independent of $\{E_k\}_{k \in \mathbb{N}}$. Conversely, starting from a strong IDT subordinator, the above construction (1) yields an exchangeable MSMVE sequence. Exchangeability means that the distribution of each finite sub-vector remains unchanged under arbitrary permutations of the components of the vector. A process $\{H_t\}_{t > 0}$ is called a strong IDT subordinator in the sense of [27] if it is right-continuous, $[0, \infty]$ -valued, non-decreasing, $H_0 = 0$, and $\lim_{t \rightarrow \infty} H_t = \infty$, a.s., and satisfies

$$\{H_t\} \stackrel{d}{=} \{H_{t/n}^{(1)} + \dots + H_{t/n}^{(n)}\}, \quad \forall n \in \mathbb{N}, \quad (2)$$

where the processes $H^{(1)}, \dots, H^{(n)}$ are iid copies of H . Property (2) corresponds to the definition of IDT processes as used in, e.g., [29] or [11], restricted to non-decreasing processes with some additional technical constraints, which guarantee that $\{X_k\}_{k \in \mathbb{N}}$ is well defined. It is well-known for strong IDT processes, see, e.g., [29], that H_t is infinitely divisible for every $t > 0$ and thus, defining $\Psi_H(x) := -\log(\mathbb{E}[\exp(-x H_1)])$ yields a so-called Bernstein function. A function $\Psi : (0, \infty) \rightarrow \mathbb{R}$ is a Bernstein function if and only if it admits a (unique) representation via

$$\Psi(x) = c + ax + \int_{(0, \infty)} (1 - e^{-xu}) \nu(du),$$

with $c \geq 0$ the so-called killing term, $a \geq 0$ the drift term, and ν a measure on $(0, \infty)$ satisfying $\int_{(0, \infty)} (1 \wedge u) \nu(du) < \infty$, the Lévy measure. In the following, we will always ignore the so-called killing term as we only consider distributions on $[0, \infty)$, i.e. we set it to zero. Bernstein functions are often extended to the domain $[0, \infty)$ setting $\Psi(0) := 0$. The set of all Lévy measures is denoted \mathcal{M} . For further information on Bernstein functions, see [39]. For the Laplace transform of the one-dimensional margins of a strong IDT subordinator, one has

$$\mathbb{E}[\exp(-x H_t)] = \exp(-t \Psi_H(x)), \quad x \geq 0.$$

Thus, for every strong IDT subordinator, there exists a Lévy subordinator with the same marginal distributions. Actually, Lévy subordinators are the best studied example of strong IDT subordinators. The marginal distribution of H_t is of relevance for the exchangeable sequence constructed in Equation (1), since $\min_{i \in I} \{X_i\} \sim \text{Exp}(\Psi_H(|I|))$ for all finite subsets $\emptyset \neq I \subset \mathbb{N}$ with cardinality $|I|$.

Theoretically, construction (1) allows to define new parametric families of MSMVE distributions (and thus stable tail dependence functions) by defining parametric families of strong IDT subordinators. Furthermore, using Lemma 1, it is possible to construct not only exchangeable sequences, but also non-exchangeable vectors, e.g. based on factor-model motivations.

Lemma 1 (Multi-factor MSMVE distributions). *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $n + 1 \in \mathbb{N}$ independent, non-decreasing strong IDT subordinators $\tilde{H}^{(i)} = \{\tilde{H}_t^{(i)}\}_{t \geq 0}$, $i = 0, \dots, n$, and an independent iid sequence E_1, \dots, E_d of exponential random variables with unit mean. Moreover, let $A = (a_{i,j}) \in \mathbb{R}^{d \times (n+1)}$ be an arbitrary matrix with non-negative entries, having at least one positive entry per row. We define the vector-valued stochastic process*

$$\mathbf{H}_t = \begin{pmatrix} H_t^{(1)} \\ H_t^{(2)} \\ \vdots \\ H_t^{(d)} \end{pmatrix} := A \cdot \begin{pmatrix} \tilde{H}_t^{(0)} \\ \tilde{H}_t^{(1)} \\ \vdots \\ \tilde{H}_t^{(n)} \end{pmatrix} = \begin{pmatrix} a_{1,0} \tilde{H}_t^{(0)} + \dots + a_{1,n} \tilde{H}_t^{(n)} \\ a_{2,0} \tilde{H}_t^{(0)} + \dots + a_{2,n} \tilde{H}_t^{(n)} \\ \vdots \\ a_{d,0} \tilde{H}_t^{(0)} + \dots + a_{d,n} \tilde{H}_t^{(n)} \end{pmatrix},$$

whose component processes are all strong IDT subordinators. Then, the random vector (X_1, \dots, X_d) defined via

$$X_k := \inf\{t > 0 : H_t^{(k)} > E_k\}, \quad k = 1, \dots, d,$$

has an MSMVE law.

Proof. See [26, Lemma 4.4]. Note that by defining the entries of the matrix A appropriately and interpreting the processes $\tilde{H}^{(i)}$ as stochastic drivers, dedicated factor models can be constructed. \square

Besides Lévy subordinators there exist only few parametric families of (non-trivial) strong IDT subordinators and those are not very well investigated, yet. The aim of the present paper is to define new classes of MSMVE distributions by defining suitable parametric families of strong IDT subordinators in a first step.

2.3 A class of strong IDT subordinators

Consider an arbitrary Lévy subordinator $\{\Lambda_t\}_{t \geq 0}$, i.e. a stochastically continuous process with independent and stationary increments, which is almost surely càdlàg and non-decreasing, see, e.g., [37, Definition 1.6], where we assume that for every $\omega \in \Omega$, $\Lambda_t(\omega)$ is right-continuous in t , non-decreasing, and $\Lambda_0(\omega) = 0$ (see, e.g. [37, p. 197]). We consider instances of the general example $H_t := \int_0^\infty f(s/t) d\Lambda_s$, with f a function fulfilling certain conditions, see Lemma 2. This general example can be found, using slightly differing notation, e.g., in [29], and in [2] as an example of an infinitely temporally selfdecomposable process. The integral could be defined using the definition of integrals with respect to independently scattered random measures by [34] or [38]. However, as we restrict ourselves to pathwise non-decreasing processes, it is possible to use pathwise the usual Lebesgue–Stieltjes integral for expressions of the form $\int_0^\infty \dots d\Lambda_s$, see, e.g., [23, Example 1.56]. This coincides a.s. with the more complex integral definition in [34, 38], so we can apply their results on properties of the integral.

Lemma 2 (A class of strong IDT subordinators). *Defining pathwise*

$$H_t := \int_0^\infty f(s/t) d\Lambda_s, \quad t > 0, \tag{3}$$

with $H_0 := 0$ for f a measurable, non-negative, non-increasing, left-continuous function, $f \not\equiv 0$, fulfilling

$$\int_0^\infty \left\{ a_\Lambda f(s) + \int_0^\infty [1 \wedge (x f(s))] \nu_\Lambda(dx) \right\} ds < \infty,$$

$$\int_0^\infty \int_0^\infty [1 \wedge (x^2 f(s)^2)] \nu_\Lambda(dx) ds < \infty,$$

where a_Λ and ν_Λ are the drift and the Lévy measure of the subordinator Λ , yields a strong IDT subordinator $H = \{H_t\}_{t \geq 0}$ in the sense of [27].

Proof. It is possible to define the integral (3) pathwise as a Lebesgue–Stieltjes integral. However, one has to admit the value $+\infty$. The two integral conditions stated above are necessary and sufficient conditions for the existence of the integral (for $t = 1$) with respect to independently scattered random measures as stated in [34, Theorem 2.7]. One can show that the existence of that integral ensures the existence of the pathwise integral, and furthermore, the two definitions coincide a.s.. \square

The next two sections are devoted to a detailed investigation of two specific families, namely families based on $f_1(s) = (1 - s)_+$ and $f_2(s) = \log^+(1/s)$. We derive the related strong IDT subordinators and the resulting MSMVE distributions.

3 Family $F1$

We examine the construction of Lemma 2 using the function $f_1(s) = (1 - s)_+ := \max\{1 - s, 0\}$, which obviously fulfills the conditions stated in Lemma 2: As we can estimate $f_1(s) \leq \mathbb{1}_{[0,1]}(s)$, it follows that the first integral expression is bounded above by $a_\Lambda + \int_0^\infty (1 \wedge x) \nu_\Lambda(dx)$, and the second integral expression is bounded above by $\int_0^\infty (1 \wedge x^2) \nu_\Lambda(dx)$, which both are finite expressions for Lévy measures. We now consider the strong IDT subordinator

$$H_t := \int_0^\infty \left(1 - \frac{s}{t}\right)_+ d\Lambda_s = \int_0^t \left(1 - \frac{s}{t}\right) d\Lambda_s, \quad t > 0, \tag{4}$$

with $H_0 := 0$. The corresponding family of distributions resulting from construction (1) is called $F1$. The process H has an alternative representation using integration by parts, namely

$$H_t := \frac{1}{t} \int_0^t \Lambda_s ds, \quad t > 0, \tag{5}$$

which can be seen as some kind of moving average of the increasing process Λ . From Equation (4) it can be seen that pathwise, H_t equals a Williamson 2-transform evaluated at $1/t$, see, e.g., [43] for the definition of Williamson d -transforms. Consequently, $H_t = \psi(1/t)$ with ψ a (random) convex and non-increasing function. It can be seen from the representation in Equation (5) that H_t equals the product of a differentiable function and a function that is a.e. differentiable, a.s.. Consequently, the paths of H are a.e. differentiable, a.s.. Furthermore,

$$\begin{aligned} H_{t+x} - H_x &= \frac{1}{t+x} \int_x^{x+t} \Lambda_s ds - \frac{t}{x(x+t)} \int_0^x \Lambda_s ds \\ &= \frac{1}{t+x} \int_x^{x+t} (\Lambda_s - \Lambda_x) ds + \frac{t\Lambda_x}{x+t} - \frac{t}{x+t} H_x \\ &\stackrel{d}{=} \frac{1}{t+x} \int_0^t \tilde{\Lambda}_s ds + \frac{t}{x+t} (\Lambda_x - H_x) = \frac{t}{x+t} \tilde{H}_t + \frac{t}{x+t} (\Lambda_x - H_x), \end{aligned}$$

where $\tilde{\Lambda}$ is an independent copy of Λ and \tilde{H} the corresponding independent copy of H . Consequently, the increments, given the path of Λ up to time x , can be decomposed into a stochastic component independent of the previous evolution and a component measurable with respect to $\mathcal{F}_x := \sigma(\Lambda_s, 0 \leq s \leq x)$. However, as the value of Λ_x can not be recovered from H_x , H is not Markovian.

3.1 Attainable marginal distributions

In a first step, we analyze possible marginal distributions of H that can arise from this construction, as we have seen above that these are of relevance for the corresponding distribution. Let Ψ_Λ be the Laplace exponent of $\{\Lambda_t\}_{t \geq 0}$, ν_Λ the corresponding Lévy measure, and a_Λ its drift term. For the resulting strong IDT subordinator H , we denote its associated Bernstein function by Ψ_H with Lévy measure ν_H and drift a_H . Let Φ denote the considered integral transform, i.e. $\Phi : \mathcal{L}(\Lambda_1) \mapsto \mathcal{L}\left(\int_0^1 (1-s) d\Lambda_s\right)$, where $\mathcal{L}(\cdot)$ denotes the law of some random variable, which we will use simultaneously on the level of considered Lévy measures $\Phi : \nu_\Lambda \mapsto \nu_H$. Furthermore, define

$$U := \{ \nu \in \mathcal{M} : \nu(dx) = g(x)dx, g \text{ non-increasing} \}.$$

The corresponding class of distributions is called “Jurek class” or class of “s-selfdecomposable distributions”, see [22], restricted to distributions on \mathbb{R}_+ . It is shown in [22] that $\Phi : \mathcal{M} \rightarrow U$ is one-to-one, i.e. in particular every Bernstein function Ψ_H possessing a non-increasing density can be attained by the given construction.

Lemma 3 (Lévy measures associated with H). (i) $\Phi : \mathcal{M} \rightarrow U$ and the mapping is one-to-one.

(ii) The Lévy density g_H of the measure ν_H is given by

$$g_H(y) = \int_y^\infty \frac{\nu_\Lambda(dx)}{x}, \quad y > 0.$$

(iii) For any non-negative, measurable function h

$$\int_0^\infty h(x) \nu_H(dx) = \int_0^\infty \frac{1}{x} \left(\int_0^x h(y) dy \right) \nu_\Lambda(dx).$$

Proof. (i) is the result in [22, Theorem 2.6] restricted to distributions on \mathbb{R}_+ . We only prove (iii), as (ii) can be proven along the same lines (see, e.g., [4, Example 6.3 (1)]). For $H_1 = \int_0^1 (1-s) d\Lambda_s$, [34, Proposition 2.6] yields

$$\Psi_H(x) = \int_0^1 \Psi_\Lambda(x(1-s)) ds, \quad x \geq 0,$$

as $f_1(s) = (1-s)_+$ is obviously integrable. With [34, Theorem 2.7(iv)], one has

$$\nu_H(B) = \int_0^1 \nu_\Lambda(B/(1-s)) ds = \int_0^1 \nu_\Lambda(B/s) ds, \quad B \in \mathcal{B}(\mathbb{R}_+). \tag{6}$$

Thus, for any non-negative, measurable function h ,

$$\begin{aligned} \int_0^\infty h(x) \nu_H(dx) &= \int_0^1 \int_0^\infty h(xs) \nu_\Lambda(dx) ds = \int_0^\infty \int_0^1 h(xs) ds \nu_\Lambda(dx) \\ &= \int_0^\infty \frac{1}{x} \left(\int_0^x h(y) dy \right) \nu_\Lambda(dx), \end{aligned}$$

applying a substitution in the last step. This proves the claim. □

Furthermore, it can be seen from Equation (6) that Φ is a so-called “Upsilon transform” in the sense of [4], with dilation measure $\gamma(dx) = \mathbb{1}_{[0,1]}dx$, from which more results can be derived, e.g., on continuity properties of the transform Φ .

Actually, it is even possible to compute the pre-image $\Phi^{-1}(g(x)dx)$ explicitly, given a non-increasing Lévy density g . Using the previous results, we can conclude that for every Lévy subordinator with marginal distributions in the Jurek class, we can find another non-decreasing process with the same marginal distributions and a.e. differentiable paths, a.s..

3.2 The corresponding MSMVE family

It is known that for all $d \geq 2$, (X_1, \dots, X_d) constructed as in Equation (1) exhibits an MSMVE distribution which corresponds to a stable tail dependence function ℓ . In the given construction, ℓ can be computed explicitly. In particular, it is a function of the Bernstein function Ψ_H . This constitutes a very flexible class of stable tail dependence functions, since one can plug in any desired Bernstein function of the Jurek class.

Theorem 1 (Constructing parametric MSMVE distributions of family $F1$). *For every Bernstein function Ψ_H with drift a_H , Lévy measure $\nu_H \in U$, and $\Psi_H(1) = 1$, the function*

$$\ell(x_1, \dots, x_d) = \frac{d}{\sum_{j=1}^d 1/x_{(j)}} \Psi_H(d) - \sum_{i=1}^{d-1} \left(\frac{d-i+1}{\sum_{j=i}^d 1/x_{(j)}} - \frac{d-i}{\sum_{j=i+1}^d 1/x_{(j)}} \right) \Psi_H \left(d-i - \sum_{j=i+1}^d x_{(i)}/x_{(j)} \right) \quad (7)$$

is a stable tail dependence function for every $d \geq 2$. A random vector (X_1, \dots, X_d) with the respective MSMVE distribution can be constructed via

$$X_k := \inf \left\{ t > 0 : E_k < \int_0^t \left(1 - \frac{s}{t} \right) d\Lambda_s \right\}, \quad k = 1, \dots, d,$$

with $\Lambda = \{\Lambda_t\}_{t \geq 0}$ a Lévy subordinator with drift $a_\Lambda = 2 a_H$ and Lévy measure $\nu_\Lambda = \Phi^{-1}(\nu_H)$, and an iid sequence $\{E_k\}_{k \in \mathbb{N}}$ of unit exponential random variables independent of Λ .

Remark 1 (ℓ when some arguments are zero). *Actually, the expression for ℓ in Theorem 1 is only defined for values $x_1, \dots, x_d > 0$. However, since the construction yields $\ell(x_1, \dots, x_d) = -\log(\mathbb{E}[\exp(-H_{x_1} - \dots - H_{x_d})])$ and $H_0 = 0$, it is obvious that the case $x_i = 0$ for at least one $i \in \{1, \dots, d\}$ has a simple solution: for $I := \{i \in \{1, \dots, d\} : x_i = 0\}$ with $k := |I|$, one has $\ell(x_1, \dots, x_d) = \ell(x_{(1)}, \dots, x_{(d)}) = \ell(0, \dots, 0, x_{(k+1)}, \dots, x_{(d)}) = \ell(x_{(k+1)}, \dots, x_{(d)})$ and $\ell(0, \dots, 0) = 0$. The same observation holds true for Theorem 2 below.*

Proof (Theorem 1). See Appendix A. □

Possible parameterizations:

We present two examples. An example with a very simple form is based on the positive α -stable case.

Example 1. $\Psi_H(x) = x^\alpha$, $\alpha \in (0, 1)$, is attainable (respectively part of the Jurek class U) as $\nu_H(dx) = g(x) dx$, with

$$g(x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}, \quad x > 0,$$

see [39, p. 218], which is a decreasing density and consequently, $\nu_H \in U$. We can compute the density f of ν_Λ as $f(x) = (1-\alpha)g(x)$, which can be checked using Lemma 3(ii). Consequently, the associated Lévy subordinator Λ is an α -stable subordinator. The resulting bivariate Pickands dependence function is

$$A(t) = t(1-t)2^{\alpha+1} + (1-t)^{1-\alpha}(1-2t)^{1+\alpha}, \quad 0 < t \leq 0.5.$$

For $\alpha \in (0, 1)$, this interpolates between complete dependence and independence as can be seen in Figure 1. The lower tail dependence coefficient is given by $\lambda_L = 2 - 2^\alpha$. Though A appears to exhibit a kink at $t = 1/2$, we know from previous computations that it is indeed differentiable.

Another simple class of Bernstein functions is based on the compound Poisson distribution. We present one specific instance.

Example 2. $\Psi_H = (1 + a)x/(x + a)$, $a > 0$, is attainable as $\nu_H(dx) = g(x) dx$, with

$$g(x) = (1 + a)a e^{-ax}, \quad x > 0,$$

which corresponds to a compound Poisson process with intensity $(1 + a)$ and $\text{Exp}(a)$ -distributed jumps. The related Lévy process is a compound Poisson process with intensity $(1 + a)$ and $\Gamma(2, a)$ -distributed jumps, as can be seen from its Lévy measure $\nu_\Lambda(dx) = (1 + a)a^2 x \exp(-ax) dx$, which can again be checked using Lemma 3(ii) (a random variable is said to be $\Gamma(c, d)$ -distributed, $c, d > 0$, if its distribution exhibits a density f of the form $f(x) = d^c/\Gamma(c) x^{c-1} e^{-dx} \mathbb{1}_{\{x>0\}}$). The resulting bivariate Pickands dependence function is

$$A(t) = (1 + a)(1 - t) \left[\frac{4t}{2 + a} + \frac{(1 - 2t)^2}{1 - 2t + a(1 - t)} \right], \quad 0 < t \leq 0.5,$$

and the lower tail dependence coefficient is given by $\lambda_L = 2/(2 + a)$, i.e. every value in $(0, 1]$ is attainable.

The number of parametric families of attainable Bernstein functions is huge. For instance, [39, pp. 218–277] list more than one hundred so-called complete Bernstein functions (for a definition, see Section 4.1), and the family of complete Bernstein functions form a proper subclass of the attainable Bernstein functions. A small selection of interesting examples can be found in Table 1. In many cases, one can also compute the corresponding Lévy subordinator. If, e.g., H_1 is distributed according to a compound Poisson distribution, the corresponding Lévy subordinator is a compound Poisson process (CP) as well. Figure 1 visualizes different attainable shapes of A .

Table 1: Possible choices for Ψ_H together with their Lévy densities. One has to add a multiplicative positive factor to adjust for $\Psi_H(1) = 1$, which is omitted here. The restrictions $\alpha \in (0, 1)$, $a > 0$, $\beta > 0$, $\eta > 0$ have to hold. All examples are also complete Bernstein functions. For a list of more than one hundred complete Bernstein functions see [39, pp. 218–277].

Name	Ψ_H	g_H	type of Λ
Stable	x^α	$\alpha/\Gamma(1 - \alpha) x^{-1-\alpha}$	Stable
CP1	$x/(x + a)$	$a \exp(-ax)$	Compound Poisson
Gamma	$\log(1 + x/\beta)$	$\exp(-ax)/x$	Sum of independent CP1 and Gamma
IG	$\sqrt{2x + \eta^2} - \eta$	$\sqrt{0.5}/\Gamma(0.5) x^{-3/2} \exp(-\eta^2/2)$	Sum of independent CP and IG

4 Family F2

We examine the construction of Lemma 2 using $f_2(s) = \log^+(1/s) := \max\{\log(1/s), 0\}$. The corresponding family of distributions resulting from construction (1) is called family F2. This approach does not yield closed form solutions for arbitrary subordinators, but it allows to find a convenient expression for the corresponding stable tail dependence function such that tractable instances can be constructed easily. What is interesting in this context is that the process H itself has an alternative representation, using integration by parts, as shown in [3, Proposition 2.4], via

$$H_t = \int_0^t \frac{\Lambda_s}{s} ds = \lim_{u \searrow 0} \int_u^t \frac{\Lambda_s}{s} ds, \tag{8}$$

where the limit is a.s.. In the context of the construction in Equation (1), this can be interpreted as an intensity model with intensity $\lambda_s := \Lambda_s/s$, $s > 0$, with $\lambda_0 = a_\Lambda$. It is defined consistently, as $\lim_{s \searrow 0} \lambda_s = a_\Lambda$ a.s., see [37, p. 351]. This is a peculiar construction, as it follows, assuming the first and second moment of Λ to exist, that $\mathbb{E}[\lambda_s] = a_\Lambda + \int_0^\infty x \nu_\Lambda(dx)$ and $\text{Var}[\lambda_s] = 1/s (\int_0^\infty x^2 \nu_\Lambda(dx))$ for $s > 0$. Thus, the variance of the intensity is exploding close to 0 and is vanishing for large s .

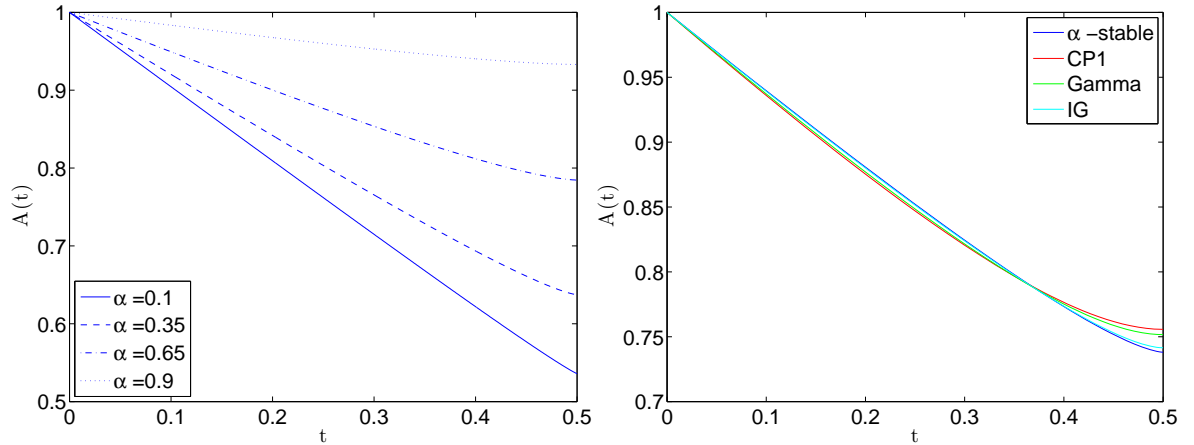


Figure 1: Left: Bivariate Pickands dependence functions for Example 1 ($\Psi_H(x) = x^\alpha$) and different values of α . Right: The bivariate Pickands dependence functions for different Ψ_H corresponding to the α -stable case, CP1, the Gamma case, and the IG case as defined in Table 1. The parameters in the example to the right are chosen such that all models exhibit a Spearman’s ρ of 0.5.

4.1 Attainable marginal distributions

From [3, Proposition 2.3] it follows that f_2 fulfills the integrability conditions in Lemma 2. Furthermore, the corresponding integral transform is well-known and thoroughly investigated in arbitrary dimensions, see [3]. We denote the transform restricted to distributions on \mathbb{R}_+ again by Φ , and define

$$BO := \{ \nu \in \mathcal{M} : \nu(dx) = g(x)dx, \text{ } g \text{ completely monotone} \},$$

the so-called “Bondesson” class, where (see, e.g., [39, Definition 1.3]) a function $g : (0, \infty) \rightarrow \mathbb{R}$ is completely monotone (c.m.) if $g \in C^\infty$ and

$$(-1)^n g^{(n)}(x) \geq 0, \quad \text{for all } n \in \mathbb{N} \cup \{0\}, x > 0.$$

The Bernstein functions corresponding to Lévy measures in BO are called complete Bernstein functions. [3] show that $\Phi(\mathcal{M}) = BO$ and that Φ is one-to-one. We provide a short proof of the first result as it is helpful for understanding the transform itself. It is based on a characterization of the class BO via complete Bernstein functions given in [39, Remark 6.4]: Ψ is the Laplace exponent of a distribution in BO if and only if it has a representation

$$\Psi(x) = ax + \int_0^\infty \frac{x}{x+t} \sigma(dt), \quad x \geq 0,$$

with σ the so-called Stieltjes measure on $(0, \infty)$, which satisfies $\int_0^\infty (1+t)^{-1} \sigma(dt) < \infty$.

Lemma 4 (Attainable marginal distributions using f_2). *Using f_2 , Ψ_H has a representation*

$$\Psi_H(x) = a_H x + \int_0^\infty \frac{x}{x+t} \sigma_H(dt), \quad x \geq 0,$$

with $a_H = a_\Lambda$ and $\sigma_H(B) := \int_0^\infty \mathbb{1}_B(1/u) \nu_\Lambda(du)$, $B \in \mathcal{B}(\mathbb{R})$. Therefore, $\Phi(\mathcal{M}) = BO$.

Proof. Using again [34, Proposition 2.6], it follows that

$$\Psi_H(x) = \int_0^\infty \Psi_\Lambda(x f_2(s)) ds, \quad x \geq 0.$$

Consequently

$$\begin{aligned} \Psi_H(x) &= \int_0^1 \Psi_\Lambda(x \log^+(1/s)) \, ds \\ &= a_\Lambda x \int_0^1 \log^+(1/s) \, ds + \int_0^1 \int_0^\infty 1 - \exp(-u x \log^+(1/s)) \, \nu_\Lambda(du) \, ds \\ &= a_\Lambda x + \int_0^\infty \int_0^1 1 - s^{ux} \, ds \, \nu_\Lambda(du) \\ &= a_\Lambda x + \int_0^\infty \frac{x}{x + 1/u} \, \nu_\Lambda(du) = a_\Lambda x + \int_0^\infty \frac{x}{x + t} \, \sigma_H(dt), \end{aligned}$$

with σ_H as defined above. Consequently, $\Phi(\mathcal{M}) \in BO$ follows from the observation that σ_H as defined above is a Stieltjes measure. $\Phi(\mathcal{M}) = BO$ follows from the observation that σ_H as defined above is a Stieltjes measure if and only if ν_Λ is a Lévy measure. This can be easily shown using basic inequalities. \square

Lemma 4 defines a direct connection between the characteristics of Ψ_H and Ψ_Λ as we can write the Stieltjes measure of H in terms of the Lévy measure of Λ . We will make use of this fact below.

4.2 The corresponding multivariate distribution

As we have seen in Lemma 4, for an arbitrary Stieltjes measures σ_H one can find a corresponding Lévy measure. We will use this fact and state the dependence function of the resulting multivariate distribution in terms of the Stieltjes measures, such that arbitrary Stieltjes measures can be plugged in. Notice that Remark 1 also applies to Theorem 2.

Theorem 2 (Constructing parametric MSMVE distributions of family F2). *For every complete Bernstein function Ψ_H with Stieltjes measure σ_H and drift a_H such that $\Psi_H(1) = 1$, the function*

$$\ell(x_1, \dots, x_d) = \sum_{i=1}^d x_{(i)} \left[a_H + \int_0^\infty \left(\prod_{j=i+1}^d \frac{x_{(j)}}{x_{(i)}} \right)^{1/s} \frac{s}{(s + d - i + 1)(s + d - i)} \sigma_H(ds) \right] \tag{9}$$

denotes a stable tail dependence function for every $d \geq 2$. A stochastic representation of an MSMVE distributions (X_1, \dots, X_d) with stable tail dependence function ℓ and unit exponential marginals is given by

$$X_k := \inf \left\{ t > 0 : E_k < \int_0^t \log \left(\frac{t}{s} \right) \, d\Lambda_s \right\},$$

with Λ a Lévy subordinator with drift $a_\Lambda = a_H$ and Lévy measure given by $\nu_\Lambda(B) := \int_0^\infty \mathbb{1}_B(1/u) \sigma_H(du)$, $B \in \mathcal{B}(\mathbb{R})$, and an iid sequence $\{E_k\}_{k \in \mathbb{N}}$ of unit exponential random variables independent of Λ .

Proof. See Appendix B. \square

At least two approaches are possible when looking for tractable specifications of family F2. As there exists a direct link between σ_H and ν_Λ , one can start from both sides. It is, for example, possible to start from σ_H corresponding to a desired Ψ_H and try to compute the expression in Theorem 2. [39] lists the Stieltjes measures for many of the known complete Bernstein functions. One could also start from a ν_Λ such that the Laplace transform of the measure $\nu_\Lambda(du)/(nu + 1)$ for $n \in \mathbb{N}$ is known in closed form. This can be seen from the third from last line of the computation in the proof of Theorem 2.

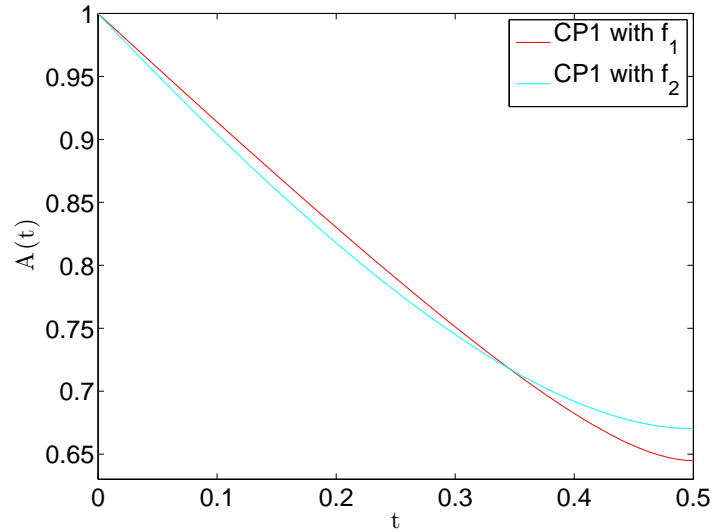


Figure 2: The bivariate Pickands dependence functions for the same Ψ_H corresponding to the CP1 case, where the approach using f_1 (Example 2) is compared to the result for f_2 (Example 3). The parameters are chosen such that both models exhibit a Spearman’s ρ of 0.75.

Possible parametrization:

We present one example starting from the Laplace exponent Ψ_H .

Example 3. $\Psi_H(x) = (1 + a)x/(x + a)$, $a > 0$ (called CP1 in Table 1) is attainable and corresponds to a compound Poisson distribution with intensity $(1 + a)$ and jump-size distribution $\text{Exp}(a)$. This coincides with Example 2 in the previous section, i.e. it is possible to construct a process H using f_2 which has the same marginal distributions as the process constructed in Example 2 using f_1 . Thus, the minima of subsets of the two different resulting MSMVE sequences have the same exponential distributions, though their multivariate distributions differ considerably. The corresponding Stieltjes measure is determined as $\sigma_H(ds) = (1 + a)\delta_a(s)$, where δ_a denotes the Dirac measure at a . It is easy to see that $\nu_\Lambda = \Phi^{-1}(\sigma_H)$ is given by $\nu_\Lambda(ds) = (1 + a)\delta_{1/a}(s)$, so Λ is a Poisson process with fixed jump-size $1/a$ and intensity $(1 + a)$. A closed-form solution for ℓ defined in Equation (9) is given by

$$\ell(x_1, \dots, x_d) = \sum_{i=1}^d x_{(i)} \left(\prod_{j=i+1}^d \frac{x_{(i)}}{x_{(j)}} \right)^{1/a} \frac{a(a+1)}{(a+d-i+1)(a+d-i)}.$$

The bivariate Pickands dependence function for $0 < t < 0.5$ can be stated as

$$A(t) = (1 - t) + t \left(\frac{t}{1 - t} \right)^{1/a} \frac{a}{a + 2}.$$

The dependence functions of this model and the one of Example 2 are compared in Figure 2. It can be observed that both approaches yield considerably different dependence functions.

5 A note on simulation

As mentioned before, the stochastic representation of (X_1, \dots, X_d) as an IDT-frailty model can be used to develop efficient simulation algorithms. When the involved Lévy subordinators are compound Poisson processes, simulating is straight-forward. Other Lévy subordinators can be approximated by compound Poisson processes or more involved schemes can be developed based on the given representation. We compare Example 2 of family F1 with Example 3 of family F2, which both yield CP1, i.e. $\Psi_H(x) = (1 + a)x/(x + a)$, $a > 0$,

as the desired (complete) Bernstein function for H . In Example 2, this corresponds to $\Lambda^{(1)}$ being a compound Poisson process with intensity $(1 + a)$ and $\Gamma(2, a)$ -distributed jumps, i.e.

$$H_t^{(1)} = \int_0^t \left(1 - \frac{s}{t}\right) d\Lambda_s^{(1)}, \quad t > 0,$$

has the desired Laplace exponent. For the family $F2$, as described in Example 3, this corresponds to $\Lambda^{(2)}$ being a Poisson process with deterministic jump-size $1/a$ and intensity $(1 + a)$, i.e.

$$H_t^{(2)} = \int_0^t \log\left(\frac{t}{s}\right) d\Lambda_s^{(2)}, \quad t > 0,$$

yields a second construction with the desired marginal distribution. Denoting by $\tau_i, i \in \mathbb{N}$, the jump times of a Poisson process with intensity $(1 + a)$, one can rewrite

$$H_t^{(1)} = \sum_{\tau_i \leq t} G_i \left(1 - \frac{\tau_i}{t}\right), \quad t \geq 0,$$

$$H_t^{(2)} = \frac{1}{a} \sum_{\tau_i \leq t} \log\left(\frac{t}{\tau_i}\right), \quad t \geq 0,$$

where $G_i, i \in \mathbb{N}$, are iid $\Gamma(2, a)$ -distributed. To illustrate the construction, sample paths are shown in Figure 3, where the same jump times are used to emphasize the differences of the resulting paths.

Based on these representations, it is clear how to sample from the construction in Equation (1). Exemplary scatterplots can be found in Figure 4, where we transformed the marginals to uniform distributions on $[0, 1]$ so that samples from the related extreme-value survival copulas are obtained for reasons of better comparability. Example 2 yields more samples close to the diagonal, which can be explained by the additional randomness introduced through the random variables G_i . High values of G_i correspond to a steep increase of $H^{(1)}$, which increases the probability of imminent triggering for both components within a short time period.

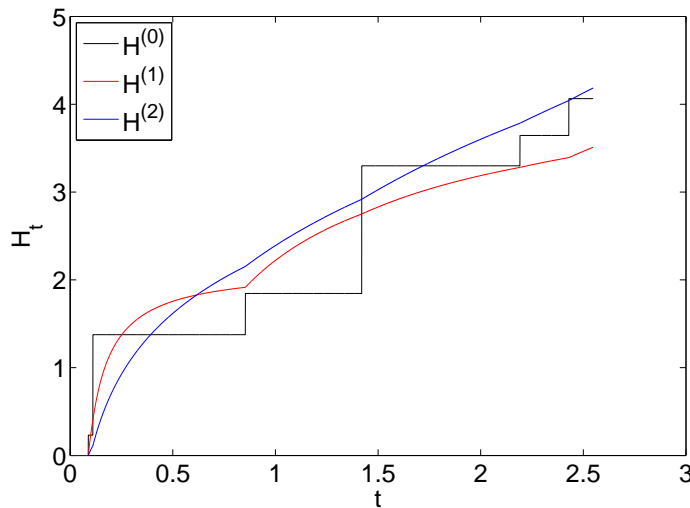
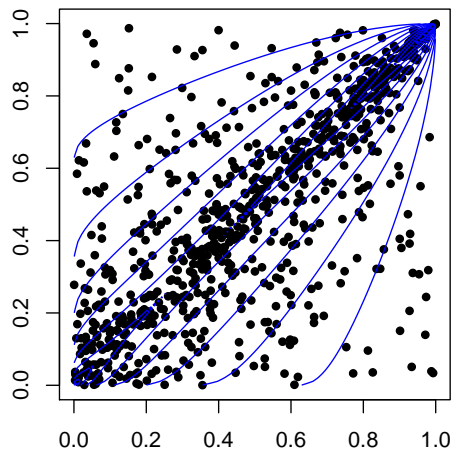


Figure 3: Simulated paths of the processes $H^{(1)}$ and $H^{(2)}$ where $a = 2$ is chosen. For comparison, a path of the simple compound Poisson process $H^{(0)}$ with $\text{Exp}(a)$ -distributed jumps is added, which has the same marginal distribution. For all processes, the same jump times are used.

Example 2: Scatterplot and density contours



Example 3: Scatterplot and density contours

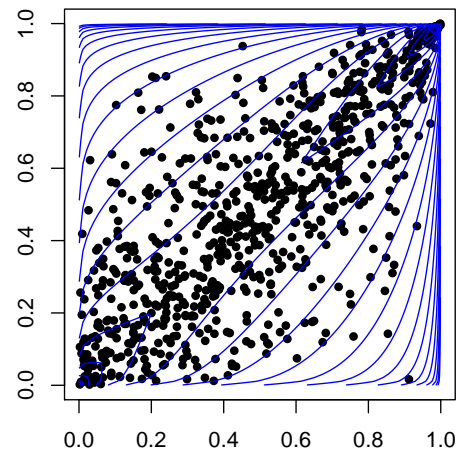


Figure 4: Scatterplots with 800 samples of the survival copulas of the MSMVE distributions generated by Example 2 and Example 3. The corresponding Pickands dependence functions are depicted in Figure 2 and the same parameters are chosen.

6 Concluding remarks

The present paper developed two new families of MSMVE distributions that give rise to many parametric models. The analysis conducted has shown that these new classes are quite flexible. Similar to [19], the aim of this paper was to introduce new classes of models, while an application of these in practice requires more detailed investigations of particular parametric models. One clear advantage of the presented models is the availability of concrete stochastic models allowing for efficient simulation even in large dimensions. Furthermore, applying these models in credit-portfolio modeling yields additional advantages.

Acknowledgments

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A Proof of Theorem 1

Theorem 1. It follows from Lemma 3 that there exists a Lévy subordinator Λ with drift $a_\Lambda = 2a_H$ and $\nu_\Lambda = \Phi^{-1}(\nu_H)$. Using this Λ in our construction, we observe

$$\begin{aligned} \mathbb{P}(X_1 > x_1, \dots, X_d > x_d) &= \exp(-\ell(x_1, \dots, x_d)) = \mathbb{E}[\exp(-H_{x_1} - \dots - H_{x_d})] \\ &= \mathbb{E} \left[\exp \left(- \sum_{i=1}^d \int_0^\infty \left(1 - \frac{s}{x_i}\right)_+ d\Lambda_s \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[\exp \left(- \int_0^\infty \sum_{i=1}^d \left(1 - \frac{s}{x_i} \right)_+ d\Lambda_s \right) \right] \\
 &= \exp \left(- \int_0^\infty \Psi_\Lambda \left(\sum_{i=1}^d \left(1 - \frac{s}{x_i} \right)_+ \right) ds \right),
 \end{aligned}$$

where in the last step, we use [34, Proposition 2.6] again. Following Remark 1, we only consider $x_1, \dots, x_d > 0$. Plugging Ψ_Λ 's representation into above expression and integrating the first summand, we find

$$\ell(x_1, \dots, x_d) = a_H \sum_{j=1}^d x_j + \int_0^{x_{(d)}} \int_0^\infty 1 - \exp \left(-u \sum_{i=1}^d \left(1 - \frac{s}{x_i} \right)_+ \right) \nu_\Lambda(du) ds. \tag{10}$$

We proceed with three helpful equalities:

$$-x_{(d)} = -\frac{1}{\sum_{j=1}^d \frac{1}{x_{(j)}}} - \sum_{i=1}^{d-1} \left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right), \tag{11}$$

which follows from a telescope argument applied to the right hand side. Furthermore,

$$x_{(d)} = \frac{d}{\sum_{j=1}^d \frac{1}{x_{(j)}}} - \sum_{i=1}^{d-1} \left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right) \left[\sum_{j=i+1}^d \frac{x_{(j)}}{x_{(j)}} - (d-i) \right], \tag{12}$$

as

$$\left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right) \left[\sum_{j=i+1}^d \frac{x_{(j)}}{x_{(j)}} - (d-i) \right] = \frac{d-i+1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{d-i}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}},$$

so again, we can use a telescope argument. Finally,

$$\sum_{j=1}^d x_j = \frac{d^2}{\sum_{j=1}^d \frac{1}{x_{(j)}}} - \sum_{i=1}^{d-1} \left(\frac{d-i+1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{d-i}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} \right) \left(d-i - \sum_{j=i+1}^d \frac{x_{(j)}}{x_{(j)}} \right), \tag{13}$$

which can be shown by rearranging

$$\left(\frac{d-i+1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{d-i}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} \right) \left(d-i - \sum_{j=i+1}^d \frac{x_{(j)}}{x_{(j)}} \right) = \frac{(d-i+1)^2}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{(d-i)^2}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - x_{(i)},$$

so again, we can use another telescope argument.

For the second term in Equation (10), we can compute, defining $x_{(0)} := 0$,

$$\begin{aligned}
 &\int_0^{x_{(d)}} \int_0^\infty 1 - \exp \left(-u \sum_{i=1}^d \left(1 - \frac{s}{x_i} \right)_+ \right) \nu_\Lambda(du) ds \\
 &= \int_0^\infty x_{(d)} - \sum_{i=1}^d \int_{x_{(i-1)}}^{x_{(i)}} \exp \left(-u \sum_{j=i}^d \left(1 - \frac{s}{x_{(j)}} \right) \right) ds \nu_\Lambda(du) \\
 &= \int_0^\infty x_{(d)} - \sum_{i=1}^d \frac{e^{-u(d-i+1)}}{u \sum_{j=i}^d \frac{1}{x_{(j)}}} \left(e^{u x_{(i)} \sum_{j=i}^d \frac{1}{x_{(j)}}} - e^{u x_{(i-1)} \sum_{j=i}^d \frac{1}{x_{(j)}}} \right) \nu_\Lambda(du) \\
 &= \int_0^\infty \frac{1}{u} \left[u x_{(d)} - x_{(d)} + \frac{e^{-u d}}{\sum_{j=1}^d \frac{1}{x_{(j)}}} \right] \nu_\Lambda(du)
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{d-1} e^u \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right) \Big] v_{\Lambda}(du) \\
\stackrel{(11)}{=} & \int_0^{\infty} \frac{1}{u} \left\{ \int_0^u x_{(d)} - \frac{d e^{-s d}}{\sum_{j=1}^d \frac{1}{x_{(j)}}} + \sum_{i=1}^{d-1} e^s \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \right. \\
& \times \left. \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right) ds \right\} v_{\Lambda}(du) \\
= & \int_0^{\infty} x_{(d)} - \frac{d e^{-u d}}{\sum_{j=1}^d \frac{1}{x_{(j)}}} + \sum_{i=1}^{d-1} e^u \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \\
& \times \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \left(\frac{1}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} - \frac{1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} \right) v_H(du)
\end{aligned}$$

where in the last step, we used Lemma 3(iii),

$$\begin{aligned}
\stackrel{(12)}{=} & \int_0^{\infty} \frac{d}{\sum_{j=1}^d \frac{1}{x_{(j)}}} (1 - e^{-d u}) - \sum_{i=1}^{d-1} \left(\frac{d-i+1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{d-i}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} \right) \\
& \times \left(1 - e^u \left[\sum_{j=i+1}^d \frac{x_{(j)} - (d-i)}{x_{(j)}} \right] \right) v_H(du).
\end{aligned}$$

From Equation (10) follows that one has to add the term $a_H \sum_{j=1}^d x_j$ to the above expression in order to compute ℓ . According to Equation (13), $a_H \sum_{j=1}^d x_j$ can be rewritten as

$$a_H \sum_{j=1}^d x_j = \frac{d}{\sum_{j=1}^d \frac{1}{x_{(j)}}} a_H d - \sum_{i=1}^{d-1} \left(\frac{d-i+1}{\sum_{j=i}^d \frac{1}{x_{(j)}}} - \frac{d-i}{\sum_{j=i+1}^d \frac{1}{x_{(j)}}} \right) a_H \left(d - i - \sum_{j=i+1}^d \frac{x_{(j)}}{x_{(j)}} \right).$$

Resorting all terms in Equation (10), we recover the corresponding expressions for Ψ_H and the claim follows. \square

B Proof of Theorem 2

Theorem 2. From Lemma 4 we know that there exists a Lévy subordinator Λ with drift $a_{\Lambda} = a_H$ and Lévy measure ν_{Λ} . Using this Λ in our frailty construction, we obtain as in Theorem 1

$$\begin{aligned}
\ell(x_1, \dots, x_d) &= \int_0^{x_{(d)}} \Psi_{\Lambda} \left(\sum_{i=1}^d \log^+ \left(\frac{x_i}{s} \right) \right) ds, \\
&= a_H \sum_{j=1}^d x_j + \int_0^{x_{(d)}} \int_0^{\infty} 1 - \exp \left(-u \left(\sum_{i=1}^d \log^+ \left(\frac{x_i}{s} \right) \right) \right) v_{\Lambda}(du) ds,
\end{aligned}$$

where following Remark 1, we consider $x_1, \dots, x_d > 0$. For the second term, we rewrite

$$\begin{aligned}
& \int_0^{\infty} \int_0^{x_{(d)}} 1 - \exp \left(-u \left(\sum_{i=1}^d \log^+ \left(\frac{x_i}{s} \right) \right) \right) ds v_{\Lambda}(du) \\
= & \int_0^{\infty} x_{(d)} - \sum_{i=1}^d \int_{x_{(i-1)}}^{x_{(i)}} \exp \left(-u \left(\sum_{j=i}^d \log \left(\frac{x_{(j)}}{s} \right) \right) \right) ds v_{\Lambda}(du)
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty x_{(d)} - \sum_{i=1}^d \int_{x_{(i-1)}}^{x_{(i)}} \left(\frac{\prod_{j=i}^d x_{(j)}}{s^{d-i+1}} \right)^{-u} ds v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} - \sum_{i=1}^d \left(\prod_{j=i}^d x_{(j)} \right)^{-u} \int_{x_{(i-1)}}^{x_{(i)}} s^{u(d-i+1)} ds v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} - \sum_{i=1}^d \left(\prod_{j=i}^d x_{(j)} \right)^{-u} \frac{x_{(i)}^{u(d-i+1)+1} - x_{(i-1)}^{u(d-i+1)+1}}{u(d-i+1)+1} v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} \frac{u}{u+1} \\
&\quad - \sum_{i=1}^{d-1} \left[\left(\prod_{j=i}^d x_{(j)} \right)^{-u} \frac{x_{(i)}^{u(d-i+1)+1}}{u(d-i+1)+1} - \left(\prod_{j=i+1}^d x_{(j)} \right)^{-u} \frac{x_{(i)}^{u[d-(i+1)+1]+1}}{u[d-(i+1)+1]+1} \right] v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} \frac{u}{u+1} - \sum_{i=1}^{d-1} \left(\prod_{j=i+1}^d x_{(j)} \right)^{-u} x_{(i)}^{u(d-i)+1} \left(\frac{1}{u(d-i+1)+1} - \frac{1}{u(d-i)+1} \right) v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} \frac{u}{u+1} - \sum_{i=1}^{d-1} x_{(i)} \left(\prod_{j=i+1}^d \frac{x_{(j)}}{x_{(i)}} \right)^{-u} \left(\frac{1}{u(d-i+1)+1} - \frac{1}{u(d-i)+1} \right) v_\Lambda(du) \\
&= \int_0^\infty x_{(d)} \frac{1}{1+s} - \sum_{i=1}^{d-1} x_{(i)} \left(\prod_{j=i+1}^d \frac{x_{(j)}}{x_{(i)}} \right)^{-1/s} \left(\frac{s}{(d-i+1)+s} - \frac{s}{(d-i)+s} \right) \sigma_H(ds) \\
&= \sum_{i=1}^d x_{(i)} \int_0^\infty \left(\prod_{j=i+1}^d \frac{x_{(j)}}{x_{(i)}} \right)^{1/s} \frac{s}{(s+d-i+1)(s+d-i)} \sigma_H(ds).
\end{aligned}$$

The claim follows. □

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