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Scattering theory of quantum systems with infinitely many degrees of freedom

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Preface

Since the early days of Quantum Mechanics, scattering theory has been a central tool for comparison of theory with experiment. Mathematical foundations of scattering theory were laid by giants of Mathematical Physics such as L.D. Faddeev or T. Kato, as discussed, e.g., in [Si18, De18]. The problem of *asymptotic completeness*, i.e., the question of particle interpretation of all states in the physical Hilbert space of the theory, emerged as one of the main problems of the mathematical scattering theory. Its solution in N-body Quantum Mechanics for particles with quadratic dispersion relations, interacting with possibly long-range forces, is an impressive chapter of the 20th century Mathematical Physics whose milestones are [En78, SiSo87, Gr90, De93]. However, if the assumption of quadratic dispersion relations is dropped, even in Quantum Mechanics the problem of complete particle interpretation is largely open beyond the two-body scattering. It is therefore not a surprise that in quantum field theory (QFT) or for quantum spin systems, where basic excitations typically have non-quadratic dispersion relations, asymptotic completeness is rather poorly understood. An additional problem for quantum systems with infinitely many degrees of freedom is a possible breakdown of the Stone-von Neumann uniqueness theorem and the resulting multitude of 'charged sectors'. This aspect undermines the conventional property of asymptotic completeness, inherited from Quantum Mechanics, and calls for more suitable concepts. It is particularly severe in the presence of electric charges and massless particles (photons) which is a one aspect of the *infrared problem.*

This work is a summary of nine papers, listed below, which form the author's habilitation project. This project focuses on scattering in relativistic and non-relativistic quantum field theories (QFT) and in quantum spin systems. The central questions include:

1. Rigorous construction of scattering states.
2. Consistent treatment of electrically charged particles (*the infrared problem*).
3. The problem of complete particle interpretation of arbitrary physical states (*asymptotic completeness*).

To make progress on these questions, methods from the theory of operator algebras, partial differential equations and spectral theory of operators on Hilbert and Banach spaces are used and further developed.

The following three core publications form the cumulative habilitation thesis. They are within the scope of objectives 1, 2, 3 above, respectively.

[BDN16] S. Bachmann, W. Dybalski and P. Naaijkens *Lieb-Robinson bounds, Arveson spectrum and Haag-Ruelle scattering theory for gapped quantum spin systems.* Ann. Henri Poincaré **17**, (2016) 1737–1791. (AHP Prize 2016).

[AD17] S. Alazzawi and W. Dybalski. *Compton scattering in the Buchholz-Roberts framework of relativistic QED.* Lett. Math. Phys. **107**, (2017) 81–106.

[DM15] W. Dybalski and J.S. Møller. *The translation invariant massive Nelson model III. Asymptotic completeness below the two-boson threshold.* Ann. Henri Poincaré **16**, (2015) 2603–2693.

Other publications within this habilitation project are listed below:

[Dy18] W. Dybalski. *Asymptotic observables in gapped quantum spin systems.* Commun. Math. Phys. **357**, (2018) 231–248.

[DG14] W. Dybalski and C. Gérard. *Towards asymptotic completeness of two-particle scattering in local relativistic QFT.* Commun. Math. Phys. **326**, (2014) 81–109.

[DG14.1] W. Dybalski and C. Gérard. *A criterion for asymptotic completeness in local relativistic QFT.* Commun. Math. Phys. **332**, (2014) 1167–1202.

[DT13] W. Dybalski and Y. Tanimoto. *Asymptotic completeness for infraparticles in two-dimensional conformal field theory.* Lett. Math. Phys. **103**, (2013) 1223–1241.

[DT12] W. Dybalski and Y. Tanimoto. *Infraparticles with superselected direction of motion in two-dimensional conformal field theory.* Commun. Math. Phys. **311**, (2012) 457–490.

[DT11] W. Dybalski and Y. Tanimoto. *Asymptotic completeness in a class of massless relativistic quantum field theories.* Commun. Math. Phys. **305**, (2011) 427–440.

In Section 1 of this summary we introduce the main concepts and methods of scattering theory in the case of quantum-mechanical potential scattering. In Sections 2-5 we summarize the content of the papers listed above, as indicated in the following table of contents.

Contents

1 Scattering in quantum mechanics

The main goal of this section is to summarize the basics of quantum mechanical scattering, presented in standard textbooks [RS3, DG]. In the last paragraph we will also explain in this simple setting an argument from [Dy18], which will be later used in Subsection 3.3.

We consider one quantum mechanical particle in an external potential. Its space of states is the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3)$ of square-integrable functions and its time evolution is governed by the Schrödinger equation

$$\mathrm{i}\partial_t \Psi_t = H\Psi_t, \quad H = -\frac{1}{2}\Delta + V(x), \tag{1.1}$$

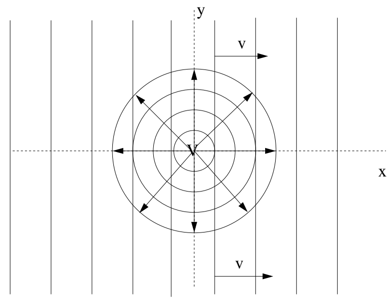


Figure 1.1. Collision of a plane wave with an obstacle.

where $\Delta := \partial_{x_1}^2 + \partial_{x_1}^2 + \partial_{x_3}^2$ is the Laplacian on $\mathbb{R}^3$, the real valued function V is the interaction potential and H is called the Hamiltonian. For a given initial condition Ψ_0, the solution to (1.1) has the form

$$\Psi_t = \mathrm{e}^{-\mathrm{i}Ht}\Psi_0. \tag{1.2}$$

Figure 1.1 schematically illustrates a possible time evolution. Here the initial condition is an incident plane wave[1] and the potential is depicted as an obstacle. For large times after the collision we expect in addition an approximately spherical scattered wave. Thus long after the collision the solution looks like a superposition of a plane wave and a spherical wave, which are both solutions of the Schrödinger equation with the free Hamiltonian $H_0 = -\Delta$. The message of Figure 1.1 is therefore that there exist states Ψ^{out} of the particle in potential V which for large times evolve like states Ψ of the free theory. In other words

$$\lim_{t\to\infty} \|\mathrm{e}^{-\mathrm{i}tH}\Psi^{\mathrm{out}} - \mathrm{e}^{-\mathrm{i}tH_0}\Psi\| = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \|\Psi^{\mathrm{out}} - \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\Psi\| = 0. \tag{1.3}$$

The states Ψ^{out} are called the **scattering states** and they form the range of the **wave operator** which is defined as

$$W^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}, \tag{1.4}$$

where the strong limit is understood. Even if the limit exist, it is not automatically a unitary operator, it may be merely an isometry. The existence of the limits above is usually proven using the **Cook's method:** Let $\Psi_t := \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\Psi$ be the approximating sequence of a scattering state. Suppose we can show that

$$\|\partial_t\Psi_t\| = \|\mathrm{e}^{\mathrm{i}tH}V\mathrm{e}^{-\mathrm{i}tH_0}\Psi\| \tag{1.5}$$

is integrable in t. Then the limit exists as it can be expressed as a convergent integral

$$\lim_{t\to\infty}\Psi_t = \int_{t_0}^{\infty}(\partial_\tau\Psi_\tau)d\tau + \Psi_{t_0}, \tag{1.6}$$

for any $t_0 \geq 0$. It is plausible from the r.h.s. of (1.5), that integrability of $\|\partial_t\Psi_t\|$ depends on decay properties of the potential. For short-range potentials, which decay faster than $|x|^{-1}$ for large $|x|$, the above argument gives the existence of the wave operator (1.4). However, in the physically important case of the Coulomb potential, with $|x|^{-1}$ decay, the strong limit in (1.4) does not exist. This is a simple example of an **infrared problem**, whose solution is well known: The construction of the

[1]Strictly speaking a plane wave is not in $L^2(R^3)$. But this point can be left aside in this introductory discussion.

wave operator must be refined, e.g., with the help of the Dollard prescription [Do64]. Instead of comparing the interacting time-evolution with the free evolution as in (1.4), we introduce a more refined comparison dynamics: It is governed by the asymptotic Hamiltonian

$$H_{\mathrm{as}}(t) := H_0 + V(-\mathrm{i}\nabla_x t), \tag{1.7}$$

where we evaluated the potential at the ballistic trajectory of the particle, which we anticipate for large t. As this Hamiltonian is time-dependent, the solution of the Schrödinger equation (1.1) is more complicated than in (1.2), namely

$$\Psi_t = \mathrm{e}^{-\mathrm{i}\int_0^t H_{\mathrm{as}}(\tau)d\tau}\Psi_0. \tag{1.8}$$

The existence of the Dollard wave operator, which has the form

$$W_{\mathrm{D}}^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}\int_0^t H_{\mathrm{as}}(\tau)d\tau}, \tag{1.9}$$

can then be shown using the Cook method.

Given the existence of the wave operators, the next problem is **asymptotic completeness**. We say that a quantum mechanical theory given by $H = -\frac{1}{2}\Delta + V(x)$ is asymptotically complete if scattering states and bound states[2] of H span the entire Hilbert space. As a bound state situation corresponds to the particle confined by the potential, the problem of proving asymptotic completeness consists in excluding 'fuzzy' configurations in which the particle in neither confined by the potential nor scattered. In other words, no matter how far it may travel, there is always a substantial probability that it will come back [En78]. Heuristically, such configurations correspond to the singular-continuous spectrum of H.

In the absence of bound states, asymptotic completeness amounts to unitarity of the wave operator (1.4) (resp. (1.9)), which may get lost in the limit $t \to \infty$. Roughly, the idea of the proof is to construct an inverse of W^{out} with the help of suitable **asymptotic observables**. The simplest and most natural asymptotic observable is the asymptotic velocity, which is given by the following strong limit

$$f^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} f(x/t)\mathrm{e}^{-\mathrm{i}tH}. \tag{1.10}$$

Let us try to prove the existence of this limit using the Cook's method. We set $f_t := \mathrm{e}^{\mathrm{i}tH} f(x/t)\mathrm{e}^{-\mathrm{i}tH}$ and compute the time derivative. This gives

$$\partial_t f_t = \mathrm{e}^{\mathrm{i}tH}\mathbf{D}f(x/t)\mathrm{e}^{-\mathrm{i}tH}, \text{ where } \mathbf{D}f(x/t) := \partial_t f(x/t) + \mathrm{i}[H, f(x/t)] \tag{1.11}$$

is the Heisenberg derivative of $t \mapsto f(x/t)$. By a straightforward computation one obtains

$$\mathbf{D}f(x/t) = -\frac{1}{t}(\nabla f)(x/t)\big((x/t) - (-\mathrm{i}\nabla_x)\big) + O(t^{-2}), \tag{1.12}$$

where the error term $O(t^{-2})$ decays in norm as t^{-2} thus is integrable. However, the leading term on the r.h.s. (1.12) exhibits only t^{-1} decay, thus we cannot apply the Cook's method directly. On the other hand, alluding to the expected ballistic motion of the particle, we expect that the average velocity x/t tends to the instantaneous velocity $-\mathrm{i}\nabla_x$ along the time evolution. This additional ingredient, called a propagation estimate, has the following form

$$\int_1^\infty \frac{dt}{t}\|(\nabla f)(x/t)\big((x/t) - (-\mathrm{i}\nabla_x)\big)\mathrm{e}^{-\mathrm{i}tH}\Psi\|^2 < \infty, \tag{1.13}$$

[2]Bound states are eigenvectors of H.

for $\Psi \in \mathfrak{H}$ of bounded energy, and allows to conclude the proof of existence of the asymptotic velocity (1.10).

Instead of discussing the particular estimate (1.13), let us recall the general idea of the **method of propagation estimates** [SiSo87]. Suppose we want to prove

$$\int_1^\infty dt \langle \Psi, \mathrm{e}^{\mathrm{i}tH} a(t) \mathrm{e}^{-\mathrm{i}tH} \Psi \rangle < \infty \tag{1.14}$$

for some **propagation observable** $t \mapsto a(t)$ s.t. $a(t) \geq 0$. The idea is to find a new propagation observable $t \mapsto b(t)$ s.t. $\sup_{t\in\mathbb{R}} \|b(t)\| < \infty$ and

$$a(t) \leq \mathbf{D}b(t), \tag{1.15}$$

possibly up to norm-integrable rest terms. By integrating both sides of this inequality along the time evolution, we easily obtain

$$\int_1^\infty dt \langle \Psi, \mathrm{e}^{\mathrm{i}tH} a(t) \mathrm{e}^{-\mathrm{i}tH} \Psi \rangle \leq 2 \sup_{t\in\mathbb{R}} \|b(t)\| + C \tag{1.16}$$

and thus the desired bound (1.14) holds.

An obvious difficulty with the method of propagation estimates consists in guessing a propagation observable b such that the inequality (1.15) holds. In some cases the required lower bound on the Heisenberg derivative (1.11) is provided by a **Mourre estimate** [Mo81]. In the present context it has the form

$$\mathbf{1}_{\mathcal{J}}(H)\mathrm{i}[H, A]\mathbf{1}_{\mathcal{J}}(H) \geq c\mathbf{1}_{\mathcal{J}}(H), \tag{1.17}$$

where $c > 0$, $A := \frac{1}{2}\{x \cdot (-\mathrm{i}\nabla_x) + \mathrm{h.c.}\}$ is the dilation operator and $\mathbf{1}_{\mathcal{J}}$ is the characteristic function of a suitable set $\mathcal{J}$ which does not contain the point spectrum of H. Another consequence of the Mourre estimate is the absence of the singular continuous spectrum.

To conclude this section we would like to point out that the method of propagation estimates is not the only available strategy for proving the existence of asymptotic observables. In [Dy18] the present author found an alternative argument, which will be used in Subsection 3.3. We restrict attention to short-range potentials and want to control the strong convergence of an asymptotic observable of the form

$$h_t := \mathrm{e}^{\mathrm{i}tH}\, \tilde{\chi} h(x/t) \tilde{\chi}\, \mathrm{e}^{-\mathrm{i}tH}. \tag{1.18}$$

Here $\chi, h \in C_0^\infty(\mathbb{R}^3)$, $\tilde{\chi} := \chi(-\mathrm{i}\nabla_x)$, $0 \notin \mathrm{supp}\, h$ and for technical reasons we also require that $\mathrm{supp}\chi \subset \mathrm{supp}\, h$ and $\mathrm{supp}\, h$ is a convex set. We start from the simple observation that for $\Psi \in \mathrm{Ran}\, W^{\mathrm{out}}$ the limit $\lim_{t\to\infty} h_t\psi$ is readily computed. Thus it suffices to consider $\Psi \in (\mathrm{Ran}\, W^{\mathrm{out}})^\perp$ and for such vectors we can write

$$\begin{aligned}
h_t\Psi &= \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\, \tilde{\chi} h(x/t + (-\mathrm{i}\nabla_x))\tilde{\chi}\, \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\Psi \\
&= \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\, \tilde{\chi} h(x/t + (-\mathrm{i}\nabla_x))\tilde{\chi}\, (W_t^* - (W^{\mathrm{out}})^*)\Psi \\
&= \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\, \tilde{\chi} h(x/t + (-\mathrm{i}\nabla_x))\tilde{\chi} \int_t^\infty ds\, (-\partial_s W_s^*)\Psi \\
&= \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\, \tilde{\chi} h(x/t + (-\mathrm{i}\nabla_x))\tilde{\chi} \int_t^\infty ds\, \mathrm{i}\mathrm{e}^{\mathrm{i}sH_0}V(x)\mathrm{e}^{-\mathrm{i}sH}\Psi,
\end{aligned} \tag{1.19}$$

where in the first step we used $h(x/t) = \mathrm{e}^{-\mathrm{i}tH_0}h(x/t + (-\mathrm{i}\nabla_x))\mathrm{e}^{\mathrm{i}tH_0}$. In the second step we applied $\Psi \in (\mathrm{Ran}\, W^{\mathrm{out}})^\perp$ which gives $(W^{\mathrm{out}})^*\Psi = 0$, and denoted by W_t the approximating sequence of the wave operator. The final expression tends to zero due to the decay of V and support properties of h, χ. Thus we obtain convergence of (1.18).

2 Asymptotic completeness in non-relativistic QFT [DM15]

Models of non-relativistic QFT (NRQFT) considered here describe massive quantum-mechanical particles ('electrons') coupled to second-quantized Bose fields, whose excitations will be called 'bosons'. Depending on the physical context, the bosons have the interpretation of photons or phonons. The models of NRQFT can generally be divided into *confined* and *translation invariant*[3]. In the former case the electrons are held in an external potential, while in the latter case only interparticle interactions are present. Since the present summary is situated at the interface between the relativistic and non-relativistic QFT, we will focus on translation invariant models in the discussion below. However, at times we will refer to confined models, whose spectral and scattering theory is better understood [DG99, BFS98.2, FGS02, Sp97, DGK13, FS12, FS12.1].

2.1 Nelson and polaron model and their energy-momentum spectrum

The Hilbert space of models of NRQFT discussed here has the structure $\mathcal{H} = \mathfrak{H} \otimes \Gamma(\mathfrak{h})$, where $\mathfrak{H} = L^2(\mathbb{R}^3)$ is the single-electron space, $\mathfrak{h} = L^2(\mathbb{R}^3)$ is the single-boson space and $\Gamma(\mathfrak{h})$ is the corresponding symmetric Fock space. A prominent class of examples of NRQFT is given by Hamiltonians of the form [Ne64, Fr73, Fr74]:

$$H = \hat{\Omega}(-\mathrm{i}\nabla_x) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega) + \lambda \int_{\mathbb{R}^3} dk\, G(k)\big(\mathrm{e}^{-\mathrm{i}kx} \otimes a^*(k) + \mathrm{e}^{\mathrm{i}kx} \otimes a(k)\big), \tag{2.1}$$

where $\hat{\Omega}, \omega$ are the dispersion relations of the (bare) electron and boson, $a^*(k), a(k)$ denote the creation and annihilation operators of bosons, and G is the form-factor which controls the interaction between the electrons and bosons. Moreover, the Hamiltonian of non-interacting bosons is given by the second quantisation operator $\mathrm{d}\Gamma(\omega) = \int_{\mathbb{R}^3} dk\, \omega(k)a^*(k)a(k)$. As for ω and G, there are two important choices: First, let $\omega(k) = \sqrt{k^2 + m_{\mathrm{f}}^2}$, $\hat{\Omega}(p) = p^2/(2m_{\mathrm{e}})$, $m_{\mathrm{e}} > 0$, $G(k) = g(k)/\sqrt{2\omega(k)}$. In this case H_{nr} is the *Nelson model*, which is a toy-model of quantum electrodynamics. We say that the Nelson model is massive (resp. massless) if $m_{\mathrm{f}} > 0$ (resp. $m_{\mathrm{f}} = 0$). The function g above implements the ultraviolet cut-off, which is usually kept fixed in NRQFT (see, however, [Ne64, BDP12, Mi12]). Second, let $\omega(k) = \text{const}$, $\hat{\Omega}(p) = p^2/(2m_{\mathrm{e}})$, $G(k) = g(k)/|k|$, $g \in S(\mathbb{R}^3)$. In this case H is the Fröhlich *polaron model* which is a physically relevant effective theory of electrons interacting with optical phonons in a dielectric crystal.

By translational invariance, the Hamiltonian (2.1) commutes with the total momentum operators $P := (-\mathrm{i}\nabla_x)\otimes 1 + 1 \otimes \mathrm{d}\Gamma(k)$, where $\mathrm{d}\Gamma(k^i) := \int_{\mathbb{R}^3} dk\, k^i a^*(k)a(k)$, $i = 1, 2, 3$, are the boson momentum operators. The joint spectral measure of the energy-momentum operators (H, P) is denoted by $E(\,\cdot\,)$. Furthermore, the Hamiltonian H has a decomposition into fiber Hamiltonians $\{H(\xi)\}_{\xi\in\mathbb{R}^3}$ at fixed momentum ξ, which are concrete operators on $\Gamma(\mathfrak{h})$, i.e.,

$$H = I^* \int_{\mathbb{R}^3}^{\oplus} d\xi\, H(\xi) I, \tag{2.2}$$

where $I : \mathcal{H} \to L^2(\mathbb{R}^3; \Gamma(\mathfrak{h}))$ is a suitable unitary map.

Let us now move on to spectral theory of the model, whose analysis was initiated in [Fr73, Fr74]. In the case of massive bosons, on which we focus here, the lower part of the spectrum of $H(\xi)$ is well understood for arbitrary values of the coupling constant λ [Mo05] (see Figure 2.1). The lower boundary of the spectrum is denoted $\Sigma^{(0)}(\xi)$ and gives the (renormalized) dispersion relation of the physical electron in its ground state. The one- and two-boson thresholds are defined by

$$\Sigma^{(1)}(\xi) := \inf_k \{\, \Sigma^{(0)}(\xi - k) + \omega(k)\,\}, \quad \Sigma^{(2)}(\xi) := \inf_{k_1,k_2} \{\, \Sigma^{(0)}(\xi - k_1 - k_2) + \omega(k_1) + \omega(k_2)\,\} \tag{2.3}$$

[3]The intermediate case of models admitting ionization will not be treated here.

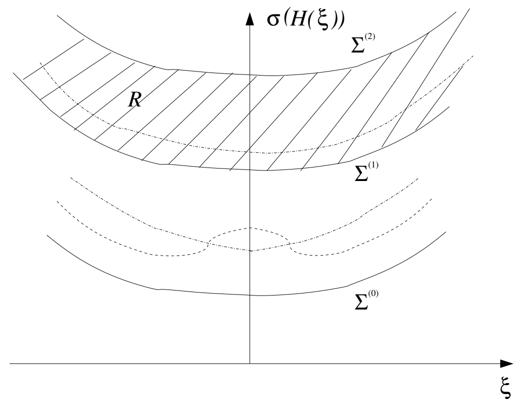


Figure 2.1. Schematic shape of the spectrum of the Nelson model.

and they define the lowest energies at which one electron and one (resp. two) bosons can coexist. $\Sigma^{(1)}(\xi)$ is the lower boundary of the essential spectrum of $H(\xi)$ and both below and above of the one-boson thresholds there may be additional branches of eigenvalues (mass-shells) corresponding to excited states of the physical electron. Together with the ground state mass-shell they constitute the pure point part of the spectrum Σ_{pp}. Furthermore, in the region R below the two-boson threshold a Mourre estimate is available [MR12]. Namely, for any $(\xi_0, E_0) \in R$ in the essential spectrum (outside of some sets of measure zero) there exist a neighbourhood N_0 of ξ_0, a neighbourhood $\mathcal{J}_0$ of E_0, and a constant $c_{\mathrm{m}} > 0$ s.t. for any $\xi \in N_0$

$$\mathbf{1}_{\mathcal{J}_0}(H(\xi))\mathrm{i}[H(\xi), \mathrm{d}\Gamma(a_{\xi_0})]\mathbf{1}_{\mathcal{J}_0}(H(\xi)) \geq c_{\mathrm{m}}\mathbf{1}_{\mathcal{J}_0}(H(\xi)). \tag{2.4}$$

Here $\mathbf{1}_{\mathcal{J}_0}$ denotes the characteristic function of $\mathcal{J}_0$, the operator $a_{\xi_0} := v_{\xi_0} \cdot \mathrm{i}\nabla_k + \mathrm{i}\nabla_k \cdot v_{\xi_0}$ is defined with the help of a suitable vector field v_{ξ_0} and $\mathrm{d}\Gamma(a_{\xi_0})$ is its second quantisation acting on $\Gamma(\mathfrak{h})$. If there is only one mass-shell of the electron in the energy-momentum spectrum, v_{ξ_0} is the unit vector in the direction of the relative velocity $\nabla\Sigma^{(0)}(\xi_0 - k) - \nabla\omega(k)$; in general it is given by a more complicated expression. A standard consequence of a Mourre estimate is the absence of singular continuous spectrum in R and a natural next question is asymptotic completeness in the same region. The latter is the main result of [DM15], which we discuss below.

2.2 Asymptotic completeness

We recall that in the energy-momentum region in question only collisions of one electron and one boson are energetically possible (in addition to single-electron states). We recall from Section 1, that proving asymptotic completeness amounts to excluding 'fuzzy' configurations in which the boson 'cannot decide' if it should stay close to the bare electron and thus contribute to the physical electron, or rather scatter to infinity. The mathematical formalism for analyzing this problem is that of extended objects. The extended Hilbert space, Hamiltonian and momentum operators are given by [HS95, DG99]

$$\mathcal{H}^{\mathrm{ex}} = \mathcal{H} \otimes \Gamma(\mathfrak{h}), \quad H^{\mathrm{ex}} = H \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega), \quad P^{\mathrm{ex}} = P \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k) \tag{2.5}$$

and the joint spectral measure of $(P^{\mathrm{ex}}, H^{\mathrm{ex}})$ will be denoted by $E^{\mathrm{ex}}(\,\cdot\,)$. The idea of the proof of asymptotic completeness, which dates back to V. Enss [En78], is to cut every physical state into two pieces: The first piece, where the bare electron and boson stay close together, is put on the first factor of the extended Hilbert space. The second piece, where the two particles travel far apart, is placed on the second factor. As the far separated particles should be essentially independent, H^{ex} and P^{ex} in (2.5) act as the free energy-momentum operators on the second factor.

To perform this cutting of states, we will use pairs of (possibly time-dependent) bounded operators j_0^t, j_∞^t on $\mathfrak{H}\otimes\mathfrak{h}$ which give rise to operators $j^t : \mathfrak{H}\otimes\mathfrak{h} \to \mathfrak{H}\otimes(\mathfrak{h}\oplus\mathfrak{h})$ defined by

$$j^t(\Psi\otimes h) := \big(j_0^t(\Psi\otimes h), j_\infty^t(\Psi\otimes h)\big). \tag{2.6}$$

With a suitable choice of j_0^t, j_∞^t, to be specified below, a state $\Psi\otimes h$, describing one bare electron and one boson, is divided into two pieces whose physical interpretation was discussed above. In order to second-quantize this operation, we denote by $U : \Gamma(\mathfrak{h}\oplus\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$ the canonical identification of the two Fock spaces. It is defined by[4]

$$U\Omega = \Omega\otimes\Omega, \quad Ua^*(h_1, h_2) = (a^*(h_1)\otimes 1 + 1\otimes a^*(h_2))U, \tag{2.7}$$

where Ω is the vacuum vector in $\Gamma(\mathfrak{h})$. After tacitly extending U from $\Gamma(\mathfrak{h}\oplus\mathfrak{h})$ to $\mathfrak{H}\otimes\Gamma(\mathfrak{h}\oplus\mathfrak{h})$, the arguments h_1, h_2 in (2.7) correspond to the two pieces in question which are distributed by U on the two tensor factors. Now the actual cutting of states at the second-quantized level is performed by the operator

$$\check{\Gamma}(j^t) : \mathcal{H} \to \mathcal{H}^{\mathrm{ex}}, \quad \check{\Gamma}(j^t) := U\Gamma(j^t), \tag{2.8}$$

where $\Gamma(j^t)|_{\mathfrak{H}\otimes(\otimes_s^n\mathfrak{h})} := \underbrace{j^t\otimes\cdots\otimes j^t}_{n}$.

After these preparations we are ready to state and discuss the main result of [DM15].

Theorem 1 [DM15] *There exists the wave operator* $W_R^{\mathrm{out}} : E^{\mathrm{ex}}(R)\mathcal{H}_+ \to E(R)\mathcal{H}$ *given by*

$$W_R^{\mathrm{out}} := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\check{\Gamma}(1,1)^*\mathrm{e}^{-\mathrm{i}tH^{\mathrm{ex}}}, \tag{2.9}$$

where $\mathcal{H}_+ := E(\Sigma_{\mathrm{pp}})\mathcal{H}\otimes\Gamma(\mathfrak{h})$ *and* Σ_{pp} *is the pure-point part of the spectrum of* (H, P) *consisting of the mass-shells of the physical electron.*

Furthermore, W_R^{out} *is a unitary, i.e.,*

$$W_R^{\mathrm{out}*}W_R^{\mathrm{out}} = E^{\mathrm{ex}}(R)|_{\mathcal{H}_+}, \tag{2.10}$$
$$W_R^{\mathrm{out}}W_R^{\mathrm{out}*} = E(R). \tag{2.11}$$

The wave operator in (2.9) is more complicated than its quantum mechanical counterpart (1.4). This can be traced back to the complicated composite structure of the physical electron in NRQFT discussed above. In the Nelson model the problems of the existence of the limit in (2.9) is a standard application of the Cook's method [DG99, FGS01, FGS04, GZ09]. In the case of the polaron model, however, the existence of this limit was non-trivial and was first established in [DM15]. Due to the constant dispersion relation of bosons (hence vanishing group velocity) it was not even clear if the picture of a propagating particle is correct. It turned out that the decisive quantity here is the dispersion relation of the relative motion of the two particles $\omega_{\mathrm{eff}}(k) = \Sigma^{(0)}(\xi - k) + \omega(k)$ which is never constant. Another interesting message of the polaron model is that a singular form-factor $G(k) \sim |k|^{-1}$ is harmless if the bosons are massive. This should be compared with the case of massless bosons, where a milder singularity $G(k) \sim |k|^{-1/2}$ causes severe infrared problems in scattering theory, which have to be handled by Dollard-type modifications [Pi05, Dy17].

The most important part of Theorem 1 is the asymptotic completeness relation (2.11). Similarly as in quantum mechanics, it says that every vector in the range of $E(R)$ is also in the range of

[4] If h is a function, $a^*(h) := \int_{\mathbb{R}^3} a^*(k)h(k)dk$.

the wave operator. Thus it is either a scattering state of one physical electron and one boson, or a physical single-electron state[5]. Compared to earlier works on asymptotic completeness for translation invariant models of NRQFT [FGS04, FGS07], the advantage of Theorem 1 is that it holds for arbitrary values of the coupling constant λ and that the spectral region of validity R is defined by the natural thresholds (2.3). An obvious limitation of Theorem 1 is the restriction to two-body scattering, which was not needed in [FGS04, FGS07].

Let us now comment on the proof of Theorem 1, assuming for simplicity that there is only one mass-shell $\Sigma^{(0)}$ of the electron in the spectrum. As mentioned in Section 1, to prove asymptotic completeness one constructs the inverse of the wave operator using various asymptotic observables. We will discuss here the problem of existence of one particular asymptotic observable, which can be seen as a variant of the asymptotic velocity (1.10). To define it, we choose a function $q \in C_0^\infty(\mathbb{R}^3)$, which is equal to one near zero, and set $q^t := q(a_{\xi_0}/t)$, where a_{ξ_0} appeared in the Mourre estimate (2.4). We set

$$Q^+ := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH(\xi)}\tilde{\chi}\Gamma(q^t)\tilde{\chi}\mathrm{e}^{-\mathrm{i}tH(\xi)}, \tag{2.12}$$

for ξ sufficiently close to ξ_0 and $\tilde{\chi} := \chi(H(\xi))$ for $\chi \in C_0^\infty(\mathbb{R}^3)$ supported below the two-boson threshold. To show the convergence in (2.12) by the Cook's method, we introduce the approximating sequence $\Phi(t) := \tilde{\chi}\Gamma(q^t)\tilde{\chi}$ and compute its Heisenberg derivative

$$\mathbf{D}\Phi(t) = \partial_t\Phi(t) + \mathrm{i}[H(\xi), \Phi(t)]. \tag{2.13}$$

A seemingly natural next step, taken in earlier work on asymptotic completeness (e.g. [DG99, FGS04]), is to evaluate the above expression to the leading order in t^{-1}, making use of the explicit expression for the Hamiltonian. Actually, this is what we did in the quantum-mechanical discussion in (1.12). However, this introduces into the computation the bare dispersion relation $\hat{\Omega}$ of the electron, rather than the physical relation $\Sigma^{(0)}$ coming from the spectrum. As a consequence the required propagation estimate (a counterpart of (1.13)) does not hold.

To bring to light the physical dispersion relation $\Sigma^{(0)}$, which may be very different from $\hat{\Omega}$ for large coupling constants, we come back to the idea of cutting the state with the help of the map $\check{\Gamma}(j^t)$ defined in (2.8). Let $j_0, j_\infty \in C^\infty(\mathbb{R}^3)$, where j_0 is supported in the region where $q_0 = 1$, and $j_0^2 + j_\infty^2 = 1$. Due to the last relation we have $\check{\Gamma}(j^t)^*\check{\Gamma}(j^t) = 1$, where $j^t_{0,\infty} := j_{0,\infty}(a_{\xi_0}/t)$. Using this, and setting $\tilde{\chi}^{\mathrm{ex}} := \chi(H^{\mathrm{ex}}(\xi))$, we obtain from (2.13)

$$\mathbf{D}\Phi(t) = \check{\Gamma}(j^t)^*\tilde{\chi}^{\mathrm{ex}}P_1^*(\Gamma(q^t)\otimes \mathbf{d}q^t)P_1\tilde{\chi}^{\mathrm{ex}}\check{\Gamma}(j^t) + O(t^{-2}), \tag{2.14}$$

where $\mathbf{d}q^t = \partial_t q^t + \mathrm{i}[\omega_{\mathrm{eff}}, q^t]$ is the Heisenberg derivative involving the correct effective dispersion relation of the electron-boson system $\omega_{\mathrm{eff}}(k) := \Sigma^{(0)}(\xi - k) + \omega(k)$ and $P_1 : \Gamma^{\mathrm{ex}}(\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\mathfrak{h}$ is a projection. With formula (2.14) at hand we are able to show convergence in (2.12) using the method of propagation estimates outlined in Section 1. To verify the key lower bound (1.15) on the Heisenberg derivative we use the Mourre estimate (2.4). Again the crucial point is that both a_{ξ_0} and the r.h.s. of (2.14) involve the physical dispersion relation $\Sigma^{(0)}$. This concludes our discussion of the proof of Theorem 1.

To elucidate the somewhat ad hoc definition of the wave operator (1) and prepare the grounds for the next section, we recall the standard concept of the asymptotic creation and annihilation operators of bosons with some wave-functions $h \in \mathfrak{h}$. They are defined as strong limits

$$a^{(*)}_{\mathrm{out}}(h) = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}a^{(*)}(\mathrm{e}^{-\mathrm{i}t\omega}h)\mathrm{e}^{-\mathrm{i}tH}, \tag{2.15}$$

[5]Single-electron states correspond to bound states from the quantum-mechanical discussion of Section 1. Differently than in quantum mechanics, here the single-electron states belong to the ranges of the wave-operators

on a certain domain in $\mathcal{H}$ and they are closely related to the wave operators. In fact, let $\Psi \in E(\Sigma_{\rm pp})\mathcal{H}$ be a physical single-electron state and h a single-boson state s.t. $\Psi \otimes a^*(h)\Omega \in \mathcal{H}^{\rm ex}$ is in the region R of the extended energy-momentum spectrum. Then it is easy to check that

$$W_R^{\rm out}(\Psi \otimes a^*(h)\Omega) = a^*_{\rm out}(h)\Psi. \tag{2.16}$$

Thus the action of the wave operator corresponds to 'adding' an asymptotic boson to the physical electron. This latter point of view, which originates from the LSZ [LSZ55] and Haag-Ruelle [Ha58, Ru62] approach to scattering theory, is very convenient in relativistic QFT.

3 Asymptotic completeness in relativistic QFT and in quantum spin systems [DG14, DG14.1, BDN16, Dy18]

3.1 Framework

In this section[6] we outline some recent progress on scattering theory and the problem of asymptotic completeness for massive relativistic QFT (RQFT) and gapped quantum spin systems, made in [DG14, DG14.1, BDN16, Dy18]. In order to treat RQFT and quantum spin systems in parallel in this review, we introduce the following general setting which collects the assumptions of direct relevance to scattering theory. From the outset, we restrict attention to RQFT in the vacuum representations and quantum spin systems in representations of translation invariant ground states.

1. We denote by Γ be the abelian group of space translations and by $\widehat{\Gamma}$ its Pontryagin dual, i.e., the momentum space. In RQFT we have $\Gamma = \mathbb{R}^d$ and $\widehat{\Gamma} = \mathbb{R}^d$, whereas for quantum spin systems $\Gamma = \mathbb{Z}^d$ and $\widehat{\Gamma} = S_1^d$, where the latter denotes the d-dimensional torus.

2. We consider a C^*-dynamical system $(\mathfrak{A}, \alpha)$, where $\mathfrak{A}$ is a C^*-algebra of observables and $\mathbb{R}\times\Gamma \ni (t,x) \mapsto \alpha_{(t,x)}$ is a group of automorphisms which describes translations of observables in space and time.

3. We assume that there is a norm-dense subalgebra $\mathfrak{B}\subset\mathfrak{A}$ of *almost-local operators*. We refrain from giving the formal definition here, merely state that for $v \in \mathbb{R}^d$ with sufficiently large norm, we have

$$\|[B_1, \alpha_{(s,vs)}(B_2)]\| = O(|s|^{-\infty}), \quad B_1, B_2 \in \mathfrak{B}, \tag{3.1}$$

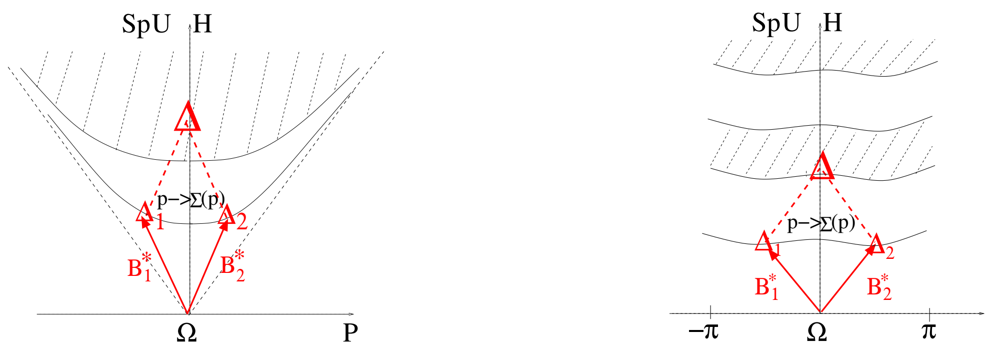
   that is the commutator decays faster than any inverse power of s. In RQFT this bound follows from locality and $|v|$ should be larger than the velocity of light, whereas in quantum spin systems it is a consequence of the Lieb-Robinson bounds [LR72] and $|v|$ should be larger than the Lieb-Robinson velocity.

4. We suppose that $\mathfrak{A}$ acts irreducibly on a Hilbert space $\mathcal{H}$ and that α is unitarily implemented on $\mathcal{H}$. That is, there is a unitary representation $\mathbb{R}\times\Gamma \ni (t,x) \mapsto U(t,x)$ s.t.

$$\alpha_{(t,x)}(A) = U(t,x)AU(t,x)^*, \quad A \in \mathfrak{A}. \tag{3.2}$$

   The spectrum of U, denoted $\mathrm{Sp}\, U$, is the support of its inverse Fourier transform[7]. It is a subset of $\mathbb{R}\times\widehat{\Gamma}$ which consists of all the possible values of the total energy and momentum of the system. The spectrum should contain a simple eigenvalue at $\{0\}$, corresponding to the vacuum

[6] This section overlaps with the author's planned contribution to the IAMP News Bulletin.
[7] We follow here the conventions from [BDN16].



(a) An exemplary spectrum in RQFT.

(b) An exemplary spectrum in a quantum spin system.

Figure 3.1. Schematic shapes of the energy-momentum spectra (a) in massive RQFT and (b) in a gapped quantum spin system. Neighbourhoods Δ_1, Δ_2 of some points on the isolated mass-shell $p \mapsto \Sigma(p)$ and the region Δ of the multiparticle spectrum illustrate the geometric situation from assumption (a) of Theorem 3. The operators B_1^*, B_2^* create from the vacuum single-particle states living in the regions Δ_1, Δ_2.

vector Ω and a smooth mass-shell $\widehat{\Gamma} \ni p \mapsto \Sigma(p)$ carrying single-particle states. Instead of stating all the assumptions on the spectrum here, we refer to Figure 3.1 for schematic shapes and remark that Σ should not be a constant function and should be isolated from the rest of the spectrum.

This is a rather abstract setting, so let us give some examples: On the relativistic side we mention the ϕ^4 models of constructive QFT in two and three spacetime dimensions for small values of the coupling constant [GJS73, SZ76, Bur77]. On the side of spin systems, the Ising model in strong transverse magnetic fields in any space dimensions satisfies all the above assumptions [BDN16, Ya04, Ya05, Po93]. We stress that the single-particle states introduced above are complicated collective excitations in these models.

3.2 Scattering states

The problem of construction of scattering states is the following[8]: Given a collection of single-particle states $\Psi_1, \dots, \Psi_N \in \mathcal{H}$, living on the mass-shell $p \mapsto \Sigma(p)$ in $\mathrm{Sp}\, U$, we would like to define a state $\Psi^{\mathrm{out}} \in \mathcal{H}$ which describes the corresponding configuration of N particles. Anticipating that asymptotically these particles should be non-interacting bosons, the state Ψ^{out} should have all the properties of the symmetrized tensor product of the constituent single-particle states Ψ_i, $i = 1, \dots, N$. Nevertheless, it should be an element of $\mathcal{H}$ and not of $\otimes_{\mathrm{s}}^N \mathcal{H}$. The first step towards the solution of this problem is suggested by the theory of Fock spaces: we pick almost-local operators $B_i^* \in \mathfrak{B}$ which create these single-particle states from the vacuum, i.e., $B_i^*\Omega = \Psi_i$. In order to select such generalized creation operators, we compute the Arveson spectrum $\mathrm{Sp}_{B_i^*}\alpha$, which is the support of the inverse Fourier transform of $(t, x) \mapsto \alpha_{(t,x)}(B_i^*)$. The key property of the Arveson spectrum is the energy-momentum transfer relation, which says that for any Borel subset $\Delta \subset \mathrm{Sp}\, U$

$$B_i^*\, E(\Delta)\mathcal{H} \subset E(\overline{\Delta + \mathrm{Sp}_{B_i^*}\alpha})\mathcal{H}, \tag{3.3}$$

where $E(\,\cdot\,)$ denotes the spectral measure of U. In particular, if $\mathrm{Sp}_{B_i^*}\alpha$ is contained in a small neighbourhood Δ_i of a point of the mass-shell (cf. Figure 3.1) then by choosing $\Delta = \{0\}$ in (3.3) we obtain that $B_i^*\Omega$ is a single-particle state. As operators with Arveson spectrum in prescribed subsets are in abundance, it is easy to obtain such generalized creation operators for a dense family of single-particle vectors.

[8] We discuss the outgoing scattering states here. The incoming case is analogous.

The next step is to define the Haag-Ruelle creation operators by smearing the above generalized creation operators with wave-packets of the particles involved:

$$B_{i,t}^*(g_{i,t}) := \int_\Gamma d\mu(x)\alpha_{(t,x)}(B_i^*)g_{i,t}(x), \quad g_{i,t}(x) := \int_{\widehat{\Gamma}} dp\, \mathrm{e}^{-\mathrm{i}\Sigma(p)t+\mathrm{i}p\cdot x}\hat{g}_i(p), \tag{3.4}$$

where $d\mu$, resp. dp, is the Haar measure on Γ, resp. $\widehat{\Gamma}$, the function $p \mapsto \Sigma(p)$ is the mass-shell appearing in the spectrum (cf. Figure 3.1) and t is the time parameter. The role of the Haag-Ruelle creation operators is to compare the interacting evolution appearing in $(t,x) \mapsto \alpha_{(t,x)}$ and the free evolution of the wave-packet at asymptotic times. This is the content of the Haag-Ruelle theorem whose relativistic variant dates back to [Ha58, Ru62]. It was adapted to quantum spin systems by S. Bachmann, P. Naaijkens and the present author in [BDN16] following the strategy of the proof from [Ar]. While a similar enterprise had been accomplished in Euclidean lattice field theory [BF91] earlier works on scattering in quantum spin systems relied more heavily on properties of particular models [Ya04.1, GS97, Ma83].

Theorem 2 *The following limits exist and are called the outgoing scattering states*

$$\Psi^{\mathrm{out}} := \lim_{t\to\infty} B_{1,t}^*(g_{1,t})\dots B_{N,t}^*(g_{N,t})\Omega. \tag{3.5}$$

The velocity supports $V(g_i) := \{\nabla\Sigma(p)\,|\,p \in \operatorname{supp}\hat{g}_i\}$ of the wave-packets g_i are assumed to be disjoint. The incoming scattering states Ψ^{in} are constructed analogously by taking the limit $t \to -\infty$.

Let us recall the main steps of the proof in order to indicate how the above general assumptions ensure the existence of multi-particle scattering states. The argument relies on the Cook's method, that is we try to make sense of the formula

$$\Psi^{\mathrm{out}} = \int_{t_0}^{\infty} (\partial_\tau \Psi_\tau)d\tau + \Psi_{t_0}, \tag{3.6}$$

where $t \mapsto \Psi_t$ is the approximating sequence of Ψ^{out} and $t_0 \geq 0$ is arbitrary. For this purpose we check that $t \mapsto \|\partial_t\Psi_t\|$ is an integrable function. This is easy to see for $N = 1$ in which case there is an exact cancellation of the interacting and the free dynamics for an arbitrary t:

$$\partial_t(B_{1,t}^*(g_{1,t})) = 0. \tag{3.7}$$

The assumption that the mass-shell is isolated from the rest of the spectrum enters crucially here. Now the case $N = 2$ is treated using the Leibniz rule and (3.7)

$$\begin{aligned}\partial_t\Psi_t &= \partial_t(B_{1,t}^*(g_{1,t}))\, B_{2,t}^*(g_{2,t})\Omega + B_{1,t}^*(g_{1,t})\, \underbrace{\partial_t(B_{2,t}^*(g_{2,t}))\Omega}_{=0} \\ &= [\partial_t(B_{1,t}^*(g_{1,t})), B_{2,t}^*(g_{2,t})]\Omega = O(t^{-\infty}),\end{aligned} \tag{3.8}$$

where in the last step we used the assumption (3.1) about the decay of commutators at large spacelike separation and the disjointness of velocity supports of the two wave-packets. This argument easily generalizes to arbitrary n, which completes this outline of the proof.

Given the scattering states, one can construct the wave operators in a standard manner. Let $\mathcal{H}_1$ be the single-particle subspace, i.e., the spectral subspace of the mass-shell $p \mapsto \Sigma(p)$, and let $\Gamma(\mathcal{H}_1)$ be the corresponding symmetric Fock space. The outgoing wave operator $W^{\mathrm{out}} : \Gamma(\mathcal{H}_1) \to \mathcal{H}$ is defined by the relation

$$W^{\mathrm{out}}(a^*(\Psi_1)\dots a^*(\Psi_N)\Omega) = \lim_{t\to\infty} B_{1,t}^*(g_{1,t})\dots B_{N,t}^*(g_{N,t})\Omega, \tag{3.9}$$

where $\Psi_i := B^*_{i,t}(g_{i,t})\Omega$ and $a^{(*)}$ are the creation/annihilation operators on $\Gamma(\mathcal{H}_1)$. By computing the scalar products of the scattering states it is easy to check that W^{out} is an isometry. The same is true for the analogously defined incoming wave operator W^{in} which enters into the definition of the scattering matrix

$$S := (W^{\mathrm{in}})^* W^{\mathrm{out}}. \tag{3.10}$$

If S is different from the identity, we say that the theory is interacting. To our knowledge, the interaction has been proven only in some two-dimensional relativistic systems [OS76, Le08, Ta14]. It is one of the central problems of RQFT to exhibit an interacting model in four-dimensional spacetime. In the context of quantum spin systems there are candidates for interacting theories in an arbitrary dimension, for example the Ising model mentioned above, but we are not aware of a proof.

3.3 Generalized asymptotic completeness

The conventional property of asymptotic completeness requires that the subspace $\mathcal{H}^{\mathrm{out}} = \mathrm{Ran}\, W^{\mathrm{out}}$, spanned by the scattering states (3.5), is in fact the full Hilbert space. That is

$$\mathcal{H}^{\mathrm{out}} = \mathcal{H}, \tag{3.11}$$

so that every vector $\Psi \in \mathcal{H}$ has an interpretation in terms of particles (cf. Figure 3.2 (a)). In the setting from Subsection 3.1 the only known examples which are interacting and asymptotically complete are certain two-dimensional relativistic models [Le08, Ta14][9]. One reason for this scarcity of examples may be the following special feature of quantum systems with infinitely many degrees of freedom: The algebra of observables $\mathfrak{A}$ may have many inequivalent representations labelled by some quantum number which we call 'charge'. The vacuum representation, we are interested in, has the charge equal to zero. Let us now consider the particle content of the region Δ of the multiparticle spectrum in Figure 3.1. Apart from the pairs of the charge-zero particles living on the mass-shell in the spectrum of U there may also be, e.g., pairs of oppositely charged particles whose single-particle constituents live in different representations. States describing such oppositely charged pairs live in the vacuum representation (as their total charge is zero), but are orthogonal to all the scattering states (3.5) of the charge-zero particles. In this case the asymptotic completeness property (3.11) fails.

As we typically do not have access to all the charged representations, it is reasonable to generalize the concept of asymptotic completeness so that it is compatible with the presence of charged pairs. The idea is illustrated on Figure 3.2 (b): Any state $\Psi \in \mathcal{H}$ should give rise to a configuration of charge-zero particles after filtering it through a particle detector which is sensitive only to such particles. Moreover, every configuration of charge-zero particles should be obtainable in this way. Arguably, such a concept should suffice to interpret physical experiments, since in the experimental reality there is always some intervening apparatus. It turns out that such generalized asymptotic completeness can be formulated and proven under the general assumptions stated in Subsection 3.1.

The first question is how to identify particle detectors in such a general mathematical formalism. This question was first asked by Araki and Haag in the setting of RQFT [AH67]. They came up with the time-dependent families of observables of the form

$$C_t := \int_\Gamma d\mu(x)\alpha_{(t,x)}(B^*B)h\left(\frac{x}{t}\right), \tag{3.12}$$

[9] We also mention partial results on $(\phi^4)_2$ [SZ76, CD82] and in certain lattice systems which do not quite fit into our framework [AB01, GS97]. There is also progress on asymptotic completeness in wedge-local QFT [DT11, Du18].

(a) Conventional asymptotic completeness. (b) Generalized asymptotic completeness.

Figure 3.2. (a) Conventional asymptotic completeness requires that every vector $\Psi \in \mathcal{H}$ is a configuration of charge-zero particles from $\mathcal{H}^{\mathrm{out}}$. (b) Generalized asymptotic completeness requires that from every vector Ψ gives rise to a configuration of charge-zero particles after filtering it through a suitable measurement apparatus. Moreover, every configuration of charge-zero particles can be obtained by this procedure.

where $B^* \in \mathfrak{B}$ is a generalised creation operator as above and $h \in C_0^\infty(\mathbb{R}^d)$. Their limits as $t \to \infty$ can be computed on scattering states $\Psi_1^{\mathrm{out}}, \Psi_2^{\mathrm{out}} \in \mathcal{H}^{\mathrm{out}}$ of bounded energy and, schematically, have the following form [AH67, Bu90]

$$\lim_{t\to\infty} \langle \Psi_1^{\mathrm{out}}, C_t \Psi_2^{\mathrm{out}} \rangle = \int_{\widehat{\Gamma}} dp\, \underbrace{\langle p|B^*B|p\rangle h(\nabla\Sigma(p))}_{\text{sensitivity of the detector}}\, \underbrace{\langle \Psi_1^{\mathrm{out}}, a_{\mathrm{out}}^*(p) a_{\mathrm{out}}(p) \Psi_2^{\mathrm{out}} \rangle}_{\text{particle density}}. \tag{3.13}$$

Here $a_{\mathrm{out}}^*(p)a_{\mathrm{out}}(p) := W^{\mathrm{out}} a^*(p)a(p)(W^{\mathrm{out}})^*$ is the asymptotic particle density in momentum space and the remaining part of the integrand above can be interpreted as the sensitivity of the detector. It should be stressed, however, that the above results do not say anything about the convergence of $\{C_t\}_{t\in\mathbb{R}}$ on states which are not in $\mathcal{H}^{\mathrm{out}}$. This question, which is essential for asymptotic completeness, has been a long-standing open problem in RQFT (see [Ha, Section VI.2.3]). First results of this sort were obtained by C. Gérard and the present author in [DG14, DG14.1] in the relativistic setting and then generalized to gapped quantum spin systems in [Dy18]. These results rely only on the general assumptions outlined in Subsection 3.1 and can be summarized as follows:

Theorem 3 *Fix a small subset Δ of the multiparticle spectrum as in Figure 3.1. Then the following strong limits exist*

$$A^{\mathrm{out}} := \lim_{t\to\infty} C_{1,t} \dots C_{N,t} E(\Delta), \quad \text{where} \quad C_{i,t} := \int_{\Gamma} d\mu(x) \alpha_{(t,x)}(B_i^* B_i) h_i\left(\frac{x}{t}\right), \tag{3.14}$$

provided that:

(a) B_i^* are generalized creation operators of single-particle states living in subsets Δ_i of $\mathrm{Sp}\, U$ *s.t. $\Delta_1 + \dots + \Delta_N \subset \Delta$ (cf. Figure 3.1).*

(b) h_i have mutually disjoint supports[10].

Denote by $[A^{\mathrm{out}}\mathcal{H}]$ the subspace of $\mathcal{H}$ spanned by the ranges of all the operators A^{out} constructed as above for different choices of Δ. Then

$$\mathcal{H}^{\mathrm{out}} = [A^{\mathrm{out}}\mathcal{H}] \oplus \mathbb{C}\Omega, \tag{3.15}$$

that is, generalised asymptotic completeness holds (cf. Figure 3.2 (b)).

We remark that assumption (a) of this theorem ensures that the detectors A^{out} annihilate configurations involving charged particles (if any) as seen a posteriori from relation (3.15). They also annihilate possible 'bound states', corresponding to embedded mass-shells passing through Δ, which

[10]We skip here some more technical restrictions on these functions.

we did not exclude by assumption. The technically most challenging part of the proof of Theorem 3 is the existence of the limit in (3.14) on arbitrary vectors of bounded energy from $\mathcal{H}$. For more contrived choices of the detectors $C_{i,t}$ this had been shown in [DG14.1] by adapting the quantum-mechanical method of propagation estimates [SiSo87]. In [Dy18] a different technique was found which applies to the usual Araki-Haag detectors, as stated above. We explained this strategy in (1.19) in the context of quantum mechanics. We outline the argument from [Dy18] in the remaining part of this section.

The strategy is to approximate A^{out} by linear combinations of rank-one operators $|\Psi^{\mathrm{out}}\rangle\langle\tilde{\Psi}^{\mathrm{out}}|$ at the level of the respective approximating sequences. In addition to the convergence, which then follows from Theorem 2, this also gives relation (3.15). For the purpose of this approximation argument, we define the mapping $a_{B_1,\dots,B_N} : \mathcal{H}_{\mathrm{c}} \to \mathcal{H} \otimes L^2(\Gamma^N)$ given by [DG14, DG14.1]

$$(a_{B_1,\dots,B_N}\Psi)(x_1,\dots,x_N) = \alpha_{x_1}(B_1)\dots\alpha_{x_N}(B_N)\Psi. \tag{3.16}$$

Here $B_1,\dots B_N \in \mathfrak{B}$ are as in Theorem 3, $\mathcal{H}_{\mathrm{c}} \subset \mathcal{H}$ is the dense domain of vectors of bounded energy and the fact that the $\mathcal{H}$-valued function on the r.h.s. of (3.16) is square-integrable follows from [Bu90]. In terms of these maps, we can write for scattering states living in Δ

$$|\Psi^{\mathrm{out}}\rangle\langle\tilde{\Psi}^{\mathrm{out}}| = \lim_{t\to\infty} a^*_{\alpha_t(\underline{B})}\big(|\Omega\rangle\langle\Omega| \otimes \mathrm{e}^{-\mathrm{i}t\underline{\Sigma}(D_{\underline{x}})}|\underline{g}\rangle\langle\underline{\tilde{g}}|\mathrm{e}^{\mathrm{i}t\underline{\Sigma}(D_{\underline{x}})}\big)a_{\alpha_t(\underline{\tilde{B}})}E(\Delta), \tag{3.17}$$

where we introduced the short-hand notation:

$$\alpha_t(\underline{B}) := (\alpha_t(B_1),\dots\alpha_t(B_N)), \tag{3.18}$$

$$\underline{\Sigma}(D_{\underline{x}}) := \Sigma(-\mathrm{i}\nabla_{x_1}) + \dots + \Sigma(-\mathrm{i}\nabla_{x_N}), \tag{3.19}$$

$$\underline{g} := (g_1,\dots,g_N) \tag{3.20}$$

and $g_i := g_{i,t=0}$ denotes the initial data of the wave-packets in (3.4). Now the approximating sequence of A^{out} can be expressed as follows

$$C_{1,t}\dots C_{n,t}E(\Delta) = a^*_{\alpha_t(\underline{B})}\big(\mathbb{1}_{\mathcal{H}} \otimes \underline{h}(\underline{x}/t)\big)a_{\alpha_t(\underline{B})}E(\Delta) + O(t^{-\infty}), \tag{3.21}$$

where $\underline{h}(\underline{x}/t) := h_1(x_1/t)\dots h_n(x_n/t)$ and the disjointness of the supports of h_i, together with (3.1), ensure the rapid decay of the error term above. Let us now compare the r.h.s. of (3.21) and (3.17). First, exploiting the presence of the projection $E(\Delta)$, assumption (a) and the energy-momentum transfer relation (3.3) we can replace $\mathbb{1}_{\mathcal{H}}$ with $|\Omega\rangle\langle\Omega|$ in (3.21). Next, since $E(\Delta)$ restricts the total momentum of the system to a compact set, we can replace $\underline{h}(\underline{x}/t)$ with $\underline{h}(\underline{x}/t)\chi(D_{\underline{x}})$, where χ is an approximate characteristic function of a sufficiently large ball. Using the trivial identity

$$\underline{h}(\underline{x}/t)\chi(D_{\underline{x}}) = \mathrm{e}^{-\mathrm{i}t\underline{\Sigma}(D_{\underline{x}})}\underline{h}(\underline{x}/t + \nabla\underline{\Sigma}(D_{\underline{x}}))\chi(D_{\underline{x}})\mathrm{e}^{\mathrm{i}t\underline{\Sigma}(D_{\underline{x}})}, \tag{3.22}$$

we incorporate the free time evolution. Finally, by approximating the compact operator $\underline{h}(\underline{x}/t + \nabla\underline{\Sigma}(D_{\underline{x}}))\chi(D_{\underline{x}})$ with finite-rank projections, we can indeed approximate the detector (3.21) by finite linear combinations of terms of the form (3.17). A careful reader may notice that the latter limit has to be exchanged with the limit $t \to \infty$. It turns out that this is not trivial and relies on assumption (b) about the disjointness of supports of h_i. This concludes the outline of the proof of Theorem 3.

4 Infrared problems in relativistic QFT [AD17]

We consider here a relativistic QFT as defined in Subsection 3.1 above, except that $\mathrm{Sp}\,U$ is not assumed to contain mass-gaps. Furthermore, we require that the C^*-algebra of observables $\mathfrak{A}$ is generated by a net of local observables $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ labelled by open bounded regions $\mathcal{O} \subset \mathbb{R}^4$. The elements of two such algebras commute if the respective regions are spacelike separated. By looking at sequences of observables localized in regions shrinking to a point one can recover pointlike localized quantum fields of the theory [Bos05]. In quantum electrodynamics (QED), in which we are interested in this section, these fields should include the Faraday tensor $F^{\mu\nu}$ and the electric current j^{μ}. The electric charge is formally given by

$$Q := \int_{\mathbb{R}^3} dx\, j^0(0,x), \tag{4.1}$$

and we start the analysis of the theory with states of zero charge. One such state is the vacuum Ω describing the empty space. The subspace $\mathcal{H}_0 := \overline{\mathfrak{A}\Omega}$ is called the vacuum sector, and we will treat it as the defining representation of QED in the following. It contains states of electrically neutral excitations as for example photons and atoms[11]. Their masses are eigenstates of the relativistic mass operator $M := \sqrt{H^2 - P^2}$, i.e., they are particles in the sense of Wigner. Scattering theory for photons in the setting of algebraic QFT was developed by Buchholz in [Bu77], exploiting the Huyghens principle. Scattering theory of atoms in the presence of photons was developed by the present author in [Dy05]; more recently the subject was revisited by Herdegen and Duch in [He14, DH15] and by Duell [Du17]. In all these works the formula for scattering states is dictated be the Haag-Ruelle theory (cf. Theorem 2), but the proof of convergence is technically more difficult than for isolated mass-shells.

Let us now proceed to representations of the algebra of observables of QED with non-zero electric charge. A conserved quantity, which can be used to classify them, is the spacelike asymptotic flux of the electric field [Bu82, KK90]

$$\phi(n) := \lim_{r\to\infty} r^2 n \cdot \mathrm{E}(0, rn), \tag{4.2}$$

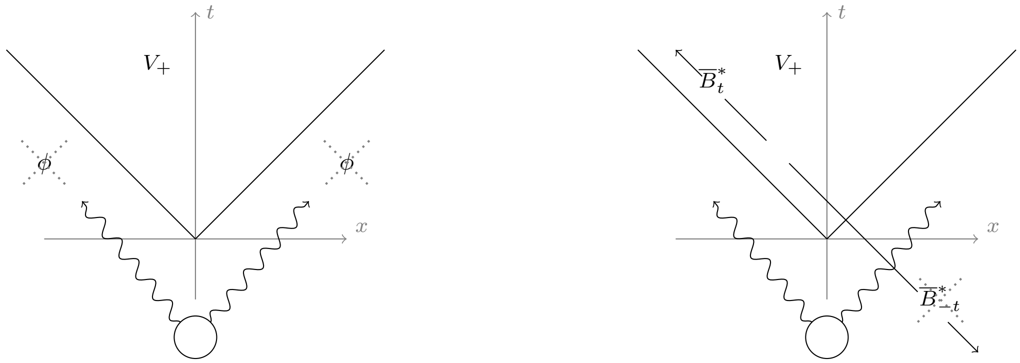
where $\mathrm{E} := (F^{0,1}, F^{0,2}, F^{0,3})$ and n is a unit vector in $\mathbb{R}^3$. Due to the Gauss Law, the integral of this quantity over a unit sphere gives the electric charge of the representation, but apart from this restriction ϕ is an arbitrary function on the unit sphere. A given representation is called an *infraparticle representation* if the limit in (4.2) exists (in a certain sense) and an *infravacuum representation* if the limit does not exist. Interestingly, the picture of the electron is quite different in these two cases [Kr82]: In the infraparticle approach the electron is a composite object consisting of an 'undressed electron[12]' and a cloud of physical soft photons correlated with its velocity. On the other hand, in the infravacuum approach the electron is expected to be an elementary object moving in a strongly fluctuating background radiation not correlated with its velocity. We remark that the relevance of asymptotic constants of motion (like (4.2)) to infrared problems has recently also been appreciated in the high-energy physics community [St17].

4.1 Infraparticle picture

At the quantum level, the requirement that the flux (4.2) exists is a selection criterion which specifies a class of representations of the algebra of observables $\mathfrak{A}$ of QED. This class contains in particular the infrared minimal representations, in the terminology of [Bu82]. They are characterized by the absence of any *background radiation field* whose decay with distance is slower than the quadratic decay of the Coulomb field. Such representations were widely used in structural analysis of infrared problems in QED [Bu82, FMS79, FMS79.1], although they suffer from several drawbacks: Firstly,

[11] As we treat here atoms from the point of view of QED, we disregard the baryon number.

[12] We stress that the 'undressed electron' is not 'bare' in the terminology from Section 2. It is a bound state of the bare electron and virtual photons responsible for the mass renormalisation. The 'dressing' discussed here is effected by physical photons.



(a) An infravacuum representation.

(b) (Non-)existence of asymptotic photon fields.

Figure 4.1 (a) An infravacuum representation of Buchholz and Roberts: highly fluctuating background radiation emitted in distant past blurs the flux ϕ. (b) Localization of the approximating sequences $\overline{B}_t^*$ of the outgoing photon creation operator and $\overline{B}_{-t}^*$ of the incoming photon creation operator.

Lorentz transformations are not unitarily implemented in any representation from this class [Bu86, FMS79.1]. Secondly, charged particles do not have sharp masses in such representations, i.e. they are infraparticles [Bu86]. This invalidates the standard Haag-Ruelle construction of scattering states, as in Theorem 2 above. An alternative approach to scattering theory of infraparticles was proposed by Buchholz, Porrmann and Stein in [BPS91]. It was used to clarify the particle content of a class of two dimensional Conformal Field Theories by the present author and Yoh Tanimoto in [DT12, DT13] as we discuss in the next section. This approach does not give, however, the wave operators or the S-matrix, aiming directly at (inclusive) collision cross-sections. In models of non-relativistic QED scattering theory for infraparticles was developed in [Pi05, CFP07]. At a heuristic level it can be linked to the Dollard formalism and the Faddeev-Kulish approach [FK70] as shown in [Dy17].

4.2 Infravacuum picture

A complementary class of representations, in which the flux (4.2) does not exist, will be called the infravacuum representations. Here the no-go theorems from [Bu86] do not hold and one can expect both covariance under Lorentz translations and sharp masses of charged particles. A class of such representations was recently introduced by Buchholz and Roberts in [BR14]. Heuristically speaking, they contain background radiation, emitted in distant past, which blurs the flux (4.2), cf. Figure 4.1 (a). By the Huyghens principle, this radiation does not enter into the future lightcone V_+ with apex at the origin. Therefore, in certain subsets of V_+, one can require that the representation is unitarily equivalent to the vacuum representation. Using this information and the existence of asymptotic photon fields in the vacuum representation [Bu77], Sabina Alazzawi and the present author constructed in [AD17] the outgoing asymptotic photon creation operators B^*_{out} in an infravacuum representation from [BR14]. In the notation from the previous section, they are (essentially) limits as $t \to \infty$ of

$$\overline{B}_t^* := \frac{1}{\ln |t|} \int_t^{t+\ln |t|} d\tau \int_{\mathbb{R}^3} dx\, \alpha_{(\tau,x)}(B^*) g_\tau(x), \tag{4.3}$$

where g is now a solution of the wave equation and the averaging in time serves to improve convergence. Assuming sharp mass of the electron, also the outgoing Compton scattering states were constructed in [AD17]. Thus the main result of [AD17] can be summarized as follows:

Theorem 4 *In a Buchholz-Roberts representation of QED admitting a sharp mass of the electron there exist Compton scattering states of the form*

$$\Psi^{\mathrm{out}} = B^*_{1,\mathrm{out}} \dots B^*_{N,\mathrm{out}} \Psi^{\mathrm{out}}_{\mathrm{el}}, \tag{4.4}$$

where the asymptotic creation operators of photons $B^*_{i,\mathrm{out}}$ are given by (4.3) and $\Psi^{\mathrm{out}}_{\mathrm{el}}$ are single-electron states. Their scalar products can be computed by standard Fock space rules, with $\Psi^{\mathrm{out}}_{\mathrm{el}}$ playing the role of the vacuum[13].

The superscript 'out' of the single-electron state $\Psi^{\mathrm{out}}_{\mathrm{el}}$ indicates that it lives in a representation localized in a *future* lightcone. It is crucial for the construction of B^*_{out} that the approximating sequence $\overline{B}^*_t$, given by formula (4.3), can be (essentially) localized in subsets of the future lightcone, as indicated in Figure 4.1 (b), and thus it does not interfere with the highly fluctuating background radiation. As also shown in Figure 4.1 (b), the incoming photon creation operators are not expected to exist in this representation as their approximating sequences collide with the background radiation. Thus to construct incoming Compton scattering states it is necessary to pass to a Buchholz-Roberts representation localized in a *backward* lightcone. As both representations act naturally on the same Hilbert space, S-matrix elements are well defined.

The key technical step of the proof of Theorem 4 consists in showing that any single-electron state $\Psi^{\mathrm{out}}_{\mathrm{el}} \in \mathcal{H}$ is a vacuum of the asymptotic photon fields, so that the scalar products of two vectors of the form (4.4) can indeed be computed by the standard Fock space rules. In order to show that $\lim_{t\to\infty} \overline{B}_t \Psi^{\mathrm{out}}_{\mathrm{el}} = 0$, we rewrite this approximating sequence in terms of the maps a_B of (3.16). This gives

$$\overline{B}_t \Psi^{\mathrm{out}}_{\mathrm{el}} = (1_{\mathcal{H}} \otimes \langle \overline{g}|) \frac{1}{\ln |t|} \int_t^{t+\ln |t|} d\tau \, \mathrm{e}^{\mathrm{i}(H+|-\mathrm{i}\nabla_x| - \omega_{m_{\mathrm{el}}}(P-\mathrm{i}\nabla_x))\tau} a_B \Psi^{\mathrm{out}}_{\mathrm{el}}, \tag{4.5}$$

where $\omega_{m_{\mathrm{el}}}(k) := \sqrt{k^2 + m_{\mathrm{el}}^2}$ is the dispersion relation of the electron, and the expression $(1_{\mathcal{H}} \otimes \langle \overline{g}|)$ denotes a certain bounded map. Given formula (4.5), the existence of $\lim_{t\to\infty} \overline{B}_t \Psi^{\mathrm{out}}_{\mathrm{el}}$ follows from the Mean Ergodic Theorem. The resulting concrete formula allows also to show that the limit is zero.

The assumption of a sharp mass of the electron, on which the above discussion relies, is presently investigated in models of non-relativistic QED by Daniela Cadamuro and the present author [CD18, CD19]. We used concrete infravacuum representations here, which were constructed by Kraus, Polley and Reents [KPR77].

5 Scattering in two-dimensional massless relativistic QFT [DT11, DT12, DT13]

5.1 Vacuum sector

We start our discussion in the setting of wedge-local QFT, which is broader than the local framework of the last two sections. All the relevant information about a vacuum representation of a wedge-local QFT is encoded in a *Borchers triple* $(\mathcal{R}, U, \Omega)$ w.r.t. a spacelike wedge $\mathcal{W}$ (cf. Figure 5.1 (a)). It is given by:

1. A von Neumann algebra $\mathcal{R} \subset B(\mathcal{H})$;

[13] This corresponds to the isometry of the wave operator.

2. A unitary representation $\mathbb{R}^{d+1} \ni x \to U(x)$ s.t.

$$\alpha_x(\mathcal{R}) = U(x)\mathcal{R}U(x)^{-1} \subset \mathcal{R} \text{ for } x \in \mathcal{W}, \tag{5.1}$$
$$\mathrm{Sp}\,\mathrm{U} \subset V_+; \tag{5.2}$$

3. A vacuum vector Ω, invariant under U, which is cyclic w.r.t. $\mathcal{R}$ and its commutant $\mathcal{R}'$. (We assume that Ω is a unique invariant vector).

It was shown in [BLS10] that for a given Borchers triple $(\mathcal{R}, U, \Omega)$ and the matrix

$$Q_\kappa := \begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}, \quad \kappa > 0, \tag{5.3}$$

one can generate a new Borchers triple $(\mathcal{R}_{Q_\kappa}, U, \Omega)$, by a Rieffel-type deformation of the algebra of observables. Namely, for $F \in \mathcal{R}$ which are smooth under translations the formal expressions

$$F_{Q_\kappa} := \int dE(q)\alpha_{Q_\kappa q}(F), \tag{5.4}$$

give bounded operators which generate the algebra $\mathcal{R}_{Q_\kappa}$. Furthermore,

$$(\mathcal{R}')_{-Q_\kappa} \subset (\mathcal{R}_{Q_\kappa})', \tag{5.5}$$

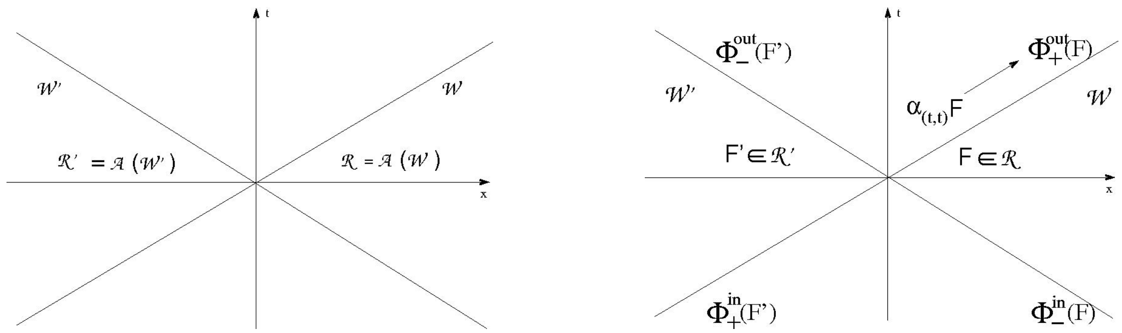
which we will use below. For massive theories it was shown that the deformation introduces interaction. Very differently than in quantum mechanics, this is effected by a modification of the algebra of observables, while the Hamiltonian remains unchanged. The interaction of the deformed theory was demonstrated as follows: Exploiting the geometric fact that two particles can be separated by two opposite wedges, two-particle Haag-Ruelle scattering states were constructed for wedge-local theories in [BBS01]. Then it was shown in [GL08] that the resulting two-particle scattering matrix of $(\mathcal{R}_{Q_\kappa}, U, \Omega)$ is non-trivial. Construction of N-particle scattering states for massive wedge-local theories, which allows to address the problem of asymptotic completeness, is a very recent result of Duell [Du18].

However, in the case of massless theories in two-dimensional spacetime two-particle scattering states suffice to demonstrate both interaction and asymptotic completeness as shown in [DT11]. Due to the dispersionless motion of solutions of the wave equation in two dimensions, the physical distinction is only between the left-running and right-running particles. Scattering theory for such 'waves' was developed in [Bu75] for local theories and then generalized to wedge-local theories by Tanimoto and the present author in [DT11]. It relies on the assumption that the wedge-local theory contains massless Wigner particles, i.e., eigenvectors of the relativistic mass operator. First, for any $F \in \mathcal{R}$ one defines asymptotic fields as the strong limits

$$\Phi_+^{\mathrm{out}}(F) := \lim_{t\to\infty} \frac{1}{\ln|t|} \int_t^{t+\ln|t|} d\tau\ \alpha_{(\tau,\tau)}(F), \tag{5.6}$$
$$\Phi_-^{\mathrm{in}}(F) := \lim_{t\to-\infty} \frac{1}{\ln|t|} \int_t^{t+\ln|t|} d\tau\ \alpha_{(\tau,-\tau)}(F) \tag{5.7}$$

which exist and are elements of $\mathcal{R}$. Operators $\Phi_-^{\mathrm{out}}(F')$, $\Phi_+^{\mathrm{in}}(F')$, where $F' \in \mathcal{R}'$, are constructed analogously (cf. Figure 5.1 (b)). Acting on the vacuum these fields generate single-particle states $\Psi_+ := \Phi_+^{\mathrm{out}}(F)\Omega$, $\Psi_- = \Phi_-^{\mathrm{out}}(F')\Omega$ and the resulting scattering states have the form

$$\Psi_+ \overset{\mathrm{out}}{\times} \Psi_- := \Phi_+^{\mathrm{out}}(F)\Phi_-^{\mathrm{out}}(F')\Omega. \tag{5.8}$$



(a) Borchers triple. (b) Asymptotic fields.

Figure 5.1. (a) The algebra $\mathcal{R} = \mathfrak{A}(\mathcal{W})$ of observables localized in the right wedge $\mathcal{W}$. (b) Asymptotic fields of a massless wedge-local theory in two dimensions.

The incoming scattering state $\Psi_+ \overset{\text{in}}{\times} \Psi_-$ is defined analogously, using the incoming fields. We say that the theory is asymptotically complete is the outgoing (and incoming) scattering states span the entire Hilbert space. We define the scattering matrix S via

$$S(\Psi_+ \overset{\text{out}}{\times} \Psi_-) = \Psi_+ \overset{\text{in}}{\times} \Psi_- \tag{5.9}$$

and say that a theory is interacting if $S \neq I$. The main result of [DT11] can be summarized as follows:

Theorem 5 *Let S be the scattering matrix of $(\mathcal{R}, U, \Omega)$ and let S_κ be the scattering matrix of $(\mathcal{R}_{Q_\kappa}, U, \Omega)$. Then*

$$S_\kappa = \mathrm{e}^{\mathrm{i}\kappa(H^2-P^2)} S. \tag{5.10}$$

In particular, a deformation of an asymptotically complete non-interacting theory is asymptotically complete and interacting.

The argument relies on (5.4), (5.5) and the following computation which relates scattering states of the defomed and undeformed theory: Let $F \in \mathcal{R}$, $F' \in \mathcal{R}'$ be smooth under translations. Then $F_{Q_\kappa} \in \mathcal{R}_{Q_\kappa}$, $F'_{-Q_\kappa} \in \mathcal{R}'_{Q_\kappa}$ and we can write

$$\begin{aligned}
\Psi_+ \overset{\text{out}}{\times}_\kappa \Psi_- &= \Phi_+^{\text{out}}(F_{Q_\kappa})\Phi_-^{\text{out}}(F'_{-Q_\kappa})\Omega \\
&= \int dE(q)\, \Phi_+^{\text{out}}(\alpha_{Q_\kappa q}(F))\Phi_-^{\text{out}}(F')\Omega \\
&= \int dE(q)\, (U(Q_\kappa q)\Psi_+) \overset{\text{out}}{\times} \Psi_- \\
&= \int dE(q)\, \mathrm{e}^{-\mathrm{i}\frac{1}{2}\kappa(H+P)(q^0-q^1)}(\Psi_+ \overset{\text{out}}{\times} \Psi_-) && (5.11) \\
&= \mathrm{e}^{-\mathrm{i}\frac{1}{2}\kappa(H^2-P^2)}(\Psi_+ \overset{\text{out}}{\times} \Psi_-), && (5.12)
\end{aligned}$$

where in (5.11) we used that $H\Psi_+ = P\Psi_+$ and the explicit form of the matrix Q_κ.

In the light of Theorem 5, to provide an example of an interacting and asymptotically complete theory, it suffices to exhibit a non-interacting and asymptotically complete one. Fortunately, the latter are in abundance. It turns out that all chiral Conformal Field Theories (CFT) have such properties. To summarize their construction, we first recall that a local net of von Neumann algebras on $\mathbb{R}$, denoted by $(\mathcal{A}_0, U_0, \Omega_0)$, consists of

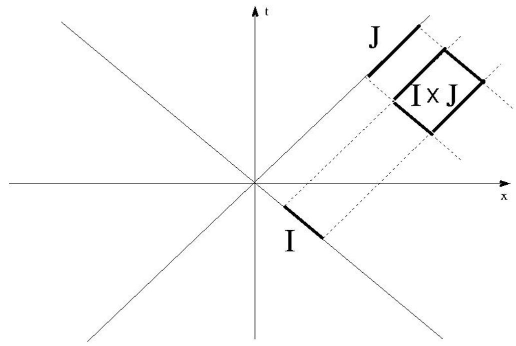


Figure 5.2. Tensor product construction of chiral CFT.

- a map $\mathbb{R} \supset I \to \mathcal{A}_0(I) \subset B(\mathcal{H})$ s.t.

$$\mathcal{A}_0(I) \subset \mathcal{A}_0(J) \text{ for } I \subset J,$$
$$[\mathcal{A}_0(I), \mathcal{A}_0(J)] = 0 \text{ for } I \cap J = \emptyset;$$

- a unitary representation $\mathbb{R} \ni s \to U_0(s)$ s.t.

$$\mathrm{Sp}\, \mathrm{U}_0 \subset \mathbb{R}_+, \tag{5.13}$$
$$U_0(s)\mathcal{A}_0(I)U_0(s)^{-1} = \mathcal{A}_0(I+s) \text{ for } s \in \mathbb{R}; \tag{5.14}$$

- a unique vacuum vector Ω_0, invariant under U_0, which is cyclic for any $\mathcal{A}_0(I)$.

Now we take two such nets and think of the two real lines as light-rays in $\mathbb{R}^2$. A chiral net of von Neumann algebras on $\mathbb{R}^2$ is then given by the tensor product construction (see Figure 5.2):

$$\mathfrak{A}(I \times J) := \mathcal{A}_0(I) \otimes \mathcal{A}_0(J), \tag{5.15}$$
$$U(t,x) := U_0((\sqrt{2})^{-1}(t-x)) \otimes U_0((\sqrt{2})^{-1}(t+x)), \tag{5.16}$$
$$\Omega := \Omega_0 \otimes \Omega_0.$$

Setting $\mathcal{R} := \mathfrak{A}(\mathcal{W})$ we obtain a Borchers triple $(\mathcal{R}, U, \Omega)$. By a simple application of the scattering theory discussed above one can show that this theory is non-interacting and asymptotically complete [DT11]. The latter property may be surprising, since chiral CFT are known to have 'charged sectors' which, as we discussed in Section 3, should be in conflict with asymptotic completeness. The solution of this apparent paradox is in the composite structure of the single-particle states in two-dimensional massless theories which is a consequence of the dispersionless motion. Differently than in the massive case or in higher dimensions, the single-particle subspace here always carries a highly reducible representation of the Poincaré group.

5.2 Charged sectors

In this section we consider theories which are local and relativistic but do not contain a vacuum vector or Wigner single-particle states. Standard examples are chiral CFT in charged representations. In this case the scattering theory discussed in the previous subsection does not apply. An alternative is the theory of particle weights from [BPS91], which relies on the concept of Araki-Haag detectors (3.12). The key observation is that these asymptotic observables may be non-zero even in the absence of Wigner particles and should carry information about more exotic excitations, e.g. infraparticles.

Let us briefly summarize the theory of particle weights from [BPS91]. Let $\mathcal{L}_0 \subset \mathfrak{A}$ be the subspace of almost-local and energy-decreasing operators, which contains, in particular, the generalised annihilation operators B of Section 3. We denote by $\mathcal{L} := \mathfrak{A}\mathcal{L}_0$ the resulting left ideal. For $L_1, L_2 \in \mathcal{L}$ and $\Psi \in \mathcal{H}$ of bounded energy the sequences

$$\psi^{(t)}(L_1, L_2) = \frac{1}{\ln|t|}\int_t^{t+\ln|t|} d\tau \int_{\mathbb{R}^d} dx\, \langle \Psi, \alpha_{(\tau,x)}(L_1^* L_2)\Psi\rangle, \tag{5.17}$$

have limit points ψ^{out} as $t \to \infty$. These limit points are examples of *particle weights* and can be seen as unbounded states (weights) on the algebra $\mathfrak{A}$. The GNS construction applies to particle weights as follows: The representation space is given by

$$\mathcal{H}_{\psi^{\mathrm{out}}} := (\,\mathcal{L}/\{\, L \in \mathcal{L} \,|\, \psi^{\mathrm{out}}(L, L) = 0\})^{\mathrm{cpl}} \tag{5.18}$$

and its generic elements are denoted by $|L\rangle$, $L \in \mathcal{L}$. It is a Hilbert space w.r.t. the scalar product $\langle L_1|L_2\rangle := \psi^{\mathrm{out}}(L_1, L_2)$. The GNS representation acts on this space in a natural manner

$$\pi_{\psi^{\mathrm{out}}}(A)|L\rangle := |AL\rangle, \quad A \in \mathfrak{A} \tag{5.19}$$

Typically this representation is highly reducible. By abstract decomposition theory we can express it as a direct integral of irreducible representations. Correspondingly, ψ^{out} can be decomposed into a statistical mixture of *pure particle weights* [Po04.1, Po04.2]

$$\psi^{\mathrm{out}} = \int_X d\mu(\lambda)\, \psi_\lambda, \tag{5.20}$$

where X is a certain measure space. We dispense here with an abstract definition of (pure) particle weights. We merely remark that in the context of massive theories decomposition (5.20) coincides with the Araki-Haag formula (3.13). In this case the pure particle weights ψ_λ are given by the plane wave configurations of the electron $\langle q_\lambda| \cdot |q_\lambda\rangle$. Such interpretation also applies in the present general setting and for any pure particle weight ψ_λ there is a canonically associated *characteristic four-momentum* (q_λ^0, q_λ). Denoting by π_{ψ_λ} the GNS representation of a pure particle weight ψ_λ, we say that a theory exhibits *velocity superselection* if

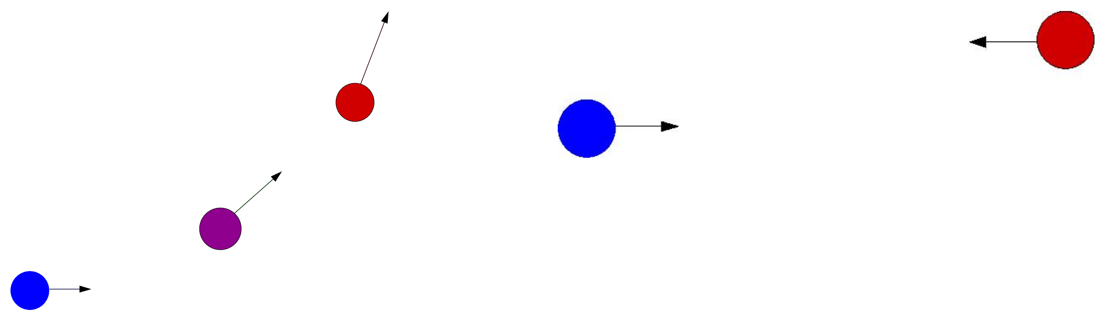
$$\frac{q_\lambda}{q_\lambda^0} \neq \frac{q_{\lambda'}}{q_{\lambda'}^0} \Rightarrow \pi_{\psi_\lambda} \not\simeq \pi_{\psi_{\lambda'}}, \tag{5.21}$$

for all $\lambda, \lambda' \in X$. (Cf. Figure 5.3 for a graphical illustration).

Of course velocity superselection does not occur in purely massive theories, as the GNS representations of particle weights $\langle q_\lambda| \cdot |q_\lambda\rangle$ are just copies of the defining vacuum representation. However, for the electron in QED velocity superselection is expected in the infraparticle approach. Here different velocities of the electron correspond to different values of the flux (4.2) which labels distinct sectors. This effect should be responsible for the absence of the sharp mass of the electron, which we discussed in Section 4: As plane wave configurations of the electron with different velocities 'live' in distinct sectors, they cannot be superposed into normalizable eigenvectors of the mass operator. These facts are rigorously established in models of non-relativistic QED, where velocity superselection has, however, a different mathematical formulation [CFP07].

First relativistic examples, for which velocity superselection holds, were found by Tanimoto and the present author in [DT12]. These models are chiral CFT in charged representations, constructed as follows: Let $(\mathcal{A}_0, V_0, \Omega_0)$ be a local net of von Neumann algebras on $\mathbb{R}$ with an internal $\mathbb{Z}_2$-symmetry implemented by a unitary W. We note that the *fixed-point subnet*

$$\mathcal{A}_{0,\mathrm{ev}}(I) := \{\, A \in \mathcal{A}_0(I) \,|\, WAW^* = A\,\} \tag{5.22}$$



(a) Velocity superselection for the electron in QED. (b) Velocity superselection for infraparticles in chiral CFT.

Figure 5.3. Graphical representation of velocity superselection. Colour indicates the superselection sector to which the given plane wave configuration belongs.

has two invariant subspaces: $\mathcal{K}_{\mathrm{ev}} := [\mathcal{A}_{0,\mathrm{ev}}\Omega_0]$, $\mathcal{K}_{\mathrm{odd}} := \mathcal{K}_{\mathrm{ev}}^{\perp}$. We set

$$\hat{\mathcal{A}}_0(I) := \mathcal{A}_{0,\mathrm{ev}}(I)|_{\mathcal{K}_{\mathrm{odd}}} \quad \text{and} \quad \hat{V}_0(s) := V_0(s)|_{\mathcal{K}_{\mathrm{odd}}}. \tag{5.23}$$

The chiral net $(\hat{\mathfrak{A}}, \hat{U})$ we are interested in results from two copies of $(\hat{\mathcal{A}}_0, \hat{V}_0)$ via the tensor product construction (5.15), (5.16). The main result of [DT12] can be summarized as follows:

Theorem 6 *The chiral net $(\hat{\mathfrak{A}}, \hat{U})$, resulting from two copies of $(\hat{\mathcal{A}}, \hat{V})$, describes excitations with* superselected velocity *(cf. Figure 5.3 (b)). More precisely, for any Ψ from a dense domain D there exists a decomposition*

$$\psi^{\mathrm{out}} = \int_X d\mu(\lambda)\, \psi_\lambda \tag{5.24}$$

such that:

(a) $\frac{q_\lambda}{q_\lambda^0} = 1 \quad \Rightarrow \pi_{\psi_\lambda}(\hat{\mathfrak{A}}) = \mathcal{A}_{\mathrm{ev}}|_{\mathcal{K}_{\mathrm{odd}}} \otimes \mathcal{A}_{\mathrm{ev}}|_{\mathcal{K}_{\mathrm{ev}}}$,

(b) $\frac{q_{\lambda'}}{q_{\lambda'}^0} = -1 \Rightarrow \pi_{\psi_{\lambda'}}(\hat{\mathfrak{A}}) = \mathcal{A}_{\mathrm{ev}}|_{\mathcal{K}_{\mathrm{ev}}} \otimes \mathcal{A}_{\mathrm{ev}}|_{\mathcal{K}_{\mathrm{odd}}}$.

The two representations above are not equivalent due to a different structure of the energy-momentum spectrum.

To conclude this section, we remark that Araki-Haag detectors can also be used to establish a weak variant of asymptotic completeness for infraparticles in charged sectors of chiral CFT. In [DT13] Tanimoto and the present author could show that for such theories every vector in the Hilbert space is in the range of a product of two detectors of the form

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{\ln|t|}\int_t^{t+\ln|t|} d\tau_+ \int_{\mathbb{R}} dx_+\, \alpha_{(\tau_+,x_1)}(B_1^* B_1) h_+\left(\frac{x_+}{\tau_+}\right) \\ \cdot \frac{1}{\ln|t|}\int_t^{t+\ln|t|} d\tau_- \int_{\mathbb{R}} dx_-\, \alpha_{(\tau_-,x_-)}(B_2^* B_2) h_-\left(\frac{x_-}{\tau_-}\right),\end{aligned} \tag{5.25}$$

where $h_\pm$ are supported close to velocities ± 1, respectively. Thus every state in such theories gives a non-zero response in a coincidence arrangement of detectors sensitive to a pair of excitations moving with $\pm$ velocity of light.

References

[AD17] S. Alazzawi and W. Dybalski. *Compton scattering in the Buchholz-Roberts framework of relativistic QED*. Lett. Math. Phys. **107**, (2017) 81–106.

[Ar] H. Araki. *Mathematical theory of quantum fields*. Oxford University Press, 1993.

[AH67] H. Araki and R. Haag. *Collision cross-sections in terms of local observables*. Commun. Math. Phys. **4**, (1967) 77–91.

[AB01] F. Auil and J.C.A. Barata. *Scattering and bound states in Euclidean lattice Quantum Field Theories.* Ann. Henri Poincaré **2**, (2001) 1065–1097.

[Ar82] W. Arveson. *The harmonic analysis of automorphism groups*. In Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I.,1982.D., pp. 199-269.

[BFS98.2] V. Bach, J. Fröhlich and I. M. Sigal. *Quantum electrodynamics of confined nonrelativistic particles.* Adv. Math. **137**, (1998) 299–395.

[BDP12] S. Bachmann, D.-A. Deckert and A. Pizzo. *The mass shell of the Nelson model without cut-offs.* J. Funct. Anal. **263**, (2012) 1224–1282.

[BDN16] S. Bachmann, W. Dybalski and P. Naaijkens. *Lieb-Robinson bounds, Arveson spectrum and Haag-Ruelle scattering theory for gapped quantum spin systems.* Ann. Henri Poincaré **17**, (2016) 1737–1791.

[BF91] J.C.A. Barata and K. Fredenhagen. *Particle scattering in Euclidean lattice field theories.* Commun. Math. Phys. **138**, (1991) 507–519.

[BBS01] H.-J. Borchers, D. Buchholz and B. Schroer, *Polarization-free generators and the S-matrix.* Commun. Math. Phys. **219**, (2001) 125-140.

[Bos05] H. Bostelmann. *Phase space properties and the short distance structure in quantum field theory.* J. Math. Phys. **46**, (2005) 052301.

[Bu75] D. Buchholz. *Collision theory for waves in two dimensions and a characterization of models with trivial S-matrix.* Commun. Math. Phys. **45**, (1975) 1–8.

[Bu77] D. Buchholz. *Collision theory for massless bosons*. Commun. Math. Phys. **52**, (1977) 147–173.

[Bu82] D. Buchholz. *The physical state space of quantum electrodynamics*. Commun. Math. Phys. **85**, (1982) 49-71.

[Bu86] D. Buchholz. *Gauss' law and the infraparticle problem.* Phys. Lett. B **174**, (1986) 331–334.

[Bu90] D. Buchholz. *Harmonic analysis of local operators*. Commun. Math. Phys. **129**, (1990) 631–641.

[BPS91] D. Buchholz, M. Porrmann and U. Stein. *Dirac versus Wigner: Towards a universal particle concept in quantum field theory.* Phys. Lett. B **267**, (1991) 377-381.

[Bu94] D. Buchholz. *On the manifestations of particles.* In: Mathematical Physics Towards the 21st Century. Proceedings Beer-Sheva 1993, R.N. Sen and A. Gersten Eds., Ben-Gurion University of the Negev Press 1994.

[BR14] D. Buchholz and J.E. Roberts. *New light on infrared problems: sectors, statistics, symmetries and spectrum.* Commun. Math. Phys **330**, (2014) 935-972.

[Bur77] C. Burnap. *Isolated one particle states in boson quantum field theory models.* Ann. of Phys. **104**, (1977) 184–196.

[BLS10] D. Buchholz, G. Lechner and S. Summers. *Warped convolutions, Rieffel deformations and the construction of Quantum Field Theories.* Commun. Math. Phys. **304**, (2010) 95–123.

[CD18] D. Cadamuro and W. Dybalski. *Relative normalizers of automorphism groups, infravacua and the problem of velocity superselection in QED.* `arXiv:1807.07919`.

[CD19] D. Cadamuro and W. Dybalski. *Curing velocity superselection in non-relativistic QED by restriction to a lightcone.* `arXiv:1902.09478`.

[CFP07] T. Chen, J. Fröhlich and A. Pizzo. *Infraparticle scattering states in non-relativistic QED: I. The Bloch-Nordsieck paradigm.* Commun. Math. Phys. **294**, (2010) 761–825.

[CD82] M. Combescure and F. Dunlop. *Three-body asymptotic completeness for $P(\phi)_2$ models.* Commun. Math. Phys. **85**, (1982) 381–418.

[DH15] P. Duch and A. Herdegen. *Massless asymptotic fields and Haag-Ruelle theory.* Lett. Math. Phys. **105**, (2015) 245–277.

[De93] J. Dereziński. *Asymptotic completeness of long-range N-body quantum systems.* Ann. of Math. **138**, (1993) 427–476.

[De18] J. Dereziński. *Review of Ludwig Faddeev Memorial Volume. A Life in Mathematical Physics.* IAMP News Bull., pages 68–76, January 2019.

[DG] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems.* Springer 1997.

[DGK13] W. De Roeck, M. Griesemer and A. Kupiainen. *Asymptotic completeness for the massless spin-boson model.* Advances in Mathematics **268**, (2015) 62–84.

[Do64] J.D. Dollard. *Asymptotic convergence and the Coulomb interaction.* J. Math. Phys. **5**, (1964) 729–738.

[Du17] M. Duell. *Strengthened Reeh-Schlieder property and scattering in quantum field theories without mass-gaps.* Commun. Math. Phys. **352**, (2017) 935–966.

[Du18] M. Duell. *N-particle scattering in relativistic wedge-local quantum field theory.* Commun. Math. Phys. **364**, (2018) 203–232.

[Dy05] W. Dybalski. *Haag-Ruelle scattering theory in presence of massless particles.* Lett. Math. Phys. **72**, (2005) 27–38.

[Dy18] W. Dybalski. *Asymptotic observables in gapped quantum spin systems.* Commun. Math. Phys. **357**, (2018) 231–248.

[DG14] W. Dybalski and C. Gérard. *Towards asymptotic completeness of two-particle scattering in local relativistic QFT.* Commun. Math. Phys. **326**, (2014) 81–109.

[DG14.1] W. Dybalski and C. Gérard. *A criterion for asymptotic completeness in local relativistic QFT.* Commun. Math. Phys. **332**, (2014) 1167–1202.

[Dy17] W. Dybalski. *From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons.* Nuclear Physics B **925**, 455–469, (2017).

[DG99] J. Dereziński and C. Gérard. *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians.* Rev. Math. Phys. **11**, (1999) 383–450.

[DM15] W. Dybalski and J.S. Møller. *The translation invariant massive Nelson model: III. Asymptotic completeness below the two-boson threshold.* Ann. Henri Poincaré **16**, (2015) 2603–2693.

[DT11] W. Dybalski and Y. Tanimoto. *Asymptotic completeness in a class of massless relativistic quantum field theories.* Commun. Math. Phys. **305**, (2011) 427–440.

[DT12] W. Dybalski and Y. Tanimoto. *Infraparticles with superselected direction of motion in two-dimensional conformal field theory.* Commun. Math. Phys. **311**, (2012) 457–490.

[DT13] W. Dybalski and Y. Tanimoto. *Asymptotic completeness for infraparticles in two-dimensional conformal field theory.* Lett. Math. Phys. **103**, (2013) 1223–1241.

[En78] V. Enss. *Asymptotic completeness for quantum mechanical potential scattering.* Commun. Math. Phys. **61**, (1978) 285–291.

[FK70] L. D. Faddeev and P. P. Kulish. *Asymptotic conditions and infrared divergences in quantum electrodynamics.* Theor. Math. Phys. **5**, (1970) 153–166.

[FS12.1] J. Faupin and I. M. Sigal. *On Rayleigh scattering in non-relativistic quantum electrodynamics.* Commun. Math. Phys. **328**, (2014) 1199–1254.

[FS12] J. Faupin and I. M. Sigal. *On quantum Huygens principle and Rayleigh scattering.* `arXiv:1202.6151v1`.

[Fr73] J. Fröhlich. *On the infrared problem in a model of scalar electrons and massless, scalar bosons.* Ann. Inst. H. Poincaré Sect. A (N.S.) **19**, (1973) 1–103.

[Fr74] J. Fröhlich. *Existence of dressed one electron states in a class of persistent models.* Fortschr. Phys. **22**, (1974) 158–198.

[FMS79] J. Fröhlich, G. Morchio and F. Strocchi. *Charged sectors and scattering states in quantum electrodynamics.* Ann. Phys. **119**, (1979) 241–284.

[FMS79.1] J. Fröhlich, G. Morchio and F. Strocchi. *Infrared problem and spontaneous breaking of the Lorentz group in QED.* Phys. Lett. B **89**, (1979) 61–64.

[FGS01] J. Fröhlich, M. Griesemer and B. Schlein. *Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field.* Adv. Math. **164**, (2001) 349–398.

[FGS02] J. Fröhlich, M. Griesemer and B. Schlein. *Asymptotic completeness for Rayleigh scattering.* Ann. Henri Poincaré **3**, (2002) 107–170.

[FGS04] J. Fröhlich, M. Griesemer and B. Schlein. *Asymptotic completeness for Compton scattering.* Commun. Math. Phys. **252**, (2004) 415–476.

[FGS07] J. Fröhlich, M. Griesemer and B. Schlein. *Rayleigh scattering at atoms with dynamical nuclei*. Commun. Math. Phys. **271**, (2007) 387–430.

[GJS73] J. Glimm, A. Jaffe and T. Spencer. *The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions: Part I. Physics of quantum field models. Part II. The cluster expansion.* In: Constructive quantum field theory. (Erice, 1973), G. Velo, A. S. Wightman (eds.). Berlin, Heidelberg, New York: Springer 1973.

[Gr90] G. M. Graf. *Asymptotic completeness for N-body short-range quantum systems: a new proof.* Commun. Math. Phys. **132** (1990), 73–101.

[GS97] G. M. Graf and D. Schenker. *2-magnon scattering in the Heisenberg model.* Ann. Inst. H. Poincaré Phys. Théor. **67** (1997) 91–107.

[GZ09] M. Griesemer and H. Zenk. *Asymptotic electromagnetic fields in non-relativistic QED: The problem of existence revisited*. J. Math. Anal. Appl. **354**, (2009) 339–346.

[GL08] H. Grosse and G. Lechner. *Wedge-local quantum fields and non-commutative Minkowski space.* JHEP **0711**, (2007) 12-39.

[Ha58] R. Haag. *Quantum field theories with composite particles and asymptotic conditions*. Phys. Rev. **112**, (1958) 669–673.

[Ha] R. Haag. *Local quantum physics*. Second edition. Springer, 1996.

[He14] A. Herdegen. *Infraparticle problem, asymptotic fields and Haag-Ruelle theory.* Ann. Henri Poincaré **15**, (2014) 345–367.

[HS95] M. Hübner and H. Spohn. *Radiative decay: Non-perturbative approaches.* Rev. Math. Phys. **07**, (1995) 363–387.

[KK90] D. L. Karatas and K.L. Kowalski. *Noether's theorem for local gauge transformations.* Am. J. Phys. **58**, (1990) 123–147.

[Kr82] K. Kraus. *Aspects of the infrared problem in quantum electrodynamics.* Found. Phys. **13**, (1983) 701–713.

[KPR77] K. Kraus, L. Polley and G. Reents. *Models for infrared dynamics. I. Classical currents.* Ann. Inst. H. Poincaré **26**, (1977) 109–162.

[Le08] G. Lechner. *Construction of quantum field theories with factorizing S-matrices*. Commun. Math. Phys. **277**, (2008) 821–860.

[LSZ55] H. Lehmann, K. Symanzik and W. Zimmermann. *Zur Formulierung quantisierter Feldtheorien*. Il Nuovo Cimento **1**, (1955) 205-225.

[LR72] E.H. Lieb and D.W. Robinson. *The finite group velocity of quantum spin systems*. Commun. Math. Phys. **28** (1972) 251–257.

[Ma83] V.A. Malyshev. *Elementarnoe vvedenie v matematicheskuyu fiziku beskonechnochastichnykh sistem [in Russian]*. Lektsii dlya Molodykh Uchenykh [Lectures for Young Scientists], R17-83-363. Ob'ed. Inst. Yadernykh Issled., Dubna, 1983.

[Mi12] T. Miyao. *Monotonicity of the polaron energy.* Rep. Math. Phys. **74**, (2014) 379–398.

[Mo81] E. Mourre. *Absence of singular continuous spectrum for certain selfadjoint operators.* Commun. Math. Phys. **78**, (1981) 519–567.

[Mo05] J. S. Møller. *The translation invariant massive Nelson model: I. The bottom of the spectrum.* Ann. Henri Poincaré **6**, (2005) 1091–1135.

[MR12] J. S. Møller and M. G. Rasmussen. *The translation invariant massive Nelson model: II. The continuous spectrum below the two-boson threshold.* Ann. Henri Poincaré **14**, (2013) 793–852.

[Ne64] E. Nelson. *Interaction of non-relativistic particles with a quantized scalar field.* J. Math. Phys. **5**, (1964) 1190.

[OS76] K. Osterwalder and R. Sénéor. *The scattering matrix is non-trivial for weakly coupled $P(\phi)_2$ models.* Helv. Phys. Acta **49**, (1976) 525–535.

[Pi05] A. Pizzo. *Scattering of an infraparticle: the one particle sector in Nelson's massless models.* Ann. Henri Poincaré **4**, (2005) 553–606.

[Po93] M. Pokorny. *Continuous spectrum in the ground state of two spin-1/2 models in the infinite-volume limit.* J. Stat. Phys. **72**, (1993) 381–403.

[Po04.1] M. Porrmann. *Particle weights and their disintegration I.* Commun. Math. Phys. **248**, (2004) 269-304.

[Po04.2] M. Porrmann. *Particle weights and their disintegration II.* Commun. Math. Phys. **248**, (2004) 305-333.

[RS3] M. Reed and B. Simon. *Methods of modern mathematical physics. Part III: Scattering theory.* Academic Press, 1979.

[Ru62] D. Ruelle. *On the asymptotic condition in quantum field theory.* Helv. Phys. Acta **35**, (1962) 147–163.

[Sp97] H. Spohn. *Asymptotic completeness for Rayleigh scattering.* J. Math. Phys. **38**, (1997) 2281.

[St17] A. Strominger. *Lectures on the infrared structure of gravity and gauge theory.* `arXiv:1703.05448`.

[SZ76] T. Spencer and F. Zirilli. *Scattering states and bound states in $\lambda P(\phi)_2$.* Commun. Math. Phys. **49**, (1976) 1–16.

[SiSo87] I. M. Sigal and A. Soffer. *The N-particle scattering problem: asymptotic completeness for short-range systems.* Ann. of Math. **126**, (1987) 35–108.

[Si18] B. Simon. *Tosio Kato's work on non-relativistic quantum mechanics. A brief report.* IAMP News Bull., pages 6–25, January 2018.

[Ta14] Y. Tanimoto. *Construction of two-dimensional quantum field models through Longo-Witten endomorphisms.* Forum Math. Sigma 2, 2014.

[Ya04] D.A. Yarotsky. *Perturbations of ground states in weakly interacting quantum spin systems.* J. Math. Phys. 45, (2004) 2134.

[Ya04.1] D.A. Yarotsky. *Quasi-particles in weak perturbations of non-interacting quantum lattice systems.* Preprint, arXiv:math-ph/0411042.

[Ya05] D.A. Yarotsky. *Ground states in relatively bounded quantum perturbations of classical lattice systems*. Commun. Math. Phys. **261**, (2005) 799–819.