

# Stability results for the reconstruction of binary pictures from two projections <sup>☆</sup>

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## Abstract

In the present paper we mathematically prove several stability results concerning the problem of reconstructing binary pictures from their noisy projections taken from two directions. Stability is a major requirement in practice, because projections are often affected by noise due to the nature of measurements. Reconstruction from projections taken along more than two directions is known to be a highly unstable task. Contrasting this result we prove several theorems showing that reconstructions from two directions closely resemble the original picture when the noise level is low and the original picture is uniquely determined by its projections.

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**Keywords:** Discrete tomography; Stability; Image reconstruction; Uniqueness; Projections

## 1. Introduction

A *binary picture* can be considered as an  $n$  by  $n$  array of pixels (or resolution cells) that are coloured either black or white. A *projection* of a picture is an ordered set of values that are sums over the grey levels of the pixel centres (typically 1 for black and 0 for white) each taken along a line out of a set of parallel lines through the picture. The set of parallel lines is called the *projection direction*, and given some projections (usually from several projection directions) the general *reconstruction problem* is to reconstruct the original picture or at least an approximation of it.

This reconstruction task is one of the most prominent problems that is studied in the field of *Discrete Tomography* (see [2] for a survey) and arises in many different contexts, e.g., in electron microscopy [3,4], data security [5], medical

imaging [6], combinatorial optimization [7,8], combinatorics [9] or image deconvolution [10]. In pattern recognition, e.g., one often uses projections as a compressed representation of the picture, which usually is accompanied by a loss of information. If the projections, however, uniquely determine the reconstruction then there is no such loss. Thus the interest in studying the relationship between pictures and projections for image processing applications is twofold: On the one hand it is important to examine properties of the projections that reflect the effect of image processing operations such as addition, averaging and differencing, on the pictures. Such results (see [11] for an example) permit to work directly on the projections without having to reconstruct the underlying pictures. On the other hand, modifications of the projections can have drastic effects on the reconstruction and need to be studied because projections are often obtained by measuring devices, which are affected by noise. In this context the question is whether the picture reconstructed from noisy projections is a good approximation of the original picture. The reconstruction process is called *stable* if a small amount of noise can only lead to small differences in the reconstruction. This question can be either approached directly or reversely by

<sup>☆</sup> A preliminary version appeared in the Proceedings of the 12th DGCI conference [1], held in Poitiers, France, April 2005.

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showing that a large difference in the reconstruction can only occur if the difference of the noisy and noise-free projections is also large. In this paper, we consider binary pictures and projections taken from two directions, and we pursue both of the just described approaches.

Gale and Ryser [7,9] (and later Wang [12]) were the first to state necessary and sufficient conditions for the two projections to uniquely determine the reconstructed picture. Uniqueness is a reasonable requirement for a first step towards a stable reconstruction since otherwise two disjoint pictures may exist with the same projections. In the following we therefore always require that the original picture is uniquely determined by its projections.

In Section 3, we show that the reconstruction problem is stable if the difference in the projections (called *projection error*), measured in the  $\ell_1$ -norm, is less or equal to 2 (Theorem 19). This is in striking contrast to the unstable behaviour of the reconstruction from more than two projections [13]. Note that a similar discrepancy holds for the algorithmic complexity: While reconstruction from two projections can be solved in polynomial time it is well-known that it is  $\mathbb{NP}$ -hard when more than two projections are taken [14].

It remains an open question whether a larger projection error still permits stable reconstructions. However, we show in Section 4 (Theorem 27) that a similar combinatorial reasoning leads to provable stability results when the requirement of uniqueness for the original picture is weakened to the assumption that in every reconstruction there exists a suitable number of invariant points [15]. Indeed, the upper bound for the difference in the pictures depends on the number of invariant points. In Section 5, we determine a lower bound on the projection error in the case where the original and reconstructed pictures are disjoint. As a result we obtain that a smaller projection error has to lead to a non-empty intersection of both pictures.

We close the introduction by giving some pointers to the literature. Reports on the stable behaviour of reconstruction algorithms that incorporate a-priori knowledge can be found in [16–19]. To our knowledge the only sources containing theoretically proven stability results are [16,17,20,21]. The relevant results in these papers concerning projections from two directions either address reconstructions with smaller projection errors [20] or they require that even the reconstruction is uniquely determined by its projections [17,21]. Our Theorem 19 is a generalization of Theorem 5.1.18 in [16] and of Theorem 17 in [1]. More about the reconstruction of binary pictures from noise-free projections in the context of  $(0,1)$ -matrices can be found in [22]. Invariant sets have been intensively studied, e.g., by [15,23–25].

**2. Notations and statement of the problems**

Since we consider projections taken from two directions we can assume, without loss of generality, that these two directions are the horizontal and the vertical ones, and we will refer to them as the set of row and column sums.

Let  $\mathbb{Z}^2$  be the set of points with integer coordinates. A binary picture on  $\mathbb{Z}^2$  can be represented by a 0/1-valued function  $f$  whose value is 0 for all but finitely many points of  $\mathbb{Z}^2$  corresponding to the black pixels (see e.g., [26]). Then, a binary picture is a member of  $\mathcal{F}^2 := \{F \subset \mathbb{Z}^2 : F \text{ is finite}\}$ , i.e., it is a (finite) *lattice set*. The function  $f$  is the characteristic function of  $F$ , usually denoted by  $\mathbf{1}_F$ , i.e.,  $\mathbf{1}_F(i,j) = 1$  if  $(i,j) \in F$ , and  $\mathbf{1}_F(i,j) = 0$ , otherwise. Let us consider the horizontal and vertical directions. Let  $\mathcal{R} := \{(x,y) \in \mathbb{Z}^2 : x_{\min} \leq x \leq x_{\max}, y_{\min} \leq y \leq y_{\max}\}$ , where  $x_{\min} = \min_{(x,y) \in F} \{x\}$ ,  $y_{\min} = \min_{(x,y) \in F} \{y\}$ ,  $x_{\max} = \max_{(x,y) \in F} \{x\}$ ,  $y_{\max} = \max_{(x,y) \in F} \{y\}$ , denote the smallest rectangle containing  $F$ . The projection of  $F$  along the  $i$ th row of  $\mathcal{R}$ , the *row sum*, is given by  $r_i := \sum_{j \in \mathbb{Z}} \mathbf{1}_F(j,i)$ , and the projection of  $F$  along the  $i$ th column of  $\mathcal{R}$ , the *column sum*, is given by  $c_i := \sum_{j \in \mathbb{Z}} \mathbf{1}_F(i,j)$ .

We use the convention that the rows and columns are numbered starting from the left-upper corner of  $\mathcal{R}$  (see Fig. 1a). Since  $F$  is finite we can assemble the non-zero  $r_i$ 's and  $c_i$ 's into two vectors (the *row and column sum vectors*), that we denote by  $R$  and  $C$ , respectively. The  $\ell_1$ -norm of a vector  $v$ , being the sum of the absolute values of its components, will be denoted by  $\|v\|_1$ ; the cardinality of  $F$  will be denoted by  $|F|$ ; and the symmetric difference of two pictures  $F$  and  $F'$  by  $F \Delta F'$ .

We are interested in theoretical results giving stability guarantees if the row and column sums contain ‘‘noise’’, i.e., we have to compare an original set  $F$  and a possible reconstruction  $F' \in \mathcal{F}^2$  by taking  $|F \Delta F'|$ , and compare the difference in their projections, called *projection error*, by  $\|R - R'\|_1 + \|C - C'\|_1$  (where  $R'$  and  $C'$  denote row and column sums of  $F'$ ). In doing so we assume that the  $i$ th entries of  $R$  and  $R'$  refer to the same row, which can be achieved by inserting zero-entries. The same assumption is made on the entries of  $C$  and  $C'$ . We note in passing that the symmetric difference can be interpreted as  $\|\mathbf{1}_F - \mathbf{1}_{F'}\|_1$  in an appropriate function space.

We will provide answers to the following problems.

**Problem 1.** Let  $F \in \mathcal{F}^2$  be uniquely determined by its row and column projections  $R$  and  $C$ . Determine a sharp upper bound for  $\max_{F'} |F \Delta F'|$  where  $F'$  fulfills  $\|R - R'\|_1 + \|C - C'\|_1 \leq 2$ .

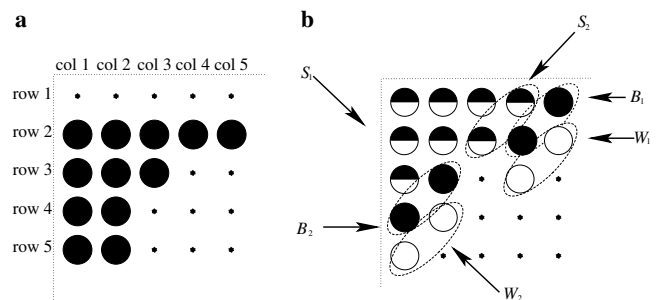


Fig. 1. An illustration of sets  $F$  (black points) and  $F'$  (white points). Points of  $F \cap F'$  are coloured half white and half black. (a) The enumeration of rows and columns. (b) An example showing  $F$  and  $F'$  as in Lemma 15.

We also consider a slightly more general problem where  $F$  is not uniquely determined, but contains invariant points. Furthermore, we investigate:

**Problem 2.** Let  $F \in \mathcal{F}^2$  be uniquely determined by its row and column projections  $R$  and  $C$ . Find a (close) lower bound for  $\max_{F'} (\|R - R'\|_1 + \|C - C'\|_1)$ , where  $R'$  and  $C'$  are row and column projections of  $F'$  fulfilling  $|F| = |F'|$  and  $|F \Delta F'| = 2|F|$ .

Finally, some remarks about the assumption of uniqueness: We can change the coordinates of each point in  $\mathcal{R}$  by first permuting the columns so that  $C$  is a non-increasing vector, and then permuting the rows so that  $R$  is also a non-increasing vector. This is a one-to-one function on the points of the rectangle yielding a triangular shape when the set is uniquely determined by its projections (it is a *maximal matrix* with non-increasing row sums, see [9]). So, we shall assume that  $F$  has a triangular shape, as in Fig. 1a.

**Remark 3.** In summary, we assume in the rest of the paper that  $F$  is a maximal matrix with sorted rows (or equivalently, row and column sum vectors are monotone and uniquely determine  $F$ ).

A last remark concerning the notation. If  $p$  is a point of  $F$ , lying in the  $i$ th row and  $j$ th column, we write  $\text{row}(p) = i$  and  $\text{col}(p) = j$ . Sometimes we do not distinguish between the row itself and its index.

### 3. Problem 1: a fixed projection error

The main part of this section comprises the results of Sections 3–5 from [1]. However, Theorem 19 is a generalized version of its counterpart ([1], Theorem 17), as will be discussed at the end of this section.

#### 3.1. Preliminary remarks

Unless stated otherwise we will always assume  $|F| = |F'|$  and  $\|R - R'\|_1 + \|C - C'\|_1 = 2$ . We will drop these requirements only in the proof of Theorem 19. Assuming a projection error of 2 means that the error occurs on exactly two lines of a single direction. This follows from the assumption  $|F| = |F'|$ . Indeed the sum of the projection values along the rows (or columns) equals the cardinality of the set to be reconstructed (cf. [16,20]), and so if there is exactly one line where the error is  $-1$  (which will in the following mean that there is one point more of  $F' \setminus F$  than of  $F \setminus F'$  on the line), then there exists exactly one line with a  $+1$ -error (meaning that there is one point more of  $F \setminus F'$  than of  $F' \setminus F$  on the line). Furthermore, by possibly rotating  $F$  and  $F'$  we may assume that the error occurs along horizontal lines (rows). If  $F'$  contains a single point on a horizontal line, then we can assume without loss of generality that it is row 1. Therefore, we assume in this section that the error occurs in exactly two rows (consequently in no columns).

**Remark 4.** Let  $p$  be any point of  $F' \setminus F$  and let  $q$  be any point of  $F \setminus F'$ .

- (a) From the shape of  $F$ , we have:
  - if  $\text{col}(p) = \text{col}(q)$ , then  $\text{row}(p) > \text{row}(q)$ ;
  - if  $\text{row}(p) = \text{row}(q)$ , then  $\text{col}(p) > \text{col}(q)$ .
- (b) Since no column error occurs: the point  $p$  exists if and only if the point  $q$  exists, with  $\text{col}(p) = \text{col}(q)$ . Similarly, in the rows in which no error occurs:  $p$  exists if and only if  $q$  exists, with  $\text{row}(p) = \text{row}(q)$ .

**Lemma 5.** Let  $i$  be the row where the  $-1$  error occurs, and let  $j$  be the row where the  $+1$  error occurs. Then  $i > j$ .

**Proof.** Suppose that  $i < j$ ; then we show that there is an infinite sequence of points in  $F \Delta F'$  starting with a point  $p$  of  $F' \setminus F$  in the  $i$ th row. Since the  $-1$  error occurs in the  $i$ th row, at least a point  $p$  of  $F' \setminus F$  exists in this row. By Remark 4 (a) and (b), for any  $p$ , a point  $q$  of  $F \setminus F'$  exists such that  $\text{col}(p) = \text{col}(q)$  and  $i = \text{row}(p) > \text{row}(q)$ . Since there is no error in any row with index less than  $i$ , we conclude with Remark 4 (a) that there exists also a point  $p'$  of  $F' \setminus F$  in  $\text{row}(q)$ , then again a point  $q'$  of  $F \setminus F'$ , etc., all of which lie in a row with index less than  $i$ . This leads to an infinite sequence of points in  $F \Delta F'$ , which is not possible.  $\square$

#### 3.2. Staircases

**Definition 6.** Let  $A, B \in \mathcal{F}^2$  with  $A \cap B = \emptyset$ . A staircase  $T = (t_1, \dots, t_m)$  according to the columns is a sequence of an even number  $m > 0$  of distinct points  $t_{2i+1} \in A$  and  $t_{2i+2} \in B$  for  $0 \leq i \leq \frac{m}{2} - 1$  such that

- (i)  $\text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$  and  $\text{row}(t_{2i+1}) > \text{row}(t_{2i+2})$  for  $0 \leq i \leq \frac{m}{2} - 1$ ;
- (ii)  $\text{row}(t_{2i}) = \text{row}(t_{2i+1})$  and  $\text{col}(t_{2i}) < \text{col}(t_{2i+1})$  for  $1 \leq i \leq \frac{m}{2} - 1$ .

The definition of a staircase according to the rows is obtained by exchanging the words “row” by “column” and “ $A$ ” by “ $B$ ”.

A staircase can be interpreted geometrically. It is a rook path of alternating points of  $A$  and  $B$  with end points  $t_1$  and  $t_m$ . We refer to  $(t_{2i}, t_{2i+1})$  as a *horizontal step* and to  $(t_{2i+1}, t_{2i+2})$  as a *vertical step* of  $T$ . Since  $A$  and  $B$  are finite, every staircase has a finite number of points. A staircase that is not a proper subset of another staircase is called a *maximal staircase*. Below we will show that there is only one single maximal staircase if  $A = F' \setminus F$  and  $B = F \setminus F'$ . The points of  $A$  will be called *white points*, and the points of  $B$  will be called *black points*.

#### 3.3. Technical lemmas

In the following we will speak about staircases in  $F \Delta F'$ , which implicitly means that  $A = F' \setminus F$  and  $B = F \setminus F'$ . Now,

we are going to show that the symmetric difference of  $F$  and  $F'$  is a maximal staircase in  $F\Delta F'$ .

**Remark 7.** Notice that for every  $p \in F\Delta F'$  there exists a staircase  $T = (t_1, \dots, t_m)$  (possibly constituted by two points) such that  $p$  is an element of  $T$ . This can be easily deduced from Remark 4.

**Lemma 8.** Any two maximal staircases in  $F\Delta F'$  have the same starting point and the same end point.

**Proof.** Let  $T_1 = (t_1, \dots, t_m)$  and  $T_2 = (s_1, \dots, s_n)$  be any two maximal staircases in  $F\Delta F'$ . We are going to show that  $t_1 = s_1$  and  $t_m = s_n$ . Since the  $-1$  error occurs in exactly one row (say  $i$ ), it follows that  $\text{row}(t_1) = \text{row}(s_1) = i$ . If  $\text{col}(t_1) \neq \text{col}(s_1)$ , a black point, say  $q$ , exists in the  $i$ th row. By Remark 4, a white point  $p$  exists such that  $\text{col}(p) = \text{col}(q)$  and  $\text{row}(p) > i$ . Since there is no error in any row  $k > i$  this leads to an infinite sequence of points in  $F\Delta F'$ . Analogously, since the  $+1$  error occurs in exactly one row (say  $j$ ), we have  $\text{row}(t_m) = \text{row}(s_m) = j$ , and  $\text{col}(t_m) = \text{col}(s_n)$  because otherwise (with Remark 4) there is an infinite sequence of black and white points in  $F\Delta F'$ .  $\square$

**Remark 9.** By Lemma 8 every maximal staircase starts in a white point  $t_1$  and ends in a black point  $t_m$ . So, there is exactly one white point in  $\text{row}(t_1)$  and  $\text{col}(t_1)$  and one black point in  $\text{row}(t_m)$  and  $\text{col}(t_m)$ . Moreover there is no black or white point outside the rectangle that is made up of the rows between  $\text{row}(t_1)$  and  $\text{row}(t_m)$ , and the columns between  $\text{col}(t_1)$  and  $\text{col}(t_m)$ .

**Lemma 10.** Any two maximal staircases in  $F\Delta F'$  coincide.

**Proof.** Let  $T_1 = (t_1, \dots, t_m)$  and  $T_2 = (s_1, \dots, s_n)$  be any two maximal staircases according to the columns in  $F\Delta F'$ . We are going to show that  $m = n$  and  $t_i = s_i$  for  $i = 1, \dots, m$ . Since  $t_1 = s_1$  there is no white point other than  $t_1$  in  $\text{col}(t_1)$  by Remark 9. It follows that exactly one black point lies on this column, that is,  $t_2 = s_2$ . Analogously, we can conclude that  $t_{m-1} = s_{n-1}$ . Consider now  $T_1 \setminus \{t_1, t_m\}$  and  $T_2 \setminus \{s_1, s_n\}$ . They are two staircases according to the rows. Proceeding as before, we conclude that  $t_3 = s_3$  and  $t_{m-2} = s_{n-2}$ . We repeat the procedure alternately on a staircase according to the rows and one according to the columns until an empty set is achieved. So,  $t_i = s_i$  for  $i = 1, \dots, m$ .  $\square$

The previous lemmas prove the following proposition.

**Proposition 11.** The points of  $F\Delta F'$  constitute a maximal staircase.

### 3.4. Bounds

In this section, we give an upper bound for the number of points on any maximal staircase, when we fix  $F$  but may vary  $F'$ . This gives a sharp bound on  $|F\Delta F'|$  since the maximal staircase contains exactly the points of  $F\Delta F'$ .

Let  $T = (t_1, \dots, t_m)$  denote this staircase, and let  $\mathcal{R} = \{1, \dots, a\} \times \{1, \dots, b\}$  be the rectangle containing  $F$  having non-empty rows and columns. Clearly, there is at most one point  $t_1$  of  $T$  outside of  $\mathcal{R}$ , and for this point we have  $1 \leq \text{col}(t_1) \leq b$ , while all the other points of  $T$  are inside of  $\mathcal{R}$ . Without loss of generality we can assume that  $F' \subseteq \{0, \dots, a\} \times \{1, \dots, b\}$ . So, we have  $R \in \mathbb{N}_0^a$  and  $C \in \mathbb{N}_0^b$ .

**Proposition 12.** Let  $R \in \mathbb{N}_0^a$  and  $C \in \mathbb{N}_0^b$  uniquely determine  $F \in \mathcal{F}^2$ . Then,

$$\max_{F' \in \mathcal{F}^2(F)} |F\Delta F'| \leq 2 \min(a+1, b).$$

**Proof.** Since  $F\Delta F'$  forms a staircase  $T$ , it is immediately clear that an upper bound for the symmetric difference can be obtained by counting two times the number of vertical steps in  $T$  that, in turn, is less than  $b$ . But another bound is given by adding 2 (for including  $t_1$  and  $t_m$ ) to two times the number of horizontal steps. Since  $t_1$  is the only point that can lie outside  $\mathcal{R}$ , we have  $\max_{F' \in \mathcal{F}^2(F)} |F\Delta F'| \leq \min(2a+2, 2b) = 2 \min(a+1, b)$ .  $\square$

**Proposition 13.** Under the same hypothesis of Proposition 12, let  $l$  be the number of pairwise different row sums of  $F$ . Then,

$$\max_{F' \in \mathcal{F}^2(F)} |F\Delta F'| \leq 2l.$$

**Proof.** Clearly,  $|T|$  equals two times the number of vertical steps in  $T$ . Since by definition of staircases,  $\text{row}(t_{2i}) = \text{row}(t_{2i+1})$  and  $\text{col}(t_{2i+2}) = \text{col}(t_{2i+1})$  with  $t_{2i}, t_{2i+2} \in F \setminus F'$  and  $t_{2i+1} \in F' \setminus F$ , we know that the number of points of  $F$  in  $\text{row}(t_{2i})$  is less than the number of points of  $F$  in  $\text{row}(t_{2i+2})$ . So the maximal number of vertical steps in any staircase  $T$  for  $F$  equals  $l$ .  $\square$

The next two lemmas provide lower bounds on  $F\Delta F'$ . These are used later to show that the derived bounds are sharp.

**Lemma 14.** For every  $n \in \mathbb{N}$  there exist  $F, F' \in \mathcal{F}^2$  with  $|F| = |F'| = \frac{1}{2}n(n+1)$  such that  $F\Delta F'$  is a staircase with  $2n$  points.

**Proof.** Taking the sets  $F = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq n+1-i\}$  and  $F' = \{(i, j) : 1 \leq i \leq n-1, 1 \leq j \leq n-i\} \cup \{(i, n+2-i) : 2 \leq i \leq n+1\}$  one can easily verify that the desired properties are fulfilled.  $\square$

**Lemma 15.** For every  $k, n \in \mathbb{N}$  with  $0 < k < n$ , there exist  $F, F' \in \mathcal{F}^2$  with  $|F| = |F'| = \frac{1}{2}n(n-1) + k$  such that  $F\Delta F'$  is a staircase with  $2(n-1)$  points.

**Proof.** We define (see Fig. 1b)

$$\begin{aligned} S_1 &= \{(i, j) : 1 \leq i \leq n-2, 1 \leq j \leq n-i-1\}, & S_2 &= \{(i, n-i) : 1 \leq i \leq k\}, \\ B_1 &= \{(i, n-i+1) : 1 \leq i \leq k\}, & B_2 &= \{(i, n-i) : k+1 \leq i \leq n-1\}, \\ W_1 &= \{(i+1, n-i+1) : 1 \leq i \leq k\}, & W_2 &= \{(i+1, n-i) : k+1 \leq i \leq n-1\}. \end{aligned}$$

Then,  $F = S_1 \cup S_2 \cup B_1 \cup B_2$  and  $F' = S_1 \cup S_2 \cup W_1 \cup W_2$  are sets with  $|F| = |F'| = \frac{1}{2}n(n-1) + k$ . It is easy to see that  $F$  has a triangular shape. The points  $F \Delta F' = B_1 \cup B_2 \cup W_1 \cup W_2$  form a staircase with  $2(n-1)$  points, namely  $T = (p_1, q_2, p_2, q_2, \dots, p_{n-2}, q_{n-2})$  with

$$p_i = \begin{cases} (n+1-i, i) \in W_2 & : 1 \leq i \leq n-k-1 \\ (n-i+1, i+1) \in W_1 & : n-k \leq i \leq n-1 \end{cases}$$

and

$$q_i = \begin{cases} (n-i, i) \in B_2 & : 1 \leq i \leq n-k-1 \\ (n-i, i+1) \in B_1 & : n-k \leq i \leq n-1. \quad \square \end{cases}$$

The next lemma is used in the following for bounding the number of different consecutive row sums for a given set  $F$ .

**Lemma 16.** For any  $n+j$  integers  $r_1 \geq \dots \geq r_{n+j} \geq 1$  with  $n \in \mathbb{N}$ ,  $j \in \mathbb{N}_0$  and

- (i)  $j \geq 1$  and  $\sum_{i=1}^{n+j} r_i = \frac{1}{2}n(n+1)$ , or
- (ii)  $j = 0$  and  $\sum_{i=1}^{n+j} r_i < \frac{1}{2}n(n+1)$

there are at most  $n-1$  pairwise different  $r_i$ 's.

**Proof.** Suppose there are more than  $n-1$  pairwise different  $r_i$ 's, which means that in  $r_1 \geq \dots \geq r_{n+j}$  we have at least  $n-1$  times a strict inequality. This implies  $r_i \geq n-i+r_{n+j}$  for  $1 \leq i \leq n-1$ , and  $r_i \geq r_{n+j}$  for  $n \leq i \leq n+j$ . Summation yields the contradiction

$$\begin{aligned} \sum_{i=1}^{n+j} r_i &\geq (n+j)r_{n+j} + n(n-1) - \sum_{i=1}^{n-1} i \\ &\geq n+j+n(n-1) - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1) + j. \end{aligned}$$

□

**Lemma 17.** Let  $F \in \mathcal{F}^2$  with  $|F| = \frac{1}{2}n(n+1)$  for an  $n \in \mathbb{N}$ . Then, we have  $\max_{F' \in \mathcal{F}^2(F)} |F \Delta F'| = 2n$ .

**Proof.** By Lemma 14, we have  $\max_{F' \in \mathcal{F}^2(F)} |F \Delta F'| \geq 2n$ . If  $F$  has  $n+j$  non-empty rows, where  $j \geq 1$ , then we have by Lemma 16 (i) at most  $n-1$  different consecutive row sums. This leads only, by Proposition 13, to a staircase with at most  $2(n-1)$  points. If  $F$  has less than  $n+1$  non-empty rows, then we again conclude (by Proposition 12) that any staircase contains at most  $2n$  points. □

**Lemma 18.** Let  $F \in \mathcal{F}^2$  with  $\frac{1}{2}n(n-1) < |F| < \frac{1}{2}n(n+1)$  for an  $n \in \mathbb{N}$ . Then,  $\max_{F' \in \mathcal{F}^2(F)} |F \Delta F'| = 2(n-1)$ .

**Proof.** Because of Lemma 15, we have  $\max_{F' \in \mathcal{F}^2(F)} |F \Delta F'| \geq 2(n-1)$  for any  $F$  with  $\frac{1}{2}n(n-1) < |F| < \frac{1}{2}n(n+1)$ . If  $F$  has  $n+j$  non-empty rows ( $j \geq 0$ ), then we have, by Lemma 16 (ii), at most  $n-1$  different row sums. Consequently, by Proposition 13, this leads to a staircase with at most  $2(n-1)$  points. If  $F$  has less than  $n$  non-empty rows, then we conclude again (by Proposition 12) that any staircase contains at most  $2(n-1)$  points. □

Now, we can summarize the results in the following theorem.

**Theorem 19.** Given  $F, F' \in \mathcal{F}^2$  with row sum vector  $R, R'$  and column sum vector  $C, C'$ , respectively. Suppose the following properties are fulfilled:

- (i)  $F$  is uniquely determined by its row and column sums  $R$  and  $C$ ;
- (ii)  $\|R - R'\|_1 + \|C - C'\|_1 \leq 2$ .

Then, we have

$$|F \Delta F'| \leq \begin{cases} 2n+1 & : \text{if } |F| = \frac{1}{2}n(n+1) \text{ for an } n \in \mathbb{N} \\ 2n-1 & : \text{if } \frac{1}{2}n(n-1) < |F| < \frac{1}{2}n(n+1) \text{ for an } n \in \mathbb{N}. \end{cases} \quad (1)$$

These bounds are sharp and imply

$$|F \Delta F'| \leq 2\sqrt{2|F|} + \frac{1}{4}. \quad (2)$$

**Proof.** Let us define  $\alpha(F) := 2n+1$  if  $|F| = \frac{1}{2}n(n+1)$  with an  $n \in \mathbb{N}$ , and  $\alpha(F) := 2n-1$  if  $\frac{1}{2}n(n-1) < |F| < \frac{1}{2}n(n+1)$  with an  $n \in \mathbb{N}$ . Thus (1) reads as  $|F \Delta F'| \leq \alpha(F)$ . Using the triangle inequalities  $\|R\|_1 - \|R'\|_1 \leq \|R - R'\|_1$  and  $\|C\|_1 - \|C'\|_1 \leq \|C - C'\|_1$  together with the fact that  $\|R\|_1 = \|C\|_1 = |F|$  and  $\|R'\|_1 = \|C'\|_1 = |F'|$ , we observe that the assumption

$$2|F| - |F'| \leq \|R - R'\|_1 + \|C - C'\|_1 \leq 2$$

of Property (ii) can only be fulfilled if either  $|F| = |F'|$  or  $\|R - R'\|_1 = 1$ . We treat both cases separately.

Let  $|F| = |F'|$ . The projection error  $\|R - R'\|_1 + \|C - C'\|_1$  has to be an even number, as we already remarked in Section 3.1. Assertion (1) follows immediately if  $\|R - R'\|_1 + \|C - C'\|_1 = 0$  since then we have  $F = F'$  (because  $F$  is uniquely determined). If  $\|R - R'\|_1 + \|C - C'\|_1 = 2$  then we obtain  $|F \Delta F'| \leq \alpha(F) - 1$  from Lemmas 17 and 18, and thus (1).

Let  $|F| = |F'| + 1$ . By the upper triangle inequalities, we can only have  $\|R - R'\|_1 + \|C - C'\|_1 = 2$ , and more specifically  $\|R - R'\|_1 + \|C - C'\|_1 = 1$ , i.e., there is exactly one row and one column containing a projection error, namely a +1 error. Let  $p \in \mathbb{Z}^2$  denote the point where these two error lines intersect.

- (a) Suppose  $p \notin F'$ . Then  $F'' := \{p\} \cup F'$  has the same cardinality as  $F$  and no projection error (compared with  $F$ ). Since  $F$  is uniquely determined we must have  $F = F''$ , i.e.,  $|F \Delta F'| = 1 \leq \alpha(F)$ .
- (b) Suppose  $p \in F'$ . We set  $F'' := \{q\} \cup F'$  where  $q$  is an arbitrary point of  $q \in F \setminus F'$  lying on the vertical line through  $p$ . Such a  $q$  must exist, otherwise the vertical error on this line would be less than +1. Compared with  $F'$  we thus reduced the column sum error (from +1 to 0), and hereby we introduced a new -1 error

on the horizontal line through  $q$ . So we are in the case that  $|F'| = |F|$  and  $\|R - R'\| + \|C - C'\| = 2 + 0$ , which we considered already above, so we have  $|F\Delta F'| \leq \alpha(F) - 1 \Rightarrow |F\Delta F'| \leq \alpha(F)$ .

In both cases (a) and (b), we therefore established (1).

The case  $|F| + 1 = |F'|$  follows analogously to the just treated case of  $|F| = |F'| + 1$ . In summary, we proved that (1) holds in all cases.

If  $|F| = \frac{1}{2}n(n+1)$  we have  $2n+1 = 2\sqrt{2|F| + \frac{1}{4}}$ . For  $|F| = \frac{1}{2}n(n-1) + k$  with  $k \in \mathbb{N}$  we have  $2|F| = (n - \frac{1}{2})^2 + \frac{1}{4} + 2k$ , thus  $2n - 1 = 2\sqrt{2|F| - 1/4 - 2k}$ . In any case we obtain from (1) that  $|F\Delta F'| \leq 2\sqrt{2|F| + \frac{1}{4}}$  holds.

The constructions given in Lemmas 14 and 15 show that for any prescribed cardinality  $c \in \mathbb{N}$  there exist  $F, F' \in \mathcal{F}^2$  with  $|F| = |F'| = c$  and  $|F\Delta F'| = \alpha(F) - 1$ . Adding an arbitrary point  $q \notin F \cup F'$  on the +1 error row to obtain  $F'' := \{q\} \cup F'$  still gives  $\|R - R''\|_1 + \|C - C''\|_1 = 2$  but  $|F\Delta F''| = |F\Delta F'| + 1 = \alpha(F)$ . This shows that (1) is sharp.  $\square$

A stability result of a similar type has been given in [1] (Theorem 17). However, Theorem 19 generalizes the previous result in several ways. If one interprets  $F$  as the original binary picture, then Theorem 19 states that if one knows in advance that  $F$  is uniquely determined by its row and column sums (without actually knowing  $F$ ), and that the error in the measurements (projection error) is not larger than 2, then a certain upper bound is given – guaranteeing that any reconstruction  $F'$  cannot differ too much from the original picture. Theorem 17 of [1] was more restrictive in the sense that prior to reconstruction one had to know additionally the cardinality of  $F$  (cf. Theorem 17 (ii), [1]), and one had to know that a measurement error really occurred (with norm 2). Clearly, the requirements of Theorem 19 are more realistic. As for the bound (1) it is notable that the right hand side only increased by 1, when compared to Theorem 17 of [1].

We remark that there is a different interpretation of Theorem 19. Suppose nothing is known about the original picture  $F$ , and the reconstruction yields a picture  $F'$  that is uniquely determined by its row and column sums (which can be easily checked by Ryser’s criterion [9]). Then, Theorem 19 gives the guarantee that  $F$  does not differ too much from  $F'$  (if the projection error is not larger than 2).

**4. Problem 1: a generalized version**

In this section, we study the stability problem under the weaker condition that the projections do not uniquely determine the set  $F$ , but we have some “invariant” points. In this section, we assume first that  $|F| = |F'|$  and  $\|R - R'\|_1 + \|C - C'\|_1 = 2$ .

Let  $\mathcal{U}(R, C)$  denote the class containing lattice sets having row and column sum vectors  $R$  and  $C$ . The class is *normalized* if  $R$  and  $C$  do not

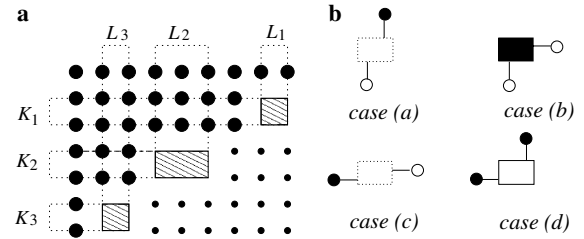


Fig. 2. The illustrations used in Section 4. (a) The typical form of the lattice sets in  $\mathcal{U}(R, C)$  with invariant points. (b) The four possible configurations of Remark 24.

determine  $F$ , then more than one set belongs to  $\mathcal{U}(R, C)$ . In this context it is meaningful to study the case where  $\mathcal{U}(R, C)$  has some *invariant points* (these are points belonging to every set in  $\mathcal{U}(R, C)$ , or to none of these sets).

It is well-known ([22]) that the normalized class  $\mathcal{U}(R, C)$  has invariant points if and only if the lattice sets in  $\mathcal{U}(R, C)$  are of the form illustrated in Fig. 2a. To be more precise, let  $\mathcal{R} = \{1, \dots, a\} \times \{1, \dots, b\}$  be the rectangle containing  $F$ ; there exist pairwise disjoint subsets  $K_1, \dots, K_h \subseteq \{1, \dots, a\}$  and pairwise disjoint subsets  $L_1, \dots, L_h \subseteq \{1, \dots, b\}$  such that  $\mathcal{R} \setminus \cup_{u=1}^h K_u \times L_u$  contains only invariant points. For example, the black points in the Fig. 2a are invariant points belonging to every set in  $\mathcal{U}(R, C)$  (also called 1-invariant points), while the smaller dots indicate invariant points not belonging to any set of  $\mathcal{U}(R, C)$  (also called 0-invariant points).

**Remark 20.** We assume that  $\mathcal{U}(R, C)$  is normalized,  $|F| = |F'|$ , and the projection error is 2, so again we assume that the error occurs in two rows.

Let  $p$  be any point of  $F' \setminus F$  and let  $q$  be any point of  $F \setminus F'$ . Clearly, statements (a) and (b) of Remark 4 hold when  $p$  or  $q$  is not in  $\cup_{u=1}^h K_u \times L_u$ .

**Remark 21.** Consider a so called non-trivial component  $K_u \times L_u, u \in \{1, \dots, h\}$ , and suppose that no error occurs on the lines intersecting this component.

- Let  $q \in F \setminus F'$  with  $\text{row}(q) \in K_u$  and  $\text{col}(q) \notin L_u$  (implying that  $q$  is an 1-invariant point). Then, there exists either one point  $p' \in F' \setminus F$  such that  $\text{row}(p') \in K_u$  and  $\text{col}(p') \notin L_u$ , or one point  $q' \in F \setminus F'$  such that  $\text{col}(q') \in L_u$  and  $\text{row}(q') \notin K_u$ . Otherwise we would obtain a contradiction to the assumption about the shape of  $\mathcal{U}(R, C)$ , and the assumption that there is no error in the rows and columns intersecting  $K_u \times L_u$ .
- Similarly, let  $p \in F' \setminus F$  with  $\text{row}(p) \in K_u$  and  $\text{col}(p) \notin L_u$  (meaning that  $p$  is a 0-invariant point). Then, there exists either one point  $q' \in F \setminus F'$  such that  $\text{row}(q') \in K_u$  and  $\text{col}(q') \notin L_u$ , or one point  $p' \in F' \setminus F$  such that  $\text{col}(p') \in L_u$  and  $\text{row}(p') \notin K_u$ .

Suppose now that exactly one error occurs on a line intersecting  $K_u \times L_u$ . From the assumptions about the shape of  $\mathcal{U}(R, C)$  there follows:

- If the  $-1$  error occurs, then there exists  $q' \in F \setminus F'$  such that  $\text{col}(q') \in L_u$  and  $\text{row}(q') \notin K_u$ ;
- If the  $+1$  error occurs, then there exists  $p' \in F' \setminus F$  such that  $\text{row}(p') \in K_u$  and  $\text{col}(p') \notin L_u$ .

From the previous remark there easily follows that, if not both rows  $i$  (the row with  $-1$  error) and  $j$  (the row with  $+1$  error) intersect the same non-trivial component, then we have  $i > j$  as in Lemma 5.

**Definition 22.** Let  $A, B \in \mathcal{F}^2$  with  $A \cap B = \emptyset$ . An  $(A, B)$ -switching component (or a switching component for short) is a sequence of an even number  $m > 0$  of distinct points  $t_1, \dots, t_m$  such that  $t_{2i+1} \in A, t_{2i+2} \in B, \text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$  for  $0 \leq i \leq \frac{m}{2} - 1, \text{row}(t_{2i+2}) = \text{row}(t_{2i+3})$  for  $0 \leq i \leq \frac{m}{2} - 2, \text{and row}(t_1) = \text{row}(t_m)$ .

**Definition 23.** Let  $\mathcal{R}$  be a rectangle of size  $a \times b$  containing the disjoint sets  $A, B \in \mathcal{F}^2$ , and let  $K_1, \dots, K_h$  and  $L_1, \dots, L_h$  be pairwise disjoint subsets of  $\{1, \dots, a\}$  and  $\{1, \dots, b\}$ , respectively. An almost-staircase  $T = (t_1, \dots, t_m)$  according to the columns in  $\mathcal{R}$  is a sequence of an even number of  $m > 0$  distinct points  $t_{2i+1} \in A, t_{2i+2} \in B, 0 \leq i \leq \frac{m}{2} - 1$  such that:

- (i) For every  $i \in \mathbb{N}_0$  with  $0 \leq i \leq \frac{m}{2} - 1$  it holds that  $\text{col}(t_{2i+1}) = \text{col}(t_{2i+2})$  and, if  $t_{2i+1} \notin \cup_{u=1}^h K_u \times L_u$  or  $t_{2i+2} \notin \cup_{u=1}^h K_u \times L_u$ , then  $\text{row}(t_{2i+1}) > \text{row}(t_{2i+2})$ ;
- (ii) For every  $i \in \mathbb{N}_0$  with  $0 \leq i \leq \frac{m}{2} - 2$  it holds that  $\text{row}(t_{2i+2}) = \text{row}(t_{2i+3})$  and if  $t_{2i+2} \notin \cup_{u=1}^h K_u \times L_u$  or  $t_{2i+3} \notin \cup_{u=1}^h K_u \times L_u$ , then  $\text{col}(t_{2i+2}) < \text{col}(t_{2i+3})$ ;
- (iii) For every  $i \in \mathbb{N}$  with  $1 \leq i \leq m$ , we have that  $t_i$  is no member of an  $(A, B)$ -switching component.

An almost-staircase is a staircase “almost everywhere” except that the properties in (i) and (ii) of Definition 6 are relaxed for points in components  $K_u \times L_u$ .

**Remark 24.** If for a staircase  $T = (t_1, \dots, t_m)$  and a component  $K_u \times L_u$  we have  $\{t_1, \dots, t_m\} \cap (K_u \times L_u) = \{t_i, \dots, t_j\}$  with  $i > 1$  and  $j < m$  then, by definition of almost-staircases, the following cases can arise:

- If  $t_i \in B$ , then  $t_{i-1} \in A$ , and (a)  $\text{col}(t_j) = \text{col}(t_{j+1})$  implies  $t_{j+1} \in B$ , whereas (b)  $\text{row}(t_j) = \text{row}(t_{j+1})$  implies  $t_{j+1} \in A$ ;
- If  $t_i \in A$ , then  $t_{i-1} \in B$ , and (c)  $\text{row}(t_j) = \text{row}(t_{j+1})$  implies  $t_{j+1} \in B$  whereas (d)  $\text{col}(t_j) = \text{col}(t_{j+1})$  implies  $t_{j+1} \in A$ .

If  $i = 1$ , then either case (a) or (b) occurs, and if  $j = m$ , then either case (a) or (d) occurs. Fig. 2b illustrates the four configurations: the three kinds of rectangles represent  $K_u \times L_u$ .

The rectangle of configuration (b) in Fig. 2b is coloured black because the number of black points inside is greater than the number of white points; the rectangle of configu-

ration (d) is coloured white because the number of black points inside is smaller than the number of white points, and finally the rectangles of configurations (a) and (c) have dotted edges because the numbers of white and black points inside are the same. Again, a maximal almost-staircase is an almost staircase that is no proper subset of another almost-staircase.

**Lemma 25.** Any two maximal almost-staircases in  $F \Delta F'$  have the same starting point and the same end point.

**Proof.** We just stress the differences in the proof of Lemma 8 by using the same notations. The case that remains to be considered is the following:  $\text{row}(t_1) = \text{row}(s_1) = i \in K_u \times L_u, \text{col}(t_1) > \text{col}(s_1)$ , and a black point  $q$  on row  $i$  is in  $K_u \times L_u$ , and this point is not in a switching component (otherwise the error would be too large). Since there is no error on the columns, a white point exists such that  $\text{col}(p) = \text{col}(q)$  but we cannot claim that  $\text{row}(p) > \text{row}(q)$ . Anyway, points alternate each other such that this sequence visits a black point to the left of the rectangle or a white point to the bottom of the rectangle (leading to an infinite sequence), or it infinitely alternates within  $K_u \times L_u$ , or forms a switching component with  $q$ . In all cases, this is a contradiction to the assumptions.  $\square$

**Lemma 26.** Any two maximal almost-staircases in  $F \Delta F'$  coincide.

**Proof.** The proof follows as in Lemma 10. Indeed the case to analyse is that of  $t_1 = s_1 \in K_u \times L_u$ . One can easily show that if there is another white point in  $\text{col}(t_1)$ , then it belongs to the same  $K_u \times L_u$ , so proving that  $t_2 = s_2$ . Similarly, one deduces that  $t_{m-1} = s_{m-1}$ . This allows to apply the procedure used in the proof of Lemma 10.  $\square$

Again, every  $p \in F \Delta F'$  (outside of a switching component) is contained in an almost-staircase that is possibly constituted by two points (see Remarks 21 and 24). Because of the shape of  $F$  and  $F'$  we also have that points in a switching component can only lie in a single component  $K_u \times L_u$ .

Let  $\mathcal{P} := \{p \in F \cup F' : p \text{ lies in an } (F, F') \text{ - switching component}\}$  and  $T_{\max}$  denote the maximal almost-staircase. Then, we can summarize the results as follows:

**Theorem 27.** Given  $F, F' \in \mathcal{F}$  with row sum vector  $R, R'$  and column sum vector  $C, C'$ , respectively. Suppose the following properties are fulfilled

- (i)  $\mathcal{U}(R, C)$  has invariant points;
- (ii)  $|F| = |F'|$  and
- (iii)  $\|R - R'\|_1 + \|C - C'\|_1 = 2$ .

Then,  $F \Delta F' = T_{\max} \cup \mathcal{P}$ .

This theorem can be used to obtain a bound (similar to Section 3.4) that depends only on  $\sum_{u=1}^h |K_u \times L_u|, R$ , and on  $C$ .

## 5. Problem 2: a fixed symmetric difference

In the previous sections we investigated a fixed upper bound on the projection error and asked for reconstructions that differ most from the original set. We reverse this approach and ask for a lower bound on the projection error when the original and reconstructed sets are disjoint. Such a bound is again a stability result in the following sense: If, in practice, one can guarantee that the projection error introduced by the measuring process is under this bound, then such a result (as Corollary 30) ensures that the original and reconstructed sets are not disjoint. Focusing here on  $F \cap F' = \emptyset$  means that we investigate a worst case scenario for the reconstruction.

**Lemma 28.** *Let the non-increasing vectors  $R = (r_1, \dots, r_{k_1}) \in \mathbb{N}_0^{k_1}$  and  $C = (c_1, \dots, c_{k_2}) \in \mathbb{N}_0^{k_2}$  uniquely determine  $F \in \mathcal{F}$ . For any  $F' \in \mathcal{F}^2$  with  $|F'| = |F|$ ,  $F' \cap F = \emptyset$  and row and column sum vectors  $R' \in \mathbb{N}_0^{k_1}$ ,  $C' \in \mathbb{N}_0^{k_2}$ , respectively, it holds that*

$$\|R - R'\|_1 + \|C - C'\|_1 \geq \sum_{i \in \{1, \dots, k_1\} \text{ with } 2r_i - r_1 \geq 0} (2r_i - r_1) + \sum_{i \in \{1, \dots, k_2\} \text{ with } 2c_i - c_1 \geq 0} (2c_i - c_1)$$

**Proof.** We define  $H := \mathbb{Z} \setminus \{\text{row}(p) : p \in F\}$  and  $V := \mathbb{Z} \setminus \{\text{col}(p) : p \in F\}$ , the sets of row and column indices, respectively, that contain no point of  $F$ . First, we show for every column  $i \in \{1, \dots, k_2\}$  with  $2c_i - c_1 \geq 0$  the following inequality:

$$|c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(i, j)| + \sum_{j \in H} \mathbf{1}_{F'}(i, j) \geq 2c_i - c_1. \quad (3)$$

We assume  $2c_i - c_1 \geq 0$ , and denote  $c'_i := \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(i, j)$ . Then we distinguish two cases

(i) If  $c_i + c'_i \leq c_1$  then we obtain

$$\begin{aligned} & |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(i, j)| + \sum_{j \in H} \mathbf{1}_{F'}(i, j) \\ & \geq |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(i, j)| = |c_i - c'_i| \stackrel{(\star_1)}{=} c_i - c'_i \\ & \stackrel{(\star_2)}{\geq} c_i - c_1 + c_i = 2c_i - c_1. \end{aligned}$$

The equality  $(\star_1)$  holds because of  $c_i + c'_i \leq c_1 \leq 2c_i \Rightarrow c'_i \leq c_i$ . The inequality  $(\star_2)$  holds, because  $c'_i \leq c_1 - c_i$ . Therefore, we have shown (3).

(ii) If  $c_i + c'_i > c_1$  then, because of the maximality of  $A$ , we have

$$|\{p \in F' \setminus F : \text{col}(p) = i\}| \geq c_i + c'_i - c_1 > 0.$$

This means that there are  $c_i + c'_i - c_1$  rows  $j \in H$  such that  $\mathbf{1}_{F'}(i, j) = 1$ . Thus,

$$\begin{aligned} & |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(i, j)| + \sum_{j \in H} \mathbf{1}_{F'}(i, j) = |c_i - c'_i| + c_i + c'_i - c_1 \\ & \geq 2c_i - c_1, \end{aligned}$$

showing (3).

In the same way as we proved (3) in both cases (i) and (ii), one can obtain a similar inequality for  $i \in \{1, \dots, k_1\}$  with  $2r_i - r_1 \geq 0$ , namely

$$|r_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{j \in V} \mathbf{1}_{F'}(j, i) \geq 2r_i - r_1. \quad (4)$$

Let  $I := \{i \in \{1, \dots, k_1\} : 2r_i - r_1 \geq 0\}$  and  $J := \{i \in \{1, \dots, k_2\} : 2c_i - c_1 \geq 0\}$ . Using  $I \cap H = J \cap V = \emptyset$  we obtain:

$$\begin{aligned} & \|R - R'\|_1 + \|C - C'\|_1 \geq \\ & \sum_{i \in I} |r_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in H} |\sum_{j \in \mathbb{Z}} \mathbf{1}_F(j, i) - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| \\ & \quad + \sum_{i \in J} |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in V} |\sum_{j \in \mathbb{Z}} \mathbf{1}_F(j, i) - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| \\ & = \sum_{i \in I} |r_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in H} \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i) \\ & \quad + \sum_{i \in J} |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in V} \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i) \\ & \stackrel{(\star_3)}{\geq} \sum_{i \in I} |r_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in I} \sum_{j \in H} \mathbf{1}_{F'}(j, i) \\ & \quad + \sum_{i \in J} |c_i - \sum_{j \in \mathbb{Z}} \mathbf{1}_{F'}(j, i)| + \sum_{i \in J} \sum_{j \in V} \mathbf{1}_{F'}(j, i) \\ & \stackrel{(\star_4)}{\geq} \sum_{i \in I} (2r_i - r_1) + \sum_{i \in J} (2c_i - c_1) \end{aligned}$$

Here  $(\star_3)$  holds because of  $I, J \subset \mathbb{Z}$ , and  $(\star_4)$  because of (3) and (4). This concludes the proof.  $\square$

Now we are able to prove the following theorem.

**Theorem 29.** *Given  $F, F' \in \mathcal{F}^2$  with row sum vector  $R, R'$  and column sum vector  $C, C'$ , respectively. Suppose the following properties are fulfilled*

- (i)  $F$  is uniquely determined by its row and column sums  $R$  and  $C$ ;
- (ii)  $|F| = |F'|$ ;
- (iii)  $F \cap F' = \emptyset$ .

Then, we have:

$$\|R - R'\|_1 + \|C - C'\|_1 \geq [2\sqrt{|F|}].$$

**Proof.** From Lemma 28, we immediately obtain the (weaker) bound

$$\|R - R'\|_1 + \|C - C'\|_1 \geq r_1 + c_1.$$

This is a bound for a prescribed set  $F$ . To establish a bound for every uniquely determined set  $F$  with cardinality  $|F|$  we have to take the minimum of all realizable  $r_1 + c_1$ 's taken over all uniquely determined sets with fixed cardinality  $|F|$ . For every uniquely determined set  $F$  we have  $c_1 r_1 \geq |F|$  (notice that the  $c_i$  and  $r_i$  are ordered). For arbitrary  $c_1 r_1 \geq 0$  with  $c_1 r_1 \geq |F|$  we have

$$c_1 + r_1 \geq 2\sqrt{c_1 r_1} \geq 2\sqrt{|F|},$$



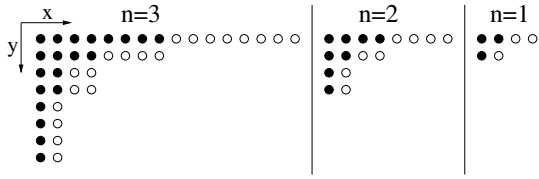


Fig. 3. An illustration showing the sets  $F$  (black points) and  $F'$  (white points) of Remark 31 for  $n = 3, 2, 1$ .

which means that

$$\|R - R'\|_1 + \|C - C'\|_1 \geq 2\sqrt{|F|},$$

proving the theorem.  $\square$

A direct consequence of this theorem is the following stability result.

**Corollary 30.** *Given any two sets  $F, F' \in \mathcal{F}^2$  with  $|F|=|F'|$ , where  $F$  is uniquely determined by its row and column sums  $R$  and  $C$ . If  $\|R - R'\|_1 + \|C - C'\|_1 < [2\sqrt{|F|}]$  then  $F \cap F' \neq \emptyset$ .*

**Remark 31.** Theorem 29 gives a lower bound on the projection error of two disjoint sets  $F$  and  $F'$ , which is asymptotically of order  $O(\sqrt{|F|})$ . This bound might be improved, however we show in the following that an asymptotically lower bound of  $O(|F|)$  is not achievable. For every  $n \in \mathbb{N}$  a construction of two disjoint sets  $F, F' \in \mathcal{F}^2$  is given, where  $F$  is uniquely determined and  $|F| = |F'| = 2^n + n2^{n-1}$ , but for which no  $c \in \mathbb{R}_+$  (independent from  $n$ ) exists with  $c|F| \leq \|R - R'\|_1 + \|C - C'\|_1$ , showing that there is no  $O(|F|)$  bound. For  $n \in \mathbb{N}$  the construction is follows: Let  $M := \{(j,1): j \in \{1, \dots, 2^n\}\}$  and for  $i \in \{0, \dots, n-1\}$  let  $M_i := \{(j,k): 2^i < k \leq 2^{i+1}, j \in \{1, \dots, 2^{n-i-1}\}\}$ , then

$$F := M \cup \bigcup_{i=0}^{n-1} M_i, F' := ((2^n, 0) + M) \cup \bigcup_{i=0}^{n-1} ((2^{n-i-1}, 0) + M_i).$$

It can be easily seen that  $|F| = |F'| = 2^n + n2^{n-1}$  and  $\|R - R'\|_1 + \|C - C'\|_1 = 2^{n+1}$  (the projection error occurs only in columns, and sums up to  $1 \cdot 2^n + 2^n \cdot 1$  – see Fig. 3). Since

$$\frac{\|R - R'\|_1 + \|C - C'\|_1}{|F|} = \frac{2^{n+1}}{2^n + n2^{n-1}} = \frac{4}{2+n}$$

and  $\frac{4}{2+n} < c$  for every  $c \in \mathbb{R}_+$ , provided that we choose  $n$  large enough, we conclude that there is no fixed  $c$  fulfilling

$$c \leq \frac{\|R - R'\|_1 + \|C - C'\|_1}{|F|} \iff c|F| \leq \|R - R'\|_1 + \|C - C'\|_1$$

for all  $|F|$ , proving that there is no  $O(|F|)$  bound on the projection error.

## 6. Conclusion

We proved several stability results related to the problem of reconstructing binary pictures from two projections under

the assumption that the original picture is uniquely determined by its projections. In particular, we showed that any reconstruction resembles the original picture quite closely, at least when the projection error is at most 2. This complements the results of [13] and is in striking contrast to the instabilities for the reconstruction from more than two projections. Furthermore, we demonstrated that by similar techniques one can obtain stability results for the more general case where the original picture contains invariant points. A necessary condition on the size of the projection error was given in order to lead to a reconstruction that is disjoint from the original picture. It remains an open question to what extent the stability persists in the case of larger projection errors.

## Acknowledgement

The first author was supported (while finishing the paper) by a Feodor Lynen fellowship of the Alexander von Humboldt Foundation (Germany).

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