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Absolute continuity and weak uniform mixing of random walk in dynamic random environment

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Abstract

We prove results for random walks in dynamic random environments which do not require the strong uniform mixing assumptions present in the literature. We focus on the "environment seen from the walker"-process and in particular its invariant law. Under general conditions it exists and is mutually absolutely continuous to the environment law. With stronger assumptions we obtain for example uniform control on the density or a quenched CLT. The general conditions are made more explicit by looking at hidden Markov models or Markov chains as environment and by providing simple examples.

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1 Introduction and main results

1.1 Background and motivation

We study the asymptotic behaviour of a class of random walks (X_t) on $\mathbb{Z}^d$ whose transition probabilities depend on another process, the random environment. Such models play an important role in the understanding of disordered systems and serve as natural generalisations of the classical simple random walk model for describing transport processes in inhomogeneous media.

These types of random walks, which are called random walks in random environment, can be split into two broad areas, static and dynamic environments. In static environments the environment is created initially and then stays fixed in time. In dynamic environments the environment instead evolves over time. Note that a dynamic environment in $\mathbb{Z}^d$ can always be reinterpreted as a static environment in $\mathbb{Z}^{d+1}$ by turning time into an additional space dimension.

A major interest in dynamic environments are their often complicated space-time dependency structure. Typically, in order to show that the random walk is diffusive, one

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looks for some way to guarantee that the environment is “forgetful” and random walk increments are sufficiently independent on large time scales.

One approach to this are various types of mixing assumptions on the environment. By now, general results are known for Markovian environments which are uniformly mixing with respect to the starting configuration (see Avena, den Hollander, and Redig [3] and Redig and Völlering [25]). For this, the rate at which the dynamic environment converges towards its equilibrium state plays an important role.

On the other hand, models where the dynamical environment has non-uniform mixing properties serve as a major challenge and are still not well understood. Opposite to the diffusive behaviour known for uniform mixing environments, it has been conjectured that (X_t) may be sub- or super-diffusive for certain non-uniform mixing environments, see Avena and Thomann [2]. Though some particular examples yielding diffusive behaviour have recently been studied by rigorous methods, e.g. Deuschel, Guo, and Ramirez [12], Hilário, den Hollander, Sidoravicius, dos Santos, and Teixeira [17], Huveneers and Simenhaus [19] and Mountford and Vares [23], these results are model specific and/or perturbative in nature. No general theory has so far been developed.

In this article we provide a new approach for determining limiting properties of random walks in dynamic random environment, in particular about the invariant law of the “environment as seen from the walker”-process. Under general mixing assumptions, we prove the existence of an invariant measure mutually absolutely continuous with respect to the random environment (Theorems 1.2 and 1.5). Our mixing assumptions are considerably weaker than the uniform mixing conditions present in the literature (e.g. cone mixing) and do not require the environment to be Markovian.

An important feature of our approach is that it can also be applied to dynamics with non-uniform mixing properties. Examples include an environment given by Ornstein-Uhlenbeck processes and the supercritical contact process.

Knowledge about the invariant measures for the “environment as seen from the walker”-process yield limit laws for the random walk itself. One immediate application of our approach is a strong law of large numbers for the random walk. Further applications include a quenched CLT based on Dolgopyat, Keller, and Liverani [15], Theorem 1, considerably relaxing its requirements.

Our key observation is an expansion of the “environment as seen from the walker”-process (Theorem 3.1). This expansion enables us to separate the contribution of the random environment to the law of the “environment as seen from the walker”-process from that of the transition probabilities of the random walk.

We also show stability under perturbations of the environment or of the jump kernel of the random walk. Under a strong uniform mixing assumption, we obtain uniform control on the Radon-Nikodym derivative of the law of the “environment as seen from the walker”-process with respect to the environment, irrespective of the choice of the jump kernel of the random walker.

Outline

In the next two subsections we give a precise definition of our model and present our main results, Theorem 1.2 and Theorem 1.5. Section 2 is devoted to examples and applications thereof. In Section 3 we derive the aforementioned expansion, and present results on stability and control on the Radon-Nikodym derivative. Proofs are postponed until Section 4.

1.2 The model

In this subsection we give a formal definition of our model. In short, (X_t) is a random walk in a translation invariant random field with a deterministic drift in a fixed coordinate direction.

The environment

Let $d \in \mathbb{N}$ and let $\Omega := E^{\mathbb{Z}^{d+1}}$ where E is assumed to be a finite set. We assign to the space Ω the standard product σ-algebra $\mathcal{F}$ generated by the cylinder events. For $\Lambda \subset \mathbb{Z}^{d+1}$, we denote by $\mathcal{F}_\Lambda$ the sub-σ-algebra generated by the cylinders of Λ. For the forward half-space $\mathbb{H} := \mathbb{Z}^d \times \mathbb{Z}_{\geq 0}$ we write $\mathcal{F}_{\geq 0}$ for $\mathcal{F}_{\mathbb{H}}$.

By $\mathcal{M}_1(\Omega)$ we denote the set of probability measures on $(\Omega, \mathcal{F})$. We call $\eta \in \Omega$ the *environment* and denote by $\mathbb{P} \in \mathcal{M}_1(\Omega)$ its law. A particular class of environments contained in our setup are path measures of a stochastic process (η_t) whose state space is $\Omega_0 := E^{\mathbb{Z}^d}$. To emphasise this, for $\eta \in \Omega$ and $(x,t) \in \mathbb{Z}^d \times \mathbb{Z}$, we often write $\eta_t(x)$ for the value of η at (x,t).

We assume throughout that $\mathbb{P}$ is measure preserving with respect to translations, that is, for any $x \in \mathbb{Z}^d$, $t \in \mathbb{Z}$,

$$\mathbb{P}(\cdot) = \mathbb{P}(\theta_{x,t}\cdot), \tag{1.1}$$

where $\theta_{x,t}$ denotes the shift operator $\theta_{x,t}\eta_s(y) = \eta_{s+t}(y+x)$. Furthermore, we assume that $\mathbb{P}$ is *ergodic in the time direction*, that is, all events $B \in \mathcal{F}$ for which $\theta_{o,1}B := \{\omega \in \Omega \colon \theta_{o,-1}\omega \in B\} = B$ are assumed to satisfy $\mathbb{P}(B) \in \{0,1\}$. Here, $o \in \mathbb{Z}^d$ denotes the origin.

Remark 1.1. By considering the environment as a hidden Markov model, we present in Subsection 2.2-2.4 an approach where E is allowed to be a general Polish space.

The random walk

The *random walk* (X_t) is a process on $\mathbb{Z}^d$. We assume w.l.o.g. that $X_0 = o$. The transition probabilities of (X_t) is assumed to depend on the state of the environment as seen from the random walk. That is, given $\eta \in \Omega$, then the evolution of (X_t) is given by

$$\begin{aligned} &P_\eta(X_0 = o) = 1 \\ &P_\eta(X_{t+1} = y + z \mid X_t = y) = \alpha(\theta_{y,t}\eta, z), \end{aligned} \tag{1.2}$$

where $\alpha \colon \Omega \times \mathbb{Z}^d \to [0,1]$ satisfies $\sum_{z \in \mathbb{Z}^d} \alpha(\eta, z) = 1$ for all $\eta \in \Omega$. The law of the random walk, $P_\eta \in \mathcal{M}_1((\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}})$, where we have conditioned on the entire environment, is called the *quenched* law. We denote its σ-algebra by $\mathcal{G}$. Further, for $\mathbb{P} \in \mathcal{M}_1(\Omega)$, we denote by $P_{\mathbb{P}} \in \mathcal{M}_1\left(\Omega \times (\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}\right)$ the joint law of (η, X), that is,

$$P_{\mathbb{P}}(B \times A) = \int_B P_\eta(A) d\mathbb{P}(\eta), \quad B \in \mathcal{F}, A \in \mathcal{G}. \tag{1.3}$$

The marginal law of $P_{\mathbb{P}}$ on $(\mathbb{Z}^d)^{\mathbb{Z}_{\geq 0}}$ is the *annealed* (or averaged) law of (X_t).

We assume that the transition probabilities of (X_t) only depend on the environment within a finite region around its location. That is, there exist $R \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$

$$\alpha(\eta, z) - \alpha(\sigma, z) = 0 \text{ whenever } \sigma \equiv \eta \text{ on } [-R,R]^d \times \{0\}. \tag{1.4}$$

Further, define

$$\mathcal{R} := \{y \in \mathbb{Z}^d \colon \sup_{\eta \in \Omega} \alpha(\eta, y) > 0\} \tag{1.5}$$

as the jump range of the random walker, which we assume to be finite and to contain o. By possibly enlarging R we can guarantee that

$$\sup_{y \in \mathbb{Z}^d} \{\| y \|_1 : y \in \mathcal{R}\} \leq R. \tag{1.6}$$

Lastly, we say that (X_t) is *elliptic in the time direction* if

$$\alpha(\eta, o) > 0, \quad \forall\, \eta \in \Omega. \tag{1.7}$$

If, after replacing o with y, (1.7) holds for all $y \in \mathcal{R}$, then we say that (X_t) is *elliptic*.

The environment process

"The environment as seen from the walker"-process is of importance for understanding the asymptotic behaviour of the random walk itself, but it is also of independent interest. This process, which is given by

$$(\eta_t^{EP}) := (\theta_{X_t, t} \eta), \quad t \in \mathbb{Z}_{\geq 0}, \tag{1.8}$$

is called the *environment process*. Note that (η_t^{EP}) is a Markov process on Ω under P_η, $\eta \in \Omega$, with initial distribution $\mathbb{P}$.

1.3 Main results

In this subsection we present our main results about the asymptotic behaviour of (X_t) and (η_t^{EP}). However, before stating our first theorem we need to introduce some more notation.

Recall (1.5) and let

$$\Gamma_k := \left\{(\gamma_{-k}, \gamma_{-k+1}, ..., \gamma_0) \colon \gamma_i \in \mathbb{Z}^d, \gamma_i - \gamma_{i-1} \in \mathcal{R}, -k \leq i < 0, \gamma_0 = o\right\} \tag{1.9}$$

be the set of all possible backwards trajectories from $(o, 0)$ of length k. For $\gamma \in \Gamma_k$ and $\sigma \in \Omega$, denote by

$$A_{-k}^{-m}(\gamma, \sigma) := \bigcap_{i=-k}^{-m} \left\{\theta_{\gamma_i, -i}\eta \equiv \sigma_i \text{ on } [-R, R]^d \times \{0\}\right\}, \quad 1 \leq m \leq k, \tag{1.10}$$

the event that an element $\eta \in \Omega$ equals σ in the R-neighbourhood along the path $(\gamma_k, \ldots, \gamma_{-m})$. $A_{-k}^{-1}(\gamma, \sigma)$ is the event that the path of the environment observed by the random walk equals σ if the random walk moves along the path γ. Given $\gamma \in \Gamma_k$, denote by

$$\mathcal{A}_{-k}^{-m}(\gamma) := \left\{A_{-k}^{-m}(\gamma, \sigma) \colon \sigma \in \Omega \text{ and } \mathbb{P}(A_{-k}^{-m}(\gamma, \sigma)) > 0\right\} \tag{1.11}$$

the set of all possible observations along the path $(\gamma_k, \ldots, \gamma_{-m})$. We write $\mathcal{A}_{-\infty}^{-m}$ for the set of events $\bigcup_{k \geq m, \gamma \in \Gamma_k} \mathcal{A}_{-k}^{-m}(\gamma)$. If $m = 1$ we simply write $\mathcal{A}_{-\infty}$.

Further, denote by $\mathcal{C} := \{(x, t) \in \mathbb{H} \colon \| x \|_1 \leq (R+1)t\}$ the forward cone with centre at $(o, 0)$ and slope proportional to $R + 1$. For $j \in \mathbb{Z}$, denote by $\mathcal{C}(j) := \mathcal{C} \cap \theta_{o,j}\mathbb{H}$ and let $\mathcal{F}_\infty^\infty := \bigcap_{j \in \mathbb{N}} \mathcal{F}_{\mathcal{C}(j)}$ be the tail-σ-algebra with respect to $\mathcal{F}_{\mathcal{C}}$.

Theorem 1.2 (Existence of an ergodic measure for the environment process)**.** *Assume that* $\mathbb{P} \in \mathcal{M}_1(\Omega)$ *satisfies*

$$\lim_{l\to\infty} \sup_{B\in\mathcal{F}_{\mathcal{C}(l)}} \sup_{A\in\mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| = 0. \tag{1.12}$$

Then there exists $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$ *invariant under* (η_t^{EP}) *satisfying* $\mathbb{P}^{EP} = \mathbb{P}$ *on* $\mathcal{F}_\infty^\infty$*.*

If (X_t) *is elliptic in the time direction and* $\mathbb{P}$ *is ergodic in the time direction, then* $\mathbb{P}^{EP}$ *is ergodic with respect to* (η_t^{EP})*. Moreover, for any* $\mathbb{Q} \ll \mathbb{P}$ *on* $\mathcal{F}_\infty^\infty$*,*

$$\frac{1}{t}\sum_{s=0}^{t-1} P_{\mathbb{Q}}(\eta_s^{EP} \in \cdot) \text{ converges weakly towards } \mathbb{P}^{EP} \text{ as } t \to \infty. \tag{1.13}$$

Remark 1.3. There is a certain freedom in the ellipticity and the ergodicity assumptions in Theorem 1.2. For instance, the statement still holds if, for some $k \in \mathbb{N}$, the walker has a positive probability to return to o after k time steps, uniformly in the environment. The definitions can also be modified to require ellipticity and ergodicity with respect other directions $(y,1) \in \mathbb{Z}^{d+1}$, with $y \in \mathcal{R}$ (instead of in direction $(o,1)$). On the other hand, both ellipticity and ergodicity in the time direction are natural assumptions if $\mathbb{P}$ is the path measure of some stochastic process.

Corollary 1.4 (Law of large numbers)**.** *Assume that* $\mathbb{P} \in \mathcal{M}_1(\Omega)$ *is ergodic in the time direction and satisfies* (1.12)*, and that* (X_t) *is elliptic in the time direction. Then there exists* $v \in \mathbb{R}^d$ *such that* $\lim_{t\to\infty} \frac{1}{t}X_t = v$*,* $P_{\mathbb{P}} - a.s.$

Condition (1.12) is a considerably weaker mixing assumption than the cone mixing condition introduced by Comets and Zeitouni [11] (see Condition $\mathcal{A}_1$ therein) and used in Avena, den Hollander, and Redig [3] in the context of random walks in dynamic random environment. For comparison, note that cone mixing is equivalent to taking the supremum over events $A \in \mathcal{F}_{<0} := \mathcal{F}_{\mathbb{Z}^{d+1}\setminus\mathbb{H}}$ in (1.12). That Condition (1.12) is strictly weaker can already be seen in the case where $\mathbb{P}$ is i.i.d. with respect to space; see Theorem 2.1. Further examples where Condition (1.12) improve on the classical cone mixing condition are given in Section 2 and include dynamic random environments with non-uniform mixing properties.

Under a slightly stronger mixing assumption on the environment we obtain more information about $\mathbb{P}^{EP}$. For this, denote by $\Lambda(l) := \{x \in \mathbb{H} \colon \|x\|_1 \geq l\}$, $l \in \mathbb{N}$, where $\|\cdot\|_1$ denotes the l_1 distance from $(o,0)$, and let $\mathcal{F}_{\geq 0}^\infty := \bigcap_{l\in\mathbb{N}} \mathcal{F}_{\Lambda(l)}$ be the tail-σ-algebra with respect to $\mathcal{F}_{\geq 0}$.

Theorem 1.5 (Absolute continuity)**.** *Let* $\phi\colon \mathbb{N} \to [0,1]$ *be such that*

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}} \sup_{A\in\mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq \phi(l), \tag{1.14}$$

with $\lim_{l\to\infty} \phi(l) = 0$*. Then* $\mathbb{P}^{EP} = \mathbb{P}$ *on* $\mathcal{F}_{\geq 0}^\infty$ *(with* $\mathbb{P}^{EP}$ *as in Theorem 1.2) and*

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}} |\mathbb{P}(B) - \mathbb{P}^{EP}(B)| \leq \phi(l). \tag{1.15}$$

Furthermore, if (X_t) *in addition is elliptic, then* $\mathbb{P}$ *and* $\mathbb{P}^{EP}$ *are mutually absolutely continuous on* $(\Omega, \mathcal{F}_{\geq 0})$*.*

Knowing that the environment process converges toward an ergodic measure, it is well known how to apply martingale technics in order to deduce an annealed functional central limit theorem. However, it may happen that the covariance matrix is trivial. In Redig and Völlering [25] it was shown that the covariance matrix is non-trivial in a rather general setting when the environment is given by a Markov process satisfying a certain uniform mixing assumption. It is an interesting question whether (X_t) satisfies an annealed functional central limit theorem with non-trivial covariance matrix under the weaker mixing assumption of (1.12).

To obtain a quenched central limit theorem is a much harder problem and is only known in a few cases for random walks in dynamic random environment, see e.g. Bricmont and Kupiainen [9], Deuschel, Guo, and Ramirez [12], Dolgopyat and Liverani [14] and Dolgopyat, Keller, and Liverani [15]. In [15], Theorem 1, a quenched central limit theorem was proven under technical conditions on both the environment and the environment process. One important condition there was that the environment process has an invariant measure mutually continuous with respect to the invariant measure of the environment. By Theorem 1.5 above this condition is fulfilled. Combining this result with rate of convergence estimates obtained in [25], we conclude a quenched central limit theorem for a large class of uniformly mixing environments.

Corollary 1.6 (Quenched central limit theorem)**.** *Assume that (η_t) is a Markov chain on $E^{\mathbb{Z}^d}$. For $\sigma, \omega \in \Omega_0$ let $\widehat{P}_{\sigma,\omega}$ be a coupling of (η_t) started from $\sigma, \omega \in \Omega_0$ respectively and satisfying, for some $c, C \in (0,\infty)$,*

$$\sup_{\sigma,\omega\in\Omega} \widehat{P}_{\sigma,\omega}(\eta_t^{(1)}(o) \neq \eta_t^{(2)}(o)) \leq Ce^{-ct}. \tag{1.16}$$

Furthermore, assume that (η_t) satisfies Conditions (A3)-(A4) in [15] and that (X_t) is elliptic. Then, there is a non-trivial $d \times d$ matrix Σ such that for P_μ-a.e. environment history (η_t)

$$\frac{X_N - Nv}{\sqrt{N}} \text{ converges weakly towards } \mathcal{N}(0,\Sigma) \quad \mathbb{P}_{(\eta_t)} - a.s., \tag{1.17}$$

where $\mu \in \mathcal{M}_1(\Omega_0)$ is the unique ergodic measure with respect to (η_t).

Conditions (A3)-(A4) in [15] are mixing assumptions on the dynamic random environment (η_t). Condition (A3) is a (weak) mixing assumption on μ, whereas Condition (A4) ensures that (η_t) is "local". For the precise definitions we refer to [15], page 1681.

In [15], Theorem 2, the statement of Corollary 1.6 was proven in a perturbative regime. Corollary 1.6 extends their result as there are no restrictions (other than ellipticity) on the transition probabilities of the random walk. We expect that Corollary 1.6 can be further improved to a functional CLT assuming only a polynomial decay in (1.16).

2 Examples and applications

In this section we present examples of environments which satisfy the conditions of Theorem 1.2 and Theorem 1.5. Particular emphasis is put on environments associated to a hidden Markov model for which we can improve on the necessary mixing assumptions.

2.1 Environments i.i.d. in space

The influence of the dimension on required mixing speeds is somewhat subtle. On the one hand, the random walk observes only a local area, and, in the case of conservative particle systems like the exclusion process, one can expect that in high dimensions information about observed particles in the past diffuses away. On the other hand, the higher dimension, the more sites the random walk can potentially visit in a fixed time. Furthermore, a comparison with a contact process or directed percolation gives an argument that information can spread easier in higher dimensions, hence observations along the path of the random walk could have more influence on future observations if the dimension increases.

This problem becomes significantly easier when the environment is assumed to be i.i.d. in space, that is $\mathbb{P} = \bigtimes_{x\in\mathbb{Z}^d} \mathbb{P}_o$, and $\mathbb{P}_o \in \mathcal{M}_1(E^{\mathbb{Z}})$ is the law of $(\eta_t(x))_{t\in\mathbb{Z}}$ for any $x \in \mathbb{Z}^d$.

Theorem 2.1. *Assume that* $\mathbb{P} = \bigtimes_{x\in\mathbb{Z}^d} \mathbb{P}_o$ *and that*

$$\sum_{t\geq 1} \sup_{B\in\mathcal{G}_{\geq t}, A\in\mathcal{G}_{<0}} |\mathbb{P}_o(B \mid A) - \mathbb{P}_o(B)| < \infty, \tag{2.1}$$

where $\mathcal{G}_{\geq t}$ *(*$\mathcal{G}_{<0}$*) is the* σ*-algebra of* $E^{\mathbb{Z}}$ *generated by the values after time* t *(before time 0) with respect to* $\mathbb{P}_o$*. Then* (1.14) *holds.*

Observe that (2.1) does not depend on the dimension. This is in contrast to the cone mixing condition of Comets and Zeitouni [11], where an additional factor t^d inside the sum of (2.1) is required. In Subsection 2.3 we present a class of environments which have arbitrary slow polynomial mixing, thus showing that Theorem 2.1 yields an essential improvement.

2.2 Hidden Markov models

When $\mathbb{P}$ is the path measure of a stochastic process (η_t) evolving on Ω_0, the results of Subsection 1.3 can be improved. In this subsection we discuss in detail the case where the random environment is governed by a hidden Markov model.

The environment (η_t) is a hidden Markov model if it is given via a function of a Markov chain (ξ_t). To be more precise, let $\widetilde{E}$ be a Polish space, $\widetilde{\Omega}_0 = \widetilde{E}^{\mathbb{Z}^{\tilde{d}}}$ with $\tilde{d} \geq d$, and $\widetilde{\Omega} = \widetilde{\Omega}_0^{\mathbb{Z}}$. Denote by $\tilde{\mathcal{F}}$ the corresponding σ-algebra. We assume that the Markov chain (ξ_t) is defined on $\widetilde{\Omega}$ with law $\widetilde{\mathbb{P}}_\xi$ and is ergodic with law $\widetilde{\mu} \in \mathcal{M}_1(\widetilde{\Omega}_0)$. Here $\xi \in \widetilde{\Omega}_0$ denotes the starting configuration. Let $\Phi : \widetilde{\Omega}_0 \to \Omega_0 = E^{\mathbb{Z}^d}$ be a translation invariant map and let $\eta_t = \Phi(\xi_t)$. We call (η_t) a *hidden Markov model*, which has μ as the induced measure on Ω_0 as invariant measure. We assume throughout that Φ is of finite range, that is, the function $\Phi(\cdot)(o)$ is $\tilde{\mathcal{F}}_\Lambda$-measurable for some $\Lambda \subset \mathbb{Z}^{\tilde{d}}$ finite.

Remark 2.2. When $\tilde{E}$ is finite, the canonical choice of Φ is the identity map. However, our setup opens for more sophisticated choices. One example is the projection map. For instance, if $\tilde{d} > 1$ and $d = 1$, one can consider the hidden Markov model given by $\eta_t(x) = \xi_t(x, 0, \ldots, 0)$. In other words, the random walk only observes the environment in one coordinate.

Condition (1.14) in Theorem 1.5 is an infinite volume condition which can be hard to verify by direct computation. The next result yields a sufficient condition which only needs to be checked for single site events. For its statement, we first introduce the concept of $\mathbb{P} \in \mathcal{M}_1(\Omega)$ having *finite speed of propagation*.

Definition 2.3. *We say that* $\mathbb{P} \in \mathcal{M}_1(\Omega)$ *has finite speed of propagation if the following holds: for some* $\alpha > 0$*, and for each* $A \in \mathcal{F}_{<0}$ *and* $A' \in \mathcal{F}_{\Lambda(\alpha t,t)}$*, where* $\Lambda(\alpha t, t) := \{(x, s) \in \mathbb{H}\colon \| x \|_1 \geq \alpha t, 0 < s \leq t\}$*, there is a coupling* $\widehat{\mathbb{P}}_{A,A'}$ *of* $\mathbb{P}(\cdot \mid A, A')$ *and* $\mathbb{P}(\cdot \mid A)$ *such that*

$$\sum_{t\geq 1} t^d \sup_{A\in\mathcal{F}_{<0}, A'\in\mathcal{F}_{\Lambda(\alpha t,t)}} \widehat{\mathbb{P}}_{A,A'} \left(\eta_t^1(o) \neq \eta_t^2(o)\right) < \infty. \tag{2.2}$$

Furthermore, any such coupling satisfies $\widehat{\mathbb{P}}_{A,A'}(\cdot) = \widehat{\mathbb{P}}_{\theta_{x,s}A,\theta_{x,s}A'}(\theta_{x,s}\cdot)$ *for all* $(x, s) \in \mathbb{Z}^{d+1}$*, where* $\omega \in \theta_{x,s}A$ *if and only if* $\theta_{-x,-s}\omega \in A$*.*

Finite speed of propagation is a natural assumption for many physical applications. Note that, for many interacting particle systems there is a canonical coupling given by the so-called graphical representation coupling.

Corollary 2.4. *Assume that* (ξ_t) *has finite speed of propagation and that*

$$\sum_{t\geq 1} t^d \sup_{A\in\mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A}\left(\Phi(\xi^1)_0(o) \neq \Phi(\xi^2)_0(o)\right) < \infty. \tag{2.3}$$

Then (1.14) *holds for* $(\eta_t) = \Phi(\xi_t)$.

Remark 2.5. The measure $\widehat{\mathbb{P}}_{\Omega,A}$ denotes the coupling of $\mathbb{P}(\cdot)$ and $\mathbb{P}(\cdot \mid A)$.

Corollary 2.4 follows by a slightly more general statement, see Theorem 4.4. This approach can also be used in cases where (ξ_t) does not have finite speed of propagation. In such cases, (2.3) is sufficient for (1.12) to hold. Observe also that, by applying the projection map introduced in Remark 2.2, the dimensionality dependence in Condition (2.3) can be replaced by the dimensionality of the range of the random walk.

Markovian environment

If $\mathbb{P}$ is the path measure of a Markov chain (η_t), we can weaken the mixing assumption. In such cases, we consider α as a function from $\Omega_0 \times \mathbb{Z}^d$. Because the Markov property allows us to look at the invariant measure of the environment process just at a time 0 instead of in the entire upper half-space $\mathbb{H}$, we have the following mixing condition. Here we denote by $\mathcal{F}_{=0}^{\infty}$ the tail-σ-algebra of $\mathcal{F}_{=0} := \mathcal{F}_{\mathbb{Z}^d\times\{0\}}$.

Theorem 2.6. *Assume that* (η_t) *is a Markov chain with ergodic invariant measure* $\mu \in \mathcal{M}_1(\Omega_0)$. *Further, assume that the path measure* $\mathbb{P}_\mu \in \mathcal{M}_1(\Omega)$ *has finite speed of propagation and that*

$$\sum_{t\geq 1} t^{d-1} \sup_{A\in\mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A}\left(\eta_0^1(o) \neq \eta_0^2(o)\right) < \infty. \tag{2.4}$$

Then there exists $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$ *invariant for the Markov chain* (η_t^{EP}) *such that* μ^{EP} *agrees with* μ *on* $\mathcal{F}_{=0}^{\infty}$. *If in addition* (X_t) *is elliptic, then* μ^{EP} *and* μ *are mutually absolutely continuous and* μ^{EP} *is ergodic with respect to* (η_t^{EP}).

It is important to note that (2.4) (as well as (2.3)) does not require (η_t) to be uniquely ergodic. However, if for every $\sigma, \xi \in \Omega_0$, there is a coupling $\widehat{\mathbb{P}}_{\sigma,\xi}$ of $\mathbb{P}_\sigma$ and $\mathbb{P}_\xi$ which satisfies the finite speed of propagation property and

$$\sum_{t=1}^{\infty} t^{d-1} \sup_{\sigma,\xi\in\widetilde{\Omega}_0} \widehat{\mathbb{P}}_{\sigma,\xi}(\eta_t^1(o) \neq \eta_t^2(o)) < \infty, \tag{2.5}$$

then it follows, under the assumptions of Theorem 2.6, that (η_t^{EP}) is uniquely ergodic.

Equation (2.5) should be compared with Assumption 1a in [25], that is;

$$\int_0^{\infty} t^{(d)} \sup_{\eta,\xi} \hat{\mathbb{E}}_{\eta,\xi}\rho(\eta_t^1(o), \eta_t^2(o))dt < \infty, \tag{2.6}$$

where $\rho\colon E \times E \to [0,1]$ is the distance function. Their assumption was used to show (among others) the existence of $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$ invariant and ergodic for the Markov chain (η_t^{EP}), see Lemma 3.2 therein. Note in particular that Assumption (2.6) has $t^{(d)}$ inside the integral, whereas (2.5) only requires $t^{(d-1)}$ inside the sum.

2.3 Polynomially mixing environments

As example of environments which fully utilise the polynomial mixing assumption of Theorem 2.1 and Corollary 2.4, we consider layered environments. These were already considered in [25] for the same purpose, but since we are in a different setting we use the setting of hidden Markov models.

The idea of layered environments is that, given a summable sequence $(b_n) \subset (0,1)$, for each layer n, the process $(\xi_t(\cdot,n))_{t\in\mathbb{Z}_{\geq 0}}$ is an uniform exponentially mixing Markov chain on $[-1,1]$ with an exponential relaxation rate b_n, and independent layers. For simplicity, in this example, we choose $\xi_t(\cdot,n)$ to be i.i.d. spin flips, that is, for each $x \in \mathbb{Z}^{\tilde{d}}$,

$$\xi_{t+1}(x,n) = \begin{cases} \xi_t(x,n), & \text{with probability} \quad 1-b_n; \\ \text{Unif}[-1,1], & \text{with probability} \quad b_n; \end{cases} \tag{2.7}$$

independent for all x, n, t. In other words, at each time step the spin retains its old value with probability $1-b_n$ and chooses uniformly on $[-1,1]$ with probability b_n.

In the context of the previous subsection we thus have $\widetilde{E} := [-1,1]^{\mathbb{N}}$. We further choose $\widetilde{d} = d \geq 1$, $E = \{0,1\}$ and set, for a summable sequence $(a_n) \subset (0,1)$,

$$\Phi(\xi)(x) = \mathbb{1}_{\sum_{n=1}^{\infty} a_n \xi(x,n) > 0}. \tag{2.8}$$

The behaviour of this kind of processes is then determined by the two sequences (a_n) and (b_n). When $a_n = \frac{1}{2}n^{-\alpha}$, $b_n = \frac{1}{2}n^{-\beta}$ for some $\alpha, \beta > 1$, we have the following bound on the mixing of (η_t).

Theorem 2.7. *There are constants* $0 < c_1 < c_2 < \infty$ *so that*

$$c_1 t^{\frac{-\alpha+1}{\beta}} \leq \sup_{\xi,\sigma} \| \mathbb{P}_\xi(\eta_t(0) \in \cdot) - \mathbb{P}_\sigma(\eta_t(0) \in \cdot) \|_{TV} \leq c_2 t^{\frac{-\alpha+1}{\beta}} (\log t)^{\frac{\alpha-1}{\beta}}. \tag{2.9}$$

Here $\|\cdot\|_{TV}$ *is the total variation distance between the two distributions. In particular, if* $\alpha > \beta + 1$, *then* (1.14) *holds.*

2.4 Independent Ornstein-Uhlenbeck processes

With the approach of environments as hidden Markov models, we can also allow for unbounded state spaces where the environment does not mix uniformly, as long as the random walk transition function is simple enough. Here we choose an underlying environment of independent Ornstein-Uhlenbeck processes $(\xi_t^x)_{t\in\mathbb{R}}$ for each site $x \in \mathbb{Z}^d$, and the jump rates depend only on the signs, that is,

$$\eta_t(x) = \text{sign}(\xi_t^x) := 1 - 2\mathbb{1}_{\xi_t^x < 0}, \quad t \in \mathbb{Z}. \tag{2.10}$$

To state the example more formally, we have $\tilde{E} = \mathbb{R}$ and $E = \{-1,1\}$, and

$$d\xi_t^x = -\xi_t^x dt + dW_t^x, \tag{2.11}$$

where $(W_t^x)_{t\in\mathbb{R}}$, $x \in \mathbb{Z}^d$, are independent two-sided Brownian motions. The stationary measure of ξ_t^x is a normal distribution, and $\tilde{\mu}$ is the product measure of normal distributions.

Theorem 2.8. *Let* $(\xi_t)_{t\in\mathbb{R}}$ *be an Ornstein-Uhlenbeck process and* $\widetilde{\mathbb{P}}$ *the two-sided path measure in stationarity. There are constants* $c, C > 0$ *so that*

$$\left\| \widetilde{\mathbb{P}}(\xi_t \in \cdot \mid A) - \widetilde{\mathbb{P}}(\xi_t \in \cdot) \right\|_{TV} \leq Ce^{-ct} \tag{2.12}$$

for all $t \geq 0$ *and any* A *of the form* $A = \{\text{sign}(\xi_{-t_k}) = a_k, 1 \leq k \leq n\}$, (t_k) *increasing sequence with* $t_1 = 0$ *and* $a_k \in \{-1,1\}$, n *arbitrary. In particular,* (1.14) *holds for* (η_t).

2.5 The contact process

As a second example of an environment with non-uniform space-time correlations and which do not satisfy the cone mixing property of [11], we consider the contact process (η_t) on $\{0,1\}^{\mathbb{Z}^d}$ with infection parameter $\lambda \in (0,\infty)$.

The contact process is one of the simplest interacting particle systems exhibiting a phase transition. That is, there is a critical threshold $\lambda_c(d) \in (0,\infty)$, depending on the dimension d, such that the following holds: if $\lambda \le \lambda_c(d)$, then the contact process is uniquely ergodic with the measure concentrating on the configuration where all sites equal to 0 as invariant measure. On the other hand, for all $\lambda > \lambda_c(d)$, the contact process is not uniquely ergodic. In particular, it has a non-trivial ergodic invariant measure, denoted here by $\bar{\nu}_\lambda$, also known as the upper invariant measure. As a general reference, and for a precise description of the contact process, we refer to Liggett [20].

Random walks on the contact process have recently been studied by den Hollander and dos Santos [18] and Mountford and Vares [23], where the one-dimensional random walk (i.e. on $\mathbb{Z}$) was shown to behave diffusively for all $\lambda > \lambda_c(1)$. See also Bethuelsen and Heydenreich [8] for some results in general dimensions.

The next theorem sheds new light on the behaviour of the environment process and the random walk for this model on $\mathbb{Z}^d$ with $d \ge 2$. In the theorem we make use of the projection map, as introduced in Remark 2.2. That is, we assume $\tilde{d} \ge 2$ and denote by $(\eta_t) = \phi((\xi_t))$ the projection of (ξ_t) onto the 1-dimensional lattice such that, for $x \in \mathbb{Z}$ and $t \in \mathbb{Z}$, we set $\eta_t(x) = \xi_t(x,0,\dots,0)$.

Theorem 2.9. *Let $\tilde{d} \ge 2$ and let (ξ_t) be the contact process with parameter $\lambda > \lambda_c(\tilde{d})$ started from $\bar{\nu}_\lambda$. Further, let (η_t^{EP}) be the environment process corresponding to the process $(\eta_t) = \phi((\xi_t))$. Then* (1.14) *holds for (η_t).*

Theorem 2.9 can be extended to higher dimensional projections by following the same approach. The proof strategy of Theorem 2.9 also applies to a larger class of models which satisfy the so-called *downward FKG* property; see Theorem 4.5.

3 Understanding the environment process

3.1 Expansion of the environment process

In this subsection, we present a key observation for understanding the environment process and for the proofs of Theorem 1.2 and Theorem 1.5.

Intuitively, the distribution of (η_t^{EP}) should converge to an invariant measure, say $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$, which describes asymptotic properties. To obtain $\mathbb{P}^{EP}$ and show that it is absolutely continuous with respect to $\mathbb{P}$, we start by interpreting the law of η_t^{EP}, $P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$, as an approximation. With this point of view, t becomes the present time. Going from t to $t+1$ thus means that we look one step further into the past. To reinforce this point of view, we denote by $\mathbb{P}^{-k} := \mathbb{P}_{\mathbb{P}}^{-k} \in \mathcal{M}_1(\Omega \times (\mathbb{Z}^d)^{\mathbb{Z}_{\ge -k}})$ the joint law of the environment $\mathbb{P}$ and random walk $(X_t)_{t \ge -k}$ so that $X_0 = o$. That is, for $k \in \mathbb{N}$,

$$\mathbb{P}^{-k}\left((\eta, X) \in (B_1, B_2)\right) := P_{\mathbb{P}}\left(\eta_{k+\cdot}^{EP} \in B_1, (X_{k+\cdot} - X_k) \in B_2\right). \tag{3.1}$$

For events $B \in \mathcal{F}$, we use the shorthand notation $\mathbb{P}^{-k}(B)$ for the probability that $\mathbb{P}^{-k}((\eta, X) \in (B, (\mathbb{Z}^d)^{\mathbb{Z}_{\ge -k}}))$.

Theorem 3.1 (Expansion of the environment process)**.** *For any $k \ge 1$ and $B \in \mathcal{F}$,*

$$\mathbb{P}^{-k}(B) = \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right), \tag{3.2}$$

where $P\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right) := \prod_{i=-k}^{-1} \alpha(\hat{\sigma}_i, \gamma_{i+1} - \gamma_i)$, and $\hat{\sigma} \in \Omega$ is any environment so that $A_{-k}^{-1}(\gamma, \hat{\sigma}) = A_{-k}^{-1}$.

Proof. We can rewrite $\mathbb{P}^{-k}(B)$ as follows;

$$\begin{aligned}\mathbb{P}^{-k}(B) &= \sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}^{-k}(B,\ X_{-k,\dots,0}=\gamma, A_{-k}^{-1})\\ &= \sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \Big[\mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\dots,0}=\gamma)\\ &\qquad\qquad \mathbb{P}^{-k}(X_{-k,\dots,0}=\gamma, A_{-k}^{-1})\Big].\end{aligned}$$

By definition,

$$\begin{aligned}\mathbb{P}^{-k}(X_{-k,\dots,0}=\gamma, A_{-k}^{-1}) &= P_{\mathbb{P}}(X_{0,\dots,k}=\gamma-\gamma_{-k}, \eta_k^{EP}\in A_{-k}^{-1})\\ &= P_{\mathbb{P}}(X_{0,\dots,k}=\gamma-\gamma_{-k}, \theta_{-\gamma_{-k},k}\eta\in A_{-k}^{-1})\\ &= P(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1})\mathbb{P}(\theta_{-\gamma_{-k},k}\eta\in A_{-k}^{-1})\\ &= P(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1})\mathbb{P}(A_{-k}^{-1}),\end{aligned}$$

where the last equality holds since first the law of the environment is translation invariant. Similarly,

$$\mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\dots,0}=\gamma) = \mathbb{P}(B \mid A_{-k}^{-1}).$$

□

The sum in Expansion (3.2) represents all the possible pasts of the random walk and the corresponding observed environments from time $-k$ to -1. There are two key features with this expansion.

First, it separates the contribution to (η_t^{EP}) of the random walk from that of the random environment. Indeed, the rightmost term in the sum, i.e. $P\left(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1}\right)$, can be calculated directly from the transition probabilities of (X_t). On the other hand, the leftmost term in the sum, i.e. $\mathbb{P}\left(B, A_{-k}^{-1}\right)$, only involves the random environment $\mathbb{P}$.

A second key feature of (3.2) is that it serves as a (formal) expression for the Radon-Nikodym derivate of $P_{\mathbb{P}}(\eta_k^{EP}\in\cdot)$ with respect to $\mathbb{P}$. Indeed, (3.2) yields that for any $B\in\mathcal{F}$ with $\mathbb{P}(B)>0$,

$$\frac{P_{\mathbb{P}}(\eta_k^{EP}\in B)}{\mathbb{P}(B)} = \sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \frac{\mathbb{P}\left(B, A_{-k}^{-1}\right)}{\mathbb{P}(B)} P\left(X_{-k,\dots,0}=\gamma \mid A_{-k}^{-1}\right). \tag{3.3}$$

3.2 Stability

It is also of interest to compare the effect of changing the environment $\mathbb{P}$ or the transition probabilities $\alpha\colon \Omega\times\mathbb{Z}^d\to[0,1]$ on the behaviour of the environment process. Our next result gives sufficient conditions for the environment process to be stable with respect to perturbations of both these parameters. This result follows as another consequence of the expansion in Theorem 3.1.

To state the theorem precisely, denote by $(\mathbb{P}^n)_{n\geq1}$ a family of measures on $\mathcal{M}_1(\Omega)$ and let $\left(\alpha_n\colon \Omega\times\mathbb{Z}^d\to[0,1]\right)_{n\geq1}$ be a collection of transition probabilities. Consider for each $n\in\mathbb{N}$ the corresponding environment process, $(\eta_t^{EP(n)})$, and let $\mathbb{P}^{EP(n)}\in\mathcal{M}_1(\Omega)$ be a measure invariant under $(\eta_t^{EP(n)})$.

Theorem 3.2. *Assume that the following holds.*

a) $\epsilon(n)=\sup_{m>n}\sup_{\eta\in\Omega,y\in\mathbb{Z}^d}|\alpha_m(\eta,y)-\alpha_n(\eta,y)|\downarrow 0$ *as* $n\to\infty$.

b) $\mathbb{P}^n \implies \mathbb{P}\in\mathcal{M}_1(\Omega)$ *weakly as* $n\to\infty$.

c) $P_{\mathbb{P}^n}(\eta_t^{EP(n)} \in \cdot) \implies \mathbb{P}^{EP(n)}$ *weakly as* $t \to \infty$*, uniformly in* n.

Then both $\mathbb{P}^{EP(n)}$ *and* $P_{\mathbb{P}}(\eta_t^{EP})$ *converge weakly towards* $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$*. In particular,* $\mathbb{P}^{EP}$ *is invariant with respect to* (η_t^{EP})*.*

Condition c) in Theorem 3.2 is a strong uniform assumption. If the $\mathbb{P}^n$'s are path measures of Markov chains $(\eta_t^{(n)})$, this condition can be replaced by the assumption that the environment process (η_t^{EP}), i.e., after taking $n \to \infty$, is uniquely ergodic. For this, recall notation from Section 2.2 and let $\mu^{EP(n)} \in \mathcal{M}_1(\Omega_0)$ be an invariant measure with respect to $(\eta_t^{EP(n)})$.

Theorem 3.3. *Let* (η_t) *be a Markov chain and assume that the following holds.*

a) $\epsilon(n) = \sup_{m>n} \sup_{\eta\in\Omega, y\in\mathbb{Z}^d} |\alpha_m(\eta,y) - \alpha_n(\eta,y)| \downarrow 0$ *as* $n \to \infty$.

b') $\mathbb{P}_\sigma^n \implies \mathbb{P}_\sigma \in \mathcal{M}_1(\Omega)$ *for every starting configuration* $\sigma \in \Omega_0$.

c') (η_t^{EP}) *is uniquely ergodic with invariant measure* $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$.

Then $\mu^{EP(n)} \implies \mu^{EP}$ *weakly.*

Remark 3.4. Theorem 3.3 does only require that the limiting process (η_t^{EP}) is uniquely ergodic. In particular, the processes $(\eta_t^{EP(n)})$ do not need to be uniquely ergodic. As an example of the latter, one can consider the case where $(\eta_t^{(n)})$ is the contact process with parameter $\lambda(n) \downarrow \lambda_c$ and $\inf_{\eta\in\Omega} \alpha_n(\eta, o) \uparrow 1$.

Theorem 3.3 gives a generalisation of Theorem 3.3 in [25]. There they showed continuity for the environment process with respect to changes of the transition probabilities of the random walk, assuming that Assumption 1a therein to hold (which we also stated in (2.6)). Theorem 3.3 yields a similar continuity result which in addition allow for changes in the dynamics of the environment (η_t). Moreover, unique ergodicity is a weaker assumption than the mixing assumption given by (2.5), as we have already seen in Subsection 2.1 and Subsection 2.2.

3.3 Estimating the Radon-Nikodym derivative

We end this section with an alternative route for proving the existence of an invariant measure for the environment process which is absolutely continuous with respect to the underlying environment. An advantage of this approach is that it implies bounds on the Radon-Nikodym derivative.

Theorem 3.5. *Assume that, for some* $M_1 \in (0,\infty)$,

$$\sup_{B\in\mathcal{F}_{\geq 0}} \sup_{A_l\in\mathcal{A}_{-l}^{-1}} \left| \frac{\mathbb{P}(B \mid A_l)}{\mathbb{P}(B)} - 1 \right| \leq M_1, \quad \forall\, l \in \mathbb{N}\,. \tag{3.4}$$

Then there is a $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$*, invariant under* (η_t^{EP})*, and* $\mathbb{P}^{EP} \ll \mathbb{P}$ *on* $(\Omega, \mathcal{F}_{\geq 0})$*. Moreover, the corresponding Radon-Nikodym derivative is bounded by* M_1 *in the* L_∞*-norm. Furthermore, if for some* $M_2 \in (0,\infty)$,

$$\sup_{B\in\mathcal{F}_{\geq 0}} \sup_{A_l\in\mathcal{A}_{-l}^{-1}} \left| \frac{\mathbb{P}(B)}{\mathbb{P}(B \mid A_l)} - 1 \right| \leq M_2, \quad \forall\, l \in \mathbb{N}\,. \tag{3.5}$$

Then $\mathbb{P} \ll \mathbb{P}^{EP}$ *and the corresponding Radon-Nikodym derivative is bounded by* M_2 *in the* L_∞*-norm.*

Remark 3.6. Mutually absolute continuity can also be shown without requiring (3.5) to hold. In particular, if (3.4) holds and (X_t) is elliptic in the time direction, it can be shown that $\mathbb{P} \ll \mathbb{P}^{EP}$. Under these assumptions it moreover follows that $\mathbb{P}^{EP}$ is ergodic and that $\left(t^{-1}\sum_{k=1}^t P_{\mathbb{P}}(\eta_k^{EP} \in \cdot)\right)_{t\geq 1}$ converges weakly towards $\mathbb{P}^{EP}$.

Mixing assumptions of the type (3.4) and (3.5) are typically much stronger than mixing assumptions as in Theorems 1.2 and 1.5. Nevertheless, we believe that Theorem 3.5 is applicable to a wide range of models and is not restricted to the uniform mixing case. However, it seems difficult to verify (3.4) and (3.5) for concrete examples unless strong mixing assumptions are made.

One class of examples to which Theorem 3.5 applies are Gibbs measures in the high-temperature regime satisfying the Dobrushin-Shlosman strong mixing condition (as considered in Rassoul-Agha [24] for RWRE models); see Theorem 1.1 (in particular, Condition IIId) in Dobrushin and Shlosman [13]. Another class of environments are certain monotone Gibbs measures for which Alexander [1] proved (see Theorems 3.3 and 3.4 therein) that weak mixing implies ratio mixing. In particular, the models considered there satisfy (3.4) and (3.5) throughout the uniqueness regime. We also mention the method of disagreement percolation, which is particularly useful for models with hard-core constraints, see van den Berg and Maes [6].

In the case of dynamic random environments which in addition are reversible with respect to time, typically, the methods described above for random fields can be adapted to yield similar bounds. In Section 4.5 we introduce a new class of dynamic random environments satisfying (3.4), allowing for non-reversible dynamics. We comment next on the scope of this approach.

Our approach is by means of disagreement percolation and applies to *discrete-time finite-range Markov chain* (η_t). In fact, we shall need more than subcriticality of the ordinary disagreement process. For what we believe to be technical reasons, we will introduce what we call the *strong disagreement percolation coupling*. This is a triple $(\eta^1_t, \eta^2_t, \xi_t)$ where (η^1_t, η^2_t) is a coupling of $\mathbb{P}_{\eta^1_0}$ and $\mathbb{P}_{\eta^2_o}$, $\xi_t(x) = 0$ implies $\eta^1_t(x) = \eta^2_t(x)$, and η^1 and ξ are independent. That is, the disagreement process ξ and the process η^1 are independent. This independence is a stronger assumption than regular disagreement percolation and the strong disagreement percolation process is subcritical for models at "very high-temperature". We refer to Section 4.5 for a precise construction of the strong disagreement percolation coupling and a proof of the following theorem.

Theorem 3.7 (Strong disagreement percolation)**.** *Suppose the strong disagreement percolation process is subcritical. Then there exists $\delta < 1$ and $C > 0$ so that for any $B \in \mathcal{F}_{=0}$,*

$$\sup_{\substack{A^{-1}_{-(k+1)} \in \mathcal{A}^{-1}_{-(k+1)}, A^{-1}_{-k} \in \mathcal{A}^{-1}_{-k} \\ A^{-1}_{-k} \subset A^{-1}_{-(k+1)}}} \sup_{B \in \mathcal{F}_{=0}} \left| \left[\frac{\mathbb{P}\left(B \mid A^{-1}_{-k-1}\right)}{\mathbb{P}\left(B \mid A^{-1}_{-k}\right)} \right]^{\pm 1} - 1 \right| \le C\delta^k. \tag{3.6}$$

Theorem 3.7 implies that the environment process (η^{EP}_t) has a unique invariant distribution, $\mu^{EP} \in \mathcal{M}_1(\Omega_0)$. In particular, μ^{EP} is absolutely continuous with respect to the (necessarily unique) invariant measure of (η_t), denoted by $\mu \in \mathcal{M}_1(\Omega_0)$. As a further consequence, we obtain uniform control on the Radon-Nikodym derivative.

Corollary 3.8 (Uniform control on the Radon-Nikodym derivative)**.** *Assume that the environment (η_t) has a strong disagreement percolation coupling which is subcritical. Then μ^{EP} and μ are mutually absolutely continuous. Moreover, there exists a constant $M \in (0, \infty)$, depending only on the environment, such that $\left\| \frac{d\mu^{EP}}{d\mu} \right\|_\infty \le M$ and $\left\| \frac{d\mu}{d\mu^{EP}} \right\|_\infty \le M$.*

Subcriticality of the strong disagreement coupling is a much stronger assumption than the uniform mixing assumption in (2.5). For comparison with other coupling methods, consider for concreteness the stochastic Ising model with inverse temperature $\beta > 0$ (see e.g. [10] for a definition). This model satisfies (2.5) for all $\beta < \beta_c$, where

β_c is the critical inverse temperature. On the other hand, it has a subcritical strong disagreement coupling whenever

$$\beta < \frac{1}{4d} \log\big(\frac{2d}{2d-1}\big) < \beta_c. \tag{3.7}$$

For comparison, this condition is better (with a factor 2) compared with the disagreement percolation coupling introduced in [10] (see Equation (11) therein).

Remark 3.9. The estimate in (3.7) is valid for antiferromagnetic models and models with a magnetic field, as also considered in [10]. In particular, the strong disagreement percolation method is not restricted to monotone environments.

4 Proofs

In this section, we present the proofs of the theorems given in the previous sections. In Subsection 4.1 we give the proofs of theorems in Section 1.3. Proofs of theorems in Section 2 are given in Subsection 4.2. In the remaining subsections we present proofs of theorems from Section 3. In particular, Subsection 4.5 introduces the strong disagreement coupling and contains the proof Theorem 3.7.

4.1 Proof of main results

The main application of the expansion in Theorem 3.1 for the proofs of Theorems 1.2 and 1.5 is the following lemma.

Lemma 4.1. *Let $\Lambda \subset \mathbb{Z}^{d+1}$. For $B \in \mathcal{F}_\Lambda$ and $k \in \mathbb{N}$ we have that*

$$|P_{\mathbb{P}}(\eta_k^{EP} \in B) - \mathbb{P}(B)| \le \sup_{A \in \mathcal{A}_{-\infty}} |\mathbb{P}(B \mid A) - \mathbb{P}(B)|. \tag{4.1}$$

Proof. Let $l \in \mathbb{N}$ and consider any $B \in \mathcal{F}_\Lambda$. From Theorem 3.1 we have that for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\big|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\big| \\
=&\Big| \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\Big| \\
=&\Big| \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B \mid A_{-k}^{-1}\right) \mathbb{P}^{-k}\left(X_{-k,\dots,0} = \gamma, A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\Big| \\
\le& \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \big|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\big| \mathbb{P}^{-k}\left(X_{-k,\dots,0} = \gamma, A_{-k}^{-1}\right) \\
\le& \sup_{A \in \mathcal{A}_{-\infty}} \big|\mathbb{P}\left(B \mid A\right) - \mathbb{P}\left(B\right)\big|,
\end{aligned}$$

where in the second last and the last inequality we used the fact that

$$\sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}^{-k}\left(X_{-k,\dots,0} = \gamma, A_{-k}^{-1}(\gamma,\sigma)\right) = 1.$$

□

Proof of Theorem 1.2. Consider a sequence (t_k) and a sequence of measures given by $\mathbb{Q}_k := \frac{1}{t_k}\sum_{t=0}^{t_k-1} P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$ that converges weakly to $\mathbb{Q} \in \mathcal{M}_1(\Omega)$. By standard compactness arguments such a sequence exists and, moreover, any such limiting measure $\mathbb{Q}$ is invariant for (η_t^{EP}). A proof of the last claim is e.g. given in [24]; see page 1457 in the proof of Theorem 3 therein.

Since (1.12) is assumed to hold, it follows by Lemma 4.1 that for any $l \in \mathbb{N}$ and $B \in \mathcal{F}_{\mathcal{C}(l)}$,

$$|\mathbb{Q}_k(B) - \mathbb{P}(B)| \leq \phi(l), \tag{4.2}$$

for some $\phi\colon\ \mathbb{N} \to [0,1]$ such that $\lim_{l\to\infty} \phi(l) = 0$. As this estimate is uniform in k, we claim that (4.2) also holds when $\mathbb{Q}_k$ is replaced by $\mathbb{Q}$. To see this, consider the space of measures measurable with respect to $(\Omega, \mathcal{F}_{\mathcal{C}(l)})$. The ball of radius $\phi(l)$ around $\mathbb{P}$ (in the total variation sense) is compact in the topology of weak convergence by the Banach-Alaoglu-Theorem. Here we use that the space is compact, the dual of the continuous bounded functions are finite signed measures equipped with the total variation norm, and the weak convergence of measures is the weak-* convergence in this functional-analytic setting. Since the ball is compact it is closed, and any limit point $\mathbb{Q}$ of the sequence $\mathbb{Q}_k$ is also inside the ball. Hence $|\mathbb{Q}(B) - \mathbb{P}(B)| \leq \phi(l)$ for any $B \in \mathcal{F}_{\mathcal{C}(l)}$ and consequently, since $\lim_{l\to\infty} \phi(l) = 0$, we have $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty^\infty$.

We continue with the proof that $\mathbb{Q}$ is ergodic with respect to (η_t^{EP}), by following the proof of Theorem 2ii) in [24]. Denote by $\mathcal{I} \subset \mathcal{F}$ the σ-algebra consisting of those events invariant under the evolution of (η_t^{EP}). Further, let f be any local bounded function on Ω and define $g = \mathbb{E}_{\mathbb{Q}}\left(f \mid \mathcal{I}\right)$. Birkhoffs ergodic theorem implies that,

$$P_\eta\left(\lim_{n\to\infty} n^{-1}\sum_{m=1}^{n} f(\eta_m^{EP}) = g(\eta)\right) = 1, \quad \text{for } \mathbb{Q}\text{-a.e. } \eta \in \Omega.$$

Using that $\mathbb{Q}$ is invariant and that g is harmonic, we have

$$\begin{aligned}
&\sum_{|x|\leq R}\int \alpha(\eta,x)(g(\eta) - g(\theta_{x,1}\eta))^2\mathbb{Q}(d\eta)\\
=&\int g^2(\eta)\mathbb{Q}(d\eta) - 2\int g(\eta)\sum_{|x|\leq R}\alpha(\eta,x)g(\theta_{x,1}\eta)\mathbb{Q}(d\eta)\\
&\qquad + \int \sum_{|x|\leq R}\alpha(\eta,x)(g(\theta_{x,1}\eta))^2\mathbb{Q}(d\eta)\\
=&0.
\end{aligned}$$

In particular, since (X_t) is elliptic in the time direction, $g = g \circ \theta_{o,1}$, $\mathbb{Q}$-a.s.

Next, for each $t \in \mathbb{N}$, denote by $B_t \subset \{X_i = o \text{ for all } i \in \{0,\dots,t\}\}$ the event that the random walk does not move in the time-interval $[0,t]$, irrespectively of the environment. Since (X_t) is elliptic in the time direction, B_t has strictly positive probability and can be taken independently of the environment. Further, define

$$\bar{g}(\eta) := \limsup_{n\to\infty} \frac{1}{\mathbb{Q}(B_t)}\int_{B_t} n^{-1}\sum_{m=1}^{n} f(\eta_m^{EP})dP_\eta. \tag{4.3}$$

Then, because of (4.1), we know that $g = \bar{g}$, $\mathbb{Q}$-a.s. Further, using the above mentioned independence property, and by possibly taking t large, we note that $\bar{g}$ is $\mathcal{C}(k)$-measurable for any $k \in \mathbb{N}$. Consequently, the same holds for g, and hence g is $\mathcal{F}_\infty^\infty$-measurable. Furthermore, since $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty^\infty$, this implies that (4.1) holds $\mathbb{P}$-a.s., and that $g = g \circ \theta_{o,1}$, $\mathbb{P}$-a.s. As $\mathbb{P}$ is ergodic with respect to $\theta_{o,1}$, it moreover follows that g is constant $\mathbb{P}$-a.s., and hence also $\mathbb{Q}$-a.s. Since f was an arbitrary local bounded function, we conclude from this that $\mathcal{I}$ is trivial and thus that $\mathbb{Q}$ is ergodic with respect to (η_t^{EP}).

To conclude the proof we also note that (1.13) holds. Indeed, since $\mathbb{Q}$ was an arbitrary (sub) sequence of $(\mathbb{Q}_k)$, all the estimates above are valid for any such limiting measure. In particular, each of these limiting measures equal $\mathbb{P}$ on $\mathcal{F}_\infty^\infty$, and consequently, they are all ergodic and equal on $\mathcal{I}$. Thus, they are the same, and we conclude that (1.13) holds with respect to $\mathbb{P}$, where we call the limiting measure $\mathbb{P}^{EP}$. Initialising (η_t^{EP}) with any other probability measure, absolute continuous with respect to $\mathbb{P}$ on $\mathcal{F}_\infty^\infty$, the exact same argument as outlined above applies, from which we conclude (1.13) and the proof of Theorem 1.2. □

Proof of Corollary 1.4. The claim is an (almost direct) application of ergodicity and that $\mathbb{P} = \mathbb{P}^{EP}$ on $\mathcal{F}_\infty^\infty$. Indeed, let $D(\eta) := \sum_{z\in\mathbb{Z}^d} z\alpha(\eta, z)$ be the local drift of the random walker in environment η. A direct consequence of Theorem 1.2 is that

$$P_{\mathbb{P}^{EP}}\left(\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} D(\theta_{X_k,k}\eta) = v\right) = 1, \tag{4.4}$$

where $v = \int D(\eta)P_{\mathbb{P}^{EP}}(d\eta)$. By using that $\mathbb{P} = \mathbb{P}^{EP}$ on $\mathcal{F}_\infty^\infty$, it follows that this also holds with respect to $P_{\mathbb{P}}$. Now, note that $M_n = X_n - \sum_{m=0}^{n-1} D(\theta_{X_k,k}\eta)$ is a martingale with bounded increments under P_η. Therefore $P_\eta(\lim_{n\to\infty} n^{-1}M_n = 0) = 1$ which together with (4.4) implies the law of large numbers. □

We next turn to the proof of Theorem 1.5. The following lemma is essentially copied from [7].

Lemma 4.2. *Assume $\mathbb{P}^{EP}$ is invariant with respect to (η_t^{EP}) and that $\mathbb{P}^{EP}$ and $\mathbb{P}$ restricted to $(\Omega, \mathcal{F}_{\geq 0})$ are not singular. Assume (X_t) is elliptic. Then there exists $\mathbb{P}_c^{EP} \in \mathcal{M}_1(\Omega)$, invariant for (η_t^{EP}) and mutually absolutely continuous to $\mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$.*

Proof of Lemma 4.2. Consider the (unique) Lebesgue decomposition of $\mathbb{P}^{EP}$ with respect to $\mathbb{P}$ restricted to $(\Omega, \mathcal{F}_{\geq 0})$. That is, let

$$\mathbb{P}^{EP}(B) = \alpha\mathbb{P}_c^{EP}(B) + (1-\alpha)\mathbb{P}_s^{EP}(B), \quad \forall\, B \in \mathcal{F}_{\geq 0}, \tag{4.5}$$

where $\mathbb{P}_c^{EP} \ll \mathbb{P}$ and $\mathbb{P}_s^{EP} \perp \mathbb{P}$ on $(\Omega, \mathcal{F}_{\geq 0})$. By assumption, we know that $\alpha > 0$. If $\alpha = 1$, the statement is immediate. Thus, assume $\alpha \in (0, 1)$. In a first step, observe that $(\theta_{y,1} \circ \mathbb{P}^{EP})_c = \theta_{y,1} \circ \mathbb{P}_c^{EP}$ for every $y \in \mathcal{R}$. This follows from translation invariance of $\mathbb{P}$ which implies that taking the continuous part with respect to $\mathbb{P}$ is the same as taking the continuous part with respect to $\theta_{y,1} \circ \mathbb{P}$. The same is true for the singular part $\mathbb{P}_s^{EP}$.

Note that, since E is finite and (X_t) is finite range, we have that ellipticity in fact implies uniform ellipticity. That is, there is an $\epsilon > 0$ such that

$$\inf_{y\in\mathcal{R}} \inf_{\eta\in\Omega} \alpha(\eta, y) \geq \epsilon > 0. \tag{4.6}$$

In particular, by invariance of $\mathbb{P}^{EP}$,

$$\mathbb{P}^{EP} = \sum_{y\in\mathcal{R}} \alpha(\cdot, y)\theta_{y,1} \circ \mathbb{P}^{EP} \geq \epsilon \sum_{y\in\mathcal{R}} \theta_{y,1} \circ \mathbb{P}^{EP} \tag{4.7}$$

and therefore $\theta_{y,1} \circ \mathbb{P}^{EP} \ll \mathbb{P}^{EP}$ for every $y \in \mathcal{R}$. By using first ellipticity and then $(\theta_{y,1} \circ \mathbb{P}^{EP})_c = \theta_{y,1} \circ \mathbb{P}_c^{EP}$ we have

$$\begin{aligned}\mathbb{P}_c^{EP} &= \left(\sum_{y\in\mathcal{R}} \alpha(\cdot, y)\theta_{y,1} \circ \mathbb{P}^{EP}\right)_c = \sum_{y\in\mathcal{R}} \alpha(\cdot, y)\left(\theta_{y,1} \circ \mathbb{P}^{EP}\right)_c \\ &= \sum_{y\in\mathcal{R}} \alpha(\cdot, y)\theta_{y,1} \circ \mathbb{P}_c^{EP},\end{aligned} \tag{4.8}$$

which means that $\mathbb{P}_c^{EP}$ is invariant for (η_t^{EP}).

Let $f = \frac{d\mathbb{P}_c^{EP}}{d\mathbb{P}}$ and define $B = \{\eta \in \Omega \colon f(\eta) > 0\}$. As a consequence of (4.8) we have

$$f(\eta) \geq \epsilon \sum_{y \in \mathcal{R}} f(\theta_{y,1}\eta), \tag{4.9}$$

and, in particular, $\eta \in B$ implies $\theta_{y,1}\eta \in B$, $y \in \mathcal{R}$. In particular, B is invariant under $\theta_{o,1}$, and by ergodicity of $\mathbb{P}$ this is a $0-1$ event. Since by assumption $\alpha > 0$ we have $\mathbb{P}(B) = 1$ and therefore $\mathbb{P} \ll \mathbb{P}_c^{EP}$ on $(\Omega, \mathcal{F}_{\geq 0})$. □

Lemma 4.3. *Let $\Lambda \subset \mathbb{Z}^{d+1}$ finite and fix $\sigma \in E^{\Lambda^c}$. Let $\mathbb{P}(\cdot \mid \sigma)$ and $\mathbb{P}^{-k}(\cdot \mid \sigma)$ be the regular conditional probabilities of $\mathbb{P}$ and $\mathbb{P}^{-k}$ on E^Λ given σ. Then, for $B \in \mathcal{F}_\Lambda$ and $k \geq 1$,*

$$\mathbb{P}^{-k}(B \mid \sigma) = \sum_{\gamma \in \Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1} \mid \sigma\right) P\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right). \tag{4.10}$$

Proof. The proof is mostly as for the unconditional expansion. Additionally we use the following equalities:

$$\begin{aligned}\mathbb{P}^{-k}(B \mid A_{-k}^{-1}, X_{-k,\dots,0} = \gamma, \sigma) &= P_{\mathbb{P}}(\theta_{-\gamma_{-k},k} B \mid \theta_{-\gamma_{-k},k} A_{-k}^{-1}, \theta_{-\gamma_{-k},k}\sigma) \\ &= \mathbb{P}(B \mid A_{-k}^{-1}, \sigma)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^{-k}(X_{-k,\dots,0} = \gamma, A_{-k}^{-1} \mid \sigma) \\ =& P_{\mathbb{P}}(\eta_k^{EP} \in A_{-k}^{-1}, X_{0,\dots,k} = \gamma - \gamma_{-k} \mid \theta_{X_k,k}\sigma) \\ =& P(X_{0,\dots,k} = \gamma - \gamma_{-k} \mid A_{-k}^{-1})\mathbb{P}(\theta_{-\gamma_{-k},k}\eta \in A_{-k}^{-1} \mid \theta_{-\gamma_{-k},k}\sigma) \\ =& P(X_{0,\dots,k} = \gamma - \gamma_{-k} \mid A_{-k}^{-1})\mathbb{P}(\eta \in A_{-k}^{-1} \mid \sigma).\end{aligned}$$

Take also note that summation should only include events A_{-k}^{-1} which have positive probability with respect to the conditional law given σ. □

Proof of Theorem 1.5. By applying the same line of reasoning as in the proof of Theorem 1.2, it easy to see that (1.15) holds as a consequence of (1.14) and Lemma 4.1. In particular, there is a measure $\mathbb{Q}$ invariant under (η_t^{EP}) such that (1.15) holds, and consequently $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_{\geq 0}^{\infty}$. We focus on the proof that $\mathbb{Q}$ and $\mathbb{P}$ are mutually absolutely continuous under the additional assumption that (X_t) is elliptic.

Since (1.15) holds, there is an $l \in \mathbb{N}$ such that $\sup_{B \in \mathcal{F}_{\Lambda(l)}} |\mathbb{Q}(B) - \mathbb{P}(B)| < 1$. In particular, $\mathbb{P}$ and $\mathbb{Q}$ are not singular on $\mathcal{F}_{\Lambda(l)}$. In order to conclude that $\mathbb{Q}$ and $\mathbb{P}$ are not singular on $\mathcal{F}_{\geq 0}$ we make use of Lemma 4.3 and the assumption that $|E| < \infty$. Indeed, for any $\sigma \in E^{\Lambda(l)}$ we have by Lemma 4.3 that $\mathbb{P}^{-k}(\cdot \mid \sigma) \ll \mathbb{P}(\cdot \mid \sigma)$. Further, since E is finite any local function is continuous and hence we also have $\mathbb{Q}(\cdot \mid \sigma) \ll \mathbb{P}(\cdot \mid \sigma)$. And since $\mathbb{Q}$ is non-singular on $\Lambda(l)$ with respect to $\mathbb{P}$ it has non-trivial continuous part and corresponding density on $\Lambda(l)$. Thus, we now also have shown that conditioned on $\Lambda(l)$ the measure $\mathbb{Q}$ has a density inside $\mathbb{H} \setminus \Lambda(l)$. It hence follows that $\mathbb{Q}$ is not singular with respect to $\mathbb{P}$ on $\mathbb{H}$. As a consequence of Lemma 4.2 and that $\mathbb{Q}$ and $\mathbb{P}$ are not singular on $\mathbb{H}$ we conclude that, when (X_t) is elliptic, there is a measure $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$ invariant under (η_t^{EP}) such that $\mathbb{P}^{EP}$ and $\mathbb{P}$ are mutually absolutely continuous on $\mathcal{F}_{\geq 0}$. □

Proof of Corollary 1.6. Corollary 1.6 is an application of Theorem 1 in [15]. In order to fulfil the requirements of their theorem six conditions needs to be satisfied, i.e. (A0)-(A5) therein. Our main contribution is that Condition (A1) is satisfied when (1.16) holds. This follows as a direct consequence of Theorem 1.5 (see also Theorem 2.6). Furthermore, that Conditions (A0) holds under (1.16) follows partly by the mixing assumptions on μ. Moreover, that also μ^{EP} and μ are mixing is a consequence of Theorem 3.4 in [25] which yields exponential rate of convergence for environment process under the assumption that (1.16) holds. By the same reasoning, Theorem 3.4 in [25] also implies that Condition (A2) holds true. Lastly, we note that Conditions (A3) and (A4) are true by assumption and Condition (A5) is satisfied since (X_t) is elliptic. □

4.2 Proof of examples

4.2.1 Proof of Theorem 2.1

Proof. For $k \in \mathbb{N}$, let $\gamma \in \Gamma_k$ and consider $A \in \mathcal{A}_{-k}^{-1}(\gamma)$. Since A consists of a fixed observation of the environment along the path γ we can write $A = \bigcap_{x\in\mathbb{Z}^d} A_x$, where A_x is the observation on the line $\{(x,s)\colon s\in\mathbb{Z}\}$. Without change of notation we also treat A_x as an event on the space $E^{\mathbb{Z}}$. Denote by $\widehat{\mathbb{P}}_x$ the optimal coupling (in the sense of total variation distance) of $\mathbb{P}_o(\cdot \mid A_x)$ and $\mathbb{P}_o(\cdot)$, and by $\widehat{\mathbb{P}} = \bigtimes_{x\in\mathbb{Z}^d} \widehat{\mathbb{P}}_x$. The product structure of $\mathbb{P}$ plus the fact that A is given by the intersection of the events A_x gives us that $\widehat{\mathbb{P}}$ is a coupling of $\mathbb{P}(\cdot \mid A)$ and $\mathbb{P}(\cdot)$.

For $l \in \mathbb{N}$, let $B \in \mathcal{F}_{\Lambda(l)}$. We have that

$$\begin{aligned}
|\mathbb{P}(B \mid A) - \mathbb{P}(B)| &\le \widehat{\mathbb{P}}\left(\exists (x,s)\in\Lambda(l)\colon \eta_s^1(x) \neq \eta_s^2(x)\right)\\
&\le \sum_{x\in\mathbb{Z}^d} \widehat{\mathbb{P}}\left(\exists s\ge 0\colon (x,s)\in\Lambda(l) \text{ and } \eta_s^1(x)\neq\eta_s^2(x)\right)\\
&\le \sum_{t<0}\ \sum_{(x,t)\in\gamma+[-R,R]^d\times\{0\}}\ \sup_{B\in\mathcal{G}_{\ge 0\vee(l-|x|)}} |\mathbb{P}_o(B|A_x) - \mathbb{P}(B)|\,.
\end{aligned}$$

The last line follows from the fact that $\eta^1(x)$ and $\eta^2(x)$ can only differ if the site x is part of the observation A, since otherwise $A_x = E^{\mathbb{Z}}$. Condition (2.1) thus ensures that the sum in the last line is finite. In particular, the sum converges to 0 as $l\to\infty$. This shows that $\mathbb{P}$ satisfies (1.14). □

4.2.2 Proof of Corollary 2.4

Corollary 2.4 follows by a slightly stronger statement, which we state and prove first.

Theorem 4.4. *Assume that $\mathbb{P}\in\mathcal{M}_1(\Omega)$ has finite speed of propagation and that*

$$\sum_{t\ge 1} t^d \sup_{A'\in\mathcal{A}_{-\infty}^{-t}} \widehat{\mathbb{P}}_{\Omega,A'}\left(\eta_0^1(o)\neq\eta_0^2(o)\right) < \infty. \tag{4.11}$$

Then $\mathbb{P}$ satisfies the conditions of Theorem 1.5.

Proof of Theorem 4.4. Let $l\in\mathbb{N}$ and consider $B\in\mathcal{F}_{\Lambda(l)}$. By Theorem 1.5, it is sufficient to obtain uniform estimates of the form $|\mathbb{P}(B\mid A) - \mathbb{P}(B)| \le \phi(l)$, where $A\in\mathcal{A}_{-\infty}$, and where $\phi(l)$ approaches 0 as $l\to\infty$. For this, we first note that

$$\begin{aligned}
|\mathbb{P}\left(B\mid A\right) - \mathbb{P}\left(B\right)| &\le \widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x)\neq\eta_t^2(x) \text{ for some } (x,t)\in\Lambda(l)\right)\\
&\le \sum_{(x,t)\in\Lambda(l)} \widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x)\neq\eta_t^2(x)\right).
\end{aligned}$$

Thus, it suffices to control $\widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x)\neq\eta_t^2(x)\right)$ for each $(x,t)\in\Lambda(l)$. For this, fix $(x,t)\in\Lambda(l)$ such that $\|\,(x,t)\,\|_1 \ge \alpha s$ for some $s\ge 0$. Further, let $A'\in\mathcal{A}_{-\infty}^{-s}$ be such that

$A' \cap A = A$, and denote by $\tilde{\mathbb{P}}_{A,A',\Omega}$ a measure on $\Omega \times \Omega \times \Omega$ such that

$$\tilde{\mathbb{P}}_{A,A',\Omega}(\eta^1 \in \cdot, \eta^2 \in \cdot, \Omega) = \widehat{\mathbb{P}}_{A,A'}(\eta^1 \in \cdot, \eta^2 \in \cdot);$$
$$\tilde{\mathbb{P}}_{A,A',\Omega}(\eta^1 \in \cdot, \Omega, \eta^3 \in \cdot) = \widehat{\mathbb{P}}_{A,\Omega}(\eta^1 \in \cdot, \eta^3 \in \cdot);$$
$$\tilde{\mathbb{P}}_{A,A',\Omega}(\Omega, \eta^2 \in \cdot, \eta^3 \in \cdot) = \widehat{\mathbb{P}}_{A',\Omega}(\eta^2 \in \cdot, \eta^3 \in \cdot).$$

We then have that

$$\begin{aligned}\widehat{\mathbb{P}}_{A,\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) =& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^3(x))\\ \leq& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^2(x) \text{ or } \eta_t^2(x) \neq \eta_t^3(x))\\ \leq& \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^1(x) \neq \eta_t^2(x)) + \tilde{\mathbb{P}}_{A,A',\Omega}(\eta_t^2(x) \neq \eta_t^3(x))\\ =& \widehat{\mathbb{P}}_{A,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) + \widehat{\mathbb{P}}_{A',\Omega}\left(\eta_t^1(x) \neq \eta_t^2(x)\right),\end{aligned}$$

Furthermore, it holds that

$$\begin{aligned}&\widehat{\mathbb{P}}_{A,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right) + \widehat{\mathbb{P}}_{\Omega,A'}\left(\eta_t^1(x) \neq \eta_t^2(x)\right)\\ \leq& \sup_{A \in \mathcal{F}_{<0}, A' \in \mathcal{F}_{\Lambda(\alpha s,s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta_s^1(o) \neq \eta_s^2(o)\right) + \sup_{A'' \in \mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{\Omega,A''}\left(\eta_0^1(o) \neq \eta_0^2(o)\right),\end{aligned}$$

since the finite speed of propagation coupling is invariant with respect to translations of the conditioning and the argument. Thus, by the analysis above, we obtain that

$$\begin{aligned}|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq& \left(\sum_{\substack{(x,t) \in \Lambda(l)\\ \| (x,t) \|_1 = \alpha s}} \sup_{A \in \mathcal{F}_{<0}, A' \in \mathcal{F}_{\Lambda(\alpha s,s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta_s^1(o) \neq \eta_s^2(o)\right)\right)\\ &+ \left(\sum_{\substack{(x,t) \in \Lambda(l)\\ \| (x,t) \|_1 = \alpha s}} \sup_{A'' \in \mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{A'',\Omega}\left(\eta_0^1(x) \neq \eta_0^2(x)\right)\right). \qquad (4.12)\end{aligned}$$

To conclude the proof, we note that the number of site in $\mathbb{H}$ at distance αs from the origin is of order s^d. Thus, due to (2.2) the first sum on the r.h.s. of (4.12) converges towards 0 as l approaches ∞. Similarly, by applying (4.11), also the second sum on the r.h.s. of (4.12) converges towards 0 as l approaches ∞. From this we conclude the proof. □

Proof of Corollary 2.4. The proof of Corollary 2.4 follows along the lines of the proof of Theorem 4.4, by making use of the finite speed of propagation property and (2.3). First note that, for any $B \in \mathcal{F}_{\Lambda(l)}$ and $A \in \mathcal{A}_{-\infty}$,

$$\begin{aligned}|\mathbb{P}(B \mid A) - \mathbb{P}(B)| =& \left|\widetilde{\mathbb{P}}(\Phi(\xi) \in B \mid \xi \in \Phi^{-1}A) - \widetilde{\mathbb{P}}(\Phi(\xi) \in B)\right|\\ \leq& \widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x) \text{ for some } (x,t) \in \Lambda(l)\right)\\ \leq& \sum_{(x,t) \in \Lambda(l)} \widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x)\right).\end{aligned}$$

Thus, it suffices control $\widehat{\mathbb{P}}_{\Omega,\Phi^{-1}A}\left(\eta_t^1(x) \neq \eta_t^2(x)\right)$ for each $(x,t) \in \Lambda(l)$. and to show that the latter term above approaches 0 as $l \to \infty$. For this, since Φ is assumed to be finite range, we note that the finite speed of propagation property (ξ_t) transfers to events of the form $\Phi^{-1}A$. Thus, by considering a coupling $\widetilde{\mathbb{P}}_{\Phi^{-1}A,\Phi^{-1}A',\Omega}$, similar to the coupling in the proof of Theorem 4.4, and where $A' \in \mathcal{A}_{-\infty}^{-s}$ and $A' \cap A = A$. Further, by applying the estimates (2.2) and (2.3), we may proceed by the same line of reasoning as in the proof of Theorem 4.4, from which we conclude the proof. □

4.2.3 Proof of Theorem 2.6

Proof of Theorem 2.6. Denote by μ^{EP} any limiting measure of the Cesaro means $\mu_n^{EP} := \frac{1}{n}\sum_{k=1}^n \mathbb{P}_\mu(\eta_k^{EP} \in \cdot) \in \mathcal{M}_1(\Omega_0)$ (by possibly taking subsequential limits) and note that μ^{EP} is invariant with respect to (η_t^{EP}).

We first show that μ^{EP} agrees with μ on $\mathcal{F}_{=0}^{\infty}$. Let $l \in \mathbb{N}$ and consider any $B \in \mathcal{F}_{\Lambda_0(l)}$ with $\Lambda_0(l) := \{(x,0)\colon \|(x,0)\|_1 \geq l\}$. Similar to the proof of Theorem 4.4, it follows that, for any $A \in \mathcal{A}_{-\infty}$,

$$|\mathbb{P}(B \mid A) - \mathbb{P}(B)| \leq \left(\sum_{x\in\Lambda_0(l)} \sup_{A\in\mathcal{F}_{<0}, A'\in\mathcal{F}_{\Lambda(\alpha s,s)}} \widehat{\mathbb{P}}_{A,A'}\left(\eta^1_{\alpha s}(o) \neq \eta^2_{\alpha s}(o)\right)\right) + \left(\sum_{x\in\Lambda_0(l)} \sup_{A''\in\mathcal{A}_{-\infty}^{-\alpha s}} \widehat{\mathbb{P}}_{A'',\Omega}\left(\eta^1_0(o)\neq\eta^2_0(o)\right)\right).$$

Since $\mathbb{P}$ has finite speed of propagation, the first term converges to 0 as l approaches ∞. For the second term, note that the number of sites in $\mathbb{Z}^d$ at distance t from the origin is of order t^{d-1}. Thus, by (2.4), also the second term converges to 0 as $l\to\infty$. This yields that μ^{EP} agrees with μ on $\mathcal{F}_{=0}^{\infty}$, and that μ^{EP} and μ are non-singular on $(\Omega, \mathcal{F}_{\Lambda_0(l)})$ for all $l\in\mathbb{N}$ sufficiently large.

Next, assume in addition that (X_t) is elliptic. By Lemma 4.3 and an argument as in the proof of Theorem 1.5, we conclude that μ and μ^{EP} are non-singular on $(\Omega, \mathcal{F}_{=0})$. From this, we conclude that there is probability measure $\hat{\mu}^{EP}$, invariant under μ and such that μ and $\hat{\mu}^{EP}$ are mutually absolutely continuous. This follows analogous to the proof of Theorem 1.5 by making use of (a slight adaptation of) Lemma 4.2 and the assumption that (X_t) is elliptic. Consequently, the path measure of (η_t^{EP}) initialised from $\hat{\mu}^{EP}$, denoted by $\mathbb{P}^{EP}\in\mathcal{M}_1(\Omega,\mathcal{F})$, is mutually absolutely continuous to $\mathbb{P}$ on $(\Omega,\mathcal{F}_{\geq 0})$. Thus, since ellipticity implies ellipticity in the time direction, and since μ is ergodic under (η_t) we conclude that $\hat{\mu}^{EP}$ is ergodic under (η_t^{EP}), as follows similar to the proof of ergodicity in Theorem 1.2. □

4.2.4 Proof of Theorem 2.7

We next prove that the environments constructed in Subsection 2.3 have arbitrary slow polynomial mixing.

Proof of Theorem 2.7. First we will show the upper bound, by choosing a particular coupling. The natural coupling of $\mathbb{P}_\xi$ and $\mathbb{P}_\sigma$ is that $\xi_t^1(x,n)$ and $\xi_t^2(x,n)$ share the resampling events of probability b_n, so that after the first resampling, the spins are identical. Note that this coupling can naturally be extended to an arbitrary number of initial configurations. If we denote by ξ_t^σ the configuration at time t when started in σ, we have under this coupling

$$\xi_t^{-\mathbf{1}}(x,n) \leq \xi_t^\sigma(x,n) \leq \xi_t^{+\mathbf{1}}(x,n)$$

and hence $\eta_t^{-\mathbf{1}}(x) \leq \eta_t^\sigma(x) \leq \eta_t^{+\mathbf{1}}(x)$ for all t,x,n,σ. In particular it follows that

$$\widehat{\mathbb{P}}_{\xi,\sigma}(\eta_t^1(0)\neq\eta_t^2(0)) \leq \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}(\eta_t^1(0)=1) - \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}(\eta_t^2(0)=1).$$

Let $R_t := \{n\in\mathbb{N} : \xi_t^1(0,n) = \xi_t^2(0,n)\}$. We have

$$\mathbb{P}_{\pm\mathbf{1}}(\eta_t(0)=1) = \widehat{\mathbb{P}}_{\mathbf{1},-\mathbf{1}}\left(\sum_{n\in R_t} a_n\xi_t^1(0,n) > \mp\sum_{n\in R_t^c} a_n\right), \tag{4.13}$$

and hence,

$$\widehat{\mathbb{P}}_{\xi,\sigma}(\eta_t^1(0) \neq \eta_t^2(0)) \leq \widehat{\mathbb{E}} \int_{-\sum_{n\in R_t^c} a_n}^{\sum_{n\in R_t^c} a_n} f_{R_t}(x)dx \tag{4.14}$$
$$= \widehat{\mathbb{E}} \sum_{n\in R_t^c} a_n \int_{-1}^{1} f_{R_t}(\sum_{n\in R_t^c} a_n x)dx,$$

where f_R is the density of $\sum_{n\in R} a_n Y_n$ and $(Y_n)_n$ are i.i.d. uniform $[-1,1]$ distributed. A simple convolution of the individual densities shows that $f_R \leq \min_{n\in R}(2a_n)^{-1}$, hence the above is bounded by

$$\widehat{\mathbb{E}}\left((\min_{n\in R_t} a_n)^{-1} \sum_{n\in R_t^c} a_n\right) = \sum_{k\in\mathbb{N}} \widehat{\mathbb{P}}(\min R_t = k) a_k^{-1} \left(\sum_{n<k} a_n\right) \widehat{\mathbb{E}}\left(\sum_{n>k} \mathbb{1}_{n\in R_t^c} a_n\right)$$
$$= \sum_{k\in\mathbb{N}} \left(\prod_{n<k}(1-b_n)^t\right)\left(1-(1-b_k)^t\right) a_k^{-1}\left(\sum_{n<k} a_n\right)\left(\sum_{n>k} a_n(1-b_n)^t\right). \tag{4.15}$$

To obtain polynomial decay, we choose $a_n = \frac{1}{2}n^{-\alpha}$ and $b_n = \frac{1}{2}n^{-\beta}$. Then we can find some constant $C > 0$ so that

$$\sum_{k\in\mathbb{N}} \left(\prod_{n<k}(1-b_n)^t\right)\left(1-(1-b_k)^t\right) a_k^{-1}\left(\sum_{n<k} a_n\right) \leq C.$$

With this and $1 - b_n \leq e^{-b_n}$,

$$(4.15) \leq C\sum_{n\in\mathbb{N}} a_n(1-b_n)^t \leq \sum_{n\leq (t/\log t^2)^{\frac{1}{\beta}}} a_n e^{-b_n t} + \sum_{n>(t/\log t^2)^{\frac{1}{\beta}}} a_n e^{-b_n t}$$
$$\leq \sum_{n\leq (t/\log t^2)^{\frac{1}{\beta}}} a_n t^{-\log t} + \sum_{n>(t/\log t^2)^{\frac{1}{\beta}}} a_n$$
$$\leq c_2 t^{\frac{-\alpha+1}{\beta}} (\log t)^{\frac{\alpha-1}{\beta}}. \tag{4.16}$$

For a lower bound, we use (4.13) plus the fact that (4.14) is an equality for $\xi = +\mathbf{1}$ and $\sigma = -\mathbf{1}$, so that we have

$$\sup_{\xi,\sigma} \| \mathbb{P}_\xi(\eta_t(0) \in \cdot) - \mathbb{P}_\sigma(\eta_t(0)\in\cdot) \|_{TV} = \widehat{\mathbb{E}} \sum_{n\in R_t^c} a_n \int_{-1}^{1} f_{R_t}(\sum_{n\in R_t^c} a_n x)dx.$$

The density f_R has a unique local maximum at 0 and its support is $[-\sum_{n\in R} a_n, \sum_{n\in R} a_n]$, so that we can lower bound the integral by replacing f_{R_t} with $(2\sum_{n\in R_t} a_n)^{-1}$:

$$(4.16) \geq \widehat{\mathbb{E}}_{\mathbf{1},-\mathbf{1}}(\sum_{n\in R_t} a_n)^{-1} \sum_{n\in R_t^c} a_n \geq (\sum_{n\in\mathbb{N}} a_n)^{-1} \sum_{n\in\mathbb{N}} a_n(1-b_n)^t$$
$$\geq (\sum_{n\in\mathbb{N}} a_n)^{-1} \sum_{n\geq t^{-\beta}} a_n(1-\frac{1}{2}t^{-1})^t \geq c_1 t^{\frac{-\alpha+1}{\beta}}. \qquad \square$$

4.2.5 Proof of Theorem 2.8

We continue with the proof of Theorem 2.8 and study random walks on an Ornstein-Uhlenbeck process.

Proof of Theorem 2.8. Fix n, a sequence t_k and $a \in \{-1,1\}^n$. Define the additional events $A_1 = \{\operatorname{sign}(\xi_{-t_k}) = 1, 1 \le k \le n\}$ and $\overline{A} = \{\operatorname{sign}(\xi_t) = 1, t \le 0\}$.

We will use the following sequence of stochastic domination:

$$\mathbb{P}(\xi_0 \in \cdot \mid A) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid A_1) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid \overline{A}). \tag{4.17}$$

Here $\mathbb{P}(\xi_0 \in \cdot \mid \overline{A})$ is the limit of $\mathbb{P}(\xi_0 \in \cdot \mid \operatorname{sign}(\xi_s) = 1, -T \le s \le 0)$ as $T \to \infty$, which exists and has Lebesgue-density $x \exp(-\frac{1}{2}x^2)$ on $[0,\infty)$ (see [22]). The argument for the stochastic domination in (4.17) is based on the following fact: Let Y^1 and Y^2 be two diffusions given by $dY_t^i = b_t^i(Y_t^i)dt + \sigma dW_t$. If $b_t^1 \le b_t^2$ and $\mathcal{L}(Y_0^1) \preccurlyeq \mathcal{L}(Y_0^2)$, then

$$\mathcal{L}(Y_t^1) \preccurlyeq \mathcal{L}(Y_t^2) \qquad \forall\, t \ge 0. \tag{4.18}$$

To apply this to the first stochastic domination in (4.17) holds, let $-t_l$ is the biggest time point with $a_l = -1$. Clearly $\mathbb{P}(\xi_{-t_l} \in \cdot|A) \preccurlyeq \mathbb{P}(\xi_{-t_l} \in \cdot|A_1)$. Furthermore, after $-t_l$ the events A and A_1 agree past $-t_l$, that means that after t_l we condition on the same event. This conditioning changes the drift to some new and time-inhomogeneous drift, for which only the initial law varies, and by (4.18) we obtain the stochastic domination.

For the second stochastic domination, we use (4.18) and the fact that conditioning the Ornstein-Uhlenbeck process on $\overline{A}$ further increases the drift compared to condition on A_1 (with the convention that the drift is $+\infty$ for $x \le 0$ when conditioning on $\overline{A}$).

An analogous bound to (4.17) holds in the other direction when we condition the process to be negative, and $\mathbb{P}(\xi_0 \in \cdot|\underline{A}) = \mathbb{P}(-\xi_0 \in \cdot|\overline{A})$. Together this implies

$$\mathbb{P}(|\,\xi_0\,| \in \cdot \mid A) \preccurlyeq \mathbb{P}(\xi_0 \in \cdot \mid \overline{A}). \tag{4.19}$$

A bound on the total variation is then given by a coupling:

$$\|\,\mathbb{P}(\xi_t \in \cdot \mid A) - \mathbb{P}(\xi_t \in \cdot)\,\|_{TV} \le \int \widehat{\mathbb{P}}_{x,y}(\tau > t)\pi_A(dx,dy),$$

where $\widehat{\mathbb{P}}_{x,y}$ is a coupling of two OU-processes ξ_t^1 and ξ_t^2 starting in x and y and π_A is any coupling of $\mathbb{P}(\xi_0 \in \cdot \mid A)$ with a normal distribution, and τ is the coupling time.

We take $\widehat{\mathbb{P}}_{x,y}$ to be the coupling where the driving Brownian motions are perfectly negatively correlated until the processes are coupled. Then the difference D_t is an OU-process satisfying

$$dD_t = -D_t dt + 2dW_t \quad \text{and} \quad D_0 = x - y.$$

The coupling time τ is τ_0, the first hitting time of 0 of D_t. Note that the coupling time increases if $|\,x-y\,|$ increases, in particular when we replace $|\,x-y\,|$ by $|\,x\,|+|\,y\,|$. With this fact, choosing π_A to be the independent coupling, and (4.19) we get

$$\int \widehat{\mathbb{P}}_{x,y}(\tau > t)\pi_A(dx,dy) \le \int_0^\infty \int_0^\infty \mathbb{P}_{x+y}(\tau_0 > t) x e^{-\frac{x^2}{2}} \frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dxdy.$$

To conclude the proof we use the fact that that $\mathbb{P}_{x+y}(\tau_0 > r + \log(x+y))$ is exponentially small in r.

The claim that this example satisfies the conditions of Theorem 1.5 is now a simple computation by telescoping over all sites in B and using the fact that the last time a site $x \in \mathbb{Z}^d$ could be observed is $-|x|/R$, where R is the interaction range of the jump kernel α. ☐

4.2.6 Proof of Theorem 2.9

In this subsection we present the proof of Theorem 2.9. Before doing so, we first introduce some definitions and prove a general theorem, Theorem 4.5, from which Theorem 2.9 follows.

Let $E = \{0,1\}$ and associate to the space Ω the partial ordering such that $\xi \le \eta$ if and only if $\xi(x) \le \eta(x)$ for all $x \in \mathbb{Z}^{d+1}$. An event $B \in \mathcal{F}$ is said to be *increasing* if $\xi \le \eta$ implies $1_B(\xi) \le 1_B(\eta)$. If $\xi \le \eta$ implies $1_B(\xi) \ge 1_B(\eta)$ then B is called *decreasing*. For $\mathbb{P}, \mathbb{Q} \in \mathcal{M}_1(\Omega)$, we say that $\mathbb{P}$ *stochastically dominates* $\mathbb{Q}$ if $\mathbb{Q}(B) \le \mathbb{P}(B)$ for all $B \in \mathcal{F}$ increasing. Furthermore, a measure $\mathbb{P} \in \mathcal{M}_1(\Omega)$ is *positively associated* if it satisfies $\mathbb{P}(B_1 \cap B_2) \ge \mathbb{P}(B_1)\mathbb{P}(B_2)$ for any two increasing events $B_1, B_2 \in \mathcal{F}$. Following [21], we say that $\mathbb{P}$ is *downward FKG* if, for every finite $\Lambda \subset \mathbb{Z}^{d+1}$, the measure $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ on } \Lambda)$ is positively associated.

Theorem 4.5. *Let $\mathbb{P} \in \mathcal{M}_1(\Omega)$ be downward FKG and assume that there exists $\phi\colon \mathbb{N} \to [0,1]$ such that for all $(x,s) \in \Lambda(l)$ and all $\gamma \in \bigcup_{k\ge1} \Gamma_k$,*

$$\mathbb{P}\left(\eta_s(x) = 1 \mid \eta \equiv 0 \textit{ along } \gamma\right) \ge \mathbb{P}\left(\eta_0(o) = 1\right) - \phi(l), \tag{4.20}$$

$$\mathbb{P}(\eta_s(x) = 1 \mid \eta \equiv 1 \textit{ along } \gamma) \le \mathbb{P}(\eta_0(o) = 1) + \phi(l). \tag{4.21}$$

If $\sum_{l\ge1} l^d \phi(l) < \infty$, then the conditions of Theorem 1.5 are satisfied.

Remark 4.6. In the above theorem, and throughout this section, we write "$\eta \equiv i$ along γ", where $i \in \{0,1\}$ and $\gamma \in \Gamma := \bigcup_{k\ge1} \Gamma_k$, for the event that $\{\eta_s(x) = i \;\forall\, (x,s) \in \gamma + [-R,R]^d \times \{0\}\}$.

Proof of Theorem 4.5. Let $B \in \mathcal{F}$. For any $k \in \mathbb{N}$, we have similar to the proof of Lemma 4.1 that

$$\begin{aligned}
&\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \\
\le& \sum_{\gamma\in\Gamma_k} \sum_{A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)} \Bigg[\left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \\
&+ \left|\mathbb{P}\left(B\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \Bigg] \mathbb{P}^{-k}\left(X_{-k,\dots,0} = \gamma, A_{-k}^{-1}\right).
\end{aligned}$$

We next show that, under (4.20) and (4.21),

$$\sup_{B\in\Lambda(l)} \left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B \mid \eta \equiv 0 \text{ along } \gamma\right)\right| \to 0, \quad \text{as } l \to \infty.$$

Fix $\gamma \in \Gamma_k$ and $A_{-k}^{-1} \in \mathcal{A}_{-k}^{-1}(\gamma)$. Since $\mathbb{P}$ is downward FKG, it is the case that $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ along } \gamma)$ is stochastically dominated by $\mathbb{P}(\cdot \mid A_{-k}^{-1})$. Hence, by Strassens Theorem, there exists a coupling $\widehat{\mathbb{P}}_{0,1}$ of $\mathbb{P}(\cdot \mid \eta \equiv 0 \text{ along } \gamma)$ and $\mathbb{P}(\cdot \mid A_{-k}^{-1})$ such that $\widehat{\mathbb{P}}_{0,1}\left(\eta^1 \le \eta^2\right) = 1$. We moreover have that, for all $B \in \mathcal{F}_{\Lambda(l)}$, $l \in \mathbb{N}$,

$$\begin{aligned}
&|\mathbb{P}(B \mid \eta \equiv 0 \text{ along } \gamma) - \mathbb{P}(B \mid A_{-k}^{-1})| \\
\le& \widehat{\mathbb{P}}_{0,1}\left(\eta^1 \ne \eta^2 \text{ on } \Lambda(l)\right) \\
\le& \sum_{(x,s)\in\Lambda(l)} \widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x) \ne \eta_s^2(x)\right) \\
=& \sum_{(x,s)\in\Lambda(l)} \widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x) = 0, \eta_s^2(x) = 1\right) \\
=& \sum_{(x,s)\in\Lambda(l)} \left(\widehat{\mathbb{P}}_{0,1}(\eta_s^1(x) = 0) - \widehat{\mathbb{P}}_{0,1}(\eta_s^2(x) = 0)\right).
\end{aligned}$$

Furthermore, since $\mathbb{P}$ is downward FKG, we know that

$$\begin{aligned}
&\widehat{\mathbb{P}}_{0,1}\left(\eta_s^1(x)=0\right)-\widehat{\mathbb{P}}_{0,1}\left(\eta_s^2(x)=0\right)\\
=&\mathbb{P}\left(\eta_s(x)=0\mid \eta\equiv 0 \text{ along } \gamma\right)-\mathbb{P}\left(\eta_s(x)=0\mid A_{-k}^{-1}\right)\\
\le&\mathbb{P}\left(\eta_s(x)=0\mid \eta\equiv 0 \text{ along } \gamma\right)-\mathbb{P}\left(\eta_s(x)=0\mid \eta\equiv 1 \text{ along } \gamma\right)
\end{aligned}$$

As a consequence, by using (4.20) and (4.21), we obtain by the derivations above that

$$\sup_{B\in\mathcal{F}_{\Lambda(l)}}|\mathbb{P}(B\mid \eta\equiv 0 \text{ along } \gamma)-\mathbb{P}(B\mid A_{-k}^{-1})|\le C\sum_{t\ge l}t^d\phi(t), \tag{4.22}$$

for some constant $C\in(0,\infty)$. By a word by word adaptation of this argument, replacing $\mathbb{P}\left(B\mid A_{-k}^{-1}\right)$ by $\mathbb{P}\left(B\right)$, it can similarly be shown that

$$\sup_{B\in\Lambda(l)}|\mathbb{P}\left(B\right)-\mathbb{P}\left(B\mid \eta\equiv 0 \text{ along } \gamma\right)|\le C\sum_{t\ge l}t^d\phi(t). \tag{4.23}$$

Substituting the estimates from (4.22) and (4.23) into the first inequality of this proof, and using that $\lim_{l\to\infty}\sum_{t\ge l}t^d\phi(t)=0$, we obtain that the conditions of Theorem 1.5 are satisfied. □

We continue with the proof of Theorem 2.9.

Proof of Theorem 2.9. Let (ξ_t) be the contact process on $\mathbb{Z}^{\tilde d}$ with $\tilde d\ge 1$ and $\lambda>\lambda_c(\tilde d)$. This process is known to satisfy the downward FKG property, as shown by [5], Theorem 3.3 (see also Lemma 2.1 in [4]). Thus, for the proof of Theorem 2.9, it is sufficient to show that (4.20) and (4.21) holds. In fact, it is sufficient to show that the estimates of Theorem 4.5 hold for sites (o,s) with $s\in\mathbb{Z}_{\ge 0}$. To see this, recall the graphical representation of the contact process (see p. 32-34 in [20]). Since the spread of information is bounded by a Poisson process with rate $2d\lambda$, it is evident that the finite speed of propagation property holds, and thus that Corollary 2.4 applies.

That (4.21) holds for the contact process is now a simple application of the graphical representation and the fact that the contact process started from all sites equal to 1 converges exponentially fast towards the upper invariant measure. See [20], Theorem 1.2.30, and the remark directly after for estimates of the latter. In particular, (4.21) holds with $\phi(l)$ exponentially decaying in l. Note that, this estimate holds for (ξ_t), that is, without applying the projection map.

In order to conclude a similar estimate for (4.20), on the other hand, we restrict to the projection of (ξ_t) onto the one dimensional lattice. In this case, (4.20), again with $\phi(l)$ exponentially decaying in l, is a direct application of [4], Theorem 1.7. Thus, by Theorem 4.5, we conclude that the conditions of Theorem 1.5 are satisfied. □

Remark 4.7. The statement of Theorem 2.9 can be extended to projection maps from $\mathbb{Z}^{\tilde d}$ to $\mathbb{Z}^{\tilde d}_{\tilde d-1}:=\mathbb{Z}^{\tilde d-1}\times\{0\}$ for any $\tilde d\ge 2$ and $\lambda>\lambda_c(\tilde d)$. Indeed, Theorem 1.7 in [4] still holds in this generality.

4.3 Proof of Theorem 3.2 and Theorem 3.3

Proof of Theorem 3.2. We first show continuity with respect to $\left(\mathbb{P}^{EP(n)}\right)$. Let $\epsilon>0$, and let $m\le n$ with $n,m\in\mathbb{N}$. For $\Lambda\subset\mathbb{Z}^{d+1}$ finite and $B\in\mathcal{F}_\Lambda$ we have that, for every $t\in\mathbb{N}$,

$$\begin{aligned}
|\mathbb{P}^{EP(m)}(B)-\mathbb{P}^{EP(n)}(B)|\le&|\mathbb{P}^{EP(m)}(B)-P_{\mathbb{P}_m}(\eta_t^{EP(m)}\in B)|\\
&+|\mathbb{P}^{EP(n)}(B)-P_{\mathbb{P}_n}(\eta_t^{EP(n)}\in B)|\\
&+|P_{\mathbb{P}_n}(\eta_t^{EP(n)}\in B)-P_{\mathbb{P}_m}(\eta_t^{EP(m)}\in B)|.
\end{aligned}$$

By Assumption c) we can fix t such that the sum of the first two terms is less than $\epsilon/2$. By the uniformity assumption, this bound holds irrespectively of m and n. It thus remains to show that also the third term can be made smaller than $\epsilon/2$ by possibly taking m large. To this end, we use the expansion in (3.2), and note that

$$\begin{aligned}
&P_{\mathbb{P}_m}(\eta_t^{EP(m)} \in B) \\
&= \sum_{\substack{\gamma\in\Gamma_t \\ A_{-t}^{-1}\in\mathcal{A}_{-t}^{-1}(\gamma)}} \mathbb{P}_m\left(B, A_{-t}^{-1}\right) P_m\left(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}\right) \\
&= \sum_{\substack{\gamma\in\Gamma_t \\ A_{-t}^{-1}\in\mathcal{A}_{-t}^{-1}(\gamma)}} \left(\mathbb{P}_n(B, A_{-t}^{-1}) \pm \delta_{1,m}(t)\right)\left(P_n(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}) \pm \delta_{2,m}(t)\right),
\end{aligned}$$

where, due to a) and b), the error terms $\delta_{1,m}(t)$ and $\delta_{2,m}(t)$ approaches 0 as $m \to \infty$. In particular, again since

$$P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) = \sum_{\substack{\gamma\in\Gamma_t \\ A_{-t}^{-1}\in\mathcal{A}_{-t}^{-1}(\gamma)}} \mathbb{P}_n\left(B, A_{-t}^{-1}\right) P_n\left(X_{-t,\dots,0} = \gamma \mid A_{-t}^{-1}\right),$$

by taking m large enough we can guarantee that

$$|P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) - P_{\mathbb{P}_m}(\eta_t^{EP(n)} \in B)| < \epsilon/2.$$

Since this bound holds for all $n \geq m$ it follows that $(\mathbb{P}^{EP(m)}(B))$ is a Cauchy sequence and hence converges to a limit. Moreover, since B and Λ were arbitrary, this is true for any local local event $B \in \mathcal{F}$. This implies that $\mathbb{P}^{EP(m)}$ converges weakly to $\mathbb{P}^{EP}$ for some $\mathbb{P}^{EP} \in \mathcal{M}_1(\Omega)$.

We next proceed with the proof of $P_{\mathbb{P}}(\eta_t^{EP}\cdot) \implies \mathbb{P}^{EP}$, where $\mathbb{P}^{EP}$ is the limiting measure above. Let $\epsilon > 0$ and $B \in \mathcal{F}$ local. For any $n \in \mathbb{N}$, we have that

$$\begin{aligned}
|\mathbb{P}^{EP}(B) - P_{\mathbb{P}}(\eta_t^{EP} \in B)| \leq\ & |\mathbb{P}^{EP}(B) - \mathbb{P}^{EP(n)}(B)| \\
& + |\mathbb{P}^{EP(n)}(B) - P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B)| \\
& + |P_{\mathbb{P}_n}(\eta_t^{EP(n)} \in B) - P_{\mathbb{P}}(\eta_t^{EP} \in B)|.
\end{aligned}$$

Fix t such that the second term is smaller than $\epsilon/3$. This we can do by applying Assumption c). Next, by taking n large the first term can be made smaller then $\epsilon/3$ as well since $\mathbb{P}^{EP(n)} \implies \mathbb{P}^{EP}$, as we have shown above. For the third term we can proceed as in for the proof of $\mathbb{P}^{EP(n)} \implies \mathbb{P}^{EP}$ above. Indeed, since t is fixed, we can use that $\mathbb{P}_n \implies \mathbb{P}$ and that $\epsilon(n) \downarrow 0$ together with the finite range assumption of the random walk. Hence we may take n so large that also the third term is less that $\epsilon/3$. Since $\epsilon > 0$ was taken arbitrary, this shows that $P_{\mathbb{P}}(\eta_t^{EP} \in B) \to \mathbb{P}^{EP}(B)$ as $t \to \infty$. Since $B \in \mathcal{F}$ was an arbitrary local event, we conclude that $P_{\mathbb{P}}(\eta_t^{EP} \in \cdot)$ converges weakly towards $\mathbb{P}^{EP}(\cdot)$. As a necessary consequence, it also follows that by standard arguments that $\mathbb{P}^{EP}$ is invariant with respect to (η_t^{EP}). ☐

Proof of Theorem 3.3. Let $\mathbb{P}_\sigma$ be the path measure of (η_t) when started from $\sigma \in \Omega_0$ and assume that (η_t^{EP}) is uniquely ergodic with invariant measure $\mu^{EP} \in \mathcal{M}_1(\Omega)$. We have that, for any $B \in \mathcal{F}_\Lambda$, $\Lambda \subset \mathbb{Z}^d \times \{0\}$ finite, and any $t \in \mathbb{N}$,

$$\begin{aligned} \left|\mu^{EP}(B) - \mu^{EP(n)}(B)\right| &= \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right| \\ &\leq \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right| \\ &+ \left|t^{-1}\sum_{k=1}^{t}\left[P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right]\right|. \end{aligned} \tag{4.24}$$

Since (η_t^{EP}) is uniquely ergodic, it follows by classical arguments that

$$\sup_{\sigma,\omega\in\Omega_0}\left|t^{-1}\sum_{k=1}^{t}\left[P_\sigma(\eta_k^{EP} \in B) - P_\omega(\eta_k^{EP} \in B)\right]\right| \to 0$$

as t approaches ∞ (see e.g. Theorem 4.10 in [16]). Hence, by taking t large we can assure that the first term of the r.h.s. of (4.24) is less than $\epsilon/2$. Next, for the second term, we have that, for any fixed $t > 0$,

$$t^{-1}\sum_{k=1}^{t}\left|P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) - P_{\mu^{EP(n)}}(\eta_k^{EP(n)} \in B)\right| \to 0 \text{ as } n \to \infty.$$

This follows similarly as in the proof of Theorem 3.2. Indeed, for each $k \in \{1,\dots,t\}$, we have that

$$\begin{aligned} &P_{\mu^{EP(n)}}(\eta_k^{EP} \in B) \\ =& \sum_{\substack{\gamma\in\Gamma_k \\ A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)}} \mathbb{P}_{\mu^{EP(n)}}\left(B, A_{-k}^{-1}\right) P_n\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right) \\ =& \sum_{\substack{\gamma\in\Gamma_k \\ A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)}} \left(\mathbb{P}_{\mu^{EP}}(B, A_{-t}^{-1}) \pm \delta_{1,n}(t)\right) \\ &\qquad\qquad \cdot\left(P(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}) \pm \delta_{2,n}(t)\right), \end{aligned}$$

where both the error terms $\delta_{1,n}(t)$ and $\delta_{2,n}(t)$ approaches 0 as $n \to \infty$. Thus, by taking n sufficiently large we can assure that the second term on the r.h.s. of (4.24) is less than $\epsilon/2$. From this we conclude that $\left|\mu^{EP}(B) - \mu^{EP(n)}(B)\right| < \epsilon$ for all n large. Since B and Λ were arbitrary chosen, we hence conclude the proof. □

4.4 Proof of Theorem 3.5

Proof of Theorem 3.5. The main part of the proof goes along the same lines as the proof of Theorem 1.2. The main difference is an estimate which is similar to Lemma 4.1 and which we present next. Let $B \in \mathcal{F}_{\geq 0}$. We have that, for any $k \in \mathbb{N}$,

$$\begin{aligned} &\left|P_{\mathbb{P}}\left(\eta_k^{EP} \in B\right) - \mathbb{P}\left(B\right)\right| \\ =&\left|\sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \mathbb{P}\left(B, A_{-k}^{-1}\right) P\left(X_{-k,\dots,0} = \gamma \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\right| \\ \leq& \sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)} \left|\mathbb{P}\left(B \mid A_{-k}^{-1}\right) - \mathbb{P}\left(B\right)\right| \mathbb{P}^{-k}\left(X_{-k,\dots,0} = \gamma, A_{-k}^{-1}\right), \end{aligned}$$

where in the last equality we used the fact that,

$$\sum_{\gamma\in\Gamma_k}\sum_{A_{-k}^{-1}\in\mathcal{A}_{-k}^{-1}(\gamma)}\mathbb{P}^{-k}\left(X_{-k,\dots,0}=\gamma, A_{-k}^{-1}(\gamma,\sigma)\right)=1.$$

Consequently, by the bound in (3.4), we conclude that, for any $k\in\mathbb{N}$,

$$\left|P_{\mathbb{P}}\left(\eta_k^{EP}\in B\right)-\mathbb{P}\left(B\right)\right|\le M_1\mathbb{P}(B),\quad \forall\, B\in\mathcal{F}_{\ge 0}. \tag{4.25}$$

In particular, $P_{\mathbb{P}}\left(\eta_k^{EP}\in\cdot\right)\ll\mathbb{P}$ on $\mathcal{F}_{\ge0}$ and $\frac{dP_{\mathbb{P}}\left(\eta_k^{EP}\in\cdot\right)|_{\mathcal{F}_{\ge0}}}{d\mathbb{P}(\cdot)|_{\mathcal{F}_{\ge0}}}\le M_1$.

Let $\mathbb{Q}\in\mathcal{M}_1(\Omega)$ be a limiting measure of the sequence $\left(t^{-1}\sum_{k=1}^t P_{\mathbb{P}}(\eta_k^{EP}\in\cdot)\right)_{t>0}$, by possibly taking sub-sequential limits. Then, by means of weak convergence, since the space of M_1-bounded functions on a compact space form a compact space, and the limit of bounded measurable functions is measurable, (4.25) immediately transfers to $\mathbb{Q}$. Consequently, we have $\mathbb{Q}\ll\mathbb{P}$ on $\mathcal{F}_{\ge0}$ and $\frac{d\mathbb{Q}|_{\mathcal{F}_{\ge0}}}{d\mathbb{P}|_{\mathcal{F}_{\ge0}}}\le M_1$. This concludes the first part.

Next, assume that (3.5) holds from which it follows that, for every $B\in\mathcal{F}_{\ge0}$,

$$|\mathbb{P}(B)-\mathbb{P}(B\mid A_k)|\le M_2\mathbb{P}(B\mid A_k),\quad\forall\, A_k\in\mathcal{A}_{-kl}^{-1},\ k\in\mathbb{N},$$

Similarly to how we obtained (4.25), we hence conclude that, for any $k\in\mathbb{N}$,

$$\left|P_{\mathbb{P}}\left(\eta_k^{EP}\in B\right)-\mathbb{P}\left(B\right)\right|\le M_2P_{\mathbb{P}}\left(\eta_k^{EP}\in B\right),\quad\forall\, B\in\mathcal{F}_{\ge0}.$$

From this estimate, and using the same argument as for the proof of the first part, we hence conclude that $\mathbb{P}\ll\mathbb{Q}$ and that $\frac{d\mathbb{P}|_{\mathcal{F}_{\ge0}}}{d\mathbb{Q}|_{\mathcal{F}_{\ge0}}}\le M_2$. □

4.5 Strong disagreement percolation

4.5.1 Basic disagreement percolation

For simplicity we assume that $E=\{0,1\}$ and that the environment (η_t) is a translation invariant nearest neighbour probabilistic cellular automaton (PCA). Further, let $c_i(\eta):=\mathbb{P}_\eta(\eta_1(o)=i), i=0,1$. By the nearest neighbour property, $c_i(\eta)=c_i(\xi)$ if $\eta(x)=\xi(x)$ for all $|\,x\,|\le 1$.

The evolution of the PCA can be constructed by a sequence $(U_t(x))_{x\in\mathbb{Z}^d,t\ge1}$ of i.i.d. $[0,1]$-uniform variables in an iterative way: given η_t, $\eta_{t+1}(x):=\mathbb{1}_{U_{t+1}(x)\le c_1(\theta_x\eta_t)}$, $x\in\mathbb{Z}^d$. Here θ_x is the shift on $\mathbb{Z}^d$, that is, for $\eta\in\Omega_0$ we have $(\theta_x\eta)(y)=\eta(y+x)$, $y\in\mathbb{Z}^d$

This construction allows for coupling of $\mathbb{P}_{\eta^1}$ and $\mathbb{P}_{\eta^2}$, the graphical construction coupling, by using the same set of $[0,1]$-uniform i.i.d. variables $(U_t(x))$. The starting point of disagreement percolation is the observation that the value of $\eta_{t+1}(x)$ is sometimes independent of η_t, namely if either $U_{t+1}(x)<c_-:=\inf_{\eta\in\Omega}c_1(\eta)$ or $U_{t+1}(x)>c_+:=\sup_{\eta\in\Omega}c_1(\eta)$. This allows the environment to forget information, which can be encoded in the coupling. The disagreement percolation is then the triple $(\eta_t^1,\eta_t^2,\xi_t)_{t\ge0}$, where η_t^i is constructed from the initial configuration η^i and the $(U_t(x))$, and ξ_t is given by $\xi_0(x)=\mathbb{1}_{\eta^1(x)\neq\eta^2(x)}$ and for $t>0$;

$$\xi_t(x)=\begin{cases}1, & U_t(x)\in[c_-,c_+]\text{ and }\exists\, y,|\,y-x\,|\le1:\xi_{t-1}(y)=1;\\ 0, & \textit{otherwise.}\end{cases} \tag{4.26}$$

The name disagreement percolation comes from the fact that $\xi_t(x)=0$ implies $\eta_t^1(x)=\eta_t^2(x)$ and (ξ_t) is a directed site percolation process with percolation parameter $p=c_+-c_-$. We denote the law of this so constructed triple $(\eta_t^1,\eta_t^2,\xi_t)_{t\ge0}$ by $\widehat{\mathbb{P}}_{\eta^1,\eta^2}$.

Definition 4.8. *If $p = c_+ - c_- < p_c$, where p_c is the critical value of directed site percolation in $\mathbb{Z}^d$, then we say that the disagreement percolation $\widehat{\mathbb{P}}$ is subcritical.*

Remark 4.9. This coupling can be improved by looking at more information. For example the site percolation model does not use the total number of neighbours which satisfy $\xi_{t-1}(y) = 1$, only that the indicator that this number is positive. By taking this information into account when deciding based on whether $\xi_t(x)$ should be 1 or 0 the range of PCA where the disagreement percolation is subcritical can be extended.

Remark 4.10. If the disagreement percolation coupling is subcritical, then necessarily there is a uniquely ergodic measure for the process (η_t), as follows by standard coupling arguments and comparison with subcritical directed site percolation.

4.5.2 Disagreement percolation and backward cones

The disagreement percolation we introduced in the previous subsection is a way to control the influence of the initial configuration on the future, by giving an upper bound on the space-time points which depend on differences in the initial configurations. In the context of this article we want something slightly different, namely to control the influence of a backwards cone. With this in mind we construct a different version of the disagreement percolation coupling.

Denote by $\mathbb{P}_\mu^{-\infty}$ the law of $(\eta_t)_{t\in\mathbb{Z}}$ under the stationary law μ and by $(U_t(x))_{t\in\mathbb{Z},x\in\mathbb{Z}^d}$ the i.i.d. uniform $[0,1]$ variables of the corresponding graphical construction. Denote by $C_b := \{(x,t) \in \mathbb{Z}^d \times \{..., -1, 0\} : |\,x\,| \le |\,t\,|\}$ the infinite backward cone with tip at $(0,0)$ and by $\mathcal{C}_b := \sigma(\eta_t(x) : (x,t) \in C_b) = \sigma(U_t(x) : (x,t) \in C_b)$ the σ-algebra generated by the sites which lie in the cone C_b.

Let $A, B \in \mathcal{C}_b$. We now construct the disagreement percolation process $(\eta_t^1, \eta_t^2, \xi_t)_{t\in\mathbb{Z}}$ with law $\widehat{\mathbb{P}}_{A,B}$, where η^1 has law $\mathbb{P}_\mu^{-\infty}(\cdot|A)$ and η^2 has law $\mathbb{P}_\mu^{-\infty}(\cdot|B)$. The idea is almost the same as in Subsection 4.5.1, the only difference is on the cone C_b. On C_b, we draw $(\eta_t^1(x))_{(x,t)\in C_b}$ from $\mathbb{P}_\mu^{-\infty}(\cdot|A)$, independently $(\eta_t^2(x))_{(x,t)\in C_b}$ from $\mathbb{P}_\mu^{-\infty}(\cdot|B)$, and set $\xi_t(x) = 1$ for $(x,t) \in C_b$. Outside C_b, (η_t^1, η^2, ξ) evolves like the basic disagreement percolation coupling by using the same $(U_t(x))_{(x,t)\in C_b^c}$. As the evolution outside C_b is the same as the basic disagreement percolation, the definition of subcriticality remains the same.

Lemma 4.11. *Suppose the disagreement percolation is subcritical. Then the environment satisfies* (1.14)*.*

Proof. Let $A \in \mathcal{C}_b$ be arbitrary and let $\widehat{\mathbb{P}}_A$ be the disagreement percolation coupling of $\mathbb{P}_\mu^{-\infty}(\cdot|A)$ and $\mathbb{P}_\mu^{-\infty}$. We then have for any $B \in \mathcal{F}_{\Lambda(l)}$, $l \ge 1$,

$$\left|\,\mathbb{P}_\mu^{-\infty}(B|A) - \mathbb{P}_\mu^{-\infty}(B)\,\right| \le \widehat{\mathbb{P}}_A(\exists (x,t) \in \Lambda(l) \colon \xi_t(x) = 1), \tag{4.27}$$

which is exponentially small in l and independent of the choice of B and A. □

4.5.3 Strong disagreement percolation

We say that $(\eta_t^1, \eta_t^2, \xi_t)$ is a strong disagreement percolation coupling if $\xi_t(x) = 0$ implies $\eta_t^1(x) = \eta_t^2(x)$ and η^1 and ξ are independent. This independence is a stronger assumption than regular disagreement percolation.

Lemma 4.12. *Suppose $p^* := \max((c_+ - c_-)/c_+, (c_+ - c_-)/(1 - c_-)) < p_c$. Then there exist strong versions of the disagreement percolation couplings in Sections 4.5.1 and 4.5.2.*

Proof. The basic concept of the construction is similar to the regular disagreement percolation. The difference is that we no longer use a single $U_t(x)$ to build the processes $(\eta_t^1(x), \eta_t^2(x), \xi_t(x))$ from $(\eta_{t-1}^1, \eta_{t-1}^2, \xi_{t-1})$. Instead we take three $[0,1]$-uniform i.i.d random variables $(U_t(x)^1, U_t(x)^2, U_t(x)^3)$. We then set

$$\begin{aligned} \eta_t^1(x) &= \mathbb{1}_{U_t^1(x) \le c_1(\theta_{-x}\eta_{t-1}^1)}; \\ \xi_t(x) &= \mathbb{1}_{U_t^3(x) \le p^*} \mathbb{1}_{\exists y, |\, y-x\,| \le 1 : \xi_{t-1}(y)=1}; \\ \eta_t^2(x) &= \begin{cases} \eta_t^1(x), & \xi_t(x) = 0; \\ 1, & U_t^2(x) \le \frac{c_1(\theta_{-x}\eta_{t-1}^2) - (1-p^*)c_1(\theta_{-x}\eta_{t-1}^1)}{p^*} \text{ and } \xi_t(x) = 1; \\ 0, & otherwise. \end{cases} \end{aligned} \tag{4.28}$$

The choice of p^* guarantees that $\frac{c_1(\theta_{-x}\eta_{t-1}^2) - (1-p^*)c_1(\theta_{-x}\eta_{t-1}^1)}{p^*} \in [0,1]$, and a direct computation shows that the probability that $\eta_t^2(x) = 1$ is $c_1(\theta_{-x}\eta_{t-1}^2)$. □

4.5.4 Proof of Theorem 3.7 and Corollary 3.8

Proof of Theorem 3.7. The proof is based on a coupling argument. Let

$$C_{-k} := \{(x,t) \in \mathbb{Z}^d \times \{..., -k-1, -k\} : |\, x - \gamma_{-k}\,| \le |\, t-k\,|\} \tag{4.29}$$

be the infinite backwards cone with tip at $(\gamma_{-k}, -k)$. We construct iteratively the random variables $(\eta_t(x)^{1,m}, \eta_t(x)^{2,m}, \xi_t(x)^m) \in E \times E \times \{0,1\}, (x,t) \in C_{-k-1+m}$, and $H^m \in \mathbb{N}$, and denote their law by $\widetilde{\mathbb{P}}^m$. We start with $\eta_t(x)^{1,0}$ and $\eta_t(x)^{2,0}, \xi_t(x)^0$ chosen independently from $\mathbb{P}_\mu^{-\infty}(\cdot)$ and $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k-1}^{-k-1}\right)$ restricted to the cone C_{-k-1+m} and set $\xi_t^0(x) = 1$ for $(x,t) \in C_{-k-1}$, and $H^0 = 0$.

Given $\widetilde{\mathbb{P}}^m$, let $\widetilde{\mathbb{P}}^{m,*}$ be the extension of $\widetilde{\mathbb{P}}^m$ to the cone C_{-k+m} based on the strong disagreement percolation coupling, that is

$$(\eta_t^{1,m}(x), \eta_t^{2,m}(x), \xi_t^m(x))_{(x,t) \in C_{-k+m} \setminus C_{-k-1+m}} \tag{4.30}$$

are distributed according to the evolution described in (4.28).

The general strategy is as follows: We want to condition the measure $\widetilde{\mathbb{P}}^{m,*}$ on the event $\{\eta^{1,m} \in A_{-k}^{-k+m}, \eta^{2,m} \in A_{-k-1}^{-k+m}\}$. Observe that, on $\xi_{-k+m}(\gamma_{-k+m}) = 0$, the events $\eta^{1,m} \in A_{-k+m}^{-k+m}$ and $\eta^{2,m} \in A_{-k+m}^{-k+m}$ are equivalent. This is the good case. The bad case is when $\xi_{-k}(\gamma_{-k}) = 1$. In this event, we simply restart the coupling procedure by coupling $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k-1}^{-k}\right)$ and $\mathbb{P}_\mu^{-\infty}\left(\cdot \,\middle|\, A_{-k}^{-k}\right)$ independently. The role of H^m is to keep track of the number of iterations since the last time we had to reset and try again.

Define

$$\begin{aligned} q_m(h) :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m}) = 0 \,\middle|\, \eta^{1,m} \in A_{-k}^{-k+m}, H^m = h\right) \\ & \wedge \widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m}) = 0 \,\middle|\, \eta^{2,m} \in A_{-k-1}^{-k+m}, H^m = h\right); \\ Q_m^1 :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m}) = 0 \,\middle|\, \eta^{1,m} \in A_{-k}^{-k+m}\right) - \sum_{h \ge 0} q_m(h); \\ Q_m^2 :=& \widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}^m(\gamma_{-k+m}) = 0 \,\middle|\, \eta^{2,m} \in A_{-k-1}^{-k+m}\right) - \sum_{h \ge 0} q_m(h). \end{aligned}$$

Let $B^1, B^2, D \in \sigma(C_{-k+m})$. We now define $\widetilde{\mathbb{P}}^{m+1}$ based on $\widetilde{\mathbb{P}}^m$ by

$$\begin{aligned} &\widetilde{\mathbb{P}}^{m+1}\left(\eta^{1,m+1} \in B^1, \eta^{2,m+1} \in B^2, \xi^{m+1} \in D, H^{m+1} = h+1\right) \\ &:= q_m(h)\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1, \eta^{2,m} \in B^2, \xi^m \in D, H^m = h \;\middle|\; \right. \\ &\qquad\qquad \left. \eta^{1,m} \in A_{-k+m}^{-k+m}, \eta^{2,m} \in A_{-k+m}^{-k+m}, \xi_{-k+m}^m(\gamma_{-k+m}) = 0\right) \end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\mathbb{P}}^{m+1}\left(\eta^{1,m+1} \in B^1, \eta^{2,m+1} \in B^2, \xi^{m+1} \in D, H^{m+1} = 0\right) \\
&:= \Big[\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1 \;\mid\; \eta^1_{-k+m} \in A^{-k+m}_{-k}, \xi^m_{-k+m}(\gamma_{-k+m}) = 0\right) Q^1_m \\
&+ \widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in B^1, \xi^m_{-k+m}(\gamma_{-k+m}) = 1 \;\middle|\; \eta^{1,m} \in A^{-k+m}_{-k}\right)\Big] \frac{1}{1-\sum_{h\geq 0} q_m(h)} \\
&\cdot \Big[\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in B^2 \;\middle|\; \eta^{1,m}_{-k+m} \in A^{-k+m}_{-k}, \xi^m_{-k+m}(\gamma_{-k+m}) = 0\right) Q^2_m \\
&+ \widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in B^2, \xi^m_{-k+m}(\gamma_{-k+m}) = 1 \;\middle|\; \eta^{2,m} \in A^{-k+m}_{-k}\right)\Big] \mathbb{1}_{1 \text{ on } C_{-k+m} \in D}.
\end{aligned}$$

A direct computation shows that we have $\widetilde{\mathbb{P}}^{m+1}(\eta^{1,m+1} \in B^1) = \mathbb{P}^{-\infty}_{\mu}(B^1 \mid A^{-k+m}_{-k})$ and $\widetilde{\mathbb{P}}^{m+1}(\eta^{2,m+1} \in B^2) = \mathbb{P}^{-\infty}_{\mu}(B^2 \mid A^{-k+m}_{-k})$, assuming that $\widetilde{\mathbb{P}}^m$ satisfies the corresponding properties. Therefore $\widetilde{\mathbb{P}}^k$ extended to all space-time points using the strong disagreement percolation construction (4.28) is a coupling of $\mathbb{P}^{-\infty}_{\mu}(\cdot|A^{-1}_{-k})$ and $\mathbb{P}^{-\infty}_{\mu}(\cdot|A^{-1}_{-k-1})$. We call this coupling $\widetilde{\mathbb{P}}^*$ and drop the super-index k from the random variables. By construction of the coupling,

$$\widetilde{\mathbb{P}}^*(\cdot|H = h) = \widehat{\mathbb{P}}_{A^{-1-h}_{-k}, A^{-1-h}_{-k-1}}\left(\cdot \;\middle|\; \eta^1 \in A^{-1}_{-h}, \xi_{-i}(\gamma_{-i}) = 0, i = 1, ..., h\right)$$

where $\widehat{\mathbb{P}}_{A^{-1-h}_{-k}, A^{-1-h}_{-k-1}}$ is the strong disagreement percolation coupling starting from the cone C_{-1-h}. In particular, η^1 and ξ are independent. Denote by $G := \{\xi_0(x) = 0 \,\forall\, x \in \mathbb{Z}^d\}$ the good event that the disagreement process has become extinct by time 0. We have

$$\frac{\mathbb{P}^{-(k+1)}_{\mu}\left(B \mid A^{-1}_{-k-1}\right)}{\mathbb{P}^{-k}_{\mu}\left(B \mid A^{-1}_{-k}\right)} \geq \frac{\widetilde{\mathbb{P}}^*\left(\eta^2_0 \in B, G\right)}{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B\right)} = \frac{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B, G\right)}{\widetilde{\mathbb{P}}^*\left(\eta^1_0 \in B\right)} = \widetilde{\mathbb{P}}^*(G).$$

Reversing the roles of η^1 and η^2, we also have

$$\frac{\mathbb{P}^{-(k+1)}_{\mu}\left(B \mid A^{-1}_{-k-1}\right)}{\mathbb{P}^{-k}_{\mu}\left(B \mid A^{-1}_{-k}\right)} \leq \widetilde{\mathbb{P}}^*(G)^{-1}.$$

Since $\widetilde{\mathbb{P}}(G^c|H = h)$ is exponentially small in h, we have completed the proof once we show that $\widetilde{\mathbb{P}}^*(H \leq h)$ is exponentially small in k for a fixed h. To see this, we look at H^m in more detail. Since H^m either increases by one or is reset to 0, (H^m) is a time-inhomogeneous house-of-cards process with transition probability $\mathbb{P}(H^{m+1} = h+1|H^m = h) = q_m(h)$. We have that $q_m(h)$ equals

$$\begin{aligned}
&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0, \eta^{1,m} \in A^{-k+m}_{-k+m} \;\middle|\; H^m = h\right)}{\max\left(\widetilde{\mathbb{P}}^{m,*}(\eta^{1,m} \in A^{-k+m}_{-k+m} \;\middle|\; H^m = h), \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m} \in A^{-k+m}_{-k+m} \;\middle|\; H^m = h)\right)} \\
=&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; H^m = h\right)}{\max\left(1, \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m} \in A^{-k+m}_{-k+m} \;\middle|\; H^m = h)(\widetilde{\mathbb{P}}^{m,*}(\eta^{1,m} \in A^{-k+m}_{-k+m} \;\middle|\; H^m = h))^{-1}\right)} \\
=&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; H^m = h\right)}{\max\left(1, \widetilde{\mathbb{P}}^{m,*}(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; H^m = h) + \widetilde{\mathbb{P}}^{m,*}(\eta^{2,m}/\eta^{1,m})\right)} \\
\geq&\frac{\widetilde{\mathbb{P}}^{m,*}\left(\xi^m_{-k+m}(\gamma_{-k+m}) = 0 \;\middle|\; H^m = h\right)}{1 + (\inf_{i,\eta} c_i(\eta))^{-1}\left(1 - \widetilde{\mathbb{P}}^{m,*}\left(\xi_{-k+m}(\gamma_{-k+m}) = 0 \mid H^m = h\right)\right)},
\end{aligned}$$

where in the last equality before the inequality we denoted by

$$\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m}/\eta^{1,m}\right) := \frac{\widetilde{\mathbb{P}}^{m,*}\left(\eta^{2,m} \in A^{-k+m}_{-k+m}, \xi^{m}_{-k+m}(\gamma_{-k+m}) = 1 \mid H^m = h\right)}{\widetilde{\mathbb{P}}^{m,*}\left(\eta^{1,m} \in A^{-k+m}_{-k+m} \mid H^m = h\right)}.$$

Note that $\inf_{i,\eta} c_i(\eta) > 0$, since $\inf_{i,\eta} c_i(\eta) = 0$ implies $p^* = 1 > p_c$. Conditioned on $H^m = h$ the probability of $\xi^m_{-k+m}(\gamma_{-k+m}) = 0$ is larger than the probability that there is no percolation path from $C_{-k+m-h-1}$ to $(\gamma_{-k+m}, -k+m)$, which converges exponentially fast to 1 in h. Therefore there are constants $c_1, c_2 > 0$ so that $1 - q_m(h) \le c_1 e^{-c_2 h}$. We also have $q_m(h) \ge (1-p^*)\inf_{i,\eta} c_i(\eta)$. Those two facts imply that $\widetilde{\mathbb{P}}^*(H \le h) \le c_3 e^{-c_4(k-c_5 h)}$ for some constants $c_3, c_4, c_5 > 0$. □

Corollary 3.8 follows as a direct consequence of Theorems 3.5 and 3.7.

Proof of Corollary 3.8. Let $l \in \mathbb{N}$ and consider $A_l \in \mathcal{A}^{-1}_{-l}$. By telescoping, for any $B \in \mathcal{F}_{=0}$, we have by Theorem 3.7 that

$$\left| \frac{\mathbb{P}_\mu(B \mid A_l)}{\mu(B)}^{\pm} - 1 \right| \le \left[\prod_{i=1}^{k} (1 + C\delta^i) \right]. \tag{4.31}$$

Setting $M := \prod_{i=1}^{\infty}(1 + C\delta^i)$, and noting that the statement (and proof) of Theorem 3.5 holds when $\mathcal{F}_{\ge 0}$ is replaced by $\mathcal{F}_{=0}$, we conclude the proof. □

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