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Discrete-to-continuum limits and stochastic homogenization of ferromagnetic surface energies

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Abstract

In this thesis we investigate the asymptotic behavior of ground states of discrete ferromagnetic Potts-type energy functionals when the number of interacting particles diverges. In the static case we assume that the position of the particles is given by a stationary stochastic lattice. We prove that the discrete energies Γ-converge to a homogeneous surface integral defined on partitions representing the energy of magnetic domain walls. Under ergodicity assumptions the limit energy turns out to be deterministic. We also include the case of Dirichlet boundary conditions and phase constraints. In the second part we prove an analogue result for models of magnetic thin films, where the particles are located close to a lower dimensional subspace. For the sake of simplicity we just consider two phases as in the classical Ising model. In an example we further investigate how the distribution of the particles - and consequently the thickness of the magnetic film - influences the limit energy. In the dynamic case we prove some results concerning the curvature-driven motion of discrete ferromagnetic interfaces under small random interactions.

Zusammenfassung

In dieser Arbeit betrachten wir Energiefunktionale, deren Struktur einer ferromagnetischen Potts-Energie ähnelt. Uns interessiert das asymptotische Verhalten von Grundzuständen, wenn die Anzahl interagierender Teilchen divergiert. Im statischen Fall nehmen wir an, dass die Position der Teilchen durch ein stationäres stochastisches Gitter gegeben ist. Via Γ-Konvergenz erhalten wir im Grenzwert ein homogenes Oberflächenintegral über Partitionen, welches die Energie der magnetischen Grenzflächen darstellt. Für ergodische Gitter ist diese Energie deterministisch. Ferner demonstrieren wir die Konvergenz von Dirichlet-Randwertproblemen und Phasennebenbedingungen. Im zweiten Teil beweisen wir analoge Ergebnisse für Modelle dünner magnetischer Filme, in denen die Teilchen nahe einem niederdimensionalen Unterraum liegen. Einfachheitshalber untersuchen wir hierbei nur Zwei-Phasen-Modelle wie etwa das Ising-Modell. In einem Beispiel diskutieren wir, wie die Verteilung der Teilchen - und damit die Dicke des magnetischen Films - den Grenzwert beeinflusst. Im dynamischen Fall zeigen wir einige Resultate über die krümmungsabhängige Evolution diskreter, ferromagnetischer Oberflächen mit schwachen zufälligen Wechselwirkungen.

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Chapter 1

Introduction

Many mathematical models in materials science use a continuum description of matter. Since Cauchy's time scientists try to understand how these continuum models can be derived from microscopic atomistic models in the limit the number of particles diverges. The motivation behind this effort is at least two-fold. On the one hand it gives a justification of the continuum theory as a coarse-grained version of the more natural atomistic model. On the other hand, finding mathematically rigorous connections between atomistic and continuum theories can also yield efficient methods to discretize the continuum models and thus simplifies their numerical analysis. A lot of discrete models use variables defined on a periodic lattice, mostly $\mathbb{Z}^d$ or more general Bravais lattices. This corresponds to the modeling assumption that particles self-assemble into a crystalline order, a fact that indeed is expected to happen for lots of materials and low temperatures. A rigorous mathematical proof of such a result seems to be very difficult. We refer to [36, 52] for a possible formulation of the problem and for partial results in two and three dimensions. Besides mathematical difficulties, for some materials crystallization is not expected to occur, even at low temperature. Such phenomena are the starting point for the main part of this thesis, where we investigate particle systems entailing a certain degree of disorder. In particular our studies are motivated by the structure of so called magnetic polymer composites. During the last decades these materials, synthesized by embedding magnetic particles into a polymer matrix, have gained popularity mostly due to their biomedical applications (we refer to [53] for an elementary introduction to the subject). Here we just collect the basic facts we use to build up our models. It is common to assume that the magnetic particles are distributed randomly in the polymer matrix. Hence there is no periodic structure but only stochastic homogeneity. We will model the position of these particles as the realization of a stationary point process. However when producing these materials chemists are interested in gaining more structural properties. For several reasons the particles are sometimes coated with one or even two surrounding layers. First of all coating prevents the particles from dissolving in the polymer matrix. In addition one may suppress physical effects like for instance dipole-dipole interactions and moreover one can endow the same material with different properties like magnetism and luminescence. For our modeling it is important to observe that such a coating leads to a minimal distance between the magnetic particles. Further assuming the particles to be distributed in such a way that there are no arbitrarily large empty regions in the matrix, we come to the definition of *admissible sets of points* that we use in this thesis as a reference configuration on which we define our variables. Admissible sets of points are (possibly random) countable sets $\mathcal{L} \subset \mathbb{R}^d$ such that

(i) there exists $R > 0$ with $\mathrm{dist}(x, \mathcal{L}) < R$ for all $x \in \mathbb{R}^d$;

(ii) there exists $r > 0$ with $|x - y| \geq r$ for all $x, y \in \mathcal{L}$, $x \neq y$.

We remark that these kinds of sets, sometimes also called Delone-sets, are quite standard in statistical mechanics (see [47]) and have been used for the first time in the context of discrete-to-continuum limits in [15] and later in [4] to derive continuum models of nonlinear elasticity from polymer physics.

In this thesis we focus on magnetic models. We restrict the analysis to the simplest kind of magnetization taking only finitely many values, that means to every particle we assign a precise magnetization out of a discrete set $\mathcal{M} = \{m_1, \dots, m_q\}$ with $q \geq 2$. The choice of the set $\mathcal{M}$ can already be seen as a result of a coarse-graining of a finer microscopic model. Within this picture our magnetization in fact corresponds to a quantity averaged on some mesoscale. To be more precise, the size of the magnetic particles sitting on the points of an admissible set is in the range of nanometers. Under the hypothesis that each particle itself is made out of a ferromagnetic material with crystalline structure (this time on the atomic scale), we assume that the periodic ordering of atoms favors finitely many easy axes for the magnetization. This phenomenon, called magnetocrystalline anisotropy is known as one source of magnetic anisotropy. We further assume that this anisotropy effect is dominant so that the average magnetization of one (nano)particle takes only finitely many values. These values are the elements of the set $\mathcal{M}$. Our models aim at describing the interaction between the nanoparticles and not the atoms. For simplicity we take into account only ferromagnetic pairwise exchange interactions between the nanoparticles as in the classical Potts model with no external magnetic field. We remark that this is compatible with coatings that suppress dipole-dipole interactions as these are weak when the particles are not too close. Up to renormalization the energy of a magnetization $u : \mathcal{L} \to \mathcal{M}$ then reads as

$$\tilde{E}(u) = \sum_{x,y\in\mathcal{L}} c(x,y)|u(x) - u(y)|^2,$$

where the interaction coefficients are nonnegative and depend on the position of the interacting particles (later on we will prove our results for slightly more general energies). Precise assumptions on $c(x, y)$ will be given in the next chapters. Note that such an energy favors constant magnetizations. In order to make the discrete energy finite for all configurations we introduce a scale, which can be seen as the average distance between particles. First we fix a regular reference domain D. Physically this set corresponds to the union of the polymer matrix and the embedded particles. In order to investigate the limit when the number of magnetic particles in the polymer matrix diverges, we scale the point set by a small parameter $\varepsilon > 0$ and define the set $\mathcal{L}_\varepsilon(D) = \varepsilon\mathcal{L} \cap D$. For magnetizations $u : \mathcal{L}_\varepsilon(D) \to \mathcal{M}$ we then consider the energy restricted to the finitely many particles in $\mathcal{L}_\varepsilon(D)$ by setting

$$E_\varepsilon(u, D) = \sum_{x,y\in\mathcal{L}_\varepsilon(D)} \varepsilon^{d-1} c(x/\varepsilon, y/\varepsilon)|u(x) - u(y)|^2.$$

Note that the scaling factor ε^{d-1} corresponds to a surface scaling and is chosen here as we aim at investigating interfacial type energies as a way to prove the formation of Weiss domains (regions of constant magnetization) via constrained minimization as discussed below.

We intend to pass rigorously from the microscopic many-particle model above to a continuum version. This will be achieved in a limit procedure when $\varepsilon \to 0$ which is, under some regularity assumptions on ∂D, equivalent to taking the thermodynamic limit as it is usually done in statistical physics (see [44]). At this point we should mention that we neglect all entropic effects and just focus on energy minimization. Physically this corresponds to the zero temperature regime which we tacitly assume for our model. We will devote one short section to discuss how asymptotic energy minimization can be related to a zero-temperature limit in the sense of large deviation principles for Gibbs-measures. Now how can we pass to a limit continuum energy? Note that the domain of the functionals $E_\varepsilon(\cdot, D)$ varies with ε. Hence it makes no sense to investigate any pointwise limit. In order to bypass this issue, we use the notion of Γ-convergence which is appropriate to study the convergence of global minimization problems. In order to keep this thesis almost self-contained, we will give a very short introduction to this notion of variational convergence in Section 2.1. If the reader is not familiar with this kind of convergence, for this introduction it suffices to view the limit continuum energy as an effective energy. Roughly speaking, when the lattice spacing gets finer and finer ($\varepsilon \to 0$), we can replace the discrete environment by a continuum, the magnetization by a new variable defined on the continuum and the energy E_ε by this effective limit energy in a way such that the error we commit when we compute global minima in the discrete and continuum setting tends to zero when $\varepsilon \to 0$.

In order to model the stochastic homogeneity of the particle positions we further let the admissible point set be generated a random variable $\omega \mapsto \mathcal{L}(\omega)$ which we assume to be stationary. This means that the probability distribution does not change if we shift the random variable in space or, in formulas, $\mathcal{L}$ and $\mathcal{L} + z$ have the same statistics for every $z \in \mathbb{Z}^d$.

We now give an overview of the main content of the thesis. To keep the introduction easy to read we will slightly oversimplify the setting. A more detailed introduction at the beginning of each chapter will restore the mathematical rigor.

Chapter 2 will contain some preliminary results on Γ-convergence, functions of bounded variation and probability theory. Those build the basis for the main part of the thesis. Here we want to highlight a new continuity result for functionals defined on Caccioppoli partitions.

The discrete-to-continuum analysis of the random energies $E_\varepsilon(\omega)$ will be contained in Chapter 3 and it is the main part of this thesis. Let us very briefly describe the result and the strategy of the proof. Using purely deterministic arguments, we first show that under some decay and coercivity conditions on the interaction coefficients every possible Γ-limit of $E_\varepsilon(\omega)$ (that is to say up to subsequences) is finite only for functions $u \in BV(D, \mathcal{M})$, where it takes the form

$$E(\omega)(u, D) = \int_{S_u \cap D} \phi(\omega; x, u^+, u^-, \nu_u)\, \mathrm{d}\mathcal{H}^{d-1}.$$

Here S_u denotes the discontinuity set of u and $\nu_u = \nu_u(x)$ is the corresponding normal vector at $x \in S_u$. The functions u^+ and u^- stand for the traces of u at S_u (see Section 2.2 for details). From a physical point of view this means that asymptotically the particles form Weiss domains of finite perimeter and the energy needed to form these domains is given by integration of a surface tension ϕ over the magnetic domain walls, that are the interfaces where the magnetization jumps. As we see from the above formula, in general the (random) surface tension depends on the position and orientation of the domain wall as well as the value of the magnetization on both sides of the interface. While the proof of this structure result is quite standard and relies only on the geometric assumptions on the point set $\mathcal{L}(\omega)$, integral representation theorems and the abstract methods of Γ-convergence, the delicate issue is to prove that if the point process is also stationary, then the Γ-limit indeed exists. The reason is that, assuming that the coefficients are of the form $c(x,y) = c(x-y)$, the well-known blow-up formulas for surface integrands yield a stochastic process parameterized by cubes that is stationary, but fails to be subadditive so that the usual arguments for stochastic homogenization used for instance in the pioneering paper [33] or in [4] fail in our setting. However we can define a subadditive process parameterized on lower-dimensional cubes contained in certain hyperplanes. A rather delicate probabilistic argument shows that this is enough to conclude existence of the limit energy. Our analysis further reveals that for stationary admissible point sets the integrand ϕ is independent of the spatial variable, so that the energy may be written as

$$E_{\mathrm{hom}}(\omega)(u,D) = \int_{S_u\cap D} \phi_{\mathrm{hom}}(\omega;u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^{d-1}.$$

We also provide an asymptotic homogenization formula for the integrand. By standard arguments, assuming also ergodicity the energy becomes deterministic. In the stationary case we further obtain results about the convergence of solutions of minimization problems with boundary conditions or with restrictions on the average magnetizations (phase constraints). The latter means that we prescribe the cardinality of the set $\{u = m_i\}$ for all i. Observe that physically such a prescription makes sense either when the particles can flip their magnetization but the overall distribution is conserved or when the particles are allowed to move in the polymer matrix. In this (actually time-dependent) setting our results should be interpreted as a long-time behavior after the system has reached equilibrium. Both suitable boundary conditions and phase constraints lead to the formation of Weiss domains. Having in mind the applications to magnetic polymer composites, as an interesting example we investigate the case of stationary, ergodic and isotropic point sets. Making the further restriction that the coefficients depend only on $|x-y|$, in this case the limit energy turns out to be isotropic, too. We close Chapter 3 with an interpretation of the limit energy as a rate functional for the large deviation principle for associated Gibbs-measures when the temperature vanishes at a certain rate simultaneously when the number of particles diverges.

We stress that our strategy to prove homogenization differs from the one used for continuum functionals with linear growth treated in [1] (which cannot be used in our setting). We think it yields deeper insight into stochastic homogenization problems in non-smooth function classes. In this sense the main contribution of this thesis consists of developing a technique to deal with stochastic homogenization of functionals defined on (discrete) sets of finite perimeter. We believe that our approach can be extended to treat stochastic homogenization in multi-scale problems where both bulk and surface contributions appear.

In the second major part of this thesis, contained in Chapter 4, we treat a so-called dimension reduction problem in random environments. Coming back to the model of magnetic composite materials, here we assume that the polymer matrix is almost a lower dimensional object with a thickness proportional to the average particle distance ε. Such a cluster of points should be understood as a model for magnetic thin films. We have in mind polymeric magnets manufactured to create thin films made of few layers. Under external magnetic fields they form Weiss domains whose wall energy is influenced by the roughness of the film which in turn is a result of the physical and chemical properties of the specific material at use. Despite a fairly large amount of experimental results has been done in order to relate the roughness of the surface to the interfacial wall energy of the domains (see for instance [41]), no rigorous mathematical result has appeared so far in this direction. We aim at beginning such an investigation in this thesis.

We work in the same framework as before, but we restrict the analysis to ferromagnetic Ising-type models. Mathematically this means that for this chapter we restrict the analysis to the case $\mathcal{M} = \{\pm 1\}$ and, given $u : \mathcal{L} \to \{\pm 1\}$, the unscaled energy takes the form

$$\tilde{F}(u) = \sum_{x,y\in\mathcal{L}} c(x,y)|u(x)-u(y)|^2 = 2\sum_{x,y\in\mathcal{L}} c(x,y)|u(x)-u(y)|.$$

We remark that the restriction to two phases is made mainly to avoid too many parameters. Indeed, it can be shown that similar results hold for a general finite set $\mathcal{M}$. The main difference to Chapter 3 is contained in the geometry of the point set $\mathcal{L}$ described below.

The system is supposed to be thin in the sense that the nodes of the matrix are within a small distance, of the order of the average distance between the nodes themselves, from a lower-dimensional subspace. For applications the most relevant case consists of a plane in a three dimensional environment. In presence of an external magnetic field or of proper boundary conditions, the ferromagnetic coupling induces the system to form mesoscopic Weiss domains. Again we aim at performing the limit of the energy as the average distance between the magnetic cells, say ε, goes to zero with respect to the macroscopic size of the system. This time such a limit will have two main effects: as in the first part it will allow us to describe the original discrete system as a continuum while at the same time it will reduce its dimension from 3 to 2 (or more in general from d to k with $2 \leq k < d$). We describe the polymeric matrix as a random network whose nodes $\mathcal{L} \subset \mathbb{R}^d$ form a *thin admissible lattice*, meaning that there exists $k \in \mathbb{N}$ with $2 \leq k < d$ and $M > 0$ such that

$$\operatorname{dist}(x, \mathbb{R}^k) \leq M \quad \forall x \in \mathcal{L}$$

and that it is admissible according to our standard definition adapted to the lower-dimensional space:

(i) There exists $R > 0$ such that $\operatorname{dist}(x, \mathcal{L}) < R$ for all $x \in \mathbb{R}^k$;

(ii) there exists $r > 0$ such that $|x - y| \geq r$ for all $x \neq y$, $x, y \in \mathcal{L}$.

Also for these point sets we assume that they are generated by a random variable $\omega \mapsto \mathcal{L}(\omega)$. However, in the case of a thin stochastic lattice we restrict the stationarity assumption to the lower-dimensional subspace meaning that $\mathcal{L}$ and $\mathcal{L} + z$ have the same statistics only for $z \in \mathbb{Z}^k$. This implies that the point set can fluctuate in the orthogonal complement. For instance one may think of a magnetic thin film whose particle density decreases in the third direction.

Similar to the first major part of the thesis we perform the asymptotic analysis when we scale the thin lattice by a small parameter $\varepsilon > 0$. However, for taking into account the energetic contribution for the creation of magnetic interfaces, the correct scaling is now ε^{k-1}. Hence, up to a factor of 2, the scaled (random) discrete energies for a magnetization $u : \varepsilon\mathcal{L} \to \{\pm 1\}$ can be written as

$$F_\varepsilon(\omega)(u, D) = \sum_{\substack{x,y \in \varepsilon\mathcal{L}(\omega) \\ x,y \in P_k^{-1}(D)}} \varepsilon^{k-1} c(x/\varepsilon, y/\varepsilon)|u(x) - u(y)|,$$

where now $D \subset \mathcal{A}^R(\mathbb{R}^k)$ is a flat reference domain and P_k denotes the projection onto $\mathbb{R}^k$. The strategy for analyzing the asymptotic behavior of the energies F_ε is quite similar to the full-dimensional problem. First we show that under suitable growth assumptions on the interaction coefficients every possible Γ-limit is finite exactly on $BV(D, \{\pm 1\})$ and for such magnetizations it can be written as a surface integral

$$F(\omega)(u, D) = \int_{S_u \cap D} \phi(\omega; x, \nu) \, \mathrm{d}\mathcal{H}^{k-1}.$$

In a second step we turn our attention to the homogenization problem for stationary thin lattices and prove that in this case the Γ-limit indeed exists and is given by a spatially homogeneous functional

$$F_{\mathrm{hom}}(\omega)(u,D) = \int_{S_u\cap D} \phi_{\mathrm{hom}}(\omega;\nu)\,\mathrm{d}\mathcal{H}^{k-1}$$

which is deterministic under ergodicity assumptions. We just sketch some of the arguments especially when the proof consists of projecting on the flat space and adapting the techniques from Chapter 3. Beside the general analysis we again obtain results about the presence of Dirichlet-type boundary conditions or sharp phase constraints. While all of these results essentially extend what we already obtained for the full-dimensional problem, we close Chapter 4 with the analysis of a model for random deposition of magnetic particles onto a flat substrate which departs from the content of Chapter 3. Roughly speaking, we build a thin stochastic lattice as follows: for each time step and each position $z \in \mathbb{Z}^2$, independently with probability $p \in (0,1)$ we let a magnetic particle fall onto z and stack them over each other if there was already a particle at z from previous time steps. This model entails fine dependencies of the surface tension $\phi_{\mathrm{hom}}(\nu)$ on the geometry of the thin lattice. While a complete analysis of this model seems out of reach, we investigate how the surface tension of the limit continuum model behaves when we let the particles deposit infinitely many times. To this end we need to prove a weak version of existence of plane-like minimizers as in [29] for periodic dimension reduction problems which may be of independent interest.

In the third part of the thesis, contained in Chapter 5, we try to leave aside the static picture and include dynamical effects in order to describe the curvature-driven motion of magnetic domain walls. In contrast to existing random dynamical schemes for spin models as for instance Glauber dynamics, we aim at a purely variational scheme. As in our setting the space of configurations is discrete, we cannot define a gradient flow. Therefore we use the notation of so-called minimizing movement schemes, which makes sense even for highly non-smooth energies. We remark that we prefer to not call this model dynamic as it is build by time-dependent minimization. In the sense of geometric motions of interfaces, minimizing movements were first introduced by Almgren, Taylor and Wang in 1993 to study curvature-driven flows (see [8]). In a nutshell it can be described as follows: Given a fixed time step $\tau > 0$ and an initial set A_0, one constructs recursively a sequence of sets $(A_k^\tau)_k$ minimizing an energy functional of the form

$$E^{\tau}(A, A^{\tau}_{k-1}) = \int_{\mathcal{F}A} \varphi(\nu(x))\,\mathrm{d}\mathcal{H}^{d-1} + \frac{1}{2\tau}\int_{A\Delta A^{\tau}_{k-1}} \mathrm{dist}(x, \partial A^{\tau}_{k-1})\,\mathrm{d}x, \qquad (1.1)$$

where $\nu(x)$ is the normal vector at the point x in the reduced boundary $\mathcal{F}A$ (see Chapter 2 for a precise definition) and φ is a suitable surface density. The basic idea behind this approach is the following: While minimizing the perimeter-type functional shrinks the set, the bulk term forces the boundary of the minimizer to be close to the boundary of the previous set. Passing to the limit as $\tau \to 0$ for the piecewise constant interpolations one obtains a time dependent family $A(t)$ of sets that evolves by a weighted curvature (depending on φ), provided the initial set A_0 is regular enough and φ is elliptic and smooth. In the isotropic case one obtains the well-known motion by mean curvature.

In the recent paper [24] Braides, Gelli and Novaga applied the above minimizing movement scheme within a deterministic, discrete environment. In this setting the environment is the scaled two-dimensional lattice $\varepsilon\mathbb{Z}^2$. The surface term in (1.1) is replaced by a discrete interfacial energy which, in its simplest form, is derived from the classical nearest neighbor Ising model and can be written formally as

$$P_{\varepsilon}(u) = \frac{1}{4}\sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \\ |i-j|=1}} \varepsilon|u(\varepsilon i) - u(\varepsilon j)|. \qquad (1.2)$$

Note that the energy in (1.2) takes into account only nearest neighbor interactions. Therefore it can be interpreted as the perimeter of the set $\{u = +1\}$ (hence the notation P_{ε}). The distance-function in the bulk term in (1.1) is replaced by a discrete version of the l^{∞}-distance to the boundary. From a physical point of view this setup can be seen as a simplified model to describe the motion of the boundaries of level sets given by spin variables $u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$. Those are precisely the magnetic domain walls at the discrete level. The identification of spin variables and $+$-level sets can be used to compare the discrete version with its continuum analogue. Since the discrete perimeter inherits the anisotropy of the lattice, this minimizing movement scheme is related to crystalline motions, where φ is not smooth (see [7, 13, 30] in the continuum case). We stress that already in [24] the authors observed that the asymptotic behavior of the discrete flows depends heavily on the scaling between ε, τ when $\varepsilon, \tau \to 0$ simultaneously. The critical scaling is $\varepsilon \sim \tau$, where pinning effects due do discreteness as well as a quantized crystalline motion can occur.

In this thesis we start studying the effect of a random discrete environment on the continuum limit flow. Of course the natural approach in the spirit of Chapters 3 and 4 would be to replace the periodic lattice in the definition of P_ε by the random lattice $\varepsilon\mathcal{L}(\omega)$ with suitable short-range interactions. This seems to be a very challenging problem. Thus we start with a much simpler problem and associate very small random perturbations directly to the periodic lattice model, that means we will study the minimizing movement of a random discrete perimeter of the form

$$P_\varepsilon^\omega(u) = \frac{1}{4} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \\ |i-j|=1}} \varepsilon(1 + \varepsilon c_{ij}(\omega))|u(\varepsilon i) - u(\varepsilon j)|.$$

For the precise assumptions on the random field c_{ij} and further discussion we refer to Chapter 5. Note that in this scaling the random perturbations are very small as they are scaled by ε. Nevertheless it turns out that they may influence at least the velocity of the limit motion. The reason why we don't let the bulk term be affected by the randomness as well, comes from the physical interpretation we give to this model as motion of aligned spins and differs from lattice particle models: While the interaction between particles may be affected by some random noise deriving from microscopic fluctuations, the energy to flip a spin should be constant, depending only on how many boundary layers are flipped in one time step. The interpretation of the bulk term in the energy in this setting is the following: Flipping the first layer of spins costs the least energy while the following layers are energetically more expensive. Of course this interpretation makes sense only if one can prove that in presence of randomness sets shrink by flipping spins close to the boundary by a certain number of layers. This is the case in the deterministic setting considered in [24].

For the sake of simplicity we investigate the evolution when the initial set is a coordinate rectangle. We prove that under stationarity and quantified mixing assumptions as well as a suitable uniform bound on the random field c_{ij}, the limit motion law is deterministic and coincides with the quantized crystalline flow obtained in [24]. This however depends strongly on the fact that the random field is stationary with respect to the translation group on $\mathbb{Z}^2$. Later on we show that the velocity changes if we restrict stationarity to a subgroup of the form $m\mathbb{Z}^2$ with $m \geq 2$. Anyhow, we stress that our results should be seen as a stability result of the deterministic problem rather than an exhaustive description of the possible effects of randomness on the limit flow. Indeed, randomness can influence the motion drastically. For example, by the results of Section 3.5, when we replace the square lattice $\mathbb{Z}^2$ by a suitable isotropic stochastic lattice $\mathcal{L}(\omega)$, then, up to a multiplicative constant, the discrete perimeters Γ-converge to the one-dimensional Hausdorff measure. Thus, with an appropriate choice of discrete distance, one should not expect a crystalline motion anymore in the limit but rather motion by mean curvature, at least if $\varepsilon << \tau$ and the initial sets $A_0^{\varepsilon,\tau}$ converge to a smooth set. To highlight possible difficulties even in this very weak random setting, we provide an example of stationary, ergodic perturbations that indicate strong non-uniqueness effects dropping the mixing hypothesis. Moreover we briefly discuss what might happen when we consider random fields c_{ij} with a generic L^∞-bound, where partial results are available.

Chapter 2

Mathematical preliminaries

In this chapter we collect several results from the calculus of variations and probability theory that we will use in the sequel. Let us start to fix some notation that we will use throughout this thesis.

For a real number $y \in \mathbb{R}$, we let $\lfloor y \rfloor$ be its integer part and $\lceil y \rceil := \lfloor y \rfloor + 1$. We fix the dimension $d \geq 2$. By $|\cdot|$ we denote the Euclidean norm on $\mathbb{R}^d$. If $B \subset \mathbb{R}^d$ is a Borel set we denote by $|B|$ its Lebesgue measure and by $\mathcal{H}^m(B)$ its m-dimensional Hausdorff measure. We write $B_\delta(x)$ for the open, euclidean ball around x with radius δ. Given an open set $D \subset \mathbb{R}^d$ we denote by $\mathcal{A}(D)$ the family of all bounded open subsets of D and by $\mathcal{A}^R(D)$ the family of those sets in $\mathcal{A}(D)$ which have a Lipschitz boundary. Moreover, we set $\dim_{\mathcal{H}}(\cdot)$ as the Hausdorff dimension and $\mathrm{d}_{\mathcal{H}}(A, B)$ as the Hausdorff metric between two sets A, B. The symmetric difference of two sets A, B is denoted by $A\Delta B$. Let $\nu_1, \ldots, \nu_d$ be a orthonormal basis of $\mathbb{R}^d$. We define the cube

$$Q_\nu = \{x \in \mathbb{R}^d : \ |\langle x, \nu_i \rangle| < \frac{1}{2} \quad \forall i\}$$

and, for $x \in \mathbb{R}^d, \rho > 0$, we set $Q_\nu(x, \rho) := x + \rho\, Q_\nu$. As usual, for $1 \leq p < +\infty$ we denote by $L^p(D)$ the Lebesgue-spaces of integrable functions while L^∞ denotes the measurable and essentially bounded functions. Whenever necessary we specify the co-domain. In the context of probability theory we denote by $\mathbb{E}[X]$ the first moment of a random variable X. In the proofs $C > 0$ denotes a generic constant that can change every time it appears.

2.1 Γ-convergence in a nutshell

This section contains a brief overview of the results that are used in this thesis. We refer to the monographs [18, 32] for an exhaustive treatment of the subject including the proofs which we omit. Although Γ-convergence can be defined for functions on general topological spaces (see [32]), for the sake of simplicity here we define it on a metric space X. We consider a sequence of functions $F_n : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ denotes the extended real line.

Definition 2.1 (Γ-convergence)**.** *The sequence $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge to $F : X \to \overline{\mathbb{R}}$, if for every $x \in X$ the following two conditions hold:*

(i) ($\liminf$-inequality) For every sequence $x_n \to x$ we have

$$F(x) \le \liminf_n F_n(x_n).$$

(ii) (recovery sequence) There exists a sequence $x_n \to x$ with

$$F(x) \ge \limsup_n F_n(x_n).$$

The function F is called the Γ-limit of the sequence $\{F_n\}_n$ and we write $F = \Gamma\text{-}\lim_n F_n$. Among the many other characterizations of the Γ-limit there is one which we will use later on.

Lemma 2.2. *The following two statements are equivalent:*

(i) $F = \Gamma\text{-}\lim_n F_n$,

(ii) $F(x) = \inf\{\liminf_n F_n(x_n) : x_n \to x\} = \inf\{\limsup_n F_n(x_n) : x_n \to x\}$.

This characterization leads to the definition of the $\Gamma\text{-}\liminf$ and $\Gamma\text{-}\limsup$. We set

$$(\Gamma\text{-}\liminf_n F_n)(x) = \inf\{\liminf_n F_n(x_n) : x_n \to x\},$$
$$(\Gamma\text{-}\limsup_n F_n)(x) = \inf\{\limsup_n F_n(x_n) : x_n \to x\}.$$

Note that these two quantities are always well-defined. Moreover we have the following Urysohn-property (see Proposition 1.44 in [18]).

Lemma 2.3. *The sequence F_n Γ-converges to F if and only if every subsequence of F_n contains a further subsequence that Γ-converges to F.*

In general, Γ-convergence can't be used to define a topology. For example, every constant sequence of functionals Γ-converges to its lower semicontinuous envelope. Indeed, any Γ-limit is lower semicontinuous as shown by [18, Proposition 1.28]:

Proposition 2.4. *Both the* $\Gamma\text{-}\liminf_n F_n$ *and the* $\Gamma\text{-}\limsup_n F_n$ *are lower semicontinuous functions with respect to the convergence in* X.

Another important property of Γ-convergence is its compactness, provided the space X is enough regular. The following result can be found in Proposition 1.42 in [18].

Proposition 2.5. *Let* X *be a separable metric space and* $F_n : X \to \overline{\mathbb{R}}$. *Then, up to subsequences, there exists* $\Gamma\text{-}\lim_n F_n$.

The next theorem is often called the fundamental theorem of Γ-convergence (see also Theorem 1.21 in [18] for a slightly stronger version). It shows why Γ-convergence is very useful for studying minimum problems in the calculus of variations. Let us first introduce the notion of equicoercivity. We remark that the definition slightly varies in the literature.

Definition 2.6. *A sequence* $F_n : X \to \overline{\mathbb{R}}$ *is called equicoercive, if every sequence* $\{x_n\} \subset X$ *with* $\sup_n F_n(x_n) < +\infty$ *is precompact in* X.

Theorem 2.7. *Let* $F_n : X \to \overline{\mathbb{R}}$ *be equicoercive and assume that there exists* $F = \Gamma\text{-}\lim_n F_n$. *Then* F *attains its minimum on* X *and*

$$\min_X F = \lim_n \inf_X F_n.$$

Moreover, if $\{x_n\}_n$ *is a converging sequence such that* $\lim_n F_n(x_n) = \lim_n \inf_X F_n$, *then its limit is a minimizer of* F.

2.2 Functions of bounded variation

We will embed discrete energy functionals into Lebesgue spaces. However, due to the surface scaling we will obtain higher regularity for the domain of the limit energy, namely $u \in BV$. In this section we give the definition and recall basic facts about functions of bounded variation. Except when stated otherwise, the following results can be found in [11]. We assume throughout this section that $O \subset \mathbb{R}^d$ is an open set.

Definition 2.8. *A function* $u \in L^1(O)$ *is a function of bounded variation, if there exists a finite vector-valued Radon measure* μ *on* O *such that for any* $\varphi \in C_c^\infty(O, \mathbb{R}^d)$ *it holds*

$$\int_O u \,\mathrm{div}\varphi \,\mathrm{d}x = -\int_O \langle \varphi, \mu \rangle.$$

In this case we write $u \in BV(O)$ *and* $Du = \mu$ *is the distributional derivative of* u. *A function* $u \in L^1(O, \mathbb{R}^N)$ *belongs to* $BV(O, \mathbb{R}^N)$ *if every component belongs to*

$BV(O)$. In this case Du denotes the matrix-valued Radon measure consisting of the distributional derivatives of each component.

The spaces $BV_{\mathrm{loc}}(O)$ and $BV_{\mathrm{loc}}(O,\mathbb{R}^N)$ are defined as usual. The space $BV(O,\mathbb{R}^N)$ becomes a Banach space endowed with the norm $\|u\|_{BV(O,\mathbb{R}^N)} = \|u\|_{L^1(O,\mathbb{R}^N)}+|Du|(O)$, where $|Du|$ denotes the total variation measure of Du. When O is a bounded Lipschitz domain, then $BV(O,\mathbb{R}^N)$ is compactly embedded in $L^1(O,\mathbb{R}^N)$. In order to deal with convergence, we say that a sequence u_n converges weakly* to u if $u_n \to u$ in $L^1(O,\mathbb{R}^N)$ and $Du_n \overset{*}{\rightharpoonup} Du$ in the sense of measures. We say that u_n converges strictly to u if $u_n \to u$ in $L^1(O,\mathbb{R}^N)$ and $|Du_n|(O) \to |Du|(O)$. Note that strict convergence implies weak*-convergence and that for Ω with Lipschitz boundary norm-bounded sequences in $BV(O,\mathbb{R}^N)$ are compact with respect to weak*-convergence, but not necessarily with respect to strict convergence.

We say that a Lebesgue-measurable set $E \subset \mathbb{R}^d$ has finite perimeter in O if its characteristic function $\mathbb{1}_E$ belongs to $BV(O)$. We say that it has locally finite perimeter in O if $\mathbb{1}_E \in BV_{\mathrm{loc}}(O)$. Let O' be the largest open set such that E has locally finite perimeter in O'. The reduced boundary $\mathcal{F}E$ of E is defined as

$$\mathcal{F}E := \left\{ x \in O' \cap \operatorname{supp}|D\mathbb{1}_E| :\ \nu_E(x) = \lim_{\rho\to 0} \frac{D\mathbb{1}_E(B_\rho(x))}{|D\mathbb{1}_E|(B_\rho(x))} \text{ exists and } |\nu_E(x)| = 1 \right\}.$$

Then it holds that $|D\mathbb{1}_E| = \mathcal{H}^{d-1} \llcorner \mathcal{F}E$ and ν_E can be interpreted as a measure theoretic inner normal vector (see also Theorem 3.59 in [11]).

Now we state some fine properties of BV-functions. To this end, we need some definitions. A function $u \in L^1(O,\mathbb{R}^N)$ is said to have an approximate limit at $x \in O$ whenever there exists $z \in \mathbb{R}^N$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho(x)} |u(y) - z| \,\mathrm{d}y = 0.$$

We remark that the definition of approximate limits varies in the literature. We let $S_u \subset O$ be the set, where u has no approximate limit. Now we introduce so called approximate jump points. Given $x \in O$ and $\nu \in S^{d-1}$ we set

$$\begin{cases} B_\rho^+(x,\nu) = \{y \in B_\rho(x) :\ \langle y-x,\nu\rangle > 0\}, \\ B_\rho^-(x,\nu) = \{y \in B_\rho(x) :\ \langle y-x,\nu\rangle < 0\}. \end{cases}$$

We say that $x \in O$ is an approximate jump point of u if there exist $a \neq b \in \mathbb{R}^N$ and

$\nu \in S^{d-1}$ such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^+(x,\nu)} |u(y) - a| \, \mathrm{d}y = \lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^-(x,\nu)} |u(y) - b| \, \mathrm{d}y = 0.$$

Note that the triplet (a, b, ν) is determined uniquely up to the change to $(b, a, -\nu)$. We denote it by $(u^+(x), u^-(x), \nu_u(x))$. We let J_u be the set of approximate jump points of u. Then the triplet (u^+, u^-, ν_u) can be chosen as a Borel function on the Borel set J_u. If $u \in BV(O, \mathbb{R}^N)$ it can be shown that $\mathcal{H}^{d-1}(S_u \backslash J_u) = 0$. Denoting the density of the absolutely continuous part of Du with respect to the Lebesgue measure by ∇u, we can decompose Du as

$$Du(B) = \int_B \nabla u \, \mathrm{d}x + \int_{J_u \cap B} (u^+(x) - u^-(x)) \otimes \nu_u(x) \, \mathrm{d}\mathcal{H}^{d-1} + D^c u(B),$$

where $D^c u$ is the so-called Cantor part of Du.

Now we are in a position to give a meaning to traces which we need for proving Lemma 2.17 below. Given any set E of finite perimeter in O and $u \in BV(O, \mathbb{R}^N)$, then for $\mathcal{H}^{d-1}$-almost every $x \in \mathcal{F}E \cap O$ there exist two values $u^+_{|\mathcal{F}E}(x)$ and $u^-_{|\mathcal{F}E}(x)$, called the inner and outer traces at x, such that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^+(x,\nu_E(x))} |u^+_{|\mathcal{F}E}(x) - u(y)| \, \mathrm{d}y = \lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho^-(x,\nu_E(x))} |u^-_{\mathcal{F}E}(x) - u(y)| \, \mathrm{d}y = 0.$$

One can show that $u^+_{|\mathcal{F}E}(x) = u^-_{|\mathcal{F}E}(x)$ agree with the approximate limit of u at x if $x \in (\mathcal{F}E \cap O) \backslash S_u$ and that $(u^+_{|\mathcal{F}E}(x), u^-_{|\mathcal{F}E}(x)) \in \{(u^+(x), u^-(x)), (u^-(x), u^+(x))\}$ for $\mathcal{H}^{d-1}$-almost every $x \in \mathcal{F}E \cap J_u$. While the first fact is easy to verify, the second one uses that $\nu_E(x) = \pm \nu_u(x)$ for $\mathcal{H}^{d-1}$-almost every $x \in \mathcal{F}E \cap J_u$.

From now on we assume that O is a bounded open set. Given a finite set $\mathcal{M} = \{m_1, \ldots, m_q\} \subset \mathbb{R}^N$ we define $BV(O, \mathcal{M})$ as the space of those functions $u \in BV(O, \mathbb{R}^N)$ such that $u(x) \in \mathcal{M}$ almost everywhere. As an immediate consequence of the coarea formula applied to each component of u, it follows that all level sets $E_i := \{u = m_i\}$ have finite perimeter in O. Moreover, the total variation and the surface measure of S_u are given by

$$\begin{aligned} |Du| &= \frac{1}{2} \sum_{i=1}^{q} \sum_{j \neq i} |m_i - m_j| \mathcal{H}^{d-1}(\mathcal{F}E_i \cap \mathcal{F}E_j \cap O), \\ \mathcal{H}^{d-1}(S_u) &= \frac{1}{2} \sum_{i=1}^{q} \mathcal{H}^{d-1}(\mathcal{F}E_i \cap O) = \frac{1}{2} \sum_{i=1}^{q} |D\mathbb{1}_{E_i}|. \end{aligned}$$

Remark 2.9. As the total variation is lower semicontinuous with respect to $L^1(O)$-convergence, we deduce that the functional $BV(O,\mathcal{M}) \ni u \mapsto \mathcal{H}^{d-1}(S_u)$ is lower semicontinuous with respect to convergence in $L^1(O)$, too.

In the main part of the thesis we will use a compactness result for BV-functions taking values in a finite set. Theorem 4.25 in [11] states a more general version valid only for Lipschitz domains, but in our setting it holds for any open set by interior approximation with Lipschitz domains.

Theorem 2.10. *Let $O \subset \mathbb{R}^d$ be open and let $u_k \in BV(O,\mathcal{M})$ be such that*

$$\sup_k \mathcal{H}^{d-1}(S_{u_k}) < +\infty.$$

Then there exists a subsequence u_k (not relabeled) and $u \in BV(O,\mathcal{M})$ such that $u_k \to u$ strongly in $L^1(O)$.

On the other hand we need some density results for more regular functions in order to use approximation arguments. We found that strict convergence is often not enough to ensure the continuity properties we look for. For example, when $u_n \to u$ strictly in $BV(O,\mathcal{M})$ then in general we cannot conclude that $\mathcal{H}^{d-1}(S_{u_n}) \to \mathcal{H}^{d-1}(S_u)$. In contrast the last property will be enough for our approximation purposes. The class defined below will be our prototypical class for approximations.

Definition 2.11. *A d-dimensional polyhedral set in $\mathbb{R}^d$ is a bounded open set $E \subset \mathbb{R}^d$ with Lipschitz boundary such that its boundary is contained in the union of finitely many affine hyperplanes. A function $u \in BV(O,\mathcal{M})$ is called a polyhedral function if $u = \sum_{i=1}^q m_i \mathbb{1}_{E_i}$ with pairwise disjoint sets E_i and each of them is a d-dimensional polyhedral set in $\mathbb{R}^d$.*

The following density result (more precisely a slightly different version, but the proof contains it) is proven in [12, Lemma 3.1].

Theorem 2.12. *Let $u \in BV(O,\mathcal{M})$. Then there exists a sequence $u_n \in BV(O,\mathcal{M})$ of polyhedral functions such that $u_n \to u$ in $L^1(O)$ and $\mathcal{H}^{d-1}(S_{u_n}) \to \mathcal{H}^{d-1}(S_u)$. Moreover one can assume that $\mathcal{H}^{d-1}(S_{u_n} \cap \partial O) = 0$.*

A more general approximation result can be found in the recent paper [22]. As a consequence of Lemma 2.14 the above convergence implies strict convergence.

Another important tool for the proofs contained in this thesis will be the following integral representation theorem proven in [16].

Theorem 2.13. *Let $\mathcal{F} : BV(O,\mathcal{M}) \times \mathcal{A}(O) \to [0,+\infty)$ satisfy for every $(u,A) \in BV(O,\mathcal{M}) \times \mathcal{A}(O)$ the following hypotheses:*

(i) *$\mathcal{F}(u,\cdot)$ is the restriction to $\mathcal{A}(O)$ of a Radon measure;*

(ii) *$\mathcal{F}(u,A) = \mathcal{F}(v,A)$ whenever $u = v$ almost everywhere on $A \in \mathcal{A}(O)$;*

(iii) *$\mathcal{F}(\cdot,A)$ is $L^1(O)$-lower semicontinuous;*

(iv) *there exists $c > 0$ such that*

$$\frac{1}{c}\mathcal{H}^{d-1}(S_u \cap A) \leq \mathcal{F}(u,A) \leq c\mathcal{H}^{d-1}(S_u \cap A).$$

Then for every $u \in BV(O,\mathcal{M})$ and $A \in \mathcal{A}(O)$

$$\mathcal{F}(u,A) = \int_{S_u \cap A} g(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1},$$

with

$$g(x_0,a,b,\nu) = \limsup_{\rho\to 0} \frac{m(u_{x_0,a,b,\nu}, Q_\nu(x_0,\rho))}{\rho^{d-1}},$$

where

$$u_{x_0,a,b,\nu}(x) := \begin{cases} a & \text{if } \langle x - x_0, \nu\rangle > 0, \\ b & \text{otherwise,} \end{cases}$$

and for any $(v,A) \in BV(O,\mathcal{M}) \times \mathcal{A}(O)$ we set

$$m(v,A) = \inf\{\mathcal{F}(w,A) : w \in BV(A,\mathcal{M}),\ w = v \text{ in a neighborhood of } \partial A\}.$$

We close this section with several approximation theorems for sets and a new continuity result for functionals defined on partitions. For the proof we use minimal liftings in BV as in [46] (see also [39]).

Lemma 2.14. *Let $u_n, u \in BV(O,\mathcal{M})$ be such that $u_n \to u$ in $L^1(O)$ and such that $\mathcal{H}^{d-1}(S_{u_n}) \to \mathcal{H}^{d-1}(S_u)$ and let g be bounded and continuous on $O \times \mathcal{M}^2 \times S^{d-1}$. Then*

$$\lim_n \int_{S_{u_n}\cap O} g(x,u_n^+,u_n^-,\nu_{u_n})\,\mathrm{d}\mathcal{H}^{d-1} = \int_{S_u \cap O} g(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}.$$

Proof. To reduce notation, we set $F(u) = \int_{S_u\cap O} g(x,u^+,u^-,\nu_u)$. We will just prove upper semicontinuity. The general result then follows applying upper semicontinuity to the functional $-F$. Moreover, by adding a constant which is compatible due to our assumptions we can assume that $g \geq 0$. For an arbitrary $v \in BV(O, \mathcal{M})$ we define for $|Dv|$-almost every $x \in O$ the vector measure λ_x via its action on functions $\varphi \in C_0(\mathbb{R}^N)$ by

$$\int_{\mathbb{R}^N} \varphi(y) \, \mathrm{d}\lambda_x(y) = \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x) \int_0^1 \varphi(\theta v^+(x) + (1-\theta)v^-(x)) \, \mathrm{d}\theta.$$

This is well-defined although v^+ and v^- are unique only up to permutation. To reduce notation, we write $v^\theta = \theta v^+ + (1-\theta)v^-$. Since v^+, v^- are $|Dv|$-measurable, using Fubini's theorem one can show that for any $\varphi \in C_0(O \times \mathbb{R}^N)$ the mapping

$$x \mapsto \int_{\mathbb{R}^N} \varphi(x, y) \, \mathrm{d}\lambda_x(y)$$

is $|Dv|$-measurable and essentially bounded. Hence we can define the generalized product $\mu[v] = |Dv| \otimes \lambda_x$ again by its action on $C_0(O \times \mathbb{R}^N)$ setting

$$\int_{O \times \mathbb{R}^N} \varphi(x, y) \, \mathrm{d}\mu[v](x, y) = \int_O \int_{\mathbb{R}^N} \varphi(x, y) \, \mathrm{d}\lambda_x(y) \, \mathrm{d}|Dv|(x);$$

see also Definition 2.27 in [11]. We next claim that up to a negligible set it holds that

$$\frac{\mathrm{d}\mu[v]}{\mathrm{d}|\mu[v]|}(x, y) = \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x). \tag{2.1}$$

Indeed, Corollary 2.29 in [11] yields $|\mu[v]| = |Dv| \otimes |\lambda_x|$. As the defining formula for the generalized product extends to integrable functions, we infer that

$$\begin{aligned}\int_{O \times \mathbb{R}^N} \varphi(x, y) \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x) \, \mathrm{d}|\mu[v]|(x, y) &= \int_O \int_{\mathbb{R}^N} \varphi(x, y) \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x) \, \mathrm{d}|\lambda_x|(y) \, \mathrm{d}|Dv|(x) \\ &= \int_O \int_{\mathbb{R}^N} \varphi(x, y) \, \mathrm{d}\lambda_x(y) \, \mathrm{d}|Dv|(x) = \int_{O \times \mathbb{R}^N} \varphi(x, y) \, \mathrm{d}\mu[v](x, y),\end{aligned}$$

where we have used that $\lambda_x = \frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)|\lambda_x|$. Hence (2.1) follows by uniqueness of the polar decomposition of measures. Because of (2.1) and the generalized product structure of $|\mu[v]|$, by an approximation argument it holds that

$$\begin{aligned}\int_{O\times\mathbb{R}^N} f(x,y,\frac{\mathrm{d}\mu[v]}{\mathrm{d}|\mu[v]|}(x,y))\,\mathrm{d}|\mu[v]|(x,y) &= \int_O\int_{\mathbb{R}^\mathbb{N}} f(x,y,\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x)\,\mathrm{d}|\lambda_x|(y)\,\mathrm{d}|Dv|(x)\\ &= \int_O\int_0^1 f(x,v^\theta,\frac{\mathrm{d}Dv}{\mathrm{d}|Dv|}(x))\,\mathrm{d}\theta\,\mathrm{d}|Dv|(x)\end{aligned} \tag{2.2}$$

for every nonnegative function $f \in C(O\times\mathbb{R}^N\times S^{N\times d-1})$. In [46] it was proven that if $v_n \to v$ strictly in $BV(O,\mathbb{R}^N)$, then $\mu[v_n]\to\mu[v]$ weakly* as measures and $|\mu[v_n](O\times\mathbb{R}^N)| \to |\mu[v](O\times\mathbb{R}^N)|$. The idea now is to apply the classical Reshetnyak continuity theorem (see for instance [45, 50]) with an appropriate f and a strictly converging sequence. To this end we transform the set $\mathcal{M}$ so that averages of the jump functions $u^\pm$ encode the values of the traces and such that the convergence assumptions yield strict convergence. Recall that $q = \#\mathcal{M}$. We define the mapping $T:\mathcal{M}\to\mathbb{R}^q$ via $T(m_i) = e_i$. Next we construct the function f. Given $i<j$ we consider the set

$$L_{ij} = \{\lambda T(m_i) + (1-\lambda)T(m_j):\ \lambda\in(1/4,3/4)\}.$$

Observe that by construction of the set $T(\mathcal{M})$ it holds $L_{ij}\cap L_{kl}=\emptyset$ whenever $\{i,j\}\neq\{k,l\}$. Given $\delta>0$ we next choose a cut-off function $\theta_{ij}^\delta : [T(m_i),T(m_j)]\to[0,1]$ such that $\theta_{ij}^\delta = 1$ on L_{ij} and $\theta_{ij}^\delta(x)=0$ if $\mathrm{dist}(x,L_{ij})\geq\delta$. Set $f_\delta\in C(O\times\mathbb{R}^q\times S^{q\times d-1})$ as any continuous nonnegative extension of the function

$$f_\delta(x,u,\xi) = \frac{\theta_{ij}^\delta(u)}{\sqrt{2}\mathcal{H}^1(L_{ij})}g(x,m_i,m_j,\frac{\xi^Te_1}{|\xi^Te_1|})|\xi^Te_1| \quad \text{if } u\in[T(m_i),T(m_j)].$$

First observe that this is well-defined due to the ordering $i<j$ (also in the case $\xi^Te_1 = 0$ as g is bounded). Moreover, for δ small enough such an extension exists by the properties of the cut-off function. Now for any $T(u)\in BV(O,T(\mathcal{M}))$, with a suitable orientation of the normal vector, for $|DT(u)|$-almost every $x\in O$ it holds that

$$\begin{aligned}&\frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x) = \frac{1}{\sqrt{2}}\sum_{i<j}(T(m_i)-T(m_j))\otimes\nu_u(x)\mathbb{1}_{\mathcal{F}E_i\cap\mathcal{F}E_j}(x),\\ &|DT(u)| = \sqrt{2}\sum_{i<j}\mathcal{H}^{d-1}\llcorner(\mathcal{F}E_i\cap\mathcal{F}E_j),\end{aligned}$$

where $E_i = \{u = m_i\}$. Therefore we can rewrite with a nonnegative error $\mathcal{O}(\delta)$

$$\begin{aligned}\int_O \int_0^1 & f_\delta(x, T(u)^\theta, \frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x))\,\mathrm{d}\theta\,\mathrm{d}|DT(u)|(x) \\ &= \int_O \sum_{i<j} g(x, m_i, m_j, \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\llcorner(\mathcal{F}E_i \cap \mathcal{F}E_j) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u \cap O) \\ &= F(u) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u \cap O).\end{aligned}$$

If u_n, u are as in the claim, then $T(u_n) \to T(u)$ in $L^1(O)$ and moreover $|DT(u_n)| = \sqrt{2}\mathcal{H}^{d-1}(S_{u_n} \cap O) \to \sqrt{2}\mathcal{H}^{d-1}(S_u \cap O) = |DT(u)|$, so that $T(u_n)$ converges strictly to $T(u)$. Hence we conclude from (2.2) and the classical Reshetnyak continuity theorem applied to the measures $\mu[T(u_n)], \mu[T(u)]$ that

$$\begin{aligned}\limsup_n F(u_n) &\leq \lim_n \int_O \int_0^1 f_\delta(x, T(u_n)^\theta, \frac{\mathrm{d}DT(u_n)}{\mathrm{d}|DT(u_n)|}(x))\,\mathrm{d}\theta\,\mathrm{d}|DT(u_n)|(x) \\ &= \int_O \int_0^1 f_\delta(x, T(u)^\theta, \frac{\mathrm{d}DT(u)}{\mathrm{d}|DT(u)|}(x))\,\mathrm{d}\theta\,\mathrm{d}|DT(u)|(x) \\ &\leq F(u) + \mathcal{O}(\delta)\mathcal{H}^{d-1}(S_u \cap O).\end{aligned}$$

The claim follows by the arbitrariness of δ. □

The following result on Lipschitz domains can be found in [35].

Theorem 2.15. *Let $A \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary. Given $\eta > 0$ let $A^\eta := \{x \in \mathbb{R}^d : \operatorname{dist}(x, A) < \eta\}$. Then, for η small enough, A^η is again a Lipschitz domain.*

Remark 2.16. Applying Theorem 2.15 to $B_R(0)\backslash A$ with R large enough, we obtain the same result for the set $A_\eta := \{x \in A : \operatorname{dist}(x, \partial A) > \eta\}$.

The last approximation result will be useful to treat the convergence of boundary value problems.

Lemma 2.17. *Let $A \subset\subset B$ be both bounded open sets with Lipschitz boundary. Given $v, w \in BV(A, \mathcal{M})$ such that $\mathcal{H}^{d-1}(S_w \cap \partial A) = 0$ we set $u = \mathbb{1}_A v + (1 - \mathbb{1}_A)w$. Then there exists a sequence $A_n \subset\subset A$ of sets of finite perimeter (not depending on B) such that $u_n := \mathbb{1}_{A_n} v + (1 - \mathbb{1}_{A_n})w$ converges to u in $L^1(B)$ and additionally $\mathcal{H}^{d-1}(S_{u_n} \cap B) \to \mathcal{H}^{d-1}(S_u \cap B)$.*

Proof. As in the proof of Lemma 2.14 we make use of the mapping $T : \mathcal{M} \to \mathbb{R}^q$ defined by $T(m_i) = e_i$. As a special case of Proposition 4.1 in [48], applied to the bounded BV-function $\alpha := T(w) - T(v)$, for every $\varepsilon > 0$ we find an open set A_ε of finite perimeter such that $A_\varepsilon \subset\subset A$, $|A\backslash A_\varepsilon| \leq \varepsilon$ and

$$\int_{\mathcal{F}A_\varepsilon} |\alpha^+_{|\mathcal{F}A_\varepsilon}| \,\mathrm{d}\mathcal{H}^{d-1} \leq \int_{\partial A} |\alpha^+_{|\mathcal{F}A}| \,\mathrm{d}\mathcal{H}^{d-1} + \varepsilon. \tag{2.3}$$

By refining in a trivial way the argument in [48], the sets A_ε can be constructed in a way that for all $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$

$$\{x \in A : \ \mathrm{dist}(x, \partial A) > \delta\} \subset A_\varepsilon. \tag{2.4}$$

We show that the sets A_ε fulfill the required properties. As a first step we claim that $T(u_\varepsilon)$ converges strictly to $T(u)$. It is easy to see that $T(u_\varepsilon)$ converges to $T(u)$ in $L^1(B)$. By lower semicontinuity of the total variation it is enough to show that

$$\limsup_{\varepsilon \to 0} |DT(u_\varepsilon)|(B) \leq |DT(u)|(B). \tag{2.5}$$

By definition we have $|DT(u_\varepsilon)|(B\backslash\overline{A}) = |DT(u)|(B\backslash\overline{A})$, so that we can reduce the analysis to $\overline{A}$. By Theorem 3.84 in [11] it holds that

$$DT(u_\varepsilon) = DT(v) \llcorner A_\varepsilon^{(1)} + DT(w) \llcorner A_\varepsilon^{(0)} + (T(v)^+_{|\mathcal{F}A_\varepsilon} - T(w)^-_{|\mathcal{F}A_\varepsilon}) \otimes \nu \, \mathcal{H}^{d-1} \llcorner \mathcal{F}A_\varepsilon,$$

where in general $A_\varepsilon^{(t)}$ is defined for $t \in [0,1]$ via

$$A_\varepsilon^{(t)} = \left\{ x \in \mathbb{R}^d : \ \lim_{\rho \to 0} \frac{|A_\varepsilon \cap B_\rho(x)|}{|B_\rho(x)|} = t \right\}.$$

Since $A_\varepsilon \subset\subset A$ and A_ε is open we infer $A_\varepsilon^{(1)} \subset A$ and $A_\varepsilon^{(0)} \subset \mathbb{R}^d \backslash A_\varepsilon$, so that

$$\begin{aligned}
|DT(u_\varepsilon)|(\overline{A}) \leq & |DT(v)|(A) + |DT(w)|(\overline{A}\backslash A_\varepsilon) + \int_{\mathcal{F}A_\varepsilon} |T(v)^+_{|\mathcal{F}A_\varepsilon} - T(w)^-_{|\mathcal{F}A_\varepsilon}| \,\mathrm{d}\mathcal{H}^{d-1} \\
\leq & |DT(v)|(A) + |DT(w)|(\overline{A}\backslash A_\varepsilon) + \int_{\mathcal{F}A_\varepsilon} |T(w)^+_{|\mathcal{F}A_\varepsilon} - T(w)^-_{|\mathcal{F}A_\varepsilon}| \,\mathrm{d}\mathcal{H}^{d-1} \\
& + \int_{\mathcal{F}A_\varepsilon} |T(v)^+_{|\mathcal{F}A_\varepsilon} - T(w)^+_{|\mathcal{F}A_\varepsilon}| \,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

By assumption on w we have $\mathcal{H}^{d-1}(S_w \cap \partial A) = 0$, so that by (2.4) the second and the third term vanish when $\varepsilon \to 0$. For the fourth one we use (2.3) and infer

$$\begin{aligned}\limsup_{\varepsilon\to 0} |DT(u_\varepsilon)|(\overline{A}) &\leq |DT(v)|(A) + \int_{\partial A} |T(v)^+_{|\mathcal{F}A} - T(w)^+_{|\mathcal{F}A}| \,\mathrm{d}\mathcal{H}^{d-1} \\ &= |DT(v)|(A) + \int_{\partial A} |T(v)^+_{|\mathcal{F}A} - T(w)^-_{|\mathcal{F}A}| \,\mathrm{d}\mathcal{H}^{d-1} \\ &= |DT(u)|(\overline{A}),\end{aligned}$$

where we used that inner and outer trace of $T(w)$ agree for $\mathcal{H}^{d-1}$-almost every $x \in \partial A$. By the structure of $T(\mathcal{M})$ strict convergence implies that

$$\mathcal{H}^{d-1}(S_{T(u_\varepsilon)} \cap B) = \frac{1}{\sqrt{2}}|DT(u_\varepsilon)| \to \frac{1}{\sqrt{2}}|DT(u)| = \mathcal{H}^{d-1}(S_{T(u)} \cap B).$$

As for every $u \in BV(O, \mathcal{M})$ it holds that $\mathcal{H}^{d-1}(S_u \cap O) = \mathcal{H}^{d-1}(S_{T(u)} \cap O)$ and L^1-convergence is trivial, we conclude the proof. □

2.3 Probabilistic ergodic theory

In this section we provide a short introduction to the probabilistic framework we will use in this thesis. We denote by the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ an arbitrary probability space Ω with σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$. We always assume that $\mathcal{F}$ is complete.

Definition 2.18. *Let $m \in \mathbb{N}$. We say that a family $\{\tau_z\}_{z\in\mathbb{Z}^m}, \tau_z : \Omega \to \Omega$, is an additive group action on Ω if*

$$\tau_{z_1+z_2} = \tau_{z_2} \circ \tau_{z_1} \quad \forall\, z_1, z_2 \in \mathbb{Z}^m.$$

Such an additive group action is called measure preserving if

$$\mathbb{P}(\tau_z B) = \mathbb{P}(B) \quad \forall\, B \in \mathcal{F},\, z \in \mathbb{Z}^m.$$

Moreover $(\tau_z)_{z\in\mathbb{Z}^m}$ is called ergodic if, in addition, for all $B \in \mathcal{F}$ we have

$$(\tau_z(B) = B \quad \forall\, z \in \mathbb{Z}^m) \quad \Rightarrow \quad \mathbb{P}(B) \in \{0, 1\}.$$

For $m \in \mathbb{N}$ we further set $\mathcal{I}_m = \{[a, b) : a, b \in \mathbb{Z}^m,\, a \neq b\}$, where $[a, b) := \{x \in \mathbb{R}^m : a_i \leq x_i < b_i \;\forall\, i\}$ denotes a m-dimensional half-open interval. Next we introduce the notion of regular families and discrete subadditive stochastic processes:

Definition 2.19. *Let $\{I_n\}_n \subset \mathcal{I}_m$ be a family of sets. Then $\{I_n\}_n$ is called regular if there exists another family $\{I'_n\}_n \subset \mathcal{I}_m$ and a constant $C > 0$ such that*

(i) $I_n \subset I'_n \quad \forall n$,

(ii) $I'_n \subset I'_m \quad$ *whenever* $n < m$,

(iii) $0 < \mathcal{H}^m(I'_n) \leq C\,\mathcal{H}^m(I_n) \quad \forall n$.

Moreover, if $\{I'_n\}$ can be chosen such that $\mathbb{R}^m = \bigcup_n I'_n$, then we write $\lim_{n\to\infty} I_n = \mathbb{R}^m$.

Definition 2.20. *A function $\mu : \mathcal{I}_m \to L^1(\Omega)$ is said to be a discrete subadditive stochastic process if the following properties hold $\mathbb{P}$-almost surely:*

(i) for every $I \in \mathcal{I}_m$ and for every finite partition $(I_j)_{j\in J} \subset \mathcal{I}_m$ of I we have

$$\mu(I, \omega) \leq \sum_{j\in J} \mu(I_j, \omega).$$

(ii) $\inf\left\{\frac{1}{\mathcal{H}^m(I)}\int_\Omega \mu(I, w)\,\mathrm{d}\mathbb{P}(\omega) : \ I \in \mathcal{I}_m\right\} > -\infty$.

One of the key ingredients to prove the stochastic homogenization results contained in this thesis will be the following pointwise ergodic theorem (see Theorem 2.7 in [2]).

Theorem 2.21. *Let $\mu : \mathcal{I}_m \to L^1(\Omega)$ be a discrete subadditive stochastic process and let $\{I_n\}_n$ be a regular family in $\mathcal{I}_m$ such that $\lim_n I_n = \mathbb{R}^m$. If μ is stationary with respect to a measure preserving group action $(\tau_z)_{z\in\mathbb{Z}^m}$, that means*

$$\forall\, I \in \mathcal{I}_m,\ \forall\, z \in \mathbb{Z}^m : \quad \mu(I + z, \omega) = \mu(I, \tau_z\omega) \quad \textit{almost surely},$$

then there exists $\Phi : \Omega \to \mathbb{R}$ such that, for $\mathbb{P}$-almost every ω,

$$\lim_{n\to+\infty} \frac{\mu(I_n, \omega)}{\mathcal{H}^m(I_n)} = \Phi(\omega).$$

Another important tool will be the Birkhoff ergodic theorem which is a special case of Theorem 2.21 for which we can say more about the structure of the limit. First let us recall the definition of the conditional expectation of a random variable $X : \Omega \to \mathbb{R}$. Let $\mathcal{F}_0 \subset \mathcal{F}$ be another σ-algebra. Then (a version of) the conditional expectation $\mathbb{E}[X|\mathcal{F}_0] : \Omega \to \mathbb{R}$ is any $\mathcal{F}_0$-measurable function such that, for all $B \in \mathcal{F}_0$, it holds

$$\int_B X\,\mathrm{d}\mathbb{P} = \int_B \mathbb{E}[X|\mathcal{F}_0]\,\mathrm{d}\mathbb{P}.$$

Due to the Radon-Nikodym theorem the conditional expectation always exists for $X \in L^1(\Omega)$ and it is unique up to null sets. For further properties we refer to Theorem 8.14 in [42]. Using the conditional expectation, Birkhoff's ergodic theorem (see for instance Theorem 2.3 in [43]) reads as follows:

Theorem 2.22. *Let $X \in L^1(\Omega)$ and $\tau : \Omega \to \Omega$ be a measure preserving map. Denoting by $\mathcal{F}_\tau$ the σ-algebra of τ-invariant sets, then, $\mathbb{P}$-almost surely, it holds that*

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} X(\tau^k \omega) = \mathbb{E}[X|\mathcal{F}_\tau](\omega).$$

In Chapter 5 we will need a quantitative version of Birkhoff's ergodic theorem. Therefore we need the notion of α-mixing sequences.

Definition 2.23. *Given a sequence of random variables $X_j : \Omega \to \mathbb{R}$ and a set of indices $I \subset \mathbb{N}$ we define $\mathcal{F}_I = \sigma(X_j : j \in I)$ as the σ-algebra generated by the random variables $(X_j)_{j\in I}$. The sequence X_j is said to be α-mixing if there exists a sequence $\alpha(n) \to 0$ such that for all sets $I_1, I_2 \subset \mathbb{N}$ with $\operatorname{dist}(I_1, I_2) \geq n$ it holds that*

$$\sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_{I_1},\, B \in \mathcal{F}_{I_2}\} \leq \alpha(n).$$

Similar to independent random variables (which are α-mixing with $\alpha(n) = 0$), α-mixing allows for quantitative estimates for the error probabilities in the law of large numbers. We will need the following polynomial decay theorem for bounded α-mixing sequences, proved by Berbee in [14].

Theorem 2.24. *Let $p > 1$ and X_j be an α-mixing sequence of random variables bounded by 1 such that $\mathbb{E}[X_j] = 0$ for all j. If*

$$\sum_{n\geq 1} n^{p-2}\alpha(n) < +\infty,$$

then, for all $\delta > 0$,

$$\sum_{n\geq 1} n^{p-2}\mathbb{P}\left(\sup_{k\geq n} |S_k/k| > \delta\right) < +\infty,$$

where $S_k = \sum_{j=1}^{k} X_j$.

Chapter 3

Domain formation in magnetic polymer composites

This chapter slightly differs from the results we published in [5]. To be more precise, we extend them to spin systems taking finitely many values instead of just ± 1. Moreover we include the analysis of phase constraints and finally provide an interpretation in the spirit of statistical mechanics. What follows can be seen as the starting point of a rigorous mathematical study of the discrete-to-continuum variational description of magnetic polymer composites (see Chapter 1), focusing on their magnetic properties. Our aim is to prove that, modeling the interaction of the magnetic particles via a classical Potts model (see [54]) on a disordered lattice, their (surface scaled) microscopic interaction energy leads to the formation of Weiss domains as the average distance between the particles vanishes. As explained below in this introduction, in order to tackle this problem we regard it as a stochastic homogenization problem in the space of functions of bounded variation in $\mathbb{R}^d$ where we are able to extend some of the results obtained in the Sobolev setting in the pioneering paper [33].

The modeling of magnetic polymer composite materials at a small (micro or nano) scales requires the modeling of two main objects: a polymer matrix containing the magnetic particles and an interaction energy between those particles (see [53] and reference therein for a beginner's guide to this topics).

The polymer matrix The particles embedded in the polymer matrix can be modeled as a random network having the cross-linked molecules as nodes. As motivated in the introduction, we will suppose the nodes of the network to satisfy some minimal geometric assumption uniformly in the randomness. More precisely we will suppose the set of the nodes of the network to form what we call an *admissible stochastic lattice* according to the definition below.

Definition 3.1. *Let $\mathcal{L} \subset \mathbb{R}^d$ be a countable set of points. $\mathcal{L}$ is called admissible if*

(i) there exists $R > 0$ such that $\operatorname{dist}(x, \mathcal{L}) < R$ for all $x \in \mathbb{R}^d$;

(ii) there exists $r > 0$ such that $|x - y| \geq r$ for all $x, y \in \mathcal{L}, \quad x \neq y$.

Roughly speaking the assumptions rule out cluster points as well as arbitrary big holes in the network. Then, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$ is called an *admissible stochastic lattice* if $\mathcal{L}(\omega)$ is admissible with constants r, R uniformly with respect to a set of full probability. Note that our assumptions on the admissibility of a stochastic lattice rule out many point processes well known in probability theory and are instead motivated by the structural assumptions explained in the introduction.

The magnetic energy To every stochastic lattice $\mathcal{L}(\omega)$ we associate a Voronoi tessellation $\mathcal{V}(\mathcal{L}(\omega))$. Given this tessellation, we define the set of nearest neighboring points, namely $\mathcal{NN}(\omega)$, as the set of those pairs of points of the stochastic lattice $\mathcal{L}(\omega)$ which share a $(d-1)$-dimensional edge of the associated Voronoi tessellation. Let $D \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and $\varepsilon > 0$ be a small parameter (the limit $\varepsilon \to 0$ will be referred to as the continuum limit). We assume that the magnetic state of the particles in D is described by a classical spin variable $u : \varepsilon\mathcal{L}(\omega) \cap D \to \mathcal{M}$, where $\mathcal{M} = \{m_1, \ldots, m_q\} \subset \mathbb{R}^N$ with $q \geq 2$. We model the interactions between the spins via an pairwise interaction energy. The energy model we consider allows all the particles to interact and may distinguish between short-range interactions, which are the interactions between the nearest-neighbor particles, and long-range interactions. The total energy of the system for a given configuration u has the form

$$E_\varepsilon(\omega)(u, D) = \sum_{\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}(\omega) \cap D} \varepsilon^{d-1} g_\varepsilon^\omega(x, y, u(\varepsilon x), u(\varepsilon y)).$$

The randomness in the interactions $g_\varepsilon^\omega : \mathbb{R}^{2d} \times \mathcal{M}^2 \to [0, +\infty)$ consists in the distinction between neighboring particles and long-range interactions, that means the functions are of the type

$$g_\varepsilon^\omega(x, y, m_i, m_j) = \begin{cases} g_{\varepsilon,nn}(x, y, m_i, m_j) & \text{if } (x, y) \in \mathcal{NN}(\omega), \\ g_{\varepsilon,lr}(x, y, m_i, m_j) & \text{otherwise.} \end{cases}$$

For our analysis we assume that there exist $c > 0$ and a decreasing function J_{lr} :

$[0, +\infty) \to [0, +\infty)$ with

$$\int_{\mathbb{R}^d} J_{lr}(|x|)|x| \,\mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$, all $x, y \in \mathbb{R}^d$ and all $m_i, m_j \in \mathcal{M}$

$$\begin{aligned} c|m_i - m_j| \leq & g_{\varepsilon,nn}(x, y, m_i, m_j) \leq J_{lr}(|x-y|)|m_i - m_j|, \\ 0 \leq & g_{\varepsilon,lr}(x, y, m_i, m_j) \leq J_{lr}(|x-y|)|m_i - m_j|. \end{aligned}$$

As the average distance between the nodes of the network $\varepsilon\mathcal{L}(\omega)$ is of order ε, the prefactor ε^{d-1} in the energy has the meaning of a surface scaling, so that $E_\varepsilon(\omega)(u)$ is the magnetic energy per unit surface of the network $\varepsilon\mathcal{L}(\omega) \cap D$ when the magnetization field is u. Taking into account the assumptions above, the atomic system we consider is the surface scaling of a ferromagnetic type system with bounded short-range and summable long-range interactions.

The continuum energy In the limit as ε tends to 0 the ferromagnetic behavior of the system will favor the formation of a partition of D into random (ω-dependent) Weiss domains described, in the continuum limit, as sets of finite perimeter with fixed magnetization. The interaction energy between the Weiss domains will depend on the randomness via the stochasticity of the polymer matrix in which the magnetic particles are embedded. The issue of the dependence of the macroscopic continuum energy of the domains on the randomness of the matrix is tackled in the framework of stochastic homogenization as explained below. In this context, as a byproduct of our analysis, one could see our main result as a generalization of a recent theorem by Braides and Piatnitski in [25] (see also Remark 3.20).
We work in the variational framework of Γ-convergence (see Section 2.1). To this end we identify the field u with its piecewise-constant interpolation taking the value $u(x)$ on the Voronoi cell centered at x and we regard the energies as defined on $L^1(D, \mathcal{M})$. The Γ-limit is performed in this space. In Theorem 3.10, we prove that, for fixed $\omega \in \Omega$, up to subsequences, the family $E_\varepsilon(\omega)$ Γ-converges with respect to the $L^1(D)$-topology to a continuum energy $E(\omega) : L^1(D) \to [0, +\infty]$ which is finite only on $BV(D, \mathcal{M})$ where it takes the form

$$E(\omega)(u) = \int_{S_u \cap D} \phi(\omega; x, u^+, u^-, \nu_u) \,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.1}$$

For the notation used in (3.1) we refer to Section 2.2. The result is proved via the abstract methods of Gamma-convergence and makes use of the integral representation results contained in Theorem 2.13. We explore the dependence of the continuum energy on the randomness induced by the stochastic lattice in Theorem 3.19. Here we assume that the stochastic lattice is stationary, that is, for all $z \in \mathbb{Z}^d$, $\mathcal{L}(\omega)$ and $\mathcal{L}(\omega) + z$ have the same statistics and that the interactions have a special structure in the sense that there exist two functions $g_{nn}, g_{lr} : \mathbb{R}^d \times \mathcal{M}^2 \to [0, +\infty)$ such that

$$\begin{aligned} g_{\varepsilon,nn}(x, y, m_i, m_j) &= g_{nn}(y - x, m_i, m_j), \\ g_{\varepsilon,lr}(x, y, m_i, m_j) &= g_{lr}(y - x, m_i, m_j). \end{aligned} \tag{3.2}$$

These assumptions, which play the same role as periodicity in the case of a deterministic periodic lattice treated in [6], turn the problem of the characterization of the continuum limit energy into a stochastic homogenization problem.

In Theorem 3.19 we prove that the functionals $E_\varepsilon(\omega)$ Γ-converge with respect to the $L^1(D)$-topology to the functional $E_{\text{hom}}(\omega) : L^1(D) \to [0, +\infty]$ which is finite on $BV(D, \mathcal{M})$ where it takes the form

$$E_{\text{hom}}(\omega)(u) = \int_{S_u \cap D} \phi_{\text{hom}}(\omega; u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1}.$$

Here, for $\mathbb{P}$-almost every ω and for all $m_i, m_j \in \mathcal{M}$, $\nu \in S^{d-1}$, the value of the function $\phi_{\text{hom}}(\omega; m_i, m_j, \nu)$ is given by an asymptotic homogenization formula. If $\mathcal{L}$ is ergodic the limit energy is deterministic and its energy density $\phi_{\text{hom}}(m_i, m_j, \nu)$ is obtained by averaging the previous homogenization formula over the probability space:

$$\phi_{\text{hom}}(m_i, m_j, \nu) = \int_\Omega \phi_{\text{hom}}(\omega; m_i, m_j, \nu_u) \, \mathrm{d}\mathbb{P}(\omega).$$

The proof of this result is quite delicate and makes use of two main ingredients: the abstract methods of Γ-convergence and the subadditive ergodic theorem by Akcoglu and Krengel in [2]. The combination of these two results in the framework of discrete-to-continuum limits was one of the key ideas in the proof of the main result in [4] drawing some ideas from the pioneering paper [33]. It consists in proving that the sequence of minimum problems characterizing the energy density of the Γ-limit at a certain point and in a given direction agrees (up to lower order terms) with a sequence of subadditive stochastic processes for which the main result in [2] applies. It is at this point that one strongly uses the assumptions on the stationarity of the lattice together with (3.2). This step of the proof is the most delicate one and cannot be solved by the same arguments as in the Sobolev case considered in [4]. Instead it requires new arguments and the generalization to higher dimensions of the translation invariance of the first passage percolation formula obtained in [25, Proposition 2.10].

A further important issue in the theory of magnetic polymer composite materials is the dependence of the macroscopic energy on the random geometry of the network. We consider this problem in Section 3.5 where we remark that if the stochastic lattice, besides satisfying the previous assumptions, is also isotropic in law, that is to say that $\mathcal{L}(\omega)$ and $R\mathcal{L}(\omega)$ have the same statistics for all $R \in SO(d)$, and the functions g_{nn} and g_{lr} are functions of the distances between points, then the limit energy density is isotropic, which means that $\phi_{\rm hom}(m_i, m_j, \nu) = \phi_{\rm hom}(m_i, m_j)$. An example of stochastic lattice sharing this isotropy in law is the random parking process studied from the point of view of homogenization theory in [38].

This chapter is organized as follows: Section 3.1 is devoted to basic notation, the definition of the class of energies we consider and to preliminary results regarding our functional setting. In Section 3.2 we prove a compactness and integral representation result for our functionals for a fixed realization of the random lattice. In Section 3.3 we prove the main result of this chapter that is the Γ-convergence of random discrete energies. Section 3.4 deals with the Γ-convergence of the discrete energies in presence of boundary conditions. Section 3.5 is devoted to applications and generalizations of the previous results to phase constraints. In the final section we show how the continuum limit energy can be connected to large deviation principles for Gibbs measures when the temperature vanishes at a certain rate.

3.1 The model and preliminary results

In this section we give a precise definition of the energies we consider and collect some preliminary results.

To start with, let us introduce the functional-analytic and probabilistic framework.

Definition 3.2. *A random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$, $\omega \mapsto \mathcal{L}(\omega) = \{\mathcal{L}(\omega)(i)\}_{i\in\mathbb{N}}$ is called a stochastic lattice. We say that $\mathcal{L}$ is admissible if $\mathcal{L}(\omega)$ is admissible in the sense of Definition 3.1 and the constants r, R can be chosen independent of ω $\mathbb{P}$-almost surely. The stochastic lattice $\mathcal{L}$ is said to be stationary if there exists a measure preserving group action $(\tau_z)_{z\in\mathbb{Z}^d}$ on Ω such that, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\mathcal{L}(\tau_z\omega) = \mathcal{L}(\omega) + z.$$

If in addition $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $\mathcal{L}$ is called ergodic, too.

Definition 3.3. *Let $\mathcal{L}$ be a stochastic lattice. We denote by $\mathcal{V}(\omega)$ the Voronoi tessel-*

lation of $\mathbb{R}^d$ *associated with* $\mathcal{L}(\omega)$*, that is* $\mathcal{V}(\omega) := \{\mathcal{C}(x)\}_{x\in\mathcal{L}(\omega)}$*, where*

$$\mathcal{C}(x) := \{z \in \mathbb{R}^d : \ |z-x| \leq |z-y| \ \forall y \in \mathcal{L}(\omega)\}.$$

The next lemma contains all the information on the Voronoi cells that we will need throughout this thesis. We outline its simple proof for readers' convenience.

Lemma 3.4. *Let* $\mathcal{L}(\omega)$ *be an admissible set of points with constants* r, R *as in Definition 3.1. Then there exist constants* $C > 0$ *depending only on* r, R *such that, for all* $x \in \mathcal{L}$*,*

(i) $B_{\frac{r}{2}}(x) \subset \mathcal{C}(x) \subset B_R(x)$*,*

(ii) $\#\{y \in \mathcal{L}(\omega) : \ \mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset\} \leq C$*,*

(iii) $\mathcal{H}^{d-1}(\mathcal{C}(x) \cap \mathcal{C}(y)) \leq C \quad \forall y \in \mathcal{L}(\omega)\backslash\{x\}$*.*

Proof. (i) For $y \in \mathcal{L}(\omega)\backslash\{x\}$ we have $|x-y| \geq r$, which implies $|z-x| \leq |z-y|$ for all $z \in B_{\frac{r}{2}}(x)$. By definition the first inclusion in (i) holds. Now suppose that there exists $z \in \mathcal{C}(x)$ such that $|z-x| \geq R$. Since $\mathcal{L}(\omega)$ is admissible, there exists $y \in \mathcal{L}(\omega)$ such that $|z-y| < R$. It follows that $R \leq |z-x| \leq |z-y| < R$, leading to a contradiction.
(ii) Note that (i) implies that if $\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset$, then $|x-y| \leq 2R$. Using an elementary covering argument it is now easy to see that it suffices to take $C = \left(1 + \frac{4R}{r}\right)^d$.
(iii) By (i) the diameter of the set $\mathcal{C}(x)\cap\mathcal{C}(y)$ is bounded by $2R$ and the set is contained in a $(d-1)$-dimensional affine subspace so that we can take $C = (2R)^{d-1}\omega_{d-1}$, where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$. □

Let $D \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $\mathcal{L}$ be an admissible set of points according to Definition 3.1. Making use of the Voronoi tessellation we introduce the notion of nearest neighbors.

Definition 3.5. *The set of nearest neighbors of* $\mathcal{L}(\omega)$ *is defined by*

$$\mathcal{NN}(\omega) := \{(x,y) \in \mathcal{L}(\omega)^2 : \ \dim_{\mathcal{H}}(\mathcal{C}(x) \cap \mathcal{C}(y)) = d-1\}.$$

For technical reasons we will need the following measurability property of the nearest neighbors relationship.

Lemma 3.6. *Let $\mathcal{L}$ be an admissible stochastic lattice. Then for any $i, j \in \mathbb{N}$, $i \neq j$, the function*

$$\omega \mapsto \mathcal{N}_{ij}(\omega) := \begin{cases} 1 & \text{if } (\mathcal{L}(\omega)_i, \mathcal{L}(\omega)_j) \in \mathcal{NN}(\omega), \\ 0 & \text{otherwise} \end{cases}$$

is $\mathcal{F}$-measurable.

Proof. For $0 < r < R$, we denote by $\Sigma_{r,R}$ the space of all admissible sets of points with corresponding constants r, R. Since $\mathcal{F}$ is a complete σ-algebra, we can assume that $\mathcal{L}(\omega) \in \Sigma_{r,R}$ for all $\omega \in \Omega$. Given $i, j \in \mathbb{N}$, we prove that the set of all $\mathcal{L} = (\mathcal{L}_i)_{i\in\mathbb{N}} \in \Sigma_{r,R}$ such that $\mathcal{L}_i$ and $\mathcal{L}_j$ are nearest neighbors is measurable. Note that $\mathcal{L}_i$ and $\mathcal{L}_j$ are nearest neighbors if and only if

$$\exists x \in \mathbb{R}^d : \quad |x - \mathcal{L}_i| = |x - \mathcal{L}_j| < |x - \mathcal{L}_k| \quad \forall k \neq i, j.$$

Let us take a countable collection $\{B_n\}_n$ of connected sets that form a basis of the norm topology in $\mathbb{R}^d$. Using the fact that $\mathcal{L} \in \Sigma_{r,R}$ on the one hand and the intermediate value theorem on the other hand one can check that the above characterization is equivalent to

$$\mathcal{L} \in \bigcup_{n\in\mathbb{N}} \Big(\{Y \in \Sigma_{r,R} : \sup_{v\in B_n} |Y_i - v| - |Y_j - v| \geq 0, \inf_{v\in B_n} |Y_i - v| - |Y_j - v| \leq 0\} \\ \cap \bigcap_{k\in\mathbb{N}\setminus\{i,j\}} \{Y \in \Sigma_{r,R} : \sup_{v\in B_n} |Y_i - v| - |Y_k - v| < 0\} \Big)$$

The last set is a countable union of product-measurable sets, whence measurable. The claim follows by measurability of the stochastic lattice. □

We are now ready to introduce the most general class of discrete energies we are going to consider in this chapter. For fixed $\varepsilon > 0$ and $u : \varepsilon\mathcal{L}(\omega) \to \mathcal{M}$, we set

$$E_\varepsilon(\omega)(u) := E_\varepsilon(\omega)(u, D),$$

where for every $A \in \mathcal{A}(\mathbb{R}^d)$ we define the localized energy

$$E_\varepsilon(\omega)(u, A) = \sum_{\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}(\omega)\cap A} \varepsilon^{d-1} g_\varepsilon^\omega(x, y, u(\varepsilon x), u(\varepsilon y)).$$

We assume that the functions $g^\omega_\varepsilon : \mathbb{R}^{2d} \times \mathcal{M}^2 \to [0, +\infty)$ are of the type

$$g^\omega_\varepsilon(x, y, m_i, m_j) = \begin{cases} g_{\varepsilon,nn}(x, y, m_i, m_j) & \text{if } (x, y) \in \mathcal{NN}(\omega), \\ g_{\varepsilon,lr}(x, y, m_i, m_j) & \text{otherwise} \end{cases}$$

with functions $g^\varepsilon_{nn}, g^\varepsilon_{lr} : \mathbb{R}^{2d} \times \mathcal{M}^2 \to [0, +\infty)$ fulfilling the following conditions:

Hypothesis 1 There exist $c > 0$ and a decreasing function $J_{lr} : [0, +\infty) \to [0, +\infty)$ with

$$\int_{\mathbb{R}^d} J_{lr}(|x|)|x| \,\mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$, all $x, y \in \mathbb{R}^d$ and all $m_i, m_j \in \mathcal{M}$

$$\begin{aligned} c|m_i - m_j| \leq & g^\varepsilon_{nn}(x, y, m_i, m_j) \leq J_{lr}(|x - y|)|m_i - m_j|, \\ 0 \leq & g^\varepsilon_{lr}(x, y, m_i, m_j) \leq J_{lr}(|x - y|)|m_i - m_j|. \end{aligned}$$

As it is customary in the context of the discrete-to-continuum variational limit, with the aim of exploiting Γ-convergence, we identify each function $u : \varepsilon\mathcal{L}(\omega) \to \mathcal{M}$ with its constant interpolation on each scaled Voronoi cell. Setting

$$\mathcal{PC}_\varepsilon(\omega) := \{u : \mathbb{R}^d \to \mathcal{M} : \ \forall \mathcal{C} \in \mathcal{V}(\omega),\ u_{|\varepsilon\mathcal{C}} \text{ is constant}\} \subset L^1(D), \tag{3.3}$$

we can consider the functionals $E_\varepsilon(\omega) : L^1(D) \to [0, +\infty]$ defined as

$$E_\varepsilon(\omega)(u) := \begin{cases} E_\varepsilon(\omega)(u, D) & \text{if } u \in \mathcal{PC}_\varepsilon(\omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.4}$$

Since we want to apply the abstract methods of Γ-convergence we also need to define local versions of the energies $E_\varepsilon(\omega)$ and of its $\Gamma\text{-}\liminf$ and $\Gamma\text{-}\limsup$ as $\varepsilon \to 0$.

Definition 3.7. *For $A \in \mathcal{A}^R(\mathbb{R}^d)$, let $E_\varepsilon(\omega)(\cdot, A) : L^1(D) \to [0, +\infty]$ be defined by*

$$E_\varepsilon(\omega)(u, A) = \begin{cases} \sum_{\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}(\omega) \cap A} \varepsilon^{d-1} g^\omega_\varepsilon(x, y, u(\varepsilon x), u(\varepsilon y)) & \textit{if } u \in \mathcal{PC}_\varepsilon(\omega), \\ +\infty & \textit{otherwise.} \end{cases}$$

Furthermore we set

$$E'(\omega)(u,A) := \Gamma(L^1(D)) - \liminf_{\varepsilon\to 0} E_\varepsilon(\omega)(u,A),$$
$$E''(\omega)(u,A) := \Gamma(L^1(D)) - \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u,A).$$

Remark 3.8. One can show that if $\mathcal{L}(\omega)$ is admissible, then for $A \in \mathcal{A}^R(D)$ it holds

$$E'(\omega)(u,A) = \Gamma(L^1(A)) - \liminf_{\varepsilon\to 0} E_\varepsilon(\omega)(u,A),$$
$$E''(\omega)(u,A) = \Gamma(L^1(A)) - \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u,A)$$

for every $u \in L^1(D)$. Thus both functionals are also $L^1(A)$-lower semicontinuous which implies that they are local in the sense of Theorem 2.13 (ii).

Following some ideas in [4] we introduce an auxiliary deterministic square lattice on which we will conveniently rewrite the energies E_ε. This lattice will turn out to be a convenient way in order to provide uniform (with respect to the stochastic variable) estimates on the discrete energies.

Setting $r' = \frac{r}{\sqrt{d}}$ it follows that for all $\alpha \in r'\mathbb{Z}^d$ it holds $\#\{\mathcal{L}(\omega)\cap\{\alpha+[0,r')^d\}\} \leq 1$. We now set

$$\begin{aligned} \mathcal{Z}_{r'}(\omega) :=& \{\alpha \in r'\mathbb{Z}^d :\ \#\left(\mathcal{L}(\omega)\cap\{\alpha+[0,r')^d\}\right) = 1\},\\ x_\alpha :=& \mathcal{L}(\omega)\cap\{\alpha+[0,r')^d\}, \quad \alpha \in \mathcal{Z}_{r'}(\omega) \end{aligned} \tag{3.5}$$

and, for $\xi \in r'\mathbb{Z}^d$, $U \subset \mathbb{R}^d$ and $\varepsilon > 0$,

$$R_\varepsilon^\xi(U) := \{\alpha :\ \alpha, \alpha+\xi \in \mathcal{Z}_{r'}(\omega),\ \varepsilon x_\alpha, \varepsilon x_{\alpha+\xi} \in U\}.$$

We can then rewrite the localized energy as

$$E_\varepsilon(\omega)(u,A) = \sum_{\xi\in r'\mathbb{Z}^d}\ \sum_{\alpha\in R_\varepsilon^\xi(A)} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})).$$

Remark 3.9. The monotonicity of the function J_{lr} in Hypothesis 1 allows to transfer the decay properties to the discrete environment as follows: For every $\delta > 0$ there exists L_δ such that

$$\sum_{|\xi|>L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \leq \delta,$$

where $\hat{\xi} \in \xi + [-r', r']^d$ is such that $|\hat{\xi}| = \operatorname{dist}([0,r')^d, [0,r')^d + \xi)$. Note that with this definition it holds that $J_{lr}(|x_\alpha - x_{\alpha+\xi}|) \leq J_{lr}(|\hat{\xi}|)$.

3.2 Integral representation

We want to make use of Theorem 2.13 to identify the structure of possible Γ-limits. The following theorem is the main result of this section.

Theorem 3.10. *Let $\mathcal{L}(\omega)$ be admissible and assume Hypothesis 1. For every sequence $\varepsilon \to 0^+$ there exists a subsequence ε_n such that the functionals $E_{\varepsilon_n}(\omega)$ defined in* (3.4) *Γ-converge with respect to the strong $L^1(D)$-topology to a functional $E(\omega) : L^1(D) \to [0, +\infty]$ of the form*

$$E(\omega)(u) = \begin{cases} \int_{S_u} \phi(\omega; x, u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in BV(D, \mathcal{M}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover a local version of the statement above holds: For all $u \in BV(D, \mathcal{M})$ and all $A \in \mathcal{A}^R(D)$

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(\omega)(u, A) = \int_{S_u \cap A} \phi(\omega; x, u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1}.$$

Note that no stationarity is needed for the above result. The proof of Theorem 3.10 will be given later. At first we prove several propositions that allow us to apply Theorem 2.13. The next two propositions ensure that the limit energy is finite only for $u \in BV(D, \mathcal{M})$.

Proposition 3.11. *Let $\mathcal{L}(\omega)$ be admissible and assume Hypothesis 1. If $A \in \mathcal{A}(D)$ and $u \in L^1(D)$ are such that $E'(\omega)(u, A) < +\infty$, then $u \in BV(A, \mathcal{M})$ and*

$$E'(\omega)(u, A) \geq c \, \mathcal{H}^{d-1}(S_u \cap A)$$

for some positive deterministic constant c independent of A and u.

Proof. Let $\mathcal{PC}_\varepsilon(\omega) \ni u_\varepsilon \to u$ in $L^1(D)$ be such that $\liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) < +\infty$. Given $\eta > 0$, we set $A_\eta = \{x \in A : \ \mathrm{dist}(x, \partial A) > \eta\}$. Note that, for ε small enough,

$$S_{u_\varepsilon} \cap A_\eta \subset \bigcup_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in A \\ u_\varepsilon(\varepsilon x) \neq u_\varepsilon(\varepsilon y)}} \varepsilon \left(\mathcal{C}(x) \cap \mathcal{C}(y) \right),$$

so that, by Lemma 3.4 and Hypothesis 1, we have the estimate

$$\mathcal{H}^{d-1}(S_{u_\varepsilon} \cap A_\eta) \leq C \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in A}} \varepsilon^{d-1} g_{\varepsilon,nn}(x, y, u(\varepsilon x), u(\varepsilon y)) \leq C E_\varepsilon(\omega)(u_\varepsilon, A).$$

Theorem 2.10 now implies that $u \in BV(A_\eta, \mathcal{M})$ and, since the bound on the measure of the jump set is uniform in η, we conclude that $u \in BV(A, \mathcal{M})$. For $\varepsilon \to 0$ we get $E'(\omega)(u, A) \geq c\,\mathcal{H}^{d-1}(S_u \cap A_\eta)$ by lower semicontinuity (see Remark 2.9). Letting $\eta \to 0$ yields the claim. □

Before we derive the necessary upper bound for Theorem 2.13, we prove an auxiliary lemma that asserts that on convex domains we can essentially control the long-range interactions by considering only neighboring particles.

Lemma 3.12. *Let $\mathcal{L}(\omega)$ be admissible and let $K \subset \mathcal{A}(\mathbb{R}^d)$ be convex. Set $K^{3R\varepsilon} = \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) < 3R\varepsilon\}$. Then there exists a constant C depending only on r, R in Definition 3.1 such that for every $\xi \in r'\mathbb{Z}^d$ and every $u \in \mathcal{PC}_\varepsilon(\omega)$ it holds*

$$\sum_{\alpha \in R_\varepsilon^\xi(K)} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})) \leq C J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in K^{3R\varepsilon}}} g_\varepsilon^\omega(x, y, u(\varepsilon x), u(\varepsilon y)),$$

where $\hat{\xi}$ is defined in Remark 3.9.

Proof. Let $\alpha \in R_\varepsilon^\xi(K)$. Given $\delta > 0$, let us consider the collection of rays

$$\mathcal{G}_\delta(x_\alpha, x_{\alpha+\xi}) = \{x' + \lambda(x_{\alpha+\xi} - x_\alpha) : \lambda \in [0,1],\, x' \in B_\delta(x_\alpha)\}. \tag{3.6}$$

By dimensional arguments there exists at least one ray $g_{\alpha,\xi} \subset \mathcal{G}_\delta(x_\alpha, x_{\alpha+\xi})$ that passes only through neighboring Voronoi cells. Consider then the set $P_{\alpha,\xi} = \{x \in \mathcal{L}(\omega) : g_{\alpha,\xi} \cap \mathcal{C}(x) \neq \emptyset\}$. By construction, choosing δ small enough we can number it as $P_{\alpha,\xi} = \{x_\alpha = x^0, \ldots, x^N = x_{\alpha+\xi}\}$ such that $(x^i, x^{i+1}) \in \mathcal{NN}(\omega)$. Moreover, from convexity of K and Lemma 3.4 it follows that $P_{\alpha,\xi} \subset \frac{1}{\varepsilon}K^{3R\varepsilon}$ again for δ small enough. As $\mathcal{M}$ is finite, by the bounds of Hypothesis 1 and the triangle inequality it holds

$$\begin{aligned} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, u(\varepsilon x_\alpha), u(\varepsilon x_{\alpha+\xi})) &\leq J_{lr}(|\hat{\xi}|)|u(\varepsilon x_\alpha) - u(\varepsilon x_{\alpha+\xi})| \\ &\leq J_{lr}(|\hat{\xi}|) \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ x,y \in \frac{1}{\varepsilon}K^{3R\varepsilon} \cap P_{\alpha,\xi}}} |u(\varepsilon x) - u(\varepsilon y)| \\ &\leq J_{lr}(|\hat{\xi}|) \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ x,y \in \frac{1}{\varepsilon}K^{3R\varepsilon} \cap P_{\alpha,\xi}}} g_\varepsilon^\omega(x, y, u(\varepsilon x), u(\varepsilon y)). \end{aligned} \tag{3.7}$$

Now given $(x, y) \in \mathcal{NN}(\omega) \cap \frac{1}{\varepsilon}K^{3R\varepsilon}$ we set

$$G_\varepsilon^\xi(x, y) = \{\alpha \in R_\varepsilon^\xi(K) : \{x, y\} \cap P_{\alpha,\xi} \neq \emptyset\}.$$

Note that if $\alpha \in G^\xi_\varepsilon(x,y)$, by the construction of $P_{\alpha,\xi}$ and Lemma 3.4 there exists $C>0$ such that

$$x_\alpha \in \{z+t\xi:\ |z-x|\leq C,\, |t|\leq C\}$$

and therefore $\#G^\xi_\varepsilon(x,y)\leq C|\xi|$ again by Lemma 3.4. The claim now follows by summing (3.7) over all $\alpha\in R^\xi_\varepsilon(K)$. □

Remark 3.13. Lemma 3.12 remains valid on non-convex domains if one replaces $K^{2R\varepsilon}$ by $K^{(|\xi|+7R)\varepsilon}$ as the constructed path is contained in this set.

Proposition 3.14. *Let $\mathcal{L}(\omega)$ be admissible and assume Hypothesis 1. Then there exists a deterministic constant $C>0$ such that for all $u\in BV(D,\mathcal{M})$ and $A\in\mathcal{A}^R(D)$,*

$$E''(\omega)(u,A)\leq C\,\mathcal{H}^{d-1}(S_u\cap A).$$

Proof. For the time being we assume that $u_{|A}\in BV(A,\mathcal{M})$ is a polyhedral function in the sense of Definition 2.11 and such that $\mathcal{H}^{d-1}(S_u\cap\partial A)=0$. We define $u_\varepsilon\in\mathcal{PC}_\varepsilon(\omega)$ by its values on $\varepsilon\mathcal{L}(\omega)$ via $u_\varepsilon(\varepsilon x)=u(\varepsilon x)$. Then, due to Lemma 3.4 we have $u_\varepsilon\to u_{|A}$ in $L^1(A)$. Given $\delta>0$, we choose $L_\delta>0$ as in Remark 3.9. For fixed $\eta>0$ we set again $A^\eta=A+B_\eta(0)$. Now for all $\xi\in r'\mathbb{Z}^d$ such that $|\xi|\leq L_\delta$ we apply Lemma 3.12 and Remark 3.13 and deduce that for ε small enough we have

$$\sum_{\alpha\in R^\xi_\varepsilon(A)} g^\omega_\varepsilon(x_\alpha,x_{\alpha+\xi},u_\varepsilon(x_\alpha),u(\varepsilon x_{\alpha+\xi}))\leq C J_{lr}(|\hat{\xi}|)|\xi|\sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in A^\eta}} g^\omega_\varepsilon(x,y,u_\varepsilon(x),u_\varepsilon(y)).$$

Let us observe that Hypothesis 1 and Lemma 3.4 further imply that if $(x,y)\in\mathcal{NN}(\omega)$, then $g^\omega_\varepsilon(x,y,m_i,m_j)\leq C|m_i-m_j|$. Hence for ε small enough the regularity of S_u and the integrability assumption from Hypothesis 1 yield

$$\begin{aligned}\sum_{|\xi|\leq L_\delta}\sum_{\alpha\in R^\xi_\varepsilon(A)}\varepsilon^{d-1}g^\omega_\varepsilon(x_\alpha,x_{\alpha+\xi},u_\varepsilon(x_\alpha),u(\varepsilon x_{\alpha+\xi}))&\leq C\sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in A^\eta}}\varepsilon^{d-1}|u(\varepsilon x)-u(\varepsilon y)|\\ &\leq C\mathcal{H}^{d-1}(S_u\cap A^\eta).\end{aligned}\tag{3.8}$$

For interactions where $|\xi|>L_\delta$ we use again Lemma 3.12 on a large cube containing A and that u is a polyhedral function so that we obtain the weaker bound

$$\sum_{|\xi|>L_\delta}\sum_{\alpha\in R^\xi_\varepsilon(A)}\varepsilon^{d-1}g^\omega_\varepsilon(x_\alpha,x_{\alpha+\xi},u(\varepsilon x_\alpha),u(\varepsilon x_{\alpha+\xi}))\leq C\sum_{|\xi|>L_\delta}J_{lr}(|\hat{\xi}|)|\xi|\,\mathcal{H}^{d-1}(S_u).$$

Using (3.8) and the previous inequality, Remark 3.8 and the definition of L_δ imply

$$E''(\omega)(u,A) \leq C\,\mathcal{H}^{d-1}(S_u \cap A^\eta) + C\mathcal{H}^{d-1}(S_u)\,\delta.$$

Now the arbitrariness of δ and η yield

$$E''(\omega)(u,A) \leq C\,\mathcal{H}^{d-1}(S_u \cap A). \tag{3.9}$$

where we have used that $\mathcal{H}^{d-1}(S_u \cap \partial A) = 0$ by assumption.

For a general function $u \in BV(D,\mathcal{M})$ let us consider $u_{|A} \in BV(A,\mathcal{M})$. By Theorem 2.12 there exists a sequence of polyhedral functions $u_n \in BV(A,\mathcal{M})$ such that $u_n \to u_{|A}$ in $L^1(A)$, $\mathcal{H}^{d-1}(S_{u_n} \cap A) \to \mathcal{H}^{d-1}(S_u \cap A)$ and $\mathcal{H}^{d-1}(S_{u_n} \cap \partial A) = 0$. Then u_n satisfies the assumptions of the first part of the proof. By the $L^1(A)$-lower semicontinuity of the Γ-lim sup and (3.9) it follows that

$$E''(\omega)(u,A) \leq C \liminf_n \mathcal{H}^{d-1}(S_{u_n} \cap A) = C\,\mathcal{H}^{d-1}(S_u \cap A).$$

□

Now we prove a weak subadditivity property for $E''(\omega)(u,\cdot)$.

Proposition 3.15. *Let $\mathcal{L}(\omega)$ be admissible and assume Hypothesis 1. Then, for every $A, B \in \mathcal{A}^R(D)$, every $A' \subset \mathcal{A}^R(D)$ such that $A' \subset\subset A$ and every $u \in BV(D,\mathcal{M})$,*

$$E''(\omega)(u, A' \cup B) \leq E''(\omega)(u,A) + E''(\omega)(u,B).$$

Proof. Without loss of generality let $E''(\omega)(u,A)$ and $E''(\omega)(u,B)$ be finite. Let $u_\varepsilon, v_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ both converge to u in $L^1(D)$ such that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon, A) = E''(\omega)(u,A), \quad \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(v_\varepsilon, B) = E''(\omega)(u,B).$$

Step 1 Extensions to convex domains
Let Q_D be a large cube containing $\overline{D}$. As $D \in \mathcal{A}^R(D)$, by extension we may assume that $u \in BV_{\rm loc}(\mathbb{R}^d,\mathcal{M})$. We first show that we can modify u_ε and v_ε on $\varepsilon\mathcal{L}(\omega)\backslash A$ and $\varepsilon\mathcal{L}(\omega)\backslash B$ respectively, such that they converge to u on $L^1(Q_D)$ and such that they have equibounded energy on Q_D. We will demonstrate the argument for u_ε. Take another cube Q' such that $Q_D \subset\subset Q'$. Arguing as in the proof of Proposition 3.14 we find a sequence $\tilde{u}_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ such that $\tilde{u}_\varepsilon \to u$ in $L^1(Q')$ and $\limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(\tilde{u}_\varepsilon, Q') \leq$

$C\mathcal{H}^{d-1}(S_u \cap Q')$. We then set $\bar{u} \in \mathcal{PC}_\varepsilon(\omega)$ as

$$\bar{u}(\varepsilon x) = \mathbb{1}_A(\varepsilon x) u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_A(\varepsilon x))\tilde{u}_\varepsilon(\varepsilon x).$$

Then $\bar{u}_\varepsilon \to u$ in $L^1(Q_D)$ and applying Lemmas 3.4 and 3.12 combined with Hypothesis 1 yields

$$\begin{aligned} E_\varepsilon(\omega)(\bar{u}_\varepsilon, Q_D) \leq & C \sum_{\xi \in r'\mathbb{Z}^d} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x, \varepsilon y \in Q'}} g_\varepsilon^\omega(x, y, \bar{u}_\varepsilon(\varepsilon x), \bar{u}_\varepsilon(\varepsilon y)) \\ \leq & C\left(E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(\tilde{u}_\varepsilon, Q'\backslash A) + \frac{1}{\varepsilon}|\partial A + B_{4R\varepsilon}(0)|\right). \end{aligned}$$

The first and second term remain bounded by construction, while the third term converges to a multiple of the Minkowski content of ∂A which agrees with $\mathcal{H}^{d-1}(\partial A)$ as $A \in \mathcal{A}^R(D)$.

Step 2 Energy estimates

Given $\delta > 0$ we choose L_δ as in Remark 3.9. Fix $d' \leq \frac{1}{2}\mathrm{dist}(A', \partial A)$ and let $N_\varepsilon := \lfloor \frac{d'}{2\varepsilon L_\delta} \rfloor$. For $k \in \mathbb{N}$ we define

$$A_{\varepsilon,k} := \{x \in A : \ \mathrm{dist}(x, A') < 2k\varepsilon L_\delta\}.$$

We let $w_\varepsilon^k \in \mathcal{PC}_\varepsilon(\omega)$ be the interpolation defined by

$$w_\varepsilon^k(\varepsilon x) = \mathbb{1}_{A_{\varepsilon,k}}(\varepsilon x) u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_{A_{\varepsilon,k}}(\varepsilon x)) v_\varepsilon(\varepsilon x).$$

Note that for each fixed $k \in \mathbb{N}$, $w_\varepsilon^k \to u$ in $L^1(D)$. Now we set

$$S_k^{\xi,\varepsilon} := \{x = y + t\,\xi' : \ y \in \partial A_{\varepsilon,k},\ |t| \leq \varepsilon, \xi' \in \xi + [-r', r']^d\} \cap (A \cup B).$$

For $k \leq N_\varepsilon$ it can easily be verified that

$$\begin{aligned} E_\varepsilon(\omega)(w_\varepsilon^k, A' \cup B) \leq & E_\varepsilon(\omega)(u_\varepsilon, A_{\varepsilon,k}) + E_\varepsilon(\omega)(v_\varepsilon, B\backslash A_{\varepsilon,k}) \\ & + \sum_{\xi \in r'\mathbb{Z}^d} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \underbrace{\varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, w_\varepsilon^k(\varepsilon x_\alpha), w_\varepsilon^k(\varepsilon x_{\alpha+\xi}))}_{=:\rho_k^{\xi,\varepsilon}(\alpha,\omega)} \\ \leq & E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(v_\varepsilon, B) + \sum_{\xi \in r'\mathbb{Z}^d} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \rho_k^{\xi,\varepsilon}(\alpha, \omega). \quad (3.10) \end{aligned}$$

We now split the interactions depending on L_δ. If $|\xi| > L_\delta$, we use Lemma 3.12.

Since $A \cup B \subset\subset Q_D$, we deduce that,

$$\sum_{|\xi|>L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \rho_k^{\xi,\varepsilon}(\alpha,\omega) \leq C \sum_{|\xi|>L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in Q_D}} \varepsilon^{d-1} g_\varepsilon^\omega(x,y,w_\varepsilon^k(\varepsilon x), w_\varepsilon^k(\varepsilon y)).$$

Obviously we have $Q_D \subset A_{\varepsilon,k} \cup Q_D \backslash A_{\varepsilon,k}$. Nearest neighbor interactions between those two sets are contained in $S_k^{\xi,\varepsilon}$ for $|\xi| \leq 4R$. Therefore we can further estimate the last inequality via

$$\begin{aligned}\sum_{|\xi|>L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \rho_k^{\xi,\varepsilon}(\alpha,\omega) \leq C\delta\Big(& E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(v_\varepsilon, Q_D) \\ & + \sum_{|\xi|\leq L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \rho_k^{\xi,\varepsilon}(\alpha,\omega)\Big). \end{aligned} \tag{3.11}$$

Now we treat the interactions when $|\xi| \leq L_\delta$. Consider any points $\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}(\omega)$. If $w_\varepsilon^k(\varepsilon x) \neq w_\varepsilon^k(\varepsilon y)$ then either $\varepsilon x, \varepsilon y \in A_{\varepsilon,k}$, $\varepsilon x, \varepsilon y \notin A_{\varepsilon,k}$ or $\varepsilon x \in A_{\varepsilon,k}$ but $\varepsilon y \notin A_{\varepsilon,k}$ (the other case can be treated similar). In the last case we have a contribution only if $u_\varepsilon(\varepsilon x) \neq v_\varepsilon(\varepsilon y)$. Then either $u_\varepsilon(\varepsilon y) = v_\varepsilon(\varepsilon y)$ or $g_\varepsilon^\omega(x,y,u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon y)) \leq C|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|$. Summarizing all cases we obtain the inequality

$$\begin{aligned}\rho_k^{\xi,\varepsilon}(\alpha,\omega) \leq & \varepsilon^{d-1} g_\varepsilon^\omega(x,y,u_\varepsilon(\varepsilon x), u_\varepsilon(\varepsilon y)) + \varepsilon^{d-1} g_\varepsilon^\omega(x,y,v_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon y)) \\ & + C\varepsilon^{d-1}|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|.\end{aligned}$$

By our construction we have $S_k^{\varepsilon,\xi} \subset (A_{\varepsilon,k+1} \backslash A_{\varepsilon,k-1}) =: S_k^\varepsilon$. We deduce that

$$\begin{aligned}\sum_{|\xi|\leq L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})} \rho_k^{\xi,\varepsilon}(\alpha,\omega) \leq & E_\varepsilon(\omega)(u_\varepsilon, S_k^\varepsilon) + E_\varepsilon(\omega)(v_\varepsilon, S_k^\varepsilon) \\ & + C_\delta \sum_{\substack{y\in\mathcal{L}(\omega) \\ \varepsilon y \in S_k^\varepsilon}} \varepsilon^{d-1}|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|,\end{aligned}$$

where C_δ depends only on L_δ. Observe that every point can only lie in at most two sets $S_{k_1}^\varepsilon, S_{k_2}^\varepsilon$. Thus averaging combined with (3.11), Step 1 and the last inequality yields

$$\begin{aligned} I_\varepsilon := &\frac{1}{N_\varepsilon}\sum_{k=1}^{N_\varepsilon}\sum_{\xi\in r'\mathbb{Z}^d}\sum_{\alpha\in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})}\rho_k^{\xi,\varepsilon}(\alpha,\omega)\leq\frac{2}{N_\varepsilon}\sum_{k=1}^{N_\varepsilon}\sum_{|\xi|\leq L_\delta}\sum_{\alpha\in R_\varepsilon^\xi(S_k^{\xi,\varepsilon})}\rho_k^{\xi,\varepsilon}(\alpha,\omega)+C\delta \\ \leq&\frac{4}{N_\varepsilon}\left(E_\varepsilon(\omega)(u_\varepsilon,Q_D)+E_\varepsilon(\omega)(v_\varepsilon,Q_D)\right)+C_\delta\sum_{\substack{y\in\mathcal{L}(\omega)\\ \varepsilon y\in D}}\varepsilon^d|u_\varepsilon(\varepsilon y)-v_\varepsilon(\varepsilon y)|+C\delta \\ \leq&\frac{C}{N_\varepsilon}+C_\delta\|u_\varepsilon-v_\varepsilon\|_{L^1(Q_D)}+C\delta. \end{aligned}$$

For every $\varepsilon>0$ let $k_\varepsilon\in\{1,\dots,N_\varepsilon\}$ be such that

$$\sum_{\xi\in r'\mathbb{Z}^d}\sum_{\alpha\in R_\varepsilon^\xi(S_{k_\varepsilon}^{\xi,\varepsilon})}\rho_{k_\varepsilon}^{\xi,\varepsilon}(\alpha,\omega)\leq I_\varepsilon \tag{3.12}$$

and set $w_\varepsilon:=w_\varepsilon^{k_\varepsilon}$. Note that w_ε still converges to u strongly in $L^1(D)$. Hence, using (3.10), (3.12) and again the first step, we conclude that

$$E''(\omega)(u,A'\cup B)\leq\limsup_{\varepsilon\to0}E_\varepsilon(\omega)(w_\varepsilon,A'\cup B)\leq E''(\omega)(u,A)+E''(\omega)(u,B)+C\,\delta.$$

The arbitrariness of δ proves the claim. □

Proof of Theorem 3.10. From Propositions 3.14 and 3.15 it follows by standard arguments that $E''(\omega)(u,\cdot)$ is inner regular on $\mathcal{A}^R(D)$ (see, for example, Proposition 11.6 in [23]). Therefore, given a sequence $\varepsilon_n\to0^+$ we can use the compactness property of Γ-convergence (Proposition 2.5) to construct a subsequence ε_n (not relabeled) such that

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(\omega)(u,A)=:\tilde{E}(\omega)(u,A)$$

exists for every $(u,A)\in L^1(D)\times\mathcal{A}^R(D)$. By Proposition 3.11 we further know that $\tilde{E}(\omega)(u,A)$ is finite only if $u\in BV(A,\mathcal{M})$. We extend $\tilde{E}(u,\cdot)$ to $\mathcal{A}(D)$ setting

$$E(\omega)(u,A):=\sup\{\tilde{E}(\omega)(u,A'):\ A'\subset\subset A,\ A'\in\mathcal{A}^R(D)\}.$$

In order to complete the proof it is enough to show that $E(\omega)$ satisfies the assumptions of Theorem 2.13. Again by standard arguments $E(\omega)(u,\cdot)$ fulfills the assumptions of the De Giorgi-Letta criterion (see Theorem 1.62 in [37]), so that $E(\omega)(u,\cdot)$ is the trace of a Borel measure. Since this Borel measure is finite on D by Proposition 3.14, it is indeed a Radon measure (Proposition 1.60 in [37]). The locality property follows from Remark 3.8. By the properties of Γ-limits we know that $\tilde{E}(\omega)(\cdot,A)$ is $L^1(D)$-

lower semicontinuous and so is $E(\omega)(\cdot, A)$ as the supremum. The growth conditions (iv) in Theorem 2.13 follow from the Propositions 3.11 and 3.14 which still hold for $E(\omega)$ in place of $\tilde{E}(\omega)$. The local version of the theorem is a direct consequence of our construction. □

3.3 Homogenization for stationary lattices

So far we considered energies defined on a fixed realization of a possibly non-periodic network. In this section we exploit the stationarity assumption to prove that in this case the Γ-limit exists along any sequence and that the integrand ϕ in Theorem 3.10 does not depend on x. In order to prove existence of the limit functional, we make use of stochastic homogenization. To this end, we need to introduce discrete boundary conditions. At this stage we are not concerned about the physically most relevant ones. Rather we try to keep the arguments as short as possible. In particular we will introduce another scale η that describes the (macroscopic) thickness of the discrete boundary and consider first the limit $\varepsilon \to 0$ and then $\eta \to 0$. We refer to Section 3.4 for a finer analysis when the thickness depends also on ε.

We now set the precise framework. Given $\eta > 0$ and $A \in \mathcal{A}^R(D)$, we define

$$\partial_\eta A = \{x \in A : \ \mathrm{dist}(x, \partial A) \leq \eta\}.$$

We restrict the class of admissible boundary data to functions $u_0 \in BV_{\mathrm{loc}}(\mathbb{R}^d, \mathcal{M})$ that are well-prepared in the sense that, setting $u_{\varepsilon,0} \in \mathcal{PC}_\varepsilon(\omega)$ as $u_{\varepsilon,0}(\varepsilon x) = u_0(\varepsilon x)$ it holds

$$\begin{aligned} &\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_{\varepsilon,0}, B) \leq C\mathcal{H}^{d-1}(S_{u_0} \cap \overline{B}), \\ &u_{\varepsilon,0} \to u_0 \text{ in } L^1(D), \qquad \mathcal{H}^{d-1}(S_{u_0} \cap \partial A) = 0 \end{aligned} \tag{3.13}$$

with C uniformly for $B \in \mathcal{A}^R(\mathbb{R}^d)$. Observe that by the proof of Proposition 3.14 we allow for any polyhedral function such that $\mathcal{H}^{k-1}(S_{u_0} \cap \partial A) = 0$, but more generally it suffices that the level sets are Lipschitz sets. The discrete boundary conditions are defined via the class

$$\mathcal{PC}^\eta_{\varepsilon,u_0}(\omega, A) := \{u \in \mathcal{PC}_\varepsilon(\omega) : \ u(\varepsilon x) = u_0(\varepsilon x) \text{ if } \varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap \partial_\eta A\}.$$

Using a similar notation as in Theorem 2.13 we define the quantities

$$\begin{aligned} m_\varepsilon^\eta(\omega)(u_0,A) &= \inf\{E_\varepsilon(\omega)(v,A):\ v\in\mathcal{PC}^\eta_{\varepsilon,u_0}(\omega,A)\},\\ m(\omega)(u_0,A) &= \inf\{E(\omega)(v,A):\ v=u_0 \text{ in a neighbourhood of } \partial A\},\end{aligned}$$

where the limit functional $E(\omega)$ is given (up to subsequences) by Theorem 3.10. Note that the mapping $\eta\mapsto m_\varepsilon^\eta(\omega)(u_0,A)$ is non-decreasing. We have the following auxiliary result.

Lemma 3.16. *Let ε_n and $E(\omega)$ be as in Theorem 3.10. Then it holds that*

$$\lim_{\eta\to 0}\liminf_n m^\eta_{\varepsilon_n}(\omega)(u_0,A) = \lim_{\eta\to 0}\limsup_n m^\eta_{\varepsilon_n}(\omega)(u_0,A) = m(\omega)(u_0,A).$$

Proof. First note that by monotonicity the limits for $\eta\to 0$ are well-defined. Moreover, by the first assumption in (3.13) we have that $m^\eta_\varepsilon(\omega)(u_0,A)$ is equibounded. For any $n\in\mathbb{N}$ let $u_n\in\mathcal{PC}^\eta_{\varepsilon_n,u_0}(\omega,A)$ be such that $m^\eta_{\varepsilon_n}(\omega)(u_0,A)=E_{\varepsilon_n}(\omega)(u_n,A)$. By Proposition 2.5 we know that, up to a subsequence (not relabeled), $u_n\to u$ in $L^1(A)$ and, using Lemma 3.4 (i) and again (3.13) it follows easily that $u=u_0$ on $\partial_\eta A$. Extending u we can assume that u is admissible in the infimum problem defining $m(\omega)(u_0,A)$ and using Theorem 3.10 we obtain

$$m(\omega)(u_0,A)\le E(\omega)(u,A)\le \liminf_n E_{\varepsilon_n}(\omega)(u_n,A)\le \liminf_n m^\eta_{\varepsilon_n}(\omega)(u_0,A).$$

As η was arbitrary, we conclude that $m(\omega)(u_0,A)\le \lim_{\eta\to 0}\liminf_n m^\eta_{\varepsilon_n}(\omega)(u_0,A)$.

In order to prove the remaining inequality, given $\gamma>0$ we let $u\in BV(A,\mathcal{M})$ be such that $u=u_0$ in a neighborhood of ∂A and $E(\omega)(u,A)\le m(\omega)(u_0,A)+\gamma$. Let $u_n\in\mathcal{PC}_{\varepsilon_n}(\omega)$ be a recovery sequence for u. In particular it satisfies

$$\limsup_n E_{\varepsilon_n}(\omega)(u_n,A) = E(\omega)(u,A). \tag{3.14}$$

We will modify u_n such that it fulfills the discrete boundary conditions. The argument will be similar to the one used in the proof of Proposition 3.15. Therefore we just sketch the argument. First we can avoid the abstract extension argument and instead redefine $u_n(\varepsilon_n x)=u_0(\varepsilon_n x)$ for all $\varepsilon_n x\notin A$. Due to the boundary conditions of u, Theorem 2.15 and Remark 2.16 there exist sets $A_1\subset\subset A_2\subset\subset A$ such that $A_1,A_2\in\mathcal{A}^R(D)$ and

$$\begin{aligned} &u=u_0 \text{ on } A\backslash A_1,\\ &\mathcal{H}^{d-1}(S_{u_0}\cap\partial A_1)=0.\end{aligned}\tag{3.15}$$

First we choose L_δ as in Remark 3.9 and fix $d' \leq \frac{1}{2}\mathrm{dist}(A_1, \partial A_2)$. Setting $N_n = [\frac{d'}{2\varepsilon_n L_\delta}]$, for $k \in \mathbb{N}$ we introduce the sets

$$A_{n,k} := \{x \in A : \ \mathrm{dist}(x, A_1) < 2k\varepsilon_n L_\delta\}.$$

We further define $u_n^k \in \mathcal{PC}_{\varepsilon_n}(\omega)$ via interpolation as

$$u_n^k(\varepsilon_n x) = \mathbb{1}_{A_{n,t_k}}(\varepsilon_n x) u_n(\varepsilon_n x) + (1 - \mathbb{1}_{A_{n,t_k}}(\varepsilon_n x)) u_0(\varepsilon_n x).$$

Note that $u_n^k \in \mathcal{PC}^\eta_{\varepsilon_n, u_0}(\omega, A)$ for some $\eta > 0$ independent of n and $k \leq N_n$. Repeating the arguments from the proof of Proposition 3.15 we find $k_n \in \{1, \dots, N_n\}$ and the corresponding sequence $u_n^{k_n} \in \mathcal{PC}^\eta_{\varepsilon_n, u_0}(\omega, A)$ with

$$\limsup_n E_{\varepsilon_n}(\omega)(u_n^{k_n}, A) \leq E(\omega)(u, A) + C\,\mathcal{H}^{d-1}(S_{u_0} \cap (A \backslash \overline{A_1})) + C\,\delta,$$

where we have also used that, by (3.13) and (3.15),

$$\limsup_n E_{\varepsilon_n}(\omega)(u_{\varepsilon_n,0}, A \backslash \overline{A_1}) \leq C\,\mathcal{H}^{d-1}(S_{u_0} \cap (A \backslash \overline{A_1})).$$

As δ was arbitrary, by the choice of u we obtain

$$\begin{aligned}\lim_{\eta \to 0} \limsup_n m^\eta_{\varepsilon_n}(\omega)(u_0, A) &\leq \limsup_n E_{\varepsilon_n}(\omega)(u_n^{k_n}, A) \\ &\leq m(\omega)(u_0, A) + \gamma + C\mathcal{H}^{d-1}(S_{u_0} \cap A \backslash \overline{A_1}).\end{aligned}$$

The claim now follows letting first $A_1 \uparrow A$ and then $\gamma \to 0$. □

In view of Theorem 2.13 and the previous lemma we can further characterize the Γ-limits of the family $E_\varepsilon(\omega)$ by investigating the quantities $m^\eta_\varepsilon(u_0, Q)$ for suitably oriented cubes and $u_0 = u_{x,m_i,m_j,\nu}$. Due to the decay assumptions of Hypothesis 1 it will be enough to consider truncated interactions. To this end we define for fixed $L \in \mathbb{N}$ the quantity

$$m_1^{\eta,L}(\omega)(u_0, A) = \inf\{E_1^L(\omega)(u, A) : \ u \in \mathcal{PC}^\eta_{1,u_0}(\omega, A)\}, \tag{3.16}$$

where $E_1^L(\omega)$ denotes the energy with interaction range L, that is we replace the functions $g^\omega_\varepsilon(x, y, m_i, m_j)$ by $g^\omega_\varepsilon(x, y, m_i, m_j)\mathbb{1}_{|x-y| \leq L}$. Moreover, to shorten notation we set $u^{ij}_{x,\nu} = u_{x,m_i,m_j,\nu}$. The following estimate will be used several times.

Lemma 3.17. *Let $Q = Q_\nu(z, \rho) \subset \mathbb{R}^d$ be a cube and let $\{Q_k = Q_\nu(z_k, \rho_k)\}_k$ be a finite family of disjoint cubes with the following properties:*

(i) $\min_k \rho_k \geq 4L$,

(ii) $z_k - z_1 \in \nu^\perp$,

(iii) $\operatorname{dist}(z_1, \nu^\perp + z) \leq \frac{1}{4}\min_k \rho_k$,

(iv) $\bigcup_k Q_k \subset Q$,

(v) $\operatorname{dist}(\partial \bigcup_k Q_k, \partial Q) > \eta$ *or* $z_k - z \in \nu^\perp$.

Then there exists $C = C_L > 0$ *such that, for all* $i \neq j$ *and* $\eta \geq L$

$$m_1^{\eta,L}(\omega)(u_{z,\nu}^{ij}, Q) \leq \sum_k m_1^{\eta,L}(\omega)(u_{z_k,\nu}^{ij}, Q_k) + C\mathcal{H}^{d-1}\Big(\Big(Q\backslash\bigcup_k \overline{Q_k}\Big) \cap (\nu^\perp + z)\Big)$$
$$+ C\sum_k \Big(\mathcal{H}^{d-2}\left((\partial Q_k \backslash \partial Q) \cap (\nu^\perp + z_1)\right) + \mathcal{H}^{d-1}(\partial Q_k \cap S_\nu(z, z_1))\Big),$$

where $S_\nu(z, z_1)$ *is the infinite (maybe flat) stripe enclosed by the two hyperplanes* $\nu^\perp + z$ *and* $\nu^\perp + z_1$.

Proof. During this proof, given $y \in \mathbb{R}^d$, we denote by $P_{\nu,y}$ the projection onto the affine space $\nu^\perp + y$. Moreover, for any k we let u_k be an admissible minimizer defining $m_1^{\eta,L}(\omega)(u_{z_k,\nu}^{ij}, Q_k)$. By assumptions (ii) and (v), the function $v : \mathcal{L}(\omega) \to \mathcal{M}$ defined as

$$v(x) = \begin{cases} u_k(x) & \text{if } x \in \overline{Q_k} \text{ for some } k, \\ u_{z,\nu}(x) & \text{otherwise} \end{cases}$$

is well-defined and belongs to $\mathcal{PC}^\eta_{1,u_{z,\nu}^{ij}}(\omega, Q)$. For $x, y \in \mathcal{L}(\omega) \cap Q$ with $|x - y| \leq L$, we abbreviate two scenarios:

(I) $x \in \overline{Q_k}$ and $y \in \overline{Q_l}$ for $k \neq l$ or $x, y \in \partial Q_k$,

(II) $x \in Q\backslash \bigcup_k \overline{Q_k}$ and $y \in \overline{Q_k}$ for some k.

By (iv) and Hypothesis 1 we can estimate

$$m_1^{\eta,L}(\omega)(u_{z,\nu}^{ij}, Q) \leq E_1^L(\omega)(v, Q) \leq \sum_k m_1^{\eta,L}(\omega)(u_{z_k,\nu}^{ij}, Q_j) + E_1^L(\omega)\Big(v, Q\backslash\bigcup_k \overline{Q_k}\Big)$$
$$+ C \sum_{\substack{|x-y|\leq L \\ \text{(I) or (II) hold}}} |v(x) - v(y)|. \tag{3.17}$$

We start with estimating the contribution of $x, y \in Q\backslash \bigcup_k \overline{Q_k}$. Suppose that $v(x) \neq v(y)$. In this case x and y lie on different sides of the hyperplane $\nu^\perp + z$. Then it holds true that $P_{\nu,z}(x) \in Q\backslash\bigcup_k \overline{Q_k}$, otherwise assumptions (i) and (iii) would imply

$$L \geq |x-y| \geq |x-P_{\nu,z}(x)| \geq \frac{\rho_k}{2} - \frac{\rho_k}{4} \geq 2L.$$

Thus $\mathrm{dist}(x,(Q\backslash\bigcup_k \overline{Q_k})\cap(\nu^\perp+z)) \leq L$ and, using Lemma 3.4, it follows that

$$E_1^L(\omega)\Big(v, Q\backslash\bigcup_k \overline{Q_k}\Big) \leq C\mathcal{H}^{k-1}\Big(\Big(Q\backslash\bigcup_k \overline{Q_k}\Big)\cap(\nu^\perp+z)\Big). \tag{3.18}$$

Next we have to control the interactions in Case (I). Given such x,y with $|x-y|\leq L$, we know that by the definition of v, the boundary conditions on the smaller cubes and (ii) that $v(x) = u^{ij}_{z_1,\nu}(x)$ and $v(y) = u^{ij}_{z_1,\nu}(y)$, so that if they contribute to the energy we conclude that x and y must lie on different sides of the hyperplane $\nu^\perp + z_1$. We deduce that $|P_{\nu,z_1}(x)-x| \leq L$. Since by (iv) the ray $[P_{\nu,z_1}(x), P_{\nu,z_1}(y)]$ intersects the $(d-2)$-dimensional set $(\partial Q_k\backslash\partial Q)\cap(\nu^\perp+z_1)$, it follows that

$$\mathrm{dist}\left(x,(\partial Q_k\backslash\partial Q)\cap(\nu^\perp+z_1)\right) \leq 2L.$$

Again, by Lemma 3.4 and the above inequality we easily can derive the estimate

$$\sum_{\substack{|x-y|\leq L\\ \text{(I) holds}}} |v(x)-v(y)| \leq C\sum_k \mathcal{H}^{d-2}\left((\partial Q_k\backslash\partial Q)\cap(\nu^\perp+z_1)\right). \tag{3.19}$$

It remains to estimate the contributions coming from Case (II). For such x,y with $|x-y|\leq L$, due to the boundary conditions on the smaller cubes, a positive energy contribution implies $u_{z,\nu}(x) \neq u_{z_1,\nu}(y)$. Thus the ray $[x,y]$ intersects ∂Q_k in (at least) one point x_k and also $S_\nu(z,z_1)$ in (at least) one point x_S. Let us denote by $x_{k,S}$ the projection of x_S onto the facet of the cube Q_k containing x_k. As this facet cannot be parallel to $\nu^\perp$ by (i) and (iii), it holds $x_{k,S}\in\partial Q_k\cap S_\nu(z,z_1)$ and

$$|x-x_{k,S}| \leq |x-x_S| + |x_S - x_{k,S}| \leq L + |x_S - x_k| \leq 2L,$$

which yields the estimate

$$\mathrm{dist}(x,\partial Q_k\cap S_\nu(z,z_1)) \leq 2L. \tag{3.20}$$

This set may be not $(d-1)$-dimensional in the second possibility of (v). In this case one can bound the interactions by the right hand side of (3.18). Otherwise, using

(3.20) we obtain the estimate

$$\sum_{\substack{|x-y|\le L\\ \text{(IV holds)}}} |v(x)-v(y)| \le C\sum_k \mathcal{H}^{d-1}(\partial Q_k\cap S_\nu(z,z_1)). \tag{3.21}$$

In any case the claim now follows from (3.17), (3.18), (3.19) and (3.21). □

Remark 3.18. Lemma 3.17 still holds if we replace cubes by boxes of the type $I_\nu(z,\{\rho_i\}_i)=z+\{x\in\mathbb{R}^d:\ |\langle x,\nu_i\rangle|<\frac{\rho_i}{2}\}$. Then the cubes Q_j are replaced by the collection $I_j=I_\nu(z_j,\{\rho_i^j\}_i)$ and in the assumptions (i) and (iii) we have to replace ρ_j by $\min_i\rho_i^j$.

As we are interested in proving a stochastic homogenization result, we suppose from now on that there exist functions $g_{nn},g_{lr}:\mathbb{R}^d\times\mathcal{M}^2\to[0,+\infty)$ such that

$$\begin{aligned} g^{\varepsilon}_{\varepsilon,nn}(x,y,m_i,m_j)&=g_{nn}(y-x,m_i,m_j),\\ g_{\varepsilon,lr}(x,y,m_i,m_j)&=g_{lr}(y-x,m_i,m_j). \end{aligned}\tag{3.22}$$

The next theorem is the main result of this section.

Theorem 3.19. *Let $\mathcal{L}$ be a stationary stochastic lattice and let g_{nn} and g_{lr} satisfy Hypothesis 1 with the additional structure property (3.22). For $\mathbb{P}$-almost every ω and for all $m_i,m_j\in\mathcal{M}$ and $\nu\in S^{d-1}$ there exists*

$$\phi_{\mathrm{hom}}(\omega;m_i,m_j,\nu)=\inf_{\eta>0}\limsup_{t\to+\infty}\frac{1}{t^{d-1}}\inf\left\{E_1(\omega)(u,tQ_\nu):\ u\in\mathcal{PC}^{\eta t}_{1,u_{0,m_i,m_j,\nu}}(\omega,tQ_\nu)\right\}.$$

The functionals $E_\varepsilon(\omega)$ Γ-converge with respect to the $L^1(D)$-topology to the functional $E_{\mathrm{hom}}(\omega):L^1(D)\to[0,+\infty]$ defined by

$$E_{\mathrm{hom}}(\omega)(u)=\begin{cases}\int_{S_u}\phi_{\mathrm{hom}}(\omega;u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u\in BV(D,\mathcal{M}),\\ +\infty & \text{otherwise.}\end{cases}$$

If $\mathcal{L}$ is ergodic, then $\phi_{\mathrm{hom}}(\cdot,m_i,m_j,\nu)$ is constant almost surely.

Remark 3.20. The result above holds with the same proof if instead of considering stochastic lattices one takes the deterministic lattice $\mathbb{Z}^d$ and random stationary interactions $g^\omega_{i,j}$ for all $i,j\in\mathbb{Z}^d$ such that $\mathbb{P}$-almost surely, it holds that

$$\begin{aligned} c|m-m'|\le g^\omega_{i,j}(m,m')&\le J_{lr}(|i-j|)|m-m'| \qquad \text{if } |i-j|=1,\\ 0\le g^\omega_{i,j}&\le J_{lr}(|i-j|)|m-m'| \end{aligned}$$

with c and J_{lr} as in Hypothesis 1. In this setting the assumption on the long range interactions could be weakened a little bit as in [6]. Here the stochastic group action acts on the coefficients via

$$g^{\omega}_{i+z,j+z} = g^{\tau_z\omega}_{i,j}.$$

In this setting an analogous result has been obtained in [25] in the case $\mathcal{M} = \{\pm 1\}$ and a two-dimensional system with nearest-neighbors ergodic interactions.

Proof of Theorem 3.19. Fix any sequence $\varepsilon_n \to 0$. According to Theorem 3.10, for all $\omega \in \Omega$ such that $\mathcal{L}(\omega)$ is admissible, there exists a (ω-dependent) subsequence ε_{n_k} such that

$$\Gamma\text{-}\lim_k E_{\varepsilon_{n_k}}(\omega)(u,A) = \int_{S_u\cap A} \phi(\omega;x,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^{d-1}$$

for all $u \in BV(D,\mathcal{M})$ and every $A \in \mathcal{A}^R(D)$. According to Theorem 2.13 and Lemma 3.16, for any $x \in D, m_i, m_j \in \mathcal{M}$ and $\nu \in S^{d-1}$ it holds that

$$\begin{aligned}\phi(\omega;x,m_i,m_j,\nu) &= \limsup_{\rho\to 0}\frac{1}{\rho^{d-1}} m(\omega)(u^{ij}_{x,\nu},Q_\nu(x,\rho))\\ &= \limsup_{\rho\to 0}\frac{1}{\rho^{d-1}}\lim_{\eta\to 0}\limsup_k m^{\eta}_{\varepsilon_{n_k}}(\omega)(u^{ij}_{x,\nu},Q_\nu(x,\rho)).\end{aligned}$$

If we change the variables via $t_k = \varepsilon_{n_k}^{-1}$ and $v(x) = u(t_k^{-1}x)$, the above characterization reads

$$\phi(\omega;x,m_i,m_j,\nu) = \limsup_{\rho\to 0}\lim_{\eta\to 0}\limsup_k \frac{1}{(\rho t_k)^{d-1}} m_1^{\eta t_k}(\omega)(u^{ij}_{t_k x,\nu}, t_k Q_\nu(x,\rho)).$$

Except for the claim on ergodicity, due to the Urysohn property of Γ-convergence (Lemma 2.3) it is enough to show that for a set of full probability the limit in ρ can be neglected and the remaining limits do not depend on x or the subsequence t_k. We divide the proof into several steps.

Step 1 Truncating the range of interactions

First we show that it suffices to consider the case of finite range interactions. We argue that it is enough to prove that there exists $\phi^L_{\mathrm{hom}}(\omega;m_i,m_j,\nu)$ and a set Ω_L of full probability such that for all $\omega \in \Omega_L$, $x \in D$, every cube $Q_\nu(x,\rho)$ and every sequence $t_k \to +\infty$ it holds

$$\phi^L_{\mathrm{hom}}(\omega;m_i,m_j,\nu) = \lim_{\eta\to 0}\limsup_k \frac{1}{(\rho t_k)^{d-1}} m_1^{\eta t_k,L}(\omega)(u^{ij}_{t_k x,\nu}, t_k Q_\nu(x,\rho)), \tag{3.23}$$

where $m_1^{\eta t_k,L}(\omega)$ is defined in (3.16). Indeed, if (3.23) is proven, then for all $\omega \in \bigcap_L \Omega_L$ we find a configuration $v_k^L \in \mathcal{PC}_1(\omega)$ with the correct boundary conditions (extended to the whole space) that minimizes $E_1(\omega)(\cdot, t_kQ_\nu(x,\rho))$. Using Lemma 3.12 we obtain the estimate

$$\begin{aligned}
0 &\leq \frac{m_1^{\eta t_k}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho)) - m_1^{\eta t_k,L}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho))}{(\rho t_k)^{d-1}} \\
&\leq \frac{E_1(\omega)(v_k^L, t_kQ_\nu(x,\rho)) - E_1^L(\omega)(v_k^L, t_kQ_\nu(x,\rho))}{(\rho t_k)^{d-1}} \\
&\leq \frac{C}{(\rho t_k)^{d-1}} \sum_{|\xi|>L} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ x,y\in(t_kQ_\nu(x,\rho))^{3R}}} g_1^\omega(x-y, v_k^L(x), v_k^L(y)).
\end{aligned}$$

The inner sum can be bounded by the energy plus interactions close to $\partial t_kQ_\nu(x,\rho)$. Due to the boundary conditions these are of order $(\rho t_k)^{d-2}$. Using the trivial bound $m_1^\eta(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho)) \leq C(\rho t_k)^{d-1}$ obtained by testing $u^{ij}_{t_kx,\nu}$, we deduce that

$$0 \leq \frac{m_1^{\eta t_k}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho)) - m_1^{\eta t_k,L}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho))}{(\rho t_k)^{d-1}} \leq C \sum_{|\xi|>L} J_{lr}(|\hat{\xi}|)|\xi|.$$

Due to the integrability assumption of Hypothesis 1, we infer that $\phi^L_{\mathrm{hom}}(\omega; m_i, m_j, \nu)$ is a Cauchy-sequence with respect to L and moreover, in combination with (3.23), we deduce that

$$\lim_L \phi^L_{\mathrm{hom}}(\omega; m_i, m_j, \nu) = \lim_{\eta\to 0} \limsup_k \frac{1}{(\rho t_k)^{d-1}} m_1^{\eta t_k}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho))$$

exists, is independent of x, ρ and the sequence t_k. Therefore it remains to show (3.23). For clarity of the argument we first consider an auxiliary problem where we replace the varying boundary width ηt_k by L. As an intermediate result we show that there exists

$$\phi^L_{ij}(\omega;\nu) = \lim_k \frac{1}{(\rho t_k)^{d-1}} m_1^{L,L}(\omega)(u^{ij}_{t_kx,\nu}, t_kQ_\nu(x,\rho)) \tag{3.24}$$

and this limit does not depend on x, ρ and the sequence t_k.

Step 2 Existence of ϕ^L_{ij} for $x = 0$ and rational directions

Fix $L \in \mathbb{N}$. We have to show that, for $\mathbb{P}$-almost every $\omega \in \Omega$ and every $\nu \in S^{d-1}$, there exists the limit in (3.24). We start with the case $x = 0$ and $\nu \in S^{d-1} \cap \mathbb{Q}^d$. For this choice we can use the subadditive ergodic theorem in $(d-1)$-dimensions.

Substep 2.1 Defining a stochastic process

We need a few preliminaries: Given $\nu \in S^{d-1}$ there exists an orthogonal matrix $A_\nu \in \mathbb{R}^{d\times d}$ such that $A_\nu e_d = \nu$, the mapping $\nu \mapsto A_\nu e_i$ is continuous on $S^{d-1}\backslash\{-e_d\}$ and if $\nu \in \mathbb{Q}^d$ then $A_\nu \in \mathbb{Q}^{d\times d}$ (see for instance the construction in [56]). We now fix a rational direction $\nu \in S^{d-1} \cap \mathbb{Q}^d$. Then there exists an integer $M = M(\nu) > 4L$ such that $MA_\nu(z,0) \in \mathbb{Z}^d$ for all $z \in \mathbb{Z}^{d-1}$. We define a discrete stochastic process (see Definition 2.20). To $I = [a_1, b_1) \times \cdots \times [a_{d-1}, b_{d-1}) \in \mathcal{I}_{d-1}$ we associate the set $Q_I \subset \mathbb{R}^d$ defined by

$$Q_I := MA_\nu\Big(\operatorname{int} I \times (-\frac{s_{\max}}{2}, \frac{s_{\max}}{2})\Big),$$

where $s_{\max} = \max_i |b_i - a_i|$. Then we define the process $\mu : \mathcal{I}_{d-1} \to L^1(\Omega)$ as

$$\mu(I,\omega) := \inf\Big\{E_1^L(\omega)(v, Q_I) : \, v \in \mathcal{PC}^L_{1,u_{0,\nu}^{ij}}(\omega, Q_I)\Big\} + C_\mu \mathcal{H}^{d-2}(\partial I), \tag{3.25}$$

where C_μ is a constant to be chosen later. We first have to show that $\mu(I,\cdot)$ is a $L^1(\Omega)$-function. Testing the $\mathcal{PC}_1(\omega)$-interpolation of $u_{0,\nu}^{ij}$ as candidate in the infimum problem, one can use the growth assumptions from Hypothesis 1 and Lemma 3.4 to show that there exists a constant $C > 0$ such that

$$\mu(I,\omega) \leq CM^{d-1}\mathcal{H}^{d-1}(I) \tag{3.26}$$

for all $I \in \mathcal{I}_{d-1}$ and almost every $\omega \in \Omega$. Hence $\mu(I,\cdot)$ is essentially bounded. Measurability is a consequence of Lemma 3.6. Indeed, using that lemma, up to standard arguments it remains to show that the infimum in the definition of $\mu(I,\omega)$ preserves measurability. Note that the discrete constraints near the boundary can be replaced by a measurable penalty term of the form

$$\sum_{i\geq 1} C|v(\mathcal{L}(\omega)_i) - u_{0,\nu}(\mathcal{L}(\omega)_i)| \mathbb{1}_{\{y\in\mathbb{R}^d :\, \operatorname{dist}(y,\partial I)\leq L\}}(\mathcal{L}(\omega)_i),$$

where C is large enough to dominate the right-hand side of (3.26). Finally we minimize over the first k coordinates of vectors in $\mathcal{M}^{\mathbb{N}}$ (the others being constantly m_1) and then let $k \to +\infty$ to see that $\mu(I,\omega)$ can be written as the pointwise limit of measurable functions. Note that the limit exists since only finitely many points of the stochastic lattice are contained in Q_I.

We continue with proving lower-dimensional stationarity of the process. Let $z \in \mathbb{Z}^{d-1}$. Note that $Q_{I-z} = Q_I - z_\nu^M$, where $z_\nu^M := MA_\nu(z,0) \in \nu^\perp \cap \mathbb{Z}^d$. Moreover, by the stationarity of $\mathcal{L}$ it holds that $v \in \mathcal{PC}^L_{1,u_{0,\nu}^{ij}}(\omega, Q_{I-z})$ if and only if $u(\cdot) = v(\cdot - z_\nu^M) \in \mathcal{PC}^L_{1,u_{0,\nu}^{ij}}(\tau_{z_\nu^M}\omega, Q_I)$. Now if the couple (x,y) is taken into account for $\mu(I-z,\omega)$, then the points $x' := x + z_\nu^M$ and $y' = y + z_\nu^M$ are points of the lattice $\mathcal{L}(\tau_{z_\nu^M}\omega)$ and are taken into account for $\mu(I, \tau_{z_\nu^M}\omega)$. Furthermore $(x,y) \in \mathcal{NN}(\omega)$ if and only if $(x',y') \in \mathcal{NN}(\tau_{z_\nu^M}\omega)$. By (3.22) and shift invariance of the Hausdorff measure we conclude that $\mu(I-z,\omega)=\mu(I,\tau_{z_\nu^M}\omega)$. Setting $\tilde{\tau}_z=\tau_{-z_\nu^M}$ we obtain a measure preserving group action on $\mathbb{Z}^{d-1}$ such that $\mu(I,\tilde{\tau}_z\omega)=\mu(I+z)(\omega)$, that is stationarity.

To show subadditivity, let $I\in\mathcal{I}_{d-1}$ and let $\{I_j\}_{j\in J}\subset\mathcal{I}_{d-1}$ be a finite disjoint family such that $I=\bigcup_{j\in J}I_j$. Note that Q_I and the family $\{Q_{I_j}\}_{j\in J}$ fulfill the assumptions of Lemma 3.17 (in the sense of Remark 3.18). We conclude

$$m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_I)\leq\sum_{j\in J}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_{I_j})+C\sum_{j\in J}\mathcal{H}^{d-2}((\partial Q_{I_j}\backslash\partial Q_I)\cap\nu^\perp).$$

Applying the definition of $\mu(I,\omega)$ yields

$$\begin{aligned}\mu(I,\omega)&=m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_I)+C_\mu\mathcal{H}^{d-2}(\partial Q_I\cap\nu^\perp)\\&\leq\sum_{j\in J}\mu(I_j,\omega)+(C-C_\mu)\sum_{j\in J}\mathcal{H}^{d-2}((\partial Q_{I_j}\backslash Q_I)\cap\nu^\perp),\end{aligned}$$

which yields subadditivity provided we choose $C_\mu\geq C$. Property (ii) in Definition 2.20 is trivial since $\mu(I,\omega)$ is always nonnegative. By Theorem 2.21 there exists $\phi_{ij}^L(\omega;\nu)$ such that almost surely, for rational directions $\nu\in S^{d-1}$, it holds

$$\phi_{ij}^L(\omega;\nu)=\lim_{k\to+\infty}\frac{1}{(2Mk)^{d-1}}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,2Mk)),$$

where we used that the term $C_\mu\mathcal{H}^{d-2}(\partial I)$ is negligible for the limit in k.

Substep 2.2 From integer sequences to all sequences

Next we consider an arbitrary sequence $t_k\to+\infty$. From the previous step we know that

$$\phi_{ij}^L(\omega;\nu)=\lim_{k\to+\infty}\frac{1}{(2M\lfloor t_k\rfloor)^{d-1}}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,2M\lfloor t_k\rfloor))$$

exists almost surely. To shorten notation we set $\Lambda_k=2Mt_k$ and $\lambda_k=2M\lfloor t_k\rfloor$. For k large enough, we can apply Lemma 3.17 to the cube $Q_\nu(0,\Lambda_k)$ and singleton family $\{Q_\nu(0,\lambda_k)\}$ and obtain

$$\begin{aligned}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,\Lambda_k))\leq&m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,\lambda_k))+\mathcal{H}^{d-2}(\partial(Q_\nu(0,\lambda_k))\cap\nu^\perp)\\&+C\mathcal{H}^{d-1}((Q_\nu(0,\Lambda_k)\backslash\overline{Q_\nu(0,\lambda_k)})\cap\nu^\perp)\\\leq&m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,\lambda_k))+C\Lambda_k^{d-2},\end{aligned}$$

which yields

$$\limsup_{k\to+\infty}\frac{1}{\Lambda_k^{d-1}}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,\Lambda_k))\leq\phi_{ij}^L(\omega;\nu).\tag{3.27}$$

Similar one can prove that

$$\phi_{ij}^L(\omega;\nu) \le \liminf_{k\to+\infty} \frac{1}{\Lambda_k^{d-1}} m_1^{L,L}(\omega)(u_{0,\nu}^{ij}, Q_\nu(0,\Lambda_k)). \tag{3.28}$$

Combining (3.27) and (3.28) yields almost surely the existence of the limit for arbitrary sequences.

Substep 2.3 Shift invariance in the probability space

Up to extracting a countable union of null sets we may assume that the limit defining $\phi_{ij}^L(\omega;\nu)$ exists for all rational directions ν. We next prove that the function $\omega \mapsto \phi_{ij}^L(\omega;\nu)$ is invariant under the entire group action $\{\tau_z\}_{z\in\mathbb{Z}^d}$. This will be important to treat the ergodic case but also for the shift invariance in the physical space. Given $z \in \mathbb{Z}^d$ there exists $R = R(L,z) > 0$ such that for all $t > 0$

$$Q_\nu(0,t) \subset Q_\nu(-z, R+t), \quad 4L \le \operatorname{dist}(\partial Q_\nu(0,t), \partial Q_\nu(-z,R+t)). \tag{3.29}$$

Similar to the stationarity of the stochastic process we have

$$\begin{aligned}\phi_{ij}^L(\tau_z\omega;\nu) &\le \limsup_{t\to+\infty} \frac{1}{(R+t)^{d-1}} m_1^{L,L}(\omega)(u_{-z,\nu}^{ij}, Q_\nu(-z,R+t)) \\ &= \limsup_{t\to+\infty} \frac{1}{t^{d-1}} m_1^{L,L}(\omega)(u_{-z,\nu}^{ij}, Q_\nu(-z,R+t)).\end{aligned}$$

Due to (3.29) we can apply Lemma 3.17 to the cube $Q_\nu(-z,R+t)$ and the singleton family $\{Q_\nu(0,t)\}$ and deduce that there exists a constant $C = C(R,z)$ such that

$$m_1^{L,L}L(\omega)(u_{-z,\nu}^{ij}, Q_\nu(-z,R+t)) \le m_1^{L,L}(\omega)(u_{0,\nu}^{ij}, Q_\nu(0,t)) + Ct^{d-2}.$$

Hence we get $\phi_{ij}^L(\tau_z\omega;\nu) \le \phi_{ij}^L(\omega;\nu)$. The other inequality can be proven similar so that the limit indeed exists (which we implicitly assumed with our notation) and, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\phi_{ij}^L(\tau_z\omega;\nu) = \phi_{ij}^L(\omega;\nu). \tag{3.30}$$

In particular the function $\omega \mapsto \phi_{ij}^L(\omega;\nu)$ is measurable with respect to the σ-algebra $\mathcal{J}$ of invariant sets, that is

$$\mathcal{J} := \{A \in \mathcal{F} :\ \mathbb{P}(A\Delta\tau_z A) = 0 \quad \forall z \in \mathbb{Z}^d\}.$$

Step 3 Shift invariance in the physical space

In this step we prove the existence of the limit defining $\phi_{ij}^L(\omega;\nu)$ when we blow up a cube not centered in the origin. We further show that it agrees with the one already considered. We first take a cube $Q_\nu(x,\rho)$ with rational direction ν, $x\in\mathbb{Z}^d\backslash\{0\}$ and $\rho\in\mathbb{Q}$. Given $\varepsilon>0$ we define the events

$$\mathcal{Q}_K := \left\{\omega\in\Omega:\ \sup_{t\geq\frac{K}{2}}\left|(t\rho)^{1-d}m_1^{L,L}(\omega)(u_{0,\nu}^{ij},Q_\nu(0,t\rho))-\phi_{ij}^L(\omega;\nu)\right|\leq\varepsilon\right\}.$$

By Step 2 we know that the function $\mathbb{1}_{\mathcal{Q}_K}$ converges almost surely to $\mathbb{1}_\Omega$ when $K\to+\infty$. Let us denote by $\mathcal{J}_x$ the σ-algebra of invariant sets for the measure preserving map τ_x. Fatou's lemma for the conditional expectation yields

$$\mathbb{1}_\Omega=\mathbb{E}[\mathbb{1}_\Omega|\mathcal{J}_x]\leq\liminf_{K\to+\infty}\mathbb{E}[\mathbb{1}_{\mathcal{Q}_K}|\mathcal{J}_x]. \tag{3.31}$$

By (3.31), given $\delta>0$, almost surely we find $K_0=K_0(\omega,\delta)$ such that

$$1\geq\mathbb{E}[\mathbb{1}_{\mathcal{Q}_{K_0}}|\mathcal{J}_x](\omega)\geq1-\delta.$$

Due to Theorem 2.22, almost surely, there exists $k_0=k_0(\omega,\delta)$ such that, for any $k\geq\frac{k_0}{2}$,

$$\left|\frac{1}{k}\sum_{i=1}^k\mathbb{1}_{\mathcal{Q}_{K_0}}(\tau_{ix}\omega)-\mathbb{E}[\mathbb{1}_{\mathcal{Q}_{K_0}}|\mathcal{J}_x](\omega)\right|\leq\delta.$$

Note that the set we exclude will be a countable union of null sets provided ε is rational.

For fixed $k\geq\max\{k_0,K_0\}$ we denote by R the maximal integer such that for all $i=k+1,\dots,k+R$ we have $\tau_{ix}(\omega)\notin\mathcal{Q}_{K_0}$. In order to bound R let $\tilde{k}$ be the number of unities in the sequence $\{\mathbb{1}_{\mathcal{Q}_{K_0}}(\tau_{ix}(\omega))\}_{i=1}^k$. By definition of R we have

$$\delta\geq\left|\frac{\tilde{k}}{k+R}-\mathbb{E}[1_{\mathcal{Q}_{K_0}}|\mathcal{J}_x](\omega)\right|=\left|1-\mathbb{E}[1_{\mathcal{Q}_{K_0}}|\mathcal{J}_x](\omega)+\frac{\tilde{k}-k-R}{k+R}\right|\geq\frac{R+k-\tilde{k}}{k+R}-\delta.$$

Since $k-\tilde{k}\geq0$ and without loss of generality $\delta\leq\frac{1}{4}$, this provides an upper bound by $R\leq4k\delta$.

So for any $k\geq\max\{k_0,K_0\}$ and $\tilde{R}=6k\delta$ we find $l_k\in[k+1,k+\tilde{R}]$ such that $\tau_{l_kx}(\omega)\in\mathcal{Q}_{K_0}$. Then by stationarity and (3.30) we have for all $t\geq\frac{K_0}{2}$ that

$$\left|(t\rho)^{1-d}m_1^{L,L}(\omega)(u_{-l_kx,\nu}^{ij},Q_\nu(-l_kx,t\rho))-\phi_{ij}^L(\omega;\nu)\right|\leq\varepsilon. \tag{3.32}$$

We define $\beta_k=k+c_L\rho^{-1}|x|(l_k-k)$, where $c_L\in\mathbb{N}$ is chosen such that $Q_\nu(-kx,k\rho)\subset Q_\nu(-l_kx,\beta_k\rho)$ and $\operatorname{dist}(\partial Q_\nu(-kx,k\rho),\partial Q_\nu(-l_kx,\beta_k\rho))>L$. Observe that such c_L

exists as $l_k - k \geq 1$. Then each face of the cube $Q_\nu(-kx, k\rho)$ has at most the distance $(\beta_k - k)\rho = c_L|x|(l_k - k)$ to the corresponding face in $Q_\nu(-l_k x, \beta_k\rho)$. Now, for k large enough, we can apply Lemma 3.17 to the cube $Q(-l_k x, \beta_k \rho)$ and the singleton family $\{Q_\nu(-kx, k\rho)\}$ to obtain

$$\begin{aligned}\frac{m_1^{L,L}(\omega)(u^{ij}_{-l_k x,\nu}, Q_\nu(-l_k x, \beta_k\rho))}{(\beta_k\rho)^{d-1}} &\leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-kx,\nu}, Q_\nu(-kx, k\rho))}{(\beta_k\rho)^{d-1}} + C\tilde{R}(\beta_k\rho)^{-1} \\ &\leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-kx,\nu}, Q_\nu(-kx, k\rho))}{(k\rho)^{d-1}} + 6C\delta. \qquad (3.33)\end{aligned}$$

On the other hand we can define $\theta_k = k - c_L'\rho^{-1}|x|(l_k - k)$ for a suitable $c_L' \in \mathbb{N}$ and deduce from a similar reasoning that

$$\frac{m_1^{L,L}(\omega)(u^{ij}_{-kx,\nu}, Q_\nu(-kx, k\rho))}{(k\rho)^{d-1}} \leq \frac{m_1^{L,L}(\omega)(u^{ij}_{-l_k x,\nu}, Q_\nu(-l_k x, \theta_k\rho))}{(\theta_k\rho)^{d-1}} + 6C\delta. \qquad (3.34)$$

Now if δ is small enough (depending only on x, L and ρ) we have $\beta_k \geq \theta_k \geq \frac{k}{2} \geq \frac{K_0}{2}$. Combining (3.33), (3.34) and (3.32) we infer

$$\limsup_{k\to+\infty} \left| \frac{m_1^{L,L}(\omega)(u^{ij}_{-kx,\nu}, Q_\nu(-kx, k))}{k^{d-1}} - \phi^L_{ij}(\omega;\nu) \right| \leq 6C\delta + \varepsilon,$$

which yields the claim in (3.24) for $Q_\nu(x,\rho)$ with $x \in \mathbb{Z}^d$ and rational ν and ρ. The extension to arbitrary sequences $t_k \to +\infty$ (and thus to rational centers x) can be achieved again by Lemma 3.17 comparing first the minimal energy on the two cubes $Q_\nu(\lfloor t_k \rfloor x, \lfloor t_k \rfloor \rho)$ and $Q_\nu(\lfloor t_k \rfloor x, t_k\rho)$ similar to Substep 2.2 and then the energy on the latter cube with the one on $Q_\nu(t_k x, t_k\rho)$ as in Substep 2.3. Eventually the convergence of irrational ρ follows from the estimate

$$m_1^{L,L}(\omega)(u^{ij}_{t_k x,\nu}, Q_\nu(t_k x, t_k\rho)) \leq m_1^{L,L}(\omega)(u^{ij}_{t_k x,\nu}, Q_\nu(t_k x, t_k(\rho-\delta)) + Ct_k\delta(t_k\rho)^{d-2},$$

which is a consequence of Lemma 3.17 applied to the cube $Q_\nu(t_k x, t_k\rho)$ and the singleton family $\{Q_\nu(t_k x, t_k(\rho-\delta))\}$, when one neglects lower order terms. Choosing $0 < \delta_n \to 0$ such that $\rho - \delta_n \in \mathbb{Q}$ then yields

$$\limsup_k \frac{m_1^{L,L}(\omega)(u^{ij}_{t_k x,\nu}, Q_\nu(t_k x, t_k\rho))}{(t_k\rho)^{d-1}} \leq \phi^L_{ij}(\omega;\nu).$$

Using the same argument for the cube $Q_\nu(t_k x, t_k(\rho+\delta))$ and the family $\{Q_\nu(t_k x, t_k\rho)\}$ we find that the limit exists and agrees with $\phi^L_{ij}(\omega;\nu)$. Finally, for irrational centers we can use again a perturbation argument based on Lemma 3.17 as we did for proving (3.33) and (3.34). We omit the details.

Step 4 From rational to irrational directions

Now we extend the convergence from rational directions to all $\nu \in S^{d-1}$. As the argument is purely geometric, we assume without loss of generality that $x = 0$. First note that the set of rational directions is dense in S^{d-1} (as the inverse of the stereographic projection maps rational points to rational directions). Given $\nu \in S^{d-1}$ and a sequence $t_k \to +\infty$ we define

$$\overline{\phi}^L_{ij}(\omega;\nu) = \limsup_{k\to+\infty} \frac{1}{t_k^{d-1}} m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t_k)),$$
$$\underline{\phi}^L_{ij}(\omega;\nu) = \liminf_{k\to+\infty} \frac{1}{t_k^{d-1}} m_1^{L,L}(\omega)(u^{ij}_{0,\nu}, Q_\nu(0,t_k)).$$

Let $\nu \in S^{d-1}\backslash\mathbb{Q}^d$. By the construction of the matrix A_ν in Substep 2.1 we can assume that there exists a sequence of rational directions ν_n such that $A_{\nu_n} \to A_\nu$. Therefore, given $\delta > 0$ we find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following properties hold:

(i) $Q_\nu(0,(1-2\delta)) \subset\subset Q_{\nu_n}(0,1-\delta) \subset\subset Q_\nu(0,1)$,

(ii) $0 < \mathrm{d}_{\mathcal{H}}(\nu^\perp \cap B_2(0), \nu_n^\perp \cap B_2(0)) \leq \delta$.

Now we argue similar as in the proof of Lemma 3.17. For fixed $n \geq n_0$ and $k \in \mathbb{N}$ let $u_{n,k} \in \mathcal{PC}_1(\omega)$ be an admissible minimizer for $m_1^{L,L}(\omega)(u^{ij}_{0,\nu_n}, Q_{\nu_n}(0,(1-\delta)t_k))$. We define a test function $v_k \in \mathcal{PC}_1(\omega)$ setting

$$v_k(x) := \begin{cases} u_{n,k}(x) & \text{if } x \in Q_{\nu_n}(0,(1-\delta)t_k), \\ u^{ij}_{0,\nu}(x) & \text{otherwise.} \end{cases}$$

Note that if $x, y \in Q_\nu(0,t_k)\backslash Q_{\nu_n}(0,(1-\delta)t_k)$ are such that $|x-y| \leq L$ and $v_k(x) \neq v_k(y)$, then by the choice of n_0 and (i), for n large enough we have

$$\mathrm{dist}\left(x, (Q_\nu(0,t_k)\backslash Q_\nu(0,(1-2\delta)t_k)) \cap \nu^\perp\right) \leq L. \tag{3.35}$$

If $x \in Q_\nu(0,t_k)\backslash Q_{\nu_n}(0,(1-\delta)t_k)$ and $y \in Q_{\nu_n}(0,(1-\delta)t_k)$ with $|x-y| \leq L$ and $v_k(x) \neq v_k(y)$, then, for n large enough one can show that by (ii) either x or y must lie in the cone

$$\mathcal{K}(\nu,\nu_n) = \{x \in \mathbb{R}^d : \langle x,\nu\rangle \cdot \langle x,\nu_n\rangle \leq 0\}.$$

As the ray $[x,y]$ intersects $\partial Q_{\nu_n}(0,(1-\delta)t_k)$, we conclude that

$$\operatorname{dist}(x,(\mathcal{K}(\nu,\nu_n)+B_L(0))\cap\partial Q_{\nu_n}(0,(1-\delta)t_k))\leq L. \tag{3.36}$$

By (i) it holds that $v_k\in\mathcal{PC}^L_{1,u^{ij}_{0,\nu}}(\omega,Q_\nu(0,t_k))$ for k large enough. From (3.35), (3.36) and the choice of n_0 we deduce that for n large enough

$$m_1^{L,L}(\omega)(u^{ij}_{0,\nu},Q_\nu(0,t_k)\leq m_1^{L,L}(\omega)(u^{ij}_{0,\nu_n},Q_{\nu_n}(0,(1-\delta)t_k))+C\delta t_k^{d-1}.$$

Dividing the last inequality by t_k^{d-1} and passing to the right subsequence of t_k we deduce

$$\overline{\phi}^L_{ij}(\omega;\nu)\leq\phi^L_{ij}(\omega;\nu_n)+C\delta.$$

Letting first $n\to+\infty$ and then $\delta\to 0$ yields $\overline{\phi}^L_{ij}(\omega;\nu)\leq\liminf_n\phi^L_{ij}(\omega;\nu_n)$. By a similar argument we can also prove that $\limsup_n\phi^L_{ij}(\omega;\nu_n)\leq\underline{\phi}^L_{ij}(\omega;\nu)$. Hence we get almost surely the existence of the limit in (3.24) for all directions ν and the limit does not depend on x,ρ and the sequence t_k.

Step 5 Proof of (3.23)

We claim that $\phi^L_{ij}(\omega;\nu)=\phi^L_{\mathrm{hom}}(\omega;m_i,m_j,\nu)$. By the preceding steps this would conclude the proof. First observe that by monotonicity it is enough to show that $\phi^L_{\mathrm{hom}}(\omega;m_i,m_j,\nu)\leq\phi^L_{ij}(\omega;\nu)$. Let $t_k\to+\infty$ and fix a cube $Q_\nu(x,\rho)$. By a trivial extension argument, for η small enough (depending on ρ) it holds that

$$m_1^{\eta t_k,L}(\omega)(u^{ij}_{t_kx,\nu},Q(t_kx,t_k\rho))\leq m_1^{L,L}(\omega)(u^{ij}_{t_kx,\nu},Q(t_kx,t_k\rho-\eta t_k))+C\eta t_k^{d-1}.$$

Dividing by $(t_k\rho)^{d-1}$ and letting first $k\to+\infty$ and then $\eta\to 0$ we obtain the claim.

When the group action is ergodic, the additional statement in Theorem 3.19 follows easily from (3.30) since in this case all the functions $\omega\mapsto\phi^L_{ij}(\omega;\nu)$ are constant and so is the pointwise limit when $L\to+\infty$. □

Remark 3.21. As the integrand of the Γ-limit is independent of x, we deduce in particular that for any fixed m_i,m_j and $\mathbb{P}$-almost every $\omega\in\Omega$ the function $\mathbb{R}^d\ni\nu\mapsto\phi_{\mathrm{hom}}(\omega;m_i,m_j,\frac{\nu}{|\nu|})|\nu|$ is convex (this follows for instance from Theorem 3.1 in [9]). In particular we obtain local Lipschitz continuity with respect to ν. In the ergodic case this implies that the whole limit energy is deterministic.

Finally we give a simpler expression for the limit integrand, where only one limit procedure is necessary. Moreover, in the next section we prove a result that shows that the formulas for the limit integrand are independent of the choice of basis vectors for the cube Q_ν (see Remark 3.25).

Corollary 3.22. *Let l_ε be sequence satisfying (3.37). Under the assumptions from Theorem 3.19, almost surely it holds that*

$$\phi_{\rm hom}(\omega;m_i,m_j,\nu)=\lim_{t\to+\infty}\frac{1}{t^{d-1}}\inf\left\{E_1(\omega)(u,tQ_\nu):\ u\in\mathcal{PC}^{l_{1/t}}_{1,u_{0,m_i,m_j,\nu}}(\omega,tQ_\nu)\right\}.$$

Proof. As $l_{1/t}\leq\eta t$ for fixed $\eta>0$ and t large enough, $\phi_{\rm hom}(\omega;m_i,m_j,\nu)$ dominates the limes superior of the right hand side. On the other hand, again by monotonicity we have

$$\liminf_{t\to+\infty}\left\{E_1(\omega)(u,tQ_\nu):\ u\in\mathcal{PC}^{l_{1/t}}_{1,u_{0,m_i,m_j,\nu}}(\omega,tQ_\nu)\right\}\geq\phi^L_{ij}(\omega;\nu).$$

Letting $L\to+\infty$, by Steps 1 and 5 of the proof of Theorem 3.19 we obtain that the limit exists and agrees with $\phi_{\rm hom}(\omega;m_i,m_j,\nu)$. □

3.4 Convergence of boundary value problems

In this section we consider the convergence of minimum problems under Dirichlet-type boundary conditions. In contrast to the rather artificial boundary conditions used in Section 3.3 we now consider boundary conditions depending only on the microscopic ε-scale. Due to possible long range interactions the boundary still needs to blow up in some weak sense.

Let us fix $A\in\mathcal{A}^R(D)$. Still we restrict the analysis to the case when the boundary data u_0 is well-prepared in the sense of (3.13).

We define a discrete trace constraint as follows: Let $l_\varepsilon>0$ be such that

$$\lim_{\varepsilon\to0}l_\varepsilon=+\infty,\quad \lim_{\varepsilon\to0}l_\varepsilon\varepsilon=0. \tag{3.37}$$

As in Section 3.3 we set $\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\omega,A)$ as the space of those u that agree with u_0 at the discrete boundary of A, namely

$$\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\omega,A):=\{u\in\mathcal{PC}_\varepsilon(\omega):\ u(\varepsilon x)=u_0(\varepsilon x)\text{ if }{\rm dist}(\varepsilon x,\partial A)\leq l_\varepsilon\varepsilon\}.$$

In contrast to Lemma 3.16 we now prove a full Γ-convergence result. For $\varepsilon>0$ and $l_\varepsilon>0$ we consider the restricted functional $E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\cdot,A):L^1(D)\to[0,+\infty]$ defined as

$$E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u,A):=\begin{cases}E_\varepsilon(\omega)(u,A) & \text{if } u\in\mathcal{PC}^{l_\varepsilon\varepsilon}_{\varepsilon,u_0}(\omega,A),\\ +\infty & \text{otherwise.}\end{cases} \tag{3.38}$$

The following convergence result holds true:

Theorem 3.23. *Under the assumptions from Theorem 3.19, for every set $A \in \mathcal{A}^R(D)$ such that $A \subset\subset D$ the functionals $E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\cdot, A)$ defined in (3.38) Γ-converge with respect to the strong $L^1(A)$-topology to the functional $E_{u_0}(\omega)(\cdot, A) : L^1(D) \to [0, +\infty]$ that is finite only for $u \in BV(A, \mathcal{M})$ where it takes the form*

$$E_{u_0}(\omega)(u, A) = \int_{S_u \cap A} \phi_{\text{hom}}(\omega; u^+, u^-, \nu_u) \, \mathrm{d}\mathcal{H}^{d-1} + \int_{\partial A} \phi_{\text{hom}}(\omega; u^+, u_0^-, \nu_x) \, \mathrm{d}\mathcal{H}^{d-1},$$

where ν_x denotes a suitably orientated normal vector to ∂A.

Proof. By Proposition 3.11 we know that any Γ-limit can be finite only for $u \in BV(A, \mathcal{M})$. We now split the argument in two steps:
Step 1 Proof of the lim inf-inequality.
Without loss of generality let $u_\varepsilon \to u$ in $L^1(A)$ be such that

$$\liminf_{\varepsilon} E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u_\varepsilon, A) < C.$$

Passing to a subsequence we can assume that $u_\varepsilon \in \mathcal{PC}^{l_\varepsilon \varepsilon}_{\varepsilon,u_0}(\omega, A)$. Given $\delta > 0$, by (3.13) and choosing appropriate level sets of the signed distance function of ∂A, Theorem 2.15 and Remark 2.16 yield Lipschitz sets $A_1 \subset\subset A \subset\subset A_2$ such that

$$\begin{aligned} &\mathcal{H}^{d-1}(S_{u_0} \cap \partial A_1) = \mathcal{H}^{d-1}(S_{u_0} \cap \partial A_2) = 0, \\ &\mathcal{H}^{d-1}(S_{u_0} \cap (A_2 \backslash \overline{A_1})) \leq \delta. \end{aligned}$$

Recall that $u_{\varepsilon,0} \in \mathcal{PC}^{l_\varepsilon \varepsilon}_{\varepsilon,u_0}(\omega, A)$ denotes the function defined by $u_{\varepsilon,0}(\varepsilon x) = u_0(\varepsilon x)$. By (3.13) we have $u_{\varepsilon,0} \to u_0$ in $L^1(D)$ and, by the choice of A_1 and A_2,

$$\limsup_{\varepsilon} E_\varepsilon(\omega)(u_{\varepsilon,0}, A_2 \backslash \overline{A_1}) \leq C\,\delta. \tag{3.39}$$

Next we define $\tilde{u}_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ by

$$\tilde{u}_\varepsilon(\varepsilon x) = \mathbb{1}_A(\varepsilon x) u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_A(\varepsilon x)) u_{\varepsilon,0}(\varepsilon x).$$

Let us observe that $\tilde{u}_\varepsilon \to \mathbb{1}_A u + (1 - \mathbb{1}_A) u_0$ in $L^1(D)$. Setting

$$R^\xi_\varepsilon := \{\alpha \in R^\xi_\varepsilon(A_2) : \ \varepsilon x_\alpha \in A,\, \varepsilon x_{\alpha+\xi} \notin A \text{ or } \varepsilon x_\alpha \notin A,\, \varepsilon x_{\alpha+\xi} \in A\},$$

for all $L \in \mathbb{N}$ the truncated energies satisfy

$$\begin{aligned} E_\varepsilon^L(\omega)(\tilde{u}_\varepsilon, A_2) \leq & E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(u_{\varepsilon,0}, A_2\backslash\overline{A_1}) \\ & + \sum_{|\xi|\leq L}\sum_{\alpha\in R_\varepsilon^\xi} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, \tilde{u}_\varepsilon(\varepsilon x_\alpha), \tilde{u}_\varepsilon(\varepsilon x_{\alpha+\xi})). \end{aligned} \tag{3.40}$$

For interactions where $|\xi| \leq L$ and ε small enough, we have that $R_\varepsilon^\xi \subset A_2\backslash\overline{A_1}$. Moreover, if $l_\varepsilon > L + 2r$, we can use the boundary conditions on u_ε to infer

$$\sum_{|\xi|\leq L}\sum_{\alpha\in R_\varepsilon^\xi} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, \tilde{u}_\varepsilon(\varepsilon x_\alpha), \tilde{u}_\varepsilon(\varepsilon x_{\alpha+\xi})) \leq E_\varepsilon(\omega)(u_{\varepsilon,0}, A_2\backslash\overline{A_1}).$$

From Theorem 3.19 applied to the truncated energies $E_\varepsilon^L(\omega)$ and the estimates (3.39) and (3.40) we conclude

$$E_{\mathrm{hom}}^L(\omega)(\mathbb{1}_A u + (1-\mathbb{1}_A)u_0, A_2) \leq \liminf_\varepsilon E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u_\varepsilon, A) + C\,\delta.$$

Now letting $A_2 \downarrow \overline{A}$, then $\delta \to 0$ and finally $L \to +\infty$ the $\liminf$-inequality follows by Step 1 of the proof of Theorem 3.19 and monotone convergence.

Step 2 Proof of the $\limsup$-inequality.

We start assuming that $u = u_0$ in a neighborhood of ∂A. In this case exactly the same construction as in the proof of Lemma 3.16 yields

$$\Gamma\text{-}\limsup_\varepsilon E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u, A) \leq E_{\mathrm{hom}}(\omega)(u, A).$$

Note that (3.37) guarantees that the constructed sequence fulfills the correct boundary conditions for ε small enough.

Now given any $u \in BV(A, \mathcal{M})$ let u_n be the sequence given by Lemma 2.17 with $v = u$ and $w = u_0$ and let $A' \in \mathcal{A}^R(D)$, $A \subset\subset A'$. By lower semicontinuity, the previous reasoning and Lemma 2.14 combined with Remark 3.21 we have

$$\begin{aligned} \Gamma\text{-}\limsup_\varepsilon E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u, A) &\leq \liminf_n \left(\Gamma\text{-}\limsup_\varepsilon E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u_n, A)\right) \\ &\leq \liminf_n E_{\mathrm{hom}}(\omega)(u_n, A') = E_{\mathrm{hom}}(\omega)(\mathbb{1}_A u + (1-\mathbb{1}_A)u_0, A'). \end{aligned}$$

Letting $A' \downarrow \overline{A}$ yields the upper bound. □

Corollary 3.24. *Let $A \in \mathcal{A}^R(D)$, $A \subset\subset D$. Under the assumptions of Theorem 3.19, the following holds almost surely:*

(i)

$$\lim_{\varepsilon}\left(\inf_{u\in BV(A,\mathcal{M})} E^{l_\varepsilon}_{\varepsilon,u_0}(u,A)\right) = \min_{u\in BV(A,\mathcal{M})} E_{u_0}(\omega)(u,A).$$

(ii) Moreover, if $(u_\varepsilon)_\varepsilon$ is a converging sequence with respect to $L^1(A)$ such that

$$E^{l_\varepsilon}_{\varepsilon,u_0}(u_\varepsilon,A) = \inf_{u\in BV(A,\mathcal{M})} E^{l_\varepsilon}_{\varepsilon,u_0}(u,A) + o(1),$$

then its limit is a minimizer of $E_{u_0}(\omega)(\cdot,A)$.

Proof. The statement follows from the general theory of Γ-convergence (in particular Theorem 2.7) and Theorem 3.23 since the functionals are equicoercive in $L^1(A)$. $\square$

Remark 3.25. (i) If we have only a finite range of interactions, which means that $g^\omega(x,y,m_i,m_j) = 0$ for $|x-y| \geq L$ then (3.37) can be weakened to $l_\varepsilon \geq L$.

(ii) Note that if $u_0 = u_{0,m_i,m_j,\nu}$ and $A = Q_\nu$, then it holds that

$$\min_{u\in BV(Q_\nu,\mathcal{M})} E_{u_0}(\omega)(u,Q_\nu) = E_{u_0}(\omega)(u_0,Q_\nu) = \phi_{\text{hom}}(\omega;m_i,m_j,\nu).$$

The first equality is a consequence of so-called BV-ellipticity and the fact that $E_{u_0}(\omega)$ coincides with the L^1-relaxation of $E(\omega)$ when we put Dirichlet-boundary conditions. We refer to [10] for more details as this is not the main subject of this thesis. The above identity and Corollaries 3.22 and 3.24 imply that the limit integrand does not depend on the orientation of the basis for $\nu^\perp$.

3.5 Examples and generalizations

In this section, motivated by the applications, we provide a nontrivial example of a stochastic lattices that is admissible in our setting. Then we generalize the results obtained hitherto to the presence of phase constraints.

Random parking and isotropy

It is a very challenging problem to relate the symmetries of the stochastic lattice to those of the limit energy density. It has already been observed in the periodic setting in [3] that for Ising spin systems the discrete symmetries of the periodic lattice induce anisotropies in the limit. In the stochastic setting, on the other hand, one can imagine that the anisotropy of a single realization of the stochastic lattice may be averaged out by ergodicity. This is indeed the case of another interesting and more involved probabilistic setup: the so-called random parking model investigated in [38]. This model provides an admissible stochastic lattice that is stationary, ergodic and in addition stationary with respect to rotations in the following sense: for all $R \in SO(d)$ there exists a measure preserving group action $\tau_R : \Omega \to \Omega$ such that

$$\mathcal{L}(\tau_R\omega) = R\mathcal{L}(\omega). \tag{3.41}$$

If the discrete energy densities are isotropic in the spatial variable, that means (with a slight abuse of notation)

$$g_{nn}(z, m_i, m_j) = g_{nn}(|z|, m_i, m_j), \quad g_{lr}(z, m_i, m_j) = g_{lr}(|z|, m_i, m_j), \tag{3.42}$$

then one expects the limit energy to be isotropic, too. Indeed, the following theorem holds.

Theorem 3.26. *Let $\mathcal{L}$ be a stationary (with respect to both translations and rotations), ergodic stochastic lattice and let g_{nn}, g_{lr} satisfy Hypothesis 1 and (3.42). Then $\nu \mapsto \phi_{\text{hom}}(m_i, m_j, \nu)$ is constant and the Γ-limit of the functionals $E_\varepsilon(\omega)$ is given by*

$$E(u) = \begin{cases} \int_{S_u} \phi_{\text{hom}}(u^+, u^-)\,\mathrm{d}\mathcal{H}^{d-1} & \text{if } u \in BV(D, \mathcal{M}), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.19 and Remark 3.21 it suffices to prove that $\phi_{\text{hom}}(m_i, m_j, R\nu) = \phi_{\text{hom}}(m_i, m_j, \nu)$ for all $R \in SO(d)$. Since τ_R is measure preserving, by dominated convergence, Corollary 3.22 and a change of variables it holds

$$\begin{aligned} \phi_{\text{hom}}(m_i, m_j, R\nu) &= \lim_t \frac{1}{t^{d-1}} \mathbb{E}\left[\inf\{E_1(\cdot)(u, tQ_{R\nu}) : \, u \in \mathcal{PC}^{l_{1/t}}_{1,u_{0,m_i,m_j,R\nu}}(\cdot, tQ_{R\nu})\}\right] \\ &= \lim_t \frac{1}{t^{d-1}} \mathbb{E}\left[\inf\{E_1(\tau_{R^T}\cdot)(u, tQ_\nu) : \, u \in \mathcal{PC}^{l_{1/t}}_{1,u_{0,m_i,m_j,\nu}}(\tau_{R^T}\cdot, tQ_\nu)\}\right] \\ &= \lim_t \frac{1}{t^{d-1}} \mathbb{E}\left[\inf\{E_1(\cdot)(u, tQ_\nu) : \, u \in \mathcal{PC}^{l_{1/t}}_{1,u_{0,m_i,m_j,\nu}}(\cdot, tQ_\nu)\}\right] \\ &= \phi_{\text{hom}}(m_i, m_j, \nu), \end{aligned}$$

where we used (3.42) from the first to the second line as well as Remark 3.25 (ii). □

Results for phase constraints

We finally consider the effect on the asymptotic behavior of the energy when we fix the number of particles taking the magnetization m_i for all i. More precisely, given a family of natural numbers $V_\varepsilon = \{V_{i,\varepsilon}\}_{i=1}^q$, we set

$$\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega) := \{u \in \mathcal{PC}_\varepsilon(\omega) :\ \#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap D :\ u(\varepsilon x) = m_i\} = V_{i,\varepsilon}\}\,.$$

Beside the natural compatibility condition $\sum_i V_{i,\varepsilon} = \#(\varepsilon\mathcal{L}(\omega) \cap D)$, we assume that, for all $i = 1, \ldots, q$, there exists

$$\lim_{\varepsilon \to 0} \frac{V_{i,\varepsilon}}{\#(\varepsilon\mathcal{L}(\omega) \cap D)} = V_i > 0. \tag{3.43}$$

Note that we exclude that case that $V_i = 0$ for some i. This case contains some non-trivial aspects which are related to the concept of (B)-convexity studied in [10]. Such conditions are not necessarily fulfilled for our discrete energies. Of course the extreme case $V_{i,\varepsilon} = 0$ for all ε can be treated by changing the set $\mathcal{M}$ (and then probably also the continuum limit). Moreover we stress that, in contrast to the previous results, in this section it is crucial that $d \geq 2$. In one dimension there are easy counterexamples that show that volume constraint may change the limit energy of the constant states.

In order to include the phase constraints in the energy, we define the functional $E_\varepsilon^{V_\varepsilon}(\omega) : L^1(D) \to [0, +\infty]$ as

$$E_\varepsilon^{V_\varepsilon}(\omega)(u) = \begin{cases} E_\varepsilon(\omega)(u) & \text{if } u \in \mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.44}$$

The next lemma characterizes the limits of L^1-converging sequences in $\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega)$.

Lemma 3.27. *For $\mathbb{P}$-almost every $\omega \in \Omega$ the following statement holds true: If $u_\varepsilon \in \mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega)$ is a sequence such that $u_\varepsilon \to u$ in $L^1(D)$ for some $u \in BV(D, \mathcal{M})$, then for all $i = 1, \ldots, q$*

$$|\{u = m_i\}| = V_i|D|.$$

Proof. We define a random family of nonnegative functions $\mu_\varepsilon(\omega) \in L^\infty(D)$ via

$$\mu_\varepsilon(\omega) = \sum_{x \in \mathcal{L}(\omega) \cap \frac{D}{\varepsilon}} \frac{1}{|\mathcal{C}(x)|} \mathbb{1}_{\varepsilon\mathcal{C}(x)}.$$

Due to Lemma 3.4, for a set of full probability the family $\mu_\varepsilon(\omega)$ is equibounded with respect to ε. Hence we may assume that, up to subsequences, $\mu_\varepsilon(\omega) \overset{*}{\rightharpoonup} \mu(\omega)$ in $L^\infty(D)$. In order to identify the limit, we use the ergodic theorem. To this end, we note that $\mu : \mathcal{I}_d \to L^1(\Omega)$ defined as $\mu(I, \omega) := \#(\mathcal{L}(\omega) \cap I)$ is a stationary, additive stochastic process. Applying Theorem 2.21, we know that there exists a set of full probability Ω' such that for each fixed set $I \in \mathcal{I}_d$ and all $w \in \Omega'$ there exists the limit

$$\mu^\infty(\omega) = \frac{1}{n^d|I|}\#(\mathcal{L}(\omega) \cap nI).$$

By standard arguments one can extend this convergence to all sequences and then to half-open cubes with rational vertices. Finally by an approximation the convergence holds for arbitrary sequences and any half-open cube $Q \subset D$. By additivity this result then can be extended to every set $A \in \mathcal{A}^R(D)$ by interior and exterior approximation with finite unions of half-open cubes. Moreover, by Lemma 3.4 it follows that

$$\left|\int_Q \mu_\varepsilon(\omega)(x)\,\mathrm{d}x - \varepsilon^d \mu(Q/\varepsilon,\omega)\right| \leq C\mathcal{H}^{d-1}(Q)\varepsilon.$$

Hence $\mu_\varepsilon(\omega) \overset{*}{\rightharpoonup} \mu^\infty(\omega)$. In particular the limit is a constant function on D. Note that by Lemma 3.4

$$\left|\frac{V_{i,\varepsilon}}{\#(\varepsilon\mathcal{L}(\omega)\cap D)} - \int_D \frac{\mathbb{1}_{\{u_\varepsilon = m_i\}}(x)\mu_\varepsilon(\omega)(x)}{\varepsilon^d\#(\varepsilon\mathcal{L}(\omega)\cap D)}\,\mathrm{d}x\right| \leq C\frac{\mathcal{H}^{d-1}(\partial D)}{|D|}\varepsilon.$$

As the function $\mathbb{1}_{\{u_\varepsilon=m_i\}}$ converges to $\mathbb{1}_{\{u=m_i\}}$ in $L^1(D)$, combining weak*/strong convergence, we obtain in the limit that

$$V_i = \frac{1}{\mu^\infty(\omega)|D|}\int_D \mathbb{1}_{\{u=m_i\}}(x)\mu^\infty(\omega)\,\mathrm{d}x = \frac{|\{u=m_i\}|}{|D|}.$$

□

Now we can prove a full Γ-convergence result for the constrained energies.

Theorem 3.28. *Under the assumptions of Theorem 3.19 and (3.43), for $\mathbb{P}$-almost every $\omega \in \Omega$ the functionals $E_\varepsilon^{V_\varepsilon}(\omega)$ Γ-converge with respect to the $L^1(D)$-topology to the functional $E_{\mathrm{hom}}^V(\omega) : L^1(D) \to [0,+\infty]$ defined by*

$$E_{\mathrm{hom}}^V(\omega)(u) = \begin{cases} E_{\mathrm{hom}}(\omega)(u) & \text{if } u \in BV(D,\mathcal{M}) \text{ and } |\{u=m_i\}| = V_i|D| \text{ for all } i,\\ +\infty & \text{otherwise.}\end{cases}$$

Proof. The lim inf-inequality is an immediate consequence of Lemma 3.27 and the Γ-convergence result of Theorem 3.19 as $E_\varepsilon^{V_\varepsilon}(\omega)(u) \geq E_\varepsilon(\omega)(u)$ for all $u \in \mathcal{PC}_\varepsilon(\omega)$.

In order to proof the upper bound, we first have to find a suitable dense class. For the moment let us assume that $u \in BV(D,\mathcal{M})$ satisfies the volume constraint and that each level set $\{u = m_i\}$ contains an interior point. In particular, in each level set we find q disjoint open balls $B_\eta(x_i^j) \subset\subset \{u = m_i\}$. By Theorem 3.19 there exists a sequence $u_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ such that $u_\varepsilon \to u$ in $L^1(D)$ and

$$\lim_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon) = E_{\hom}(\omega)(u). \tag{3.45}$$

Arguing similar to the proof of Proposition 3.15, up to reducing η we may assume that $u_\varepsilon(\varepsilon x) = m_i$ for all $\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap B_\eta(x_i^j)$ and that u_ε has equibounded energy on a larger cube Q_D containing $\overline{D}$. For each i let us set $V'_{i,\varepsilon} = \#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap D : u_\varepsilon(\varepsilon x) = m_i\}$. As $u_\varepsilon \to u$ in $L^1(D)$, we can repeat the arguments from the proof of Lemma 3.27 and obtain that

$$\lim_{\varepsilon\to 0} \frac{V'_{i,\varepsilon} - V_{i,\varepsilon}}{\#(\varepsilon\mathcal{L}(\omega) \cap D)} = 0. \tag{3.46}$$

We now adjust the sequence u_ε so that it belongs to $\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega)$. This will be done locally on the balls $B_\eta(x_i^j)$. First we change the values on $B_\eta(x_1^1)$ and $B_\eta(x_2^1)$ so that the sequence satisfies the constraint for $i = 1$. In general, for $i < q$ we change the sequence on $B_\eta(x_i^i)$ and $B_\eta(x_{i+1}^i)$ so that it satisfies the constraints for all $j \leq i$. At the end the constraint for $i = q$ follows by the compatibility assumption. Each modification will be such that L^1-convergence and convergence of the energies is conserved. We will provide the construction only for the first step. In what follows we consider the case $V'_{1,\varepsilon} > V_{1,\varepsilon}$. We set $h_\varepsilon = (V'_{1,\varepsilon} - V_{1,\varepsilon})^{\frac{1}{d}}$. Without loss of generality we may assume that $h_\varepsilon \to +\infty$, otherwise we change u_ε on a set of lattice points with diverging cardinality much less than ε^{1-d} that is contained in the complement of the union of balls $B_\eta(x_i^j)$. Using the integrability assumption from Hypothesis 1 it can be verified that this operation still yields a recovery sequence with the same properties.

Observe that (3.46) and Lemma 3.4 imply that

$$\lim_{\varepsilon\to 0} h_\varepsilon \varepsilon = 0. \tag{3.47}$$

From the proof of Lemma 3.27 we already know that, almost surely, we can write

$$q_1^\omega(h_\varepsilon) := \#\{x \in \mathcal{L}(\omega) \cap Q_{e_1}(x_1^1, \mu^\infty(\omega)^{-1}h_\varepsilon)\} = h_\varepsilon^d + h_\varepsilon^{d-1}\gamma_\varepsilon,$$

for some sequence $\gamma_\varepsilon = \gamma_\varepsilon(\omega, x_1^1)$ such that $\lim_{\varepsilon\to 0} \frac{\gamma_\varepsilon}{h_\varepsilon} = 0$. In the following we assume that $\gamma_\varepsilon \leq 0$, but with a similar argument we can also treat the case $\gamma_\varepsilon > 0$. As $\mathcal{L}(\omega)$ is admissible in the sense of Definition 3.1, using Lemma 3.4 one can show that for some appropriate $c = c(R) > 0$ it holds true that

$$\frac{1}{C}h_\varepsilon^{d-1} \leq q_1^\omega(h_\varepsilon + n + c) - q_1^\omega(h_\varepsilon + n) \leq Ch_\varepsilon^{d-1}$$

for any $0 \leq n \leq h_\varepsilon$. In particular, there exists $n_\varepsilon = \mathcal{O}(\gamma_\varepsilon)$ and a nonnegative equibounded c_ε such that

$$q_1^\omega(h_\varepsilon + n_\varepsilon) = h_\varepsilon^d + c_\varepsilon h_\varepsilon^{d-1}. \tag{3.48}$$

Using (3.47) we can find a set $G_\varepsilon \subset B_\eta(x_2^1)$ such that $\#\left(G_\varepsilon \cap \varepsilon\mathcal{L}(\omega)\right) = c_\varepsilon h_\varepsilon^{d-1}$. To reduce notation let us write $Q_\varepsilon := Q_{e_1}(x_1^1, \mu^\infty(\omega)^{-1}\varepsilon(h_\varepsilon + n_\varepsilon))$. We define

$$\bar{u}_\varepsilon(\varepsilon x) = \begin{cases} m_2 & \text{if } \varepsilon x \in Q_\varepsilon, \\ m_1 & \text{if } \varepsilon x \in G_\varepsilon, \\ u_\varepsilon(\varepsilon x) & \text{otherwise.} \end{cases}$$

By (3.47) it holds $Q_\varepsilon \subset\subset B_\eta(x_1^1)$ for ε small enough and therefore $\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap D : \bar{u}_\varepsilon(\varepsilon x) = m_1\} = V_{1,\varepsilon}$ as well as $\bar{u}_\varepsilon \to u$ in $L^1(D)$. We now estimate the energy. From Hypothesis 1 we deduce

$$\begin{aligned} E_\varepsilon(\omega)(\bar{u}_\varepsilon) \leq & E_\varepsilon(\omega)(u_\varepsilon) + C\sum_{\xi\in r'\mathbb{Z}^d} J_{lr}(|\hat{\xi}|)\#(G_\varepsilon \cap \varepsilon\mathcal{L}(\omega))\varepsilon^{d-1} \\ & + \sum_{\xi\in r'\mathbb{Z}^d} \sum_{\substack{\alpha\in R_\varepsilon^\xi(D) \\ \varepsilon[x_\alpha,x_{\alpha+\xi}]\cap\partial Q_\varepsilon\neq\emptyset}} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, \bar{u}_\varepsilon(\varepsilon x_\alpha), \bar{u}_\varepsilon(\varepsilon x_{\alpha+\xi})). \end{aligned} \tag{3.49}$$

It remains to bound the last term as the second one vanishes by (3.47) and integrability of J_{lr}. We split the interactions according to Remark 3.9. By Lemma 3.12 and Hypothesis 1, for ε small enough we have by construction

$$\begin{aligned} &\sum_{|\xi|\leq L_\delta} \sum_{\substack{\alpha\in R_\varepsilon^\xi(D) \\ \varepsilon[x_\alpha,x_{\alpha+\xi}]\cap\partial Q_\varepsilon\neq\emptyset}} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}, \bar{u}_\varepsilon(\varepsilon x_\alpha), \bar{u}_\varepsilon(\varepsilon x_{\alpha+\xi})) \\ &\leq C\sum_{|\xi|\leq L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y)\in\mathcal{NN}(\omega) \\ \varepsilon x,\varepsilon y\in B_\eta(x_1^1)}} \varepsilon^{d-1}|\bar{u}_\varepsilon(\varepsilon x) - \bar{u}_\varepsilon(\varepsilon y)| \leq C\mathcal{H}^{d-1}(\partial Q_\varepsilon) \leq C(\varepsilon h_\varepsilon)^{d-1}, \end{aligned} \tag{3.50}$$

so that the left hand side vanishes when $\varepsilon \to 0$. To control the remaining interactions, recall that u_ε has finite energy on the larger cube Q_D. Hence Lemma 3.12 and

Hypothesis 1 yield

$$\begin{aligned}
&\sum_{|\xi|>L_\delta} \sum_{\substack{\alpha\in R_\varepsilon^\xi(D)\\ \varepsilon[x_\alpha,x_{\alpha+\xi}]\cap\partial Q_\varepsilon\neq\emptyset}} \varepsilon^{d-1} g_\varepsilon^\omega(x_\alpha,x_{\alpha+\xi},\bar u_\varepsilon(\varepsilon x_\alpha),\bar u_\varepsilon(\varepsilon x_{\alpha+\xi}))\\
&\leq C\delta \sum_{\substack{(x,y)\in\mathcal{NN}(\omega)\\ \varepsilon x,\varepsilon y\in Q_D}} \varepsilon^{d-1}|\bar u_\varepsilon(\varepsilon x)-\bar u_\varepsilon(\varepsilon y)|\\
&\leq C\delta\left(E_\varepsilon(\omega)(u_\varepsilon,Q_D)+\mathcal{H}^{d-1}(\partial Q_\varepsilon)+\#(G_\varepsilon\cap\varepsilon\mathcal{L}(\omega))\varepsilon^{d-1}\right)\leq C\delta.
\end{aligned}$$

As $\delta>0$ was arbitrary, we infer from (3.45), (3.49) and (3.50) that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(\bar u_\varepsilon)=\limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon)=E_{\mathrm{hom}}(\omega)(u).$$

The case when $V_\varepsilon'\leq V_\varepsilon$ can be treated by an almost symmetric argument. Repeating this construction for the remaining phases as described at the beginning of this proof, we obtain

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon^{V_\varepsilon}(\omega)(u)=E_{\mathrm{hom}}(\omega)(u).$$

Now for a general $u\in BV(D,\mathcal{M})$ such that $|\{u=m_i\}|=V_i|D|$, the statement follows by density. This procedure is classical (see [9, 12]), but we display it for the reader's convenience. As all level sets have positive measure, for each i we find a point $x_i\in S_u$ and a second index $j\neq i$ such that the density of the two level sets $\{u=m_i\}$ and $\{u=m_j\}$ equals $\frac{1}{2}$. Then one takes a small ball $B_\rho(x_i)$ and divides it into concentric (possibly degenerate) annuli A_l^i such that $|A_l^i|=|B_\rho(x_i)\cap\{u=m_l\}|$. We define

$$u_\rho(x)=\begin{cases} m_l & \text{if } x\in A_l^i \text{ for some } i \text{ and } l,\\ u(x) & \text{otherwise.}\end{cases}$$

Observe that u_ρ satisfies the assumptions from the first part as A_i^i has non-empty interior for ρ small enough. Moreover u_ρ converges to u in $L^1(D)$ and $\mathcal{H}^{d-1}(S_{u_\rho})\leq\mathcal{H}^{d-1}(S_u)+Cq^2\rho^{d-1}$. By the first part of the proof, Lemma 2.14 and lower semicontinuity of the Γ-$\limsup$ we infer

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} E_\varepsilon^{V_\varepsilon}(\omega)(u)\leq\liminf_{\rho\to 0} E_{\mathrm{hom}}(\omega)(u_\rho)=E_{\mathrm{hom}}(\omega)(u).$$

□

3.6 The Γ-limit as rate functional for vanishing temperature

In this section we prove in a very special setting the folklore statement about Γ-convergence and physical models at zero temperature. More precisely, we show that when the temperature depends also on the parameter ε, then in a certain scaling regime the Γ-limits derived in this chapter are exactly the rate functionals for large deviation principles of the Gibbs measures associated to the discrete energies $E^{l_\varepsilon}_{\varepsilon,u_0}$. We restrict the analysis to the case of Dirichlet-type boundary conditions as it is the most interesting one. Similar results can be proved when there are no boundary conditions. We stress that our result is just an example and that the techniques can be used only for one special scaling regime between temperature and lattice spacing. Moreover, since this section is not a building block for this thesis, we refer the reader to the monograph [44] for a more detailed introduction to statistical mechanics.

Fix $A \in \mathcal{A}^R(D)$ and let $E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u,A)$ be defined as in (3.38) with the boundary data u_0 satisfying (3.13). As usual in the context of statistical physics, using the unscaled Hamiltonian we define a (random) probability measure on $L^1(A,\mathcal{M})$, called Gibbs-measure, via

$$\mu_\varepsilon = \mu_\varepsilon(\omega,A,\beta) = \frac{1}{Z_{\varepsilon,u_0}(\omega,A,\beta)} \sum_{\varphi:\varepsilon\mathcal{L}(\omega)\cap A\to\mathcal{M}} \exp\left(-\beta\varepsilon^{1-d}E^{l_\varepsilon}_{\varepsilon}(\omega)(\varphi,A)\right)\delta_\varphi, \quad (3.51)$$

where $Z_{\varepsilon,u_0}(\omega,A,\beta)$ is a normalization factor, sometimes called partition function and $\beta = \frac{1}{k_B T}$ denotes the inverse temperature. .

To investigate the asymptotic behavior of the Gibbs measures, we need a discretized version of the $L^1(A)$-topology. Given $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathcal{M})$ we consider the following interpolation: recall that for $x \in \mathcal{L}(\omega)$ we denote by $\mathcal{C}(x)$ its Voronoi cell. We let

$$u_\varepsilon(\varepsilon x) = \mathrm{Proj}_{\mathcal{M}}\left(\frac{1}{|\varepsilon\mathcal{C}(x)|}\int_{\varepsilon\mathcal{C}(x)} u(y)\,\mathrm{d}y\right), \quad (3.52)$$

where $\mathrm{Proj}_{\mathcal{M}}$ means the projection on the nearest element of $\mathcal{M}$ (if this is not unique we chose any). We may embed u_ε in $\mathcal{PC}_\varepsilon(\omega)$ and by Lebesgue's differentiation theorem we have that $u_\varepsilon \to u$ pointwise almost everywhere and hence in $L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathcal{M})$. Then, given $\varepsilon,\kappa > 0$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathcal{M})$, we define a discrete $L^1(A)$-neighborhood of u setting

$$\mathcal{N}(u,A,\varepsilon,\kappa) := \left\{\varphi : \varepsilon\mathcal{L}(\omega)\cap A \to \mathcal{M} : \sum_{\varepsilon x\in A}\varepsilon^d|u_\varepsilon(\varepsilon x) - \varphi(\varepsilon x)| \le \kappa\right\}$$

with u_ε defined in (3.52). Note that by Lemma 3.4 this term controls the L^1-norm except for a negligible contribution near ∂A so that, given $\kappa > 0$ and a sequence $\varphi_\varepsilon \to u$ in $L^1(D)$, for ε small enough we have $\varphi_\varepsilon \in \mathcal{N}(u, A, \varepsilon, \kappa)$. Vice versa, given a ball $B_\eta(u) \subset L^1(D, \mathcal{M})$, for ε, κ small enough we have $\mathcal{N}(u, A, \varepsilon, \kappa) \subset B_\eta(u)$.

In order to measure the local distribution of the configurations with respect to their energy we consider the functional

$$F^{\beta}_{\varepsilon,\kappa}(\omega)(u, A) := -\varepsilon^d \log\Bigg(\sum_{\varphi\in\mathcal{N}(u,A,\varepsilon,\kappa)} \exp\left(-\beta\varepsilon^{1-d} E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi, A)\right)\Bigg). \tag{3.53}$$

In what follows we consider the case $\beta = \beta_\varepsilon = \frac{\bar{\beta}}{\varepsilon}$. In particular, this yields a zero temperature limit since $\lim_{\varepsilon\to 0}\beta_\varepsilon = +\infty$. For our analysis we introduce the functionals

$$F'_\kappa(\omega)(u, A) := \liminf_{\varepsilon\to 0} F^{\beta_\varepsilon}_{\varepsilon,\kappa}(\omega)(u, A), \qquad F''_\kappa(\omega)(u, A) := \limsup_{\varepsilon\to 0} F^{\beta_\varepsilon}_{\varepsilon,\kappa}(\omega)(u, A).$$

Remark 3.29. As both $F'_\kappa(u)$, $F''_\kappa(u)$ are increasing when $\kappa \to 0$, we can consider the pointwise limit when $\kappa \to 0$.

The following proposition will be the main ingredient to prove the large deviation principle for the Gibbs-measures.

Proposition 3.30. *Let $\mathcal{L}(\omega)$ be admissible. Then there exists a constant $C > 0$ such that, for all κ, ε small enough*

$$0 \le \varepsilon^d \log\left(\#\mathcal{N}(u, A, \varepsilon, \kappa)\right) \le \varepsilon^d \log(C\kappa\varepsilon^{-d}) + C\kappa\log(C\kappa^{-1}).$$

In particular $\lim_{\kappa\to 0}\lim_{\varepsilon\to 0}\varepsilon^d \log(\#\mathcal{N}(u, A, \varepsilon, \kappa)) = 0$.

Proof. Note that by definition one can change the value of u_ε at most on $\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil$ lattice points to obtain a function $\varphi \in \mathcal{N}(u, A, \varepsilon, \kappa)$, where $C_{\mathcal{M}}^{-1} = \min_{m_i\neq m_j}|m_i - m_j|$. Moreover, by Lemma 3.4 the number of lattice points in A can be bounded by

$$\frac{1}{C}|A|\varepsilon^{-d} \le a_\varepsilon := \#(\varepsilon\mathcal{L}(\omega)\cap A) \le C|A|\varepsilon^{-d}.$$

Thus we can estimate

$$\#\mathcal{N}(u, A, \varepsilon, \kappa) \le \Bigg(\sum_{k=0}^{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil} \binom{a_\varepsilon}{k} q^k\Bigg).$$

For small κ (independent of ε) the binomial coefficients are monotone increasing, so that we can bound the last expression by

$$\Big(\sum_{k=0}^{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil}\binom{a_\varepsilon}{k}q^k\Big)\leq \lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil\binom{a_\varepsilon}{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil}q^{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil}$$
$$\leq \lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil\left(\frac{a_\varepsilon\exp(1)q}{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil}\right)^{\lceil C_{\mathcal{M}}\kappa\varepsilon^{-d}\rceil}\leq C\kappa\varepsilon^{-d}\left(\frac{C}{\kappa}\right)^{C\kappa\varepsilon^{-d}},$$

where we used the estimate $\binom{n}{k}\leq\left(\frac{n\cdot e}{k}\right)^k$. Taking logarithms and multiplying both sides by ε^d we deduce

$$\varepsilon^d\log(\#\mathcal{N}(u,A,\varepsilon,\kappa))\leq\varepsilon^d\log(C\kappa\varepsilon^{-d})+C\kappa\log(\frac{C}{\kappa}).$$

On the other hand, as $\#\mathcal{N}(u,A,\varepsilon,\kappa)\geq 1$, the logarithm is nonnegative and we get the claim. □

The previous proposition essentially rules out entropic contributions as can be seen in the following corollary.

Corollary 3.31. *Let $\beta_\varepsilon=\frac{\bar\beta}{\varepsilon}$ for some $\bar\beta>0$. Under the assumptions of Theorem 3.23, for every $u\in L^1(A,\mathcal{M})$ it holds*

$$\lim_{\kappa\to0}F'_\kappa(\omega)(u,A)=\lim_{\kappa\to0}F''_\kappa(\omega)(u,A)=\bar\beta E_{u_0}(\omega)(u,A).$$

Proof. Note that by an easy estimate and the structure of β_ε we have

$$\min_{\varphi\in\mathcal{N}(u,A,\varepsilon,\kappa)}\bar\beta E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi,A)\geq F^{\beta_\varepsilon}_{\varepsilon,\kappa}(\omega)(u,A)\geq\min_{\varphi\in\mathcal{N}(u,A,\varepsilon,\kappa)}\bar\beta E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi,A)$$
$$-\varepsilon^d\log(\#\mathcal{N}(u,A,\varepsilon,\kappa)).$$

By the definition of Γ-convergence and the neighborhood-type properties of the sets $\mathcal{N}(u,A,\varepsilon,\kappa)$ it holds that

$$\lim_{\kappa\to0}\liminf_{\varepsilon\to0}\min_{\varphi\in\mathcal{N}(u,A,\varepsilon,\kappa)}E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi,A)=\Gamma\text{-}\liminf_{\varepsilon\to0}E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u,A),$$
$$\lim_{\kappa\to0}\limsup_{\varepsilon\to0}\min_{\varphi\in\mathcal{N}(u,A,\varepsilon,\kappa)}E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi,A)=\Gamma\text{-}\limsup_{\varepsilon\to0}E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u,A).$$

Therefore the claim follows by Theorem 3.23 and Proposition 3.30. □

Now we are in a position to prove a large deviation principle for the Gibbs measures. We just recall the definition and refer to [34] for a detailed introduction to the subject.

Definition 3.32. *Let X be a topological space and $\mathcal{B}_X$ the Borel sets. A family of probability measure μ_ε on $(X, \mathcal{B}_X)$ satisfies a large deviation principle with rate function $\mathcal{I} : X \to [0, +\infty]$ and speed $\alpha_\varepsilon \downarrow 0$ if the following conditions hold: $\mathcal{I}$ is lower semicontinuous and, for all open sets $U \subset X$ and all closed sets $A \subset X$,*

(i) $\liminf_{\varepsilon\to 0} \alpha_\varepsilon \log(\mu_\varepsilon(U)) \geq -\inf_{x\in U} \mathcal{I}(x)$,

(ii) $\limsup_{\varepsilon\to 0} \alpha_\varepsilon \log(\mu_\varepsilon(A)) \leq -\inf_{x\in A} \mathcal{I}(x)$.

Remark 3.33. (i) When the measures μ_ε are exponentially tight as in Lemma 3.34, then it is enough to establish the second inequality in the large deviation principle only for compact instead of closed sets (Lemma 1.2.18 in [34]).

(ii) If the rate functional is coercive, then it is also called a good rate functional.

We have the following preliminary tightness result:

Lemma 3.34. *Let $\beta_\varepsilon = \frac{\bar{\beta}}{\varepsilon}$ and let $\mathcal{L}(\omega)$ be admissible. If Hypothesis 1 holds, then the measures μ_ε defined in (3.51) are exponentially tight, that is for every $N > 0$ there exists a compact set $K_N \subset L^1(A, \mathcal{M})$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(L^1(A, \mathcal{M})\backslash K_N)) \leq -N.$$

Proof. Fix $N > 0$. Using Hypothesis 1 and Lemma 3.4, for all $\varphi \in \mathcal{PC}_\varepsilon(\omega)$, we have

$$E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi, A) \geq \frac{1}{C}\,\mathcal{H}^{d-1}(S_\varphi \cap A) - C\varepsilon^{d-1}\#\{\varepsilon x :\, \varepsilon\mathcal{C}(x) \cap \partial A \neq \emptyset\}.$$

Since A has a Lipschitz boundary it follows by the existence of the outer Minkowski content that

$$\#\{\varepsilon x :\, \varepsilon\mathcal{C}(x) \cap \partial A \neq \emptyset\} \leq C\varepsilon^{1-d}\mathcal{H}^{d-1}(\partial A).$$

We infer that for ε small enough it holds

$$\frac{1}{C}\mathcal{H}^{d-1}(S_\varphi \cap A) - C\mathcal{H}^{d-1}(\partial A) \leq E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi, A). \tag{3.54}$$

Therefore we define the compact set K_N as

$$K_N := \{\varphi \in BV(A, \mathcal{M}) :\, \frac{1}{C}\mathcal{H}^{d-1}(S_\varphi \cap A) \leq \frac{N}{\bar{\beta}} + C\mathcal{H}^{d-1}(\partial A)\}.$$

Then for all $\varphi \notin K_N$ we have $\bar{\beta}E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\varphi, A) > N$ (possibly $+\infty$). On the other hand, we can control the logarithm of the partition function by

$$F^{\beta_\varepsilon}_{\varepsilon,1}(\omega)(u, A) \geq -\varepsilon^d \log(Z_{\varepsilon,u_0}(\omega, A, \beta_\varepsilon)). \tag{3.55}$$

In particular the left hand side term in (3.55) is bounded from above when $\varepsilon \to 0$. We deduce

$$\limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(L^1(A,\mathcal{M})\backslash K_N)) \leq \limsup_{\varepsilon\to 0} \varepsilon^d \log(q)a_\varepsilon - N - \varepsilon^d \log(Z_{\varepsilon,u_0}(\omega,A,\beta_\varepsilon))$$
$$\leq -N + C,$$

proving the claim of exponential tightness. □

Now we can prove the large deviation principle for the measures μ_ε defined in (3.51).

Theorem 3.35. *Let the inverse temperature be given by $\beta_\varepsilon = \frac{\bar{\beta}}{\varepsilon}$. Under the assumptions from Theorem 3.23, almost surely the random probability measures $\mu_\varepsilon(\omega, A, \beta_\varepsilon)$ satisfy a large deviation principle with speed ε^d and the good rate functional $\mathcal{I}(\omega) : L^1(A,\mathcal{M}) \to [0,+\infty]$ that is finite only on $BV(A,\mathcal{M})$, where it is defined as*

$$\mathcal{I}(\omega)(u) = \bar{\beta}E_{u_0}(\omega)(u,A) - \inf_{v\in BV(A,\mathcal{M})} \bar{\beta}E_{u_0}(\omega)(v,A).$$

Proof. For the moment let us pass to a subsequence (not relabeled) such that

$$\lim_{\varepsilon\to 0} \varepsilon^d \log(Z_{\varepsilon,u_0}(\omega,A,\beta_\varepsilon)) = c.$$

Such a subsequence exists due to (3.55) and the lower bound $-\varepsilon^d \log(Z_{\varepsilon,u_0}(\omega,A,\beta_\varepsilon)) \geq -\varepsilon^d \log(q)a_\varepsilon$, which holds by nonnegativity of the discrete energy. Now we follow the general idea of [34, Theorem 4.1.11]. Let $U \subset L^1(A,\mathcal{M})$ be open. For every $u \in U$ let $\kappa_0 > 0$ be such that $B_\kappa(u) \subset U$ for all $\kappa < \kappa_0$. Then there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ it holds $\mathcal{N}(u,A,\varepsilon,\kappa) \subset U$ for all $\kappa < \kappa_0$. By Corollary 3.31 we have

$$\liminf_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(U)) \geq \liminf_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(\mathcal{N}(u,A,\varepsilon,\kappa)))$$
$$= -F''_\kappa(\omega)(u,A) - c \geq -\bar{\beta}E_{u_0}(\omega)(u,A) - c.$$

Since $u \in U$ was arbitrary, we infer

$$\liminf_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(U)) \geq -\inf_{u\in U} \bar{\beta}E_{u_0}(\omega)(u,A) - c \tag{3.56}$$

By Lemma 3.34 and Remark 3.33 we can restrict the analysis of the upper bound on compact sets. Let $C \subset L^1(A,\mathcal{M})$ be compact. We define the δ-truncation of the limit energy. Given $\delta > 0$, we set

$$E^\delta(u) := \min\{\bar{\beta}E_{u_0,\delta}(\omega)(u,A) - \delta, \frac{1}{\delta}\}.$$

By Proposition 3.31, for fixed δ and $u \in C$, there exists $\kappa > 0$ such that

$$-\limsup_{\varepsilon} \varepsilon^d \log(\mu_\varepsilon(\mathcal{N}(u, A, \varepsilon, \kappa))) \geq E^\delta(u) + c.$$

We now chose κ' small enough (depending on u and δ) such that $B_{\kappa'}(u) \cap \mathcal{PC}_\varepsilon(\omega) \subset \mathcal{N}(u, A, \varepsilon, \kappa)$. By compactness we find a finitely many $u_1, \ldots, u_m \in C$ such that $C \subset \bigcup_{i=1}^m B_{\kappa'}(u_i)$. By the construction we deduce that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(C)) &= \limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_{\varepsilon_j}(C \cap \mathcal{PC}_\varepsilon(\omega))) \\
&\leq \limsup_{\varepsilon\to 0} \varepsilon^d \log\left(\sum_{i=1}^m \mu_\varepsilon(\mathcal{N}(u_i, A, \varepsilon, \kappa))\right) \\
&= \max_i \limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(\mathcal{N}(u_i, A, \varepsilon, \kappa))) \\
&\leq -\min_i E^\delta(u_i) - c \leq -\inf_{u\in C} E^\delta(u) - c.
\end{aligned}$$

Letting $\delta \to 0$ we get

$$\limsup_{\varepsilon\to 0} \varepsilon^d \log(\mu_\varepsilon(C)) \leq -\inf_{u\in C} \bar{\beta} E_{u_0}(\omega)(u, A) - c. \tag{3.57}$$

Since $u \mapsto E_{u_0}(\omega)(u, A)$ is lower semicontinuous, this proves the large deviation principle for the rate functional $\bar{\beta} E_{u_0}(\omega)(u, A) + c$. But necessarily we have $c = -\inf_{v\in BV(A,\mathcal{M})} \bar{\beta} E_{u_0}(\omega)(v, A)$ by testing the open and closed set $L^1(A, \mathcal{M})$. Thus passing to another subsequence was not needed. By Proposition 3.11 the rate functional is coercive, whence a good rate functional. □

Remark 3.36. The reader might notice that during this analysis we did not use the boundary conditions except for identifying the correct Γ-limit. Hence similar results can be proved for phase constraints with the help of Theorem 3.28.

Chapter 4

Dimension reduction for non-periodic spin systems

In this chapter we study dimension reduction problems in random environments. As mentioned in the introduction, our motivation comes from so-called polymeric magnetic thin films made of few layers. Under external magnetic fields they form Weiss domains whose domain wall energy in general depends on the thickness and the roughness of the film, which in turn is a result of the physical and chemical properties of the specific material at use. There has been already some experimental effort to find the relation between film thickness and interfacial domain wall energy for different ferromagnetic materials (see [41] and references therein), but no rigorous explanation has appeared so far in this direction. Among the reasons for such an unsatisfactory analysis we single out one which has a geometric flavor: depositing magnetic particles on a substrate to obtain a thin film leads to disordered arrangements of particles and rough film surfaces which makes it very difficult to formulate a right ansatz leading to the correct (and simpler) continuum model. In this chapter we look at this problem from a different perspective: we choose a simple Ising-type model for a thin film obtained by random deposition of magnetic particles on a flat substrate, for which the geometric part of the problem is still non trivial, and propose an ansatz-free variational analysis of such a film. Combining Γ-convergence and percolation theory we finally obtain a rigorous explanation of the relation between film thickness and domain-wall energy in some asymptotic regime.

As in Chapter 3 we start with defining a polymeric matrix made of magnetic cells and then we introduce an interaction energy between those cells (see [53] and reference therein for further details). The polymeric matrix of such a system can be seen as a random network whose nodes are the cross-linked molecules of the 3-d polymeric magnet. The molecules are supposed to entail the local magnetic properties of the system and to interact as magnetic elementary cells via a ferromagnetic Ising-type coupling. The system is supposed to be thin in the sense that the nodes of the matrix are within a small distance, of the order of the average distance between the nodes themselves, from a 2-d plane. In presence of an external magnetic field or of proper boundary conditions, the ferromagnetic coupling induces the system to form mesoscopic Weiss domains.

We aim at upscaling the system above from its microscopic description to a mesoscopic one in a variational setting. As in Chapter 3 this consists in performing the limit of its energy as the average distance between the magnetic cells, say ε, goes to zero with respect to the macroscopic size of the system. This will lead to two main effects: we can pass from the original discrete system to a continuum model while at the same time it will reduce its dimension from 3 to 2 (more general from d to k with $2 \leq k < d$).

Using the same model as in the previous chapter we describe the polymeric matrix as a random network whose nodes $\mathcal{L} \subset \mathbb{R}^d$ form a thin admissible stochastic lattice, meaning that the matrix is thin, so that there exists $k \in \mathbb{N}$ with $2 \leq k < d$ and $M > 0$ such that

$$\operatorname{dist}(x, \mathbb{R}^k) \leq M \quad \forall x \in \mathcal{L}$$

and that it is admissible according to the following definition: We say that $\mathcal{L}$ is an admissible set of points if the following two requirements are satisfied:

(i) There exists $R > 0$ such that $\operatorname{dist}(x, \mathcal{L}) < R$ for all $x \in \mathbb{R}^k$;

(ii) there exists $r > 0$ such that $|x - y| \geq r$ for all $x, y \in \mathcal{L}$, $x \neq y$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$ is called an admissible stochastic lattice if, uniformly with respect to $\omega \in \Omega$, $\mathcal{L}(\omega)$ is an admissible set of points.

Associating to such a lattice a Voronoi tessellation $\mathcal{V}(\omega)$, one introduces the set of nearest neighbors $\mathcal{NN}(\omega)$ as the set of those pairs of points in $\mathcal{L}(\omega)$ whose Voronoi cells share a $(d-1)$-dimensional facet. This allows us to distinguish between long- and short-range interactions introducing the following (ω-dependent) interaction coefficients

$$c_\varepsilon(x, y) = \begin{cases} c^\varepsilon_{nn}(x, y) & \text{if } (x, y) \in \mathcal{NN}(\omega), \\ c^\varepsilon_{lr}(x, y) & \text{otherwise,} \end{cases}$$

that we assume to be nonnegative and to satisfy the following coerciveness and growth assumptions.

There exist $c > 0$ and a decreasing function $J_{lr} : [0, +\infty) \to [0, +\infty)$ with

$$\int_{\mathbb{R}^k} J_{lr}(|x|)|x| \,\mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$ and all $x, y \in \mathbb{R}^d$,

$$c \leq c^{\varepsilon}_{nn}(x, y) \leq J_{lr}(|x - y|), \quad c^{\varepsilon}_{lr}(x, y) \leq J_{lr}(|x - y|).$$

Given $D \subset \mathbb{R}^k$ and denoted by $P_k : \mathbb{R}^d \to \mathbb{R}^k$ the projection onto $\mathbb{R}^k$, for a given configuration $u : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ we consider the energy per unit ($(k-1)$-dimensional) surface of D to have the ferromagnetic Ising form (see also [3, 5]) given by

$$E_{\varepsilon}(\omega)(u) = \sum_{\substack{x,y \in \mathcal{L}(\omega) \\ \varepsilon x, \varepsilon y \in P_k^{-1} D}} \varepsilon^{k-1} c_{\varepsilon}(x, y)|u(\varepsilon x) - u(\varepsilon y)|.$$

As the sets $\varepsilon\mathcal{L}(\omega)$ will eventually shrink to a k-dimensional set, we conveniently describe the system in terms of an average spin order parameter $Pu : \varepsilon P_k\mathcal{L}(\omega) \to [-1, 1]$ defined on the k-dimensional set $\varepsilon P_k\mathcal{L}(\omega)$ by

$$Pu(z) := \frac{1}{\#\left(P_k^{-1}(z) \cap \varepsilon\mathcal{L}(\omega)\right)} \sum_{\varepsilon x \in P_k^{-1}(z) \cap \varepsilon\mathcal{L}(\omega)} u(\varepsilon x).$$

We then embed the energies $E_{\varepsilon}(\omega)$ in $L^1(D)$ by identifying Pu with a function piecewise constant on the cells of the Voronoi tessellation of $\varepsilon P_k\mathcal{L}(\omega)$, define the convergence $u_{\varepsilon} \to u$ in D in the sense that the piecewise constant functions Pu_{ε} converge to u strongly in $L^1(D)$ and perform the Γ-convergence analysis with respect to this notion of convergence (see Section 4.1 for further details).

In Theorem 4.10 we prove a compactness and integral representation result for the Γ-limit $E(\omega)$ of $E_{\varepsilon}(\omega)$, stating that, up to subsequences, the limit is finite only on $BV(D, \{\pm 1\})$, where it takes the integral form

$$E(\omega)(u) = \int_{S_u} \phi(\omega; x, \nu_u) \,\mathrm{d}\mathcal{H}^{k-1}.$$

Here again S_u is the jump set of u and $\nu_u \in S^{k-1}$ its measure theoretic inner normal. In this formula one interprets $\phi(\omega; \cdot)$ as the domain-wall interaction energy (per unit $(k-1)$-dimensional area) between Weiss domains.

The dependence of such an energy on the randomness of the lattice is studied in Section 4.3 in the context of stochastic homogenization assuming the thin random lattice to be stationary (or ergodic) in the directions of the flat subspace to which it is close to and the interaction coefficients to be invariant under translation in these directions. More precisely we assume that there exists a measure preserving group action $(\tau_z)_{z\in\mathbb{Z}^k}$ on Ω such that, almost surely in Ω, $\mathcal{L}(\tau_z\omega) = \mathcal{L}(\omega) + z$ (if in addition $(\tau_z)_{z\in\mathbb{Z}^k}$ is ergodic, then $\mathcal{L}$ is said to be ergodic, too) and that there exist functions $c_{nn}, c_{lr} : \mathbb{R}^k \times \mathbb{R}^{2(d-k)} \to [0, +\infty)$ such that, setting $\Delta_k(x, y) = (y_1 - x_1, \dots, y_k - x_k, x_{k+1}, y_{k+1}, \dots, x_d, y_d)$, it holds

$$c^\varepsilon_{nn}(x, y) = c_{nn}(\Delta_k(x, y)), \quad c^\varepsilon_{lr}(x, y) = c_{lr}(\Delta_k(x, y))$$

In Theorem 4.17 we prove that under this additional assumption combined with stationarity (or ergodicity) in the sense specified above, the Γ-limit of $E_\varepsilon(\omega)$ as $\varepsilon \to 0$ exists and is finite only on $BV(D, \{\pm 1\})$, where it takes the spatially homogeneous form

$$E_{\mathrm{hom}}(\omega)(u) = \int_{S_u} \phi_{\mathrm{hom}}(\omega; \nu_u)\, \mathrm{d}\mathcal{H}^{k-1}.$$

The energy density is given by an asymptotic homogenization formula which is averaged in the probability space under ergodicity assumptions on $\mathcal{L}$, thus turning the stochastic domain wall energy into a deterministic one.

Similar to Chapter 3 the result is proved by the abstract methods of Gamma-convergence. The proof makes use of two main ingredients: the integral representation results in Theorem 2.13 and the subadditive ergodic theorem by Akcoglu and Krengel. Section 4.4 is devoted to boundary value problems while Section 4.5 extends the result above to the case of a volume constraint on the phase.

An interesting issue in the theory of thin magnetic polymer composite materials is the dependence of the domain wall energy on the random geometry of the polymer matrix. We devote the second part of this chapter to this problem. We consider a specific model of a discrete system in which the stochastic lattice is generated by the random deposition of magnetic particles on a two-dimensional flat substrate. For simplicity we limit ourselves to a simple deposition model with vertical order (more general deposition model are left for future investigations) and limit the magnetic interactions to have finite range. We are interested in the dependence of the domain wall energy on the average thickness of the thin film. Even if a complete picture seems to be out of reach we are able to attack the problem in the asymptotic cases when the thickness of the film is either small or very large.

More specifically we model the substrate (where the particles are deposited) by taking a two-dimensional deterministic lattice, which we choose for simplicity as $\mathcal{L}^0 = \mathbb{Z}^2 \times \{0\}$. We then consider an independent random field $\{X_i^p\}_{i\in\mathbb{Z}^3}$, where the X_i^p are Bernoulli random variables with $\mathbb{P}(X_i^p = 1) = p \in (0,1)$. For fixed $M \in \mathbb{N}$ we construct a random point set as follows:

$$\mathcal{L}_p^M(\omega) := \left\{ (i_1, i_2, i_3) \in \mathbb{Z}^3 : \; 0 \le i_3 \le \sum_{k=1}^{M} X^p_{(i_1,i_2,k)}(\omega) \right\},$$

which means that we successively deposit particles M times independently onto the flat lattice $\mathcal{L}^0$ and stack them over each other (the point set constructed is stationary with respect to translations in $\mathbb{Z}^2$ and ergodic). Moreover, given $u : \varepsilon\mathcal{L}_p^M(\omega) \to \{\pm 1\}$, we consider an energy of the form

$$E^p_{\varepsilon,M}(\omega)(u,A) = \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega)\\ \varepsilon P_2(x),\varepsilon P_2(y)\in A}} \varepsilon c(x-y)|u(\varepsilon x) - u(\varepsilon y)|,$$

where the interactions $c : \mathbb{R}^3 \to [0,+\infty)$ are of finite range, bounded above and coercive on nearest-neighbors. Then we are essentially in the regime of our hypotheses. As a result Theorem 3.19 guarantees the existence of a surface tension, say $\phi^p_{\rm hom}(M;\nu)$ given by an asymptotic cell formula.

The main issue now is the dependence of $\phi^p_{\rm hom}(M;\nu)$ on p and M.

A first result in this direction is proved in Proposition 4.25 where we show that, for every direction $\nu \in S^1$, the wall energy density is linear in the average thickness pM as $M \to +\infty$, that is

$$\lim_{M\to+\infty} \frac{\phi^p_{\rm hom}(M;\nu)}{pM} = \phi^1(\nu), \tag{4.1}$$

with $\phi^1(\nu)$ given in Lemma 4.24 being the wall energy per unit thickness of the deterministic problem obtained for $p = 1$.
A second and more delicate result is contained in Theorem 4.27 and concerns a percolation type phenomenon which can be roughly stated as follows: when the deposition probability p is sufficiently low (below a certain critical percolation threshold) the domain wall energy is zero for M small enough. At this stage it is worth noticing that our energy accounts for the interactions between the deposited particles and the substrate. On one hand this assumption might be questionable from a physical point of view in the case one assumes to grow thin films on neutral media, thus expecting

the properties of the film to be independent of the substrate. On the other hand removing such an interaction leads to a dilute model similar the one considered in [26]. An adaption of this analysis would require a lot of additional work like the extension of fine percolation results to the (range 1)-dependent case which goes far beyond the scopes of the present chapter (see also Remark 4.26 (i)). We prove the percolation result for nearest-neighbor positive interactions. Setting the interaction with the substrate to be $\eta > 0$ we can prove that if $(1-p)^M > p_{site}$ (here $p_{site} \approx 0.593$ is the critical site percolation threshold in $\mathbb{Z}^2$), the limit energy $\phi_{\rm hom}^{p,\eta}(M;\nu)$ is bounded above (up to a constant) by η. This result suggests the absence of a positive domain wall energy in the thin film on a neutral substrate ($\eta = 0$ case). In the limit as M diverges (4.1) holds with $\phi_{\rm hom}^{p,\eta}(M;\nu)$, which is independent of η. This shows that the contribution of the first layer does not affect the asymptotic average domain wall energy as expected. The proof of these results needs the extension to the dimension reduction framework of a result by Caffarelli-de la Lave [29] about the existence of plane-like minimizers for discrete systems subject to periodic Ising-type interactions at the surface scaling. This is contained in the last section.

As a final remark, we mention that we prove all our results in the case when the flat object is at least two-dimensional. Most of the results can be extended to one-dimensional objects (with the proof being much simpler), except the ones contained in Section 4.5 which fail in dimension one as can be seen by simple examples and the percolation-type phenomenon in Section 4.6 as no percolation can occur in (essentially) one-dimensional lattices.

4.1 Modeling discrete disordered flat sets and spin systems

This section is devoted to the precise description of the model we are going to study. We start with some notation we are going to use in the sequel. We stress that in this chapter the notation slightly differs from the previous one.

As we are concerned with dimension reduction, there will be two dimensions k and d with $2 \leq k < d$. Given a measurable set $A \subset \mathbb{R}^k$, here we denote by $|A|$ its k-dimensional Lebesgue measure, while more generally $\mathcal{H}^m(A)$ means again the m-dimensional Hausdorff measure. Given $x \in \mathbb{R}^k$ and $\delta > 0$, we let $B_\delta(x)$ be the open ball around x with radius r. By $|x|$ we mean the usual euclidean norm. If it is clear from the context we will use the same notation as above also in $\mathbb{R}^d$ (otherwise we will indicate the dimension by sub/superscript indices). Given an open set $D \subset \mathbb{R}^k$

we denote by $\mathcal{A}(D)$ the family of all bounded open subsets of D and by $\mathcal{A}^R(D)$ the family of those sets in $\mathcal{A}(D)$ with Lipschitz boundary. Given an orthonormal basis $\nu_1, \ldots, \nu_k$ of $\mathbb{R}^k$, we define the k-dimensional open cube

$$Q_\nu = \{x \in \mathbb{R}^k : \ |\langle x, \nu_i \rangle| < \frac{1}{2} \quad \forall i\}$$

and, for $x \in \mathbb{R}^k, \rho > 0$, we set $Q_\nu(x, \rho) := x + \rho Q_\nu$. We denote by $P_k : \mathbb{R}^d \to \mathbb{R}^k$ the projection onto $\mathbb{R}^k$.

We want to describe random particle systems, where the particles themselves are located very close to a lower dimensional subspace. To this end we make the following assumptions: Let $\mathcal{L} \subset \mathbb{R}^d$ be a countable set. We assume that there exists $M > 0$ such that, after identifying $\mathbb{R}^k \sim \mathbb{R}^k \times \{0\}^{d-k}$, we have

$$\operatorname{dist}(x, \mathbb{R}^k) \leq M \quad \forall x \in \mathcal{L}. \tag{4.2}$$

Moreover, adapting ideas from the previous chapter, we assume that the point set is regular in the following sense:

Definition 4.1. *A countable set $\mathcal{L} \subset \mathbb{R}^d$ is a thin admissible lattice if (4.2) holds and*

(i) $\exists\, R > 0$ *such that* $\operatorname{dist}(x, \mathcal{L}) \leq R \quad \forall x \in \mathbb{R}^k$,

(ii) $\exists\, r > 0$ *such that* $|x - y| \geq r \quad \forall x, y \in \mathcal{L},\, x \neq y$.

In what follows we introduce the probabilistic framework. As usual we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a complete σ-algebra $\mathcal{F}$.

Definition 4.2. *A random variable $\mathcal{L} : \Omega \to (\mathbb{R}^d)^{\mathbb{N}}$, $\omega \mapsto \mathcal{L}(\omega) = \{\mathcal{L}(\omega)_i\}_{i \in \mathbb{N}}$ is called a stochastic lattice. We say that $\mathcal{L}$ is a thin admissible stochastic lattice if $\mathcal{L}(\omega)$ is a thin admissible lattice in the sense of Definition 4.1 and the constants M, r, R can be chosen independent of ω $\mathbb{P}$-almost surely. The stochastic lattice $\mathcal{L}$ is said to be stationary if there exists a measure preserving group action $(\tau_z)_{z \in \mathbb{Z}^k}$ on Ω such that, for $\mathbb{P}$-almost every $\omega \in \Omega$,*

$$\mathcal{L}(\tau_z \omega) = \mathcal{L}(\omega) + z.$$

If in addition $(\tau_z)_{z \in \mathbb{Z}^k}$ is ergodic, then $\mathcal{L}$ is called ergodic, too.

We associate to a realization of a thin stochastic lattice a truncated Voronoi tessellation $\mathcal{V}(\omega)$, where the corresponding d-dimensional cells $\mathcal{C} \in \mathcal{V}(\omega)$ are defined by

$$\mathcal{C}(x) := \{z \in \mathbb{R}^k \times [-2M, 2M]^{d-k} : \ |z - x| \leq |z - x'| \quad \forall x' \in \mathcal{L}(\omega)\}$$

and we introduce the set of nearest neighbors accordingly setting

$$\mathcal{NN}(\omega) := \{(x,y) \in \mathcal{L}^2 : \dim_{\mathcal{H}}(\mathcal{C}(x) \cap \mathcal{C}(y)) = d-1\}.$$

As usual in the passage from atomistic to continuum theories we scale the point set $\mathcal{L}(\omega)$ by a small parameter $\varepsilon > 0$. Let us fix a k-dimensional reference set $D \in \mathcal{A}^R(\mathbb{R}^k)$. Given $A \in \mathcal{A}^R(D)$ and $u : \varepsilon\mathcal{L} \to \{\pm 1\}$, we consider a localized (on A) Ising-type energy

$$E_\varepsilon(\omega)(u,A) = \sum_{\substack{x,y\in\mathcal{L}(\omega)\\ \varepsilon x,\varepsilon y\in P_k^{-1}A}} \varepsilon^{k-1} c_\varepsilon^\omega(x,y)|u(\varepsilon x)-u(\varepsilon y)|,$$

where the (ω-dependent) interaction coefficients distinguish between long and short-range interactions via

$$c_\varepsilon(x,y) = \begin{cases} c_{nn}^\varepsilon(x,y) & \text{if } (x,y)\in\mathcal{NN}(\omega),\\ c_{lr}^\varepsilon(x,y) & \text{otherwise.}\end{cases}$$

For our analysis, qualitatively similar to Chapter 3, we make the following assumptions on the measurable functions $c_{nn}^\varepsilon, c_{lr}^\varepsilon : \mathbb{R}^d\times\mathbb{R}^d \to [0,+\infty)$:

Hypothesis 2 There exist $c > 0$ and a decreasing function $J_{lr} : [0,+\infty) \to [0,+\infty)$ with

$$\int_{\mathbb{R}^k} J_{lr}(|x|)|x|\,\mathrm{d}x = J < +\infty$$

such that, for all $\varepsilon > 0$ and all $x, y \in \mathbb{R}^d$,

$$c \le c_{nn}^\varepsilon(x,y) \le J_{lr}(|x-y|), \quad c_{lr}^\varepsilon(x,y) \le J_{lr}(|x-y|).$$

As the sets $\varepsilon\mathcal{L}(\omega)$ shrink to a k-dimensional set when ε vanishes, we want to define a convergence of discrete variables on shrinking domains. To this end we define the averaged and projected spin variable $Pu : \varepsilon P_k\mathcal{L}(\omega) \to [-1,1]$ via

$$Pu(\varepsilon z) := \frac{1}{\#\left(P_k^{-1}(z)\cap\mathcal{L}(\omega)\right)} \sum_{x\in P_k^{-1}(z)\cap\mathcal{L}(\omega)} u(\varepsilon x).$$

The projected lattice $P_k\mathcal{L}(\omega) \subset \mathbb{R}^k$ obviously inherits property (i) from Definition 4.1, but (ii) might fail after projection. Nevertheless, due to (4.2) the projected lattice is still locally finite with a uniform bound on the number of points in a set. In particular, there exists a constant $C = C_{\mathcal{L}} > 0$ such that, given a set $A \in \mathcal{A}(D)$ with $|\partial A| = 0$, for ε small enough it holds that

$$\varepsilon^k \#\{\varepsilon z \in \varepsilon P_k \mathcal{L} \cap A\} \leq C|A|. \tag{4.3}$$

We now associate a usual k-dimensional Voronoi tessellation $\mathcal{V}_k(\omega) = \{\mathcal{C}_k(z)\}$ in $\mathbb{R}^k$ to the lattice $P_k\mathcal{L}(\omega)$ and we identify Pu with a piecewise constant function of the class

$$\mathcal{PC}_\varepsilon(\omega) := \{v : \mathbb{R}^k \to [-1,1] : \ v_{|\varepsilon \mathcal{C}_k(z)} \text{ is constant } \quad \forall z \in P_k\mathcal{L}(\omega)\}.$$

This construction is sketched in Figure 4-1. Note that we can embed $\mathcal{PC}_\varepsilon(\omega)$ in $L^1(D)$ since the intersection of two Voronoi cells always has zero k-dimensional Lebesgue measure. To deal with convergence of sequences $u_\varepsilon : \varepsilon\mathcal{L} \to \{\pm 1\}$, we adopt the idea of [20]. We will see in Section 4.5 that this notion of convergence is indeed meaningful for variational problems in a random environment.

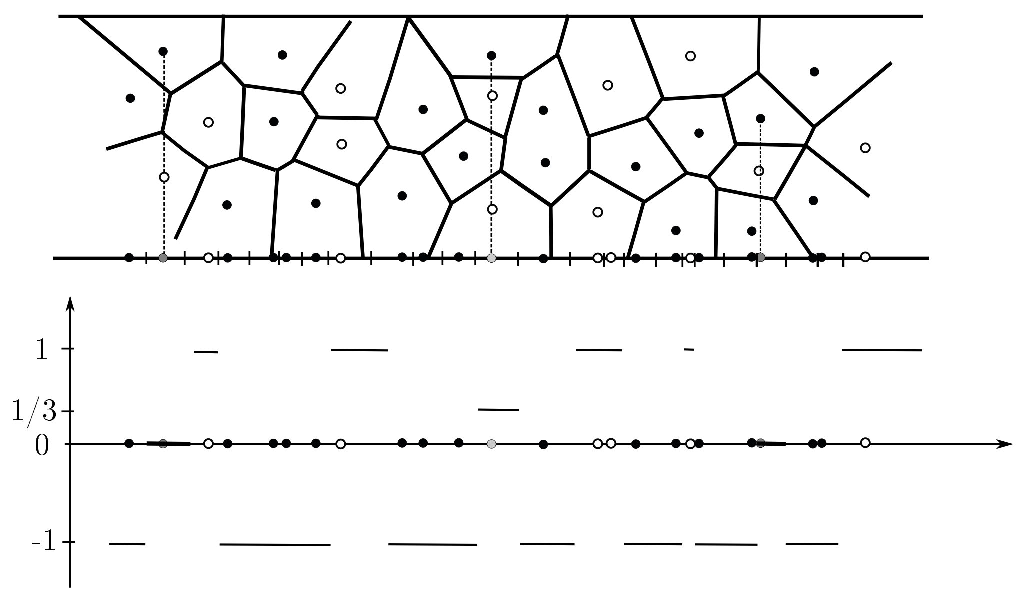

Figure 4-1: Construction of the averaging interpolation in the simple case $d = 2$, $k = 1$. Above: A portion of the truncated Voronoi diagram subordinated to the stochastic lattice $\mathcal{L}(\omega)$ represented by the dots. The values of the variable u on this set can be read by the color code of the dots: black for $u = -1$ and white for $u = +1$. At the bottom of the Voronoi diagram the projected points $P_1\mathcal{L}(\omega)$ and the values of the variable Pu. According to the definition $Pu \in [-1,1]$ (range reflected by the scale of gray in the figure). The dashed lines indicate the exceptional set of projection points where $|P(u)| \neq 1$. Below: the piecewise-constant function on the Voronoi intervals subordinated to $P_1\mathcal{L}(\omega)$.

Definition 4.3. *Let $A \in \mathcal{A}(D)$. We say that a sequence $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ converges in A to $u : A \to \mathbb{R}$ if the piecewise constant functions Pu_ε converge to u in $L^1(A)$.*

For our variational analysis we also need to introduce the lower and upper Γ-limits $E'(\omega), E''(\omega) : L^1(D) \times \mathcal{A}^R(D) \to [0, +\infty]$ setting

$$E'(\omega)(u, A) := \inf\{\liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) : \; u_\varepsilon \to u \text{ in } D\},$$
$$E''(\omega)(u, A) := \inf\{\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) : \; u_\varepsilon \to u \text{ in } D\}.$$

Remark 4.4. The functionals $E'(\omega), E''(\omega)$ are not Γ-lower/upper limits in the usual sense since they are not defined on the same space as $E_\varepsilon(\omega)$. However, if we define the functionals $\tilde{E}_\varepsilon(\omega) : L^1(D) \times \mathcal{A}^R(D) \to [0, +\infty]$ as

$$\tilde{E}_\varepsilon(\omega)(u, A) := \begin{cases} \inf_v E_\varepsilon(\omega)(v, A) & \text{if } u = Pv \text{ for some } v : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}, \\ +\infty & \text{otherwise,} \end{cases}$$

then it is easy to see that $E'(\omega), E''(\omega)$ agree with the Γ-lower/upper limit of $\tilde{E}_\varepsilon(\omega)$ in the strong $L^1(D)$-topology. Therefore we will refer to the equality of $E'(\omega)$ and $E''(\omega)$ as Γ-convergence. Moreover, one can show that

$$E'(\omega)(u, A) = \inf\{\liminf_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) : \; u_\varepsilon \to u \text{ in } A\},$$
$$E''(\omega)(u, A) = \inf\{\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) : \; u_\varepsilon \to u \text{ in } A\}.$$

By the properties of Γ-convergence this implies that both functionals $u \mapsto E'(\omega)(u, A)$ and $u \mapsto E''(\omega)(u, A)$ are $L^1(A)$-lower semicontinuous and therefore local in the sense of Theorem 2.13 (ii).

We now prove several properties of the convergence introduced in Definition 4.3. We start with an equicoercivity property.

Lemma 4.5. *Assume Hypothesis 2 holds. Let $A \in \mathcal{A}(D)$ and let $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ be such that*

$$\sup_\varepsilon E_\varepsilon(\omega)(u_\varepsilon, A) < +\infty.$$

Then, up to subsequences, the functions Pu_ε converge strongly in $L^1(A)$ to some $u \in BV(A, \{\pm 1\})$.

Proof. Let us fix $A' \subset\subset A$. We start by estimating the measure of the set $\{|Pu_\varepsilon| \neq 1\} \cap A'$. Note that if $|Pu_\varepsilon(\varepsilon z)| \neq 1$ for some $z \in P_k\mathcal{L}(\omega)$ such that $\varepsilon\mathcal{C}_k(z) \cap A' \neq \emptyset$, then there exist $x_1, x_2 \in P_k^{-1}(z) \cap \mathcal{L}(\omega)$ such that $u_\varepsilon(\varepsilon x_1) \neq u_\varepsilon(\varepsilon x_2)$. As a preliminary step we show that we can find a path of nearest neighbors in $\mathcal{L}(\omega)$ joining x_1 and x_2 which does not vary too much from the straight line between x_1 and x_2. To this end, fix $0 < \delta << 1$ and consider the collection of segments

$$\mathcal{G}_\delta(x_1, x_2) = \{x + \lambda(x_2 - x_1) :\ x \in B_\delta(x_1),\, 0 \leq \lambda \leq 1\}. \tag{4.4}$$

Since the number of d-dimensional Voronoi cells $\mathcal{C}(x) \in \mathcal{V}(\omega)$ such that $\mathcal{C}(x) \cap \mathcal{G}_\delta \neq \emptyset$ is uniformly bounded, by a dimensional argument there must exist a line $g^* = \{x^* + \lambda(x_2 - x_1) : 0 \leq \lambda \leq 1\} \subset \mathcal{G}_\delta(x_1, x_2)$ such that whenever it passes from some $\mathcal{C}(x)$ to some $\mathcal{C}(x')$, then $(x, x') \in \mathcal{NN}(\omega)$. The path connecting x_1 and x_2 is then given by the set $G(x_1, x_2) := \{x \in \mathcal{L} :\ g^* \cap \mathcal{C}(x) \neq \emptyset\}$, provided that δ is small enough. Observe that there exist $x, x' \in G(x_1, x_2)$ such that $(x, x') \in \mathcal{NN}(\omega)$ and $u_\varepsilon(\varepsilon x) \neq u_\varepsilon(\varepsilon x')$. From the coercivity assumption in Hypothesis 2, we thus deduce that each path contributes to the energy and by the local construction of the paths any couple $(x, x') \in \mathcal{NN}(\omega)$ can be used by only finitely many $z \in P_k\mathcal{L}(\omega)$ for their paths. We infer that

$$\varepsilon^{k-1}\#\{\varepsilon z :\ \varepsilon\mathcal{C}_k(z) \cap A' \neq \emptyset,\, |Pu_\varepsilon(\varepsilon z)| \neq 1, \} \leq CE_\varepsilon(\omega)(u_\varepsilon, A) \leq C, \tag{4.5}$$

where we have used that $\varepsilon G(x_1, x_2) \subset (P_k^{-1}A) \cap \varepsilon\mathcal{L}(\omega)$ for ε small enough. Since the measure of a Voronoi cell in $P_k\mathcal{L}(\omega)$ can be bounded uniformly by a constant, via rescaling we deduce that

$$|\{|Pu_\varepsilon| \neq 1\} \cap A'| \leq C\varepsilon. \tag{4.6}$$

We continue bounding the total variation $|DPu_\varepsilon|(A')$. Since Pu_ε is equibounded and piecewise constant, it is enough to provide a bound for $\mathcal{H}^{k-1}(S_{Pu_\varepsilon} \cap A')$. Note that the jump set S_{Pu_ε} is contained in the facets of the Voronoi cells of the lattice $\varepsilon P_k\mathcal{L}(\omega)$. Since $\mathcal{L}(\omega)$ is thin admissible in the sense of Definition 4.1 and property (i) is preserved by projection, for each such facet F it holds that

$$\mathcal{H}^{k-1}(F) \leq C\varepsilon^{k-1}.$$

As the number of nearest neighbors in the lattice $P_k\mathcal{L}(\omega)$, denoted by $\mathcal{NN}_k(\omega)$, is equibounded once we fix one node, for ε small enough we conclude

$$\begin{aligned}\mathcal{H}^{k-1}(S_{Pu_\varepsilon}\cap A') \leq & C\varepsilon^{k-1}\#\{\varepsilon z\in\varepsilon P_k\mathcal{L}(\omega):\ \varepsilon\mathcal{C}_k(z)\cap A'\neq\emptyset,\ |Pu_\varepsilon(\varepsilon z)|\neq 1\}\\ & + C\varepsilon^{k-1}\#\{(z,z')\in\mathcal{NN}_k(\omega):\ |Pu_\varepsilon(\varepsilon z)-Pu_\varepsilon(\varepsilon z')|=2,\\ & \qquad \varepsilon z,\varepsilon z'\in A'+B_{R\varepsilon}(0)\}.\end{aligned}$$

Due to (4.5) it suffices to estimate the last term. Again, given $\varepsilon z,\varepsilon z'\in A'+B_{R\varepsilon}(0)$ such that $(z,z')\in\mathcal{NN}_k(\omega)$, we find a path $x^0\in P_k^{-1}(z),x^1,...,x^N\in P_k^{-1}(z')$ such that $(x^i,x^{i+1})\in\mathcal{NN}(\omega)$ and the number of couples $(z,z')\in\mathcal{NN}_k(\omega)$ that use the same $(x,y)\in\mathcal{NN}(\omega)$ for their path is equibounded. Reasoning as in the first part of the proof we find that

$$\begin{aligned}&\varepsilon^{k-1}\#\{(z,z')\in\mathcal{NN}_k(\omega):\ |Pu_\varepsilon(\varepsilon z)-Pu_\varepsilon(\varepsilon z')|=2,\ \varepsilon z,\varepsilon z'\in A'+B_{R\varepsilon}(0)\}\\ &\leq CE_\varepsilon(\omega)(u_\varepsilon,A)\leq C.\end{aligned}$$

By Theorem 2.10 and (4.6), there exists a subsequence (not relabeled) such that $Pu_\varepsilon\to u$ in $L^1(A')$ for some $u\in BV(A',\{\pm1\})$. Since A' was arbitrary, the claim follows by a diagonal argument combined with equi-integrability. □

We will also use the following auxiliary result about the convergence introduced in Definition 4.3.

Lemma 4.6. *Let $A\in\mathcal{A}(D)$ and let $u_\varepsilon,v_\varepsilon:\varepsilon\mathcal{L}(\omega)\to\{\pm1\}$ both converge in A to u in the sense of Definition 4.3 and assume both have equibounded energy on A. Then*

$$\lim_{\varepsilon\to0}\sum_{\substack{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\\ \varepsilon P_k(x)\in A}}\varepsilon^k|u_\varepsilon(\varepsilon x)-v_\varepsilon(\varepsilon x)|=0.$$

Proof. Fix a set $A'\subset\subset A$ such that $A'\in\mathcal{A}^R(D)$. Applying (4.3) and equiboundedness it is enough to show that

$$\lim_{\varepsilon\to0}\sum_{\substack{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\\ \varepsilon P_k(x)\in A'}}\varepsilon^k|u_\varepsilon(\varepsilon x)-v_\varepsilon(\varepsilon x)|=0.$$

Using the fact that $u_\varepsilon,v_\varepsilon$ both have finite energy in A, we can argue as in the derivation of (4.5) to show that

$$\#\{\varepsilon x\in P_k^{-1}(A')\cap\varepsilon\mathcal{L}(\omega):\ Pu_\varepsilon(\varepsilon P_k(x))\neq u_\varepsilon(\varepsilon x)\text{ or }Pv_\varepsilon(\varepsilon P_k(x))\neq v_\varepsilon(\varepsilon x)\}\leq C\varepsilon^{1-k}.$$

Inserting this estimate and using that $\mathcal{L}(\omega)$ satisfies (4.2) we obtain

$$\sum_{\substack{\varepsilon x\in\varepsilon\mathcal{L}(\omega)\\ \varepsilon P_k(x)\in A'}}\varepsilon^k|u_\varepsilon(\varepsilon x)-v_\varepsilon(\varepsilon x)|\leq C\sum_{\substack{\varepsilon z\in\varepsilon P_k\mathcal{L}(\omega)\\ \varepsilon z\in A'}}\varepsilon^k|Pu_\varepsilon(\varepsilon z)-Pv_\varepsilon(\varepsilon z)|+C\varepsilon.$$

Thus it is enough to control the last sum. As the Voronoi cells in the projected lattice may become degenerate, we can use only bounds on the number of cells. To this end fix $L>1$ large enough such that, for all $z_L\in L\mathbb{Z}^k$, we have

$$1\leq\#\big(\varepsilon P_k\mathcal{L}(\omega)\cap(\varepsilon z_L+[0,L\varepsilon)^k)\big)\leq C. \tag{4.7}$$

Define $I_\varepsilon:=\{z_L\in L\mathbb{Z}^k:\ (\varepsilon z_L+[0,L\varepsilon)^k)\cap A'\neq\emptyset\}$ and subdivide this set again via

$$\begin{aligned}
I_\varepsilon^1&:=\{z_L\in I_\varepsilon:\ Pu_\varepsilon\text{ is not constant on }\varepsilon z_L+[0,L\varepsilon)^k\},\\
I_\varepsilon^2&:=\{z_L\in I_\varepsilon:\ Pv_\varepsilon\text{ is not constant on }\varepsilon z_L+[0,L\varepsilon)^k\},\\
I_\varepsilon^3&:=I_\varepsilon\backslash(I_\varepsilon^1\cup I_\varepsilon^2).
\end{aligned}$$

Since every scaled k-dimensional Voronoi cell $\varepsilon\mathcal{C}_k(z)$ can only intersect finitely many cubic cells $\varepsilon z_L+[0,L\varepsilon)^k$ with a uniform bound on the cardinality, we can again use the energy bound in A and argue as for (4.5) to conclude that

$$\#(I_\varepsilon^1\cup I_\varepsilon^2)\leq C\varepsilon^{1-k}. \tag{4.8}$$

Combining (4.7) and (4.8) we infer from the definition of the set I_ε^3 that

$$\begin{aligned}
\sum_{\substack{\varepsilon z\in\varepsilon P_k\mathcal{L}(\omega)\\ \varepsilon z\in A'}}\varepsilon^k|Pu_\varepsilon(\varepsilon z)-Pv_\varepsilon(\varepsilon z)|&\leq C\varepsilon+\sum_{z_L\in I_\varepsilon^3}\sum_{\substack{\varepsilon z\in\varepsilon P_k\mathcal{L}(\omega)\\ \varepsilon z\in\varepsilon z_L+[0,L\varepsilon)^k}}\varepsilon^k|Pu_\varepsilon(\varepsilon z)-Pv_\varepsilon(\varepsilon z)|\\
&\leq C\varepsilon+C\sum_{z_L\in I_\varepsilon^3}\int_{\varepsilon z_L+[0,L\varepsilon)^k}|Pu_\varepsilon(s)-Pv_\varepsilon(s)|\,\mathrm{d}s\leq C\varepsilon+C\|Pu_\varepsilon-Pv_\varepsilon\|_{L^1(A)}.
\end{aligned}$$

This concludes the proof, since the last term tends to 0 by assumption. □

As in Chapter 3 we introduce an auxiliary deterministic square lattice to control the long-range interactions. Recall that with $r'=\frac{r}{\sqrt{d}}$ it follows that $\#\{\mathcal{L}(\omega)\cap\{\alpha+[0,r')^d\}\}\leq 1$ for all $\alpha\in r'\mathbb{Z}^d$. Defining $\mathcal{Z}_{r'}(\omega)$ and x_α as in (3.5), for $\xi\in r'\mathbb{Z}^d$, $U\subset\mathbb{R}^k$ and $\varepsilon>0$, we set

$$R_\varepsilon^\xi(U):=\{\alpha:\ \alpha,\alpha+\xi\in\mathcal{Z}_{r'}(\omega),\,\varepsilon x_\alpha,\varepsilon x_{\alpha+\xi}\in P_k^{-1}U\}.$$

Note that by (4.2), enlarging M if necessary, it is enough to consider $\xi \in r'\mathbb{Z}^d_M := r'\mathbb{Z}^d \cap (\mathbb{R}^k \times [-2M, 2M]^{d-k})$. We can then rewrite the localized energy as

$$E_\varepsilon(\omega)(u, A) = \sum_{\xi \in r'\mathbb{Z}^d_M} \sum_{\alpha \in R^\xi_\varepsilon(A)} \varepsilon^{k-1} c^\omega_\varepsilon(x_\alpha, x_{\alpha+\xi}) |u(\varepsilon x_\alpha) - u(\varepsilon x_{\alpha+\xi})|.$$

Remark 4.7. With the help of the auxiliary lattice $r'\mathbb{Z}^d$ we can strengthen the estimate (4.3) to tubular neighborhoods of flat boundaries in the following sense: There exists a constant $C = C(M, R, r) > 0$ with the following property: Given $A \in \mathcal{A}^R(\mathbb{R}^k)$, for all ε small enough it holds that

$$\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) : \operatorname{dist}(\varepsilon P_k(x), \partial A) \leq 2R\varepsilon\} \leq C\varepsilon^{1-k}\mathcal{H}^{k-1}(\partial A). \tag{4.9}$$

In this estimate we can replace $2R$ by any other constant. Then of course C changes.

The following lemma is the equivalent version of Lemma 3.12 for thin lattices. We omit the proof as it is almost identical.

Lemma 4.8. *Let $B \subset \mathcal{A}(\mathbb{R}^k)$ be convex and $B^\varepsilon := \{x \in \mathbb{R}^k : \operatorname{dist}(x, B) < 3(R + M)\varepsilon\}$. Then there exists a constant C depending only on r, R, M in Definition 4.1 such that for every $\xi \in r'\mathbb{Z}^d_M$ and every $u : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ it holds*

$$\sum_{\alpha \in R^\xi_\varepsilon(B)} |u(\varepsilon x_\alpha) - u(\varepsilon x_{\alpha+\xi})| \leq C|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in P_k^{-1} B^\varepsilon}} |u(\varepsilon x) - u(\varepsilon y)|.$$

Remark 4.9. Observe that we can write

$$\{\xi \in r'\mathbb{Z}^d_M\} = \bigcup_{\substack{z \in r'\mathbb{Z}^{d-k} \\ |z|_\infty \leq 2M}} \{\xi = (\xi_k, z_1, \ldots, z_{d-k}) : \xi_k \in r'\mathbb{Z}^k\}.$$

Hence we deduce from Hypothesis 2 that, given $\delta > 0$, there exists $L_\delta > 0$ such that

$$\sum_{\substack{\xi \in r'\mathbb{Z}^d_M \\ |\xi| > L_\delta}} J_{lr}(|\hat{\xi}|)|\xi| \leq \delta, \tag{4.10}$$

where $\hat{\xi} \in \xi + [-r', r']^d$ is such that $|\hat{\xi}| = \operatorname{dist}([0, r')^d, [0, r')^d + \xi)$. As in Chapter 3 this decay property along with Lemma 4.8 will be crucial to control the long-range interactions. However note that L_δ in general depends on M.

4.2 Integral representation on the flat set

Our first aim is to characterize all possible variational limits of energies $E_\varepsilon(\omega)$ that satisfy Hypothesis 2. As for the case $k = d$ treated in Chapter 3, we make use of the integral representation results of Theorem 2.13, but due to the lower-dimensional structure of the problem we apply it for k-dimensional sets.

The following theorem is the main result of this section.

Theorem 4.10. *Let $\mathcal{L}(\omega)$ be thin admissible and let c^ε_{nn} and c^ε_{lr} satisfy Hypothesis 2. For every sequence $\varepsilon \to 0^+$ there exists a subsequence ε_n such that the functionals $E_{\varepsilon_n}(\omega)$ Γ-converge with respect to the convergence of Definition 4.3 to a functional $E(\omega) : L^1(D) \to [0, +\infty]$ of the form*

$$E(\omega)(u) = \begin{cases} \int_{S_u} \phi(\omega; x, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1} & \text{if } u \in BV(D, \{\pm 1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover a local version of the statement above holds: For all $u \in BV(D, \{\pm 1\})$ and all $A \in \mathcal{A}^R(D)$

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(\omega)(u, A) = \int_{S_u \cap A} \phi(\omega; x, \nu_u)\, \mathrm{d}\mathcal{H}^{k-1}.$$

Again we remark that this result is of deterministic nature. The proof of Theorem 4.10 will be given later. At first we use the standard machinery that allows us to apply Theorem 2.13.

Throughout this section we tacitly assume that $\mathcal{L}(\omega)$ is a thin admissible point set. Let us start with the growth condition (iv) of Theorem 2.13. Using the $L^1(A)$-lower semicontinuity of the total variation (which reduces to $u \mapsto 2\mathcal{H}^{k-1}(S_u \cap A)$ for $u \in BV(A, \{\pm 1\})$, one can use the same argument as for Lemma 4.5 to prove the following lower bound for $E'(\omega)(u, A)$:

Proposition 4.11. *Assume that Hypothesis 2 holds and let $A \in \mathcal{A}(D)$. Then $E'(\omega)(u, A) < +\infty$ only if $u \in BV(A, \{\pm 1\})$ and there exists a constant $c > 0$ independent of A such that*

$$E'(\omega)(u, A) \geq \frac{1}{c}\mathcal{H}^{k-1}(S_u \cap A).$$

As a next step we provide a suitable upper bound for $E''(\omega)(u, A)$.

Proposition 4.12. *Assume that Hypothesis 2 holds. Then there exists a constant*

$c > 0$ such that, for all $A \in \mathcal{A}^R(D)$ and all $u \in BV(D, \{\pm 1\})$,

$$E''(\omega)(u, A) \leq c\,\mathcal{H}^{k-1}(S_u \cap A).$$

Proof. For the moment let us assume that $F := \{u = +1\}$ is a polyhedral set in $\mathbb{R}^k$ such that

$$\mathcal{H}^{k-1}(\partial F \cap \partial A) = 0. \tag{4.11}$$

We define a sequence $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ setting

$$u_\varepsilon(\varepsilon x) := \begin{cases} +1 & \text{if } \varepsilon P_k(x) \in F, \\ -1 & \text{otherwise.} \end{cases}$$

Then it is straightforward to check that $u_\varepsilon \to u$ in the sense of Definition 4.3. Given $\delta > 0$, we set $A^\delta = A + B_\delta(0)$ and L_δ be as in (4.10). For $|\xi| \leq L_\delta$, we can argue as in the proof of Lemma 3.12 to show that, for ε small enough, it holds that

$$\begin{aligned} \sum_{\alpha \in R_\varepsilon^\xi(A)} \varepsilon^{k-1} |u_\varepsilon(\varepsilon x_\alpha) - u_\varepsilon(\varepsilon x_{\alpha+\xi})| &\leq C|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in P_k^{-1} A^\delta}} \varepsilon^{k-1} |u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)| \\ &\leq C|\xi| \mathcal{H}^{k-1}(\partial F \cap A^\delta), \end{aligned} \tag{4.12}$$

where we used Remark 4.7 for the last estimate. On the other hand, when $|\xi| > L_\delta$ we apply Lemma 4.8 on the whole space to deduce for any $\varepsilon > 0$ the weaker bound

$$\sum_{\alpha \in R_\varepsilon^\xi(\mathbb{R}^k)} \varepsilon^{k-1} |u_\varepsilon(\varepsilon x_\alpha) - u_\varepsilon(\varepsilon x_{\alpha+\xi})| \leq C|\xi| \mathcal{H}^{k-1}(\partial F). \tag{4.13}$$

Summing (4.12) and (4.13) over all $\xi \in r'\mathbb{Z}^d_M$ combined with the bounds of Hypothesis 2 and the definition of L_δ yields for ε small enough

$$E_\varepsilon(\omega)(u_\varepsilon, A) \leq C\Bigg(\sum_{\xi \in r'\mathbb{Z}^d_M} J_{lr}(|\hat{\xi}|)|\xi|\Bigg)\mathcal{H}^{k-1}(S_u \cap A^\delta) + C\delta\mathcal{H}^{k-1}(\partial F).$$

Using the integrability assumption from Hypothesis 2 we infer that

$$E''(\omega)(u, A) \leq \limsup_{\varepsilon} E_\varepsilon(\omega)(u_\varepsilon, A) \leq c\,\mathcal{H}^{k-1}(S_u \cap A^\delta) + C\delta\mathcal{H}^{k-1}(\partial F).$$

Due to (4.11), the claim follows by letting $\delta \to 0$.

The general case can be proven by density. Indeed, for every $u \in BV(A, \{\pm 1\})$

there exists a sequence $u_n \in BV_{loc}(\mathbb{R}^k, \{\pm 1\})$ with $\{u_n = +1\}$ being a polyhedral set in $\mathbb{R}^k$ satisfying (4.11) and such that $u_n \to u$ in $L^1(A)$ as well as $\mathcal{H}^{k-1}(S_{u_n} \cap A) \to \mathcal{H}^{k-1}(S_u \cap A)$ (see Theorem 2.12). The claim then follows by $L^1(A)$-lower semicontinuity of $u \mapsto E''(\omega)(u, A)$, which is a consequence of Remark 4.4. □

As usual for applying integral representation theorems we next establish a weak subadditivity property of $A \mapsto E''(\omega)(u, A)$. Although the basic idea is quite similar to the one of Proposition 3.15 we provide all details as this is the most important step for proving the integral representation.

Proposition 4.13. *Let c^{ε}_{nn} and c^{ε}_{lr} satisfy Hypothesis 2. Then, for every $A, B \in \mathcal{A}^R(D)$, every $A' \subset \mathcal{A}^R(D)$ such that $A' \subset\subset A$ and every $u \in BV(D, \{\pm 1\})$,*

$$E''(\omega)(u, A' \cup B) \leq E''(\omega)(u, A) + E''(\omega)(u, B).$$

Proof. We already know from Proposition 4.12 that $E''(\omega)(u, A)$ and $E''(\omega)(u, B)$ are both finite. Let $u_\varepsilon, v_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ both converge to u in the sense of Definition 4.3 such that

$$\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) = E''(\omega)(u, A), \quad \limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(v_\varepsilon, B) = E''(\omega)(u, B). \quad (4.14)$$

Step 1 Extensions to convex domains
Let Q_D be a large cube containing $\overline{D}$. As $D \in \mathcal{A}^R(D)$, by extension we may assume that $u \in BV_{\mathrm{loc}}(\mathbb{R}^k, \{\pm 1\})$. We aim at modifying u_ε and v_ε on $\varepsilon\mathcal{L} \backslash P_k^{-1}(A)$ and $\varepsilon\mathcal{L} \backslash P_k^{-1}(B)$ respectively, such that they converge to u in D and such that they have equibounded energy on the larger set Q_D. We will demonstrate the argument for u_ε. Take another cube Q' such that $Q_D \subset\subset Q'$. Arguing as in the proof of Proposition 4.12 we find a sequence $\tilde{u}_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ such that $\tilde{u}_\varepsilon \to u$ in Q' and $\limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(\tilde{u}_\varepsilon, Q') \leq C\mathcal{H}^{k-1}(S_u \cap Q')$. We then set $\bar{u}_\varepsilon \in \mathcal{PC}_\varepsilon(\omega)$ as

$$\bar{u}(\varepsilon x) = \mathbb{1}_A(P_k(\varepsilon x)) u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_A(P_k(\varepsilon x)))\tilde{u}_\varepsilon(\varepsilon x).$$

Then $\bar{u}_\varepsilon \to u$ in Q_D and applying Lemma 4.8 combined with Hypothesis 2 and (4.7) yields that, for ε small enough,

$$\begin{aligned} E_\varepsilon(\omega)(\bar{u}_\varepsilon, Q_D) \leq & C \sum_{\xi \in r'\mathbb{Z}^d_M} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in P_k^{-1}(Q')}} \varepsilon^{k-1} |\bar{u}_\varepsilon(\varepsilon x) - \bar{u}_\varepsilon(\varepsilon y)| \\ \leq & C \left(E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(\tilde{u}_\varepsilon, Q' \backslash A) + \mathcal{H}^{k-1}(\partial A) \right). \end{aligned}$$

The first and second term remain bounded by construction.

Step 2 Energy estimates

Given $\delta > 0$ we choose L_δ such that (4.10) holds. Fix $d' \leq \frac{1}{2}\mathrm{dist}(A', \partial A)$ and let $N_\varepsilon := \lfloor \frac{d'}{2\varepsilon L_\delta} \rfloor$. For $j \in \mathbb{N}$ we define

$$A_{\varepsilon,j} := \{x \in A : \ \mathrm{dist}(x, A') < 2j\varepsilon L_\delta\}.$$

We let $w_\varepsilon^j \in \mathcal{PC}_\varepsilon(\omega)$ be the interpolation defined by

$$w_\varepsilon^j(\varepsilon x) = \mathbb{1}_{A_{\varepsilon,j}}(P_k(\varepsilon x))u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_{A_{\varepsilon,j}}(P_k(\varepsilon x)))v_\varepsilon(\varepsilon x).$$

Note that $w_\varepsilon^j \to u$ in D in the sense of Definition 4.3. Let us set

$$S_j^{\xi,\varepsilon} := \{x = y + t\,P_k(\xi') : \ y \in \partial A_{\varepsilon,j},\ |t| \leq \varepsilon, \xi' \in \xi + [-r', r']^d\} \cap (A \cup B).$$

For $j \leq N_\varepsilon$ it follows from the definition that

$$\begin{aligned} E_\varepsilon(\omega)(w_\varepsilon^j, A' \cup B) \leq & E_\varepsilon(\omega)(u_\varepsilon, A_{\varepsilon,j}) + E_\varepsilon(\omega)(v_\varepsilon, B \backslash A_{\varepsilon,j}) \\ & + \sum_{\xi \in r'\mathbb{Z}^d_M} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \underbrace{\varepsilon^{k-1} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |w_\varepsilon^j(\varepsilon x_\alpha) - w_\varepsilon^j(\varepsilon x_{\alpha+\xi})|}_{=:\rho_j^{\xi,\varepsilon}(\alpha,\omega)} \\ \leq & E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(v_\varepsilon, B) + \sum_{\xi \in r'\mathbb{Z}^d_M} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha, \omega). \end{aligned} \tag{4.15}$$

We now split the interactions depending on L_δ. If $|\xi| > L_\delta$, we use Lemma 4.8. Since $A \cup B \subset\subset Q_D$, we deduce that

$$\sum_{|\xi| > L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha, \omega) \leq C \sum_{|\xi| > L_\delta} J_{lr}(|\hat{\xi}|)|\xi| \sum_{\substack{(x,y) \in \mathcal{NN}(\omega) \\ \varepsilon x, \varepsilon y \in P_k^{-1} Q_D}} \varepsilon^{k-1} |w_\varepsilon^j(\varepsilon x) - w_\varepsilon^j(\varepsilon y)|.$$

Obviously we have $P_k^{-1} Q_D \subset P_k^{-1} A_{\varepsilon,j} \cup P_k^{-1}(Q_D \backslash A_{\varepsilon,j})$. Nearest neighbor interactions between those two sets are contained in $P_k^{-1}(S_k^{\xi,\varepsilon})$ for some $\xi \in r'\mathbb{Z}^d_M$ with $|\xi| \leq 4(R + M)$. By coercivity we can therefore further estimate the last inequality via

$$\begin{aligned} \sum_{|\xi| > L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha, \omega) \leq C\delta \Big(& E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(v_\varepsilon, Q_D) \\ & + \sum_{|\xi| \leq L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha, \omega) \Big). \end{aligned} \tag{4.16}$$

Now we treat the interactions when $|\xi| \leq L_\delta$. Consider any points $\varepsilon x, \varepsilon y \in \varepsilon\mathcal{L}(\omega)$. A straightforward computation yields

$$|w_\varepsilon^j(\varepsilon x) - w_\varepsilon^j(\varepsilon y)| \leq |u_\varepsilon(\varepsilon x) - u_\varepsilon(\varepsilon y)| + |v_\varepsilon(\varepsilon x) - v_\varepsilon(\varepsilon y)| + 2|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|.$$

By our construction we have $S_j^{\varepsilon,\xi} \subset (A_{\varepsilon,j+1} \backslash A_{\varepsilon,j-1}) =: S_j^\varepsilon$. We deduce that

$$\begin{aligned}\sum_{|\xi|\leq L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha,\omega) \leq & E_\varepsilon(\omega)(u_\varepsilon, S_j^\varepsilon) + E_\varepsilon(\omega)(v_\varepsilon, S_j^\varepsilon) \\ & + C_\delta \sum_{\substack{y\in\mathcal{L}(\omega) \\ \varepsilon P_k(y)\in S_j^\varepsilon}} \varepsilon^{k-1}|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)|,\end{aligned}$$

where C_δ depends only on L_δ. Observe that by definition every point can be in at most two sets $S_{j_1}^\varepsilon, S_{j_2}^\varepsilon$. Thus averaging combined with (4.16), Step 1 and the last inequality yields

$$\begin{aligned} I_\varepsilon := \frac{1}{N_\varepsilon}\sum_{j=1}^{N_\varepsilon} \sum_{\xi\in r'\mathbb{Z}_M^d} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha,\omega) \leq \frac{2}{N_\varepsilon}\sum_{j=1}^{N_\varepsilon} \sum_{|\xi|\leq L_\delta} \sum_{\alpha\in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \rho_j^{\xi,\varepsilon}(\alpha,\omega) + C\delta \\ \leq \frac{4}{N_\varepsilon}\left(E_\varepsilon(\omega)(u_\varepsilon, Q_D) + E_\varepsilon(\omega)(v_\varepsilon, Q_D)\right) + C_\delta \sum_{\substack{y\in\mathcal{L}(\omega) \\ \varepsilon y\in D}} \varepsilon^d|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)| + C\delta \\ \leq \frac{C}{N_\varepsilon} + C_\delta \sum_{\substack{y\in\mathcal{L}(\omega) \\ \varepsilon y\in D}} \varepsilon^k|u_\varepsilon(\varepsilon y) - v_\varepsilon(\varepsilon y)| + C\delta. \end{aligned}$$

Due to Step 1 we can apply Lemma 4.6 to deduce that $\limsup_{\varepsilon\to 0} I_\varepsilon \leq C\delta$. For every $\varepsilon > 0$ let $j_\varepsilon \in \{1, \ldots, N_\varepsilon\}$ be such that

$$\sum_{\xi\in r'\mathbb{Z}_M^d} \sum_{\alpha\in R_\varepsilon^\xi(S_{j_\varepsilon}^{\xi,\varepsilon})} \rho_{j_\varepsilon}^{\xi,\varepsilon}(\alpha,\omega) \leq I_\varepsilon \tag{4.17}$$

and set $w_\varepsilon := w_\varepsilon^{j_\varepsilon}$. Note that as a convex combination w_ε still converges to u in D. Hence, using (4.14), (4.15) and (4.17), we conclude that

$$E''(\omega)(u, A' \cup B) \leq \limsup_{\varepsilon\to 0} E_\varepsilon(\omega)(w_\varepsilon, A' \cup B) \leq E''(\omega)(u, A) + E''(\omega)(u, B) + C\delta.$$

The arbitrariness of δ proves the claim. □

Proof of Theorem 4.10. Having in mind Remark 4.4 the argument is identical to the full-dimensional case treated in Section 3.2. The only difference is that due to the additional symmetry property $E_\varepsilon(\omega)(u,A)=E_\varepsilon(\omega)(-u,A)$ and the fact that u takes only values in $\{\pm 1\}$ we infer that the limit energy obtained by Theorem 2.13 is independent of the traces u^+,u^-. $\square$

4.3 Homogenization for stationary thin lattices

In this section we improve the results from Theorem 4.10 using the randomness, in particular the stationarity assumption, of the stochastic lattice. To this end we need again a convergence result about discrete boundary condition that holds without any knowledge of homogenization. To this end we consider the flat analogue of the artificial boundary conditions used in Section 3.3. Fix a macroscopic value η. For $A\in\mathcal{A}^R(D)$ we set

$$\partial_\eta A=\{x\in A:\ \operatorname{dist}(x,\partial A)\leq\eta\}.$$

We let u_0 be a well-prepared boundary data, that means $u_0\in BV_{loc}(\mathbb{R}^k,\{\pm 1\})$ and setting $u_{\varepsilon,0}\in\mathcal{PC}_\varepsilon(\omega)$ as $u_{\varepsilon,0}(\varepsilon x)=u_0(P_k(\varepsilon x))$, we have $u_{\varepsilon,0}\to u_0$ in D and

$$\limsup_{\varepsilon\to 0}E_\varepsilon(\omega)(u_{\varepsilon,0},B)\leq C\mathcal{H}^{k-1}(S_{u_0}\cap\overline{B}),\qquad \mathcal{H}^{k-1}(S_{u_0}\cap\partial A)=0. \tag{4.18}$$

with C uniformly for $B\in\mathcal{A}^R(\mathbb{R}^k)$. Let us introduce the set of configurations with discrete boundary value u_0 as

$$\mathcal{PC}^\eta_{\varepsilon,u_0}(\omega,A):=\{u:\varepsilon\mathcal{L}(\omega)\to\{\pm 1\}:\ u(\varepsilon x)=u_0(P_k(\varepsilon x))\text{ if }P_k(\varepsilon x)\in\partial_\eta A\}.$$

In order to characterize the integrand of the limit functional in Theorem 4.10, we first analyze the connection between the minimum values

$$\begin{aligned}
m^\eta_\varepsilon(\omega)(u_0,A)&=\inf\{E_\varepsilon(\omega)(v,A):\ v\in\mathcal{PC}^\eta_{\varepsilon,u_0}(\omega,A)\},\\
m(\omega)(u_0,A)&=\inf\{E(\omega)(v,A):\ v=u_0\text{ in a neighbourhood of }\partial A\},
\end{aligned}$$

where the limit functional $E(\omega)$ is given (up to subsequences) by Theorem 4.10. Arguing similar to Lemma 3.16 (see also the proof of Theorem 4.19), we obtain the following characterization.

Lemma 4.14. *Let ε_n and $E(\omega)$ be as in Theorem 4.10. Then it holds that*

$$\lim_{\eta\to 0}\liminf_n m^\eta_{\varepsilon_n}(\omega)(u_0,A)=\lim_{\eta\to 0}\limsup_n m^\eta_{\varepsilon_n}(\omega)(u_0,A)=m(\omega)(u_0,A).$$

In order to prove a homogenization result we suppose from now on that there exist functions $c_{nn}, c_{lr} : \mathbb{R}^k \times \mathbb{R}^{2(d-k)} \to [0, +\infty)$ such that, setting $\Delta_k(x, y) = (y_1 - x_1, \dots, y_k - x_k, x_{k+1}, y_{k+1}, \dots, x_d, y_d)$, it holds

$$c^\varepsilon_{nn}(x, y) = c_{nn}(\Delta_k(x, y)), \quad c^\varepsilon_{lr}(x, y) = c_{lr}(\Delta_k(x, y)). \tag{4.19}$$

Note that the coefficients are homogeneous only along shifts in $\mathbb{R}^k$ but may depend on the position in the orthogonal complement.

In view of Theorem 2.13 and Lemma 4.14 we can further characterize the Γ-limits of the family $E_\varepsilon(\omega)$ by investigating the quantities $m^\eta_\varepsilon(u_0, Q)$ for suitable oriented cubes and $u_0 = u^{1,-1}_{x,\nu}$. Due to the decay assumptions of Hypothesis 2 it suffices to consider truncated interactions. To this end, for fixed $L \in \mathbb{N}$ we replace the long-range coefficients by

$$c^L_{lr}(x, y) := c_{lr}(\Delta_k(x, y))\mathbb{1}_{|x-y|\le L}.$$

We denote the corresponding energy by $E^L_\varepsilon(\omega)(u, A)$. We already know from Chapter 3 that the limit energy is essentially characterized by the following quantity: For $\nu \in S^{k-1}$ and a cube $Q_\nu(x, \rho)$ we let $u_{x,\nu} := u^{1,-1}_{x,\nu}$ and we set

$$m^{\eta,L}_1(\omega)(u_{x,\nu}, Q_\nu(x, \rho)) := \inf\left\{E^L_1(\omega)(u, Q_\nu(x, \rho)) : \ u \in \mathcal{PC}^\eta_{1,u_{x,\nu}}(\omega, Q_\nu(x, \rho))\right\}. \tag{4.20}$$

The following technical auxiliary result is the analogue of Lemma 3.17. We omit its proof as it is based on exactly the same estimates combined with projection and the rescaled version of (4.7).

Lemma 4.15. *Let $Q = Q_\nu(z, \rho) \subset \mathbb{R}^k$ be a cube and let $\{Q_j = Q_\nu(z_j, \rho_j)\}_j$ be a finite family of disjoint cubes with the following properties:*

(i) $\min_j \rho_j \ge 4L$,

(ii) $z_j - z_1 \in \nu^\perp$,

(iii) $\mathrm{dist}(z_1, \nu^\perp + z) \le \frac{1}{4}\min_j \rho_j$,

(iv) $\bigcup_j Q_j \subset Q$,

(v) $\mathrm{dist}(\partial\bigcup_j Q_j, \partial Q) > \eta$ *or* $z_j - z \in \nu^\perp$.

Then there exists $C = C(L, M) > 0$ such that for all $\eta \ge L$

$$\begin{aligned} m^{\eta,L}_1(\omega)(u_{z,\nu}, Q) \le &\sum_j m^{\eta,L}_1(\omega)(u_{z_j,\nu}, Q_j) + C\mathcal{H}^{k-1}\Big(\Big(Q\backslash\bigcup_j\overline{Q_j}\Big) \cap (\nu^\perp + z)\Big) \\ &+ C\sum_j \big(\mathcal{H}^{k-2}\left((\partial Q_j\backslash\partial Q) \cap (\nu^\perp + z_1)\right) + \mathcal{H}^{k-1}(\partial Q_j \cap S_\nu(z, z_1))\big), \end{aligned}$$

where $S_\nu(z,z_1)$ is the infinite (maybe flat) stripe enclosed by the two hyperplanes $\nu^\perp+z$ and $\nu^\perp+z_1$.

Remark 4.16. Also in the flat environment Lemma 4.15 still holds if we replace cubes by boxes of the type $I_\nu(z,\{\rho_i\}_i) = z + \{x \in \mathbb{R}^k : \ |\langle x, \nu_i\rangle| < \frac{\rho_i}{2}\}$. Then the cubes Q_j are replaced by the collection $I_j = I_\nu(z_j, \{\rho_i^j\}_i)$ and in the assumptions (i) and (iii) we have to replace ρ_j by $\min_i \rho_i^j$.

The next theorem is the main result of this section.

Theorem 4.17. *Let $\mathcal{L}$ be a stationary, thin admissible stochastic lattice and let c_{nn} and c_{lr} satisfy Hypothesis 2 with the additional structure property (4.19). For $\mathbb{P}$-almost every ω and for all $\nu \in S^{k-1}$ there exists*

$$\phi_{\mathrm{hom}}(\omega;\nu) := \inf_{\eta>0} \limsup_{t\to+\infty} \frac{1}{t^{k-1}} \inf\left\{ E_1(\omega)(u, Q_\nu(0,t)) : \ u \in \mathcal{PC}^{\eta t}_{1,u_{0,\nu}}(\omega, Q_\nu(0,t))\right\}.$$

The functionals $E_\varepsilon(\omega)$ Γ-converge with respect to the convergence of Definition 4.3 to the functional $E_{\mathrm{hom}}(\omega) : L^1(D) \to [0,+\infty]$ defined by

$$E_{\mathrm{hom}}(\omega)(u) = \begin{cases} \int_{S_u} \phi_{\mathrm{hom}}(\omega;\nu_u)\,\mathrm{d}\mathcal{H}^{k-1} & \text{if } u \in BV(D,\{\pm1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

If $\mathcal{L}$ is ergodic, then $\omega \mapsto \phi_{\mathrm{hom}}(\omega,\nu)$ is constant almost surely.

Proof. Fix any sequence $\varepsilon \to 0$. According to Theorem 4.10, for all $\omega \in \Omega$ such that $\mathcal{L}(\omega)$ is admissible, there exists a (ω-dependent) subsequence ε_n such that

$$\Gamma\text{-}\lim_n E_{\varepsilon_n}(\omega)(u,A) = \int_{S_u\cap A} \phi(\omega;x,\nu)\,\mathrm{d}\mathcal{H}^{k-1}$$

for all $u \in BV(D,\{\pm1\})$ and every $A \in \mathcal{A}^R(D)$. According to Theorem 2.13 and Lemma 4.14, for any $x \in D$ and $\nu \in S^{k-1}$ it holds that

$$\begin{aligned}\phi(\omega;x,\nu) &= \limsup_{\rho\to0} \frac{1}{\rho^{k-1}} m(\omega)(u_{x,\nu}, Q_\nu(x,\rho)) \\ &= \limsup_{\rho\to0} \frac{1}{\rho^{k-1}} \lim_{\eta\to0} \limsup_n m^\eta_{\varepsilon_n}(\omega)(u_{x,\nu}, Q_\nu(x,\rho)).\end{aligned}$$

Changing the variables via $t_n = \varepsilon_n^{-1}$ and $v(x) = u(t_n^{-1}x)$, this formula turns into

$$\phi(\omega;x,\nu) = \limsup_{\rho\to0} \lim_{\eta\to0} \limsup_n \frac{1}{(\rho t_n)^{k-1}} m_1^{\eta t_n}(\omega)(u_{t_n x,\nu}, t_n Q_\nu(x,\rho)).$$

Except for the claim on ergodicity, due to the Urysohn property of Γ-convergence (recall Remark 4.4) we conclude the proof as soon as we show that for a set of full probability the limit in ρ is negligible and the other limits do not depend on x or the subsequence t_n. The argument follows the same steps as the proof of Theorem 3.19.

Step 1 Truncating the range of interactions

Using the decay assumptions from Hypothesis 2 in the form of Remark 4.9 combined with Lemma 4.8 one can show that it is enough to prove that there exists $\phi_{\mathrm{hom}}^L(\omega;\nu)$ and a set Ω_L of full probability such that for all $\omega\in\Omega_L$, $x\in D$, every cube $Q_\nu(x,\rho)$ and every sequence $t_n\to+\infty$ it holds

$$\phi_{\mathrm{hom}}^L(\omega;\nu)=\lim_{\eta\to0}\limsup_n\frac{1}{(\rho t_n)^{k-1}}m_1^{\eta t_n,L}(\omega)(u_{t_nx,\nu},t_nQ_\nu(x,\rho)),\tag{4.21}$$

where $m_1^{\eta t_n,L}(\omega)$ is defined in (4.20). Indeed, by the same reasoning as in Step 1 of the proof of Theorem 3.19 we deduce that

$$0\leq\frac{m_1^{\eta t_n}(\omega)(u_{t_nx,\nu},t_nQ_\nu(x,\rho))-m_1^{\eta t_n,L}(\omega)(u_{t_nx,\nu},t_nQ_\nu(x,\rho))}{(\rho t_n)^{k-1}}\leq C\sum_{2|\xi|>L}J_{lr}(|\hat{\xi}|)|\xi|.$$

From this estimate and Remark 4.9 we infer that $\phi_{\mathrm{hom}}^L(\omega;\nu)$ is a Cauchy-sequence with respect to L and moreover, in combination with (4.21), we deduce that

$$\lim_L\phi_{\mathrm{hom}}^L(\omega;\nu)=\lim_{\eta\to0}\limsup_j\frac{1}{(\rho t_n)^{k-1}}m_1^{\eta t_n}(\omega)(u_{t_nx,\nu},t_nQ_\nu(x,\rho))$$

exists, is independent of x,ρ and the sequence t_n. Therefore it remains to show (4.21). We first consider an auxiliary problem with fixed boundary width L. We show that there exists

$$\phi^L(\omega;\nu)=\lim_j\frac{1}{(\rho t_n)^{k-1}}m_1^{L,L}(\omega)(u_{t_nx,\nu},t_nQ_\nu(x,\rho))\tag{4.22}$$

and this limit does not depend on x,ρ and the sequence t_n.

Step 2 Existence of ϕ^L for $x=0$ and rational directions

Fix $L\in\mathbb{N}$. We have to show that, for $\mathbb{P}$-almost every $\omega\in\Omega$ and every $\nu\in S^{k-1}$, there exists the limit in (4.22). We start with the case $x=0$ and $\nu\in S^{k-1}\cap\mathbb{Q}^k$. For this choice we can use the subadditive ergodic theorem in $(k-1)$-dimensions.

Substep 2.1 Defining a stochastic process

We recall the construction used for proving Theorem 3.19: Given $\nu\in S^{k-1}$ there exists an orthogonal matrix $A_\nu\in\mathbb{R}^{k\times k}$ such that $A_\nu e_k=\nu$, the mapping $\nu\mapsto A_\nu e_i$ is continuous on $S^{k-1}\backslash\{-e_k\}$ and if $\nu\in\mathbb{Q}^k$ then $A_\nu\in\mathbb{Q}^{k\times k}$. Let us fix a rational direction $\nu\in S^{k-1}\cap\mathbb{Q}^k$ and an integer $N=N(\nu)>4L$ such that $NA_\nu(z,0)\in\mathbb{Z}^k$ for all $z \in \mathbb{Z}^{k-1}$. We now define a discrete stochastic process (see Definition 2.20). To $I = [a_1, b_1) \times \cdots \times [a_{k-1}, b_{k-1}) \in \mathcal{I}_{k-1}$ we associate the set $Q_I \subset \mathbb{R}^k$ defined by

$$Q_I := NA_\nu\Big(\operatorname{int} I \times (-\frac{s_{\max}}{2}, \frac{s_{\max}}{2})\Big),$$

where $s_{\max} = \max_i |b_i - a_i|$ is the maximal side length. Then we define the process $\mu : \mathcal{I}_{k-1} \to L^1(\Omega)$ as

$$\mu(I, \omega) := \inf \Big\{ E_1^L(\omega)(v, Q_I) : \, v \in \mathcal{PC}^L_{1,u_{0,\nu}}(\omega, Q_I) \Big\} + C_\mu \mathcal{H}^{k-2}(\partial I),$$

where C_μ is a constant to be chosen later for subadditivity. We first have to show that $\mu(I, \cdot)$ is a $L^1(\Omega)$-function. Testing the $\mathcal{PC}_1(\omega)$-interpolation of $u_{0,\nu}$ as candidate in the infimum problem we easily get

$$\mu(I, \omega) \le CN^{k-1}\mathcal{H}^{k-1}(I)$$

for all $I \in \mathcal{I}_{k-1}$ and almost every $\omega \in \Omega$. Therefore $\mu(I, \cdot)$ is essentially bounded. $\mathcal{F}$-measurability can be proven with similar arguments as for Substep 2.1 in the proof of Theorem 3.19.

Next we show lower-dimensional stationarity of the process. Let $z \in \mathbb{Z}^{k-1}$. Note that $Q_{I-z} = Q_I - z_\nu^N$, where $z_\nu^N := NA_\nu(z, 0) \in \nu^\perp \cap \mathbb{Z}^k$. Arguing as in the proof of Theorem 3.19, by the stationarity of $\mathcal{L}$ and the structure assumption (4.19) we infer $m_1^{L,L}(\omega)(u_{0,\nu}, Q_{I-z}) = m_1^{L,L}(\tau_{z_\nu^N}\omega)(u_{0,\nu}, Q_I)$. Since the Hausdorff measure is invariant under translations we conclude that $\mu(I - z, \omega) = \mu(I, \tau_{z_\nu^N}\omega)$. Setting $\tilde{\tau}_z = \tau_{-z_\nu^N}$ we obtain a measure preserving group action on $\mathbb{Z}^{k-1}$ such that $\mu(I, \tilde{\tau}_z\omega) = \mu(I + z)(\omega)$.

To show subadditivity, let $I \in \mathcal{I}_{k-1}$ and let $\{I_j\}_{j \in J} \subset \mathcal{I}_{k-1}$ be a finite disjoint family such that $I = \bigcup_{j \in J} I_j$. Note that Q_I and the family $\{Q_{I_j}\}_{j \in J}$ fulfill the assumptions of Lemma 4.15 (in the sense of Remark 4.16). We conclude

$$m_1^{L,L}(\omega)(u_{0,\nu}, Q_I) \le \sum_{j \in J} m_1^{L,L}(\omega)(u_{0,\nu}, Q_{I_j}) + C \sum_{j \in J} \mathcal{H}^{k-2}((\partial Q_{I_j} \backslash \partial Q_I) \cap \nu^\perp).$$

Inserting the definition of $\mu(I, \omega)$ then yields

$$\begin{aligned} \mu(I, \omega) &= m_1^{L,L}(\omega)(u_{0,\nu}, Q_I) + C_\mu \mathcal{H}^{k-2}(\partial Q_I \cap \nu^\perp) \\ &\le \sum_{j \in J} \mu(I_j, \omega) + (C - C_\mu) \sum_{j \in J} \mathcal{H}^{k-2}((\partial Q_{I_j} \backslash Q_I) \cap \nu^\perp), \end{aligned}$$

which yields subadditivity if we choose $C_\mu > C$. Property (ii) in Definition 2.20 is trivial since $\mu(I, \omega)$ is clearly nonnegative. By Theorem 2.21 there exists $\phi^L(\omega; \nu)$

such that almost surely, for rational directions $\nu \in S^{k-1}$, it holds

$$\phi^L(\omega;\nu) = \lim_{n\to+\infty} \frac{1}{(2Nn)^{k-1}} m_1^{L,L}(\omega)(u_{0,\nu}, Q_\nu(0,2Nn)),$$

where we used that $C_\mu \mathcal{H}^{k-2}(\partial I)$ is negligible for the limit in n.

Substep 2.2 From integer sequences to all sequences

The extension of the convergence in Substep 2.1 to arbitrary sequence is identical to the corresponding step for proving Theorem 3.19 and uses only Lemma 4.8. Hence we skip the details and just keep in mind that for set of full probability it holds

$$\phi^L(\omega;\nu) = \lim_{n\to+\infty} \frac{1}{t_n^{k-1}} m_1^{L,L}(\omega)(u_{0,\nu}, Q_\nu(0,t_n))$$

Substep 2.3 Shift invariance in the probability space

Up to extracting a countable union of null sets we know that the limit defining $\phi^L(\omega;\nu)$ exists for all rational directions ν. We next prove that the function $\omega \mapsto \phi^L(\omega;\nu)$ is invariant under the entire group action $\{\tau_z\}_{z\in\mathbb{Z}^k}$. This is essential to deal with the ergodic case but also to prove shift invariance in the physical space. Given $z \in \mathbb{Z}^k$ there exists $R = R(L,z) > 0$ such that for all $t > 0$

$$Q_\nu(0,t) \subset Q_\nu(-z,R+t), \quad 2L \leq \mathrm{dist}(\partial Q_\nu(0,t), \partial Q_\nu(-z,R+t)). \tag{4.23}$$

As for stationarity of the stochastic process we know that

$$\begin{aligned}\phi^L(\tau_z\omega;\nu) &\leq \limsup_{t\to+\infty} \frac{1}{(R+t)^{k-1}} m_1^{L,L}(\omega)(u_{-z,\nu}, Q_\nu(-z,R+t)) \\ &= \limsup_{t\to+\infty} \frac{1}{t^{k-1}} m_1^{L,L}(\omega)(u_{-z,\nu}, Q_\nu(-z,R+t)).\end{aligned}$$

Due to (4.23) we can apply Lemma 3.17 to the cube $Q_\nu(-z,R+t)$ and the singleton family $\{Q_\nu(0,t)\}$ and deduce that there exists a constant $C = C(R,z)$ such that

$$m_1^{L,L}L(\omega)(u_{-z,\nu}, Q_\nu(-z,R+t)) \leq m_1^{L,L}(\omega)(u_{0,\nu}, Q_\nu(0,t)) + Ct^{k-2}.$$

Hence we get $\phi^L(\tau_z\omega;\nu) \leq \phi^L(\omega;\nu)$. The other inequality can be proven with the same argument so that the limit indeed exists (which was already hinted in our notation) and, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\phi^L(\tau_z\omega;\nu) = \phi^L(\omega;\nu). \tag{4.24}$$

Step 3 Shift invariance in the physical space

We establish the existence of the limit defining $\phi^L(\omega;\nu)$ when the cubes are not centered in the origin. We further argue that it agrees with the one already considered. The argument is very similar to the full-dimensional case so we leave out some details. We start with a cube $Q_\nu(x,\rho)$ with rational direction ν, $x\in\mathbb{Z}^k\backslash\{0\}$ and $\rho\in\mathbb{Q}$. Given $\varepsilon>0$ and $N\in\mathbb{N}$ (different from the one of Substep 2.1) we define the events

$$\mathcal{Q}_N := \left\{\omega\in\Omega:\ \sup_{t\geq\frac{N}{2}}\left|(t\rho)^{1-k}m_1^{L,L}(\omega)(u_{0,\nu},Q_\nu(0,t\rho))-\phi^L(\omega;\nu)\right|\leq\varepsilon\right\}.$$

By Step 2 we know that the function $\mathbb{1}_{\mathcal{Q}_N}$ converges almost surely to $\mathbb{1}_\Omega$ when $N\to+\infty$. Let us denote by $\mathcal{J}_x$ the σ-algebra of invariant sets for the measure preserving map τ_x. Fix $\delta>0$. By exactly the same probabilistic argument used in Step 3 of the proof of Theorem 3.19, almost surely we find n_0,N_0 (depending on ω and δ) such that for any $n\geq\max\{n_0,N_0\}$ and $\tilde{R}=6n\delta$ there exists $l_n\in[n+1,n+\tilde{R}]$ with $\tau_{l_nx}(\omega)\in\mathcal{Q}_{N_0}$. Then, by (4.24) and stationarity, for all $t\geq\frac{N_0}{2}$ it holds that

$$\left|(t\rho)^{1-k}m_1^{L,L}(\omega)(u_{-l_nx,\nu},Q_\nu(-l_nx,t\rho))-\phi^L(\omega;\nu)\right|\leq\varepsilon. \tag{4.25}$$

Let us define $\beta_n=n+c_L\rho^{-1}|x|(l_n-n)$ with $c_L\in\mathbb{N}$ chosen such that $Q_\nu(-nx,n\rho)\subset Q_\nu(-l_nx,\beta_n\rho)$ and $\mathrm{dist}(\partial Q_\nu(-nx,n\rho),\partial Q_\nu(-l_nx,\beta_n\rho))>L$. Such c_L exists since $l_n-n\geq1$. Then each face of the cube $Q_\nu(-nx,n\rho)$ has at most the distance $(\beta_n-n)\rho=c_L|x|(l_n-n)$ to the corresponding face in $Q_\nu(-l_nx,\beta_n\rho)$. Then, for n large enough, we can apply Lemma 4.15 to the cube $Q(-l_nx,\beta_n\rho)$ and the singleton family $\{Q_\nu(-nx,n\rho)\}$ to obtain

$$\begin{aligned}\frac{m_1^{L,L}(\omega)(u_{-l_nx,\nu},Q_\nu(-l_nx,\beta_n\rho))}{(\beta_n\rho)^{k-1}}&\leq\frac{m_1^{L,L}(\omega)(u_{-nx,\nu},Q_\nu(-nx,n\rho))}{(\beta_n\rho)^{k-1}}+C\tilde{R}(\beta_n\rho)^{-1}\\&\leq\frac{m_1^{L,L}(\omega)(u_{-nx,\nu},Q_\nu(-nx,n\rho))}{(n\rho)^{k-1}}+6C\delta.\end{aligned} \tag{4.26}$$

Conversely we can define $\theta_n=n-c_L'\rho^{-1}|x|(l_n-n)$ for a suitable $c_L'\in\mathbb{N}$ and deduce by a similar argument that

$$\frac{m_1^{L,L}(\omega)(u_{-nx,\nu},Q_\nu(-nx,n\rho))}{(n\rho)^{k-1}}\leq\frac{m_1^{L,L}(\omega)(u_{-l_nx,\nu},Q_\nu(-l_nx,\theta_n\rho))}{(\theta_n\rho)^{k-1}}+6C\delta. \tag{4.27}$$

Note that if δ is small enough (depending only on x,L and ρ) we have $\beta_n\geq\theta_n\geq$

$\frac{n}{2} \geq \frac{N_0}{2}$. Combining (4.26), (4.27) and (4.25) we infer

$$\limsup_{n\to+\infty}\left|\frac{m_1^{L,L}(\omega)(u_{-nx,\nu},Q_\nu(-nx,n))}{n^{k-1}}-\phi^L(\omega;\nu)\right|\leq 6C\delta+\varepsilon,$$

which yields the claim in (4.22) for $Q_\nu(x,\rho)$ with $x \in \mathbb{Z}^k$ and rational ν and ρ. The extension to arbitrary sequences $t_n \to +\infty$ and arbitrary x, ρ is identical to Step 3 of the proof of Theorem 3.19. Hence we omit the details.

Step 4 From rational to irrational directions

Next we extend the convergence from rational direction to all $\nu \in S^{k-1}$. As the argument is purely geometric, we can assume without loss of generality that $x = 0$. Given $\nu \in S^{k-1}$ and a sequence $t_n \to +\infty$ we define

$$\begin{aligned}\overline{\phi}^L(\omega;\nu)&=\limsup_{n\to+\infty}\frac{1}{t_n^{k-1}}m_1^{L,L}(\omega)(u_{0,\nu},Q_\nu(0,t_n)),\\ \underline{\phi}^L(\omega;\nu)&=\liminf_{n\to+\infty}\frac{1}{t_n^{k-1}}m_1^{L,L}(\omega)(u_{0,\nu},Q_\nu(0,t_n)).\end{aligned}$$

Let $\nu \in S^{k-1}\backslash\mathbb{Q}^k$. By the construction of the matrix A_ν in Substep 2.1 we know that there exists a sequence of rational directions ν_j such that $A_{\nu_j} \to A_\nu$. Therefore, given $\delta > 0$ we find $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ the following two properties hold:

(i) $Q_\nu(0,(1-2\delta)) \subset\subset Q_{\nu_j}(0,1-\delta) \subset\subset Q_\nu(0,1)$,

(ii) $0 < \mathrm{d}_{\mathcal{H}}(\nu^\perp \cap B_2(0), \nu_j^\perp \cap B_2(0)) \leq \delta$.

For fixed $j \geq j_0$ and $n \in \mathbb{N}$ let $u_{n,j} \in \mathcal{PC}_1(\omega)$ be an admissible minimizer for $m_1^{L,L}(\omega)(u_{0,\nu_j}, Q_{\nu_j}(0,(1-\delta)t_n))$. We define a trial function $v_n \in \mathcal{PC}_1(\omega)$ setting

$$v_n(x) := \begin{cases} u_{n,j}(x) & \text{if } x \in Q_{\nu_j}(0,(1-\delta)t_n),\\ u_{0,\nu}(x) & \text{otherwise.}\end{cases}$$

Note that if $P_k(x), P_k(y) \in Q_\nu(0,t_n)\backslash Q_{\nu_j}(0,(1-\delta)t_n)$ are such that $|x-y| \leq L$ and $v_n(x) \neq v_n(y)$, then by the choice of j_0 and (i), for j large enough we have

$$\operatorname{dist}\big(P_k(x),(Q_\nu(0,t_n)\backslash Q_\nu(0,(1-2\delta)t_n))\cap\nu^\perp\big)\leq L. \tag{4.28}$$

If $P_k(x) \in Q_\nu(0,t_n)\backslash Q_{\nu_j}(0,(1-\delta)t_n)$ and $P_k(y) \in Q_{\nu_j}(0,(1-\delta)t_n)$ with $|x-y| \leq L$ and $v_n(x) \neq v_n(y)$, then, for j large enough one can show that by (ii) either $P_k(x)$ or

$P_k(y)$ must lie in the cone

$$\mathcal{K}(\nu,\nu_j)=\{x\in\mathbb{R}^k:\ \langle x,\nu\rangle\cdot\langle x,\nu_j\rangle\leq 0\}.$$

As the ray $[P_k(x),P_k(y)]$ intersects $\partial Q_{\nu_j}(0,(1-\delta)t_n)$, we conclude that

$$\mathrm{dist}(P_k(x),(\mathcal{K}(\nu,\nu_j)+B_L(0))\cap\partial Q_{\nu_j}(0,(1-\delta)t_n))\leq L. \tag{4.29}$$

By (i) it holds that $v_n\in\mathcal{PC}^L_{1,u_{0,\nu}}(\omega,Q_\nu(0,t_n))$ for n large enough. Having in mind (4.7), from (4.28), (4.29) and the choice of j_0 we deduce that for j large enough

$$m_1^{L,L}(\omega)(u_{0,\nu},Q_\nu(0,t_n)\leq m_1^{L,L}(\omega)(u_{0,\nu_j},Q_{\nu_j}(0,(1-\delta)t_n))+C\delta t_n^{k-1}.$$

Dividing the last inequality by t_n^{k-1} and passing to the right subsequence of t_n we deduce

$$\overline{\phi}^L(\omega;\nu)\leq\phi^L(\omega;\nu_j)+C\delta.$$

Letting first $j\to+\infty$ and then $\delta\to 0$ yields $\overline{\phi}^L(\omega;\nu)\leq\liminf_j\phi^L(\omega;\nu_j)$. By a similar argument we can also prove that $\limsup_j\phi^L(\omega;\nu_j)\leq\underline{\phi}^L(\omega;\nu)$. Hence the limit in (4.22) exists almost surely for all directions ν and it does not depend on x,ρ and the sequence t_n.

Step 5 Proof of (4.21)

Finally one can prove that $\phi^L(\omega;\nu)=\phi^L_{\mathrm{hom}}(\omega;\nu)$ as in Step 5 of the proof of Theorem 3.19.

When the group action is ergodic, the additional statement in Theorem 4.17 again follows easily from (4.24). □

Remark 4.18. As in Chapter 3 one can show that the surface tension can be obtained by one single limit procedure. Indeed, referring to (4.30) and repeating Step 1 and 5 of the proof of Theorem 4.17 it follows that

$$\phi_{\mathrm{hom}}(\omega;\nu)=\lim_{t\to+\infty}\frac{1}{t^{k-1}}\inf\{E_1(\omega)(u,Q_\nu(0,t)):\ u\in\mathcal{PC}^{l_{1/t}}_{1,u_{0,\nu}}(\omega,Q_\nu(0,t))\}.$$

Moreover lower semicontinuity implies again the convexity of the one-homogeneous extension of $\nu\mapsto\phi_{\mathrm{hom}}(\omega;\nu)$ and thus Lipschitz continuity on S^{k-1}.

4.4 Convergence of boundary value problems

In this section we consider a refined version of the convergence of minimum problems under Dirichlet-type boundary data when the width of the discrete boundary depends also on ε.

Let $A \in \mathcal{A}^R(D)$ and fix $u_0 \in BV(\mathbb{R}^k_{\mathrm{loc}}, \{\pm 1\})$. As at the beginning of Section 4.3 we assume that the boundary data is well-prepared in the sense of (4.18). Similar to Section 3.2 we define a discrete trace constraint as follows: Let $l_\varepsilon > 0$ be such that

$$\lim_{\varepsilon \to 0} l_\varepsilon = +\infty, \quad \lim_{\varepsilon \to 0} l_\varepsilon \varepsilon = 0. \tag{4.30}$$

We set $\mathcal{PC}^{l_\varepsilon \varepsilon}_{\varepsilon,u_0}(\omega, A)$ as the space of those u that agree with u_0 at the discrete boundary of A, namely

$$\mathcal{PC}^{l_\varepsilon}_{\varepsilon,u_0}(\omega, A) := \{u : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\} : \ u(\varepsilon x) = u_0(P_k(\varepsilon x)) \text{ if } \mathrm{dist}(P_k(\varepsilon x), \partial A) \leq l_\varepsilon \varepsilon\}.$$

For given $\varepsilon > 0$ and $l_\varepsilon > 0$ let us consider the restricted functional $E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\cdot, A)$: $\mathcal{PC}^{l_\varepsilon \varepsilon}_{\varepsilon,u_0}(\omega, A) \to [0, +\infty]$ defined as

$$E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u, A) := E_\varepsilon(\omega)(u, A). \tag{4.31}$$

Given $u \in BV(D, \{\pm 1\})$, we further define the extension $u_{A,0} : \mathbb{R}^k \to \{\pm 1\}$ as

$$u_{A,0}(x) := \begin{cases} u(x) & \text{if } x \in A, \\ u_0(x) & \text{otherwise.} \end{cases}$$

Since A is regular we have $u_{A,0} \in BV_{loc}(\mathbb{R}^k, \{\pm 1\})$. The following convergence result holds true:

Theorem 4.19. *Under the assumptions of Theorem 4.17, for every set $A \in \mathcal{A}^R(D)$, $A \subset\subset D$, the functionals $E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(\cdot, A)$ defined in (4.31) Γ-converge with respect to the convergence in A to the functional $E_{u_0}(\omega)(\cdot, A) : L^1(D) \to [0, +\infty]$ that is finite only for $u \in BV(A, \{\pm 1\})$, where it takes the form*

$$E_{u_0}(\omega)(u, A) = \int_{S_{u_{A,0}} \cap \overline{A}} \phi_{\mathrm{hom}}(\omega; \nu_{u_{A,0}}) \, \mathrm{d}\mathcal{H}^{k-1}.$$

Proof. We already know that the limit energy is finite only for $u \in BV(A, \{\pm 1\})$.

Lower bound: Without loss of generality let $u_\varepsilon \to u$ in A in the sense of Definition 4.3 be such that

$$\liminf_{\varepsilon} E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u_\varepsilon, A) \leq C.$$

Passing to a subsequence, we may assume that $u_\varepsilon \in \mathcal{PC}^{l_\varepsilon \varepsilon}_{\varepsilon,u_0}(\omega, A)$. We define a new sequence $v_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ by

$$v_\varepsilon(\varepsilon x) = \mathbb{1}_A(P_k(\varepsilon x))u_\varepsilon(\varepsilon x) + (1 - \mathbb{1}_A(P_k(\varepsilon x)))u_0(\varepsilon x).$$

Note that by our assumptions on u_0 we have $v_\varepsilon \to u_{A,0}$ in D in the sense of Definition 4.3. Now fix $A_1 \subset\subset A \subset\subset A_2$ such that $A_1, A_2 \in \mathcal{A}^R(D)$. Setting

$$R_\varepsilon^\xi := \{\alpha \in R_\varepsilon^\xi(A_2) : \ \varepsilon x_\alpha \in P_k^{-1}A, \, \varepsilon x_{\alpha+\xi} \notin P_k^{-1}A \text{ or vice versa}\},$$

for $L \in \mathbb{N}$ large enough the truncated energies can be estimated via

$$\begin{aligned} E_\varepsilon^L(\omega)(v_\varepsilon, A_2) \leq & E_{\varepsilon,u_0}^{l_\varepsilon}(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(u_{\varepsilon,0}, A_2\backslash\overline{A_1}) \\ & + \sum_{|\xi|\leq 2L} \sum_{\alpha \in R_\varepsilon^\xi} \varepsilon^{k-1} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |v_\varepsilon(\varepsilon x_\alpha) - v_\varepsilon(\varepsilon x_{\alpha+\xi})|, \end{aligned} \tag{4.32}$$

For interactions with $|\xi| \leq 2L$ and ε small enough, we have that $R_\varepsilon^\xi \subset A_2\backslash\overline{A_1}$. Moreover, as soon as $l_\varepsilon > 3L$ (actually $l_\varepsilon \geq L$ would be enough but requires additional notation), by the boundary conditions on u_ε we get

$$\sum_{|\xi|\leq 2L} \sum_{\alpha \in R_\varepsilon^\xi} \varepsilon^{k-1} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |v_\varepsilon(\varepsilon x_\alpha) - v_\varepsilon(\varepsilon x_{\alpha+\xi})| \leq E_\varepsilon(\omega)(u_{\varepsilon,0}, A_2\backslash\overline{A_1}).$$

From a local version (see Theorem 4.10) of Theorem 4.17 applied to the truncated energies and from (4.18) and (4.32) we infer

$$E_{\mathrm{hom}}^L(\omega)(u_{A,0}, A_2) \leq \liminf_\varepsilon E_{\varepsilon,u_0}^{l_\varepsilon}(u_\varepsilon, A) + C\mathcal{H}^{d-1}(S_{u_0} \cap \overline{A_2}\backslash A_1).$$

The lower bound follows by letting $A_2 \downarrow \overline{A}$ and $A_1 \uparrow A$ combined with (4.18) and then letting $L \to +\infty$. For the last limit recall Step 1 of the proof of Theorem 4.17.

Upper bound: We first provide a recovery sequence in the case when $u = u_0$ in a neighborhood of ∂A. Let $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ converge to u in D in the sense of Definition 4.3 such that

$$\lim_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon, A) = E_{\mathrm{hom}}(\omega)(u, A). \tag{4.33}$$

Given $\delta > 0$ we let $L_\delta > 0$ be such that (4.10) holds. Now choose regular sets $A_1 \subset\subset A_2 \subset\subset A$ such that

$$u = u_0 \quad \text{on } A\backslash\overline{A_1}. \tag{4.34}$$

The remaining argument is similar to the proof of Proposition 4.13 and therefore we only sketch it. Fix $d' \leq \frac{1}{2}\mathrm{dist}(A_1, \partial A_2)$ and set $N_\varepsilon = \lfloor \frac{d'}{2\varepsilon L_\delta} \rfloor$. For $j \in \mathbb{N}$ we introduce the sets

$$A_{\varepsilon,j} := \{x \in A : \ \mathrm{dist}(x, A_1) < 2j\varepsilon L_\delta\}.$$

We further define $u_\varepsilon^j : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ setting

$$u_\varepsilon^j(\varepsilon x) = \begin{cases} u_0(P_k(\varepsilon x)) & \text{if } P_k(\varepsilon x) \notin A_{\varepsilon,j}, \\ u_\varepsilon(\varepsilon x) & \text{otherwise.} \end{cases}$$

Splitting the interactions as usual we deduce that

$$\begin{aligned} E_\varepsilon(\omega)(u_\varepsilon^j, A) \leq & E_\varepsilon(\omega)(u_\varepsilon, A) + E_\varepsilon(\omega)(u_{\varepsilon,0}, A \backslash \overline{A_1}) \\ & + \sum_{\xi \in r'\mathbb{Z}^d_M} \varepsilon^{k-1} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |u_\varepsilon^j(\varepsilon x_\alpha) - u_\varepsilon^j(\varepsilon x_{\alpha+\xi})|, \end{aligned}$$

where we have set

$$S_j^{\xi,\varepsilon} := \{x = y + t\, P_k(\xi') : \ y \in \partial A_{\varepsilon,j},\, |t| \leq \varepsilon,\, \xi' \in \xi + [-r', r']^d\} \cap A.$$

As in the proof of Proposition 4.13, using an extension argument combined with (4.18) and (4.33) one can show that

$$\begin{aligned} & \sum_{\xi \in r'\mathbb{Z}^d_M} \varepsilon^{k-1} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |u_\varepsilon^j(\varepsilon x_\alpha) - u_\varepsilon^j(\varepsilon x_{\alpha+\xi})| \\ & \qquad \leq C\delta + C \sum_{|\xi| \leq L_\delta} \sum_{\alpha \in R_\varepsilon^\xi(S_j^{\xi,\varepsilon})} \varepsilon^{k-1} c_\varepsilon^\omega(x_\alpha, x_{\alpha+\xi}) |u_\varepsilon^j(\varepsilon x_\alpha) - u_\varepsilon^j(\varepsilon x_{\alpha+\xi})| \Big). \end{aligned}$$

To estimate the interactions with $|\xi| \leq L_\delta$, note that due to (4.34) we can use the averaging technique again to obtain $j_\varepsilon \in \{1, \dots, N_\varepsilon\}$ and the corresponding sequence $u_\varepsilon^{j_\varepsilon}$ satisfying the boundary conditions (at least for small ε because of (4.30)) such that

$$\limsup_{\varepsilon \to 0} E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u_\varepsilon^{j_\varepsilon}, A) \leq E_{\mathrm{hom}}(\omega)(u, A) + C\mathcal{H}^{k-1}(S_{u_0} \cap (\overline{A} \backslash A_1)) + C\delta,$$

where we used again (4.18). Due to the assumptions on u_0 and (4.34), we know that $u_\varepsilon^{j_\varepsilon} \to u$ on A. Letting first $\delta \to 0$ and then $A_1 \uparrow A$ we finally get

$$\Gamma\text{-}\limsup_\varepsilon E^{l_\varepsilon}_{\varepsilon,u_0}(\omega)(u, A) \leq E_{\mathrm{hom}}(\omega)(u, A) = E_{u_0}(\omega)(u, A).$$

The general case can be deduced by density using Lemmata 2.17 and 2.14. The details are already contained in the proof of Theorem 3.23. □

Remark 4.20. (i) In the case of finite range of interactions, that is $c_{lr}(x,y)=0$ for $|x-y|\geq L$, it is enough to take $l_\varepsilon \geq L$.

(ii) By Remark 4.4 the above Theorem 4.19 implies the usual convergence of minimizers in the spirit of Γ-convergence.

4.5 Phase constraints in the stationary case

In this section we will discuss the variational limit of the energies $E_\varepsilon(\omega)$ when we fix the number of lattice points where the configuration takes the value $+1$. In order to formulate the result, given $V_\varepsilon \in \mathbb{N}_0$, let us introduce the class

$$\mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega) := \{u : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\} : \; \#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D : \; u(\varepsilon x) = 1\} = V_\varepsilon\}.$$

In the sequel we will assume that there exists $V \in [0,1]$ such that

$$\lim_{\varepsilon\to 0} \frac{V_\varepsilon}{\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D\}} = V.$$

The lemma below describes how phase constraints behave for finite energy sequences.

Lemma 4.21. *For $\mathbb{P}$-almost all $\omega \in \Omega$ the following statement holds true: For all $u \in BV(D,\{\pm 1\})$ such that there exists a sequence $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ with $u_\varepsilon \to u$ in the sense of Definition 4.3 and*

$$\sup_{\varepsilon>0} E_\varepsilon(\omega)(u_\varepsilon) \leq C, \qquad \lim_{\varepsilon\to 0} \frac{\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D : \; u_\varepsilon(\varepsilon x) = 1\}}{\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}D\}} = V',$$

we have

$$\frac{1}{|D|}\int_D u \,\mathrm{d}x = 2V' - 1.$$

Proof. For $\omega \in \Omega$ we consider the sequence of nonnegative Borel measures $\gamma_\varepsilon(\omega)$ on D defined as the weighted sum of point masses

$$\gamma_\varepsilon(\omega) = \sum_{z \in P_k(\mathcal{L}(\omega)) \cap \frac{D}{\varepsilon}} \varepsilon^k \# \left(P_k^{-1}(z) \cap \mathcal{L}(\omega)\right) \delta_{\varepsilon z}.$$

As $\gamma_\varepsilon(\omega)(D) \leq C|D|$, up to subsequences we know that $\gamma_\varepsilon(\omega) \overset{*}{\rightharpoonup} \gamma(\omega)$ in the sense of measures. Let us identify the limit measure. To this end we define a discrete stochastic process $\gamma : \mathcal{I}_k \to L^1(\Omega)$ as

$$\gamma(I)(\omega) := \sum_{y \in P_k(\mathcal{L}(\omega)) \cap I} \#\left(P_k^{-1}(y) \cap \mathcal{L}(\omega)\right) = \#\left(x \in \mathcal{L}(\omega) :\ P_k(x) \in I\right). \tag{4.35}$$

It follows from (4.3) that $\gamma(I)$ is essentially bounded for every $I \in \mathcal{I}_k$. In addition it can be checked that $\gamma(I)$ is $\mathcal{F}$-measurable, thus we infer that $\gamma(I) \in L^\infty(\Omega)$. Upon redefining the group action as $\tilde{\tau}_z = \tau_{-z}$, the process γ is stationary and (sub)additive. By Theorem 2.21 there exists $\gamma_0(\omega)$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$ and all $I \in \mathcal{I}_k$ we have

$$\lim_{n \to +\infty} \frac{\gamma(nI)(\omega)}{n^k |I|} = \gamma_0(\omega).$$

It is straightforward to extend this result to all sequences $t_n \to +\infty$ and then to all half open cubes in $\mathbb{R}^k$ by a continuity argument. Now let $Q \subset D$ be any half open cube. Then by definition

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(Q) = \lim_{\varepsilon \to 0} \sum_{z \in P_k(\mathcal{L}(\omega)) \cap \frac{1}{\varepsilon} Q} \varepsilon^k \#\left(P_k^{-1}(z) \cap \mathcal{L}(\omega)\right) = \gamma_0(\omega)|Q|. \tag{4.36}$$

Given any open set $A \in \mathcal{A}(D)$, for $\delta > 0$ we consider the following interior approximation:

$$A_{\text{int}}(\delta) = \bigcup_{z \in \delta\mathbb{Z}^k :\, z + [0,\delta)^k \subset A} z + [0,\delta)^k.$$

It can be checked by monotone convergence that $\lim_{\delta \to 0} |A(\delta)| = |A|$. By (4.36) and additivity we obtain

$$\liminf_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A) \geq \liminf_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A(\delta)) = \gamma_0(\omega)|A(\delta)|.$$

Letting $\delta \to 0$ we obtain $\liminf_\varepsilon \gamma_\varepsilon(\omega)(A) \geq \gamma_0(\omega)|A|$. The Portmanteau-Theorem implies that $\gamma(\omega)(B) = \gamma_0(\omega)|B|$ for all Borel sets $B \subset D$, in particular the whole sequence converges in the sense of measures. On the other hand, if $A \in \mathcal{A}(D)$ is such that $|\partial A| = 0$, then the outer approximation

$$A_{\text{out}}(\delta) = \bigcup_{z \in \delta\mathbb{Z}^k :\, z + [0,\delta)^k \cap A \neq \emptyset} z + [0,\delta)^k$$

also fulfills $\lim_{\delta \to 0} |A(\delta)| = |A|$. Hence we conclude again by additivity and (4.36) that

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon(\omega)(A) = \gamma_0(\omega)|A| \tag{4.37}$$

for all $A \in \mathcal{A}(D)$ such that $|\partial A| = 0$. Given again $\delta > 0$, we take any polyhedral function $u_\delta \in BV_{loc}(\mathbb{R}^k, \{\pm 1\})$ such that $\|u - u_\delta\|_{L^1(D)} \leq \delta$. As u_δ is Borel-measurable, we have

$$\int_D Pu_\varepsilon \,\mathrm{d}\gamma_\varepsilon(\omega) = \int_D (Pu_\varepsilon - u_\delta)\,\mathrm{d}\gamma_\varepsilon(\omega) + \int_D u_\delta \,\mathrm{d}\gamma_\varepsilon(\omega).$$

Since u_δ is a polyhedral function, we can use (4.37) for the second term to obtain

$$\lim_{\varepsilon \to 0} \int_D u_\delta \,\mathrm{d}\gamma_\varepsilon(\omega) = \gamma_0(\omega) \int_D u_\delta \,\mathrm{d}x. \tag{4.38}$$

What concerns the first term, by (4.2) we have

$$\left| \int_D (Pu_\varepsilon - u_\delta)\,\mathrm{d}\gamma_\varepsilon(\omega) \right| \leq C \sum_{z \in P_k(\mathcal{L}(\omega)) \cap \frac{D}{\varepsilon}} \varepsilon^k |Pu_\varepsilon(\varepsilon z) - u_\delta(\varepsilon z)| \tag{4.39}$$

Now using the fact that u_ε has equibounded energy and that u_δ is a polyhedral function, one can reason as in the proof of Lemma 4.6 to show that

$$\limsup_{\varepsilon \to 0} \sum_{z \in P_k(\mathcal{L}(\omega)) \cap \frac{D}{\varepsilon}} \varepsilon^k |Pu_\varepsilon(\varepsilon z) - u_\delta(\varepsilon z)| \leq C \|u - u_\delta\|_{L^1(D)} \leq C\delta.$$

Combining the above inequality with (4.38) and (4.39) we finally obtain by the arbitrariness of δ that

$$\lim_{\varepsilon \to 0} \int_D Pu_\varepsilon \,\mathrm{d}\gamma_\varepsilon(\omega) = \gamma_0(\omega) \int_D u \,\mathrm{d}x$$

The claim now follows on observing that, by plugging in the definition and using again (4.37), it holds

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_D Pu_\varepsilon \,\mathrm{d}\gamma_\varepsilon(\omega) = &\lim_{\varepsilon \to 0} \varepsilon^k 2\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega) \cap P_k^{-1}(D) :\ u_\varepsilon(\varepsilon x) = 1\} \\ &- \lim_{\varepsilon \to 0} \varepsilon^k \#\{\varepsilon\mathcal{L}(\omega) \cap P_k^{-1}(D)\} = (2V' - 1)|D|\gamma_0(\omega).\end{aligned}$$

□

In order to incorporate the phase constraint into the functional, for $\mathbb{P}$-almost every $\omega \in \Omega$ we introduce $E_\varepsilon^{V_\varepsilon}(\omega) : \mathcal{PC}_\varepsilon(\omega) \to [0, +\infty]$ defined by

$$E_\varepsilon^{V_\varepsilon}(\omega)(u) = \begin{cases} E_\varepsilon(\omega)(u) & \text{if } u \in \mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

With the help of Lemma 4.21 we now prove the thin film analogue of Theorem 3.28.

Theorem 4.22. *Let $\mathcal{L}$ be a stationary thin admissible stochastic lattice and let c_{nn} and c_{lr} satisfy Hypothesis 2 with the additional structure property (4.19). For $\mathbb{P}$-almost every ω the functionals $E_\varepsilon^{V_\varepsilon}(\omega)$ Γ-converge with respect to the convergence of Definition 4.3 to the functional $E_{\rm hom}^V(\omega) : L^1(D) \to [0,+\infty]$ defined by*

$$E_{\rm hom}^V(\omega)(u) = \begin{cases} \int_{S_u} \phi_{\rm hom}(\omega;\nu_u)\,\mathrm{d}\mathcal{H}^{k-1} & \text{if } u \in BV(D,\{\pm 1\}),\ \frac{1}{|D|}\int_D u\,\mathrm{d}x = 2V-1, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The lower bound is an immediate consequence of Theorem 4.17 and Lemma 4.21. In order to prove the upper bound, note that due to the density result proved in [6, Lemma 7.1] and Lemma 2.14 we can reduce the analysis to the case where $u \in BV(D,\{\pm 1\})$ is a polyhedral function such that

$$\frac{1}{|D|}\int_D u\,\mathrm{d}x = 2V-1.$$

By Theorem 4.17 we can find a sequence $u_\varepsilon : \varepsilon\mathcal{L}(\omega) \to \{\pm 1\}$ such that u_ε converges to u in the sense of Definition 4.3 and

$$\lim_{\varepsilon\to 0} E_\varepsilon(\omega)(u_\varepsilon) = E_{\rm hom}(\omega)(u). \tag{4.40}$$

Repeating the argument used for proving Proposition 4.13 one can show that without loss of generality we may assume that $u_\varepsilon = u$ on $S_\eta := \{x \in D :\ \mathrm{dist}(x,S_u) > \eta\}$ for some $0 < \eta << 1$. We extend $u_\varepsilon(\varepsilon x) := u(\varepsilon x)$ for $\varepsilon x \in \varepsilon\mathcal{L}(\omega)\backslash D$. Let us set $\tilde{V}_\varepsilon = \#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D :\ u_\varepsilon(\varepsilon x) = 1\}$. Applying Lemma 4.21 we deduce that

$$\lim_{\varepsilon\to 0}\frac{\tilde{V}_\varepsilon - V_\varepsilon}{\#\{\varepsilon x \in \varepsilon\mathcal{L}(\omega)\cap P_k^{-1}D\}} = 0. \tag{4.41}$$

In the following we modify the function u_ε in the case $\tilde{V}_\varepsilon > V_\varepsilon$. We set $h_\varepsilon = (\tilde{V}_\varepsilon - V_\varepsilon)^{\frac{1}{k}}$. Without loss of generality we may assume that $h_\varepsilon \to +\infty$ (otherwise flip the value of a very slowly diverging number of particles).

Upon reducing η, we find x_0 such that $Q_{e_1}(x_0,\eta) \subset S_\eta \cap \{u = 1\}$. We already know from the proof of Lemma 4.21 that, almost surely, we can write

$$q^\omega(x_0,h_\varepsilon) := \#\{x \in \mathcal{L}(\omega) :\ P_k(x) \in Q_{e_1}(x_0,\gamma_0(\omega)^{-1}h_\varepsilon)\} = h_\varepsilon^k + h_\varepsilon^{k-1}\gamma_\varepsilon,$$

for some sequence $\gamma_\varepsilon = \gamma_\varepsilon(\omega,x_0)$ such that $\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon}{h_\varepsilon} = 0$. In the following we assume that $\gamma_\varepsilon \le 0$, but with a similar argument we can also treat the case $\gamma_\varepsilon > 0$. As $\mathcal{L}(\omega)$ is thin admissible in the sense of Definition 4.1, for some appropriate $c = c(R) > 0$ it holds true that

$$\frac{1}{C}h_\varepsilon^{k-1} \leq q^\omega(x_0, h_\varepsilon + n + c) - q^\omega(x_0, h_\varepsilon + n) \leq Ch_\varepsilon^{k-1}$$

for any $0 \leq n \leq h_\varepsilon$. In particular, there exists $n_\varepsilon = \mathcal{O}(\gamma_\varepsilon)$ and a nonnegative and equibounded c_ε such that

$$q^\omega(x_0, h_\varepsilon + n_\varepsilon) = h_\varepsilon^k + c_\varepsilon h_\varepsilon^{k-1}.$$

Now choose any set $G_\varepsilon \subset \mathbb{R}^d$ such that $P_k G_\varepsilon \subset S_\eta \cap \{u = -1\}$ and $\#(G_\varepsilon \cap \mathcal{L}(\omega)) = c_\varepsilon h_\varepsilon^{k-1}$. To reduce notation let us set $Q_\varepsilon := Q_{e_1}(x_0, \gamma_0(\omega)^{-1}\varepsilon(h_\varepsilon + n_\varepsilon))$. We define

$$\bar{u}_\varepsilon(\varepsilon x) = \begin{cases} -1 & \text{if } P_k(\varepsilon x) \in Q_\varepsilon, \\ +1 & \text{if } \varepsilon x \in G_\varepsilon, \\ u_\varepsilon(\varepsilon x) & \text{otherwise.} \end{cases}$$

By construction we have $\bar{u}_\varepsilon \in \mathcal{PC}_\varepsilon^{V_\varepsilon}(\omega)$ and, since $h_\varepsilon \varepsilon \to 0$ by (4.41), we still have that $\bar{u}_\varepsilon \to u$ in the sense of Definition 4.3. Repeating the energy estimates from the proof of the upper bound in Theorem 3.28 one can show that

$$\limsup_{\varepsilon \to 0} E_\varepsilon^{V_\varepsilon}(\omega)(\bar{u}_\varepsilon) = \limsup_{\varepsilon \to 0} E_\varepsilon(\omega)(u_\varepsilon) = E_{\text{hom}}(\omega)(u).$$

The case when $\tilde{V}_\varepsilon \leq V_\varepsilon$ can be treated by an almost symmetric argument. Hence, by the Urysohn property of Γ-convergence and Remark 4.4, we get the claim. □

4.6 A model for random deposition

The general homogenization result proved in Section 4.3 describes only the qualitative phenomenon that interfaces may form on the flat subspace. In this final section we investigate the asymptotic behavior of the limit energy as a function of the average thickness. To simplify matter, we consider a 3d to 2d dimension reduction problem in which the magnetic particles are deposited with vertical order on a two-dimensional flat substrate and interact via finite-range ferromagnetic interactions (see Remark 4.26 (ii) for a short discussion on problems for infinite range interactions). We obtain information on the dependence of the limit energy on the average thickness when the latter is very small or very large.

In order to model the substrate where the particles are deposited, we take a two-dimensional deterministic lattice, which we choose for simplicity to be $\mathcal{L}^0 = \mathbb{Z}^2 \times \{0\}$. We then consider an independent random field $\{X_i^p\}_{i\in\mathbb{Z}^3}$, where the X_i^p are Bernoulli random variables with $\mathbb{P}(X_i^p = 1) = p \in (0,1)$ and, for fixed $M \in \mathbb{N}$, we define a random point set as follows:

$$\mathcal{L}_p^M(\omega) := \left\{ (i_1, i_2, i_3) \in \mathbb{Z}^3 : \ 0 \le i_3 \le \sum_{k=1}^{M} X^p_{(i_1,i_2,k)}(\omega) \right\}, \tag{4.42}$$

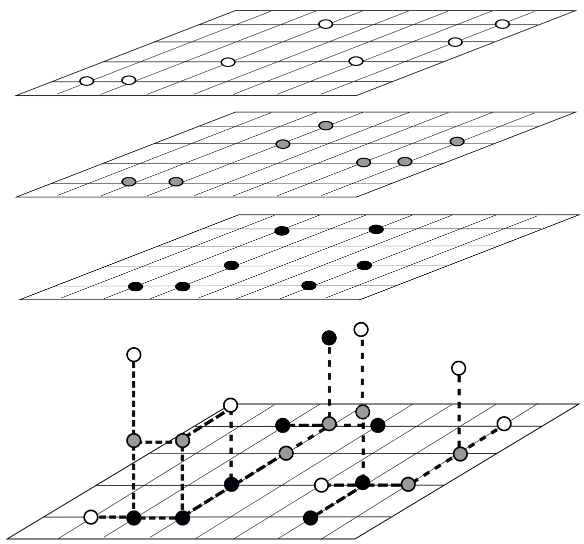

Figure 4-2: Three successive deposition steps (black, gray and white) in the construction of $\mathcal{L}_p^M(\omega)$. The dashed bonds connect nearest neighboring particles.

which means that we successively deposit particles M times independently onto the flat lattice $\mathcal{L}^0$ and stack them over each other (see Figure 4-2). Note that the point set constructed in (4.42) is stationary with respect to integer translations in $\mathbb{Z}^2$ and ergodic by the independence assumption (for a suitable group action on a product space). Given $u : \varepsilon\mathcal{L}_p^M(\omega) \to \{\pm 1\}$, we consider an energy of the form

$$E^p_{\varepsilon,M}(\omega)(u, A) = \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega) \\ P_2(x),P_2(y)\in\frac{A}{\varepsilon}}} \varepsilon c(x-y)|u(\varepsilon x) - u(\varepsilon y)|, \tag{4.43}$$

where the interaction $c : \mathbb{R}^3 \to [0, +\infty)$ fulfills

(i) $c(z) \le C$ for all $z \in \mathbb{R}^3$,

(ii) $c(z) = 0$ if $|z| \ge L$,

(iii) $c(z) \geq c_0 > 0$ if $|z| = 1$.

Remark 4.23. Coefficients satisfying the above conditions (i)-(iii) are of the form (4.19), but in general do not satisfy the coercivity condition of Hypothesis 2. However the results obtained in the first part of this chapter still hold true. This is due to the vertical order of the deposition model which makes the proof of coercivity much simpler. However note that for instance the constant in Lemma 3.12 now depends strongly on M.

Due to Remark 4.23 we can apply Theorem 3.19 and deduce that there exists the effective (deterministic) surface tension

$$\phi^p_{\rm hom}(M;\nu) := \lim_{t\to+\infty} \frac{1}{t} \inf\{E^p_{1,M}(\omega)(v, Q_\nu(0,t)) :\ v \in \mathcal{PC}^{2L}_{1,u_{0,\nu}}(\omega, Q_\nu(0,t))\},$$

where we used the alternative formula in Remark 4.18 and Remark 4.20.

We are interested in the asymptotic behavior of $\phi^p_{\rm hom}(M;\nu)$ when $M \to +\infty$. First let us define some auxiliary quantities. Given $p \in (0,1]$, $0 \leq N < M$ and $u : \mathbb{Z}^3 \to \{\pm 1\}$ we set

$$E^p_{[N,M]}(\omega)(u, A) := \sum_{\substack{x,y \in \mathcal{L}^M_p(\omega) \\ x,y \in A\times[N,M]}} c(x-y)|u(x) - u(y)|$$

and omit the dependence on ω of $E^p_{[N,M]}$ when $p = 1$. In that case, given $\nu \in S^1$ we further introduce the corresponding surface tension

$$\phi^{1,M}(\nu) = \lim_{t\to+\infty} \frac{1}{t} \inf\{E^1_{[0,M]}(u, Q_\nu(0,t)) :\ v \in \mathcal{PC}^{2L}_{1,u_{0,\nu}}(\omega, Q_\nu(0,t))\}.$$

Note that the existence of this limit follows by standard subadditivity arguments. The next lemma shows that the auxiliary surface tensions converge when $M \to +\infty$.

Lemma 4.24. *For any $\nu \in S^1$ there exists the limit*

$$\phi^1(\nu) := \lim_{M\to+\infty} \frac{1}{M} \phi^{1,M}(\nu).$$

Proof. We define a sequence $a_k = \phi^{1,k-1}(\nu)$. It is enough to show that a_k is superadditive. To reduce notation, similar to (4.20) let us introduce

$$m_{[N,M]}(u_{0,\nu}, Q_\nu(x,\rho)) := \inf\{E^1_{[N,M]}(v, Q_\nu(x,\rho)) :\ v \in \mathcal{PC}^{2L}_{1,u_{0,\nu}}(Q_\nu(x,\rho))\}.$$

Note that by periodicity $m_{[N,M]}(u_{0,\nu}, Q_\nu(x,\rho)) = m_{[N+k,M+k]}(u_{0,\nu}, Q_\nu(x,\rho))$ for every

$k \in \mathbb{N}$. For fixed $t >> 1$ one can take any admissible minimization candidate for $m_{[0,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t))$ and restrict it to the sets $Q_\nu(0,t)\times[0,M-1]$ and $Q_\nu(0,t)\times[M, M+M'-1]$ to obtain the inequality

$$\begin{aligned} m_{[0,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t)) &\geq m_{[0,M-1]}(u_{0,\nu}, Q_\nu(0,t)) + m_{[M,M+M'-1]}(u_{0,\nu}, Q_\nu(0,t)) \\ &= m_{[0,M-1]}(u_{0,\nu}, Q_\nu(0,t)) + m_{[0,M'-1]}(u_{0,\nu}, Q_\nu(0,t)), \end{aligned}$$

where we neglected the interactions between the two cubes and used periodicity in the last equality. Dividing the inequality by t and letting $t \to +\infty$, we obtain superadditivity of the sequence a_k. □

The next result shows the asymptotic behavior of the surface tension when the average number of layers pM diverges.

Proposition 4.25. *Let ϕ^1 be defined as in Lemma 4.24. For $\nu \in S^1$ it holds that*

$$\lim_{M\to+\infty} \frac{\phi^p_{\mathrm{hom}}(M;\nu)}{pM} = \phi^1(\nu).$$

Proof. Throughout this proof we assume without loss of generality that $L \in \mathbb{N}$ and we set $\mathbb{Z}^2_M := \mathbb{Z}^2 \times \{0,\dots,M\}$. Let us fix $\nu \in S^1$. We separately show two inequalities. For the moment we also fix M. Let us consider a sequence of minimizing configurations u_N such that $\lim_N \frac{1}{N} E^1_{[0,M]}(u_N, Q_\nu(0,N)) = \phi^{1,M}(\nu)$. As we show now, we can assume that u_N is a plane-like configuration as provided by Theorem 4.30. Indeed, applying this theorem we find a plane-like ground state u_ν for the energy

$$E_M(u, Q_\nu(0,N)) := \sum_{\substack{x\in\mathbb{Z}^2_M \\ P_2(x)\in Q_\nu(0,N)}} \sum_{y\in\mathbb{Z}^2_M} c(x-y)|u(x)-u(y)|.$$

To reduce notation, we introduce the stripe

$$S_\nu(N,\gamma) = \{x \in \mathbb{R}^2 :\ x \in Q_\nu(0,N),\ \mathrm{dist}(x,\nu^\perp) \leq 4(\gamma+L)\}$$

so that the energy of u_ν is concentrated on $S_\nu(N,\gamma)\times[0,M]$ with $\gamma \leq CM$ (see Theorem 4.30). For $N \in \mathbb{N}$ we define two configurations $\overline{u}_N, \tilde{u}_N : \mathbb{Z}^2_M \to \{\pm 1\}$ via

$$\overline{u}_N(x) = \begin{cases} u_{0,\nu}(P_2(x)) & \text{if } \mathrm{dist}(P_2(x), \mathbb{R}^2\backslash Q_\nu(0,N) \leq 2L, \\ u_\nu(x) & \text{otherwise} \end{cases}$$

and

$$\tilde{u}_N(x) = \begin{cases} u_\nu(x) & \text{if } \mathrm{dist}(P_2(x), \mathbb{R}^2 \setminus (Q_\nu(0,N)) \leq L, \\ u_N(x) & \text{otherwise.} \end{cases}$$

Then $\overline{u}_N$ is a plane-like configuration whose energy is still concentrated on $S_\nu(N,\gamma) \times [0,M]$. Using the boundary conditions and the finite range assumptions one can easily prove that

$$\begin{aligned} E^1_{[0,M]}(u_N, Q_\nu(0,N)) &\leq E^1_{[0,M]}(\overline{u}_N, Q_\nu(0,N)) \leq E_M(u_\nu, Q_\nu(0,N)) + CM^2 \\ &\leq E_M(\tilde{u}_N, Q_\nu(0,N)) + CM^2 \leq E^1_{[0,M]}(u_N, Q_\nu(0,N)) + 2CM^2. \end{aligned}$$

Dividing by N and letting $N \to +\infty$ we see that asymptotically we can replace u_N by the plane-like configuration $\overline{u}_N$. From now on we denote by $u_{N,M}$ a plane-like minimizer whose energy is concentrated on $S_\nu(N,\gamma) \times [0,M]$ with $\gamma \leq CM$ and such that $\phi^{1,M}(\nu) = \lim_N \frac{1}{N} E^1_{[0,M]}(u_{N,M}, Q_\nu(0,N))$. We extend $u_{N,M}$ to $\mathbb{Z}^3$ setting $u_{N,M}(x) = u_{0,\nu}(P_2(x))$ for $x_3 \notin \{0,\dots,M\}$.

For $\delta > 0$ small enough, we separate the contribution of the substrate and the first $M^p_\delta := \lceil (p+\delta)M \rceil$ random layers and estimate the remaining interactions. Using dominated convergence this leads to

$$\begin{aligned} \frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu) \leq& \frac{1}{M} \liminf_{N\to+\infty} \frac{1}{N} \mathbb{E}[E^p_{1,M}(\omega)(u_{N,M^p_\delta}, Q_\nu(0,N))] \\ \leq& \frac{1}{M} \liminf_{N\to+\infty} \frac{1}{N} \mathbb{E}[E^1_{[0,M^p_\delta]}(u_{N,M^p_\delta}, Q_\nu(0,N))] \\ &+ \frac{C}{M} \limsup_{N\to+\infty} \frac{1}{N} \mathbb{E}\left[\#\{x \in \mathcal{L}^M_p(\omega) :\ x \in S_\nu(N,\gamma) \times (M^p_\delta - L, M]\}\right] \\ \leq& \frac{1}{M}\phi^{1,M^p_\delta}(\nu) + C\mathbb{E}[\#\{x \in \mathcal{L}^p_M(\omega) :\ x \in \{(0,0)\} \times (M^p_\delta - L, M]\} \\ \leq& \frac{1}{M}\phi^{1,M^p_\delta}(\nu) + C \sum_{k=M^p_\delta - L}^{M} (k - M^p_\delta + L) \binom{M}{k} p^k (1-p)^{M-k}, \end{aligned}$$

where in the last step we have used that the probability of having k points in $\{(0,0)\} \times (M^p_\delta - L, M]$ is the same as having $k + M^p_\delta - L$ successes out of M trials in a Bernoulli experiment. In order to bound the last sum, we use Hoeffding's inequality which yields, for M large enough depending on L and δ,

$$\mathbb{P}(\sum_{i=1}^{M} X^p_{(0,0,i)} \geq k + M^p_\delta - L) \leq \mathbb{P}(\sum_{i=1}^{M} X^p_{(0,0,i)} \geq k + (p + \frac{\delta}{2})M) \leq \exp(-2M(\frac{\delta}{2} + \frac{k}{M})^2).$$

From this bound we infer the estimate

$$\sum_{k=M^p_\delta-L}^{M}(k-M^p_\delta+L)\binom{M}{k}p^k(1-p)^{M-k}\le\sum_{k=1}^{M}k\exp(-\frac{1}{2}M\delta^2)\exp(-2\delta k).$$

As the right hand side vanishes when $M\to+\infty$, by Lemma 4.24 we deduce

$$\limsup_M\frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu)\le(p+\delta)\,\phi^1(\nu).$$

As δ was arbitrary the first inequality is proven.

It remains to show the reverse inequality. Given any admissible function $v_N:\mathcal{L}^M_p(\omega)\to\{\pm1\}$ we can neglect the interactions coming from $Q_\nu(0,N)\times[M^p_{-\delta}+1,M]$ which yields the estimate

$$E^p_{1,M}(\omega)(v_N,Q_\nu(0,N))\ge E^p_{[0,M^p_{-\delta}]}(\omega)(v_N,Q_\nu(0,N)).$$

Minimizing on both sides and dividing by N, we obtain in the limit that

$$\frac{1}{M}\phi^p_{\mathrm{hom}}(M;\nu)\ge\frac{1}{M}\phi^{p,M^p_{-\delta}}(\nu). \tag{4.44}$$

Now the idea is to estimate the error when we replace $\phi^{p,M^p_{-\delta}}(\nu)$ by $\phi^{1,M^p_{-\delta}}(\nu)$. Let $u_{N,M^p_{-\delta}}$ be a plane-like sequence of configurations as in the first part of the proof. We also consider an optimal sequence $u^{p,\delta}_N=u^{p,\delta}_N(\omega)$ such that

$$\phi^{p,M^p_{-\delta}}(\nu)=\lim_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^p_{[0,M^p_{-\delta}]}(\omega)(u^{p,\delta}_N,Q_\nu(0,N))].$$

As the deterministic surface tension certainly dominates the random one, we have

$$\begin{aligned}
0&\le\phi^{1,M^p_{-\delta}}(\nu)-\phi^{p,M^p_{-\delta}}(\nu)\\
&=\lim_N\frac{1}{N}\mathbb{E}\left[E^1_{[0,M^p_{-\delta}]}(u_{N,M^p_{-\delta}},S_\nu(N,\gamma))-E^p_{[0,M^p_{-\delta}]}(\omega)(u^{p,\delta}_N(\omega),Q_\nu(0,N))\right]\\
&\le\limsup_N\frac{1}{N}\mathbb{E}\left[E^1_{[0,M^p_{-\delta}]}(u^{p,\delta}_N,S_\nu(N,\gamma))-E^p_{[0,M^p_{-\delta}]}(\omega)(u^{p,\delta}_N(\omega),S_\nu(N,\gamma))\right]\\
&\le C\limsup_N\frac{1}{N}\mathbb{E}[\#\{x\in\big(S_\nu(N,\gamma)\times[1,M^p_{-\delta}]\big)\cap\mathbb{Z}^3:\ x\notin\mathcal{L}^M_p(\omega)\}]\\
&\le CM\mathbb{E}\Big[\max\{M^p_{-\delta}-\sum_{i=1}^{M}X^p_{(0,0,i)},0\}\Big]\le CM\sum_{k=1}^{M^p_{-\delta}}k\,\mathbb{P}(M^p_{-\delta}-\sum_{i=1}^{M}X^p_{(0,0,i)}\ge k).
\end{aligned}$$

Here we used that the number of missing interactions can be estimated by the number of missing lattice points since each point can only interact with finitely many others. Now we apply again Hoeffding's inequality which yields

$$\mathbb{P}(M^p_{-\delta} - \sum_{i=1}^{M} X^p_{(0,0,i)} \geq k) \leq \mathbb{P}(M(p-\frac{\delta}{2}) - k \geq \sum_{i=1}^{M} X^p_{(0,0,i)}) \leq \exp(-2M(\frac{\delta}{2}+\frac{k}{M})^2).$$

We conclude the bound

$$\sum_{k=1}^{M^p_{-\delta}} k\mathbb{P}(M^p_{-\delta} - \sum_{i=1}^{M} X^p_{(0,0,i)} \geq k) \leq \sum_{k=1}^{M^p_{-\delta}} k\exp(-\frac{1}{2}M\delta^2)\exp(-2\delta k).$$

Again the right hand side vanishes when $M \to +\infty$ and thus

$$\lim_{M} \frac{1}{M} |\phi^{1,M^p_{-\delta}}(\nu) - \phi^{p,M^p_{-\delta}}(\nu)| = 0,$$

so that Lemma 4.24 and (4.44) imply the estimate

$$\liminf_{M\to+\infty} \frac{1}{M}\phi^p_{\text{hom}}(M;\nu) \geq \lim_{M\to+\infty} \frac{1}{M}\phi^{1,M^p_{-\delta}}(\nu) = (p-\delta)\phi^1(\nu).$$

Again the desired inequality follows by the arbitrariness of $\delta > 0$. □

Remark 4.26. (i) If we would not include the initial layer $\mathcal{L}^0$, then Proposition 4.25 would still hold. However then the surface tension may not be related to an appropriate Γ- limit as compactness of sequences with bounded energy becomes a nontrivial issue. We refer to [26] for a possible approach to this problem in the case of nearest neighbor interactions and bond percolation models.

(ii) When one tries to extend Proposition 4.25 to the case of infinite range of interactions, there occur several problems. In order to use the result for the truncated energies we would have to get an estimate between the surface tension and its truncated version. In the previous sections this was achieved using Lemma 4.8 which now depends on M as we do not have coercive interactions along Voronoi neighbors. On the other hand the argument used in the proof of Proposition 4.25 is not available for infinite range interactions.

A percolation-type phenomenon

We close this section with a result on the growth of the averaged surface tension comparing the two extreme cases where the average number of layers is very low or diverges. We let $\mathcal{L}^M_p(\omega)$ be defined as in (4.42) but restrict the analysis to nearest neighbor interactions and make them slightly non-periodic in the sense that their magnitude is very small when one of the particles belongs to the substrate $\mathcal{L}^0$. More precisely, given $0 < \eta << 1$ we consider functions of the form

$$c_\eta(x-y) = \begin{cases} 0 & \text{if } |x-y| > 1, \\ \eta & \text{if } |x-y| = 1 \text{ and } x_3 \cdot y_3 = 0, \\ c(x-y) & \text{otherwise,} \end{cases}$$

where $x \mapsto c(x)$ is strictly positive on the unit circle. Then the coefficients are of the type (4.19). We define $E^{p,\eta}_{\varepsilon,M}(\omega)$ as in (4.43) with c replaced by c_η. By Remark 4.23 we may again apply Theorem 4.17 and deduce that there exists the limit

$$\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) := \lim_{t\to+\infty} \frac{1}{t} \inf\{E^{p,\eta}_{1,M}(\omega)(v, Q_\nu(0,t)) :\ v \in \mathcal{PC}^2_{1,u_{0,\nu}}(\omega, Q_\nu(0,t))\}.$$

In contrast to Proposition 4.25, for this model we also consider the case of small M. We will show that if $p < 1 - p_{\mathrm{site}}$, where p_{site} is the critical site percolation probability on $\mathbb{Z}^2$, then it holds that

$$\phi^{p,\eta}_{\mathrm{hom}}(1;\nu) \leq C_p \eta,$$

where C_p may blow up only for $p \to 1 - p_{\mathrm{site}}$. Note that we do not claim here that p_{site} is optimal. We can actually improve the result in the sense that for all $M \in \mathbb{N}$ such that $(1-p)^M > p_{\mathrm{site}}$, we have

$$\phi^{p,\eta}_{\mathrm{hom}}(M;\nu) \leq C_{p,M}\eta.$$

This shows that when the probability is very small but finite, the surface tension can be arbitrary small depending on the strength of the interaction in the substrate layer; on the other hand we will establish an analogue of Proposition 4.25 asserting that if the average number of layers increases further, even the normalized surface tension approaches a value independent of η. This result can be interpreted as a weak version of the experimentally observed phenomenon in [41]. Before we prove the result, let us introduce the typical energy of one slice. Given $q \in (0,1]$ and $u : \mathbb{Z}^2 \to \{\pm 1\}$ we set

$$E^q_{sl}(\omega)(u,A) := \sum_{\substack{x,y \in \mathcal{L}^1_q(\omega)\setminus\mathcal{L}^0 \\ P_2(x),P_2(y)\in A}} c(x-y)|u(x)-u(y)|$$

and omit the dependence on ω if $q = 1$. We further introduce the corresponding surface tension

$$\phi_{sl}^q(\nu) = \lim_{t\to+\infty} \frac{1}{t} \inf\{E_{sl}^q(\omega)(u, Q_\nu(0,t)) : \ v(x) = u_{0,\nu}(x) \text{ if } \operatorname{dist}(x, \partial Q_\nu(0,t)) \leq 2\}.$$

Note that the existence of this deterministic limit follows again from the subadditive ergodic theorem as in the proof of Theorem 4.17, since we used condition (i) of Definition 4.1 only for passing from finite range to decaying interactions in Step 1. In general the random variables $\omega \mapsto E_{sl}^q(\omega)(u, A)$ are not defined on the same probability space but we will use them only for slices of the large set $\mathcal{L}_p^M(\omega)$.

Theorem 4.27. *Let $p \in (0,1)$ and $M \in \mathbb{N}$ be such that $(1-p)^M > p_{\text{site}}$. There exists a constant $C_{p,M}$ locally bounded for $(1-p)^M \in (p_{\text{site}}, 1)$ such that*

$$\phi_{\text{hom}}^{p,\eta}(M;\nu) \leq C_{p,M}\eta.$$

On the other hand, for any $p \in (0,1)$ it holds that

$$\lim_{M\to+\infty} \frac{\phi_{\text{hom}}^{p,\eta}(M;\nu)}{pM} = 2\Big(\big(c(e_1) + c(-e_1)\big)|\nu_1| + \big(c(e_2) + c(-e_2)\big)|\nu_2|\Big).$$

Proof. In order to prove the first statement, we start with the case $\nu = e_2$ and use results from percolation theory which show that the contribution from the random layers is negligible: For $q := (1-p)^M > p_{\text{site}}$, we consider the so-called Bernoulli site percolation on $\mathbb{Z}^2$, that is we assign independently a weight $X_i(\omega) \in \{\pm 1\}$ to all the nodes $i \in \mathbb{Z}^2$ such that $\mathbb{P}(X_i = 1) = q$. We say that $i_0, \dots, i_k \in \mathbb{Z}^2$ is an occupied path if $|i_n - i_{n+1}| = 1$ and $X_{i_n}(\omega) = 1$ for all $n = 0, \dots, k$. Theorem 11.1 in [40] yields that there exist universal constants c_j, d_j such that

$$\mathbb{P}\Big(\exists \text{ at least } c_1(q-p_{\text{site}})^{d_1} n \text{ disjoint occupied paths from } \{0\}\times[0,n] \text{ to } \{m\}\times[0,n] \\ \text{and contained in } [0,m]\times[0,n]\Big) \geq 1 - c_2(m+1)\exp(-c_3(q-p_{\text{site}})^{d_2} n).$$

Given $N \in \mathbb{N}$, we first combine this estimate with the Borel-Cantelli lemma and, using stationarity, we obtain that for almost every $\omega \in \Omega$ there exists $N_0 = N_0(\omega)$ such that for all $N \geq N_0$ we find at least $c_1(q-p_{\text{site}})^{d_1} 2\sqrt{N}$ disjoint occupied paths connecting the vertical boundary segments of the rectangle $R_N := [-\lfloor \frac{N}{2} \rfloor + 2, \lfloor \frac{N}{2} \rfloor - 2] \times [-\lceil \sqrt{N} \rceil, \lceil \sqrt{N} \rceil]$. As the paths are disjoint and are contained in R_N, at least one of them uses at most $\frac{2}{c_1}(q-p_{\text{site}})^{-d_1} N$ vertices.

Now we come back to the actual proof. By definition of the random lattice in

(4.42), using the above considerations in the layer $\mathbb{Z}^2 \times \{1\}$, for $N \geq N_0$ we can find a path connecting the vertical boundary segments of the rectangle $R_N \times \{1\}$ that is contained in $R_N \times \{1\}$ and uses at most $c_{p,M}N$ vertices with none of them belonging to $\mathcal{L}_p^M(\omega)$. This path separates $R_N \times \{1\}$ into two subregions $R_N^- \times \{1\}$ and $R_N^+ \times \{1\}$. For $N \geq N_0$ we define a (random) configuration $u_N : \mathcal{L}_p^M(\omega) \to \{\pm 1\}$ by

$$u_N(x) = \begin{cases} u_{0,e_2}(P_2(x)) & \text{if } P_2(x) \notin R_N, \\ +1 & \text{if } P_2(x) \in R_N^+, \\ -1 & \text{otherwise.} \end{cases}$$

Up to possibly exchanging the roles of $R_N^\pm$ we have that $u_N \in \mathcal{PC}^2_{1,u_{0,e_2}}(\omega, Q_{e_2}(0,N))$. Hence by definition of $\phi^{p,\eta}_{\text{hom}}(M,e_2)$ and the fact that u_N depends not on the x_3-direction, it holds that

$$\begin{aligned} \phi^{p,\eta}_{\text{hom}}(M,e_2) &\leq \liminf_{N\to+\infty} \frac{1}{N} E^{p,\eta}_{1,M}(\omega)(u_N, Q_{e_2}(0,N)) \\ &\leq \limsup_{N\to+\infty} \frac{1}{N} \sum_{\substack{x,y\in Q_{e_2}(0,N)\cap\mathbb{Z}^2 \\ |x-y|=1}} \eta|u_N(x)-u_N(y)| \\ &\quad + \limsup_{N\to+\infty} \frac{1}{N} \sum_{k=1}^{M} \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega) \\ x,y\in Q_{e_2}(0,N)\times\{k\}}} c(x-y)|u_N(x)-u_N(y)|. \end{aligned} \tag{4.45}$$

We now estimate each of the two terms on the right hand side. Concerning the second one, we observe that if $x, y \in (Q_{e_2}(0,N) \times \{k\}) \cap \mathcal{L}_p^M(\omega)$ are such that $|x-y| = 1$ and $u_N(x) \neq u_N(y)$, then either $P_2(x), P_2(y) \in \pm\frac{N}{2}e_1 + \big([-4,4] \times [-2\sqrt{N}, 2\sqrt{N}]\big)$ or, without loss of generality, $P_2(x) \in R_N^-$ and $P_2(y) \in R_N^+$. In the second case, we note that either $(P_2(x),1)$ or $(P_2(y),1)$ has to be a vertex of the path constructed above. Hence either $x \notin \mathcal{L}_p^M(\omega)$ or $y \notin \mathcal{L}_p^M(\omega)$. We conclude that these interactions don't exist and we may bound the second term via

$$\limsup_{N\to+\infty} \frac{1}{N} \sum_{k=1}^{M} \sum_{\substack{x,y\in\mathcal{L}_p^M(\omega) \\ x,y\in Q_{e_2}(0,N)\times\{k\}}} c(x-y)|u_N(x)-u_N(y)| \leq \limsup_{N\to+\infty} \frac{CM}{\sqrt{N}} = 0. \tag{4.46}$$

Applying the same arguments for the first term, we can use the fact that the separating path uses at most $c_{p,M}N$ vertices and we deduce that

$$\limsup_{N\to+\infty}\frac{1}{N}\sum_{\substack{x,y\in Q_{e_2}(0,N)\cap\mathbb{Z}^2\\|x-y|=1}}\eta|u_N(x)-u_N(y)|\leq 4c_{p,M}\eta.$$

From this estimate, the first claim in the case $\nu=e_2$ follows by (4.45) and (4.46). The above argument can be adapted easily to the cases $\nu=-e_2$ and $\nu=\pm e_1$. By L^1-lower semicontinuity, the one-homogeneous extension of $\phi_{\rm hom}^{p,\eta}$ must be convex (see Remark 4.18). For general $\nu\in S^1$ the claim then follows upon multiplying the constant by a factor $\sqrt{2}$.

In order to prove the second claim, again we show two inequalities. Given a sequence of admissible configurations u_N such that $\lim_N \frac{1}{N}E^1_{sl}(u_N,Q_\nu(0,N))=\phi^1_{\rm sl}(\nu)$, we define an admissible configuration $\overline{u}_N:\mathcal{L}_p^M(\omega)\to\{\pm1\}$ via

$$\overline{u}_N(x)=u_N(P_2(x)).$$

Arguing as in the proof of Proposition 4.25, we may assume that u_N is a plane-like configuration and its energy is concentrated in a stripe

$$S_\nu(N,\gamma)=\{x\in\mathbb{R}^2:\ x\in Q_\nu(0,N),\,\mathrm{dist}(x,\nu^\perp)\leq 4(\gamma+1)\},$$

where now γ is independent of N,M. We let $p_k=\sum_{l=k}^M\binom{M}{l}p^l(1-p)^{M-l}$ be the probability of having at least k successes out of M trials in a Bernoulli experiment. By definition and the fact that $\overline{u}_N$ is independent of x_3-coordinate, we obtain that for any $\delta>0$ small enough

$$\begin{aligned}\frac{\phi_{\rm hom}^{p,\eta}(M;\nu)}{M}\leq&\frac{1}{M}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^{p,\eta}_{1,M}(\omega)(\overline{u}_N,Q_\nu(0,N))]\\ \leq&\liminf_{N\to+\infty}\frac{1}{M}\left(\sum_{k=1}^M\frac{1}{N}\mathbb{E}[E^{p_k}_{sl}(\omega)(u_N,Q_\nu(0,N))]+\frac{C}{N}\#(\mathbb{Z}^2\cap S_\nu(N,\gamma))\right)\\ \leq&\lim_{N\to+\infty}\frac{1}{N}(p+\delta)E^1_{sl}(u_N,Q_\nu(0,N))\\ &+\liminf_{N\to+\infty}\frac{1}{M}\sum_{k>\lfloor(p+\delta)M\rfloor}^M\mathbb{E}[E^{p_k}_{sl}(\omega)(u_N,Q_\nu(0,N))]+\frac{C\gamma}{M}\\ \leq&(p+\delta)\phi^1_{\rm sl}(\nu)+\sup_{k>\lfloor(p+\delta)M\rfloor}\liminf_{N\to+\infty}\frac{1}{N}\mathbb{E}[E^{p_k}_{\rm sl}(\omega)(u_N,Q_\nu(0,N))]+\frac{C\gamma}{M}.\end{aligned}$$

Note that here the new random variables are indeed defined on the same probability space and are coupled to the variables generating the stochastic lattice $\mathcal{L}_p^M(\omega)$. As γ is independent of M, the third term vanishes when $M \to +\infty$, so that we are left to show that also the second one tends to zero. In order to estimate the second term we use the fact that u_N is a plane-like configuration which in particular implies

$$\frac{1}{N}\mathbb{E}[E_{\mathrm{sl}}^{p_k}(\omega)(u_N, Q_\nu(0,N))] = \frac{1}{N}\mathbb{E}[E_{\mathrm{sl}}^{p_k}(\omega)(u_N, S_\nu(N,\gamma))] \leq p_k C\gamma.$$

For any $k > \lfloor (p+\delta)M \rfloor$, by the law of large numbers it holds that $p_k \to 0$ when $M \to +\infty$. Hence we deduce $\limsup_M \frac{1}{M}\phi_{\mathrm{hom}}^{p,\eta}(M;\nu) \leq (p+\delta)\,\phi_{\mathrm{sl}}^1(\nu)$. As $\delta > 0$ was arbitrary, we finally obtain

$$\limsup_M \frac{1}{M}\phi_{\mathrm{hom}}^{p,\eta}(M;\nu) \leq p\,\phi_{\mathrm{sl}}^1(\nu).$$

We next show the reverse inequality. Given any admissible function $\overline{u}_N : \mathcal{L}_p^M(\omega) \to \{\pm 1\}$ we can neglect the interactions between different layers and the whole lowest layer $\mathcal{L}^0$ to obtain for $\delta > 0$ small enough the estimate

$$\begin{aligned} E_{1,M}^{p,\eta}(\omega)(\overline{u}_N, Q_\nu(0,N)) &\geq \sum_{k=1}^{M} E_{\mathrm{sl}}^{p_k}(\omega)(\overline{u}_N(\cdot,k), Q_\nu(0,N)) \\ &\geq \sum_{k=1}^{\lceil (p-\delta)M \rceil} E_{\mathrm{sl}}^{p_k}(\omega)(\overline{u}_N(\cdot,k), Q_\nu(0,N)). \end{aligned}$$

Since $\overline{u}_N(\cdot,k)$ fulfills the correct boundary condition in the k-th layer, we deduce that

$$\frac{1}{M}\phi_{\mathrm{hom}}^{p,\eta}(M;\nu) \geq (p-\delta) \inf_{k \leq \lceil (p-\delta)M \rceil} \phi_{\mathrm{sl}}^{p_k}(\nu).$$

Again by the law of large numbers for an independent Bernoulli experiment it remains to show that the function $q \mapsto \phi_{sl}^q(\nu)$ is continuous in $q = 1$, that means we can pass from a random to a deterministic lattice.

In order to prove the continuity let u_N be an optimal plane-like sequence of configurations such that $\phi_{\mathrm{sl}}^1(\nu) = \lim_N \frac{1}{N} E_{\mathrm{sl}}^1(u_N, S_\nu(N,\gamma))$ and consider an optimal (possibly random) sequence $u_N^q(\omega)$ such that

$$\phi_{\mathrm{sl}}^q(\nu) = \lim_{N \to +\infty} \frac{1}{N}\mathbb{E}[E_{\mathrm{sl}}^q(\omega)(u_N^q(\omega), Q_\nu(0,N))].$$

Similar to the proof of Proposition 4.25 we obtain

$$\begin{aligned}
0 \leq \phi^1_{\rm sl}(\nu) - \phi^q_{\rm sl}(\nu) &= \lim_N \frac{1}{N}\mathbb{E}[E^1_{\rm sl}(u_N, S_\nu(N,\gamma)) - E^q_{\rm sl}(\omega)(u^q_N(\omega), Q_\nu(0,N))] \\
&\leq \limsup_N \frac{1}{N}\mathbb{E}[E^1_{\rm sl}(u^q_N(\omega), S_\nu(N,\gamma)) - E^q_{sl}(\omega)(v^q_N(\omega), S_\nu(N,\gamma))] \\
&\leq C \lim_N \frac{1}{N}\mathbb{E}[\#\{z \in (S_\nu(N,\gamma)\cap\mathbb{Z}^2)\times\{1\} :\ z \notin \mathcal{L}^1_q(\omega)\}] = C(1-q)\gamma.
\end{aligned}$$

The estimate above clearly implies convergence of the surface tensions when $q \to 1$ which shows that $\limsup_M \frac{1}{M}\phi^{p,\eta}_{\rm hom}(M;\nu) \geq p\,\phi^1_{\rm sl}(\nu)$.

It remains to identify $\phi^1_{\rm sl}(\nu)$. We just sketch the argument (compare with the results of [3]). Any admissible configuration asymptotically has an interface containing at least $|\nu_1|$ interactions along the two directions $\pm e_1$ and $|\nu_2|$ interactions along the directions $\pm e_2$. As any couple of interacting points is counted twice with reversing direction and $|u(x) - u(y)| \in \{0, 2\}$ we find that $\phi^1_{\rm sl}(\nu) \geq 2(c(e_1) + c(-e_1))|\nu_1| + 2(c(e_2) + c(-e_2))|\nu_2|$. On the other hand a suitable discretization of a plane attains this value, hence we finished the proof as

$$\phi^1_{\rm sl}(\nu) = 2(c(e_1) + c(-e_1))|\nu_1| + 2(c(e_2) + c(-e_2))|\nu_2|.$$

□

4.7 Plane-like minimizers for one-periodic dimension reduction problems

In this final section we prove that the results about plane-like minimizers for periodic interactions in [29] can be extended to dimension reduction problems. We restrict the analysis to one-periodic interactions which is the case when the coefficients depend only on the difference as in (4.19). Moreover, we focus on the case of 3-d to 2-d since it is the most interesting one. We remark that the content of this section should be seen as an appendix to Section 4.6, although we think that the result is interesting on its own.

To fix notation, for any set $\Gamma \subset \mathbb{Z}^2$, we write $\Gamma_M = \Gamma \times (\mathbb{Z}\cap[0, M])$. In contrast to the main part of this chapter, here we consider an interaction energy that takes into account also interactions outside the domain. More precisely, given $u : \mathbb{Z}^2_M \to \{\pm 1\}$

we investigate finite range energies of the form

$$E_M(u,\Gamma) = \sum_{x\in\Gamma_M}\sum_{y\in\mathbb{Z}^2_M} c(x-y)|u(x)-u(y)|,$$

where the coefficients fulfill the following assumptions:

(i) $0 \le c(z) \le C$ for all $z \in \mathbb{R}^3$ and $\min_i c(\pm e_i) \ge c_0 > 0$,

(ii) there exists $L > 0$ such that $c(z) = 0$ for all $|z| \ge L$.

Before we state and prove the main theorem we recall some standard definitions.

Definition 4.28. *We say that* $u : \mathbb{Z}^2_M \to \{\pm 1\}$ *is a ground state for the energy* E_M *whenever* $E_M(u,\Gamma) \le E_M(v,\Gamma)$ *for all finite sets* $\Gamma \subset \mathbb{Z}^2$ *and all* $v : \mathbb{Z}^2_M \to \{\pm 1\}$ *such that* $u = v$ *on* $\{z \in \mathbb{Z}^2_M :\ \exists z' \in (\mathbb{Z}^2\backslash\Gamma)_M$ *with* $|z - z'| \le L\}$.

Remark 4.29. When u and Γ are such that $E_M(u,\Gamma) \le E_M(v,\Gamma)$ for all v such that $u = v$ on $\{z \in \mathbb{Z}^2_M :\ \exists z' \in (\mathbb{Z}^2\backslash\Gamma)_M$ with $|z-z'| \le L\}$, then the same conclusion holds for every subset $\Gamma' \subset \Gamma$. Indeed, take any v such that $u = v$ on $\{z \in \mathbb{Z}^2_M :\ \exists z' \in (\mathbb{Z}^2\backslash\Gamma')_M$ with $|z-z'| \le L\}$. Then for any couple x, y such that $x \in (\Gamma\backslash\Gamma')_M$ and $y \in \mathbb{Z}^2_M$ with $|x-y| \le L$, it holds that $u(x) = v(x)$ and $u(y) = v(y)$. Hence it follows easily that

$$E_M(u,\Gamma') - E_M(v,\Gamma') = E_M(u,\Gamma) - E_M(v,\Gamma) \le 0.$$

Using the same notation as for the stochastic group action, for $k \in \mathbb{Z}^2$ we denote by τ_k the shift operator acting on sets Γ and configurations $u : \mathbb{Z}^2_M \to \{\pm 1\}$ via

$$\tau_k\Gamma = \Gamma + k, \qquad \tau_k u(x) = u(x-(k,0)).$$

Then one can easily check that the following formula holds true:

$$E_M(\tau_k u, \tau_k\Gamma) = E_M(u,\Gamma). \tag{4.47}$$

The remaining part of this section will be devoted to the proof of the next theorem.

Theorem 4.30. *There exists* $\gamma > 0$ *such that for all* $\nu \in S^1$ *there exists a ground state* u_ν *of* E_M *such that* $u(x) \ne u(y)$ *implies* $\operatorname{dist}(x,\nu^\perp) \le \gamma$. *Such a ground state is called plane-like. Moreover we can choose* $\gamma \le CM$ *for some constant* C *independent of* ν, M.

We prove this theorem similar to [29, 31]. We first construct a particular minimizer amongst periodic configurations that enjoys several geometric properties. To this end, we need further notation.

Let us fix a rational direction $\nu \in S^1 \cap \mathbb{Q}^2$. We define the $\mathbb{Z}$-module $\mathbb{Z}_\nu = \{z \in \mathbb{Z}^2 : \langle z, \nu \rangle = 0\}$ and, given $m \in \mathbb{N}$, we let $\mathcal{F}_{m,\nu}$ be any fundamental domain of the quotient $\mathbb{Z}^2_{/m\mathbb{Z}_\nu}$. Recall that this means that for every $z \in \mathbb{Z}^2$ there exist unique $z_1 \in m\mathbb{Z}_\nu$ and $z_2 \in \mathcal{F}_{m,\nu}$ such that $z = z_1 + z_2$. Given real numbers $\beta < \gamma$, we further introduce

$$\mathcal{F}^{\beta,\gamma}_{m,\nu} = \{z \in \mathcal{F}_{m,\nu} : \langle \nu, z \rangle \in [\beta, \gamma]\}.$$

Now we define an admissible class of periodic configurations: A function $u : \mathbb{Z}^2_M \to \{\pm 1\}$ is called (m,ν)-periodic, if $u(x) = u(x + m(z, 0))$ for every $x \in \mathbb{Z}^2_M$ and every $z \in \mathbb{Z}_\nu$. We set

$$\mathcal{A}^{\beta,\gamma}_{m,\nu} = \{u \text{ is } (m,\nu)\text{-periodic, } u = 1 \text{ if } \langle P_2(z), \nu \rangle < \beta,\ u(z) = -1 \text{ if } \langle P_2(z), \nu \rangle > \gamma\}.$$

We start with an elementary lemma which shows that for periodic functions any translation yields the same energy.

Lemma 4.31. *Let u be (m,ν)-periodic and $k \in \mathbb{Z}^2$. Then it holds that*

$$E_M(\tau_k u, \mathcal{F}_{m,\nu}) = E_M(u, \mathcal{F}_{m,\nu}).$$

Proof. Given $x \in (\tau_{-k}\mathcal{F}_{m,\nu})_M$, we find $z_1(x) \in m\mathbb{Z}_\nu$ and $z_2(x) \in \mathcal{F}_{m,\nu}$ such that $P_2(x) = z_1(x) + z_2(x)$. By (m,ν)-periodicity, for any $y \in \mathbb{Z}^2_M$ it holds that

$$\begin{aligned} |u(x) - u(y)| &= |u(x - (z_1(x), 0)) - u(y - (z_1(x), 0))|, \\ c(x - y) &= c(x - (z_1(x), 0) - y + (z_1(x), 0)). \end{aligned}$$

Now assume that there exists another $x' \in (\tau_{-k}\mathcal{F}_{m,\nu})_M \backslash \{x\}$ with $\langle x - x', e_3 \rangle = 0$ and $z_2(x) = z_2(x')$. Then $\tau_k P_2(x) - \tau_k P_2(x') = z_1(x) - z_1(x') \in m\mathbb{Z}_\nu \backslash \{(0,0)\}$. As $\tau_k P_2(x), \tau_k P_2(x') \in \mathcal{F}_{m,\nu}$ this contradicts the fact that $\mathcal{F}_{m,\nu}$ is a fundamental domain. Using (4.47) we conclude by comparison that

$$E_M(\tau_k u, \mathcal{F}_{m,\nu}) = E_M(u, \tau_{-k}\mathcal{F}_{m,\nu}) \leq E_M(u, \mathcal{F}_{m,\nu}).$$

Applying the above inequality to τ_{-k} and $\tilde{u} := \tau_k u$, which is also (m,ν)-periodic, we obtain the claim. $\square$

We define the class of minimizers for the energy $E_M(\cdot, \mathcal{F}_{m,\nu})$ on $\mathcal{A}^{\beta,\gamma}_{m,\nu}$ via

$$\mathcal{M}^{\beta,\gamma}_{m,\nu} = \{u \in \mathcal{A}^{\beta,\gamma}_{m,\nu} : \ E_M(u, \mathcal{F}_{m,\nu}) \leq E_M(v, \mathcal{F}_{m,\nu}) \quad \forall v \in \mathcal{A}^{\beta,\gamma}_{m,\nu}\}.$$

As the set $\mathcal{A}^{\beta,\gamma}_{m,\nu}$ is finite, the class of minimizers is non-empty. Next we define the so-called infimal minimizer which has several useful properties.

$$u^{\beta,\gamma}_{m,\nu} := \min\{u \in \mathcal{M}^{\beta,\gamma}_{m,\nu}\} \in \mathcal{A}^{\beta,\gamma}_{m,\nu}.$$

We first show that the infimal minimizer also belongs to the class of minimizers. This follows easily from the following elementary observation (see Lemma 2.1 and also Lemma 2.3 in [31]).

Lemma 4.32. *Given any $u : \mathbb{Z}^2_M \to \{\pm 1\}$ and $\Gamma \in \mathbb{Z}^2$ finite, it holds that*

$$E_M(\min\{u, v\}, \Gamma) + E_M(\max\{u, v\}, \Gamma) \leq E_M(u, \Gamma) + E_M(v, \Gamma).$$

Iterating the above lemma finitely many times we find that $u^{\beta,\gamma}_{m,\nu} \in \mathcal{M}^{\beta,\gamma}_{m,\nu}$.

We now turn to the first property of the infimal minimizer. This is the so-called absence of symmetry breaking, which says that the infimal minimizer does not depend on the length m of the period.

Lemma 4.33. *For any $m \in \mathbb{N}$ it holds that $u^{\beta,\gamma}_{m,\nu} = u^{\beta,\gamma}_{1,\nu}$.*

Proof. We define an auxiliary configuration via $u = \min\{\tau_k u^{\beta,\gamma}_{m,\nu} : \ k \in \mathbb{Z}_\nu\}$. By elementary arguments it follows that $u \in \mathcal{A}^{\beta,\gamma}_{1,\nu}$, while Lemma 4.31 implies that $\tau_k u^{\beta,\gamma}_{m,\nu} \in \mathcal{M}^{\beta,\gamma}_{m,\nu}$ and by iterating Lemma 4.32 we obtain that $u \in \mathcal{M}^{\beta,\gamma}_{m,\nu}$. Since obviously $u \leq u^{\beta,\gamma}_{m,\nu}$, by definition of the infimal minimizer we obtain that $u = u^{\beta,\gamma}_{m,\nu}$. Moreover, as u and $u^{\beta,\gamma}_{1,\nu}$ are both $(1,\nu)$-periodic it follows that

$$E_M(u, \mathcal{F}_{1,\nu}) = \frac{1}{m} E_M(u, \mathcal{F}_{m,\nu}) \leq \frac{1}{m} E_M(u^{\beta,\gamma}_{1,\nu}, \mathcal{F}_{m,\nu}) = E_M(u^{\beta,\gamma}_{1,\nu}, \mathcal{F}_{1,\nu}). \tag{4.48}$$

In particular we deduce that $u \in \mathcal{M}^{\beta,\gamma}_{1,\nu}$ and thus $u \geq u^{\beta,\gamma}_{1,\nu}$. On the other hand (4.48) must be an equality, so that $u^{\beta,\gamma}_{1,\nu} \in \mathcal{M}^{\beta,\gamma}_{m,\nu}$ and therefore $u^{\beta,\gamma}_{1,\nu} \geq u$. This proves the claim. □

We next establish the so-called Birkhoff property of the infimal minimizer which will be the main ingredient for the proof of Theorem 4.30.

Lemma 4.34. *Let $k \in \mathbb{Z}^2$. Then it holds that*

$$\tau_k u_{1,\nu}^{\beta,\gamma} \begin{cases} \leq u_{1,\nu}^{\beta,\gamma} & \text{if } \langle k,\nu\rangle \leq 0, \\ \geq u_{1,\nu}^{\beta,\gamma} & \text{if } \langle k,\nu\rangle \geq 0. \end{cases}$$

Proof. We start with the case $\langle k,\nu\rangle \leq 0$ and define the two configurations $m_* = \min\{u_{1,\nu}^{\beta,\gamma}, \tau_k u_{1,\nu}^{\beta,\gamma}\}$ and $m^* = \max\{u_{1,\nu}^{\beta,\gamma}, \tau_k u_{1,\nu}^{\beta,\gamma}\}$. By elementary considerations one can prove that $m_* \in \mathcal{A}_{1,\nu}^{\beta+\langle k,\nu\rangle,\gamma+\langle k,\nu\rangle}$ and $m^* \in \mathcal{A}_{1,\nu}^{\beta,\gamma}$. Using Lemma 4.32 we obtain

$$\begin{aligned} E_M(m_*, \mathcal{F}_{1,\nu}) + E_M(u_{1,\nu}^{\beta,\gamma}, \mathcal{F}_{1,\nu}) &\leq E_M(m_*, \mathcal{F}_{1,\nu}) + E_M(m^*, \mathcal{F}_{1,\nu}) \\ &\leq E_M(\tau_k u_{1,\nu}^{\beta,\gamma}, \mathcal{F}_{1,\nu}) + E_M(u_{1,\nu}^{\beta,\gamma}, \mathcal{F}_{1,\nu}), \end{aligned}$$

which yields $E_M(m_*, \mathcal{F}_{1,\nu}) \leq E_M(\tau_k u_{1,\nu}^{\beta,\gamma}, \mathcal{F}_{1,\nu})$. As a next step we claim that $\tau_k u_{1,\nu}^{\beta,\gamma} = u_{1,\nu}^{\beta+\langle k,\nu\rangle,\gamma+\langle k,\nu\rangle}$. Indeed, as $\tau_k u_{1,\nu}^{\beta,\gamma} \in \mathcal{A}_{1,\nu}^{\beta+\langle k,\nu\rangle,\gamma+\langle k,\nu\rangle}$ this configuration is admissible and minimality follows from Lemma 4.31. Now assume it would not be the infimal minimizer, then also $u_{1,\nu}^{\beta,\gamma}$ cannot be the infimal minimizer as we could construct a smaller one by translation of the other infimal minimizer.

By definition of the infimal minimizer we infer that $m_* \geq \tau_k u_{1,\nu}^{\beta,\gamma}$, which proves the claim by definition of m_*. The case $\langle k,\nu\rangle \geq 0$ follows upon applying the translation τ_k to the inequality $\tau_{-k} u_{1,\nu}^{\beta,\gamma} \leq u_{1,\nu}^{\beta,\gamma}$ which holds by the first part of the proof. □

In the next lemma we deduce a powerful property of configurations satisfying the Birkhoff property.

Lemma 4.35. *Let $u : \mathbb{Z}^2_M \to \{\pm 1\}$ satisfy the Birkhoff property with respect to $\nu \in S^1 \cap \mathbb{Q}^2$, that means*

$$\tau_k u \begin{cases} \leq u & \text{if } \langle k,\nu\rangle \leq 0, \\ \geq u & \text{if } \langle k,\nu\rangle \geq 0. \end{cases}$$

Assume further that $u(x_0) = -1$ for some $x_0 \in \mathbb{Z}^2_M$. Then $u(x) = -1$ for all $x \in \mathbb{Z}^2_M$ such that $\langle x - x_0, e_3\rangle = 0$ and $\langle P_2(x - x_0), \nu\rangle \geq 0$.

Proof. Every such x can be written as $x = x_0 - (k, 0)$ with $k \in \mathbb{Z}^2$ such that $\langle k,\nu\rangle \leq 0$. Hence Lemma 4.34 implies that $u(x) = \tau_k u(x_0) \leq u(x_0) = -1$, so that $u(x) = -1$. □

We are now in a position to prove that the infimal minimizer becomes unconstrained when we take $\beta = 0$ and γ large enough. From now on we set $u_\nu^\gamma := u_{1,\nu}^{0,\gamma}$.

Lemma 4.36. *There exists $0 < \gamma_0 \leq CM$ such that for all $\gamma \geq \gamma_0$ it holds $u_\nu^\gamma(x) = -1$ for all $x \in \mathbb{Z}^2_M$ such that $\langle P_2(x), \nu\rangle \geq \gamma - \sqrt{2}$.*

Proof. By Lemma 4.35 it is enough to show that in every layer $\mathbb{Z}^2 \times \{l\}$ with $l \in \{0, \dots, M\}$ there exists some x_l such that $\langle P_2(x_l), \nu\rangle \leq \gamma - \sqrt{2}$ and $u_\nu^\gamma(x_l) = -1$. We will show that this is always the case provided γ is large enough.

Assume that there exists a layer $\mathbb{Z}^2 \times \{l\}$ such that $u_\nu^\gamma(x) = 1$ for all $x \in \mathbb{Z}^2 \times \{l\}$ with $\langle P_2(x), \nu\rangle \leq \gamma - \sqrt{2}$. We argue that in this case there must exists a second layer $\mathbb{Z}^2 \times \{l'\}$ and a point $x_{l'} \in \mathbb{Z}^2 \times \{l'\}$ with $\langle P_2(x_{l'}), \nu\rangle \leq \sqrt{2}$ and $u_\nu^\gamma(x_{l'}) = -1$. Indeed, if this is false, then the function $\tau_k u_\nu^\gamma$ with any $k \in \{0, \pm 1\}^2$ such that $\langle k, \nu\rangle < 0$ fulfills $\tau_k u_\nu^\gamma \in \mathcal{A}_{1,\nu}^{0,\gamma}$. From Lemma 4.34 we further know that $\tau_k u_\nu^\gamma \leq u_\nu^\gamma$. On the other hand, by Lemma 4.31 we have that $\tau_k u_\nu^\gamma \in \mathcal{M}_{1,\nu}^{0,\gamma}$. Hence by definition of the infimal minimizer we obtain $\tau_k u_\nu^\gamma = u_\nu^\gamma$. By the choice of k this contradicts the boundary conditions. Now applying Lemma 4.35 in the layer $\mathbb{Z}^2 \times \{l'\}$ we obtain that $u_\nu^\gamma(x) = -1$ for all $x \in \mathbb{Z}^2 \times \{l'\}$ such that $\langle P_2(x), \nu\rangle \geq \sqrt{2}$. As we will see, for fixed M this will cost too much energy.

Without loss of generality we assume that $l > l'$, the other case can be treated almost the same way. For every $r \in \{1, \dots, M\}$ there exists $x \in \mathbb{Z}^2 \times \{r\}$ such that $u_\nu^\gamma(x_r) = -1$. Let x_r be one of such points that minimizes $\langle P_2(x), \nu\rangle$ among all such points. According to Lemma 4.35 we obtain $u_\nu^\gamma(x) = -1$ for all $x \in \mathbb{Z}^2 \times \{r\}$ with $\langle P_2(x), \nu\rangle \geq \langle P_2(x_r), \nu\rangle =: p_r$. Note that

$$\left| \sum_{r=l'}^{l-1} (p_{r+1} - p_r) \right| \geq \gamma - 2\sqrt{2}. \tag{4.49}$$

On the other hand, just counting the interactions between neighboring layers, we obtain by the nearest neighbor coercivity of the interactions and (4.49) that

$$E_M(u_\nu^\gamma, \mathcal{F}_{1,\nu}) \geq c \sum_{r=1}^{M} |p_r - p_{r-1}| \geq c(\gamma - 2\sqrt{2}).$$

Testing a discretized plane as a possible minimizer, by the finite range assumption we know an a priori bound of the form $E_M(u_\nu^\gamma, \mathcal{F}_{1,\nu}) \leq CM$. Hence our assumption can only hold as long as $\gamma \leq CM$ for some constant C not depending on ν nor on M and the claim follows upon setting $\gamma_0 = 2CM$. □

Finally we bound the oscillation of the jump set of the infimal minimizer $u_\nu^{\gamma_0}$.

Lemma 4.37. *Let γ_0 be as in Lemma 4.36. Then $u_\nu^{\gamma_0} \in \mathcal{M}_{m,\nu}^{-n,\gamma_0+n}$ for any $n, m \in \mathbb{N}$.*

Proof. We first claim that $u_\nu^{\gamma_0} = u_\nu^{\gamma_0+l}$ for any $l \in \mathbb{N}$. This will be done iteratively. First note that for any $\gamma \geq \gamma_0$ it holds that $u_\nu^\gamma \in \mathcal{A}_{1,\nu}^{0,\gamma+1}$ and by Lemma 4.36 it also holds that $u_\nu^{\gamma+1} \in \mathcal{A}_{1,\nu}^{0,\gamma}$. Then

$$E_M(u_\nu^{\gamma+1}, \mathcal{F}_{1,\nu}) = E_M(u_\nu^\gamma, \mathcal{F}_{1,\nu})$$

and both are infimal minimizers. Hence they must agree. This proves the first claim.

Given an arbitrary configuration $v \in \mathcal{A}_{m,\nu}^{-n,\gamma_0+n}$ we choose a vector $k \in \mathbb{Z}^2$ such that $\langle k,\nu\rangle \geq n$ and $\langle k,\nu\rangle \in \mathbb{N}$. Then

$$\tau_k v \in \mathcal{A}_{m,\nu}^{-n+\langle k,\nu\rangle,\gamma_0+n+\langle k,\nu\rangle} \subset \mathcal{A}_{m,\nu}^{0,\gamma_0+n'}$$

with $n' \in \mathbb{N}$. Using the first claim and the Lemmata 4.31 and 4.33 we obtain that $E_M(u_\nu^{\gamma_0}, \mathcal{F}_{m,\nu}) \leq E_M(\tau_k v, \mathcal{F}_{m,\nu}) = E_M(v, \mathcal{F}_{m,\nu})$. As $u_\nu^{\gamma_0} \in \mathcal{A}_{m,\nu}^{-n,\gamma_0+n}$ we conclude. □

Proof of Theorem 4.30. First assume that $\nu \in S^1 \cap \mathbb{Q}^2$. We show that $u_\nu^{\gamma_0}$ is a ground state. To this end let $\Gamma \subset \mathbb{Z}^2$ be finite and let $v : \mathbb{Z}_M^2 \to \{\pm 1\}$ be such that $v = u_\nu^{\gamma_0}$ on $\{z \in \mathbb{Z}_M^2 : \exists z' \in (\mathbb{Z}^2\backslash\Gamma)_M \text{ with } |z - z'| \leq L\}$. Then we find $m \in \mathbb{N}$ such that, for a suitable fundamental domain, $\Gamma \subset \mathcal{F}_{m,\nu}$. By Lemma 4.37 we have that $E_M(u_\nu^{\gamma_0}, \mathcal{F}_{m,\nu}) \leq E_M(v, \mathcal{F}_{m,\nu})$ and the claim then follows by Remark 4.29.

For general directions $\nu \in S^1$ we argue by approximation. Take a sequence $\nu_j \to \nu$ of rational directions and consider the sequence $u_j := u_{\nu_j}^{\gamma_j}$, where γ_j is uniformly bounded in j. By Tychonoff's theorem we can assume that $u_j \to u$ for some $u : \mathbb{Z}_M^2 \to \{\pm 1\}$. Obviously u is a plane-like configuration. By definition of the topology, given any finite set $\Gamma \subset \mathbb{Z}^2$ we find an index j_0 such that $u_j(x) = u(x)$ for all $x \in \Gamma_M$ and all $j \geq j_0$. Since we assume a finite range of interactions, the previous convergence property easily implies that u is also a ground state and we conclude. □

Chapter 5

Motion of interfaces in random environments: the low-contrast case

5.1 Geometric minimizing movements

In 1993, Almgren, Taylor and Wang introduced the following variational scheme to study curvature-driven flows [8]. Given a fixed time step $\tau > 0$ and an initial set A_0 one constructs recursively a sequence of sets $(A_k^\tau)_k$ minimizing an energy functional of the form

$$E^\tau(A, A_{k-1}^\tau) = \int_{\mathcal{F}A} \varphi(\nu(x))\,\mathrm{d}\mathcal{H}^{d-1} + \frac{1}{2\tau}\int_{A\Delta A_{k-1}^\tau} \mathrm{dist}(x, \partial A_{k-1}^\tau)\,\mathrm{d}x, \tag{5.1}$$

where $\nu(x)$ is the (generalized) normal vector at the point x in the reduced boundary $\mathcal{F}A$ (see Section 2.2 for the definitions) and φ is a suitable surface density. The basic idea is the following: While minimizing the surface energy shrinks the set, the bulk term forces the boundary of the minimizer to be close to the boundary of the previous set. Passing to the limit as $\tau \to 0$ for the piecewise constant interpolations one obtains a time dependent family $A(t)$ of sets that evolves by a weighted curvature (depending on φ), provided the initial set A_0 is regular enough and φ is elliptic and smooth. In the case of $\varphi(\nu) = |\nu|^2$, so that $\int_{\mathcal{F}A} \varphi(\nu(x))\,\mathrm{d}\mathcal{H}^{d-1} = \mathcal{H}^{d-1}(\mathcal{F}A)$, one obtains motion by mean curvature. This minimizing movement procedure was later on exported to random environments by Yip in [55] as follows: at each discrete time step, a minimizer of the energy in (5.1) is computed and then this set is perturbed by a random diffeomorphism.

In the recent paper [24] Braides, Gelli and Novaga applied the above minimizing movement scheme within a deterministic, discrete environment. In this setting the environment is the scaled two-dimensional lattice $\varepsilon\mathbb{Z}^2$. The surface term in (5.1) is replaced by an interfacial energy, which, in its simplest form is derived from the classical nearest neighbor Ising model and can be written as

$$P_\varepsilon(u) = \frac{1}{4} \sum_{\substack{\varepsilon i, \varepsilon j \in \varepsilon\mathbb{Z}^2 \\ |i-j|=1}} \varepsilon |u(\varepsilon i) - u(\varepsilon j)|. \tag{5.2}$$

The scaling factor ε accounts for surface-type contributions. Of course this energy is not finite in general, but within the minimizing movement scheme the sum will be restricted to bounded sets. Note that the energy in (5.2) takes into account only nearest neighbor interactions. Therefore it coincides with the perimeter of the set $\{u = +1\}$ and the relationship to the continuum model is given by identifying the spin variable with this level set. The distance-function in the bulk term in (5.1) is replaced by a discrete version of the l^∞-distance to the boundary defined precisely in (5.4). From a physical point of view this setup can be seen as a simplified model to describe the motion of level sets of spin variables $u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$. Since the discrete perimeter inherits the anisotropy of the lattice, this minimizing movement scheme is related to crystalline motions, where φ is not smooth (see [7, 13, 30] for results in the continuum case). Note that the continuum limit of the energies in (5.2) is given by the crystalline perimeter, that is

$$P(u) = \int_{S_u} |\nu(x)|_1 \, \mathrm{d}\mathcal{H}^1 = \int_{\mathcal{F}\{u=1\}} |\nu(x)|_1 \, \mathcal{H}^1,$$

where $|\nu|_1$ denotes the l^1-norm of ν (see [3]). The ratio between the space and timescale ε/τ plays an important role when studying the geometric flows generated by the minimizing movement scheme as $\varepsilon, \tau \to 0$ simultaneously. We refer the reader to Chapter 8 in [19] for a parade of examples. Concerning the discrete perimeters, when $\varepsilon/\tau \to 0$ fast enough we have in particular that the motion is governed by the Γ-limit, that means one obtains the continuum motion by crystalline curvature. If, on the other hand, $\varepsilon/\tau \to +\infty$ fast enough, the motions are pinned by the presence of many local minimizers in the discrete environment. This phenomenon is similar to any gradient flow that starts in a local minimum. We remark these a priori results are abstract and that without any further analysis they are of limited use as the necessary speed of convergence/divergence might be unknown. In [24] it is shown that the critical scaling regime is $\varepsilon/\tau \sim 1$, where both pinning phenomena and a degenerate crystalline motion can occur, depending on the initial data (see also Theorem 5.3 below). If $\varepsilon/\tau \to +\infty$ then pinning occurs for all bounded regular initial sets, while in the case $\varepsilon/\tau \to 0$ all bounded sets shrink to a point in finite time and for (poly)rectangles the motion law is given by crystalline curvature.

In this chapter we start studying the effect of a random discrete environment on the continuum limit flow. We take a different approach compared to [55] and associate the random effects directly to the lattice points. In the spirit of Chapters 3 and 4 we would like to replace the periodic lattice in the definition of P_ε by the random lattice $\varepsilon\mathcal{L}(\omega)$ with suitable short-range interactions. However results for this problem seem out of reach at the moment. That is why we treat a much simpler problem and associate very small random effects directly to the periodic lattice model, that means we will study the minimizing movement of a random discrete perimeter of the form

$$P_\varepsilon^\omega(u) = \frac{1}{4}\sum_{\substack{\varepsilon i,\varepsilon j\in\varepsilon\mathbb{Z}^2\\ |i-j|=1}} \varepsilon(1+\varepsilon c_{ij}(\omega))|u(\varepsilon i)-u(\varepsilon j)|.$$

The precise assumptions on the random field c_{ij} will be stated in the next section. Note that in this scaling the random perturbations are very small, however it turns out that they can influence at least the velocity of the limit motion. At this point we refer the reader to Chapter 1 for our motivation to keep the bulk term deterministic.

We investigate the evolution of level sets that are coordinate rectangles, that are rectangles with all sides parallel to one of the coordinate axis. In Theorem 5.9 we prove that if the random field c_{ij} satisfies a suitable L^∞-bound and is stationary and α-mixing with a certain decay rate of the mixing coefficients, the limit motion law is deterministic and coincides with the quantized crystalline flow obtained in [24]. However this already depends strongly on the fact that the random field is stationary with respect to the translation group on $\mathbb{Z}^2$. In Theorem 5.20 we show that the velocity changes if we restrict stationarity to a subgroup of the form $m\mathbb{Z}^2$ with $m \geq 2$.

We stress that our results should be seen as a stability result of the deterministic problem rather than an exhaustive description of the possible effects of randomness on the limit flow. Indeed, randomness can influence the motion drastically. For example, in Theorem 3.26 we showed that if we replace the square lattice $\mathbb{Z}^2$ by the random parking process, then, up to a multiplicative constant, we can obtain discrete perimeters that Γ-converge to the one-dimensional Hausdorff measure. Thus, with an appropriate choice of discrete distance, one should not expect a crystalline motion anymore in the limit but rather motion by mean curvature, at least if $\varepsilon << \tau$ and the initial sets $A_0^{\varepsilon,\tau}$ converge to a smooth set.

To highlight some difficulties already present in this simple setting we provide an example of stationary, ergodic perturbations that indicates strong non-uniqueness effects dropping the mixing hypothesis (see Section 5.3). Moreover in this case the functional describing the pointwise movement may not converge (Example 5.6) so that the discrete velocity remains random, but still averaging over an increasing number of time steps one may obtain a homogenized limit velocity making further assumptions.

The random model

Let us introduce some notation. In this chapter we set $Q_\delta(x) = x + [-\frac{\delta}{2}, \frac{\delta}{2})^2$ as the half-open coordinate square centered at x with side length δ.

We now specify the framework for our model. As usual let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. As pointed out in the introduction, we consider the easiest type of normalized ferromagnetic energies accounting only for nearest neighbor interactions. Given $\omega \in \Omega$ and a function $u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$ we set

$$P_\varepsilon^\omega(u) = \frac{1}{4} \sum_{\substack{i,j \in \mathbb{Z}^2 \\ |i-j|=1}} \varepsilon \left(1 + \varepsilon c_{ij}(\omega)\right) |u(\varepsilon i) - u(\varepsilon j)|,$$

where the $c_{ij} : \Omega \to \mathbb{R}$ are uniformly bounded random variables satisfying a suitable α-mixing assumption specified later. Note that without loss of generality we may assume that $c_{ij} = c_{ji}$ for all $|i-j| = 1$. For notational convenience we define

$$\mathcal{A}_\varepsilon := \{A \subset \mathbb{R}^2 : \; A = \bigcup_{i \in \mathcal{I}} Q_\varepsilon(i) \text{ for some } \mathcal{I} \subset \varepsilon\mathbb{Z}^2\}.$$

Note that $\mathcal{A}_\varepsilon$ is closed under unions and intersections. Identifying a function $u : \varepsilon\mathbb{Z}^2 \to \{\pm 1\}$ with the set A given by

$$A := \bigcup_{i \in \mathbb{Z}^2 :\, u(\varepsilon i) = +1} Q_\varepsilon(i) \in A_\varepsilon,$$

we can interpret P_ε^ω as a random perimeter defined on $\mathcal{A}_\varepsilon$ via

$$P_\varepsilon^\omega(A) := P_\varepsilon^\omega(u).$$

If $\mathcal{Z}^2 := \{\xi = \frac{i+j}{2} : \; i, j \in \mathbb{Z}^2, |i-j| = 1\}$ denotes the dual lattice of $\mathbb{Z}^2$, we can rewrite the random perimeter as a sum over points on the boundary ∂A as

$$P_\varepsilon^\omega(A) = \sum_{\substack{\xi \in \mathcal{Z}^2 \\ \varepsilon\xi \in \partial A}} \varepsilon(1 + \varepsilon c_\xi(\omega)), \tag{5.3}$$

where with a slight abuse of notation we set $c_\xi(\omega) := c_{ij}(\omega)$. From now on we define the random variables indexed by the dual lattice. Given $A \in \mathcal{A}_\varepsilon$ it will be useful to define the properly scaled random perimeter also on portions of the boundary $\Gamma \subset \partial A$ setting

$$p_\varepsilon^\omega(\Gamma) = \sum_{\xi\in\mathcal{Z}^2:\, \varepsilon\xi\in\Gamma} \varepsilon c_\xi(\omega).$$

With this notion, it holds that $P_\varepsilon^\omega(A) = \mathcal{H}^1(\partial A) + \varepsilon p_\varepsilon^\omega(\partial A)$.

In order to adapt the idea of Almgren, Taylor and Wang for studying motions, we have to define a suitable discrete distance to measure how far the moved sets differ from the previous one. Since we intend to derive limit motions for sets $A \in \mathcal{A}_\varepsilon$, we restrict ourselves to define the distance with respect to sets rather than points. In order to obtain a crystalline motion we take a discrete version of the l_∞-distance (see [24]). To this end, first note that for every $x \in \mathbb{R}^2$ there exists a unique point $i \in \varepsilon\mathbb{Z}^2$ such that $x \in Q_\varepsilon(i)$. Given a set $A \subset \mathcal{A}_\varepsilon$ we define the value of the measurable function $d_\infty^\varepsilon(\cdot, \partial A) : \mathbb{R}^2 \to [0, +\infty)$ at $x \in Q_\varepsilon(i)$ by

$$d_\infty^\varepsilon(x, \partial A) := \begin{cases} \inf\{\|i-j\|_\infty :\ j \in \varepsilon\mathbb{Z}^2\backslash A\} & \text{if } i \in A, \\ \inf\{\|i-j\|_\infty :\ j \in \varepsilon\mathbb{Z}^2 \cap A\} & \text{if } i \notin A. \end{cases} \tag{5.4}$$

Observe that by definition $d_\infty^\varepsilon(x, \partial A) \in \varepsilon\mathbb{N}$.

Now we can define the total energy to be considered in the minimizing movement scheme. Given a mesh size $\varepsilon > 0$, a time step $\tau > 0$, sets $A, F \in \mathcal{A}_\varepsilon$ and $\omega \in \Omega$ we set

$$E_{\varepsilon,\tau}^\omega(A, F) := P_\varepsilon^\omega(A) + \frac{1}{\tau}\int_{A\Delta F} d_\infty^\varepsilon(x, \partial F)\,\mathrm{d}x.$$

For a fixed (possibly random) initial set $A_\varepsilon^0(\omega)$, we introduce the following discrete-in-time minimization scheme:

(i) $A_{\varepsilon,\tau}^0(\omega) := A_\varepsilon^0(\omega)$,

(ii) $A_{\varepsilon,\tau}^{k+1}(\omega)$ minimizes $A \mapsto E_{\varepsilon,\tau}^\omega(A, A_{\varepsilon,\tau}^k(\omega))$.

Note that this procedure might not be unique. The discrete flat flow is defined as the piecewise constant interpolation

$$A_{\varepsilon,\tau}(t)(\omega) := A_{\varepsilon,\tau}^{\lfloor t/\tau\rfloor}(\omega).$$

As a by-product of the analysis performed in [24], the most interesting regime is $\tau \sim \varepsilon$. Hence we assume for simplicity that

$$\tau = \gamma\,\varepsilon \quad \text{for some } \gamma > 0 \tag{5.5}$$

and omit the dependence on τ in the notation introduced above. For a complete analysis we have to require that the coefficient field satisfies the bound

$$\sup_{\xi} |c_\xi(\omega)| < \frac{1}{4\gamma} \quad \mathbb{P}\text{-almost surely.} \tag{5.6}$$

We remark that some of the results in this chapter are valid for a generic L^∞-bound but unfortunately these are not enough to characterize the motion.

Remark 5.1. Using (5.6) it is easy to see that $P_\varepsilon^\omega(A)$ has the same Γ-limit in the L^1-topology as P_ε defined in (5.2), so that it converges to the crystalline perimeter.

Now we introduce several stochastic properties of the random field $\{c_\xi\}_{\xi\in\mathcal{Z}^2}$. Given a set $I \subset \mathcal{Z}^2$ we set $\mathcal{F}_I = \sigma\,(c_\xi :\ \xi \in I)$ as the σ-algebra generated by the random variables $\{c_\xi\}_{\xi\in I}$. We recall the following definitions from ergodic theory:

Definition 5.2. *Let $\{\tau_z\}_{z\in\mathbb{Z}^2} : \Omega \to \Omega$ be a measurable, measure preserving group action. We say that the random field (c_ξ) is*

(i) stationary, *if $c_\xi(\tau_z\omega) = c_{\xi+z}(\omega) \quad \forall z \in \mathbb{Z}^2,\ \xi \in \mathcal{Z}^2$;*

(ii) ergodic, *if it is stationary and $\{\tau_z\}_z$ is ergodic.*

(iii) strongly mixing (in the ergodic sense), *if it is stationary and*

$$\lim_{|z|\to+\infty} \mathbb{P}(A \cap (\tau_z B)) = \mathbb{P}(A)\mathbb{P}(B) \quad \forall A, B \in \mathcal{F};$$

(iv) α-mixing, *if there exists a sequence $\alpha(n) \to 0$ such that for all sets $I_1, I_2 \in \mathcal{Z}^2$ with $\mathrm{dist}(I_1, I_2) \geq n$ we have*

$$\sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| :\ A \in \mathcal{F}_{I_1},\ B \in \mathcal{F}_{I_2}\} \leq \alpha(n).$$

While for static problems the above notions (i) and (ii) are often enough to prove stochastic homogenization results for variational models (see for example [4] or the results obtained in the previous chapters), in this minimizing movement setting we make use of mixing properties. More precisely, we require that the random field is α-mixing with

$$\sum_{n\geq 1}\alpha(n) < +\infty. \tag{5.7}$$

There are stronger notions of mixing in the literature, however we prefer to choose α-mixing with a certain decay rate of $\alpha(n)$ rather than some ϕ-mixing condition since the generalization of ϕ-mixing conditions to two dimensional random fields is not trivial and many of them already imply a finite range dependence assumption (see [17]). Moreover, in general α-mixing is much weaker than any kind of ϕ-mixing.

Results for deterministic models

At first let us collect some results obtained in the deterministic setting. Within a discrete, deterministic environment, the problem we are interested in has first been studied by Braides, Gelli and Novaga in [24] in the case $c_\xi(\omega) = 0$. For coordinate rectangles as initial sets they prove the following:

Theorem 5.3 (Braides, Gelli, Novaga). *Let $A^0_\varepsilon \in \mathcal{A}_\varepsilon$ be a coordinate rectangle with sides $S_{1,\varepsilon}, ..., S_{4,\varepsilon}$. Assume that A^0_ε converges in the Hausdorff metric to a coordinate rectangle A. Then, up to subsequences, $A_\varepsilon(t)$ converges locally in time to $A(t)$, where $A(t)$ is a coordinate rectangle with sides $S_i(t)$ such that $A(0) = A$ and any side S_i moves inward with velocity $v_i(t)$ given by*

$$v_i(t)\begin{cases} = \frac{1}{\gamma}\left\lfloor \frac{2\gamma}{L_i(t)}\right\rfloor & \text{if } \frac{2\gamma}{L_i(t)} \notin \mathbb{N},\\ \in \frac{1}{\gamma}\left[\left(\frac{2\gamma}{L_i(t)}-1\right), \frac{2\gamma}{L_i(t)}\right] & \text{if } \frac{2\gamma}{L_i(t)} \in \mathbb{N},\end{cases}$$

where $L_i(t) := \mathcal{H}^1(S_i(t))$ denotes the length of the side $S_i(t)$, until the extinction time when $L_i(t) = 0$.
Assume in addition that the lengths L^0_1, L^0_2 of A satisfy one of the three following conditions (we assume that $L^0_1 \leq L^0_2$):

(i) $L^0_1, L^0_2 > 2\gamma$ *(total pinning),*

(ii) $L^0_1 < 2\gamma$ *and* $L^0_2 \leq 2\gamma$ *(vanishing in finite time with shrinking velocity larger than* $1/\gamma$*),*

(iii) $L^0_1 < 2\gamma$ *such that* $2\gamma/L^0_1 \notin \mathbb{N}$ *and* $L^0_2 > 2\gamma$ *(partial pinning),*

then $A_\varepsilon(t)$ converges locally in time to $A(t)$ as $\varepsilon \to 0$, where $A(t)$ is the unique rectangle with side lengths $L_1(t)$ and $L_2(t)$ solving the following system of ordinary *differential equations*

$$\begin{cases} \frac{d}{dt}L_1(t) = -\frac{2}{\gamma}\left\lfloor \frac{2\gamma}{L_2(t)} \right\rfloor, \\ \\ \frac{d}{dt}L_2(t) = -\frac{2}{\gamma}\left\lfloor \frac{2\gamma}{L_1(t)} \right\rfloor \end{cases}$$

for almost every t with initial conditions $L_1(0) = L_1^0$ and $L_2(0) = L_2^0$.

It is the aim of this chapter to extend these results to small random perturbations of the perimeter. While in [24] more general classes of sets are studied, we restrict ourselves to rectangles as the analysis of these sets already contains the main features deriving from randomness. We mention that some effects of periodic perturbations have already been studied in [27, 49]. In [27] the authors treat the following type of high-contrast periodicity: Let $N_a, N_b \in \mathbb{N}$ and $N_{ab} = N_a + N_b$. The coefficients c_ξ are N_{ab} periodic and on the periodicity cell $0 \leq \xi_1, \xi_2 < N_{ab}$ they satisfy

$$c_\xi = \begin{cases} b & \text{if } 0 \leq \xi_1, \xi_2 \leq N_b, \\ a & \text{otherwise,} \end{cases}$$

with weights $a < b$. It is shown that minimizers avoid the b-interactions and thus the limit velocity does not depend on b but only on the geometric proportions N_a, N_b of the periodicity cell. It would be interesting to see how random interactions acting on this scale influence the minimizing sets, since without periodicity it might be impossible to take only a-interactions. However, in this thesis we take the same scaling as the periodic perturbations considered in [49]. These are so called low-contrast perturbations as they vanish when $\varepsilon \to 0$. It is shown in [49] that the right scaling to obtain also b-interactions is $b - a \sim \varepsilon$. With $a = 1$ this model corresponds to a deterministic version of (5.3). We leave the high-contrast case open for future studies. For the interested reader we mention the recent papers [21, 28], where the minimizing movements have been studied for other discrete models.

5.2 Homogenized limit motion of a rectangle

In the sequel we study the case, when the initial data A_ε^0 is a coordinate rectangle. We further assume for the rest of this chapter that

$$\sup_\varepsilon \mathcal{H}^1(\partial A_\varepsilon^0(\omega)) = C < +\infty. \tag{5.8}$$

This bound implies that, for ε small enough, the sequence chosen by the minimizing movement has equibounded perimeter. Indeed, by minimality we have

$$P_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega)) \leq E_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega), A_\varepsilon^k(\omega)) \leq E_\varepsilon^\omega(A_\varepsilon^k(\omega), A_\varepsilon^k(\omega)) = P_\varepsilon^\omega(A_\varepsilon^k(\omega)),$$

so that by induction and (5.6) we infer for ε small enough that

$$\mathcal{H}^1(\partial A_\varepsilon^k(\omega)) \leq 2P_\varepsilon^\omega(A_\varepsilon^k(\omega)) \leq 2P_\varepsilon^\omega(A_\varepsilon^0(\omega)) \leq 4\mathcal{H}^1(\partial A_\varepsilon^0(\omega)). \tag{5.9}$$

Qualitative behavior

The main result of this section ensures that coordinate rectangles remain sets of the same type as long as its sides don't degenerate to a point. As we will see later, this is enough to derive the equation of motion at a fixed time t. The argument splits into two steps. First we prove that any minimizer must be connected and second, using (5.6), we conclude that this component has to be a coordinate rectangle. The idea to prove connectedness is as follows: First we compare the energy with a fast flow of a deterministic functional to conclude that the minimizer must contain a very large rectangle. Then the remaining components are ruled out using the isoperimetric inequality.

Proposition 5.4. *Assume that $\{c_\xi\}_\xi$ fulfills (5.6). Let $\eta > 0$ and suppose $A_\varepsilon^k(\omega)$ is a coordinate rectangle which has all sides greater than η. Then, for ε small enough, $A_\varepsilon^{k+1}(\omega)$ is again a coordinate rectangle contained in $A_\varepsilon^k(\omega)$.*

Proof. As explained above we divide the proof into two steps. As the arguments are purely deterministic we drop the ω-dependence of the sets.

Step 1 Connectedness of minimizers

Let consider the minimizing movement for an auxiliary deterministic functional that turns out to evolve faster. Given $0 < \delta << 1$, we define

$$G_\varepsilon^\delta(A, F) := \mathcal{H}^1(\partial A) + \frac{\delta}{\gamma\varepsilon}\int_{A\Delta F} d_\infty^\varepsilon(x, \partial F)\,\mathrm{d}x. \tag{5.10}$$

Let us observe that for any sets $A, B, F \in \mathcal{A}_\varepsilon$ we have the (in)equalities

$$\begin{aligned} &P_\varepsilon^\omega(A\cup B) + P_\varepsilon^\omega(A\cap B) \leq P_\varepsilon^\omega(A) + P_\varepsilon^\omega(B),\\ &\int_{F\Delta(A\cap B)} d_\infty^\varepsilon(x,\partial F)\,\mathrm{d}x + \int_{F\Delta(A\cup B)} d_\infty^\varepsilon(x,\partial F)\,\mathrm{d}x = \int_{F\Delta A} d_\infty^\varepsilon(x,\partial F)\,\mathrm{d}x\\ &\qquad\qquad + \int_{F\Delta B} d_\infty^\varepsilon(x,\partial F)\,\mathrm{d}x. \end{aligned}$$

The inequality also holds for the standard perimeter, which implies the two general estimates

$$\begin{aligned} &E_\varepsilon^\omega(A\cap B,F)+E_\varepsilon^\omega(A\cup B,F)\leq E_\varepsilon^\omega(A,F)+E_\varepsilon^\omega(B,F),\\ &G_\varepsilon^\delta(A\cap B,F)+G_\varepsilon^\delta(A\cup B,F)\leq G_\varepsilon^\delta(A,F)+G_\varepsilon^\delta(B,F). \end{aligned} \tag{5.11}$$

Now let $R_\varepsilon^\delta\in\mathcal{A}_\varepsilon$ be the smallest minimizer of $G_\varepsilon^\delta(\cdot,A_\varepsilon^k)$ with respect to set inclusion. This is well-defined due to (5.11). From the analysis in [24] we know that $R_\varepsilon^\delta\subset A_\varepsilon^k$ is a coordinate rectangle and, denoting by $N_{i,\varepsilon}$ the distance between corresponding sides of R_ε^δ and A_ε^k, for ε small enough it holds that

$$\left(\frac{2\gamma}{\delta L_{i,\varepsilon}}-1\right)\varepsilon\leq N_{i,\varepsilon}\leq\left(\frac{2\gamma}{\delta L_{i,\varepsilon}}+1\right)\varepsilon,$$

where $L_{i,\varepsilon}$ denotes the length of the side $S_{i,\varepsilon}$ of A_ε^k. In particular, using (5.8), (5.9) and the assumptions on the sides of A_ε^k, we infer the two-sided bound

$$\left(\frac{\gamma}{C\delta}-1\right)\varepsilon\leq N_{i,\varepsilon}\leq\left(\frac{2\gamma}{\delta\eta}+1\right)\varepsilon. \tag{5.12}$$

We argue that $R_\varepsilon^\delta\subset A_\varepsilon^{k+1}$. Assume by contradiction that $R_\varepsilon^\delta\backslash A_\varepsilon^{k+1}\neq\emptyset$. Since (5.12) implies that

$$d_\infty^\varepsilon(x,\partial A_\varepsilon^k)\geq\left(\frac{\gamma}{C\delta}-1\right)\varepsilon\quad\forall x\in R_\varepsilon^\delta, \tag{5.13}$$

using (5.11) combined with the fact that both A_ε^{k+1} and R_ε^δ are minimizers of the corresponding functionals, we obtain

$$\begin{aligned} 0\geq&E_\varepsilon^\omega(A_\varepsilon^{k+1},A_\varepsilon^k)-E_\varepsilon^\omega(R_\varepsilon^\delta\cup A_\varepsilon^{k+1},A_\varepsilon^k)\geq E_\varepsilon^\omega(R_\varepsilon^\delta\cap A_\varepsilon^{k+1},A_\varepsilon^k)-E_\varepsilon^\omega(R_\varepsilon^\delta,A_\varepsilon^k)\\ =&\varepsilon\left(p_\varepsilon^\omega(\partial(R_\varepsilon^\delta\cap A_\varepsilon^{k+1}))-p_\varepsilon^\omega(\partial R_\varepsilon^\delta)\right)+\frac{1-\delta}{\gamma\varepsilon}\int_{R_\varepsilon^\delta\backslash A_\varepsilon^{k+1}}d_\infty^\varepsilon(x,\partial A_\varepsilon^k)\,\mathrm{d}x\\ &+\mathcal{H}^1(\partial(R_\varepsilon^\delta\cap A_\varepsilon^{k+1}))-\mathcal{H}^1(\partial R_\varepsilon^\delta)+\frac{\delta}{\gamma\varepsilon}\int_{R_\varepsilon^\delta\backslash A_\varepsilon^{k+1}}d_\infty^\varepsilon(x,\partial A_\varepsilon^k)\,\mathrm{d}x\\ =&\varepsilon\left(p_\varepsilon^\omega(\partial(R_\varepsilon^\delta\cap A_\varepsilon^{k+1}))-p_\varepsilon^\omega(\partial R_\varepsilon^\delta)\right)+\frac{1-\delta}{\gamma\varepsilon}\int_{R_\varepsilon^\delta\backslash A_\varepsilon^{k+1}}d_\infty^\varepsilon(x,\partial A_\varepsilon^k)\,\mathrm{d}x\\ &+G_\varepsilon^\delta(R_\varepsilon^\delta\cap A_\varepsilon^{k+1},A_\varepsilon^k)-G_\varepsilon^\delta(R_\varepsilon^\delta,A_\varepsilon^k)\\ \geq&\varepsilon\left(p_\varepsilon^\omega(\partial(R_\varepsilon^\delta\cap A_\varepsilon^{k+1}))-p_\varepsilon^\omega(\partial R_\varepsilon^\delta)\right)+\frac{1-\delta}{\gamma\varepsilon}\int_{R_\varepsilon^\delta\backslash A_\varepsilon^{k+1}}d_\infty^\varepsilon(x,\partial A_\varepsilon^k)\,\mathrm{d}x, \end{aligned}$$

where we used several times that $R_\varepsilon^\delta\subset A_\varepsilon^k$ to simplify the symmetric differences. In combination with (5.13), for $\delta \leq \frac{1}{2}$ the last estimate yields

$$\left(\frac{1}{2C\delta} - \frac{1}{2\gamma}\right) |R_\varepsilon^\delta \backslash A_\varepsilon^{k+1}| \leq \varepsilon \left(p_\varepsilon^\omega(\partial R_\varepsilon^\delta) - p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1}))\right). \tag{5.14}$$

In order to use this inequality, we need to analyze which boundary contributions cancel in the last difference. Given $\xi = \frac{i+j}{2} \in \mathcal{Z}^2$ we distinguish two exhaustive cases:

(I) $i \in R_\varepsilon^\delta$, $j \notin R_\varepsilon^\delta$: If $i \in A_\varepsilon^{k+1}$ we have $i \in R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$ and $j \notin R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$ which implies $\xi \in \partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})$ and thus this contribution cancels. Otherwise $i \notin A_\varepsilon^{k+1}$ and consequently $\xi \in \partial(R_\varepsilon^\delta \backslash A_\varepsilon^{k+1})$;

(II) $i \in R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$, $j \notin R_\varepsilon^\delta \cap A_\varepsilon^{k+1}$: If $j \notin R_\varepsilon^\delta$, then $\xi \in \partial R_\varepsilon^\delta$ and the contribution cancels, while $j \in R_\varepsilon^\delta$ yields $j \notin A_\varepsilon^{k+1}$ and therefore $\xi \in \partial(R_\varepsilon^\delta \backslash A_\varepsilon^{k+1})$.

From those two cases and (5.6) we infer that

$$\varepsilon p_\varepsilon^\omega(\partial R_\varepsilon^\delta) - \varepsilon p_\varepsilon^\omega(\partial(R_\varepsilon^\delta \cap A_\varepsilon^{k+1})) \leq \frac{1}{4\gamma}\varepsilon\mathcal{H}^1(\partial(R_\varepsilon^\delta \backslash A_\varepsilon^{k+1})).$$

As for all sets $A \in \mathcal{A}_\varepsilon$ we have the reverse isoperimetric inequality $\varepsilon\mathcal{H}^1(\partial A) \leq 4|A|$, we can put together the last inequality and (5.14) to deduce

$$\left(\frac{1}{2C\delta} - \frac{1}{2\gamma}\right) |R_\varepsilon^\delta \backslash A_\varepsilon^{k+1}| \leq \frac{1}{\gamma}|R_\varepsilon^\delta \backslash A_\varepsilon^{k+1}|.$$

Choosing δ small enough this yields a contradiction. Hence we proved that $R_\varepsilon^\delta \subset A_\varepsilon^{k+1}$ for δ small enough.

Next we rule out any other connected component except the one containing R_ε^δ. Note that estimate (5.12) implies

$$d_\infty^\varepsilon(x, \partial A_\varepsilon^k) \leq (\frac{2\gamma}{\delta\eta} + 1)\varepsilon \qquad \forall x \in A_\varepsilon^k \backslash R_\varepsilon^\delta. \tag{5.15}$$

Consider a connected component A of A_ε^{k+1} not containing R_ε^δ. We set $A' = A_\varepsilon^{k+1} \backslash A$. Due to (5.8) and (5.15) it holds that $|A_\varepsilon^k \cap A| \leq |A_\varepsilon^k \backslash R_\varepsilon^\delta| \leq C_{\delta,\eta}\varepsilon$. Hence, for ε small enough, we obtain from (5.6) that

$$\begin{aligned} E^\omega_\varepsilon(A^{k+1}_\varepsilon, A^k_\varepsilon) - E^\omega_\varepsilon(A', A^k_\varepsilon) \geq & (1-\frac{1}{4\gamma}\varepsilon)\mathcal{H}^1(\partial A) - \frac{1}{\gamma\varepsilon}\int_{A^k_\varepsilon \cap A} d^\varepsilon_\infty(x, \partial A^k_\varepsilon)\,\mathrm{d}x \\ \geq & \frac{\mathcal{H}^1(\partial A)}{2} - \Big(\frac{2}{\delta\eta} + \frac{1}{\gamma}\Big)|A^k_\varepsilon \cap A| \geq \frac{\mathcal{H}^1(\partial A)}{2} - |A^k_\varepsilon \cap A|^{\frac{1}{2}} \\ \geq & \frac{\mathcal{H}^1(\partial A)}{2} - |A|^{\frac{1}{2}} \geq \frac{1}{2}(1-\frac{1}{\sqrt{\pi}})\mathcal{H}^1(\partial A) > 0, \end{aligned}$$

where we used the two-dimensional isoperimetric inequality. This contradicts the minimality of A^{k+1}_ε and we conclude that A^{k+1}_ε has exactly one connected component.
Step 2 Reduction to coordinate rectangles
First note that if we replace an arbitrary set $A \in \mathcal{A}_\varepsilon$ by the set $A \cap A^k_\varepsilon$ we strictly reduce the energy if the sets are not equal. To see this, we observe that

$$\begin{aligned} E^\omega_\varepsilon(A, A^k_\varepsilon) - E^\omega_\varepsilon(A \cap A^k_\varepsilon, A^k_\varepsilon) \geq & \frac{1}{\gamma\,\varepsilon}\int_{A\backslash A^k_\varepsilon} d^\varepsilon_\infty(x, \partial A^k_\varepsilon)\,\mathrm{d}x + P^\omega_\varepsilon(A) - P^\omega_\varepsilon(A\cap A^k_\varepsilon) \\ \geq & \frac{|A\backslash A^k_\varepsilon|}{\gamma} + P^\omega_\varepsilon(A) - P^\omega_\varepsilon(A \cap A^k_\varepsilon). \end{aligned} \tag{5.16}$$

Again we need to analyze which interactions cancel due to the random perimeter difference. As A^k_ε is a coordinate rectangle, by elementary geometric considerations one can prove that $\mathcal{H}^1(\partial A) \geq \mathcal{H}^1(\partial(A \cap A^k_\varepsilon))$. On the other hand, reasoning similar to the lines succeeding (5.14) one can prove that all random interactions cancel except those coming from $\partial(A\backslash A^k_\varepsilon)$. In case this set is non-empty, by (5.6) we conclude that (5.16) can be further estimated via the strict inequality

$$E^\omega_\varepsilon(A, A^k_\varepsilon) - E^\omega_\varepsilon(A\cap A^k_\varepsilon, A^k_\varepsilon) > \frac{|A\backslash A^k_\varepsilon|}{\gamma} - \frac{1}{4\gamma}\varepsilon\mathcal{H}^{d-1}(\partial(A\backslash A^k_\varepsilon)) \geq 0,$$

where we used again the reverse isoperimetric inequality in $\mathcal{A}_\varepsilon$. Whenever A is a minimizer we obtain a contradiction which shows that $\partial(A^{k+1}_\varepsilon \backslash A^k_\varepsilon) = \emptyset$, or equivalently $A^{k+1}_\varepsilon \subset A^k_\varepsilon$.

To conclude we assume by contradiction that A^{k+1}_ε is not a coordinate rectangle. Consider then the minimal coordinate rectangle R containing A^{k+1}_ε (see Figure 5-1). Then again by elementary geometric arguments it holds that $\mathcal{H}^1(\partial A^{k+1}_\varepsilon \backslash \partial R) \geq \mathcal{H}^1(\partial R \backslash \partial A^{k+1}_\varepsilon)$. Moreover, since $A^{k+1}_\varepsilon \subset R$ one can check that

$$\Big(\partial A^{k+1}_\varepsilon \backslash \partial R\Big) \cup \Big(\partial R \backslash \partial A^{k+1}_\varepsilon\Big) = \partial(R\backslash A^{k+1}_\varepsilon). \tag{5.17}$$

Since $R \subset A^k_\varepsilon$ by the previous argument, using (5.6) the difference of the energies can be estimated by

$$
\begin{aligned}
0 \geq & E_\varepsilon^\omega(A_\varepsilon^{k+1}, A_\varepsilon^k) - E_\varepsilon^\omega(R, A_\varepsilon^k) \geq \frac{1}{\gamma}|R \backslash A_\varepsilon^{k+1}| + P_\varepsilon^\omega(A_\varepsilon^{k+1}) - P_\varepsilon^\omega(R) \\
> & \frac{1}{\gamma}|R \backslash A_\varepsilon^{k+1}| + (1 - \frac{\varepsilon}{4\gamma})\mathcal{H}^1(\partial A_\varepsilon^{k+1} \backslash \partial R) - (1 + \frac{\varepsilon}{4\gamma})\mathcal{H}^1(\partial R \backslash \partial A_\varepsilon^{k+1}) \\
\geq & \frac{1}{\gamma}|R \backslash A_\varepsilon^{k+1}| - \frac{\varepsilon}{4\gamma}\Big(\mathcal{H}^1(\partial A_\varepsilon^{k+1} \backslash \partial R) + \mathcal{H}^1(\partial R \backslash \partial A_\varepsilon^{k+1})\Big) \\
\geq & \frac{1}{\gamma}|R \backslash A_\varepsilon^{k+1}| - \frac{\varepsilon}{4\gamma}\mathcal{H}^1(\partial(R \backslash A_\varepsilon^{k+1})),
\end{aligned}
$$

where in the last inequality we used (5.17). The last term is nonnegative by the reverse isoperimetric. Thus we reach a contradiction and conclude the proof. □

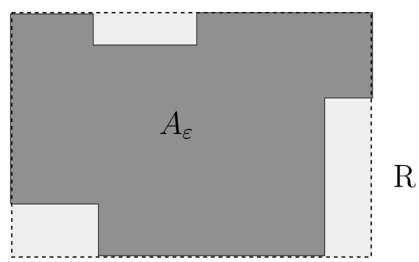

Figure 5-1: The minimal coordinate rectangle R containing a connected set $A_\varepsilon \in \mathcal{A}_\varepsilon$.

Remark 5.5. We want to comment on the L^∞-bound (5.6): First note that is optimal in the sense that if it is violated, with positive probability there could be defects at the corners of the rectangle. On the other hand, our argument for proving connectedness can be extended to a generic L^∞-bound on the random field. In order to determine the shape one then would need to use probabilistic arguments. We strongly believe (even though we did not check the argument in detail), that the law of large numbers implies that the minimizer must have the same deterministic perimeter as the minimal coordinate rectangle containing it. Moreover, the bulk term yields a control on the deviation from this minimal rectangle. However deviations can exist and that causes difficulties. In order to apply an inductive argument (which is necessary also for connectedness), one needs to control the deviation from a rectangle. However we are not able to rule out that deviations grow with the number of time steps.

Computation of the velocity

As a next step we derive a precise formula for the velocity of the discrete motion. We follow [24] and express the functional to be minimized by the distance from each side of the optimal rectangle to the corresponding side of the previous set $A_\varepsilon^k(\omega)$. Let $A_\varepsilon^{k+1}(\omega)$ be a minimizer. To reduce notation, we let $s_{i,\varepsilon}$ and $s'_{i,\varepsilon}$ $(i = 1, \dots, 4)$ be the sides of $A_\varepsilon^k(\omega)$ and $A_\varepsilon^{k+1}(\omega)$ respectively and set $l_{i,\varepsilon} = \mathcal{H}^1(s_{i,\varepsilon}^k)$. We define $N_{i,\varepsilon}^{k+1}\varepsilon$ as the distance from the side $s_{i,\varepsilon}$ to the side $s'_{i,\varepsilon}$. It can be easily shown that $A_\varepsilon^{k+1}(\omega)$ must contain the center of the previous rectangle $A_\varepsilon^k(\omega)$. Rewriting the functional $E_\varepsilon^\omega(A, A_\varepsilon^k)$ in terms of the four integer numbers $N_{i,\varepsilon}^{k+1}$, we obtain that these are minimizers of the function $\tilde{f}_\varepsilon^\omega : \mathbb{N}_0^4 \to \mathbb{R}$ defined by

$$\begin{aligned}\tilde{f}_\varepsilon^\omega(N) :=& \sum_{i=1}^4 (l_{i,\varepsilon} - 2N_i\varepsilon) + \sum_{i=1}^4 p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon v_i)\varepsilon - \varepsilon^2 e_\varepsilon^{per} + \frac{\varepsilon}{\gamma}\sum_{i=1}^4\sum_{n=1}^{N_i} l_{i,\varepsilon} n - \varepsilon^2 e_\varepsilon^{bulk} \\ =& \varepsilon\sum_{i=1}^4\left(\left(\frac{l_{i,\varepsilon}}{\varepsilon} - 2N_i\right) + p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon v_i) + \frac{(N_i+1)N_i\, l_{i,\varepsilon}}{2\gamma}\right) - \varepsilon^2(e_\varepsilon^{per} + e_\varepsilon^{bulk}),\end{aligned}$$

where $v_i \in \{\pm e_1, \pm e_2\}$ denotes the vector representing the inward motion of each side and the error terms e_ε^{per}, e_ε^{bulk} account for the fact that we neglect the shrinking effect on the random part of the energy and that we count twice the bulk part in the corners (one time with the wrong distance). For these errors we have the following bounds:

$$\begin{aligned}|e_\varepsilon^{per}| &\le \frac{2}{\gamma}\max_i N_i, \\ |e_\varepsilon^{bulk}| &\le \frac{4}{\gamma}(\max_i N_i)^3.\end{aligned} \tag{5.18}$$

We argue that the error terms are negligible as $\varepsilon \to 0$. To this end we show that $\max_i N_{i,\varepsilon}$ is equibounded with respect to ε as long as $l_{i,\varepsilon} \ge \eta > 0$ for some $\eta > 0$. Indeed, suppose without loss of generality that $N^* := \max_i N_i$ corresponds to the right vertical side $s_{i,\varepsilon}$. Let us denote by P the center of $A_\varepsilon^k(\omega)$. Then, for ε small enough, one can easily prove that

$$\{x \in A_\varepsilon^k : \frac{N}{2}\varepsilon \le \operatorname{dist}(x, s_{i,\varepsilon}) \le N\varepsilon,\ |\langle x - P, e_2\rangle| \le \frac{\eta}{4}\} \subset A_\varepsilon^k(\omega)\backslash A_\varepsilon^{k+1}(\omega).$$

Thus for the bulk term we obtain the lower bound

$$\frac{1}{\gamma\varepsilon}\int_{A_\varepsilon^k(\omega)\backslash A_\varepsilon^{k+1}(\omega)} d_\infty^\varepsilon(x, \partial A_\varepsilon^k)\,\mathrm{d}x \ge \frac{\min\{\frac{\eta}{4}, \frac{N}{2}\varepsilon\}}{\gamma}\frac{N\eta}{4}.$$

Using (5.6) and (5.8), for ε small enough, we deduce a lower bound for the random perimeter via

$$\begin{aligned}P_\varepsilon^\omega(A_\varepsilon^{k+1}(\omega)) &\geq P_\varepsilon^\omega(A_\varepsilon^k(\omega)) - 8N\varepsilon - \frac{1}{2\gamma}\varepsilon\mathcal{H}^1(\partial A_\varepsilon^k(\omega)) - \frac{2}{\gamma}\varepsilon^2 N \\ &\geq E_\varepsilon^\omega(A_\varepsilon^k(\omega), A_\varepsilon^k(\omega)) - \varepsilon\left(9N + \frac{C}{2\gamma}\right).\end{aligned}$$

Assuming that $N \geq \frac{C}{2\gamma}$, we infer that such N can't yield a minimizer as soon as

$$-10N\varepsilon + \frac{\min\{\frac{\eta}{4}, \frac{N}{2}\varepsilon\}}{\gamma}\frac{N\eta}{4} > 0. \tag{5.19}$$

From (5.19) one can easily deduce that N has to be bounded when $\varepsilon \to 0$.

It follows from (5.18) that, asymptotically, we can instead minimize the functional

$$f_\varepsilon^\omega(N) = \sum_{i=1}^{4}\left(-2N_i + p_\varepsilon^\omega(s_{i,\varepsilon} + N_i\varepsilon v_i) + \frac{1}{2\gamma}(N_i+1)N_i\, l_{i,\varepsilon}\right), \tag{5.20}$$

provided that the minimizer of the limit is unique. In particular, as in [24] each side moves independently from the remaining ones. More precisely, we have to study the minimizers of the one-dimensional random function

$$v_{i,\varepsilon}^\omega(N) := -2N + p_\varepsilon^\omega(s_{i,\varepsilon} + N\varepsilon v_i) + \frac{1}{2\gamma}(N+1)N\, l_{i,\varepsilon}. \tag{5.21}$$

The asymptotic behavior of the stochastic term in (5.21) is more involved since the segment $s_{i,\varepsilon}$ can vary along infinitely many different lattice positions as $\varepsilon \to 0$. Thus a direct application of Birkhoff's ergodic theorem to prove the existence of a limit is not possible. Indeed, in what follows we will show the existence of a stationary, ergodic system of perturbations where for at least one side $s_{i,\varepsilon}$ the term $p_\varepsilon^\omega(s_{i,\varepsilon}+N\varepsilon v_i)$ does not converge with probability 1. We will come back to this example in Section 5.3, where we show how to treat this case making some further assumptions.

Example 5.6. *Let $\gamma = 1$ and let X_i, $i \in \mathbb{Z}$ be a sequence of non-constant independent and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a measure-preserving ergodic map $\tau : \Omega \to \Omega$ such that $X_k(\omega) = X_0(\tau^k\omega)$, where τ^k denotes the k-times iterated composition of the map τ (this setup can be realized on a suitable product space with the shift operator). Moreover assume that $\|X_i\|_\infty < \frac{1}{4}$ and set $c_\xi(\omega) = X_{\lfloor \xi_1 \rfloor}(\omega)$. Then c_ξ is a stationary, ergodic random field. If the initial coordinate rectangles A_ε^0 converge in the Hausdorff metric to a coordinate rectangle*

A^0, then for at least one of the vertical sides we have that, for all $N \in \mathbb{N}_0$,

$$\mathbb{P}\left(\{\omega : \lim_{\varepsilon\to 0} v^{\omega}_{i,\varepsilon}(N) \text{ exists}\}\right) = 0.$$

Proof. Let $\varepsilon_n \to 0$. Note that for at least one vertical side the x-component of $\frac{s_{i,\varepsilon_n}}{\varepsilon_n}$ diverges to $\pm\infty$. Then so does the x-component of $\frac{s_{i,\varepsilon_n}}{\varepsilon_n} + Nv_i$. Without loss of generality we assume that these x-components form a sequence of positive numbers $\{k_n + \frac{1}{2}\}_n \to +\infty$ with $k_n \in \mathbb{N}$. Passing to a subsequence (not relabeled) we can assume that this sequence is monotone increasing. Since l_{i,ε_n} converges to the vertical side length l_i of A^0, we only have to take into account the random term. Since $A^0_{\varepsilon_n} \in \mathcal{A}_{\varepsilon_n}$, we have

$$p^{\omega}_{\varepsilon_n}(s_{i,\varepsilon_n} + N\varepsilon_n v_i) = X_{k_n}(\omega) l_{i,\varepsilon_n}.$$

Since l_{i,ε_n} converges to $l_i \neq 0$, the asymptotic behavior of $p^{\omega}_{\varepsilon_n}(s_{i,\varepsilon_n} + N\varepsilon_n v_i)$ is characterized by $X_{k_n}(\omega)$. Since these variables are non-constant and independent, it follows from Kolmogorov's 0-1 law that

$$\mathbb{P}(\{\omega : \lim_{n} X_{k_n}(\omega) \text{ exists}\}) = 0.$$

Moreover we can define the measure preserving group action $\tau_z : \Omega \to \Omega$ as

$$\tau_z\omega := \tau^{z_1}\omega.$$

From the construction of the random field, it follows immediately that $\{c_\xi\}_\xi$ is stationary. By assumption the group action is ergodic, too. □

Despite the negative result of the previous example, we now show that even in the worst case the term $p^{\omega}_{\varepsilon}(s_{i,\varepsilon} + N\varepsilon v_i)$ doesn't influence the range of possible minimizers too much. Indeed, by (5.6) we have

$$\sup_{N,N'} |p^{\omega}_{\varepsilon}(s_{i,\varepsilon} + N\varepsilon v_i) - p^{\omega}_{\varepsilon}(s_{i,\varepsilon} + N'\varepsilon v_i)| \leq \frac{1}{2\gamma} l_{i,\varepsilon},$$

while (one of) the integer minimizers for the polynomial $P(x) = -2x + \frac{l_{i,\varepsilon}}{2\gamma}(x+1)x$ is given by $x^* = \lfloor \frac{2\gamma}{l_{i,\varepsilon}} \rfloor$. We deduce the estimate

$$|P(x^* \pm 2) - P(x^*)| = \begin{cases} \frac{3l_{i,\varepsilon}}{\gamma} - 4 + \frac{2l_{i,\varepsilon}}{\gamma}x^* \geq \frac{l_{i,\varepsilon}}{\gamma}, \\ \frac{l_{i,\varepsilon}}{\gamma} + 4 - \frac{2l_{i,\varepsilon}}{\gamma}x^* \geq \frac{l_{i,\varepsilon}}{\gamma}. \end{cases}$$

We infer that for minimizing $v^\omega_{i,\varepsilon}$ we need only to consider three values, that means

$$\min_N v^\omega_{i,\varepsilon}(N) = \min\{v^\omega_{i,\varepsilon}(x^*),\, v^\omega_{i,\varepsilon}(x^*+1),\, v^\omega_{i,\varepsilon}(x^*-1)\}. \tag{5.22}$$

Thus the randomness can only cause one additional jump forwards or backwards. In order to obtain the convergence we need a stronger form of independence than ergodicity that is preserved on one-dimensional sections of $\mathbb{Z}^2$. It turns out that the α-mixing condition introduced in Section 5.1 is enough. Indeed, we have the following crucial result:

Proposition 5.7. *Assume that the random field $\{c_\xi\}_\xi$ is stationary and α-mixing such that (5.7) holds and set $\mu := \mathbb{E}[c_\xi]$. Let $\varepsilon_j \downarrow 0$. There exists a set $\Omega' \subset \Omega$ of full probability (independent of the particular sequence ε_j) such that for every $\omega \in \Omega'$ and every sequence of sides $\{S_j\}_{j\in\mathbb{N}}$ such that S_j converges in the Hausdorff metric to a segment S, we have*

$$\lim_j p^\omega_{\varepsilon_j}(S_j) = \mathcal{H}^1(S)\mu.$$

Proof. We assume that the side is a vertical side, the case of horizontal sides works the same way with another set of full measure. Moreover it is not restrictive to consider the case $\mu = 0$. To reduce notation we let $[x]^* := \lfloor x \rfloor + \frac{1}{2}$. Given $q \in \mathbb{Q} \cap (0, +\infty)$ we define the following sequences of random variables:

$$X^{q,\pm}_n(\omega) := \sup_{k \geq qn} \left| \frac{1}{2k+1} \sum_{l=-k}^{k} c_{([\pm n]^*, l)}(\omega) \right|.$$

Given $\delta > 0$, by stationarity and an elementary fact about average sums we have

$$\begin{aligned} \mathbb{P}\left(|X^{q,\pm}_n| > \delta\right) &= \mathbb{P}\Bigg(\sup_{k\geq qn} \left| \frac{1}{2k+1} \sum_{l=-k}^{k} c_{([0]^*,l)} \right| > \delta \Bigg) \\ &\leq \mathbb{P}\Bigg(\sup_{k\geq qn} \left| \frac{1}{k+1} \sum_{l=0}^{k} c_{([0]^*,l)} \right| > \delta \Bigg) + \mathbb{P}\Bigg(\sup_{k\geq qn} \left| \frac{1}{k} \sum_{l=1}^{k} c_{([0]^*,-l)} \right| > \delta \Bigg). \end{aligned} \tag{5.23}$$

Upon rescaling c_ξ we can apply Theorem 2.24 with $p = 2$ to the two bounded and

α-mixing sequences $\{c_{([0]^*,l)}\}_{l\in\mathbb{N}_0}$ and $\{c_{([0]^*,-l)}\}_{l\in\mathbb{N}}$ and deduce from (5.23) that

$$\begin{aligned}
&\sum_{n\geq 1}\mathbb{P}(|X_n^{q,\pm}|>\delta)\\
&\leq \sum_{n\geq 1}\mathbb{P}\Big(\sup_{k\geq qn}\Big|\frac{1}{k+1}\sum_{l=0}^{k}c_{([0]^*,l)}\Big|>\delta\Big)+\mathbb{P}\Big(\sup_{k\geq qn}\Big|\frac{1}{k}\sum_{l=1}^{k}c_{([0]^*,-l)}\Big|>\delta\Big)\\
&\leq \lceil q^{-1}\rceil\sum_{i\geq 1}\mathbb{P}\Big(\sup_{k\geq i}\Big|\frac{1}{k+1}\sum_{l=0}^{k}c_{([0]^*,l)}\Big|>\delta\Big)+\mathbb{P}\Big(\sup_{k\geq i}\Big|\frac{1}{k}\sum_{l=1}^{k}c_{([0]^*,-l)}\Big|>\delta\Big)<+\infty.
\end{aligned}$$

Hence by the Borel-Cantelli Lemma there exists a set of full probability Ω^q such that both $X_n^{q,+}$ and $X_n^{q,-}$ converge to 0 pointwise on Ω^q. We set $\Omega'':=\bigcap_q\Omega^q$.

Next we check that we can relate the random length of the side S_j to one of the random variables $X_n^{q,\pm}$. Let S_j converge to a segment S in the Hausdorff metric and denote by $x\in\mathbb{R}$ the x-coordinate of S. We start with the case $x>0$. Fix $\beta>0$ and let $x_j\in\mathbb{Z}+\frac{1}{2}$ be the x-component of S_j/ε_j. Then there exists $j_0=j_0(\beta)$ such that for all $j\geq j_0$ we have $x+\beta\geq\varepsilon_j x_j$ and $\varepsilon_j\#\{\xi\in S_j/\varepsilon_j\cap\mathcal{Z}^2\}\geq\mathcal{H}^1(S)-\beta$. For such j we infer that

$$\#\{\xi\in\frac{S_j}{\varepsilon_j}\cap\mathcal{Z}^2\}\geq\frac{\mathcal{H}^1(S)-\beta}{x+\beta}x_j.$$

For β small enough, there exists $q\in\mathbb{Q}$ such that $\frac{\mathcal{H}^1(S)-\beta}{x+\beta}>3q>0$. Now for every j we choose $n_j\in\mathbb{N}$ satisfying $[n_j]^*=x_j$ (we may assume that $x_j>0$ for all j). Then

$$\#\{\xi\in\frac{S_j}{\varepsilon_j}\cap\mathcal{Z}^2\}>3qn_j. \tag{5.24}$$

Let us first assume that $S=\{x\}\times\frac{1}{2}[-\mathcal{H}^1(S),\mathcal{H}^1(S)]$. For j large enough it holds

$$\#\Big(\Big\{\xi\in\frac{S_j}{\varepsilon_j}\cap\mathcal{Z}^2\Big\}\Delta\Big\{\xi=(x_j,l):\ |l|\leq\frac{\mathcal{H}^1(S_j)}{2\varepsilon_j}\Big\}\Big)\leq\frac{\beta}{\varepsilon_j}, \tag{5.25}$$

so that by (5.6) we have

$$\begin{aligned}
\Big|\sum_{\varepsilon_j\xi\in S_j}\varepsilon_j c_\xi(\omega)\Big| &= \mathcal{H}^1(S_j)\Big|\frac{1}{\#\{\varepsilon_j\xi\in S_j\}}\sum_{\varepsilon_j\xi\in S_j}c_\xi(\omega)\Big|\\
&\leq C\beta+C\Big|\frac{1}{\#\{\varepsilon_j\xi\in S_j\}}\sum_{2\varepsilon_j|l|\leq\mathcal{H}^1(S_j)}c_{(x_j,l)}(\omega)\Big|\leq C\beta+CX_{n_j}^{q,+}(\omega),
\end{aligned}\tag{5.26}$$

where we used that $\lfloor\mathcal{H}^1(S_j)/(2\varepsilon_j)\rfloor\geq qn_j$ for all but finitely many j by (5.24). Since

$\beta > 0$ is arbitrary and $X_{n_j}^{q,+}(\omega) \to 0$ for all $\omega \in \Omega''$ we conclude in this special case.

Now assume that $S = \{x\} \times [y - \mathcal{H}^1(S), y + \mathcal{H}^1(S)]$ with $y > 0$ (the other case is similar). As in Chapters 3 and 4 we need to transfer the variables pointwise with the help of the group action. For $\beta > 0$ and q as above, we define the events

$$\mathcal{Q}_N := \left\{\omega \in \Omega : \ \forall n \geq \frac{N}{2} \text{ it holds } |X_n^{q,+}(\omega)| \leq \beta\right\}.$$

By the arguments hitherto we know that the function $\mathbb{1}_{\mathcal{Q}_N}$ converges to $\mathbb{1}_\Omega$ on Ω''. Let us denote by $\mathcal{J}_{e_2}$ the (maybe non-trivial) σ-algebra of invariant sets for the measure preserving map τ_{e_2}. Fatou's lemma for the conditional expectation yields

$$\mathbb{1}_\Omega = \mathbb{E}[\mathbb{1}_\Omega|\mathcal{J}_{e_2}] \leq \liminf_{N\to+\infty} \mathbb{E}[\mathbb{1}_{\mathcal{Q}_N}|\mathcal{J}_{e_2}].$$

Hence we know that, given $\delta > 0$, almost surely we find $N_0 = N_0(\omega, \delta)$ such that

$$1 \geq \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}}|\mathcal{J}_{e_2}](\omega) \geq 1 - \delta.$$

Due to Birkhoff's ergodic theorem (see Theorem 2.22 in Chapter 2), almost surely, there exists $n_0 = n_0(\omega, \delta)$ such that, for any $m \geq \frac{1}{2}n_0$,

$$\left|\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_{\mathcal{Q}_{N_0}}(\tau_{ie_2}\omega) - \mathbb{E}[\mathbb{1}_{\mathcal{Q}_{N_0}}|\mathcal{J}_{e_2}](\omega)\right| \leq \delta.$$

Note that the set we exclude will be a countable union of null sets (depending only on the sequences $X_n^{q,\pm}$ and rational β). With a slight abuse of notation we still call the smaller set Ω''.

We now fix $\omega \in \Omega''$. For $m \geq \max\{n_0(\omega, \delta), N_0(\omega, \delta)\}$ we denote by R the maximal integer such that for all $i = m+1, \dots, m+R$ we have $\tau_{ie_2}(\omega) \notin \mathcal{Q}_{N_0}$. By exactly the same estimates used in Step 3 of the proof of Theorem 3.19 we deduce that $R \leq 4m\delta$ provided that $\delta \leq \frac{1}{4}$. So if we choose an arbitrary $m \geq \max\{n_0(\omega, \theta), N_0(\omega, \delta)\}$ and $\tilde{R} = 6m\delta$ we find $l_m \in [m+1, m+\tilde{R}]$ such that $\tau_{l_m e_2}(\omega) \in \mathcal{Q}_{N_0}$. Then we have for all $n \geq \frac{N_0}{2}$ that

$$|X_n^{q,+}(\tau_{l_m e_2}\omega)| \leq \beta. \tag{5.27}$$

For j large enough we have $\lfloor y/\varepsilon_j \rfloor \geq \max\{n_0(\omega, \delta), N_0(\omega, \delta)\}$ so that there exists $l_j \in \mathbb{N}$ satisfying (5.27) and moreover

$$|\lfloor y/\varepsilon_j \rfloor - l_j| \leq 6\delta \lfloor y/\varepsilon_j \rfloor. \tag{5.28}$$

In addition we can assume that $|\mathcal{H}^1(S)-\mathcal{H}^1(S_j)| \leq \beta$. Note that (5.28) is the analogue of (5.25). Thus from (5.27), stationarity and the definition of $X_n^{q,+}$ we deduce that

$$|p^\omega_{\varepsilon_j}(S_j)| \leq C_y(\beta+\delta) \tag{5.29}$$

for all j large enough. By the arbitrariness of β and δ we proved the claim.

The case $x < 0$ can be proved the same way using the random variables $X_n^{q,-}$ instead. It remains the case when $x = 0$. For fixed $z \in \mathbb{Z}$ we consider the following sequences of random variables:

$$Y_n^z(\omega) := \sup_{k \geq n} \left| \frac{1}{2k+1} \sum_{l=-k}^{k} c_{([z]^*,l)}(\omega) \right|.$$

With essentially the same arguments as above one can show that there exists a set Ω^z of full probability such that for every sequence of sides S_j contained in $[z]^* \times \mathbb{R}$ and all $\omega \in \Omega^z$ we have

$$p^\omega_{\varepsilon_j}(S_j) \to 0,$$

where Ω^z does not depend on the sequence ε_j. We finally set $\Omega' := \Omega'' \cap \bigcap_z \Omega^z$. Let us fix $\omega \in \Omega'$. Note that if $x = 0$, then for every subsequence of ε_j there exists a further subsequence ε_{j_k}, such that either

(i) $x_{j_k} \to \pm\infty$,

(ii) $x_{j_k} = [z]^*$ for all k and for some $z \in \mathbb{Z}$

In the first case we can use the construction for $x \neq 0$ with arbitrary $q \in \mathbb{Q} \cap (0, +\infty)$ since $\omega \in \Omega''$ and in the second case we use that $\omega \in \Omega^z$ to conclude. □

Remark 5.8. It is straightforward to check that the limit relation of Proposition 5.7 holds for convergence in probability even under the weaker assumption that both the σ-algebras invariant with respect to the two group actions τ_{e_1}, τ_{e_2} are trivial.

With Proposition 5.7 at hand we are now in a position to prove our main result.

Theorem 5.9. *Assume that the random field $\{c_\xi\}_\xi$ is stationary and α-mixing such that (5.7) holds. Then with probability 1 the following holds: Let $\varepsilon_j \downarrow 0$ and let $A^0_j(\omega) \in \mathcal{A}_{\varepsilon_j}$ be a coordinate rectangle with sides $S_{1,j}(\omega), ..., S_{4,j}(\omega)$. Assume that $A^0_j(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(\omega)$. Then we can choose a subsequence (not relabeled), such that $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is a coordinate rectangle with sides $S_i(t)(\omega)$ such that*

$A(0)(\omega) = A(\omega)$ and any side $S_i(t)(\omega)$ moves inward with velocity $v_i(t)(\omega)$ solving the following differential inclusions:

$$v_i(t)(\omega) \begin{cases} = \frac{1}{\gamma} \left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \notin \mathbb{N}, \\ \in \left[\frac{1}{\gamma} \left(\frac{2\gamma}{L_i(t)(\omega)} - 1 \right), \frac{1}{\gamma} \frac{2\gamma}{L_i(t)(\omega)} \right] & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \in \mathbb{N}, \end{cases}$$

where $L_i(t)(\omega) := \mathcal{H}^1(S_i(t)(\omega))$ denotes the length of the side $S_i(t)(\omega)$. The differential inclusions are valid until the extinction time when $L_i(t)(\omega) = 0$.

Proof. Let Ω' be the set of full probability given by Proposition 5.7. We fix $\omega \in \Omega'$. Since $A_j^0(\omega)$ converges to a coordinate rectangle we can assume that the sides of $A_j^0(\omega)$ are larger than $\eta > 0$ for some η independent of j. Therefore we can apply Proposition 5.4 for all j large enough. For fixed j and $i = 1, ..., 4$, the minimizing movement procedure yields two random sequences $L_{i,\varepsilon_j}^k(\omega)$, $N_{i,\varepsilon_j}^k(\omega)$. Let us denote by $L_i^j(t)(\omega) = L_{i,\varepsilon_j}^{\lfloor \tau_j/t \rfloor}(\omega)$ and $N_i^j(t)(\omega) = N_{i,\varepsilon_j}^{\lfloor \tau_j/t \rfloor}(\omega)$ the piecewise constant interpolations. Note that the function $L_i^j(t)(\omega)$ is decreasing in t. Set

$$t^* := \min_i \left\{ \inf\{t > 0 : \liminf_j L_i^j(t)(\omega) = 0\} \right\} \in [0, +\infty].$$

We already deduced from (5.19) that the discrete velocity, that is the distance between two corresponding sides between two time steps is equibounded by $C_\eta \varepsilon$ for some constant C_η. Thus it follows that

$$\min_i \liminf_j L_i^j(t^*)(\omega) = 0$$

and consequently $t^* > 0$. Without changing notation we consider the subsequence realizing the lim inf. Then, by monotonicity, one can verify that for all $t < t^*$ we have

$$\min_i \liminf_j L_i^j(t)(\omega) > 0.$$

Now fix $t_1 < t^*$. Taking i into account modulo 4, by construction it holds

$$\frac{L_{i,\varepsilon_j}^{k+1}(\omega) - L_{i,\varepsilon_j(\omega)}^k}{\tau} = -\frac{1}{\gamma}(N_{i-1,\varepsilon_j}^k(\omega) + N_{i+1,\varepsilon_j}^k(\omega)). \tag{5.30}$$

Hence on $[0, t_1]$ the piecewise affine interpolations $t \mapsto L_i^{j,a}(t)(\omega)$ are uniformly Lipschitz-continuous and decreasing while $N_i^j(t)(\omega)$ is locally bounded in L^∞. Thus, by a diagonal argument, we can find a further subsequence such that $L_i^j(t) \to L_i(t)$ pointwise and locally uniformly on $[0, t^*)$ for some locally Lipschitz-continuous, decreasing function $L_i(t)(\omega)$ and $N_i^j(t)(\omega)$ weakly*-converges in $L^\infty_{loc}(0,t^*)$ to some function $\tilde v_i(t)(\omega)$. It follows that, up to a subsequence, $A_{\varepsilon_j}(t)(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(t)(\omega)$ for all $0 \le t < t^*$.

We conclude the proof by computing the velocity of each side $L_i(t)(\omega)$. Again we fix $0 < t < t^*$. Then $\liminf_j L_i^j(t)(\omega) > 0$ for all i. Therefore we have that the minimizers $N^k_{i,\varepsilon_j}(\omega)$ of the functional $f^\omega_{\varepsilon_j}(N)$ introduced in (5.20) are uniformly bounded if $|k\tau_j - t|$ is small enough. Hence they converge, up to subsequences, to minimizers of the pointwise limit of $f^\omega_{\varepsilon_j}$ (this can be seen as a special case of Γ-convergence on discrete spaces). By Proposition 5.7 and the precedent discussion we know that

$$f^\omega_{\varepsilon_j}(N) \to \sum_{i=1}^4 -2N_i + L_i(t)(\omega)\mu + \frac{1}{2\gamma}(N_i+1)N_iL_i(t)(\omega).$$

A straightforward calculation shows that the minimizers are given by

$$N_i \begin{cases} = \left\lfloor \frac{2\gamma}{L_i(t)(\omega)} \right\rfloor & \text{if } \frac{2\gamma}{L_i(t)(\omega)} \notin \mathbb{N}, \\ \in \left\{ \frac{2\gamma}{L_i(t)(\omega)} - 1, \frac{2\gamma}{L_i(t)(\omega)} \right\} & \text{otherwise.} \end{cases}$$

Summing the equality (5.30) we further infer that

$$\begin{aligned} L_i^j(t)(\omega) &= L_i^j(0)(\omega) - \frac{1}{\gamma} \sum_{k=0}^{\lfloor t/\tau_j \rfloor} \tau_j(N^j_{i-1}(k\tau_j)(\omega) + N^j_{i+1}(k\tau_j)(\omega)) \\ &= L_i^j(0)(\omega) - \frac{1}{\gamma}\int_0^t \big(N^j_{i-1}(s)(\omega) + N^j_{i+1}(s)(\omega)\big)\,\mathrm{d}s + \mathcal{O}(\tau_j). \end{aligned} \tag{5.31}$$

Passing to the limit as $j \to +\infty$ in (5.31), we deduce from weak convergence that

$$L_i(t)(\omega) = L_i(0)(\omega) - \frac{1}{\gamma}\int_0^t (\tilde v_{i-1}(s)(\omega) + \tilde v_{i+1}(s)(\omega))\,\mathrm{d}s. \tag{5.32}$$

To conclude, we note that if t is such that $2\gamma/L_i(t)(\omega) \notin \mathbb{N}$, then by continuity we have that $2\gamma/L_i(t')(\omega) \notin \mathbb{N}$ for $|t-t'| \le \delta$ for some $\delta > 0$. It follows from comparing pointwise convergence with weak*-convergence that

$$\tilde v_i(t')(\omega) = v_i(t')(\omega) \quad \text{for almost all } |t-t'| \le \delta.$$

In particular $\tilde v_i(\cdot)(\omega)$ has a constant representative on $(t-\delta, t+\delta)$ so that the velocity of the side $S_i(t)(\omega)$ given by

$$\lim_{h\to 0}\frac{1}{2}\frac{L_{i-1}(t+h)(\omega)-L_{i-1}(t)(\omega)}{h}=-\frac{1}{\gamma}v_i(t)(\omega)$$

exists by (5.32) whenever $2\gamma/L_i(t)(\omega)\notin\mathbb{N}$. Note that the formula for the velocity is true because if $2\gamma/L_i(t)(\omega)\notin\mathbb{N}$, then at least in every short time interval opposite sides move with the same velocities. The claim for $2\gamma/L_i(t)(\omega)\in\mathbb{N}$ follows from well known properties of weak*-convergence (note that for these values of t the velocity may not be a classical derivative). $\square$

Following word by word the proof of [24, Theorem 2] we obtain unique limit motions in many cases:

Corollary 5.10. *Let $A^0_\varepsilon(\omega)$ and $\{c_\xi\}_\xi$ be as in Theorem 5.9. Assume in addition that the lengths $L^0_1(\omega), L^0_2(\omega)$ of $A(\omega)$ satisfy one of the three following conditions (we assume that $L^0_1(\omega)\le L^0_2(\omega)$):*

(i) $L^0_1(\omega), L^0_2(\omega)>2\gamma$ (total pinning),

(ii) $L^0_1(\omega)<2\gamma$ and $L^0_2(\omega)\le 2\gamma$ (vanishing in finite time with shrinking velocity larger than $1/\gamma$),

(iii) $L^0_1(\omega)<2\gamma$ and $2\gamma/L^0_1(\omega)\notin\mathbb{N}$, and $L^0_2(\omega)>2\gamma$ (partial pinning).

Let $\varepsilon_j\to 0$. The sequence $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is the unique coordinate rectangle with sides $S_1(t)(\omega)$ and $S_2(t)(\omega)$ such that $A(0)(\omega)=A(\omega)$ and the side lengths $L_1(t)(\omega)$ and $L_2(t)(\omega)$ solve the following differential equations for all but countably many times:

$$\begin{cases}\frac{d}{dt}L_1(t)(\omega)=-\frac{2}{\gamma}\left\lfloor\frac{2\gamma}{L_2(t)(\omega)}\right\rfloor,\\ \frac{d}{dt}L_2(t)(\omega)=-\frac{2}{\gamma}\left\lfloor\frac{2\gamma}{L_1(t)(\omega)}\right\rfloor\end{cases}$$

with initial condition $L_1(0)(\omega)=L^0_1(\omega)$ and $L_2(0)(\omega)=L^0_2(\omega)$.

Remark 5.11. Without any assumptions on the distribution of the random field, up to subsequences we can still obtain a rectangular limit motion. Due to (5.22) we can also give an estimate of the velocity via

$$v_i(t)(\omega)\in\frac{1}{\gamma}\left[\left\lfloor\frac{2\gamma}{L_i(t)(\omega)}\right\rfloor-1,\left\lfloor\frac{2\gamma}{L_i(t)(\omega)}\right\rfloor+1\right].$$

Note that the subsequence may depend on ω.

Remark 5.12. For the continuum flow it is known that rectangles always shrink to a point; see for example the more general result contained in [51, Proposition 3.1]. The same occurs for any possible limit motion in our discrete model provided the sets vanish in finite time. Indeed, assume by contradiction that $L_i(t^*)(\omega) = 0$ and $L_{i+1}(t^*)(\omega) = a > 0$. Then, for any $t < t^*$, by monotonicity of the side-lengths and the velocity estimate in Remark 5.11 there exists a constant $c > 0$ such that

$$L_i(t^*)(\omega) - L_i(t)(\omega) \geq -c(t^* - t).$$

By definition of t^* we obtain the bound $L_i(t) \leq c(t^* - t)$. Inserting this bound in the estimate of Remark 5.11 we conclude that, again for any $0 < t < t^*$ and a slightly larger constant $c > 0$,

$$L_{i+1}(t)(\omega) - L_{i+1}(0)(\omega) \leq -\int_0^t \frac{c}{t^* - s}\,\mathrm{d}s = c\log(1 - t/t^*).$$

Letting $t \uparrow t^*$ we obtain a contradiction.

5.3 Dependence on the range of stationarity

In the previous section we proved that the velocity is the same as in the unperturbed deterministic case. This fact however changes if we replace the stationarity assumption on all integer shifts τ_z to a smaller subgroup since the distributions on two neighboring points in the dual lattice can be different. In particular this highlights that the results obtained hitherto are not only due to the additional scaling of the random terms but due to homogenization.

Definition 5.13. *Let $m \in \mathbb{N}$. We say that the random field $\{c_\xi\}_\xi$ is m-stationary if*

$$c_\xi(\tau_{mz}\omega) = c_{\xi+mz}(\omega) \qquad \forall z \in \mathbb{Z}^2.$$

There are $2m$ quantities that can affect the velocity. For $i = 0, ..., m-1$ set $[i] = i + \frac{1}{2}$ and consider the following random variables:

$$c_{i,|}^{\mathrm{eff}}(\omega) := \frac{1}{m}\sum_{j=0}^{m-1} c_{([i]^*,j)}(\omega), \qquad c_{i,-}^{\mathrm{eff}}(\omega) := \frac{1}{m}\sum_{j=0}^{m-1} c_{(j,[i]^*)}(\omega).$$

To obtain the velocity of the sides we need a generalization of Proposition 5.7.

Proposition 5.14. *Assume that the random field $\{c_\xi\}_\xi$ is m-stationary and α-mixing such that (5.7) holds. Then there exists a set $\Omega' \subset \Omega$ of full probability such that for all*

$\omega \in \Omega'$ the following holds: Suppose that a vertical side S_j converges in the Hausdorff sense to a limit side S and that for all j we have that the x-component x_j of S_j/ε_j fulfills

$$x_j = i + \frac{1}{2} \mod m \quad \forall j. \tag{5.33}$$

Then it holds that

$$\lim_j p^{\omega}_{\varepsilon_j}(S_j) = \mathcal{H}^1(S)\mathbb{E}[c^{\mathit{eff}}_{i,|}].$$

Moreover the convergence is locally uniform in the following weak sense: there exists $j_0 = j_0(\omega)$ such that for all $j \geq j_0$ and all sequences of vertical sides S'_j such that (5.33) holds and $d_H(S_j, S'_j) \leq \delta$ we have

$$\left| p^{\omega}_{\varepsilon_j}(S'_j) - \mathcal{H}^1(S'_j)\mathbb{E}[c^{\mathit{eff}}_{i,|}] \right| \leq C\delta \tag{5.34}$$

for some positive constant $C > 0$ independent of S'_j.
The same statement holds for horizontal sides with the condition on the y-component and the first moment of $c^{\mathit{eff}}_{i,-}$.

Proof. The argument to show convergence is very similar to the one used in Proposition 5.7 restricted to a thinned dual lattice. We therefore only provide the main steps. We fix i as in (5.33), set $\mu_i = \mathbb{E}[c^{\mathrm{eff}}_{i,|}]$ and define the two-sided sequence of random variables $\{z_k\}_{k\in\mathbb{Z}}$ via

$$z_k(\omega) := c^{\mathrm{eff}}_{i,|}(\tau_{kme_2}\omega).$$

Note that this sequence is stationary and α-mixing such that (5.7) holds. For $q \in \mathbb{Q} \cap (0, +\infty)$ we define the following average sequences:

$$Z^{q,\pm}_n(\omega) := \sup_{k \geq qn} \left| \frac{1}{2k+1} \sum_{l=-k}^{k} z_k(\tau_{\pm nme_1}\omega) - \mu_i \right|$$

Using m-stationarity and the mixing property we can argue as in the proof of Proposition 5.7 to show that there exists a set Ω'' of full probability such that all the sequences $Z^{q,\pm}_n$ converge to 0 pointwise on Ω''. Up to minor changes the proof of convergence now is the same as for Proposition 5.7. We omit the details.

In order to prove (5.34) we have to distinguish two cases: First assume that the x-coordinate (also denoted by x) of S is positive (the case of negative x-coordinate works the same way). Then, for δ small enough (otherwise (5.34) is trivial), we have $x'_j > 0$ for j large enough depending only on S_j. The key is to show that we can compare S'_j to one of the sequences of random variables $Z^{q,+}_n$ as in the proof of Proposition 5.7, where q can be chosen only depending on the sequence S_j. Then the speed of convergence is determined by the one of $Z_n^{q,+}$ for one particular q.

We start with the case of a vertically centered side S, that means $S = \{x\} \times [-\mathcal{H}^1(S)/2, \mathcal{H}^1(S)/2]$: Given $0 < \beta << \delta$ there exists j_0 such that for all $j \geq j_0$ we have $x + \beta \geq \varepsilon_j x_j$ and $\varepsilon_j \#\{\xi \in S_j/\varepsilon_j \cap \mathcal{Z}^2\} \geq \mathcal{H}^1(S) - \beta$. Using the assumption $\mathrm{d}_H(S_j, S_j') \leq \delta$, a straightforward computation yields

$$\#\{\xi \in \frac{S_j'}{\varepsilon_j} \cap \mathcal{Z}^2\} \geq \frac{\mathcal{H}^1(S) - 2\delta - \beta}{x + \delta + \beta} x_j'$$

for all $j \geq j_0$. Therefore we have to chose $\frac{\mathcal{H}^1(S)-2\delta-\beta}{x+\delta+\beta} > 3q$ which can be done uniformly for small δ. Moreover, from our assumptions we deduce

$$\#\left(\left\{\xi \in \frac{S_j'}{\varepsilon_j} \cap \mathcal{Z}^2\right\} \Delta \left\{\xi = (x_j, l) : \ |l| \leq \frac{\mathcal{H}^1(S_j')}{2\varepsilon_j}\right\}\right) \leq \frac{4\delta}{\varepsilon_j}.$$

Assuming (5.8) we deduce that $\sup_j \mathcal{H}^1(S_j') \leq C$. Hence we can argue as in (5.26) to prove that

$$\left|p^{\omega}_{\varepsilon_j}(S_j') - \mathcal{H}^1(S_j')\mu_i\right| \leq C\delta + Z_{n_j}^{q,+}(\omega), \tag{5.35}$$

where $[n_j]^* = x_j'$. Since $|x_j - x_j'| \leq \delta/\varepsilon_j$ and $\varepsilon_j x_j \to x$, for every $n \in \mathbb{N}$ we can find j_0 (depending only on S_j) such that for all $j \geq j_0$ we have

$$x_j' \geq \frac{x/2 - \delta}{\varepsilon_j} \geq n.$$

Hence $n_j \to +\infty$ and since $Z_n^{q,+}$ converges to 0 on Ω'', (5.34) holds in this particular case.

The case of a general side $S = \{x\} \times [y - \mathcal{H}^1(S)/2, y + \mathcal{H}^1(S)/2]$ with $x, y > 0$ can be treated with the same arguments as in the derivation of (5.29) since this construction is uniformly with respect to small displacements of the limit side. We leave out the details here.

We are left with the case when $x = 0$. Again it is enough to consider a centered side S since the other cases can be deduced from this one. Let us take q small enough such that

$$\frac{\mathcal{H}^1(S) - 4\delta}{2\delta} > 3q.$$

By construction there exists j_0 such that for every sequence S_j' fulfilling the assumptions we have $\mathcal{H}^1(S_j')/2\varepsilon_j > q|x_j'|$ for all $j \geq j_0$. Thus, if x_j' is not bounded we can control the speed of convergence with the random variables $Z_n^{q,\pm}$ as in (5.35). Perhaps after enlarging j_0, we obtain that

$$|Z_j^{q,\pm}(\omega)| \leq \delta \quad \forall j \geq j_0.$$

The estimate (5.34) now follows from distinguishing the case where $|x'_j| > j_0$ for which we can use the above bound and (5.35) or $|x'_j| \leq j_0$ where we have to control finitely many sequences of random variables that converge to 0 as $S_j \to S$. □

Before we state our next theorem, let us derive a suitable expression for the velocity. We remark that due to Proposition 5.14 the argument is similar to the deterministic case treated in [27]. To reduce notation, we set $\mu_k = \mathbb{E}[c^{\mathrm{eff}}_{k,|}]$ and $\lambda_k = \mathbb{E}[c^{\mathrm{eff}}_{k,-}]$ and identify the indices modulo m whenever necessary.

We have to minimize the function $v^{\omega}_{i,\varepsilon}(N)$ given by (5.21) which is the correct one describing the velocity if the limit function as $\varepsilon_j \to 0$ has a unique minimizer. For the moment we restrict the analysis to the left vertical side. Up to a subsequence, we have that the x-component of $s_{i,\varepsilon_j}/\varepsilon_j$ is constant modulo m, that is there exists $n \in 0, ..., m-1$ such that

$$x_j^i = n + \frac{1}{2} \mod m \quad \forall j.$$

If s_{i,ε_j} converges to a limit side of length L, then by Proposition 5.14 we have that along this particular subsequence, it holds that

$$v^{\omega}_{i,\varepsilon_j}(N) \to v_i^{n,L}(N) := -2N + L\mu_k + \frac{L}{2\gamma}(N+1)N \quad \text{if } N+n = k \mod m. \tag{5.36}$$

As we will show in the following, we can define an effective velocity which does not depend on the particular subsequence. Setting $N^* = \lfloor 2\gamma/L \rfloor$, as an analogue of (5.22) we have

$$\min_N v_i^{n,L}(N) = \min\{v_i^{n,L}(N^*),\, v_i^{n,L}(N^*+1),\, v_i^{n,L}(N^*-1)\}. \tag{5.37}$$

Since a precise analysis of the minimization process is only possible provided the limit functional has a unique minimizer, let us check when this is the case. There are three equivalences that turn out to be useful to characterize the lack of uniqueness. Write $N^* = 2\gamma/L - \xi$ with $\xi \in [0,1)$ and suppose that $N^* + n = k^* \mod m$. Then it holds

$$\begin{aligned}
v_i^{n,L}(N^*) \leq v_i^{n,L}(N^*+1) \quad &\Longleftrightarrow \quad \xi \leq 1 + \gamma(\mu_{k^*+1} - \mu_{k^*}),\\
v_i^{n,L}(N^*) \leq v_i^{n,L}(N^*-1) \quad &\Longleftrightarrow \quad \xi \geq \gamma(\mu_{k^*} - \mu_{k^*-1}),\\
v_i^{n,L}(N^*+1) \leq v_i^{n,L}(N^*-1) \quad &\Longleftrightarrow \quad \xi \geq \frac{1}{2} + \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}).
\end{aligned} \tag{5.38}$$

Thus minimizers are not unique if and only if

(i) $\gamma(\mu_{k^*+1} - \mu_{k^*}) + 1 = \xi \geq \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}) + \frac{1}{2}$,

(ii) $\gamma(\mu_{k^*} - \mu_{k^*-1}) = \xi \leq \frac{\gamma}{2}(\mu_{k^*+1} - \mu_{k^*-1}) + \frac{1}{2}$,

(iii) $\xi = \frac{1}{2} = \gamma(\mu_{k^*} - \mu_{k^*-1}) = \gamma(\mu_{k^*} - \mu_{k^*+1})$,

where we left out those inequalities with no information. Due to (5.6) the third possibility cannot occur and also the inequalities in (i) and (ii) are always fulfilled since $\mu_{k+1} - \mu_{k-1} = (\mu_{k+1} - \mu_k) + (\mu_k - \mu_{k-1})$. In particular the set of side lengths where the minimization problem (5.37) has not a unique solution is discrete. The same analysis for the remaining sides yields the following singular side lengths:

$$
\begin{aligned}
\mathcal{S}_|^l &:= \{L \in (0, +\infty) : \ 2\gamma/L \in \mathbb{N}_0 + \gamma(\mu_k - \mu_{k-1}) \quad \text{for some } k\}, \\
\mathcal{S}_|^r &:= \{L \in (0, +\infty) : \ 2\gamma/L \in \mathbb{N}_0 - \gamma(\mu_k - \mu_{k-1}) \quad \text{for some } k\}, \\
\mathcal{S}_-^d &:= \{L \in (0, +\infty) : \ 2\gamma/L \in \mathbb{N}_0 + \gamma(\lambda_k - \lambda_{k-1}) \quad \text{for some } k\}, \\
\mathcal{S}_-^u &:= \{L \in (0, +\infty) : \ 2\gamma/L \in \mathbb{N}_0 - \gamma(\lambda_k - \lambda_{k-1}) \quad \text{for some } k\}.
\end{aligned}
$$

Whenever it is clear from the context, we associate to a side S_i the corresponding set $\mathcal{S}_i \in \{\mathcal{S}_|^l, \mathcal{S}_|^r, \mathcal{S}_-^d, \mathcal{S}_-^u\}$.

Now let us analyze the minimization scheme. Again we illustrate the procedure only for the left vertical side. To this end we fix $L \notin \mathcal{S}_|^l$. Setting $X_0 = x_j^i$, we will see that the motion of the corresponding left vertical side will be given locally by the following algorithm:

For $l = 0, 1, ...$ set

$$n_l := X_l - \tfrac{1}{2} \mod m,$$

$$N_{l+1} = \underset{N}{\operatorname{argmin}}\ v_i^{n_l, L}(N),$$

$$X_{l+1} := X_l + N_{l+1},$$

where $v_i^{n,L}$ is defined in (5.36). This algorithm is well-defined as $L \notin \mathcal{S}_|^l$ and gives rise to an effective velocity as shown in the lemma below:

Lemma 5.15. *There exist nonnegative integer numbers $\tilde{l}$, T, M such that $\tilde{l} + T \leq m$ and*

$$X_{l+T} - X_l = Mm \quad \forall l \geq \tilde{l}.$$

Moreover, the quotient M/T does not depend on X_0.

Proof. Observe that the quotient space $\mathbb{Z}_{/m\mathbb{Z}}$ has only m distinct elements so that the first statement holds. For the second statement we first establish a monotonicity property of the orbits with respect to the initial data X_0. To this end let $X_0 \leq X_0'$. We argue inductively. Due to (5.37) it is clear that $X_1 \leq X_1'$ in case that $X_0 = X_0'$ or $X_0' - X_0 \geq 2$. It remains the case where $X_0' - X_0 = 1$. We assume by contradiction that $X_1 > X_1'$. Writing $N^* = 2\gamma/L - \xi$, the minimizer to determine X_1 would be given by $N^* + 1$ while for X_1' minimizing yields $N^* - 1$. Using minimality one easily derives that in this case we have

$$\gamma(\mu_{k^*+1} - \mu_{k^*}) \geq \xi \geq \gamma(\mu_{k^*+1} - \mu_{k^*}) + 1,$$

where $k^* = X_0 - \frac{1}{2} + N^* \mod m$. This gives a contradiction. By iteration we obtain that $X_k \leq X_k'$ for all k. Now we argue as in Proposition 3.6 in [27] by comparing the long-time behavior of the orbits with starting points X_0, X_0' and X_0+m. For $L, l_0 \in \mathbb{N}$ we set $k = l_0 + LT(x_0)T(x_0')$. By the first part of the proof and orbit monotonicity, for l_0 large enough it holds that

$$X_{l_0} + LT(x_0')M(x_0)m \leq X_{l_0}' + LT(x_0)M(x_0')m \leq X_{l_0} + LT(x_0')M(x_0)m + m.$$

Dividing this inequality by L and letting $L \to +\infty$ yields the claim. □

Definition 5.16. *For a given type of side with length $L \notin \mathcal{S}_i$, let M_i, T_i be as in Lemma 5.15, where T_i is chosen to be minimal. The effective velocity for a side S_i is defined as a function $v_i^{\mathit{eff}} : (0, +\infty)\backslash\mathcal{S}_i \to [0, +\infty)$ by*

$$v_i^{\mathit{eff}}(L) = \frac{M_i m}{T_i}.$$

In view of Lemma 5.15, this function is well-defined.

Remark 5.17. In contrast to the deterministic environments considered in [27, 49] in our setting the effective velocity of two opposite sides can be different. However this is not due to random effects but can already be caused by a slightly more complex periodic structure as shown in the following example.

Example 5.18. *Let $m = 6$ and let c_ξ be a (maybe deterministic) field such that*

$$\mu_0 = -\frac{1}{8\gamma}, \quad \mu_1 = \mu_2 = \mu_5 = \frac{1}{8\gamma} \quad \mu_3 = \mu_4 = 0.$$

If $2\gamma/L \in (3 - \frac{1}{8}, 3)$, then the left side of a rectangle moves faster than the right side, *namely*

$$v_i^{\mathit{eff}}(L) = 3 > 2 = v_{i+2}^{\mathit{eff}}(L).$$

Proof. This follows from a straightforward computation based on the minimality criteria (5.38). Indeed, if the left side starts at $n_0 = 0$, then we have $N_1 = N_2 = 3$. If the right side starts also at $n_0 = 0$ we deduce that $N_1 = N_2 = N_3 = 2$. We leave the details of the computation to the interested reader. □

Let us now compute the pinning threshold, that is the critical side length above which a side does not move after a finite number of time steps (or equivalently $v_i^{\text{eff}}(L) = 0$). Due to (5.37) a necessary condition is given by $L > \gamma$. We then have to compare the values of $N \in \{0, 1, 2\}$. For an arbitrary starting position of a left vertical side we obtain the conditions

$$L > \frac{2\gamma}{1 + \gamma(\mu_{k+1} - \mu_k)}, \qquad L > \frac{4\gamma}{3 + \gamma(\mu_{k+2} - \mu_k)}.$$

As we can chose the index k, the pinning threshold for a left vertical side is given by

$$\overline{L}_i = \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\mu_{k+1} - \mu_k)}, \frac{4\gamma}{3 + \gamma(\mu_{k+2} - \mu_k)} \right\} \right\} > \gamma.$$

The pinning thresholds for the other sides are given by

$$\begin{aligned}
\overline{L}_{i+1} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\lambda_{k-1} - \lambda_k)}, \frac{4\gamma}{3 + \gamma(\lambda_{k-2} - \lambda_k)} \right\} \right\}, \\
\overline{L}_{i+2} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\mu_{k-1} - \mu_k)}, \frac{4\gamma}{3 + \gamma(\mu_{k-2} - \mu_k)} \right\} \right\}, \\
\overline{L}_{i+3} &= \min_k \left\{ \max \left\{ \frac{2\gamma}{1 + \gamma(\lambda_{k+1} - \lambda_k)}, \frac{4\gamma}{3 + \gamma(\lambda_{k+2} - \lambda_k)} \right\} \right\},
\end{aligned}$$

where the indices rotate clockwise. The next lemma contains some properties of the effective velocities. We remark that the same results have been obtain in [27] but we find it difficult to reproduce the argument in our slightly more complex setting. Therefore we provide a different proof.

Lemma 5.19. *The velocity functions v_i^{eff} satisfy the following properties:*

(a) v_i^{eff} is constant on each interval contained in $(0, +\infty) \backslash \mathcal{S}_i$.

(b) $v_i^{\mathit{eff}}(L) = 0$ if $L > \overline{L}_i$.

(c) $v_i^{\mathit{eff}}(\cdot)$ is non-increasing in L.

Proof. To prove the first assertion, fix an interval $I \subset (0, +\infty) \backslash \mathcal{S}_i$ and let $L \in I$. We claim that there exists an open interval I_L around L such that for all $n = 0, \dots, m-1$ and all $L' \in I_L$ the unique minimizers of $v_i^{n,L'}$ agree with the unique minimizer of $v_i^{n,L}$. As I is connected, it then follows that the minimizers are the same for all $L' \in I$ and we conclude by iteration. To prove the claim, it is enough to observe that whenever $L_j \to L$, it follows that $v_i^{n,L_j}(N) \to v_i^{n,L}(N)$ pointwise. Due to (5.37) also the minimizers are bounded. By uniqueness they converge to the minimizer of the limit function. Hence the claim follows for any fixed n and then we take a finite intersection of open intervals to conclude.

The second assertion is an immediate consequence of the definition of the pinning threshold.

To prove the monotonicity, take $L > L'$. The claim follows from the fact that, for every n, in a multi-valued sense it holds that

$$\operatorname*{argmin}_N \, v_i^{n,L}(N) \leq \operatorname*{argmin}_N \, v_i^{n,L'}(N). \tag{5.39}$$

Indeed, observe that $N(L) := \lfloor 2\gamma/L \rfloor \leq \lfloor 2\gamma/L' \rfloor =: N(L')$. Then by (5.37) it suffices to treat the two cases $N(L) = N(L')$ and $N(L) + 1 = N(L')$. In any case, again applying (5.37) there are only finitely many options for violating (5.39) that can be ruled out by a direct calculation based on a characterization as in (5.38). We omit the details. □

Due to the monotonicity proven in Lemma 5.19 (c) we can define the two following extensions of the effective velocity: Given $L_0 \in \mathcal{S}_i$ we set

$$(v_i^{\text{eff}})^{(-)}(L_0) = \lim_{L \downarrow L_0} v_i^{\text{eff}}(L), \quad (v_i^{\text{eff}})^{(+)}(L_0) = \lim_{L \uparrow L_0} v_i^{\text{eff}}(L).$$

Now we are in a position to state the main theorem for m-stationary fields under the same α-mixing hypothesis as in Theorem 5.9.

Theorem 5.20. *Assume that the random field $\{c_\xi\}_\xi$ satisfies (5.6), is m-stationary and α-mixing such that (5.7) holds. Then with probability* 1 *the following holds: Let $\varepsilon_j \downarrow 0$ and let $A_j^0(\omega) \in \mathcal{A}_{\varepsilon_j}$ be a coordinate rectangle with sides $S_{1,j}(\omega), ..., S_{4,j}(\omega)$. Assume that $A_j^0(\omega)$ converges in the Hausdorff metric to a coordinate rectangle $A(\omega)$. Then we can choose a subsequence (not relabeled), such that $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is a coordinate rectangle with sides $S_i(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$. Each side $S_i(t)(\omega)$ moves inward with velocity $v_i(t)(\omega)$*

solving the following inclusions:

$$v_i(t)(\omega) \begin{cases} = \frac{1}{\gamma} v_i^{\mathit{eff}}(L_i(t)(\omega)) & \textit{if } L_i(t)(\omega) \notin \mathcal{S}_i, \\ \in \frac{1}{\gamma}\left[(v_i^{\mathit{eff}})^{(-)}(L_i(t)(\omega)), (v_i^{\mathit{eff}})^{(+)}(L_i(t)(\omega))\right] & \textit{otherwise,} \end{cases}$$

where $L_i(t)(\omega) := \mathcal{H}^1(S_i(t)(\omega))$ *denotes the length of the side* $S_i(t)(\omega)$*. The inclusions are valid until the extinction time when* $L_i(t)(\omega) = 0$*.*

Proof. Due to Remark 5.11 we only have to derive the formula for the velocities. We fix $\omega \in \Omega'$ given by Proposition 5.14. Using the same notation as in the proof of Theorem 5.9, we have to identify the weak*-limit $\tilde{v}_i$ of $N_i^j(\cdot)$ on the interval $(0, t^*)$. Therefore we fix $t_1 \in (0, t^*)$ such that $L_i(t_1)(\omega) \notin \mathcal{S}_i$. Given $\delta > 0$ there exists an open interval $I_\delta \ni t_1$ and j_0 such that for all $j \geq j_0$

(i) $L_i^j(t)(\omega) \notin \mathcal{S}_i \quad \forall t \in I_\delta$,

(ii) $\mathrm{d}_H(S_{i,j}(t)(\omega), S_{i,j}(t_1)(\omega)) \leq \delta \quad \forall t \in I_\delta$.

Hence, by Proposition 5.14 we may assume that for $j \geq j_0$ and $t \in I_\delta$ there exists $n = n(j,t)$ such that for $L = L_i^j(t)(\omega)$

$$N_i^j(t)(\omega) = \operatorname*{argmin}_N \, v_i^{n,L}(N),$$

where $v_i^{n,L}$ is defined in (5.36). Since without loss of generality $L_i^j(t)(\omega)$ is in the same interval contained in $(0, +\infty)\backslash\mathcal{S}_i$ as $L_i^j(t_1)(\omega)$, we infer from the Lemmata 5.15 and 5.19 (a) that

$$\begin{aligned} \int_{I_\delta} \tilde{v}_i(s)(\omega)\,\mathrm{d}s &= \lim_j \int_{I_\delta} N_i^j(s)(\omega)\,\mathrm{d}s = \lim_j \sum_{k\tau_j \in I_\delta} \tau_j N_i^j(k\tau_j)(\omega) + \mathcal{O}(\tau_j) \\ &= \lim_j |I_\delta| v_i^{\mathrm{eff}}(L_i(t_1)(\omega)) + \mathcal{O}(\tau_j) = |I_\delta| v_i^{\mathrm{eff}}(L_i(t_1)). \end{aligned} \tag{5.40}$$

Dividing by $|I_\delta|$ and letting $\delta \to 0$ we obtain the claim using Lebesgue's differentiation theorem. Note that similar to the proof of Theorem 5.9 the formula for the velocity holds for every such t_1 since $\tilde{v}_i$ has a constant representative locally near t_1 so that the side positions are differentiable in the classical sense. However here we have to take the side positions and cannot deduce the velocity from the side lengths since the center might move (see Example 5.18).

It remains the case where $L_i(t_1)(\omega) \in \mathcal{S}_i$. Note that by (5.39) we still have the monotonicity of orbits, that means if $L^-, L^+ \in (0, +\infty)\backslash\mathcal{S}_i$ are in the two intervals enclosing $L_i(t_1)(\omega)$ such that $L^- < L_i(t_1)(\omega) < L^+$ and we start the algorithm for computing the effective velocity with the same initial datum choosing the minimizer arbitrarily in the case of non-uniqueness, we have

$$X_k^+ \le X_k \le X_k^-.$$

This yields

$$|I_\delta| v_i^{\mathrm{eff}}(L^+) \le \int_{I_\delta} \tilde{v}_i(s)(\omega)\, \mathrm{d}s \le |I_\delta| v_i^{\mathrm{eff}}(L^-).$$

The claim follows after dividing by $|I_\delta|$, sending $\delta \to 0$ and then taking both the limits as $L^- \uparrow L_i(t_1)(\omega)$ and $L^+ \downarrow L_i(t_1)(\omega)$ for which we use monotonicity. $\square$

Again we have several cases where a unique limit motion exists. However the equations differ since the velocity of two opposite sides may be not equal. We don't list all possible cases where there is a unique motion.

Corollary 5.21. *Let $A_\varepsilon^0(\omega)$ and $\{c_\xi\}_\xi$ be as in Theorem 5.20. Assume in addition that the lengths $L_1^0(\omega), L_2^0(\omega)$ of $A(\omega)$ satisfy one of the three following conditions (we assume that $L_1^0(\omega) \le L_2^0(\omega)$ and $\overline{L}_1 \le \overline{L}_3$ as well as $\overline{L}_2 \le \overline{L}_4$):*

(i) $L_i^0(\omega) > \overline{L}_i$ *(total pinning),*

(ii) $L_1^0(\omega) < \overline{L}_1$ *and* $L_2^0(\omega) \le \overline{L}_2$ *(vanishing in finite time),*

(iii) $\overline{L}_1 < L_1^0(\omega) < \overline{L}_3$ *and* $L_1^0(\omega) \notin \mathcal{S}_3$*, and* $L_2^0(\omega) > \overline{L}_4$ *(partial pinning).*

Then with probability 1 the following holds: Let $\varepsilon_j \to 0$. The sequence $A_{\varepsilon_j}(t)(\omega)$ converges locally in time to $A(t)(\omega)$, where $A(t)(\omega)$ is the unique coordinate rectangle with sides $S_i(t)(\omega)$ such that $A(0)(\omega) = A(\omega)$ and the side lengths $L_i(t)(\omega)$ solve the following differential equations for all but countably many times:

$$\frac{d}{dt} L_i(t)(\omega) = -\frac{1}{\gamma}\left(v_{i-1}^{\mathit{eff}}(L_{i-1}(t)(\omega)) + v_{i+1}^{\mathit{eff}}(L_{i+1}(t)(\omega)) \right)$$

with initial condition $L_1(0)(\omega) = L_1^0(\omega)$ and $L_2(0) = L_2^0$.

Proof. (i) and (ii) can be proven as in Theorem 3.2 in [24]. In Case (iii) note that the side S_3 moves inward with a strictly positive velocity bounded away from 0. Hence $L_2(t)(\omega)$ is strictly decreasing until it vanishes. Consequently $L_2(t)(\omega) \in \mathcal{S}_2 \cup \mathcal{S}_4$ only for countably many times. Moreover, as soon as $L_2(t)(\omega) < \overline{L}_4$ also the side length $L_1(t)(\omega)$ shrinks strictly since from that time on the side S_4 moves inward with positive velocity. Hence the times when $L_1(t)(\omega) \in \mathcal{S}_1 \cup \mathcal{S}_3$ are discrete, too. Note that by continuity, the values at the critical times are uniquely defined. In between these critical times, one can use general results from ODE-theory to obtain that the rectangular motion is unique. The particular form of the ODE describing the motion is a straightforward consequence of Theorem 5.20. □

An outlook for possible homogenization in time

In this last section we show that under certain assumptions the random field considered in Example 5.6 exhibits an averaged velocity as well. We don't aim at giving results in full detail since Example 5.6 only serves as a toy model and the case of only stationary perturbations seems much more involved and we are not sure if homogenization can be proved.

To be precise we generalize Lemma 5.15 in a probabilistic setting. Note that by construction of the perturbations in Example 5.6 there is nothing left to prove for horizontal sides since we may apply Proposition 5.7. For vertical sides we first hint at some possible uniqueness issues. For the moment let us neglect the terms in (5.18). Denoting by $x_j = x_j(\omega)$ the x-component of $s_{i,\varepsilon_j}(\omega)/\varepsilon_j$, for left vertical sides we have to minimize

$$v^{\omega}_{i,\varepsilon_j}(N) = -2N + \frac{l_{i,\varepsilon_j}}{2\gamma}(N+1)N + l_{i,\varepsilon_j} X_{\lfloor x_j+N \rfloor}(\omega).$$

Let us take a closer look at the non-uniqueness of minimizers. Again we set $N_j^* = \lfloor 2\gamma/l_{i,\varepsilon_j} \rfloor$ and $k_j := \lfloor x_j + N_j^* \rfloor$. Writing $N_j^* = 2\gamma/l_{i,\varepsilon_j} - \xi_j$, as in (5.38) we deduce that minimizers are not unique if

$$\xi_j \in \left\{\gamma(X_{k_j}(\omega) - X_{k_j-1}(\omega)), 1 + \gamma(X_{k_j+1}(\omega) - X_{k_j}(\omega))\right\}.$$

Without any further assumptions, as j varies this set can be dense in a whole interval as the following example shows.

Example 5.22. *Let X_k be uniformly distributed on the interval $(0,1)$. Then, by independence, for every k the random variable $Y_k = X_k - X_{k-1}$ has a triangular distribution on $(-1,1)$. Hence the sequence $(Y_{2k})_k$ is an independent and identically distributed sequence of random variables. Then*

$$\mathbb{P}\left(\omega:\ (Y_{2k}(\omega))_k \textit{ is not dense in } (-1,1)\right) = 0.$$

Indeed, given $q \in \mathbb{Q} \cap (-1,1)$ and $n \in \mathbb{N}$, from independence we infer

$$\mathbb{P}\left(\omega:\ Y_{2k}(\omega) \notin q + (-\frac{1}{n}, \frac{1}{n}) \quad \forall k\right) = 0.$$

This example indicates that a precise analysis for the limit velocity is quite difficult (of course one has to take into account also the error terms (5.18)). Instead, if we assume that X_k takes only finitely many values, then the set of side lengths $\mathcal{S}$ where the minimization problem has not a unique solution is again discrete. We now give a formal argument how one can treat this case. For side lengths with unique corresponding minimizers, we can indeed neglect (5.18). Moreover, by the same topological argument used in the proof of Lemma 5.19, the minimization does not depend on the particular side length in one interval contained in $(0,+\infty)\backslash\mathcal{S}$. However, in contrast to the mixing case, the choice of minimizers is still random. Given $L \notin \mathcal{S}$ we have to consider the following (now random) algorithm: Given a starting point $P_0 = x_j$,

for $l = 0, 1, ...$ set

$$n_l(\omega) := P_l(\omega) - \tfrac{1}{2},$$

$$N_{l+1}(\omega) = \operatorname*{argmin}_N \left\{-2N + \tfrac{L}{2\gamma}(N+1)N + LX_{n_l(\omega)+N}(\omega)\right\},$$

$$P_{l+1}(\omega) := P_l(\omega) + N_{l+1}(\omega).$$

Note that if L is below the corresponding pinning threshold (which can easily be estimated since the random variables take only finitely many values), then the sequence $(N_l)_l$ is identically distributed and has a finite range dependence. Therefore, by the strong law of large numbers, almost surely we have

$$\lim_{l_1\to+\infty} \frac{1}{l_1}\sum_{l=1}^{l_1} N_l(\omega) = \mathbb{E}[N_0].$$

Note that the limit does not depend on x_j. Moreover, as a trivial remark we can make the exceptional set independent of the starting position x_j. In order to prove that $\mathbb{E}[N_0]$ is, up to a multiplicative constant, the velocity of the left vertical side, we need to control the speed of convergence independently of x_j. This can be achieved by defining finitely many stochastic processes (for a fixed limit side) similar to the proof of Proposition 5.14 using the fast decay of error probabilities due to finite range dependence. Since we only want to give a possible outlook we don't go into details here. Finally one can argue as in the proof of Theorem 5.20 and pass to the limit in the integral in (5.40). We leave the computation to the interested reader.

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