Monte Carlo Methods Appl. **19** (2013), 107 – 141
DOI 10.1515/mcma-2013-0005

Double-barrier first-passage times of jump-diffusion processes

Lexuri Fernández, Peter Hieber and Matthias Scherer

Abstract. Required in a wide range of applications in, e.g., finance, engineering, and physics, first-passage time problems have attracted considerable interest over the past decades. Since analytical solutions often do not exist, one strand of research focuses on fast and accurate numerical techniques. In this paper, we present an efficient and unbiased Monte-Carlo simulation to obtain double-barrier first-passage time probabilities of a jump-diffusion process with arbitrary jump size distribution; extending single-barrier results by [Journal of Derivatives 10 (2002), 43–54]. In mathematical finance, the double-barrier first-passage time is required to price exotic derivatives, for example *corridor bonus certificates*, *(step) double barrier options*, or *digital first-touch options*, that depend on whether or not the underlying asset price exceeds certain threshold levels. Furthermore, it is relevant in structural credit risk models if one considers two exit events, e.g., default and early repayment.

Keywords. Double-barrier problem, first-exit time, first-passage time, Brownian bridge, corridor derivatives, barrier options, bonus certificates, first-touch options.

2010 Mathematics Subject Classification. 60G51, 91G60, 65Y20, 58J65.

1 Introduction

For continuous diffusions, the probability of first hitting a lower (or an upper) threshold has been extensively treated in the literature. Analytical solutions exist for Brownian motion on constant (see, e.g., [8, 18, 39]), on linear (see, e.g., [12, 23, 33]), or (at least in sufficient approximation) on any continuous (see, e.g., [37]) barriers. Furthermore, the case of continuously time-changed Brownian motion can in many cases be solved analytically (see, e.g., [26, 27]). However, those models cannot explain several empirical observations concerning market returns and its underlying derivative's prices (see the discussions in, e.g., [6, 9, 29]). For example, when used as firm-value process in structural credit risk models, Brownian motion implies vanishing credit spreads for bonds/CDS with short time to maturity and does not allow for fat-tailed return distributions of equity returns.

Lexuri Fernandez is supported by a FPI-UPV/EHU grant. Peter Hieber acknowledges funding by the TUM graduate school.

That motivates why jump processes have experienced considerable interest in mathematical finance. The first-passage time of jump-diffusions, however, is mathematically challenging. In the single-barrier case, the Laplace transform of the first-passage time is available, e.g., if the jump-size distribution is double-exponential (see, e.g., [3, 32]) or a spectrally one-sided Lévy process (see, e.g., [41]). Extensions to the double-barrier case exist, e.g., in the double-exponential jump-diffusion model, see [46]. For arbitrary jump-size distributions, time-dependent barriers, or extensions like, e.g., state-dependent drift, volatility, and jump size, one has to rely on numerical schemes. Some authors obtain the Laplace transform of the first-passage time by numerically solving the so-called Wiener–Hopf factorization (see, e.g., [3,4], and many others). Apart from that, many authors solve the resulting partial integro-differential equations (PIDEs) numerically (see, e.g., [7] and the references therein). Others rely on Monte-Carlo simulations. However, the standard Monte-Carlo simulation on a discrete grid (see, e.g., [40,48]) exhibits two disadvantages: First, even for 1000 discretization intervals per unit of time, we obtain a significant discretization bias. Second, computation time increases rapidly if one has to simulate on a fine grid. Several strategies to remove or reduce the discretization bias in jump-diffusion models are available in the literature. In [16] an adaptive discretization scheme for the simulation of functionals of killed Lévy processes with controlled bias is developed. Several authors focused on unbiased simulation schemes. The paper [19] presents an unbiased sampling algorithm applying a variance reduction technique called “acceptance/rejection”. The authors of [36] provide an unbiased, fast, and accurate alternative based on the so-called “Brownian bridge technique”. In their approach, one proceeds as follows: First, the jump-instants of the process in consideration as well as the process immediately before and after the jump times are simulated. In between these generated points, one has a pure diffusion with fixed endpoints. Here, the so-called Brownian bridge probabilities provide an analytical expression for the first-passage time on a given threshold. This simulation technique turns out to be (1) unbiased and (2) significantly faster than the standard Monte-Carlo simulation. It has various applications in finance and can, e.g., be used to derive efficient algorithms for pricing single barrier options in jump-diffusion models (see, e.g., [36, 43]) or regime-switching models (see, e.g., [24, 25]). An extension to the double-barrier case has been considered in, e.g., [20, 47].

We show how the Brownian bridge technique can be adapted to a large variety of exotic double barrier products. Those products are very flexible and thus allow investors to adapt to their specific hedging needs or speculative views. However, those contracts can hardly be traded without a fast and reliable pricing technique. Analytical solutions reach their limitations as they are often not flexible enough to adapt to complicated payoff streams and/or jump size distributions. To provide this flexibility for the Brownian bridge technique, we extend the existing algorithms and (1) allow to price double barrier derivatives that trigger different events depending on which barrier was hit first (a feature that is required to price, e.g., *corridor bonus certificates*) and (2) allow to evaluate payoff streams that depend on the first-passage time (a feature important in, e.g., *structural credit risk models*). Furthermore, (3) we show that time dependent barriers can easily be treated (a feature that is relevant for, e.g., *window* or *step double barrier options*). Finally, we discuss the implementation and show that – in contrast to most alternative techniques – the Brownian bridge algorithms are easy to understand and implement.

The paper is organized as follows: In Section 2, we introduce the jump-diffusion setting, an overview of the double-barrier first-passage time, and the Brownian bridge probabilities. Then, Section 3 presents the algorithms: Algorithm 1 returns the barrier hitting probabilities and the final asset value; Algorithm 2 additionally allows to evaluate expectations that depend on the first-passage time. Finally, in Section 4, the algorithms are applied to credit risk (Section 4.4), to the pricing of *corridor bonus certificates* (Section 4.3), *step double barrier options* (Section 4.2), and to *digital first-touch options* (Section 4.1). Section 5 discusses the implementation.

2 First-passage times and Brownian bridge technique

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting all required stochastic processes, we consider the jump-diffusion process

$$B_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

with drift $\mu \in \mathbb{R}$, volatility $\sigma > 0$, and initial value $B_0 = 0$, where $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion. The counting process $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda \geq 0$ and the jumps $Y = \{Y_i\}_{i\in\mathbb{N}}$ are i.i.d. with distribution $\mathbb{P}_Y$. All random objects are mutually independent.

The first-passage time on two constant barriers $b < B_0 < a$ is defined as

$$T_{ab} := \begin{cases} \inf\{t \geq 0 : B_t \notin (b, a)\}, & \text{if such a } t \text{ exists,} \\ \infty, & \text{if } B_t \text{ never exits } (b, a). \end{cases} \tag{2.1}$$

Here, T_{ab} is the first time the process B_t hits or crosses one of the two barriers a and b. Further define

$$\begin{aligned} T^+_{ab} &:= T_{ab}, \quad T^-_{ab} := \infty \quad \text{if } B_{T_{ab}} \geq a, \\ T^-_{ab} &:= T_{ab}, \quad T^+_{ab} := \infty \quad \text{if } B_{T_{ab}} \leq b. \end{aligned}$$

If the lower barrier b is hit or crossed first, the first-passage time is $T_{ab} = T_{ab}^-$; if the upper barrier a is hit or crossed first, the first-passage time is $T_{ab} = T_{ab}^+$. If not hit, T_{ab}^+ (respectively T_{ab}^-) are set to ∞.

We now present results on the first-passage times of Brownian motion, i.e. we remove the jumps and obtain

$$B_t = \mu t + \sigma W_t,$$

where the notation is the same as above. For our algorithms, we are interested in the first-passage times on some interval (t_{i-1}, t_i), where $0 \le t_{i-1} < t_i < \infty$, $i \in \mathbb{N}$. The Brownian motion at start- and endpoint of this interval is denoted by $x_{i-1} := B_{t_{i-1}} \in (b, a)$, respectively $x_i := B_{t_i} \in \mathbb{R}$. If x_{i-1} as well as x_i are known, we obtain conditional first-passage probabilities, called Brownian bridge probabilities,

$$BB_{ab}^+(t_{i-1}, t_i, x_{i-1}, x_i) := \mathbb{P}\big(t_{i-1} < T_{ab}^+ < t_i \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big),$$

$$BB_{ab}^-(t_{i-1}, t_i, x_{i-1}, x_i) := \mathbb{P}\big(t_{i-1} < T_{ab}^- < t_i \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big).$$

In the single barrier case, we find (see, e.g., [30, p. 240])

$$\lim_{a\to\infty} BB_{ab}^-(t_{i-1}, t_i, x_{i-1}, x_i) = \begin{cases} \exp\big(-\frac{2(x_{i-1}-b)(x_i-b)}{\sigma^2(t_i-t_{i-1})}\big), & \min(x_i, x_{i-1}) > b, \\ 1, & \text{else.} \end{cases}$$

Theorem 2.1 (Brownian bridge probabilities 1). *Consider a Brownian motion B_t with volatility $\sigma > 0$. Assume that $x_{i-1} := B_{t_{i-1}} \in (b, a)$ and $0 \le t_{i-1} < t_i < \infty$.*

(1) *For $x_i := B_{t_i} \in (b, a)$ we get*

$$\begin{aligned} &BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) \\ &\quad := BB_{ab}^+(t_{i-1}, t_i, x_{i-1}, x_i) + BB_{ab}^-(t_{i-1}, t_i, x_{i-1}, x_i) \\ &\quad = \sum_{n=-\infty}^{\infty} \Bigg[\exp\bigg(-\frac{2n(a-b)}{\sigma^2(t_i - t_{i-1})}\big(x_{i-1} - x_i + n(a-b)\big)\bigg) \\ &\qquad\qquad + \exp\bigg(-\frac{2\big(x_i - na + (n-1)b\big)\big(x_{i-1} - na + (n-1)b\big)}{\sigma^2(t_i - t_{i-1})}\bigg)\Bigg] - 1. \end{aligned}$$

(2) *If $x_i := B_{t_i} \notin (b, a)$, we obtain*

$$BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1.$$

(3) *If* $x_i := B_{t_i} \in (-\infty, a)$,

$$
\begin{aligned}
&BB^{+}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) \\
&\quad = \sum_{n=1}^{\infty} \Bigg[\exp\Big(-\frac{2(x_{i-1} - na + (n-1)b)\big(x_i - na + (n-1)b\big)}{\sigma^2 (t_i - t_{i-1})} \Big) \\
&\qquad\qquad - \exp\Big(-\frac{2n(a-b)}{\sigma^2 (t_i - t_{i-1})} \big(x_{i-1} - x_i + n(a-b) \big) \Big) \Bigg]. \qquad (2.2)
\end{aligned}
$$

(4) *If* $x_i := B_{t_i} \in (b, \infty)$,

$$BB^{-}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = BB^{+}_{-b\,-a}(t_{i-1}, t_i, -x_{i-1}, -x_i).$$

If $x_i > a$, *the probability of hitting the level* a *first is*

$$BB^{+}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1 - BB^{-}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i).$$

If $b > x_i$, *the probability of hitting the level* b *first is*

$$BB^{-}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 1 - BB^{+}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i).$$

If $x_{i-1} := B_{t_{i-1}} \notin (b, a)$, *we set*

$$BB^{\pm}_{ab}(t_{i-1}, t_i, x_{i-1}, x_i) = 0.$$

Proof. For a proof, we refer to [2]. In a different, sometimes more convenient, notation the formulas are displayed in [37, Remark 2]. For a proof of the expression $BB_{ab}(t_{i-1}, t_i, x_{i-1}, x_i)$, see, e.g., [18]. □

Theorem 2.2 presents the first-passage time probabilities for two barriers of a Brownian motion with drift. Those probabilities can either be obtained by renewal-type arguments together with Fourier inversion (see, e.g., [8] or Remark A.2 in the Appendix) or by risk-neutral valuation (see, e.g., [33, 34]). Both approaches typically yield closed-form expressions that include (rapidly converging) infinite series. In the literature, two different representations for those probabilities are common, they are displayed as representations (a) and (b) in Theorem 2.2. The (numerical) differences between those two representations are discussed in Section 5.

The proof of Theorem 2.2 provides a link between the two representations using the so-called *Jacobi transformation formula* (see, e.g., [45]). Furthermore, this proof explains the connection to the Brownian bridge probabilities in Theorem 2.1. The interested reader is referred to the Appendix.

Theorem 2.2 (First-passage time probabilities). *Consider a Brownian motion B_t with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. Assume that $x_{i-1} := B_{t_{i-1}} \in (b, a)$ and $0 \leq t_{i-1} < t_i < \infty$. Then, setting $BM_{ab}^{+}(t_{i-1}, t_i, x_{i-1}) := \mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i)$:*

(a) *The first representation yields*

$$
\begin{aligned}
BM_{ab}^{+}(t_{i-1}, t_i, x_{i-1}) = \sum_{n=1}^{\infty} \bigg[&\exp\Big(-\frac{\mu}{\sigma^2} k_{n-1}\Big) \Phi\Big(\frac{\mu T - k_{n-1} - a}{\sigma\sqrt{T}}\Big) \\
&- \exp\Big(-\frac{\mu}{\sigma^2}(k_n - a)\Big) \Phi\Big(\frac{\mu T - k_n + a}{\sigma\sqrt{T}}\Big) \bigg] \\
+ \sum_{n=1}^{\infty} \bigg[&\exp\Big(\frac{\mu}{\sigma^2}(k_n + b)\Big) \Phi\Big(\frac{-\mu T - k_{n-1} - a}{\sigma\sqrt{T}}\Big) \\
&- \exp\Big(\frac{\mu}{\sigma^2} k_n\Big) \Phi\Big(\frac{-\mu T - k_n + a}{\sigma\sqrt{T}}\Big) \bigg],
\end{aligned}
$$

where $k_n := 2n(a - b)$ and the standard normal cumulative distribution function is denoted by $\Phi(\cdot)$.

(b) *The second representation yields for $\mu \neq 0$*

$$
\begin{aligned}
&BM_{ab}^{+}(t_{i-1}, t_i, x_{i-1}) \\
&= \frac{\exp\big(-\frac{2\mu(b - x_{i-1})}{\sigma^2}\big) - 1}{\exp\big(-\frac{2\mu(b - x_{i-1})}{\sigma^2}\big) - \exp\big(-\frac{2\mu(a - x_{i-1})}{\sigma^2}\big)} \\
&\quad + \frac{\sigma^2 \pi}{(a - b)^2} \exp\Big(\frac{\mu(a - x_{i-1})}{\sigma^2}\Big) \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a-b)^2}} \\
&\quad \cdot \exp\Big(-\Big(\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2 n^2 \pi^2}{2(a - b)^2}\Big)(t_i - t_{i-1})\Big) \\
&\quad \cdot \sin\Big(\frac{n\pi(b - x_{i-1})}{a - b}\Big),
\end{aligned}
$$

and for $\mu = 0$

$$
\begin{aligned}
&BM_{ab}^{+}(t_{i-1}, t_i, x_{i-1}) \\
&= \frac{x_{i-1} - b}{a - b} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \exp\Big(-\frac{\sigma^2 n^2 \pi^2}{2(a - b)^2}(t_i - t_{i-1})\Big) \\
&\quad \cdot \sin\Big(\frac{n\pi(b - x_{i-1})}{a - b}\Big).
\end{aligned}
$$

Furthermore

$$\mathbb{P}(t_{i-1} < T_{ab}^- < t_i) = BM_{-b\,-a}^+(t_{i-1}, t_i, -x_{i-1}),$$
$$\mathbb{P}(t_{i-1} < T_{ab} < t_i) = \mathbb{P}(t_{i-1} < T_{ab}^+ < t_i) + \mathbb{P}(t_{i-1} < T_{ab}^- < t_i).$$

Proof. A proof of the first representation is provided in [35]; [8] give a proof of the second representation. Some details are given in the Appendix. □

In some applications, we are not only interested in the first-passage time probabilities, but also in the time $t_{i-1} < t < t_i$ this first exit event takes place. Therefore, we define first-passage time intensities[1] by

$$f_{ab}^+(t, x_{i-1}) := \mathbb{P}\big(T_{ab}^+ \in dt \mid B_{t_{i-1}} = x_{i-1}\big),$$
$$f_{ab}^-(t, x_{i-1}) := \mathbb{P}\big(T_{ab}^- \in dt \mid B_{t_{i-1}} = x_{i-1}\big).$$

Similarly, we define Brownian bridge first-passage time intensities

$$g_{ab}^+(t, x_{i-1}, x_i) := \mathbb{P}\big(T_{ab}^+ \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big),$$
$$g_{ab}^-(t, x_{i-1}, x_i) := \mathbb{P}\big(T_{ab}^- \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big).$$

Theorem 2.3 presents the resulting analytical expressions for $f_{ab}^{\pm}(t, x_{i-1})$ and $g_{ab}^{\pm}(t, x_{i-1}, x_i)$, adapting an idea by [15] and [36] to two barriers.

Theorem 2.3 (First-passage time intensities). *Consider a Brownian motion B_t with volatility $\sigma > 0$. Assume that $x_{i-1} := B_{t_{i-1}} \in (b, a)$. For $x_i := B_{t_i}$ and $t_{i-1} < t < t_i < \infty$ we get the first-passage time intensities*

$$f_{ab}^+(t, x_{i-1}) = \frac{\sigma^2\pi}{(a-b)^2} \exp\left(\frac{\mu(a - x_{i-1})}{\sigma^2}\right) \sum_{n=1}^{\infty} (-1)^n n \sin\left(\frac{\pi n(b - x_{i-1})}{a-b}\right) \cdot \exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}\right)(t - t_{i-1})\right),$$

$$f_{ab}^-(t, x_{i-1}) = \frac{\sigma^2\pi}{(a-b)^2} \exp\left(\frac{\mu(b - x_{i-1})}{\sigma^2}\right) \sum_{n=1}^{\infty} (-1)^n n \sin\left(\frac{\pi n(-a + x_{i-1})}{a-b}\right) \cdot \exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}\right)(t - t_{i-1})\right),$$

[1] Note that we use the term "intensity" instead of "density" since there is a non-zero probability that the upper, respectively lower, barrier is never hit and thus $\int_0^\infty f_{ab}^{\pm}(t, x_{i-1})\, dt \le 1$.

and the Brownian bridge first-passage time intensities

$$g^+_{ab}(t, x_{i-1}, x_i) = \frac{\sigma^2\pi}{(a-b)^2}\frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}}\exp\left(\frac{(x_i - x_{i-1})^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a)^2}{2\sigma^2(t_i - t)}\right)$$
$$\cdot \sum_{n=1}^{\infty}(-1)^n n \exp\left(-\frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}(t - t_{i-1})\right)$$
$$\cdot \sin\left(\frac{\pi n(b - x_{i-1})}{a-b}\right),$$
$$g^-_{ab}(t, x_{i-1}, x_i) = g^+_{-b\,-a}(t, -x_{i-1}, -x_i).$$

Proof. The intensities $f^+_{ab}(t, x_{i-1})$ are derived from representation (b) from Theorem 2.2. A proof of the expression $g^+_{ab}(t, x_{i-1}, x_i)$ is referred to the Appendix. An integration with respect to t over the interval $[t_{i-1}, t_i]$ yields the probabilities in Theorem 2.1. In the limit $\lim_{a\to\infty}$, $f^+_{ab}(t, x_{i-1})$ converges to an inverse Gaussian density and we obtain [36, equation (11)]. □

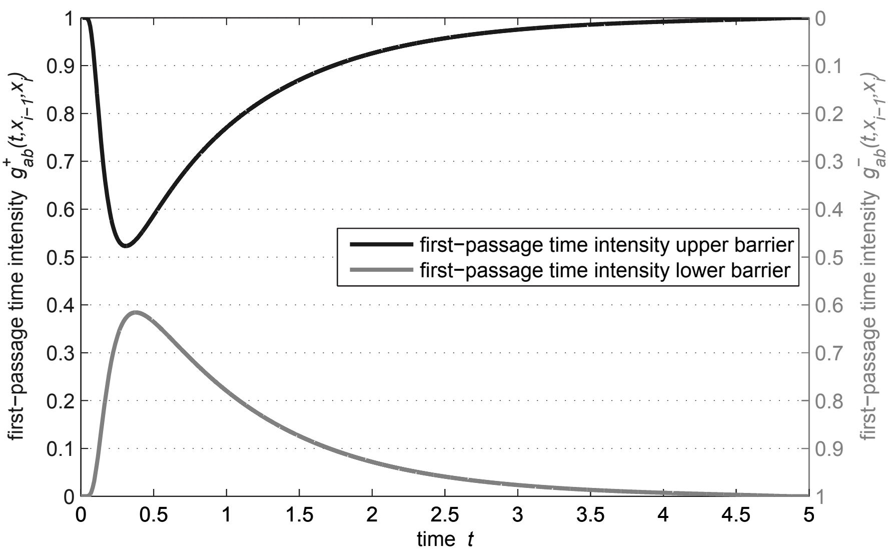

Figure 1. The first-passage time intensities $g^+_{ab}(t, x_{i-1}, x_i)$ and $g^-_{ab}(t, x_{i-1}, x_i)$ in the interval $(t_{i-1}, t_i) = (0, 5)$ using the parameters $x_{i-1} = x_i = 0, \sigma = 10\%$, upper barrier $a = \ln(1.1)$, and lower barrier $b = \ln(0.9)$.

3 Algorithms

This section introduces the Brownian bridge algorithms that allow to efficiently estimate two-sided level crossing probabilities of jump-diffusion processes, extending the single barrier case of [36]. The idea behind those algorithms is to sample first the number of jumps N_T within the time interval $[0, T]$ and the jump instants $t_1 < \cdots < t_{N_T}$. Then – remaining with a diffusion between two successive jumps – to sample the process B immediately before (denoted $B_{t_i -}, i = 1, \ldots, N_T$) and after those jump instants (denoted $B_{t_i}, i = 1, \ldots, N_T$). Between two successive jumps, the barrier hitting probabilities, respectively the first-passage time intensities, can be obtained using the results in Theorem 2.1 and Theorem 2.3.

Crossing by jump at t_{i-1}	$\mathbb{P}(T_{ab}^- = t_{i-1})$	$\mathbb{P}(T_{ab}^+ = t_{i-1})$
$B_{t_{i-1}} \geq a$	0	$\mathcal{P}_{i-1}$
$a > B_{t_{i-1}} > b$	0	0
$b \geq B_{t_{i-1}}$	$\mathcal{P}_{i-1}$	0

Crossing by diffusion in (t_{i-1}, t_i)	$\mathbb{P}(t_{i-1} < T_{ab}^- < t_i)$	$\mathbb{P}(t_{i-1} < T_{ab}^+ < t_i)$
$B_{t_i -} \in \mathbb{R}$	$\mathcal{P}_{i-1}\, \mathcal{P}_{i-1,i}^-$	$\mathcal{P}_{i-1}\, \mathcal{P}_{i-1,i}^+$

Table 1. Conditional probabilities of hitting the lower barrier b (left) and the upper barrier a (right) in the interval $[t_{i-1}, t_i)$. In the upper table the barrier is hit due to a jump in t_{i-1}, in the lower table it is hit between two successive jumps in (t_{i-1}, t_i).

Algorithm 1 returns the barrier hitting probabilities until time $T < \infty$ and samples of the final value B_T. In mathematical finance, this can be used to price derivatives with maturity T that depend on whether a lower barrier b or an upper barrier a is hit. Examples include *digital first-touch options* (see Section 4.1), *step double barrier options* (Section 4.2), or *corridor bonus certificates* (see Section 4.3). Algorithm 2 additionally allows to estimate expectations on the time T_{ab}^-, T_{ab}^+ of the first exit. This can, for example, be used in credit risk where the recovery payment usually depends on the time of the exit event (see Section 4.4).

We set $t_0 = 0$ and abbreviate the exit probabilities $\mathcal{P}_{i-1,i}^+$ and $\mathcal{P}_{i-1,i}^-$ and the survival probability $\mathcal{P}_{i-1,i}$ on the interval (t_{i-1}, t_i) by

$$\begin{aligned}
\mathcal{P}_{i-1,i}^+ &:= BB_{ab}^+(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i -}),\\
\mathcal{P}_{i-1,i}^- &:= BB_{ab}^-(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i -}),\\
\mathcal{P}_{i-1,i} &:= 1 - \left(\mathcal{P}_{i-1,i}^+ + \mathcal{P}_{i-1,i}^-\right).
\end{aligned}$$

We note from Theorem 2.1, part (2), that $\mathcal{P}_{i-1,i} = 0$ if $B_{t_i-} \notin (a, b)$. The exit and survival probabilities sum up to one, i.e.

$$\mathcal{P}^+_{i-1,i} + \mathcal{P}^-_{i-1,i} + \mathcal{P}_{i-1,i} = 1.$$

Further, denote on the interval $[0, t_i)$ the cumulated probabilities

$$\mathcal{P}_i := \prod_{k=1}^{i} \mathcal{P}_{k-1,k},$$

$$\mathcal{P}^+_i := \sum_{k=1}^{i} \mathcal{P}_{k-1}\, BB^+_{ab}(t_{k-1}, t_k, B_{t_{k-1}}, B_{t_k-}),$$

$$\mathcal{P}^-_i := \sum_{k=1}^{i} \mathcal{P}_{k-1}\, BB^-_{ab}(t_{k-1}, t_k, B_{t_{k-1}}, B_{t_k-}).$$

Since between $B_{t_{i-1}}$ and B_{t_i-} the process B behaves like a Brownian motion, the algorithm can take advantage of the Brownian bridge probabilities from Theorem 2.1. The probability of first hitting the upper (respectively lower) barrier in the interval $[t_{i-1}, t_i)$ is given in Table 1. The path may cross the barriers due to a jump (upper table) or between two successive jumps (lower table).

Algorithm 1 (Brownian bridge technique 1). This algorithm samples the first-passage time probabilities $\mathbb{P}(T^+_{ab} \leq T)$ and $\mathbb{P}(T_{ab} \leq T)$. To this end, the number of simulation runs K, the barriers a and b, and the parameters that describe the process B are required. As a second output, the algorithm generates a $K \times 3$ matrix whose columns contain for each simulation run (1) the conditional probability of hitting the upper barrier, (2) the conditional probability that the path stays within the corridor (b, a), and (3) the realized final path value. This allows to price derivatives that depend on both the first-passage time probabilities and the final asset value, for example, *corridor bonus certificates*.

(1) Repeat Steps (A)–(F) for each simulation run $k \in \{1, \ldots, K\}$[2], then continue with Step (2).

(A) Simulate the number of jumps within $[0, T]$ as $N_T \sim \mathrm{Poi}(\lambda T)$.

(B) Simulate the jump times $0 < t_1 < \cdots < t_{N_T} < T$. Conditional on N_T, these jumps are distributed as order statistics of i.i.d. Uni[0,T] random variables, see [44, p. 17].

[2] In the kth simulation run the quantities $\mathcal{BB}^+(k)$ (conditional probability of hitting the upper barrier) and $\mathcal{BB}(k)$ (conditional probability of surviving within the corridor (b, a)), and the final asset value $B_T(k)$ are sampled.

(C) Generate two independent series of random variables $b_1, \ldots, b_{N_T+1}$ and $y_1, \ldots, y_{N_T}$, independent of N_T:

$$b_i \sim \mathcal{N}\big(\mu(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\big) \quad \text{and} \quad y_i \sim \mathbb{P}_Y.$$

(D) Simulate the asset path on the grid of the jump times (set $t_{N_T+1} = T$):

$$B_{t_0} = 0, \quad B_{t_i-} = B_{t_{i-1}} + b_i, \quad \forall i \in \{1, \ldots, N_T + 1\},$$
$$B_{t_i} = B_{t_i-} + y_i, \quad \forall i \in \{1, \ldots, N_T\}.$$

(E) Compute the conditional barrier crossing probabilities between the grid points. Set $\mathcal{P}_0 := 1$ and repeat the following Steps (a)–(d) for each time step $i \in \{1, \ldots, N_T + 1\}$.

(a) Compute the probabilities

$$\mathcal{P}^+_{i-1,i} = BB^+_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-}),$$
$$\mathcal{P}_{i-1,i} = 1 - \mathcal{P}^+_{i-1,i} - BB^-_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-}),$$

and obtain $\mathcal{P}_i = \mathcal{P}_{i-1}\mathcal{P}_{i-1,i}$, $\mathcal{P}^+_i = \mathcal{P}^+_{i-1} + \mathcal{P}_{i-1}\,\mathcal{P}^+_{i-1,i}$.

(b) • If $B_{t_i-} \notin (b, a)$, set $\mathcal{BB}^+(k) = \mathcal{P}^+_i$, $\mathcal{BB}(k) = 0$ and go to (F).
• If $B_{t_i-} \in (b, a)$, continue with Step (c).

(c) If $i = N_T + 1$, set $\mathcal{BB}^+(k) = \mathcal{P}^+_{N_T+1}$, $\mathcal{BB}(k) = \mathcal{P}_{N_T+1}$ and go to (F), else continue with Step (d).

(d) Check whether a barrier crossing occurs due to a jump.
• If $B_{t_i} > a$, set $\mathcal{BB}^+(k) = \mathcal{P}_i + \mathcal{P}^+_i$, $\mathcal{BB}(k) = 0$ and go to (F).
• If $B_{t_i} < b$, set $\mathcal{BB}^+(k) = \mathcal{P}^+_i$, $\mathcal{BB}(k) = 0$ and go to (F).
• If $B_{t_i} \in (b, a)$, return to Step (E).

(F) Set $B_T(k) = B_{t_{N_T+1}-}$ and return to Step (1).

(2) Estimate the unconditional quantities in question via the sample mean of all conditional quantities over all runs, i.e.

$$\mathbb{P}(T^+_{ab} \le T) \cong \frac{1}{K}\sum_{k=1}^{K} \mathcal{BB}^+(k),$$

$$\mathbb{P}(T^-_{ab} \le T) \cong 1 - \frac{1}{K}\sum_{k=1}^{K} \mathcal{BB}^+(k) - \frac{1}{K}\sum_{k=1}^{K} \mathcal{BB}(k),$$

and return a $K \times 3$ matrix with rows $(\mathcal{BB}^+(k), \mathcal{BB}(k), B_T(k))$, where $k \in \{1, \ldots, K\}$.

Similar as in Algorithm 1, Algorithm 2 samples the number of jumps N_T in the interval $[0, T]$, the jump instants $0 < t_1 < \cdots < t_{N_T} < T$, and the process values immediately before and after the jump (Steps (1A)–(1D)). However, in contrast to this first algorithm, it additionally allows to evaluate expectations of the form $\mathbb{E}[w(\hat{T}_{ab}, B_T, \mathcal{E})]$, where $\mathcal{E} \in \{\oplus, \ominus, \emptyset\}$, that depend on the first-passage time.

Therefore, the inter-jump periods (t_{i-1}, t_i) are considered sequentially. In the ith period, the barrier crossing probabilities $v := BB^-_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-})$ and $w := BB^+_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-})$ are determined (Step (1E)(b)). A standard uniform random variable U is drawn that decides whether a barrier crossing has occurred in the interval (t_{i-1}, t_i). We use an importance sampling scheme for the explicit times of the first-passage. If the lower barrier is crossed ($U \leq v$; the upper barrier is treated similarly), the first-passage time $\hat{T}^-_{ab}$ is taken uniformly in the interval (t_{i-1}, t_i). Then, the uniform random variable $t_{i-1} < \hat{T}^-_{ab} < t_i$ is weighted according to its actual density, given by the fraction of the Brownian bridge first-passage time intensity $g^-_{ab}(T^-_{ab}, B_{t_{i-1}}, B_{t_i-})$ in the interval (t_{i-1}, t_i) (see Theorem 2.3) and v. This importance sampling weight is denoted by $p(\hat{T}^-_{ab})$. We know that

$$\int_{t_{i-1}}^{t_i} \frac{g^-_{ab}\big(t, B_{t_{i-1}}, B_{t_i-}\big)}{v}\, dt = 1 = \int_{t_{i-1}}^{t_i} \frac{1}{t_i - t_{i-1}}\, p(t)\, dt.$$

The two densities in the latter expression coincide, thus we can conclude that $p(\hat{T}^-_{ab}) = g^-_{ab}(\hat{T}^-_{ab}, B_{t_{i-1}}, B_{t_i-})(t_i - t_{i-1})/v$. We note that the importance sampling weight is on average 1, i.e. $\mathbb{E}[p(\hat{T}^-_{ab})] = 1$.

If there is no barrier crossing in the interval (t_{i-1}, t_i), we continue with the jump time t_i (Step (1E)(d)). If a crossing occurs due to a jump, the weight of this path is 1, its first-passage time is t_i. If, again, no crossing occurs, we increment i by one and repeat the whole procedure. This continues until either one of the barriers is crossed or time T is reached (Step (1E)(c)).

Note that the expected number of jumps until time T is λT. Hence, the average runtime increases about linearly in T.

Algorithm 2 (Brownian bridge technique 2). This algorithm evaluates expectations of the form $X(0) := \mathbb{E}[w(\hat{T}_{ab}, B_T, \mathcal{E})]$, where $\mathcal{E} \in \{\oplus, \ominus, \emptyset\}$, that depend on the first-passage times T_{ab}, T^+_{ab}, and T^-_{ab} and the final path value B_T. Therefore, the number of simulation runs K, the barriers a and b, and the parameters that describe the process B are required. To stress that the first-passage time is sampled by an importance sampling scheme, we use the notation $\hat{T}_{ab}$ instead of T_{ab}.

(1) Repeat Steps (A)–(E) for each simulation run $k \in \{1, \ldots, K\}$, then continue with Step (2).

(A)–(D) See Algorithm 1.

(E) Check whether a barrier crossing occurs continuously. Repeat Steps (a)–(d) for each time step $i \in \{1, \dots, N_T + 1\}$. The type of exit event is denoted by the random variable $\mathcal{E} \in \{\oplus, \ominus, \emptyset\}$. Possible exits include hitting the upper ($\mathcal{E} = \oplus$), respectively lower ($\mathcal{E} = \ominus$), barrier. In case that all sampled process values stay within the corridor (b, a), we set $\mathcal{E} = \emptyset$.

(a) Sample a random variable $U \sim \mathrm{Uni}[0, 1]$.

(b) Calculate

$$v := BB^{-}_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-}),$$
$$w := BB^{+}_{ab}(t_{i-1}, t_i, B_{t_{i-1}}, B_{t_i-}).$$

- If $U < v$, return

$$\mathcal{E}(k) = \ominus, \quad \hat{T}_{ab}(k) = t_{i-1} + (t_i - t_{i-1})U/v,$$
$$p(k, \hat{T}_{ab}(k)) = g^{-}_{ab}\big(\hat{T}_{ab}(k), B_{t_{i-1}}, B_{t_i-}\big)\,\frac{t_i - t_{i-1}}{v},$$

and return to Step (1).

- If $U > 1 - w$, return

$$\mathcal{E}(k) = \oplus, \quad \hat{T}_{ab}(k) = t_{i-1} + (t_i - t_{i-1})(1 - U)/w,$$
$$p(k, \hat{T}_{ab}(k)) = g^{+}_{ab}\big(\hat{T}_{ab}(k), B_{t_{i-1}}, B_{t_i-}\big)\frac{t_i - t_{i-1}}{w},$$

and return to Step (1). Else, continue with Step (c).

(c) If $i = N_T + 1$, set $\mathcal{E}(k) = \emptyset$, $(\hat{T}_{ab}(k), p(k, \hat{T}_{ab}(k))) = (T, 1)$, and return to Step (1), else continue with Step (d).

(d) Check whether a barrier crossing occurs due to a jump:

- If $B_{t_i} > a$, set $\mathcal{E}(k) = \oplus$, $(\hat{T}_{ab}(k), p(k, \hat{T}_{ab}(k))) = (t_i, 1)$, and return to Step (1).
- If $B_{t_i} < b$, set $\mathcal{E}(k) = \ominus$, $(\hat{T}_{ab}(k), p(k, \hat{T}_{ab}(k))) = (t_i, 1)$, and return to Step (1).
- If $B_{t_i} \in (b, a)$, return to Step (E).

(2) Set $B_T(k) = B_{t_{N_T+1}-}$, and compute the estimate

$$\mathrm{X}(0) \cong \frac{1}{K}\sum_{k=1}^{K} p(k, \hat{T}_{ab}(k))\; w\big(\hat{T}_{ab}(k), B_T(k), \mathcal{E}(k)\big),$$

where $w(\hat{T}_{ab}(k), B_T(k), \mathcal{E}(k))$ is the quantity that needs to be estimated conditional on the sampled quantities.

Remark 3.1 (Generalization to stochastic volatility). It is possible to generalize the algorithms to jump-diffusion models in the sense of [14], where the model parameters change depending on a time-homogeneous Markov chain Z with finitely many states. The process B is then given by

$$dB_t = \mu_{Z_t} dt + \sigma_{Z_t} dW_t + Y_i \, dN_t^{Z_t}, \quad B_0 = 0,$$

where $Z = \{Z_t\}_{t \geq 0} \in \{1, 2, \dots, \#\text{states}\}$ is a time-homogeneous Markov chain and $W = \{W_t\}_{t \geq 0}$ a standard Brownian motion. The parameters now depend on Z: Drift $\mu_{Z_t} \in \mathbb{R}$, volatility $\sigma_{Z_t} > 0$, and time in-homogeneous counting process $N^{Z_t} = \{N_t^{Z_t}\}_{t \geq 0}$, a Poisson process with intensity $\lambda_{Z_t} \geq 0$ at time t. The initial value is B_0; the jump-size process $Y = \{Y_i\}_{i \geq 1}$ is i.i.d. with distribution $\mathbb{P}_Y$. All processes are mutually independent.

To modify Algorithms 1 and 2, in Step (1B) not only the jump times but also the times of changes in the Markov chain Z have to be simulated. However, due to the non-homogeneity of N^{Z_t}, they have to be simulated iteratively. Conditional on those times, the algorithms can proceed as before if the jump size at state changes is set to zero.

4 Applications

In the following, we consider financial applications of the presented algorithms. Up to now, we have worked with first-passage times of Brownian motion with jumps. This process, however, allows for negative values and is thus not a reasonable model for a stock price process. We assume that there exists a risk-neutral measure $\mathbb{Q}$ and define for $t \geq 0$ the stock price process under $\mathbb{Q}$ as the exponential of a jump-diffusion process (with jumps having finite first exponential moment), i.e.

$$S_t := S_0 \exp(B_t) = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2 - \delta\right)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right),$$

where r denotes the riskless interest rate and $\delta := \lambda(\mathbb{E}_{\mathbb{Q}}[\exp(Y_1)] - 1)$ the drift adjustment due to the jumps. The other parameters and processes are defined as in Section 2.

Example 4.1 (Double-exponential jump-diffusion). One popular possibility to model jump sizes is the double-exponential distribution, see, e.g., [32]:

$$\mathbb{P}_Y(dx) = p \, \alpha_{\oplus} e^{-\alpha_{\oplus} x} \mathbb{1}_{\{x \geq 0\}} dx + (1 - p) \, \alpha_{\ominus} e^{\alpha_{\ominus} x} \mathbb{1}_{\{x < 0\}} dx, \tag{4.1}$$

where $0 \leq p \leq 1$ is the probability that a jump has positive sign. Upward jumps are exponentially distributed with parameter $\alpha_\oplus > 1$, downward jumps are exponentially distributed with parameter $\alpha_\ominus > 0$. In this model

$$\delta := \lambda\big(\mathbb{E}_{\mathbb{Q}}[\exp(Y_1)] - 1\big) = \lambda\bigg(\frac{p\alpha_\oplus}{\alpha_\oplus - 1} + \frac{(1-p)\alpha_\ominus}{\alpha_\ominus + 1} - 1\bigg).$$

The payoff of the products we are pricing depends on whether or not a continuously monitored, constant lower barrier $D < S_0$, or an upper barrier $P > S_0$, is hit until maturity T of the contract. Exploiting the strict monotonicity of the natural logarithm, we find that

$$\begin{aligned}\mathbb{P}\big(S_t \in (D,P),\ \forall t \in [0,T]\big) &= \mathbb{P}\big(B_t \in \big(\ln(D/S_0), \ln(P/S_0)\big),\ \forall t \in [0,T]\big)\\ &= \mathbb{P}\big(T_{ab} > T\big),\end{aligned}$$

where $a := \ln(P/S_0)$, $b := \ln(D/S_0)$, and $B_t := \ln(S_t/S_0)$ for all $t \geq 0$. Thus, we can express the first-passage times of the process S by the first-passage times of a Brownian motion with jumps with modified barriers.

In the current section, different applications for the Brownian bridge algorithms are analyzed. First, Section 4.1 prices *digital first-touch options*, then we consider *step double barrier options* in Section 4.2, and *corridor bonus certificates* in Section 4.3. Section 4.4 examines an application to credit risk.

4.1 Digital first-touch options

In this section, we deal with *(knock-in) digital first-touch options*, the corresponding knock-out options can be priced similarly. Options of this form provide a building block for more complex derivatives; for more details see, e.g., [3]. According to [5], *digital first-touch options* are the most liquid and actively traded exotic options on FX markets.

Digital first-touch options pay \$1 at maturity T if the stock price crosses (a) specific threshold(s). The owner of an *up-and-in* (respectively *down-and-in*) contract receives \$1 if the upper barrier P (respectively lower barrier D) is hit first. If the underlying remains within the corridor (D, P), the option expires worthless.

Under the risk-neutral measure $\mathbb{Q}$, the *up-and-in* option can be priced as

$$X^+(0) := e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{T^+_{ab} \leq T\}}\big] = e^{-rT}\,\mathbb{Q}(T^+_{ab} \leq T),$$

where $a := \ln(P/S_0)$, $b := \ln(D/S_0)$. Similarly, the price of a *down-and-in* option is given by

$$X^-(0) := e^{-rT}\,\mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{T^-_{ab} \leq T\}}\big] = e^{-rT}\,\mathbb{Q}(T^-_{ab} \leq T).$$

Example 4.2 (Digital first-touch options). Table 2 compares the prices of *(knock-in) digital first-touch options* in a double-exponential jump-diffusion model obtained by the standard Monte-Carlo simulation on a discrete grid to the Brownian bridge algorithm (Algorithm 1). The parameters considered in the simulations are: $r = 5\%$, $\sigma = 20\%$, $p = 0.5$, $\alpha_{\oplus} = \alpha_{\ominus} = 5$, and $T = 1$ (expiration date of the contract). Furthermore, we set $D = 80$, $P = 120$, and $S_0 = 100$.

In the standard Monte-Carlo algorithm, we use 250, respectively 1000, discretization steps. According to the expected number of jumps per year λ, we consider the scenarios "Black–Scholes" ($\lambda = 0$), "Low" ($\lambda = 0.5$), "Middle" ($\lambda = 2$), and "High" ($\lambda = 8$). We chose double-exponential jump diffusions as this allows us to compare the results to the Laplace transforms of the first-passage time as presented in, e.g., [32, 46]. Note that it is very easy to change the jump-size distribution in a Brownian bridge algorithm, whereas it often requires new theoretical results if one is interested in analytical solutions for the Laplace transform of the first-passage times under alternative jump size specifications.

From Table 2, we conclude that the Brownian bridge algorithm is significantly faster than the brute–force Monte-Carlo simulation on a discrete grid. Furthermore, the Brownian bridge algorithm is unbiased and thus leads to prices that are close to the exact prices by [46]. There are ways to further accelerate this algorithm: First, a parallelization of Monte-Carlo simulations can very easily be implemented. Secondly, there are many variance reduction techniques to further (significantly) accelerate the algorithm (see, e.g., [10, 31, 42] on the single barrier algorithm).

Figure 2 further examines the discretization bias of the two algorithms. Note that the discretization bias in the standard Monte-Carlo simulation is still considerably high even for 1000 discretization steps ($\triangle t = 1\text{e}{-03}$). The Brownian bridge technique returns unbiased price estimates.

4.2 (Step) Double barrier options

Double barrier options are very popular OTC derivatives and are frequently embedded in a variety of structured products. The holder receives the payoff of a standard call or put option if the stock price stays within a corridor (D, P) over the lifetime of the option (and receives 0 otherwise). For an investor, this standard contract is often not suitable: Following his risk aversion, his hedging requirements, or his market views, the investor might want to change the barrier level over time. For instance, he/she might want to widen the corridor (D, P) over time (a contract often referred to as "expanding tunnel"). A large range of exotic products aim at providing this additional flexibility, i.e. step double barrier options (the

		Black–Scholes	Jump-diffusion		
		$\lambda = 0$	Low ($\lambda = 0.5$)	Middle ($\lambda = 2$)	High ($\lambda = 8$)
StdMC (250)	$\hat{X}^+(0)$	0.3734 ± 0.0008	0.3765 ± 0.0008	0.3836 ± 0.0008	0.3815 ± 0.0008
	relative bias	4.6%	3.8%	1.7%	0.3%
	runtime	47.2s	46.1s	45.0s	41.7s
StdMC (1000)	$\hat{X}^+(0)$	0.3820 ± 0.0003	0.3838 ± 0.0003	0.3880 ± 0.0008	0.3825 ± 0.0004
	relative bias	2.2%	1.8%	1.3%	0.2%
	runtime	185.6s	183.8s	179.4s	161.7s
Brownian bridge	$\hat{X}^+(0)$	0.3907 ± 0.0002	0.3915 ± 0.0002	0.3928 ± 0.0003	0.3821 ± 0.0003
	relative bias	0.0%	0.0%	0.0%	0.0%
	runtime	18.1s	19.3s	14.3s	30.1s
Exact price	$X^+(0)$	0.3908	0.3913	0.3928	0.3822

Table 2. Prices $X^+(0) = e^{-rT}\,\mathbb{Q}(T_{ab}^+ \leq T)$ and confidence intervals at the confidence level $\alpha = 90\%$ of *(upper barrier) digital first-touch options* for different jump intensities λ. We compare the standard Monte-Carlo simulation on a discrete grid with mesh 1/250, respectively 1/1000, to the Brownian bridge algorithm (Algorithm 1) using $K = 10^6$ simulation runs. The exact value of the option was estimated by inverting the Laplace transforms presented by, e.g., [46]. The table additionally presents the bias and runtime for each algorithm. The relative bias is the relative difference between the expected simulated value $\mathbb{E}[\hat{X}^+(0)]$ divided by the true value $X^+(0)$ of the option. The computation time was calculated using Matlab 2012a on a 3.1 GHz PC.

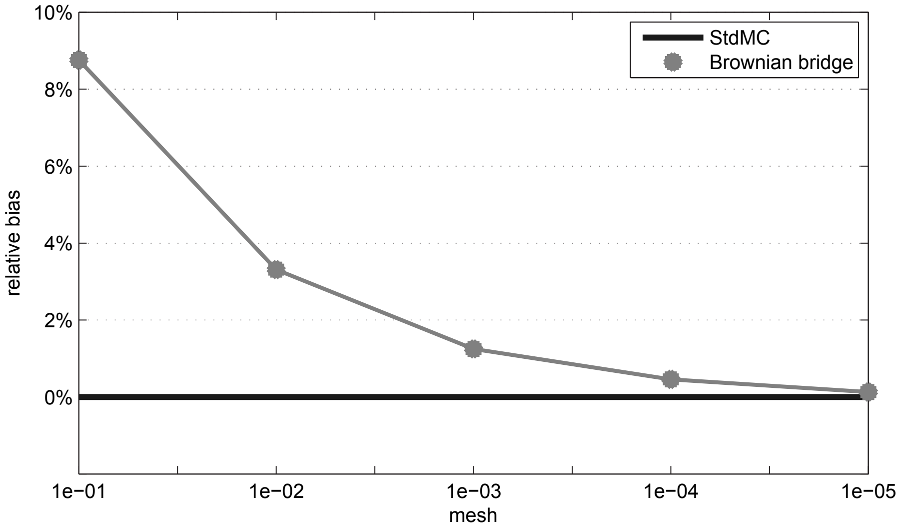

Figure 2. Bias of *(upper barrier) digital first-touch option* prices. We compare the standard Monte-Carlo simulation on a discrete grid with mesh $\triangle t \in \{$1e−01, 1e−02, 1e−03, 1e−04, 1e−05$\}$ to the Brownian bridge algorithm (Algorithm 1) using $K = 10^6$ simulation runs. The jump intensity is $\lambda = 2$; the same parameters as in Table 2 are used. The relative bias is the relative difference between the expected simulated value $\mathbb{E}[\hat{X}^+(0)]$ divided by the true value $X^+(0)$ of the option.

barriers are piecewise constant) or window double barrier options (the barriers can only be observed during certain time intervals). Many different names were created for those type of contracts, i.e. "hot-dog-option", "wedding cakes", or "onion options", for a more detailed review of traded contracts, we refer to [21] and [22]. Analytical pricing formulas for such products tend to become extremely complicated (this is already true for simple examples in the Black–Scholes model, see, e.g., the Appendix of [22]) and are not flexible enough to adapt to many different payoff streams or underlying specifications.

In the following, we introduce step double barrier options. For certain observation points $t_0 = 0 < t_1 < \cdots < t_n = T$, the barrier of an n-step double barrier option is constant over the intervals $[t_{i-1}, t_i]$, for $i = 1, 2, \ldots, n$. Denoting those ranges by $[D_i, P_i]$, a step double barrier option has the same payoff as a standard call or put if the underlying asset has stayed within the thresholds (D_i, P_i) until the maturity of the contract.

Example 4.3 (Step double barrier options). We will consider a two-step double barrier option, i.e. an "expanding tunnel" with $t_1 = 1$, $[D_1, P_1] = [60, 140]$ and

$\alpha_\oplus = \alpha_\ominus = 5$		Black–Scholes $\lambda = 0$	Jump-diffusion Low ($\lambda = 0.5$)	Middle ($\lambda = 2$)	High ($\lambda = 8$)
StdMC (250)	$\hat{X}^+(0)$	21.17 ± 0.02	16.00 ± 0.02	7.36 ± 0.01	0.48 ± 0.05
	relative bias	3.3%	3.9%	5.1%	9.6%
	runtime	43.0s	42.4s	40.0s	36.1s
StdMC (1000)	$\hat{X}^+(0)$	20.85 ± 0.01	15.72 ± 0.01	7.20 ± 0.01	0.47 ± 0.04
	relative bias	1.1%	1.9%	2.9%	4.7%
	runtime	171.1s	169.2s	159.5s	144.2s
Brownian bridge	$\hat{X}^+(0)$	20.50 ± 0.01	15.42 ± 0.01	7.03 ± 0.01	0.43 ± 0.00
	relative bias	0.0%	0.0%	0.0%	0.0%
	runtime	27.4s	30.7s	33.5s	32.3s
Exact price	$X^+(0)$	20.49	15.42	7.02	0.43

Table 3. Prices and confidence intervals at the confidence level $\alpha = 90\%$ of *two-step double barrier options* for different jump intensities λ. We compare the standard Monte-Carlo simulation on a discrete grid with mesh 1/250, respectively 1/1000, to the Brownian bridge algorithm (Algorithm 1) using $K = 10^6$ simulation runs. The exact value of the option is considered to be the one obtained by $K = 10^8$ simulation runs using the Brownian bridge pricing algorithm. The table additionally presents the bias and runtime for each algorithm. The bias is the (relative) difference between the expected simulated value $\mathbb{E}[\hat{X}^+(0)]$ and the true value $X^+(0)$ of the option. The computation time was calculated using Matlab 2012a on a 3.1 GHz PC.

$[D_2, P_2] = [50, 150]$. The strike price at $T = t_2 = 2$ is $K = 70$. The other parameters are chosen as in Example 4.2, i.e. $r = 5\%$, $\sigma = 20\%$, and $S_0 = 100$.

Again, we use a double-exponential jump-diffusion model with $p = 0.5$ and $\alpha_{\oplus} = \alpha_{\ominus} = 5$. According to the expected number of jumps per year λ, we consider the scenarios "Black–Scholes" ($\lambda = 0$), "Low" ($\lambda = 0.5$), "Middle" ($\lambda = 2$), and "High" ($\lambda = 8$).

Table 3 compares the prices of *two-step double barrier options* obtained by the standard Monte-Carlo simulation on a discrete grid and the Brownian bridge algorithm (Algorithm 1). Pricing this type of exotic options analytically tends – even for the case of a double-exponential jump size distribution – to be very tedious and also computationally challenging. In the Black–Scholes model, closed-form prices for *two-step double barrier options* are available (see the Appendix of [22]). The advantage of both Monte-Carlo algorithms presented in Table 3 is the fact that they are very flexible, easy to implement, and can easily be adapted to different payoff streams or jump-size distributions. The Brownian bridge algorithm is moreover both unbiased and faster than the standard Monte-Carlo simulation.

4.3 Corridor bonus certificates

Corridor bonus certificates provide the largest payoff if the stock price stays within a given corridor (D, P) during the lifetime of the contract and thus offer the possibility to bet on sideways markets. On different underlying assets, they are emitted by all major banks.[3]

The payoff of a *corridor bonus certificate* depends on the market value of the underlying at the expiration date T, the first-passage time events on the two barriers $D < S_0 < P$, and on an initially specified "fixed amount" F. One distinguishes the following two cases:

- If the stock prices stays within the corridor (D, P), then the owner of the certificate receives the fixed amount F at maturity T.
- If the stock price reaches one of the barriers, the payoff depends on the stock price S_T at the expiration date of the certificate.

More precisely, we get the following payoff:

$$\text{payoff}(T) = \begin{cases} F & \text{if } T_{ab} > T, \\ \min(S_T, F) & \text{if } T_{ab}^{-} \leq T, \\ \max\big(\min(2S_0 - S_T, F), 0\big) & \text{if } T_{ab}^{+} \leq T, \end{cases}$$

[3] Note that there exist different kinds of *corridor bonus certificates* with slightly different payoff structures. As an example, Société Générale emitted several certificates (e.g. ISIN: DE000SG12BS9, maturity 01/04/2013).

where $a := \ln(P/S_0)$ and $b = \ln(D/S_0)$. Note that if the upper barrier is hit, the certificate converts into a short position in the underlying S. In most standard contract specifications the resulting loss is bounded by the initial investment. Under the risk-neutral measure $\mathbb{Q}$, the price of *corridor bonus certificates* conditional on $\mathbb{P}(T_{ab}^- \le T)$, $\mathbb{P}(T_{ab}^+ \le T)$, and S_T is given by

$$\begin{aligned} B^+(0) &:= e^{-rT}\, \mathbb{E}_{\mathbb{Q}}\big[\text{payoff}(T)\big] \\ &= e^{-rT}\, \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\big[\text{payoff}(T) \,\big|\, T_{ab}^-,\, T_{ab}^+,\, S_T\big]\Big] \\ &= e^{-rT}\, \mathbb{E}_{\mathbb{Q}}\Big[F\big(1 - \mathbb{1}_{\{T_{ab}^+ \le T\}} - \mathbb{1}_{\{T_{ab}^- \le T\}}\big) \\ &\qquad\qquad + \mathbb{1}_{\{T_{ab}^+ \le T\}} \max(\min(2S_0 - S_T, F), 0) \\ &\qquad\qquad + \mathbb{1}_{\{T_{ab}^- \le T\}} \min(S_T, F)\Big]. \end{aligned}$$

This can be estimated using the triplets $(\mathcal{BB}^+(k), \mathcal{BB}(k), B_T(k))$ from Algorithm 1.

Example 4.4 (Corridor bonus certificates). Table 4 estimates the prices of *corridor bonus certificates* using Algorithm 1. The parameters used in the simulations are: $r = 1\%$, $\sigma = 10\%$, $p = 0.5$, and $T = 1$ for different values of $\alpha_\oplus$ and $\alpha_\ominus$.

	Black–Scholes	Jump-diffusion		
	$\lambda = 0$	Low ($\lambda = 0.5$)	Middle ($\lambda = 2$)	High ($\lambda = 8$)
$\alpha_\oplus = \alpha_\ominus = 10$	118.75 ± 0.00	116.88 ± 0.02	109.84 ± 0.04	82.46 ± 0.07
$\alpha_\oplus = 2\alpha_\ominus = 20$	118.75 ± 0.00	118.05 ± 0.01	114.01 ± 0.03	89.68 ± 0.07
$\alpha_\oplus = \alpha_\ominus/2 = 10$	118.75 ± 0.00	117.28 ± 0.02	112.47 ± 0.04	91.06 ± 0.07

Table 4. Prices $B^+(0)$ of *corridor bonus certificates* using Algorithm 1 for different jump size distributions (parameters $\alpha_\oplus$ and $\alpha_\ominus$) and jump size intensities λ. We choose $S_0 = 100$, $P = 140$, $D = 60$, $r = 1\%$, $T = 1$, $\sigma = 10\%$, and $F = 120$. The results have been obtained in a simulation with $K = 10^6$ trajectories.

4.4 Credit risk

Another frequent application of first-passage times are structural credit risk models. The idea behind those models is to define default as a consequence of insufficient asset values, with the result that bonds can be priced as an option on the company's assets S. One considers two possibilities that the bond spread payments cease: First, the company defaults as soon as S falls below some prespecified level $D < S_0$. Second, due to, for example, the company's desire to upgrade its credit rating or due to an initial public offering (IPO), the bond might be repaid earlier. This event is triggered as soon as S crosses an upper barrier $P > S_0$, see, e.g., [11, 13, 17].

The price of a defaultable bond with nominal value 1 and maturity T that pays (continuous) interest at a rate d and can (at any time) be recalled by the bond holder is then priced as

$$\text{Bond Price} = \text{Nominal} + \text{Interest} \tag{4.2}$$

with

$$\begin{aligned}
\text{Nominal} &= \exp(-rT)\,\mathbb{Q}(T_{ab} > T) \\
&\qquad + \mathbb{E}_{\mathbb{Q}}\Big[R\,\exp(-rT_{ab}^{-})\,\mathbb{1}_{\{T_{ab}^{-}\le T\}} + \exp(-rT_{ab}^{+})\,\mathbb{1}_{\{T_{ab}^{+}\le T\}}\Big], \\
\text{Interest} &= \mathbb{E}_{\mathbb{Q}}\Big[\int_0^{\min(T_{ab},T)} d\,\exp(-rt)\,dt\Big] \\
&= \frac{d}{r}\big(1-\exp(-rT)\big)\,\mathbb{Q}(T_{ab} > T) \\
&\qquad + \underbrace{\mathbb{E}_{\mathbb{Q}}\Big[\int_0^{T_{ab}^{+}} d\,\exp(-rt)\,dt\,\mathbb{1}_{\{T_{ab}^{+}\le T\}}\Big]}_{\text{early repayment in } T_{ab}^{+}} \\
&\qquad + \underbrace{\mathbb{E}_{\mathbb{Q}}\Big[\int_0^{T_{ab}^{-}} d\,\exp(-rt)\,dt\,\mathbb{1}_{\{T_{ab}^{-}\le T\}}\Big]}_{\text{default in } T_{ab}^{-}},
\end{aligned}$$

where r is the riskless interest rate, $a := \ln(P/S_0)$, $b := \ln(D/S_0)$, and $R \in [0,1]$ the recovery rate in case of default.

This price depends on the first-passage times T_{ab}^{+}, T_{ab}^{-} and can thus be computed using Algorithm 2. We obtain bond prices conditional on the time t and type $\mathcal{E} \in \{\oplus, \ominus, \emptyset\}$ of the exit

$$\begin{aligned}
w(t,x,\ominus) &= \frac{d}{r} + \left(R - \frac{d}{r}\right)\exp(-rt), \\
w(t,x,\emptyset) = w(t,x,\oplus) &= \frac{d}{r} + \left(1 - \frac{d}{r}\right)\exp(-rt).
\end{aligned}$$

5 Implementation

This section discusses the implementation of the infinite series in Theorems 2.1 and 2.2. Depending on $\kappa := \sigma\sqrt{(t_i - t_{i-1})/2}\exp(b-a)$, a quantity that can be interpreted as a measure of the distance to default (see [45]), Figure 3 displays the (logarithmic) absolute error of the infinite series in $BB^{+}_{ab}(\cdot)$ (Theorem 2.1, equation (2.2)) if truncated after $N \in \mathbb{N}$ terms. Our numerical results for a large number of different parameters suggest that the required values for N are extremely small, even for extraordinarily high values of κ.

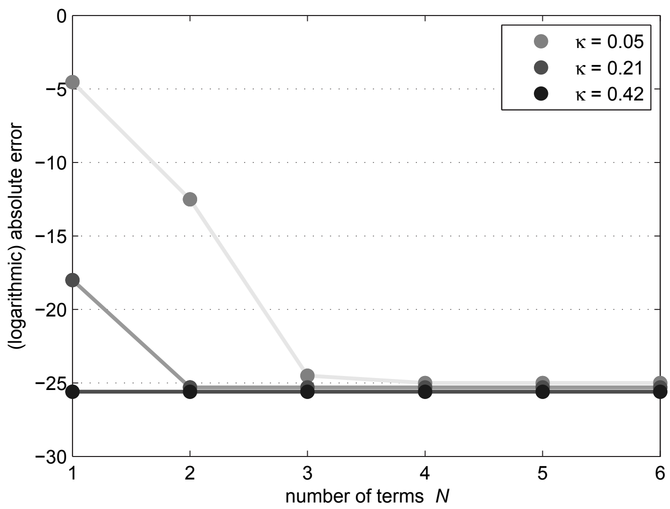

Figure 3. The (logarithmic) absolute error of the Brownian bridge probability $BB^{+}_{ab}(\cdot)$ on two barriers is computed using N terms in the infinite sum given in Theorem 2.1, equation (2.2). Different values for the distance to default $\kappa := \sigma\sqrt{(t_i - t_{i-1})/2}\exp(b-a)$ are used.

In Theorem 2.2, two different representations of the first-passage time distribution are given. The answer to the question "which representation is numerically more efficient?" depends on the parameters. The paper [45] claims that representation (a) is more suitable for small values of the distance to default κ whereas representation (b) might be preferred if κ is large. We confirm those results in Figure 4: For $\kappa = 1.00$ (left), representation (b) is obviously a better choice, for $\kappa = 0.25$ (right), representation (a) is more accurate. The convergence rate of the infinite series is extremely fast: For most parameters, $N \leq 5$ terms lead to a precision of at least $\epsilon = 1\mathrm{e}{-08}$. Error bounds for (a) are to be found in [45]; for representation (b) in [26].

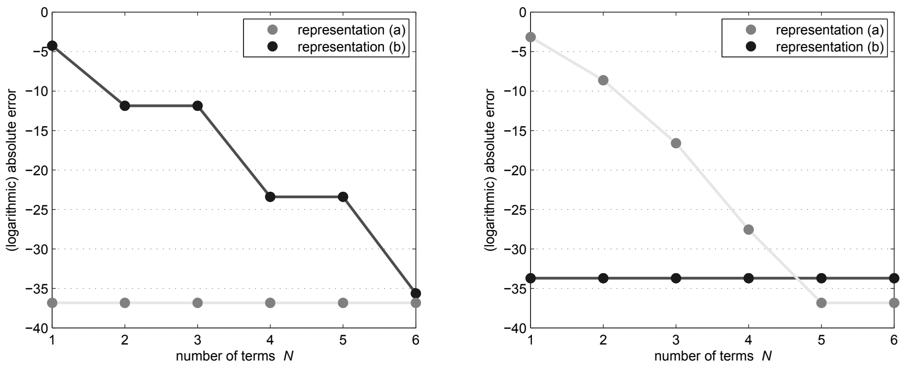

Figure 4. The (logarithmic) absolute error of the first-passage time probability on two barriers is computed using N terms in the infinite sum given in Theorem 2.2. The two representations (a) and (b) are compared using two different values of $\kappa := \sigma\sqrt{(t_i - t_{i-1})/2}\exp(b-a)$: $\kappa = 1.00$ (left; $\mu = 10\%$, $\sigma = 40\%$, $t_i - t_{i-1} = 1$, $a = \ln(1.2)$, $b = \ln(0.8)$, $B_0 = 0$), $\kappa = 0.25$ (right; $\mu = 10\%$, $\sigma = 10\%$, $t_i - t_{i-1} = 1$, $a = \ln(1.2)$, $b = \ln(0.8)$, $B_0 = 0$).

6 Conclusion

We showed how the original single barrier Brownian bridge algorithm by [36] can be extended to price many exotic double barrier derivatives. Standard Monte-Carlo simulations on a discrete grid lead to a discretization bias even for fine grids. In contrast, the Brownian bridge algorithms are unbiased and significantly faster. One big advantage of Monte-Carlo simulations is the fact that they are easy to understand and implement and they are very flexible. It is straightforward to change, e.g., the jump size distribution or to include a stochastic volatility (in contrast, most analytical approaches often need a completely new theory to account for such changes). Furthermore, payoff streams can easily be adapted to, e.g., piecewise constant or partially monitored barriers (e.g., to price *step* and *window double barrier options*). The flexibility of non-constant barriers allows to adjust the derivative's payoff to the investor's hedging needs and speculative views. A further expansion of those type of contracts on the markets is contingent on the fact that fast and reliable pricing techniques are available.

A Appendix

Proof of Theorem 2.2. In this proof, we will provide a link between the Brownian bridge probabilities (Theorem 2.1), the first-passage time densities, and the first-passage time probabilities in the two representations (a) and (b).

The integration over x_i yields the (unconditional) default probabilities presented in, e.g., [35]. We obtain[4]

$$\begin{aligned}
&\mathbb{P}(t_{i-1} < T_{ab}^{+} \le t_i) \\
&\quad = \mathbb{P}\big(a > x_i, t_{i-1} < T_{ab}^{+} < t_i\big) + \mathbb{P}\big(x_i > a, t_{i-1} < T_{ab}^{+} < t_i\big) \\
&\quad = \mathbb{P}\big(a > x_i, t_{i-1} < T_{ab}^{+} < t_i\big) \\
&\qquad\qquad + \exp\bigg(\frac{2\mu(a - x_{i-1})}{\sigma^2}\bigg)\tilde{\mathbb{P}}\big(a > x_i, t_{i-1} < T_{ab}^{+} < t_i\big) \\
&\quad = \int_{-\infty}^{a} BB_{ab}^{+}(t_{i-1}, t_i, x_{i-1}, x_i)\Big[\varphi\big(x_i - x_{i-1}; \mu(t_i - t_{i-1}), \sigma\sqrt{t_i - t_{i-1}}\big) \\
&\qquad\qquad\qquad + \exp\bigg(\frac{2\mu(a - x_{i-1})}{\sigma^2}\bigg) \\
&\qquad\qquad\qquad\qquad \cdot \varphi\big(x_i - x_{i-1}; -\mu(t_i - t_{i-1}), \sigma\sqrt{t_i - t_{i-1}}\big)\Big]dx_i,
\end{aligned}$$

where $\varphi(x; \mu, \sigma)$ denotes the density function of a normal distribution with mean μ and variance σ^2 and $BB_{ab}^{+}(t_{i-1}, t_i, x_{i-1}, x_i)$ is given in Theorem 2.1. To get

[4] The second equality holds by applying the reflection principle. We find that

$$\begin{aligned}
&\mathbb{P}\big(x_i > a, t_{i-1} < T_{ab}^{+} < t_i\big) \\
&\quad = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{1}_{\{x_i > a, t_{i-1} < T_{ab}^{+} < t_i\}}\Big] \\
&\quad = \mathbb{E}_{R}\Big[\exp\Big(\frac{\mu}{\sigma^2}(x_i - x_{i-1}) - \frac{1}{2}\frac{\mu^2}{\sigma^2}(t_i - t_{i-1})\Big)\,\mathbb{1}_{\{x_i > a, t_{i-1} < T_{ab}^{+} < t_i\}}\Big] \\
&\quad = \exp\bigg(\frac{2\mu(a - x_{i-1})}{\sigma^2}\bigg)\mathbb{E}_{R}\bigg[\exp\bigg(-\frac{\mu}{\sigma^2}(\tilde{x}_i - x_{i-1}) - \frac{1}{2}\frac{\mu^2}{\sigma^2}(t_i - t_{i-1})\bigg) \\
&\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{1}_{\{a > \tilde{x}_i, t_{i-1} < T_{ab}^{+} < t_i\}}\bigg] \\
&\quad = \exp\bigg(\frac{2\mu(a - x_{i-1})}{\sigma^2}\bigg)\tilde{\mathbb{P}}\big(a > x_i, t_{i-1} < T_{ab}^{+} < t_i\big),
\end{aligned}$$

where we used the change of measure

$$\frac{d\mathbb{P}}{dR} = \exp\bigg(\frac{(x_i - x_{i-1})\mu}{\sigma^2} - \frac{\mu^2(t_i - t_{i-1})}{2\sigma^2}\bigg),$$

respectively

$$\frac{d\tilde{\mathbb{P}}}{dR} = \exp\bigg(-\frac{(x_i - x_{i-1})\mu}{\sigma^2} - \frac{\mu^2(t_i - t_{i-1})}{2\sigma^2}\bigg),$$

and a reflection at the barrier a.

closed-form expressions for the integrals, one can show by a lengthy, but straightforward, calculation that

$$\begin{aligned}
&\int_g^h BB_{ab}^+(t_{i-1},t_i,x_{i-1},x_i)\varphi\big(x_i - x_{i-1};\mu(t_i-t_{i-1}),\sigma\sqrt{t_i-t_{i-1}}\big)\,dx_i \\
&\quad= \sum_{n=1}^{\infty}\Bigg[\exp\bigg(\frac{\mu}{\sigma^2}(-2x_{i-1}+k_n+2b)\bigg) \\
&\qquad\qquad\cdot\Phi\bigg(\frac{x_i+x_{i-1}-\mu(t_i-t_{i-1})-k_n-2b}{\sigma\sqrt{t_i-t_{i-1}}}\bigg) \\
&\qquad\qquad - \exp\bigg(\frac{\mu k_n}{\sigma^2}\bigg)\Phi\bigg(\frac{x_i-x_{i-1}-\mu(t_i-t_{i-1})-k_n}{\sigma\sqrt{t_i-t_{i-1}}}\bigg)\Bigg]_g^h,
\end{aligned}$$

where $k_n := 2(a-b)$.

In the following, we restrict ourselves to the case $\mu > 0$.[5] We obtain [35, equation (29)], i.e.

$$\begin{aligned}
&\mathbb{P}(t_{i-1} < T_{ab}^+ < t_i) \\
&= \sum_{n=1}^{\infty}\Bigg[\exp\bigg(\frac{\mu}{\sigma^2}(-2x_{i-1}+k_n+2b)\bigg)\Phi\bigg(\frac{x_{i-1}-\mu(t_i-t_{i-1})-k_{n-1}-a}{\sigma\sqrt{t_i-t_{i-1}}}\bigg) \\
&\qquad - \exp\bigg(\frac{\mu}{\sigma^2}k_n\bigg)\Phi\bigg(\frac{-x_{i-1}-\mu(t_i-t_{i-1})-k_n+a}{\sigma\sqrt{t_i-t_{i-1}}}\bigg)\Bigg] \\
&\quad + \sum_{n=1}^{\infty}\Bigg[\exp\bigg(-\frac{\mu k_{n-1}}{\sigma^2}\bigg)\Phi\bigg(\frac{x_{i-1}+\mu(t_i-t_{i-1})-k_{n-1}-a}{\sigma\sqrt{t_i-t_{i-1}}}\bigg) \\
&\qquad - \exp\bigg(-\frac{\mu}{\sigma^2}(2x_{i-1}+k_n-2a)\bigg) \\
&\qquad\qquad \cdot\Phi\bigg(\frac{-x_{i-1}+\mu(t_i-t_{i-1})-k_n+a}{\sigma\sqrt{t_i-t_{i-1}}}\bigg)\Bigg] \qquad \text{(A.1)}
\end{aligned}$$

[5] This assumption is needed to apply the geometric series in equation (A.2). The case $\mu < 0$ can be treated similarly if one substitutes n by $-n+1$ in equation (A.1) and sets

$$\begin{aligned}
&\sum_{n=-\infty}^{0}\exp\Big(-\frac{\mu}{\sigma^2}(2x_{i-1}-k_n+2a)\Big) - \sum_{n=1}^{\infty}\exp\Big(\frac{\mu k_n}{\sigma^2}\Big) \\
&\quad= \frac{\exp\big(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\big)-1}{\exp\big(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\big)-\exp\big(-\frac{2\mu(a-x_{i-1})}{\sigma^2}\big)}.
\end{aligned}$$

After resubstituting n by $-n+1$ the same final result is obtained.

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \exp\Big(\frac{\mu}{\sigma^2}(-2x_{i-1} + k_n + 2b)\Big)\Phi\Big(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - k_{n-1} - a}{\sigma\sqrt{t_i - t_{i-1}}}\Big) \\
&\quad - \sum_{n=-\infty}^{0} \exp\Big(-\frac{\mu k_{n-1}}{\sigma^2}\Big)\Phi\Big(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + k_{n-1} + a}{\sigma\sqrt{t_i - t_{i-1}}}\Big) \\
&\quad + \sum_{n=1}^{\infty} \exp\Big(-\frac{\mu k_{n-1}}{\sigma^2}\Big)\Phi\Big(-\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + k_{n-1} + a}{\sigma\sqrt{t_i - t_{i-1}}}\Big) \\
&\quad - \sum_{n=-\infty}^{0} \exp\Big(\frac{\mu}{\sigma^2}(-2x_{i-1} + k_n + 2b)\Big) \\
&\qquad\qquad \cdot \Phi\Big(-\frac{x_{i-1} - \mu(t_i - t_{i-1}) - k_{n-1} - a}{\sigma\sqrt{t_i - t_{i-1}}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \exp\Big(-\frac{\mu k_{n-1}}{\sigma^2}\Big) - \sum_{n=-\infty}^{0} \exp\Big(\frac{\mu}{\sigma^2}(-2x_{i-1} + k_n + 2b)\Big) \\
&\quad + \sum_{n=-\infty}^{\infty} \Big[\exp\Big(\frac{\mu}{\sigma^2}(-2x_{i-1} + k_n + 2b)\Big) \\
&\qquad\qquad \cdot \Phi\Big(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - k_{n-1} - a}{\sigma\sqrt{t_i - t_{i-1}}}\Big) \\
&\qquad - \exp\Big(-\frac{\mu k_{n-1}}{\sigma^2}\Big)\Phi\Big(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + k_{n-1} + a}{\sigma\sqrt{t_i - t_{i-1}}}\Big)\Big]
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
&= \frac{\exp(-\frac{2\mu(b-x_{i-1})}{\sigma^2}) - 1}{\exp(-\frac{2\mu(b-x_{i-1})}{\sigma^2}) - \exp(-\frac{2\mu(a-x_{i-1})}{\sigma^2})} \\
&\quad + \sum_{n=-\infty}^{\infty} \Big[\exp\Big(\frac{\mu}{\sigma^2}(-2x_{i-1} + k_n + 2b)\Big) \\
&\qquad\qquad \cdot \Phi\Big(\frac{x_{i-1} - \mu(t_i - t_{i-1}) - k_{n-1} - a}{\sigma\sqrt{t_i - t_{i-1}}}\Big) \\
&\qquad - \exp\Big(-\frac{\mu k_{n-1}}{\sigma^2}\Big)\Phi\Big(\frac{-x_{i-1} - \mu(t_i - t_{i-1}) + k_{n-1} + a}{\sigma\sqrt{t_i - t_{i-1}}}\Big)\Big],
\end{aligned}
$$

where the last step is – for $\mu \geq 0$ – an application of the geometric power series.

Defining

$$\alpha_n := x_{i-1} + (2n-2)b - (2n-1)a,$$

we finally obtain

$$
\begin{aligned}
&\mathbb{P}(t_{i-1} < T_{ab}^{+} < t_i) \\
&\quad = \frac{\exp\left(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\right) - 1}{\exp\left(-\frac{2\mu(b-x_{i-1})}{\sigma^2}\right) - \exp\left(-\frac{2\mu(a-x_{i-1})}{\sigma^2}\right)} \\
&\qquad + \exp\left(\frac{\mu}{\sigma^2}(a - x_{i-1})\right) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\mu\alpha_n}{\sigma^2}\right)\left[\Phi\left(\frac{\alpha_n - \mu(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}\right)\right. \\
&\qquad\qquad \left. - \exp\left(\frac{2\mu\alpha_n}{\sigma^2}\right)\Phi\left(\frac{-\alpha_n - \mu(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}\right)\right].
\end{aligned}
$$

The latter sum is related to the cumulative distribution function of an inverse Gaussian distribution

$$
F(t) = 1 - \left(\Phi\left(\frac{\alpha - \mu(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}\right) - \exp\left(\frac{2\mu}{\sigma^2}\alpha\right)\Phi\left(\frac{-\alpha - \mu(t_i - t_{i-1})}{\sigma\sqrt{t_i - t_{i-1}}}\right)\right).
$$

Using that the inverse Gaussian probability distribution function is

$$
f(t) = \frac{\alpha}{\sqrt{2\pi}\sigma t^{\frac{3}{2}}} \exp\left(-\frac{\left(\alpha - \mu(t_i - t_{i-1})\right)^2}{2\sigma^2(t_i - t_{i-1})}\right),
$$

we derive the first-passage time density as

$$
\begin{aligned}
f_{ab}^{+}(t, x_{i-1}) &:= \lim_{\triangle t \to 0} \frac{\mathbb{P}(t - \triangle t < T_{ab}^{+} < t + \triangle t)}{2\triangle t} \\
&= \exp\left(\frac{\mu}{\sigma^2}(a - x_{i-1})\right) \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\mu}{\sigma^2}\alpha_n\right) \\
&\qquad \cdot \frac{\alpha_n \exp\left(-\frac{(\alpha_n - \mu(t - t_{i-1}))^2}{2\sigma^2(t - t_{i-1})}\right)}{\sqrt{2\pi}\sigma(t - t_{i-1})^{\frac{3}{2}}} \\
&= \exp\left(\frac{\mu}{\sigma^2}(a - x_{i-1})\right) \exp\left(-\frac{\mu^2(t - t_{i-1})}{2\sigma^2}\right) \\
&\qquad \cdot \sum_{n=-\infty}^{\infty} \frac{\alpha_n \exp\left(-\frac{\alpha_n^2}{2\sigma^2(t - t_{i-1})}\right)}{\sqrt{2\pi}\sigma(t - t_{i-1})^{\frac{3}{2}}} \\
&= \exp\left(-\frac{\mu^2(t - t_{i-1})}{2\sigma^2}\right) \exp\left(\frac{\mu(a - x_{i-1})}{\sigma^2}\right) \\
&\qquad \cdot \sum_{n=-\infty}^{\infty} \frac{x_{i-1} - k_{n-1} - b}{\sqrt{2\pi\sigma^2}(t - t_{i-1})^{\frac{3}{2}}} \exp\left(-\frac{(x_{i-1} - k_{n-1} - b)^2}{2\sigma^2(t - t_{i-1})}\right)
\end{aligned}
$$

$$= \exp\left(\frac{\mu(a - x_{i-1})}{\sigma^2}\right)\frac{\sigma^2\pi}{(a-b)^2}\sum_{n=1}^{\infty}(-1)^n n \sin\left(\frac{\pi n(x_{i-1} - b)}{a-b}\right)$$
$$\cdot \exp\left(-\left(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2\sigma^2}{2(a-b)^2}\right)(t - t_{i-1})\right), \tag{A.3}$$

where the last step is an application of the *Jacobi transformation formula* (Lemma A.1, see [28]) using

$$x = \frac{x_{i-1} - b}{2(a-b)} \quad \text{and} \quad t^* = \frac{\sigma^2(t - t_{i-1})}{2(a-b)^2}.$$

Finally representation (a) for

$$\mathbb{P}(t_{i-1} < T_{ab}^+ < t_i) = \int_{t_{i-1}}^{t_i} f_{ab}^+(t, x_{i-1})\,dt$$

can be obtained by integration. Similarly the probabilities $\mathbb{P}(t_{i-1} < T_{ab}^- < t_i)$ and $\mathbb{P}(t_{i-1} < T_{ab} < t_i)$ can be derived.

The expression for the probability $\mathbb{P}(t_{i-1} < T_{ab} < t_i)$ is given in [8, p. 633][6] and in [12, p. 173]. The paper [8] uses an alternative approach by renewal-type arguments, see Remark A.2 below.

Lemma A.1 (Jacobi transformation formula). *For $x \in \mathbb{R}$, $t^* \geq 0$, we find that*

$$\frac{-2}{\sqrt{\pi t^*}}\sum_{n=-\infty}^{\infty}\frac{(x + 1/2 - n)}{t^*}\exp\left(-\frac{(x+1/2-n)^2}{t^*}\right)$$
$$= 4\pi\sum_{n=1}^{\infty}(-1)^{n+1} n \exp(-\pi^2 n^2 t^*)\sin(2\pi n x). \tag{A.4}$$

Proof. Define the Jacobi theta function as originally in [28] (or [1, p. 576] if one substitutes $q = \exp(\pi i \tau)$):

$$\vartheta_3(z|\tau) := \sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2 + 2inz).$$

Exploiting the identity by [28]

$$\vartheta_3(z|\tau) = \exp\left(-\frac{iz^2}{\pi\tau}\right)\vartheta_3\left(\frac{z}{\tau}\,\middle|\,-\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}}\sum_{n=1}^{\infty}\exp\left(-i\frac{(z-\pi n)^2}{\pi\tau}\right)$$

[6] Note that the expression in [8, p. 633] contains two typos: π^2 has to be replaced by π and $(-1)^n$ by $(-1)^{n+1}$.

using $z = (x + 1/2)\pi$ and $\tau = it^*\pi$, we obtain

$$\begin{aligned}\vartheta_3(z|\tau) &= \sum_{n=-\infty}^{\infty} \exp\big(-\pi^2 t^* n^2\big) \exp\big(2\pi i n (x + 1/2)\big) \\ &= 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp\big(-\pi^2 n^2 t^*\big) \cos\big(2\pi n x\big) \\ &= \frac{1}{\sqrt{\pi t^*}} \sum_{n=1}^{\infty} \exp\Big(-\frac{(x + 1/2 - n)^2}{t^*}\Big).\end{aligned}$$

Equation (A.4) is a derivation of this equation with respect to x. □

Remark A.2. The Laplace transforms

$$\hat{f}_{ab}^{+}(\lambda) := \int_{t_{i-1}}^{t_i} f_{ab}^{+}(t, x_{i-1}) \exp(\lambda t)\, dt$$

of the first-passage time densities (denoted by $f_{ab}^{+}(t, x_{i-1})$, $f_{ab}^{-}(t, x_{i-1})$, and $f_{ab}(t, x_{i-1})$) are much less complex than the infinite series presented in Theorem 2.2. Using some renewal-type arguments, they were first derived in [8] as

$$\hat{f}_{ab}^{+}(\lambda) = \exp\Big(\frac{\mu(a - x_{i-1})}{\sigma^2}\Big) \frac{\sinh\Big(\frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2}(b - x_{i-1})\Big)}{\sinh\Big(\frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2}(b - a)\Big)}$$

and

$$\hat{f}_{ab}^{-}(\lambda) = -\exp\Big(\frac{\mu(b - x_{i-1})}{\sigma^2}\Big) \frac{\sinh\Big(\frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2}(a - x_{i-1})\Big)}{\sinh\Big(\frac{\sqrt{\mu^2+2\sigma^2\lambda}}{\sigma^2}(b - a)\Big)}.$$

From Laplace inversion tables, for example [38, p. 295], we obtain equation (A.3).

This concludes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Adapting the idea in [15] and [36], we find that

$$\begin{aligned}g_{ab}^{+}(t, x_{i-1}, x_i) &:= \mathbb{P}\big(T_{ab}^{+} \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} x_i\big) \\ &= \frac{\mathbb{P}\big(T_{ab}^{+} \in dt, x_i \in dx \mid B_{t_{i-1}} = x_{i-1}\big)}{\mathbb{P}\big(x_i \in dx \mid B_{t_{i-1}} = x_{i-1}\big)}\end{aligned}$$

$$= \frac{\mathbb{P}\big(T_{ab}^{+} \in dt \mid B_{t_{i-1}} = x_{i-1}\big) \cdot \mathbb{P}\big(x_i \in dx \mid t = T_{ab}^{+},\ B_{t_{i-1}} = x_{i-1}\big)}{\mathbb{P}\big(x_i \in dx \mid B_{t_{i-1}} = x_{i-1}\big)}$$

$$= \frac{\mathbb{P}\big(T_{ab}^{+} \in dt \mid B_{t_{i-1}} = x_{i-1}\big) \cdot \mathbb{P}\big(x_i \in dx \mid t = T_{ab}^{+},\ B_t = a\big)}{\mathbb{P}\big(x_i \in dx \mid B_{t_{i-1}} = x_{i-1}\big)}$$

$$= \frac{f_{ab}^{+}(t, x_{i-1}) \cdot \varphi\big(x_i; a + \mu(t_i - t), \sigma\sqrt{t_i - t}\big)}{\varphi\big(x_i; x_{i-1} + \mu(t_i - t_{i-1}), \sigma\sqrt{t_i - t_{i-1}}\big)}$$

$$= \sum_{n=1}^{\infty} \frac{\sigma^2 \pi (-1)^n n}{(a-b)^2} \exp\Big(-\Big(\frac{\mu^2}{2\sigma^2} + \frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}\Big)(t - t_{i-1})\Big)$$
$$\cdot \sin\Big(\frac{\pi n (b - x_{i-1})}{a-b}\Big) \exp\Big(\frac{\mu(a - x_{i-1})}{\sigma^2}\Big)$$
$$\cdot \frac{\frac{1}{\sqrt{2\pi(t_i - t)}\sigma} \exp\Big(-\frac{(x_i - a - \mu(t_i - t))^2}{2\sigma^2(t_i - t)}\Big)}{\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}\sigma} \exp\Big(-\frac{(x_i - x_{i-1} - \mu(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})}\Big)}$$

$$= \frac{\sigma^2 \pi}{(a-b)^2} \sum_{n=1}^{\infty} (-1)^n n \exp\Big(-\frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}(t - t_{i-1})\Big) \sin\Big(\frac{\pi n (b - x_{i-1})}{a-b}\Big)$$
$$\cdot \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}} \exp\Big(-\frac{\mu^2}{2\sigma^2}(t - t_{i-1})\Big) \exp\Big(\frac{\mu(a - x_{i-1})}{\sigma^2}\Big)$$
$$\cdot \exp\Big(\frac{(x_i - x_{i-1} - \mu(t_i - t_{i-1}))^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a - \mu(t_i - t))^2}{2\sigma^2(t_i - t)}\Big)$$

$$= \frac{\sigma^2 \pi}{(a-b)^2} \frac{\sqrt{t_i - t_{i-1}}}{\sqrt{t_i - t}} \exp\Big(\frac{(x_i - x_{i-1})^2}{2\sigma^2(t_i - t_{i-1})} - \frac{(x_i - a)^2}{2\sigma^2(t_i - t)}\Big)$$
$$\cdot \sum_{n=1}^{\infty} (-1)^n n \exp\Big(-\frac{\pi^2 n^2 \sigma^2}{2(a-b)^2}(t - t_{i-1})\Big) \sin\Big(\frac{\pi n (b - x_{i-1})}{a-b}\Big),$$

where $\varphi(x; \mu, \sigma)$ denotes the density function of a normal distribution with mean μ and variance σ^2. By symmetry, the expression for

$$\mathbb{P}\big(T_{ab}^{-} \in dt \mid B_{t_{i-1}} = x_{i-1},\ B_{t_i} = x_i\big) =: g_{ab}^{-}(t, x_{i-1}, x_i)$$

is obtained. □

Bibliography

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.

[2] T. W. Anderson, A modification of the sequential probability ratio test to reduce the sample size, *The Annals of Mathematical Statistics* **31** (1960), no. 1, 165–197.

[3] S. Boyarchenko and S. Levendorskiĭ, Barrier options and touch-and-out options under regular Lévy processes of exponential type, *Annals of Applied Probability* **12** (2002), no. 4, 1261–1298.

[4] S. Boyarchenko and S. Levendorskiĭ, Valuation of continuously monitored double barrier options and related securities, *Mathematical Finance* **22** (2012), no. 2, 419–444.

[5] P. Carr and J. Crosby, A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options, *Journal of Quantitative Finance* **10** (2010), no. 10, 1115–1136.

[6] M. Chahal and J. Wang, A jump diffusion process and emerging bond and stock markets: An investigation using daily data, *Multinational Finance Journal* **1** (1997), no. 3, 169–197.

[7] R. Cont and E. Voltchkova, Integro-differential equations for option prices in exponential Lévy models, *Finance and Stochastics* **9** (2005), 299–325.

[8] D. A. Darling and A. J. F. Siegert, The first passage problem for a continuous Markov process, *The Annals of Mathematical Statistics* **24** (1953), no. 4, 624–639.

[9] S. R. Das, The surprise element: Jumps in interest rates, *Journal of Econometrics* **106** (2002), 27–65.

[10] J. DiCesare and D. McLeish, Simulation of jump diffusions and the pricing of options, *Insurance: Mathematics and Economics* **43** (2008), 316–326.

[11] P. Dobránszky and W. Schoutens, Generic Lévy one-factor models for the joint modelling of prepayment and default: Modelling LCDX, Technical report 08-03, Section of Statistics, K.U. Leuven, 2008.

[12] M. Dominé, First passage time distribution of a Wiener process with drift concerning two elastic barriers, *Journal of Applied Probability* **33** (1996), 164–175.

[13] C. Downing, R. Stanton and N. Wallace, An empirical test of a two-factor mortgage valuation model: How much do house prices matter?, *Real Estate Economics* **4** (2005), 681–710.

[14] D. Duffie, J. Pan and K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* **68** (2000), 1343–1376.

[15] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley and Sons, New York, 1966.

[16] J. E. Figueroa-López and P. Tankov, Small-time asymptotics of stopped Lévy bridges and simulation schemes with controlled bias, preprint (2012), `http://arxiv.org/pdf/1203.2355.pdf`.

[17] X. Gabaix, A. Krishnamurthy and O. Vigneron, Limits of arbitrage: Theory and evidence from the mortgage-backed securities market, *Journal of Finance* **52** (2007), no. 2, 557–595.

[18] H. Geman and M. Yor, Pricing and hedging double-barrier options: A probabilistic approach, *Mathematical Finance* **6** (1996), no. 4, 365–378.

[19] K. Giesecke and D. Smelov, Exact sampling of jump-diffusions, *Operations Research*, to appear.

[20] E. Gobet, *Advanced Monte Carlo methods for barrier and related exotic options*, in: *Handbook of Numerical Analysis. Vol. XV. Special Volume: Mathematical Modeling and Numerical Methods in Finance*, Handb. Numer. Anal. 15, Elsevier, Amsterdam (2009), 497–528.

[21] T. Guillaume, Window double barrier options, *Review of Derivatives Research* **6** (2003), 47–75.

[22] T. Guillaume, Step double barrier options, *Journal of Derivatives* **18** (2010), no. 1, 59–79.

[23] W. J. Hall, The distribution of Brownian motion on linear stopping boundaries, *Sequential Analysis* **4** (1997), 345–352.

[24] P. N. Henriksen, Pricing barrier options by a regime switching model, *Journal of Quantitative Finance* **11** (2011), no. 8, 1221–1231.

[25] P. Hieber and M. Scherer, Efficiently pricing barrier options in a Markov-switching framework, *Journal of Computational and Applied Mathematics* **235** (2010), 679–685.

[26] P. Hieber and M. Scherer, A note on first-passage times of continuously time-changed Brownian motion, *Statistics & Probability Letters* **82** (2012), no. 1, 165–172.

[27] T. R. Hurd, Credit risk modeling using time-changed Brownian motion, *International Journal of Theoretical and Applied Finance* **12** (2009), no. 8, 1213–1230.

[28] C. Jacobi, Note sur les fonctions elliptiques, *Journal für die reine und angewandte Mathematik* **3** (1828), 192–195.

[29] R. Jarrow and R. Protter, Structural versus reduced form models: A new information based perspective, *Journal of Investment Management* **2** (2004), no. 2, 1–10.

[30] M. Jeanblanc, M. Yor and M. Chesney, *Mathematical Methods for Financial Markets*, Springer-Verlag, London, 2009.

[31] M. S. Joshi and T. Leung, Using Monte Carlo simulation and importance sampling to rapidly obtain jump-diffusion prices of continuous barrier options, *Journal of Computational Finance* **10** (2007), no. 4, 93–105.

[32] S. G. Kou and H. Wang, First passage times of a jump diffusion process, *Advances in Applied Probability* **35** (2003), no. 2, 504–531.

[33] N. Kunitomo and M. Ikeda, Pricing options with curved boundaries, *Mathematical Finance* **2** (1992), no. 4, 275–298.

[34] X. S. Lin, Double barrier hitting time distributions with applications to exotic options, *Insurance: Mathematics and Economics* **23** (1998), 45–58.

[35] X. S. Lin, Laplace transform and barrier hitting time distribution, *Actuarial Research Clearing House* **1** (1999), 165–178.

[36] S. Metwally and A. Atiya, Using Brownian bridge for fast simulation of jump-diffusion processes and barrier options, *Journal of Derivatives* **10** (2002), 43–54.

[37] A. Novikov, V. Frishling and N. Kordzakhia, Approximations of boundary crossing probabilities for a Brownian motion, *Journal of Applied Probability* **36** (1999), 1019–1030.

[38] F. Oberhettinger and L. Badii, *Tables of Laplace Transforms*, Springer-Verlag, Berlin, 1973.

[39] A. Pelsser, Pricing double barrier options using Laplace transforms, *Finance and Stochastics* **4** (2000), 95–104.

[40] E. Platen and N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Stoch. Model. Appl. Probab. 64 Springer-Verlag, Berlin, 2010.

[41] L. C. G. Rogers, Evaluating first-passage probabilities for spectrally one-sided Lévy processes, *Journal of Applied Probability* **37** (2000), 1173–1180.

[42] S. M. Ross and S. Ghamami, Efficient Monte Carlo barrier option pricing when the underlying security price follows a jump-diffusion process, *Journal of Derivatives* **17** (2010), no. 3, 45–52.

[43] J. Ruf and M. Scherer, Pricing corporate bonds in an arbitrary jump-diffusion model based on an improved Brownian-bridge algorithm, *Journal of Computational Finance* **14** (2011), no. 3, 127–145.

[44] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Stud. Adv. Math. 68, Cambridge University Press, Cambridge, 1999.

[45] M. Schröder, On the valuation of double-barrier options: Computational aspects, *Journal of Computational Finance* **3** (2000), no. 4, 5–33.

[46] A. Sepp, Analytical pricing of double-barrier options under a double-exponential jump diffusion process: Applications of Laplace transform, *International Journal of Theoretical and Applied Finance* **7** (2004), no. 2, 151–175.

[47] P. V. Shevchenko, Addressing the bias in Monte Carlo pricing of multi-asset options with multiple barriers through discrete sampling, *Journal of Computational Finance* **6** (2003), no. 3, 1–20.

[48] C. Zhou, The term structure of credit spreads with jump risk, *Journal of Banking & Finance* **25** (2001), 2015–2040.

Received March 8, 2013; accepted May 30, 2013.

Author information

Lexuri Fernández, University of the Basque Country UPV/EHU,
Ekonomia Analisiaren Oinarriak II Saila,
Lehendakari Agirre Hiribidea, 83, 48015 Bilbao, Spain.
E-mail: `lexuri.fernandez@ehu.es`

Peter Hieber, Lehrstuhl für Finanzmathematik (M13),
Technische Universität München, Parkring 11, 85748 Garching-Hochbrück, Germany.
E-mail: `hieber@tum.de`

Matthias Scherer, Lehrstuhl für Finanzmathematik (M13),
Technische Universität München, Parkring 11, 85748 Garching-Hochbrück, Germany.
E-mail: `scherer@tum.de`