A Topological Fixed-Point Index Theory
for Evolution Inclusions

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Abstract. In the paper we construct a topological fixed-point theory for a class of set-valued maps which appears in natural way in boundary value problems for differential inclusions. Our construction is based upon the notion of (U, V)-approximation in the sense of Ben-El-Mechaiekh and Deguire. As applications we consider initial-value problems for nonlinear evolution inclusions of the type

$$\left. \begin{aligned} x'(t) &\in -A(t, x(t)) + F(t, x(t)) \\ x(0) &= x_0 \end{aligned} \right\}$$

where the operator A satisfies various monotonicity assumptions and F is an upper semi-continuous set-valued perturbation.

Keywords: *Fixed-point index, (U, V)-approximation, evolution inclusions*

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1. Introduction

It is well-known that various types of boundary value problems for differential inclusions may be equivalently reformulated as set-valued fixed-point problems

$$x \in \mathcal{F}(x) := S \circ N_F(x) \tag{1}$$

in appropriately choosen functional spaces (see [9] and references given there). In (1) N_F denotes the set-valued Nemytskii operator associated with the right-hand side F of the differential inclusion (see Section 4 for relevant definitions) and S is a (single-valued) operator given by the problem under consideration (S might be, e.g., an integral operator, the Green operator, mild solution operator etc.). In the present paper we would like to construct a topological fixed-point index theory for maps as considered in problem (1).

We have to encounter two different types of problems:

1. In applications of the fixed-point index theory which we have in mind the mapping S is nonlinear; hence we are not able to assume that the map $\mathcal{F}$ has convex values, even

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if (as we do assume) F and therefore also N_F has convex values. Thus we treat $\mathcal{F}$ as composition of the convex-valued map N_F and S (compare the approach in [10]).

2. Under our assumptions the set-valued Nemytskii operator is upper semi-continuous from the strong to the weak topology. This has to be counterbalanced by a stronger continuity assumption on the map S (it will be supposed that S is sequentially continuous from the weak to the strong topology). Fortunately, S has this property in various types of problems.

The fixed-point index theory will be constructed via the method of single-valued approximation. Homological techniques widely used in the fixed-point index theory of set-valued maps are not necessary. Since we have to consider the (non-metrizable) weak topology on Banach spaces, we will use the concept of (U, V)-approximability from Ben-El-Mechaiekh and Deguire [4]. This notion extends in a natural way the notion of ε-approximability used in metric spaces to topological vector spaces. Preliminary results on (U, V)-approximability will be given in Section 2. In Section 3 we define the fixed-point index theory and derive some fixed-point principles from its main properties. In the spirit of Couchouron and Kamenskii [8] we then apply our results in Section 4 in a unifying approach to various types of evolution inclusions. Examples are given in Section 5.

In the forthcoming paper [3] the fixed-point index theory presented here will be applied to vector differential inclusions involving the p-Laplacian and nonlinear boundary conditions formulated in terms of maximal monotone maps.

2. Preliminaries

Let X and Y be topological spaces. A set-valued transformation φ of X into Y will be denoted by $\varphi : X \multimap Y$. We say that φ is upper semi-continuous provided for each open set $V \subset Y$ the set $\{x \in X : \varphi(x) \subset V\}$ is open in X. Given a set-valued map $\psi : Y \multimap Z$, the composition of φ and ψ, denoted by $\psi \circ \varphi$, is the set-valued map from X into Z given by $(\psi \circ \varphi)(x) = \bigcup_{y \in \varphi(x)} \psi(y)$. In the case $X \subset Y$, a point $x \in X$ such that $x \in \varphi(x)$ is called a fixed-point of φ; the set of all fixed-points of φ is denoted by $\mathrm{Fix}\,(\varphi)$.

Let E and F be topological vector spaces. By $\mathcal{U}_E(0)$ and $\mathcal{U}_F(0)$ we denote the filter of neighborhoods of the origin in E and F, respectively. Now let $\varphi : X \multimap Y$ be a set-valued map where $X \subset E$ and $Y \subset F$, and let $U \in \mathcal{U}_E(0)$ and $V \in \mathcal{U}_F(0)$ be given. Following [4], we say that a continuous map $f : X \to Y$ is an (U, V)-approximation of φ provided

$$f(x) \in \big(\varphi[(x + U) \cap X] + V\big) \cap Y \qquad \text{for each } x \in X. \tag{2}$$

We say that φ is approximable, if for each $U \in \mathcal{U}_E(0)$ and $V \in \mathcal{U}_F(0)$ there exists an (U, V)-approximation of φ.

For given neighborhoods $U' \in \mathcal{U}_E(0)$ and $V' \in \mathcal{U}_F(0)$ such that $U' \subset U$ and $V' \subset V$ any (U', V')-approximation is also an (U, V)-approximation of φ. In the case where

E and F are normed spaces we can consider the ε-balls $B_E(0, \varepsilon)$ and $B_F(0, \varepsilon)$ [1] as neighborhoods of the origin; then any $\big(B_E(0, \varepsilon), B_F(0, \varepsilon)\big)$-approximation of φ is the usual ε-approximation on the graph of φ (see [7]).

The following Lemma generalizes [4: Proposition 3.3].

Lemma 1. *Let X, Y and Z be subsets of topological vector spaces E, F and G, respectively. Let $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ be upper semi-continuous set-valued maps with compact values and assume that X is compact. Then for each $U \in \mathcal{U}_E(0)$ and $W \in \mathcal{U}_G(0)$ there exist $U_0 \in \mathcal{U}_E(0), V_0 \in \mathcal{U}_F(0)$ and $W_0 \in \mathcal{U}_G(0)$ such that $g \circ f$ is an (U, W)-approximation of $\psi \circ \varphi$, provided that f is an (U_0, V_0)-approximation of φ and g is an (V_0, W_0)-approximation of ψ. In particular, $\psi \circ \varphi$ is approximable, provided that φ and ψ are approximable.*

Proof. Let $U \in \mathcal{U}_E(0)$ and $W \in \mathcal{U}_G(0)$ be fixed. Take $U' \in \mathcal{U}_E(0)$ closed such that $U' + U' \subset U$ and define an open and balanced $U_0 \in \mathcal{U}_E(0)$ such that $U_0 + U_0 \subset U'$. Also, let $W_0 \in \mathcal{U}_G(0)$ be such that $W_0 + W_0 \subset W$. Since X is compact, there are $x_1, \ldots, x_m \in X$ such that $\{x_i + U_0\}_{i=1}^m$ is an open cover of X. Since $\varphi[(x_i + U') \cap X]$ is compact and ψ is upper semi-continuous, it is easy to see that there exists $V_i \in \mathcal{U}_F(0)$ such that

$$\psi\big[\big(\varphi[(x_i + U') \cap X] + V_i\big) \cap Y\big] \subset \psi\big(\varphi[(x_i + U') \cap X]\big) + W_0 \tag{3}$$

for every $i \in \{1, \ldots, m\}$. Now let $V := \bigcap_{i=1}^m V_i$ and define $V_0 \in \mathcal{U}_F(0)$ such that $V_0 + V_0 \subset V$. Let f be an (U_0, V_0)-approximation of φ and g an (V_0, W_0)-approximation of ψ. We claim that $g \circ f$ is an (U, W)-approximation of $\psi \circ \varphi$. Let $x \in X$. From (2) it follows that

$$f(x) \in \big(\varphi[(x + U_0) \cap X] + V_0\big) \cap Y$$
$$g(f(x)) \in \big(\psi[(f(x) + V_0) \cap Y] + W_0\big) \cap Z.$$

Take $i \in \{1, \ldots, m\}$ such that $x \in x_i + U_0$. Then $x + U_0 \subset x_i + U'$ and thus $f(x) \in \varphi[(x_i + U') \cap X] + V_0$. It follows that

$$(f(x) + V_0) \cap Y \subset \big(\varphi[(x_i + U') \cap X] + V_i\big) \cap Y.$$

Hence,

$$g(f(x)) \in \Big(\psi\big[\big(\varphi[(x_i + U') \cap X] + V_i\big) \cap Y\big] + W_0\Big) \cap Z$$
$$\stackrel{(3)}{\subset} \big(\psi(\varphi[(x_i + U') \cap X]) + W\big) \cap Z$$
$$\subset \big(\psi \circ \varphi[(x + U) \cap X] + W\big) \cap Z$$

and the lemma is proved ∎

As an example of (U, V)-approximability we have the following

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[1] For a normed space E, a point $x \in E$ and $\varepsilon > 0$ we denote by $B_E(x, \varepsilon) = \{y \in E : \|x - y\| < \varepsilon\}$ the open ball and by $D_E(x, \varepsilon) = \{y \in E : \|x - y\| \leq \varepsilon\}$ the closed ball with center x and radius ε.

Proposition 2 (see [4, 11]). *Let X be a compact, convex subset of a Banach space E and Y a closed, convex subset of a topological vector space F. Let $\varphi : X \multimap Y$ be an upper semi-continuous mapping with compact, convex values. Then:*

(i) *φ is approximable.*

(ii) *For any $\varepsilon > 0$ and any $V \in \mathcal{U}_F(0)$ there exist $\varepsilon^* > 0$ and $V^* \in \mathcal{U}_F(0)$ such that for given $(B_E(0, \varepsilon^*), V^*)$-approximations $f, g : X \to Y$ of φ there is a homotopy $h : X \times [0, 1] \to Y$ such that $h(\cdot, 0) = f, h(\cdot, 1) = g$ and $h(\cdot, t) : X \to Y$ is a $(B_E(0, \varepsilon), V)$-approximation of φ for each $t \in [0, 1]$.*

Let F be a real Banach space. If F is endowed with the weak topology, we will denote it by (F, w). Also, given a subset Y of F, then (Y, w) will denote the weak relative topology on Y. The convergence in (F, w) (or in (Y, w)) will be denoted by $y_n \xrightarrow{w} y$.

In the next section we will need the following

Lemma 3 (compare [1]). *Let Z be a topological space, Y a weakly compact subset of a Banach space F and let $T : (Y, w) \to Z$ be a map. Then T is continuous from (Y, w) to Z if and only if T is sequentially continuous.*

Proof. Clearly, it suffices to prove the "if" part only. Let $A \subset Z$ be closed. We have to show that the set $T^{-1}(A) := \{y \in Y : T(y) \in A\}$ is weakly closed. But since Y is weakly compact, $T^{-1}(A)$ is weakly relatively compact. Thus in view of the Eberlein-Shmulian theorem, it suffices to prove that $T^{-1}(A)$ is weakly sequentially closed. So let (y_n) be a sequence in $T^{-1}(A)$ such that $y_n \xrightarrow{w} y$. Then by our assumption we see that $T(y_n) \to T(y)$. Since A is closed and $T(y_n) \in A$ we therefore get $T(y) \in A$, i.e. $y \in T^{-1}(A)$ finishing the proof ∎

3. A fixed-point index theory for a specific class of set-valued maps

In the section we develop the fixed-point index theory suitable for application to boundary value problems for differential inclusions. The construction, being similar to techniques given in [2, 12], will be carried out in two steps.

Step 1. Let X be a compact, convex subset of a Banach space E and let $\Phi : X \multimap X$ be a set-valued map with the following property: There is a closed, convex subset Y of a topological vector space F, an upper semi-continuous map $\varphi : X \multimap Y$ with compact, convex values and a continuous map $T : Y \to X$ such that $\Phi = T \circ \varphi$. In this case we say that Φ has a *decomposition* which we denote by

$$D(\Phi) : X \xrightarrow{\varphi} Y \xrightarrow{T} X. \tag{4}$$

Let W be a (relatively) open subset of X and assume that $\mathrm{Fix}\,(\Phi) \cap \partial W = \emptyset$. For the map Φ (more precisely, for the decomposition $D(\Phi)$ of Φ) we may define a fixed-point index.

First, it is easy to see that there is $\eta > 0$ such that $\mathrm{Fix}\,(f) \cap \partial W = \emptyset$ for each $f : X \to X$ being a $\big((B_E(0, \eta), (B_E(0, \eta))\big)$-approximation of Φ. Using Lemma 1, we

find $\gamma > 0$ and $V \in \mathcal{U}_F(0)$ such that if $f : X \to Y$ is a $(B_E(0, \gamma), V)$-approximation of φ, then $T \circ f$ is a $\big((B_E(0, \eta), (B_E(0, \eta))\big)$-approximation of Φ. Finally, we apply Proposition 2/(ii) and obtain $\varepsilon^* > 0$ and $V^* \in \mathcal{U}_F(0)$ such that for given $(B_E(0, \varepsilon^*), V^*)$-approximations $f, g : X \to Y$ of φ there is a homotopy $h : X \times [0, 1] \to Y$ such that $h(\cdot, 0) = f, h(\cdot, 1) = g$ and $h(\cdot, t) : X \to Y$ is a $(B_E(0, \gamma), V)$-approximation of φ for each $t \in [0, 1]$.

Define the fixed-point index for $D(\Phi)$ on W by

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{ind}_X(T \circ f, W) \tag{5}$$

where $f : X \to Y$ is an arbitrary $(B_E(0, \varepsilon), V)$-approximation of φ for $\varepsilon, 0 < \varepsilon \le \varepsilon^*$, and $V \subset V^*$. Here ind stands for the classical fixed-point index for single-valued maps on compact absolute neighborhood retracts (see [6]).

The existence of the approximation f of φ in (5) follows from Theorem 2/(i). Also, it is clear from the construction of the fixed-point index for $D(\Phi)$ in connection with application of the homotopy property of ind that the above definition is independent from the chosen (ε, V)-approximation. Finally, we see that this index is integer-valued.

In the following proposition we collect some properties of this fixed-point index. Proofs follow easily from corresponding properties of the classical fixed-point index.

Proposition 4. *Let* $\Phi : X \multimap X$ *be a given set-valued map with decomposition* $D(\Phi)$ *(see (4)) and* W *an open subset of* X *such that* $\mathrm{Fix}\,(\Phi) \cap \partial W = \emptyset$.

(i) (Additivity). *Let* $\mathrm{Fix}\,(\Phi) \cap W \subset W_1 \cup W_2$ *where* W_1 *and* W_2 *are open disjoint subsets of* W. *Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_X(D(\Phi), W_1) + \mathrm{Ind}_X(D(\Phi), W_2).$$

Particularly, $\mathrm{Ind}_X(D(\Phi), W) \ne 0$ *implies* $\mathrm{Fix}\,(\Phi) \cap W \ne \emptyset$.

(ii) (Homotopy). *Let* $\Psi : X \to X$ *be given with a decomposition* $D(\Psi) : X \xrightarrow{\psi} \multimap Y \xrightarrow{S} X$ *such that* $D(\Phi)$ *and* $D(\Psi)$ *are homotopic, i.e. there exists an upper semi-continuous map* $\eta : X \times [0, 1] \multimap Y$ *with compact, convex values such that* $\eta(\cdot, 0) = \varphi$ *and* $\eta(\cdot, 1) = \psi$, *and a continuous map* $h : Y \times [0, 1] \to X$ *such that* $h(\cdot, 0) = T$ *and* $h(\cdot, 1) = S$. *Assume also that* $x \notin \chi(x, t)$ *for every* $x \in \partial W$ *and* $t \in [0, 1]$, *where* $\chi : X \times [0, 1] \multimap X$ *is the set-valued homotopy between* Φ *and* Ψ *given by* $\chi(x, t) = h(\eta(x, t), t)$ *(observe* $\chi(\cdot, 0) = \Phi$ *and* $\chi(\cdot, 1) = \Psi$). *Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_X(D(\Psi), W).$$

(iii) (Weak normalization). *Assume that* T *in* $D(\Phi)$ *is a constant map, i.e.* $T(x) = a \notin \partial W$ *for each* $x \in Y$. *Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \begin{cases} 1 & \textit{if } a \in W \\ 0 & \textit{if } a \notin W. \end{cases}$$

(iv) (Contraction). *Let* X' *be a compact, convex subset of* E *such that* $X \overset{j}{\subset} X'$ *and let* W' *be an open subset of* X'. *Let* $\varphi' : X' \multimap Y$ *be an upper semi-continuous map*

with compact, convex values and assume that the map Φ' given by the decomposition $D: X' \xrightarrow{\varphi'} Y \xrightarrow{T} X$ has $\mathrm{Fix}\,(\Phi') \cap \partial W' = \emptyset$. Then

$$\mathrm{Ind}_{X'}(j \circ D, W') = \mathrm{Ind}_X(D|_X, W' \cap X)$$

where $D|_X : X \xrightarrow{\varphi'|_X} Y \xrightarrow{T} X$.

   (v) (Decomposition). *Let $\Phi : X \multimap X$ has two decompositions $D(\Phi)$ (see (4)) and $D'(\Phi) : X \xrightarrow{\varphi'} Y' \xrightarrow{T'} X$ such that there exists a continuous map $p : Y \to Y'$ with $\varphi' = p \circ \varphi$ and $T' \circ p = T$. Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_X(D'(\Phi), W).$$

   For subsequent applications we are interested in the following class of set-valued maps.

   Step 2: Let X be a closed convex subset of a Banach space E and let $\Phi : X \multimap X$ have a decomposition $D(\Phi)$ of the form

$$D(\Phi) : X \xrightarrow{\varphi} (Y, w) \xrightarrow{T} X. \tag{6}$$

Here we assume that Y is a closed convex subset of a Banach space F, φ is an upper semi-continuous map (since Y is endowed with the weak topology this means that φ is upper semi-continuous from the strong topology on X to the weak topology on Y) with weakly compact, convex values and T is a (in general nonlinear) sequentially continuous map (i.e. given a sequence (y_n) in Y such that $y_n \xrightarrow{w} y$, then $T(y_n) \to T(y)$ in X; thus T is weakly-strongly sequentially continuous). Let W be a bounded and (relatively) open subset of X such that $\mathrm{Fix}\,(\Phi) \cap \partial W = \emptyset$ and assume also that Φ is a compact map, i.e. Φ maps bounded sets onto relatively compact ones (this is the case if, e.g., φ maps norm bounded sets onto norm bounded sets and F is reflexive; observe also that in the present setting this assumption on Φ implies that Φ is upper semi-continuous).

   We define a fixed-point index for the decomposition $D(\Phi)$ (see (6)) as follows. Choose a compact convex set $C \subset X$ such that $\Phi(\overline{W}) \subset C$ (such set exists: take $C := \overline{\mathrm{conv}}\,\Phi(\overline{W}))$ and let $r : X \to C$ be a retraction. Next, take a convex, weakly compact set $D \subset Y$ such that $\varphi(C) \subset D$ (such set exists: take $D = \overline{\mathrm{conv}}\,\varphi(C)$; recall that by upper semi-continuity $\varphi(C)$ is weakly compact and therefore D is weakly compact in view of the Krein-Shmulian theorem). Thus we arrive at a decomposition

$$D(\Phi, C, r, D) : C \xrightarrow{\varphi} (D, w) \xrightarrow{r \circ T} C.$$

   We define a fixed-point index for the decomposition $D(\Phi)$ of the map Φ by

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_C(D(\Phi, C, r, D), W \cap C) \in \mathbb{Z} \tag{7}$$

where the right-hand side is given by formula (5).

Proposition 5. *Definition* (7) *is correct.*

Proof. The decomposition $D(\Phi, C, r, D)$ is admissible for the fixed-point index defined in (5). This follows from Lemma 3 since $r \circ T : (D, w) \to C$ is continuous by the weak compactness of D. To show that the definition is unique, let for a moment C and r be fixed and consider a decomposition

$$D(\Phi, C, r, D') : C \xrightarrow{\varphi} \circ \; (D', w) \xrightarrow{r \circ T} C$$

where D' is weakly compact, $\varphi(C) \subset D' \subset Y$. Without loss of generality we may assume $D \subset D'$ (if not, consider $D'' = \overline{\mathrm{conv}}\, D \cup D'$). But then the inclusion $i : D \hookrightarrow D'$ satisfies the requirements of the decomposition property in Proposition 4/(v) and thus we get

$$\mathrm{Ind}_C\big(D(\Phi, C, r, D), W \cap C\big) = \mathrm{Ind}_C\big(D(\Phi, C, r, D'), W \cap C\big).$$

Now let C' be a compact convex set such that $X \supset C' \supset \Phi(\overline{W})$ and let $r' : X \to C'$ be a retraction. Again we may assume that $C \overset{j}{\subset} C'$ and thus we get

$$D(\Phi, C', r', D) : C' \xrightarrow{\varphi} \circ \; (D, w) \xrightarrow{r' \circ T} C'.$$

Consider a homotopy

$$h : (D, w) \times [0, 1] \to C', \qquad h(x, t) = r'\big((1 - t)T(x) + \mathrm{tr}\,(T(x))\big).$$

Then $h(\cdot, 0) = r' \circ T$ and $h(\cdot, 1) = j \circ r \circ T$. Hence the homotopy property in Proposition 4/(ii) shows that

$$\mathrm{Ind}_{C'}\big(D(\Phi, C', r', D), W \cap C'\big) = \mathrm{Ind}_{C'}(D, W \cap C')$$

where

$$D : C' \xrightarrow{\varphi} \circ \; (D, w) \xrightarrow{j \circ r \circ T} C'.$$

Finally, an application of the contraction property in Proposition 4/(iv) finishes the proof ∎

The index from definition (7) has the following properties, whose proofs follow from Proposition 4.

Proposition 6. *Let* $\Phi : X \multimap X$ *be a compact map with decomposition* $D(\Phi)$ *(see* (6)*) and let* W *be a bounded open subset of* X *such that* $\mathrm{Fix}\,(\Phi) \cap \partial W = \emptyset$.

(i) (Additivity). *Let* $\mathrm{Fix}\,(\Phi) \cap W \subset W_1 \cup W_2$ *where* W_1 *and* W_2 *are open disjoint subsets of* W. *Then*

$$\mathrm{Ind}_X\,(D(\Phi), W) = \mathrm{Ind}_X\,(D(\Phi), W_1) + \mathrm{Ind}_X\,(D(\Phi), W_2).$$

Particularly, $\mathrm{Ind}_X\,(D(\Phi), W) \neq 0$ *implies* $\mathrm{Fix}\,(\Phi) \cap W \neq \emptyset$.

(ii) (Homotopy). *Let* $\Psi : X \multimap X$ *be given with a decomposition* $D(\Psi) : X \xrightarrow{\psi} \circ$ $(Y, w) \xrightarrow{S} X$ *such that* $D(\Phi)$ *and* $D(\Psi)$ *are homotopic, i.e. there exist an upper semi-continuous map* $\eta : X \times [0, 1] \multimap (Y, w)$ *with weakly compact, convex values such that*

$\eta(\cdot, 0) = \varphi$ and $\eta(\cdot, 1) = \psi$, and a sequentially continuous map $h : (Y, w) \times [0,1] \to X$ such that $h(\cdot, 0) = T$ and $h(\cdot, 1) = S$. Assume also that $x \notin \chi(x,t)$ for every $x \in \partial W$ and $t \in [0,1]$ where $\chi : X \times [0,1] \multimap X$ is the set-valued homotopy between Φ and Ψ and χ is a compact map. Then

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_X(D(\Psi), W).$$

(iii) (Weak normalization). *Assume that T in $D(\Phi)$ is a constant map, i.e. $T(x) = a \notin \partial W$ for each $x \in Y$. Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \begin{cases} 1 & if \ a \in W \\ 0 & if \ a \notin W. \end{cases}$$

(iv) (Decomposition). *Given another decomposition $D'(\Phi) : X \xrightarrow{\varphi'} (Y', w) \xrightarrow{T'} X$ where Y' is a closed convex subset of a Banach space F' such that there exists a sequentially continuous map $p : (Y, w) \to (Y', w)$ with $\varphi' = p \circ \varphi$ and $T' \circ p = T$. Then*

$$\mathrm{Ind}_X(D(\Phi), W) = \mathrm{Ind}_X(D'(\Phi), W).$$

From the above proposition one can derive several fixed-point principles for the maps under consideration. As an example we prove the Nonlinear Alternative and Leray-Schauder-type fixed-point theorem which will be applied in the subsequent section.

Theorem 7. *Let $R > 0$ and $\Phi : D_E(0, R) \multimap E$ be a compact map with decomposition*

$$D(\Phi) : D_E(0, R) \xrightarrow{\varphi} (Y, w) \xrightarrow{T} E$$

satisfying the assumptions of decomposition (6). Then either

$$there \ exists \ x_0, \|x_0\| = R \ and \ \lambda, 0 < \lambda < 1 \ such \ that \ x_0 \in \lambda\Phi(x_0) \qquad (8)$$

or Φ has a fixed-point.

Proof. Let $r : E \to D_E(0, R)$ be a retraction and obtain a decomposition

$$D(\Phi \circ r) : E \xrightarrow{\varphi \circ r} (Y, w) \xrightarrow{T} E.$$

Assume that Φ has no fixed-point on $\partial D_E(0, R)$ (otherwise we are done). Then

$$\mathrm{Ind}_E(D(\Phi \circ r), B_E(0, R))$$

is defined. Now let

$$h : (Y, w) \times [0,1] \to E, \qquad h(y,t) := tT(y).$$

Using h we see that the decompositions $D(\Phi \circ r)$ and $D(0) : E \xrightarrow{\varphi \circ r} (Y, w) \xrightarrow{0} E$ are homotopic and, moreover, the conditions of the homotopy property of Proposition 6/(ii) are fulfilled provided that assumption (8) is not valid. So in this case we get

$$\mathrm{Ind}_E(D(\Phi \circ r), B_E(0, R)) = \mathrm{Ind}_E(D(0), B_E(0, R)) = 1$$

by the weak normalization of Proposition 6/(iii). By the additivity property of Proposition 6/(i) we therefore find a fixed-point $x \in B_E(0, R)$ of $\Phi \circ r$ being a fixed-point of Φ since $r(x) = x$ for such x ∎

Similarly one shows

Theorem 8. *Let $\Phi : X \multimap X$ be a given set-valued map with decomposition $D(\Phi)$ (see (6)). Assume that Φ is compact and $0 \in X$. Then either the set $S = \{x \in X : x \in \beta\Phi(x) \ for \ some \ \beta \in (0,1)\}$ is unbounded or Φ has a fixed-point.*

4. Application to evolution inclusions

In this section we study an abstract fixed-point problem which can be applied to the Cauchy problem for differential inclusions in Banach spaces. Our approach being rather synthetic than analytic is similar to that of Couchouron and Kamenskii [8]. However, in [8] the compactness conditions are formulated in terms of measures of non-compactness.

In order to state this fixed-point problem we need some definitions and notations.

A set-valued mapping $F : [0, b] \times E \multimap E$ is called an *upper p-Carathéodory map* $(1 \leq p < \infty)$ provided that:

(F1) For every $t \in T = [0, b]$ and $x \in E$ assume that $F(t, x)$ is a convex, weakly compact subset of E.

(F2) For every $x \in E$ the map $F(\cdot, x) : T \multimap E$ has a strongly measurable selection.

(F3) For a.e. $t \in T$ the map $F(t, \cdot) : E \multimap (E, w)$ is upper semi-contiunuous.

(F4) For every non-empty bounded set $\Omega \subset E$ there exists $\nu = \nu(\Omega) \in L^p(T, \mathbb{R})$ such that $\|F(t, x)\| := \sup\{\|z\| : z \in F(t, x)\} \leq \nu(t)$ for a.e. $t \in T$ and every $x \in \Omega$.

With F given as above we associate the Nemytskii operator (or superposition operator)

$$N_F^p : C(T, E) \multimap L^p(T, E)$$

given by

$$N_F^p(x) = \left\{ f \in L^p(T, E) : f(t) \in F(t, x(t)) \text{ a.e. on } T \right\}$$

for each $x \in C(T, E)$.

The following Lemma is well-known (see, e.g., [16: p. 88]).

Lemma 9. *Let* $F : T \times E \multimap E$ *be an upper p-Carathéodory mapping. Then the Nemytskii operator* $N_F^p : C(T, E) \multimap (L^p(T, E), w)$ *is upper semi-continuous, and* $N_F^p(x)$ *is a non-empty, convex, weakly compact subset of* $L^p(T, E)$ *for each* $x \in C(T, E)$.

Let us now introduce a map

$$S_p : L^p(T, E) \to C(T, E)$$

satisfying the following assumptions:

(a1) There is a constant $M > 0$ such that $\|S_p(f)(t) - S_p(g)(t)\| \leq M \int_0^t \|f(s) - g(s)\| \, ds \ \ (t \in T)$ for every $f, g \in L^p(T, E)$.

(a2) If $\Gamma \subset L^p(T, E)$ and there exists a function $\mu \in L^p(T, \mathbb{R})$ such that $f(t) \leq \mu(t)$ for a.e. $t \in T$ and every $f \in \Gamma$, then $\{S_p(f) : f \in \Gamma\}$ is relatively compact in $C(T, E)$.

(a3) If $y_n = S_p(f_n)$ and $f_n \xrightarrow{w} f$ as well as $y_n \to y$, then $y = S_p(f)$.

We will be interested in solving the fixed-point problem [3]

$$x \in \mathcal{F}(x) := S_p \circ N_F^p(x). \tag{9}$$

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[3] In [8] problem (9) was studied under an additional assumption which implies the contractability of the values $\mathcal{F}(x)$. In view of the fixed-point theory developed we may dispense with this assumption. Thus our approach shows that this particular assumption is also not neccessary in [8] and we may slightly improve the results given there.

The problems we have in mind and which can be solved by (9) are the Cauchy problems

$$\left.\begin{array}{l} x'(t) \in -A(t, x(t)) + F(t, x(t)) \\ x(0) = x_0 \end{array}\right\}. \tag{10}$$

Recall that a solution of problem (10) is a solution to problem (9) where $S_p(f) \in C(T, E)$ is the (mild) solution operator to a quasi-autonomous problem of the form

$$\left.\begin{array}{l} x'(t) \in -A(t, x(t)) + f(t) \\ x(0) = x_0 \end{array}\right\} \tag{11}$$

with $f \in L^p(T, E)$. As it will become clear by subsequent examples (see Section 5) the (mild) solution operator S_p fulfills conditions $(a1) - (a3)$ under various assumptions on the operator A.

Remark 10. Consider the condition

$(*)$ If $f_n \xrightarrow{w} f$ in $L^p(T, E)$, then $S_p(f_n) \to S_p(f)$ in $C(T, E)$.

Clearly, $(*)$ implies $(a3)$. Let us show that $(*)$ also implies $(a2)$ in the case that E is reflexive. Let Γ be given as in $(a2)$ and let $x_n = S_p(f_n)$ for some $f_n \in \Gamma$ $(n \geq 1)$. From the Dunford-Pettis theorem we know that Γ is weakly relatively compact. Thus there are $f \in L^p(T, E)$ and a subsequence $f_{n_k} \xrightarrow{w} f$. By condition $(*)$, $x_{n_k} = S_p(f_{n_k}) \to S_p(f)$ and we obtain relative compactness of $\{S(f) : f \in \Gamma\}$.

We now turn to solve the above problem (9). We start with the following

Lemma 11. *Let F be an upper p-Carathéodory map and for $R > 0$ let*

$$Y_p(R) := \overline{\mathrm{conv}}\, N_F^p\big(D_{C(T,E)}(0, R)\big).$$

If S_p satisfies condition $(a2)$, then $S_p(Y_p(R))$ is a relatively compact subset of $C(T, E)$.

Proof. Let $f \in \mathrm{conv}\, N_F^p(D_{C(T,E)}(0, R))$. Then $f = \sum_{i=1}^{n} \lambda_i f_i$ where $n \geq 1$, $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $f_i \in N_F^p(x_i)$ for some $x_i \in D_{C(T,E)}(0, R)$. Let $\nu = \nu(D_E(0, R)) \in L^p(T, \mathbb{R})$ following assumption (F4). Then

$$\|f(t)\| \leq \sum_{i=1}^{n} \lambda_i \|f_i(t)\| \leq \nu(t)$$

for a.e. $t \in T$. Also, if $f \in Y_p(R)$, then there exist $f_n \in \mathrm{conv}\, N_F^p\big(D_{C(T,E)}(0, R)\big)$ $(n \geq 1)$ such that $f_n \to f$ and hence some subsequence such that $f_{n_k}(t) \to f(t)$ a.e. on T. Thus $f(t) \leq \nu(t)$ a.e. on T also for such f. Hence $S_p(Y_p(R))$ is relatively compact by assumption $(a2)$ ∎

From assumption $(a3)$ we immediately have the following

Corollary 12. *The map* $S_p : (Y_p(R), w) \to C(T, E)$ *is sequentially continuous for each* $R > 0$.

Theorem 13. *Let* $F : T \times E \longrightarrow\!\!\!\circ\, E$ *be an upper p-Carathéodory map and assume that*

(F4)′ $|F(t, x)\| \leq c(t)(1 + \|x\|)$ *a.e. on* T *and for every* $x \in E$

holds where $c \in L^p(T, \mathbb{R})$. *Let* S_p *satisfy assumptions* $(a1) - (a3)$. *Then the fixed-point problem* (9) *has a solution.*

Proof. Using assumptions $(a1)$ and $(F4)'$ together with the usual application of Gronwall's inequality we get an a priori bound for the solutions of problem (9). Indeed, let $x \in S_p \circ N_F^p(x)$. Then there is $f \in N_F^p(x)$ such that $x = S_p(f)$. Define $v = S_p(0)$. Then for every $t \in T$

$$\|x(t) - v(t)\| \leq M \int_0^t \|f(s)\| \, ds \leq M \int_0^t c(s)(1 + \|x(s)\|) \, ds \leq M\|c\|_1 + M \int_0^t \|x(s)\| \, ds$$

and therefore

$$\|x(t)\| \leq \|v\|_\infty + M\|c\|_1 + M \int_0^t \|x(s)\| \, ds.$$

It follows that for each $t \in T$

$$\|x(t)\| \leq (\|v\|_\infty + M\|c\|_1)e^{tM} \leq R := (\|v\|_\infty + M\|c\|_1)e^{bM}.$$

Now consider the diagram

$$D_{C(T,E)}(0, R+1) \xrightarrow{\ N_F^p\ }\!\!\!\circ\ (Y_p(R+1), w) \xrightarrow{\ S_p\ } C(T, E) \tag{12}$$

of the map $\mathcal{F} = S_p \circ N_F^p$ restricted to $D_{C(T,E)}(0, R+1)$. From Lemma 9 and Corollary 12 we see that (12) is a decomposition in the sense of (6) (clearly, assumption $(F4)'$ implies $(F4)$). Also, by the same argument as above using Gronwall's inequality, it is clear that given $x_0 \in C(T, E)$ with $\|x_0\|_\infty = R + 1$ and λ with $0 < \lambda < 1$, then $x_0 \notin \lambda\mathcal{F}(x_0)$. Thus by the Nonlinear Alternative (Theorem 7) it follows that problem (9) has a solution ∎

5. Examples

We now would like to list some of the cases where the above considered fixed-point problem applies.

Example 1 (M-accretive operators). Let $A : D(A) \subset E \longrightarrow\!\!\!\circ\, E$ be an m-accretive operator such that $-A$ generates a compact nonlinear semigroup of contractions. Let $x_0 \in \overline{D(A)}$ and assume that S_1 assigns to each $f \in L^1(T, E)$ the mild solution of problem (11) where $A(t, x(t)) = A(x(t))$.

Condition $(a1)$ appears as a weak form of the Benilan integral inequalities; conditions $(a2)$ and $(a3)$ are proven in Vrabie [16: pp. 44] provided that the topological dual

E^* is uniformly convex. Hence, an application of Theorem 13 shows the existence of a solution to the initial-value problem (10) in the case when F is a 1-Carathéodory map.

Let us note that Bothe [5] gave a finite-dimensional example showing that without the assumption on the geometry of E^* the result is wrong.

Example 2 (Linear operators). Let $-A$ be a densely defined linear (non-continuous) operator generating a compact C_0-semigroup $\{U(t)\}_{t \geq 0}$. Here S_1 is given by the variation of constants formula

$$S_1(f) = U(t)x_0 + \int_0^t U(t-s)f(s)\,ds.$$

Clearly, conditions $(a1) - (a3)$ are fulfilled. We infer the existence of a solution for a 1-Carathéodory map F.

Example 3 (Time-dependant subdifferentials). Let $E = H$ be a Hilbert space and $\varphi : T \times H \to \mathbb{R} \cup \{\infty\}$ a function such that, for each $t \in T$, $\varphi(t, \cdot)$ is proper, convex and lower semi-continuous. Assume also that φ satisfies the Yotsutani conditions (see [17]) and that $\varphi(t, \cdot)$ is of compact type for every $t \in T$. Let $x_0 \in \operatorname{dom} \varphi(0, \cdot)$. Then it was shown in [17] that for each $f \in L^2(T, H)$ problem (11) with $A(t, x(t)) = \partial \varphi(t, x(t))$ has a strong solution $S_2(f)$.

Clearly, S_2 satisfies condition $(a1)$ and it was shown in Papageorgiou and Papalini [14] that S_2 also fulfils condition $(*)$ from Remark 10 and hence conditions $(a2)$ and $(a3)$. Thus Theorem 13 applies and we obtain a strong solution to problem (10) for a 2-Caratheodory map $F : T \times H \longrightarrow H$.

Example 4 (Nonlinear evolution inclusions). Let (E, H, E^*) be an evolution triple of spaces where we assume that E embeds compactly into H (hence so does H into E^*). Let $A : T \times E \to E^*$ be an operator, measurable in t and monotone and hemicontinuous in x, satisfying the assumptions given in Zeidler [18: p. 771]. Let $x_0 \in H$. Then for each $f \in L^q(T, H)$ $(1 < q < \infty)$ there exists a unique solution $S_q(f) \in W_p^1(T; E, H)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ of the quasi-autonomous problem (11) (see [18]). Since $W_p^1(T; E, H) \subset C(T, H)$ we obtain a mapping $S_q : L^q(T, H) \to C(T, H)$. Now condition (a1) follows from the integration by parts formula for elements in $W_p^1(T; E, H)$, and Papageorgiou and Shahzad [15] show that S_q satisfies condition $(*)$ of Remark 10. Thus, given a q-Carathéodory map $F : T \times H \longrightarrow H$, we infer the existence of a solution to problem (10).

Of course, all the above given existence results are well-known (see [5, 13 - 16]). However, as authors use the fixed-point approach to prove these theorems, they usually show that the map $\mathcal{F}$ (given in (9)) has contractible values and apply fixed-point arguments for set-valued maps of this type. In our opinion the presented approach is more direct and simplifies this argument.

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