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Lévy-driven tempo-spatial Ornstein-Uhlenbeck processes

Bachelor's Thesis

by

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I hereby declare that I have written the Bachelor's Thesis on my own and have used no other than the stated sources and aids.

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Zusammenfassung

Wir verallgemeinern den Lévy-getriebenen Ornstein-Uhlenbeck Prozess von einem Prozess mit Zeitparameter zu einem Raum-Zeit Prozess. Dazu stellen wir eine stochastische Volterra-Integralgleichung in Raum und Zeit auf, welche ein stochastisches Integral bezüglich einer Lévy-Basis als Komponente enthält. Wir formulieren Bedingungen für die Existenz und Eindeutigkeit der Lösung und leiten eine explizite Lösungsformel her. Nachdem wir Kriterien für die Stationarität des Lösungsprozesses angeben, berechnen wir die Kovarianzstruktur im stationären Fall anhand der Lösungsformel. Die theoretischen Resultate werden von konkreten Beispielen veranschaulicht.

Summary

We extend the Lévy-driven Ornstein-Uhlenbeck process as a timewise process to time and space. This is achieved by deploying stochastic Volterra integral equations in time and space, which comprises a stochastic integral with respect to a Lévy basis. We formulate conditions for the existence and uniqueness of the solution and derive an explicit solution formula. After giving criteria for stationarity of these processes, we establish the second order structure in the stationary case by means of the solution formula. The theoretical results are illustrated by concrete examples.

Contents

1 Introduction

A Lévy-driven Ornstein-Uhlenbeck (OU) process is defined as the unique solution of the stochastic integral equation

$$X(t) = \int_0^t -\lambda X(s)\ \mathrm{d}s + \int_0^t \mathrm{d}L(s), \quad t \geq 0,$$

where L is a Lévy process, i.e. a process with independent and stationary increments and càdlàg paths (see e.g. Applebaum [1, Section 4.3.5]). The main goal of this thesis is to generalize the timewise OU process to time and space. Our method to achieve this is to generalize the defining stochastic integral equation. First we introduce a spatial component x as an element of the space $\mathbb{R}^d$ in addition to the temporal component t. Moreover, we use a multi-parameter analogue of a Lévy process in order to work in a tempo-spatial framework. Lévy bases (see Definition 2.10) have proven to be a suitable replacement for the Lévy process since they inherit most important properties of Lévy processes and there exists a tractable stochastic integration theory for them (see Rajput and Rosinski [7]).
Now let us consider the following stochastic integral equation

$$X(t,x) = \int_0^t -\lambda X(s,x)\ \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} e^{-\lambda' \|x-y\|} \Lambda(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

where Λ is a Lévy basis. The underlying idea of this model is that innovations coming from Λ at any site do not only affect the evolution of the process at this site but rather at every site in the whole space. However, the magnitude of the impact is damped depending on the distance to the site at which the innovation occurred. This is realized with an exponential function as integrand of the second integral. Again, both integrals integrate within the time interval from 0 to t in order to allow for temporal causality. That is, the past influences the present and the present does not depend on the future. Models like the previous one are studied in this thesis in the more general context of *convolution Volterra integral equations*, which are equations of the form

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y) + f(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

where μ denotes a measure on $\mathbb{R}^+ \times \mathbb{R}^d$ and f is a function on $\mathbb{R}^+ \times \mathbb{R}^d$, usually called the *forcing function*. A rather complete theory exists for these equations in the case where the forcing function is *deterministic* (see Gripenberg [4, Chapter 4]). In contrast, we study these equations under the assumption that the forcing function is a stochastic integral, allowing us to embed the above model. More precisely, f is chosen to be

$$f(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0,\ x \in \mathbb{R}^d,$$

where g is a deterministic function on $\mathbb{R}^+ \times \mathbb{R}^d$. We examine for which combinations of μ, g and Λ the stochastic convolution Volterra integral equation has a solution and employ the theory to define Lévy-driven tempo-spatial Ornstein-Uhlenbeck processes.

This thesis is structured as follows. After a brief recap of the integration theory w.r.t. Lévy bases and the deterministic theory of convolution Volterra integral equations in section 2, these two concepts are merged in section 3. Therein we solve the stochastic Volterra equation and present an explicit formula for its unique solution. Furthermore, the model above is taken up again and analyzed together with two additional models. In section 4 the question of stationarity is investigated. An affirmative answer is deduced under some mild assumptions. Finally, the second order structure in stationary cases is examined in section 5.

2 Preliminaries

This section provides the theoretical background for this thesis. We briefly review the theory on Volterra integral equations to the extent which is needed for what follows. Additionally a summary of the stochastic integration theory by Rajput and Rosinski [7] is given.

2.1 The deterministic convolution Volterra integral equation

Volterra integral equations, a special class of integral equations, naturally arise and are broadly used in applications, such as physics (cosmic ray transport models, superfluidity), materials science (viscoelasticity of materials with memory) or demography (population dynamics), see [4, Section 1.2] and the examples therein. They are named after Vito Volterra, who was one of the first to examine equations of this type. In this thesis we deal with convolution Volterra integral equations in space-time[1], these are equations of the form

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y) + f(t,x), \tag{2.1}$$

where a measure μ and a deterministic function f, called the forcing function, are given[2]. In this case a rather complete theory exists, which we present in line with Gripenberg [4]. Note that t is usually interpreted as a time parameter, likewise x its the spatial parameter.

Before we proceed to solve this equation for X, let us fix some terminology. Let S be a Borel subset of $\mathbb{R}^d$ for some dimension $d \in \mathbb{N}$, then $M(S)$ denotes the space of all signed complete Borel measures on S with finite total variation. For $\mu \in M(S)$ let $|\mu|$ be its total variation measure and $\|\mu\| = |\mu|(S)$ be its total variation norm. As a matter of fact, $M(S)$ becomes a Banach space when equipped with this norm, which will be employed later on.

Moreover, the notation $M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is used for signed measures on $\mathbb{R}^+ \times \mathbb{R}^d$ which lie in $M([0,T] \times \mathbb{R}^d)$ when restricted to $[0,T] \times \mathbb{R}^d$ for all positive T. Similarly $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ is the set of real functions on $\mathbb{R}^+ \times \mathbb{R}^d$ which are Lebesgue integrable over $[0,T] \times \mathbb{R}^d$ when restricted to $[0,T] \times \mathbb{R}^d$ for all positive T.

[1] More precisely the equation is called a deterministic linear convolution Volterra integral equation of the second kind in space-time.

[2] By $\int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y)$ we mean $\int_{[0,t]\times\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y)$. We choose this notation in order to emphasize the distinction between the time component and the space component.

Definition 2.1 For two measures $\mu, \eta \in M(\mathbb{R}^{d+1})$ the convolution $\mu * \eta$ is the completion of the measure that assigns to each Borel set $B \subset \mathbb{R}^{d+1}$ the value

$$(\mu * \eta)(B) = \int_{\mathbb{R}^{d+1}} \eta(B - z)\mu(\mathrm{d}z),$$

where $B - z = \{s - z | s \in B\}$.
For two measures $\mu, \eta \in M(\mathbb{R}^+ \times \mathbb{R}^d)$ the convolution $\mu * \eta$ is defined by firstly extending μ and η onto $\mathbb{R}^{d+1}$ via setting $\mu(B) = \mu(B \cap (\mathbb{R}^+ \times \mathbb{R}^d))$ and $\eta(B) = \eta(B \cap (\mathbb{R}^+ \times \mathbb{R}^d))$, then obtaining $\mu * \eta$ as above and finally restricting it again to $\mathbb{R}^+ \times \mathbb{R}^d$. The convolution of two measures $\mu, \eta \in M([0, T] \times \mathbb{R}^d)$ is defined analogously.
μ^{*j} denotes the $(j-1)-$fold convolution of μ by itself if this exists. By convention we set $\mu^{*0} = \delta_0$, i.e. the Dirac measure in the origin. □

Remark 2.2 The function $z \mapsto \eta(B - z)$ is Borel measurable and bounded, as a result the integral $\int_{\mathbb{R}^{n+1}} \eta(B - z)\mu(\mathrm{d}z)$ exists and the convolution $\mu * \eta$ is well-defined (see [4, p. 112]). □

The following proposition sums up some useful properties of the convolution (see [4, Section 4.1] or [8, Example 10.3]).

Proposition 2.3. *Let S be $\mathbb{R}^{n+1}$, $\mathbb{R}^+ \times \mathbb{R}^d$ or $[0, T] \times \mathbb{R}^d$ and μ, η and σ be measures in $M(S)$. Then*

1. *$\mu * \eta \in M(S)$ and $\|\mu * \eta\| \leq \|\mu\| \|\eta\|$,*

2. *$(\mu * \eta) * \sigma = \mu * (\eta * \sigma)$,*

3. *$\mu * \eta = \eta * \mu$.*

Moreover, $M(S)$ equipped with the total variation norm and the convolution product is a commutative unital Banach algebra.

In light of this proposition the next result is immediate.

Corollary 2.4. *For μ, η and $\sigma \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ it holds $\mu * \eta \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, $(\mu * \eta) * \sigma = \mu * (\eta * \sigma)$ and $\mu * \eta = \eta * \mu$.*

In addition to the convolution of two measures, we define the convolution of a function and a measure.

Definition 2.5 For a measure $\mu \in M(\mathbb{R}^+ \times \mathbb{R}^d)$ and a real-valued, measurable function h on $\mathbb{R}^+ \times \mathbb{R}^d$ the convolution $h * \mu = \mu * h$ is the function

$$(h * \mu)(u) = (\mu * h)(u) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} h(u - z)\mu(\mathrm{d}z),$$

defined for those u for which the integral exists, i.e. for those u for which the function $s \mapsto h(u-s)$ is $|\mu|$-integrable. $\square$

Similar to Proposition 2.3, some information about the convolution of a function with a measure is given in the next proposition (see [4, Section 3.6]).

Proposition 2.6. *Let μ and η be measures in $M_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and $h \in \mathrm{L}^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$. Then*

1. *$h * \mu \in \mathrm{L}^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$,*

2. *$(h * \mu) * \eta = h * (\mu * \eta)$ and $(\mu * h) * \eta = \mu * (h * \eta)$.*

3. *If additionally $\mu \in M(\mathbb{R}^+ \times \mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ is bounded, then $h * \mu \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ is bounded.*

Having established these results we are able to proceed with solving equation (2.1). A function X from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}$ is called a solution of (2.1) if the equation holds for almost all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. We cannot expect more since the forcing function f, which is usually chosen to be an element of $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, is only defined up to null sets. In certain cases, that is for special choices of μ and f, there is a unique solution and it can be expressed explicitly in terms of a measure ρ, which is related to μ. In the construction of ρ we follow the structure of the proof of [4, Thm. 3.3.1]. However, Gripenberg only deals with the purely temporal case, this is why we have to extend the proof to our tempo-spatial setting.

Theorem 2.7. *Let $\mu \in M_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\mu(\{0\} \times \mathbb{R}^d) = 0$. Then there exists a unique measure $\rho \in M_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\rho + \mu = \mu * \rho$.*

Proof. First we show that for each positive T there is a unique ρ_T in $M([0,T] \times \mathbb{R}^d)$ such that

$$\rho_T + \mu_T = \mu_T * \rho_T.$$

Here μ_T is the restriction of μ on $[0,T] \times \mathbb{R}^d$.
To show the existence of ρ_T we construct a geometric series and use a Banach space argument:
Let us consider first the special case $\|\mu_T\| = |\mu_T|([0,T] \times \mathbb{R}^d) = |\mu_T|((0,T] \times \mathbb{R}^d) < 1$, where the second equation follows from the assumptions. Define

$$\rho_m := -\sum_{j=1}^{m} \mu_T^{*j}, \quad m \in \mathbb{N},$$

then

$$\rho_m + \mu_T = -\sum_{j=1}^{m} \mu_T^{*j} + \mu_T = -\sum_{j=2}^{m} \mu_T^{*j} = \mu_T * (-\sum_{j=1}^{m-1} \mu_T^{*j}) = \mu_T * \rho_{m-1}, \quad m \in \mathbb{N}\backslash\{1\}.$$

By Proposition 2.3, we have
$$\|\mu_T^{*j}\| \le \|\mu_T\|^j,$$
thus (ρ_m) is a Cauchy sequence and converges to some $\rho_T \in M([0,T]\times\mathbb{R}^d)$ because $M([0,T]\times\mathbb{R}^d)$ is a Banach space. In addition, $\mu_T * \rho_m \to \mu_T * \rho_T$ in $M([0,T]\times\mathbb{R}^d)$ by Proposition 2.3, so that we get

$$\rho_T + \mu_T = \mu_T * \rho_T.$$

Next we show that we may always, without loss of generality, take $\|\mu_T\| < 1$. To see this consider the measure $\lambda_m(\mathrm{d}s, \mathrm{d}y) := e^{-ms}\mu_T(\mathrm{d}s, \mathrm{d}y)$ and note that for sufficiently large m we have $\|\lambda_m\| < 1$. In this case there is a unique η_m satisfying $\eta_m + \lambda_m = \lambda_m * \eta_m$ as above. But then $\rho_T(\mathrm{d}s, \mathrm{d}y) := e^{ms}\eta_m(\mathrm{d}s, \mathrm{d}y)$ satisfies

$$\begin{aligned}
\rho_T(\mathrm{d}s, \mathrm{d}y) + \mu_T(\mathrm{d}s, \mathrm{d}y) &= e^{ms}\eta_m(\mathrm{d}s, \mathrm{d}y) + e^{ms}e^{-ms}\mu_T(\mathrm{d}s, \mathrm{d}y)\\
&= e^{ms}\eta_m(\mathrm{d}s, \mathrm{d}y) + e^{ms}\lambda_m(\mathrm{d}s, \mathrm{d}y)\\
&= e^{ms}(\eta_m(\mathrm{d}s, \mathrm{d}y) + \lambda_m(\mathrm{d}s, \mathrm{d}y))\\
&= e^{ms}(\lambda_m * \eta_m)(\mathrm{d}s, \mathrm{d}y)\\
&= ([e^{ms}\lambda_m(\mathrm{d}s, \mathrm{d}y)] * [e^{ms}\eta_m(\mathrm{d}s, \mathrm{d}y)])(\mathrm{d}s, \mathrm{d}y)\\
&= (\mu_T * \rho_T)(\mathrm{d}s, \mathrm{d}y),
\end{aligned}$$

where the fifth equation follows from the definition of the convolution. Thus $\rho_T + \mu_T = \mu_T * \rho_T$.

To show uniqueness of ρ_T assume that there are ρ_T and η_T in $M([0,T]\times\mathbb{R}^d)$ with $\rho_T + \mu_T = \mu_T * \rho_T$ and $\eta_T + \mu_T = \mu_T * \eta_T$. Then

$$\rho_T = \mu_T * \rho_t - \mu_T = (\mu_T * \eta_T - \eta_T) * \rho_T - \mu_T = \eta_T * (\mu_T * \rho_T - \rho_T) - \mu_T = \eta_T * \mu_T - \mu_T = \eta_T.$$

Now, having constructed ρ_T for every positive T and noting that for every $j \in \mathbb{N}$ the restriction of ρ_{j+1} to $[0,j]\times\mathbb{R}^d$ must be equal to ρ_j by uniqueness, we define a measure ρ on $\mathbb{R}^+\times\mathbb{R}^d$ by setting $\rho = \rho_T$ on $[0,T]\times\mathbb{R}^d$ and extend it onto $\mathbb{R}^+\times\mathbb{R}^d$ via the uniqueness theorem for measures (see e.g. Billingsley [3, Thm. 3.3]). Note that it also holds $\rho \in M_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d)$ and $\rho + \mu = \mu * \rho$, which finishes the proof. □

Definition 2.8 The measure ρ in Theorem 2.7 is called *the resolvent of* μ. □

With the resolvent at our disposal we can prove the main theorem of this subsection.

Theorem 2.9. *Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d)$ with $\mu(\{0\}\times\mathbb{R}^d) = 0$. Then*

1. *for every $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d)$ there is a unique solution $X \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d)$ of* (2.1). *This solution is given by*

$$X(t,x) = f(t,x) - \int_0^t \int_{\mathbb{R}^d} f(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y), \quad (t,x) \in \mathbb{R}^+\times\mathbb{R}^d \tag{2.2}$$

or in short $X = f - f * \rho$, where ρ is the resolvent of μ.

2. *for every measurable* $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *such that* $f * \rho$, $f * \mu$ *and* $(f * \rho) * \mu$ *exist there is a unique measurable solution* $X : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *of* (2.1). *This solution is also given by* (2.2).

Proof. Let ρ be the resolvent of μ as in Theorem 2.7. Then for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ define X by (2.2) while employing Proposition 2.6. Also by Proposition 2.6, we obtain $X \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and

$$X - \mu * X = X - \mu * (f - \rho * f) = X - (\mu - \mu * \rho) * f = X + \rho * f = f,$$

thus X is a solution of (2.1).
To show uniqueness let $\tilde{X}$ be an arbitrary solution of (2.1) in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$. Then

$$\tilde{X} = f + \mu * \tilde{X} = f + (\rho * \mu - \rho) * \tilde{X} = f - \rho * (\tilde{X} - \mu * \tilde{X}) = f - \rho * f,$$

hence $\tilde{X} = X$. The proof in the second case is analogous. □

2.2 Stochastic integration w.r.t. Lévy bases

No stochasticity was involved in our previous considerations. We will later bring it in to allow for stochastic modeling. This will be done through the concept of stochastic integration with respect to so-called Lévy bases, which was proposed by Rajput and Rosinski in their seminal paper [7]. Let us briefly recall this theory. For the rest of this thesis we fix some complete probability space $(\Omega, \mathcal{F}, P)$. Also let $\mathcal{B}_{\mathrm{b}}(S)$ be the collection of all bounded Borel sets in $S \subseteq \mathbb{R}^d$.

Definition 2.10 A stochastic process $(\Lambda(B))_{B \in \mathcal{B}_{\mathrm{b}}(S)}$ is called a *random measure on*[3] $\mathcal{B}_{\mathrm{b}}(S)$ if for disjoint sets $(B_i)_{i\in\mathbb{N}}$ in $\mathcal{B}_{\mathrm{b}}(S)$ satisfying $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}_{\mathrm{b}}(S)$ we have

$$\Lambda\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \Lambda(B_i)$$

almost surely, where the r.h.s. is assumed to converge almost surely.
Further it is called *independently scattered* if $(\Lambda(B_i))_{i\in\mathbb{N}}$ are independent for disjoint $(B_i)_{i\in\mathbb{N}}$ in $\mathcal{B}_{\mathrm{b}}(S)$.
If $\Lambda(B)$ is infinitely divisible for all $B \in \mathcal{B}_{\mathrm{b}}(S)$, then Λ is called *infinitely divisible*, too.
Now a *Lévy basis* on S is an infinitely divisible independently scattered random measure on $\mathcal{B}_{\mathrm{b}}(S)$.
Moreover it is called *homogeneous* if $\mathrm{Leb}(B) = \mathrm{Leb}(\tilde{B})$ implies $\Lambda(B) \overset{\mathrm{d}}{=} \Lambda(\tilde{B})$ for all $B, \tilde{B}$ in $\mathcal{B}_{\mathrm{b}}(S)$, i.e. $\Lambda(B)$ and $\Lambda(\tilde{B})$ have the same distribution. Here Leb denotes the Lebesgue measure. □

[3] We also say "on S" instead of on "on $\mathcal{B}_{\mathrm{b}}(S)$".

In [7], the stochastic integral w.r.t. a Lévy basis is firstly defined for simple integrands and then for more general deterministic integrands. However, since we only regard homogeneous Lévy bases, the theory simplifies considerably. Let us fix a homogeneous Lévy basis Λ on S for the rest of this subsection. Due to the homogeneity and the Lévy-Khintchine formula for infinitely divisible distributions (see e.g. Applebaum [1]), the characteristic function of $\Lambda(B)$ can be written as

$$\Phi(\Lambda(B))(u) = \exp\left\{\mathrm{Leb}(B)[iub - \frac{1}{2}u^2 C + \int_{\mathbb{R}} (e^{iuz} - 1 - iu\tau(z))\nu(\mathrm{d}z)]\right\}$$

for all $u \in \mathbb{R}$ and $B \in \mathcal{B}_{\mathrm{b}}(S)$, where the truncation function τ is defined as $\tau(z) = z\mathbb{1}_{(-1,1)}(z)$. The drift term $b \in \mathbb{R}$, the Gaussian part $C \in \mathbb{R}^+$ and the Lévy measure ν on $\mathbb{R}$ are independent of the choice of B. We call (b, C, ν) the characteristic triplet of Λ. Now a simple function is a function h of the form $h = \sum_{i=1}^n w_i \mathbb{1}_{B_i}$ with real w_i and disjoint $B_i \in \mathcal{B}_{\mathrm{b}}(S)$. For those, the canonical integral over a Borel set $B \in \mathcal{B}(S)$ is defined as

$$\int_B h \; \mathrm{d}\Lambda = \sum_{i=1}^n w_i \Lambda(B \cap B_i).$$

Definition 2.11 A measurable function $h : S \to \mathbb{R}$ is called Λ-*integrable* if there exists a sequence of simple functions (h_n) such that

1. h_n converges to h Leb-a.e.,

2. $(\int_B h_n \; \mathrm{d}\Lambda)$ converges in probability for all $B \in \mathcal{B}(S)$.

In this case we define:

$$\int_B h \; \mathrm{d}\Lambda = \mathrm{P}\text{-}\lim_{n\to\infty} \int_B h_n \; \mathrm{d}\Lambda.$$

Here $\mathrm{P}\text{-}\lim_{n\to\infty}$ denotes the limit in probability. □

Remark 2.12 It can be shown, that the integral above does not depend on the approximating sequence (h_n), hence the integral is well-defined (see [7]). □

This definition does not exactly specify the class of integrable functions. Nevertheless, Rajput and Rosinski give an integrability condition in terms of the characteristic triplet of Λ in [7, Thm. 2.7]. We state it for homogeneous Lévy bases in the next proposition, which is a special case of theorem [7, Thm. 2.7].

Proposition 2.13. *A measurable function $h : S \to \mathbb{R}$ is Λ-integrable if and only if*

1. $\int_S |bh(s) + \int_{\mathbb{R}} (\tau(zh(s)) - h(s)\tau(z))\nu(\mathrm{d}z)| \; \mathrm{d}s < \infty$,

2. $\int_S C|h(s)|^2 \; \mathrm{d}s < \infty$, *3.* $\int_S \int_{\mathbb{R}} \min\{1, |zh(s)|^2\}\nu(\mathrm{d}z)\ \mathrm{d}s < \infty$.

In that case the characteristic function of $\int_S h\ \mathrm{d}\Lambda$ *can be written as*

$$\Phi\left(\int_S h\ \mathrm{d}\Lambda\right)(u) = \exp\left\{iub_h - \frac{1}{2}u^2 C_h + \int_{\mathbb{R}} (e^{iuz} - 1 - iu\tau(z))\nu_h(\mathrm{d}z)\right\},$$

where

- $b_h = \int_S (bh(s) + \int_{\mathbb{R}} (\tau(zh(s)) - h(s)\tau(z))\nu(\mathrm{d}z))\ \mathrm{d}s$,
- $C_h = \int_S C|h(s)|^2\ \mathrm{d}s$,
- $\nu_h(B) = F(\{(s,z) \in S \times \mathbb{R} | h(s)z \in B\backslash\{0\}\})$ *with* $F = \mathrm{Leb} \times \nu$.

3 The stochastic case

The *deterministic* convolution Volterra integral equation was solved in the last section. As already mentioned, we are going to implement the stochasticity in this section. This will be done by choosing the forcing function f in equation (2.1) to be a certain stochastic integral itself, namely

$$f(t,x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y), \tag{3.1}$$

or in short $f = g * \Lambda$. We present conditions for f to be well-defined. Afterwards we proceed with solving the stochastic convolution Volterra integral equation and give an explicit formula for the solution. In some cases it has itself a representation as a stochastic integral w.r.t. the same Lévy basis Λ from (3.1). This fact is extensively used in upcoming sections. The second part of this section applies these results to three concrete examples including the model which was mentioned in the introduction.

3.1 The general solution

Our first concern should be to ensure that f in equation (3.1) is well defined. For simplicity we assume Λ to have finite second moments. Then f is already well defined under the additional assumptions of boundedness and integrability of g. This result is stated more generally in the next lemma.

Lemma 3.1. *Let $S \subseteq \mathbb{R}^d$, $h : S \to \mathbb{R}$ be a bounded and integrable function and Λ be a homogeneous Lévy basis on S with finite second moments, i.e. $\Lambda(B) \in \mathrm{L}^2(\Omega, \mathcal{F}, P)$ for all $B \in \mathcal{B}_\mathrm{b}(S)$. Then h is Λ-integrable.*

Proof. Our aim is to confirm the conditions of Proposition 2.13. Let (b, C, ν) be the characteristics of Λ and $K \in \mathbb{R}^+$ be a bound on h, i.e. $|h(s)| \leq K$ for all $s \in S$. Without loss of generality let $K > 1$.

1. Let us first show, that

$$|\mathbb{1}_{(-1,1)}(xh(s)) - \mathbb{1}_{(-1,1)}(x)| \leq \mathbb{1}_{\mathbb{R}\setminus(-K^{-1},K^{-1})}(x)$$

for all $s \in S$ and $x \in \mathbb{R}$.
If $h(s) = 0$, then

$$|\mathbb{1}_{(-1,1)}(xh(s)) - \mathbb{1}_{(-1,1)}(x)| = \mathbb{1}_{\mathbb{R}\setminus(-1,1)}(x) \leq \mathbb{1}_{\mathbb{R}\setminus(-K^{-1},K^{-1})}(x).$$

If $0 < |h(s)| \leq 1$, then

$$|\mathbb{1}_{(-1,1)}(xh(s)) - \mathbb{1}_{(-1,1)}(x)| = \mathbb{1}_{(-|h(s)|^{-1},|h(s)|^{-1})\setminus(-1,1)}(x) \leq \mathbb{1}_{\mathbb{R}\setminus(-K^{-1},K^{-1})}(x).$$

Finally if $1 < |h(s)| \leq K$, then

$$|\mathbb{1}_{(-1,1)}(xh(s)) - \mathbb{1}_{(-1,1)}(x)| = \mathbb{1}_{(-1,1)\setminus(-|h(s)|^{-1},|h(s)|^{-1})}(x) \leq \mathbb{1}_{(-1,1)\setminus(-K^{-1},K^{-1})}(x) \\ \leq \mathbb{1}_{\mathbb{R}\setminus(-K^{-1},K^{-1})}(x).$$

Recalling $\tau(x) = x\mathbb{1}_{(-1,1)}(x)$, we get

$$\begin{aligned}
&\int_S |bh(s) + \int_{\mathbb{R}} (\tau(xh(s)) - h(s)\tau(x))\nu(\mathrm{d}x)| \; \mathrm{d}s \\
\leq &\int_S |bh(s)| \; \mathrm{d}s + \int_S \left| \int_{\mathbb{R}} (\tau(xh(s)) - h(s)\tau(x))\nu(\mathrm{d}x) \right| \; \mathrm{d}s \\
= &\int_S |bh(s)| \; \mathrm{d}s + \int_S \left| \int_{\mathbb{R}} h(s)x\mathbb{1}_{(-1,1)}(xh(s)) - h(s)x\mathbb{1}_{(-1,1)}(x)\nu(\mathrm{d}x) \right| \; \mathrm{d}s \\
\leq &\int_S |bh(s)| \; \mathrm{d}s + \int_S |h(s)| \int_{\mathbb{R}} |x||\mathbb{1}_{(-1,1)}(xh(s)) - \mathbb{1}_{(-1,1)}(x)|\nu(\mathrm{d}x) \; \mathrm{d}s \\
\leq &\int_S |bh(s)| \; \mathrm{d}s + \int_S |h(s)| \int_{\mathbb{R}\setminus(-K^{-1},K^{-1})} |x|\nu(\mathrm{d}x) \; \mathrm{d}s \\
= &(|b| + \int_{\mathbb{R}\setminus(-K^{-1},K^{-1})} |x|\nu(\mathrm{d}x)) \int_S |h(s)| \; \mathrm{d}s.
\end{aligned}$$

Since Λ has finite second moments, it has finite first moments. This implies that the first integral in the last line is finite (see [1, Section 2.5]). Since h is integrable on S, the second integral in the last line is finite, too.

2. As a bounded and integrable function on S, h is also square-integrable. Hence $\int_S C|h(s)|^2 \; \mathrm{d}s < \infty$.

3. $\int_S \int_{\mathbb{R}} \min\{1, |xh(s)|^2\}\nu(\mathrm{d}x) \; \mathrm{d}s \leq \int_S \int_{\mathbb{R}} |xh(s)|^2\nu(\mathrm{d}x) \; \mathrm{d}s = \int_{\mathbb{R}} |x|^2\nu(\mathrm{d}x) \int_S |h(s)|^2 \; \mathrm{d}s < \infty$ because Λ has finite second moments and h is square-integrable.

$\square$

Corollary 3.2. *Let $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded and Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Then $f(t, x)$ in* (3.1) *is well defined for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$.*

Proof. Fix $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$. Then the function $(s, y) \mapsto g(t - s, y - x)$ is bounded and integrable on $[0, t] \times \mathbb{R}^d$ by assumption. The restriction of Λ to $[0, t] \times \mathbb{R}^d$ is still homogeneous and has finite second moments. Therefore the integral $\int_0^t \int_{\mathbb{R}^d} g(t - s, x - y)\Lambda(\mathrm{d}s, \mathrm{d}y)$ exists by Theorem 3.1. $\square$

For our purposes it is often necessary to work with a measurable version of $f(t, x)$. Two stochastic processes $f(t, x)$ and $\tilde{f}(t, x)$ on $\mathbb{R}^+ \times \mathbb{R}^d$ are versions of each other if

$f(t,x) = \tilde{f}(t,x)$ a.s. for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$. From now on we will always assume f to be almost surely measurable. This is justified by the next lemma, which is a direct consequence of Lebedev [5, Thm. 1].

Lemma 3.3. *Let $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded and Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Then $f(t,x)$ as defined in* (3.1) *has a version, which is almost surely measurable.*

Having specified $f(t,x)$ in a proper way, we continue with studying the stochastic convolution Volterra integral equation

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y). \tag{3.2}$$

Similarly to the deterministic case, a stochastic process X on $\mathbb{R}^+ \times \mathbb{R}^d$ is called a solution to (3.2) if the equation holds almost surely for almost all (t,x). The strategy for solving (3.2) is to solve it ω-wise by using the deterministic theory in section 2. This is retained in the main theorem of this section:

Theorem 3.4. *Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ with $\mu(\{0\} \times \mathbb{R}^d) = 0$, $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded and Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Furthermore let f be an almost surely measurable version in equation* (3.1) *via Corollary 3.2 and Lemma 3.3. Then there is a unique (up to versions) solution of* (3.2)*. This solution is given by*

$$X(t,x) = f(t,x) - \int_0^t \int_{\mathbb{R}^d} f(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{3.3}$$

or in short $X = f - f * \rho$, where ρ is the resolvent of μ.

Proof. We begin with claiming that for all $T \in \mathbb{R}^+$ there is a $C_T \in \mathbb{R}^+$ such that

$$\sup_{[0,T]\times\mathbb{R}^d} \mathbb{E}(|f(t,x)|) \leq C_T.$$

For $t \in \mathbb{R}^+$ and $x, \tilde{x} \in \mathbb{R}^d$ it holds that

$$\begin{aligned} f(t, x+\tilde{x}) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x+\tilde{x}-y)\Lambda(\mathrm{d}s, \mathrm{d}y) \quad &= \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \tilde{x}+\mathrm{d}y) \\ \stackrel{\mathrm{d}}{=} \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y) \quad &= f(t,x). \end{aligned}$$

Therefore the distribution of $f(t,x)$ does not depend on x. Using Theorem 5.2 from Section 5 we get for all $T \in \mathbb{R}^+$

$$\sup_{[0,T]\times\mathbb{R}^d} \mathbb{E}(|f(t,x)|) \leq \sup_{[0,T]\times\mathbb{R}^d} \sqrt{\mathbb{E}(|f(t,x)|^2)} = \sup_{[0,T]\times\mathbb{R}^d} \sqrt{\mathrm{Var}(f(t,x)) + \mathbb{E}(f(t,x))^2} < \infty,$$

which proves the claim. Note that the first inequality holds by Jensen's inequality. Recalling $\rho \in M_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and invoking the measurability of f, it follows for all $t \in \mathbb{R}^+, x \in \mathbb{R}^d$ that

$$
\begin{aligned}
&\mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^d} f(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y)\right|\right) \\
\leq &\mathbb{E}\left(\int_0^t \int_{\mathbb{R}^d} |f(t-s, x-y)||\rho|(\mathrm{d}s, \mathrm{d}y)\right) \\
\leq &\int_0^t \int_{\mathbb{R}^d} \mathbb{E}(|f(t-s, x-y)|)|\rho|(\mathrm{d}s, \mathrm{d}y) \\
\leq &\int_0^t \int_{\mathbb{R}^d} C_t|\rho|(\mathrm{d}s, \mathrm{d}y) \\
= &C_t|\rho|([0,t] \times \mathbb{R}^d) < \infty.
\end{aligned}
$$

This implies that $\int_0^t \int_{\mathbb{R}^d} f(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y)$ exists and is finite almost surely for all (t, x). Analogously we have that $f * \mu$ is well defined. Now we only have to use the second part of Theorem 2.9 ω-wise to finish the proof. □

Our solution in equation (3.3) can also be represented as a stochastic integral with respect to Λ. Beforehand, we need an auxiliary Fubini type result which allows us to interchange the order of integration between a stochastic integral and a Lebesgue integral. For a proof we refer to Lebedev [5, Thm. 2].

Lemma 3.5. *Let $B \subseteq \mathbb{R}^d$, $T \in \mathbb{R}^+$, $S = [0,T] \times B$ and Λ be a homogeneous Lévy basis with finite second moments on S. Further let $\tilde{S} \subseteq \mathbb{R}^{\tilde{d}}$, $\tilde{d} \in \mathbb{N}$ and μ be a signed measure on $\tilde{S}$. If a measurable function $h : S \times \tilde{S} \to \mathbb{R}$ is such that*

1. *$h(\cdot, \tilde{s})$ is Λ-integrable for μ-almost all $\tilde{s} \in \tilde{S}$ and*

2. *$\int_{\tilde{S}}(\int_S h^2(s, \tilde{s})\ \mathrm{d}s)^{\frac{1}{2}}|\mu|(\mathrm{d}\tilde{s}) < \infty$,*

then $\int_{\tilde{S}} \int_S h(s, \tilde{s})\Lambda(\mathrm{d}s)\mu(\mathrm{d}\tilde{s})$ is well-defined, $\int_{\tilde{S}} h(s, \tilde{s})\mu(\mathrm{d}\tilde{s})$ is Λ-integrable and

$$\int_{\tilde{S}} \int_S h(s, \tilde{s})\Lambda(\mathrm{d}s)\mu(\mathrm{d}\tilde{s}) = \int_S \int_{\tilde{S}} h(s, \tilde{s})\mu(\mathrm{d}\tilde{s})\Lambda(\mathrm{d}s) \quad a.s.$$

Theorem 3.6. *Under the conditions of Theorem 3.4, the unique solution X has a version with the representation*

$$X(t, x) = \int_0^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \tag{3.4}$$

or in short $X = (g * (\delta_0 - \rho)) * \Lambda$.

Proof. Extend g onto $\mathbb{R} \times \mathbb{R}^d$ by setting $g(s,y) := 0$ for all $s < 0, y \in \mathbb{R}^d$ and fix $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Then it holds under Theorem 3.4

$$\begin{aligned}
X(t,x) &= f(t,x) - \int_0^t \int_{\mathbb{R}^d} f(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} f(t-s,x-y)\delta_0(\mathrm{d}s,\mathrm{d}y) - \int_0^t \int_{\mathbb{R}^d} f(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} f(t-s,x-y)(\delta_0 - \rho)(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} \int_0^{t-s} \int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta)(\delta_0-\rho)(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} \int_0^{t} \int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta)(\delta_0-\rho)(\mathrm{d}s,\mathrm{d}y) = (*).
\end{aligned}$$

Our intention is the application of Lemma 3.5 with the choice $S = \tilde{S} = [0,t] \times \mathbb{R}^d$. By the definition of the extension of g and Theorem 3.1, the function $(t-s-\alpha, x-y-\beta) \mapsto g(t-s-\alpha, x-y-\beta)$ is Λ-integrable for all $(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$. For this reason condition one in Lemma 3.5 is satisfied. Since g is integrable and bounded on $[0,T] \times \mathbb{R}^d$ for every positive T, it is square-integrable on $[0,T] \times \mathbb{R}^d$ for every positive T. Consequently,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} g^2(t-s-\alpha, x-y-\beta)\mathrm{d}\alpha\mathrm{d}\beta \right)^{\frac{1}{2}} |(\delta_0-\rho)|(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} \left(\int_0^{t-s} \int_{\mathbb{R}^d} g^2(t-s-\alpha, x-y-\beta)\mathrm{d}\alpha\mathrm{d}\beta \right)^{\frac{1}{2}} |(\delta_0-\rho)|(\mathrm{d}s,\mathrm{d}y) \\
&= \int_0^t \int_{\mathbb{R}^d} \|g\|_{\mathrm{L}^2([0,t-s]\times\mathbb{R}^d)} |(\delta_0-\rho)|(\mathrm{d}s,\mathrm{d}y) \leq \|g\|_{\mathrm{L}^2([0,t]\times\mathbb{R}^d)} |(\delta_0-\rho)|([0,t]\times\mathbb{R}^d) < \infty,
\end{aligned}$$

where the last inequality holds because $(\delta_0 - \rho)$ is an element of $M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$. Since the second condition is also satisfied, Lemma 3.5 implies

$$\begin{aligned}
(*) &= \int_0^t \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)(\delta_0-\rho)(\mathrm{d}s,\mathrm{d}y)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta) \\
&= \int_0^t \int_{\mathbb{R}^d} \int_0^{t-\alpha} \int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)(\delta_0-\rho)(\mathrm{d}s,\mathrm{d}y)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta) \\
&= \int_0^t \int_{\mathbb{R}^d} (g * (\delta_0-\rho))(t-\alpha, x-\beta)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta) \\
&= \int_0^t \int_{\mathbb{R}^d} (g * (\delta_0-\rho))(t-s, x-y)\Lambda(\mathrm{d}s,\mathrm{d}y),
\end{aligned}$$

which proves the theorem. □

3.2 Examples

The measure μ and the function g in equation (3.2) are specified in the following three examples, giving rise to three stochastic models in time and space. By means of the just developed theory we examine the tempo-spatial evolution of these processes depending on the underlying driving Lévy basis. In fact, all three models share the same measure μ which originates from the classical OU process. Against this background, it may seem fair to call any of these examples an *Lévy-driven tempo-spatial Ornstein-Uhlenbeck process.*

3.2.1 The first model

The first model is exactly the model mentioned in the introduction, namely

$$X(t,x) = \int_0^t -\lambda X(s,x)\ \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} e^{-\lambda'\|x-y\|}\Lambda(\mathrm{d}s, \mathrm{d}y), \tag{3.5}$$

$\lambda, \lambda' > 0$, $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Here and henceforth $\|\cdot\|$ stands for the Euclidean norm whenever the argument is a vector in $\mathbb{R}^d$. A thorough inspection of the equation reveals that the first integral on the right-hand side is copied from the classical OU process taken in time for each fixed site. This results in a mean-reverting feature like in the classical OU case. In addition, the second integral sums up all past innovations coming from Λ. As a consequence, every innovation affects every site in the whole space. The fact that both integrals have the integration borders 0 and t leads to temporal causality, i.e. the past influences the present, which does not depend on the future. The exponential function is chosen as the integrand of the second integral. It damps the magnitude of the innovations in space. If an innovation occurs at site y, then the impact of this innovation on another site x is damped exponentially in terms of the distance between x and y.
It can be seen that this model is of the form (3.2) with the identifications[4]

$$\mu = -\lambda \mathrm{Leb}_{\mathbb{R}^+} \otimes \delta_{0,\mathbb{R}^d}$$

and

$$g(s,y) = e^{-\lambda'\|y\|}, \quad (s,y) \in \mathbb{R}^+ \times \mathbb{R}^d.$$

Attempting to solve this equation through Theorem 3.6 we need to calculate the resolvent of μ first.

Theorem 3.7. *Let $\lambda > 0$ and $\mu = -\lambda \mathrm{Leb}_{\mathbb{R}^+} \otimes \delta_{0,\mathbb{R}^d}$. Then the resolvent of μ is $\rho = (\lambda e^{-\lambda t}\mathrm{d}t) \otimes \delta_{0,\mathbb{R}^d}$.*

[4] A subscript at a measure denotes the domain of this measure. Also note that $\int_0^t -\lambda X(s,x)\ \mathrm{d}s = \int_0^t -\lambda X(t-s,x)\ \mathrm{d}s$.

Proof. The measure $\mu = -\lambda \mathrm{Leb}_{\mathbb{R}^+} \otimes \delta_{0,\mathbb{R}^d}$ lies in $M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and satisfies $\mu(\{0\} \times \mathbb{R}^d) = -\lambda \mathrm{Leb}_{\mathbb{R}^+}(\{0\}) \cdot \delta_{0,\mathbb{R}^d}(\mathbb{R}^d) = -\lambda \cdot 0 \cdot 1 = 0$. Using Theorem 2.7 and since $\rho \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$, we only need to show

$$\rho + \mu = \mu * \rho. \tag{3.6}$$

Let $B = [0,T] \times \{0\}$ for some $T \in \mathbb{R}^+$. Then

$$\rho(B) = \int_0^T \lambda e^{-\lambda s} \,\mathrm{d}s = [-e^{-\lambda s}]_0^T = 1 - e^{-\lambda T}.$$

Since $\mu(B) = -\lambda T$, we obtain $\mu(B) + \rho(B) = 1 - e^{-\lambda T} - \lambda T$. Moreover, recalling Definition 2.1,

$$\begin{aligned}
\mu * \rho(B) = \rho * \mu(B) &= \int_{\mathbb{R}^+ \times \mathbb{R}^d} \mu(B - s)\rho(\mathrm{d}s) \\
&= \int_{[0,T]\times\{0\}} \mu(B - s)\rho(\mathrm{d}s) \\
&= \int_0^T -\lambda(T - s)\lambda e^{-\lambda s} \,\mathrm{d}s \\
&= 1 - e^{-\lambda T} - \lambda T.
\end{aligned}$$

The third equality holds because $\mu(B - s) = 0$ if $s \notin [0,T] \times \{0\}$. Therefore (3.6) holds for B. Now let $\tilde{B} \in \mathcal{B}(\mathbb{R}^+ \times (\mathbb{R}^d \backslash \{0\}))$, then

$$\begin{aligned}
\rho(B + \tilde{B}) + \mu(B + \tilde{B}) &= \rho(B) + \mu(B) && = \mu * \rho(B) \\
&= \int_{[0,T]\times\{0\}} \mu(B - s)\rho(\mathrm{d}s) && = \int_{\mathbb{R}^+\times\{0\}} \mu(B - s)\rho(\mathrm{d}s) \\
&= \int_{\mathbb{R}^+\times\{0\}} \mu(B + \tilde{B} - s)\rho(\mathrm{d}s) && = \int_{\mathbb{R}^+\times\mathbb{R}^d} \mu(B + \tilde{B} - s)\rho(\mathrm{d}s) \\
&= \mu * \rho(B + \tilde{B}).
\end{aligned}$$

The fifth and the sixth equality hold because μ and ρ only charge $\mathbb{R}^+ \times \{0\}$. Consequently (3.6) is also true for $B + \tilde{B}$. Now the set $\{B + \tilde{B} | B = [0,T],\ T \in \mathbb{R}^+,\ \tilde{B} \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d \backslash \mathbb{R}^+ \times \{0\})\}$ is a intersection-stable generator of $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d)$. Applying the uniqueness theorem of measures finishes the proof (see [3, Thm. 3.3]). □

Remark 3.8 Note that $\rho = (\lambda e^{-\lambda t}\mathrm{d}t) \otimes \delta_{0,\mathbb{R}^d} \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ also lies in $M(\mathbb{R}^+ \times \mathbb{R}^d)$ since

$$\|\rho\| = |\rho|(\mathbb{R}^+ \times \mathbb{R}^d) = \rho(\mathbb{R}^+ \times \mathbb{R}^d) = \int_{\mathbb{R}^+} \lambda e^{-\lambda s} \,\mathrm{d}s = 1 < \infty.$$

□

Now we have all tools to solve equation (3.5).

Theorem 3.9. *Let Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Then the unique solution to equation* (3.5) *is*

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} e^{-\lambda(t-s)-\lambda'\|x-y\|} \Lambda(\mathrm{d}s, \mathrm{d}y). \tag{3.7}$$

Proof. All conditions in Theorem 3.4 are satisfied, since g lies in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$. In light of Theorem 3.6, the only thing that is left to show is

$$(g * (\delta_0 - \rho))(t-s, x-y) = e^{-\lambda(t-s)-\lambda'\|x-y\|}.$$

Clearly, $g * \delta_0(t-s, x-y) = g(t-s, x-y) = e^{-\lambda'\|x-y\|}$. Recalling $\rho = (\lambda e^{-\lambda t}\mathrm{d}t) \otimes \delta_{0,\mathbb{R}^d}$ we compute

$$\begin{aligned} g * \rho(t-s, x-y) &= \int_0^{t-s} \int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)\rho(\mathrm{d}\alpha, \mathrm{d}\beta) \\ &= \int_0^{t-s} \int_{\mathbb{R}^d} e^{-\lambda'\|x-y-\beta\|}\rho(\mathrm{d}\alpha, \mathrm{d}\beta) \\ &= \int_0^{t-s} e^{-\lambda'\|x-y\|}\lambda e^{-\lambda\alpha}\ \mathrm{d}\alpha \\ &= e^{-\lambda'\|x-y\|}(1 - e^{-\lambda(t-s)}). \end{aligned}$$

As a consequence,

$$\begin{aligned} (g * (\delta_0 - \rho))(t-s, x-y) &= g * \delta_0(t-s, x-y) - g * \rho(t-s, x-y) \\ &= e^{-\lambda'\|x-y\|} - e^{-\lambda'\|x-y\|}(1 - e^{-\lambda(t-s)}) \\ &= e^{-\lambda(t-s)-\lambda'\|x-y\|}. \end{aligned}$$

□

Remark 3.10 The second integral in equation (3.5) is a Lévy process for fixed x. Nevertheless the solution in Theorem 3.9 is a timewise OU process for fixed x, which is clearly not a Lévy process. □

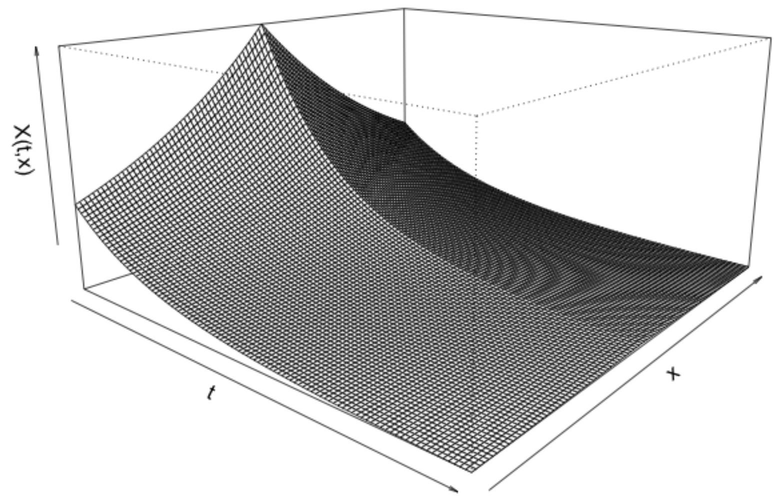

Figure 1: A sample of the tempo-spatial evolution of a single jump innovation in the first model is depicted. The peak belongs to the point of occurrence of the innovation in space-time. Exponential decay in both time and space can be observed. For pure-jump Lévy bases the first model can be understood as the superposition of numerous such jump effects.

3.2.2 The second model

In the first model, innovations of Λ at any site x have an instantaneous damped effect on other sites. In contrast to this, the next model takes a delayed impact mechanism into account, implying that a certain amount of time is needed for the propagation of the innovations from one site to another in space. This is realized by summing up the innovations only on a restricted domain of influence $A(t,x)$ rather than the whole space. These so-called ambit sets were introduced by Barndorff-Nielsen and Schmiegel in a seminal paper on ambit stochastics, see [2]. It is reasonable to assume

$$A(s,x) \subset A(t,x) \text{ for all } s < t \text{ and } A(t,x) \cap (t,\infty) \times \mathbb{R}^d = \varnothing,$$

since these conditions on the ambit sets maintain temporal causality. For simplicity, we only consider translation invariant ambit sets, i.e. ambit sets of the form

$$A(t,x) = A + (t,x).$$

A is chosen to be

$$A = \{(t,x) \in \mathbb{R}^- \times \mathbb{R}^d : \|x\| \leq c|t|\},$$

where $c > 0$ is the constant propagation velocity, see Figure 2.

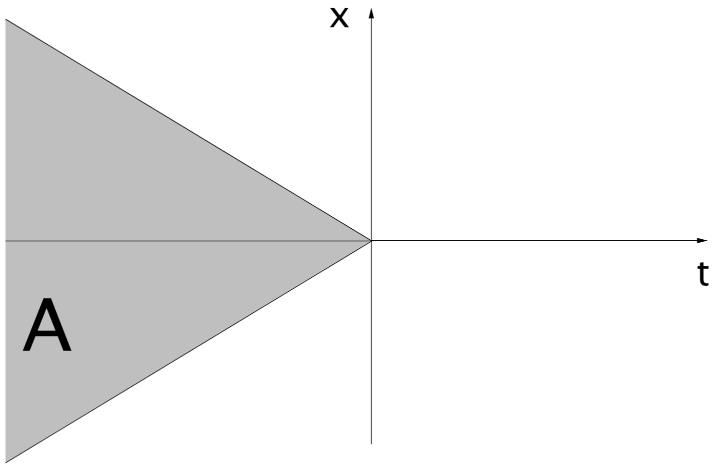

Figure 2: The ambit set $A = \{(t, x) \in \mathbb{R}^- \times \mathbb{R}^d : \|x\| \leq c|t|\}$.

Remark 3.11 It is also possible to model non-constant propagation velocity by setting $A = \{(t, x) \in \mathbb{R}^- \times \mathbb{R}^d : |x| \leq q(|t|)\}$, where q is a non-negative strictly increasing function on $[0, \infty]$ with $q(0) = 0$ (see Nguyen and Veraart [6]). For instance choose $q(|t|) = c|t|^2$ for an uniformly decelerated propagation. $\square$

The ambit sets are incorporated in our second model

$$X(t,x) = \int_0^t -\lambda X(s,x)\ \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y) e^{\frac{-\lambda\|x-y\|}{c}} \Lambda(\mathrm{d}s, \mathrm{d}y), \tag{3.8}$$

$c, \lambda > 0$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. The only difference to the first model is the appearance of the indicator function in the second integral. Notice that we also choose the parameter λ' in the first model to be $\frac{\lambda}{c}$ in the second model. Although this replacement is not necessary, it simplifies the solution formula. As before we aim to solve this equation. The measure μ is here the same as in the first model, which spares us the calculation of the resolvent ρ.

Theorem 3.12. *Let Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Then the unique solution to equation* (3.8) *is*

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y) e^{-\lambda(t-s)} \Lambda(\mathrm{d}s, \mathrm{d}y). \tag{3.9}$$

Proof. First of all we have

$$\mathbb{1}_{A(t,x)}(s,y) = \mathbb{1}_{-A}(t-s, x-y).$$

Therefore
$$g(s,y) = \mathbb{1}_{-A}(s,y)e^{\frac{-\lambda\|y\|}{c}}, \quad (s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$$
lies in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$. Once again all conditions in Theorem 3.4 are satisfied and via Theorem 3.6 we only have to show

$$(g * (\delta_0 - \rho))(t-s, x-y) = \mathbb{1}_{A(t,x)}(s,y)e^{-\lambda(t-s)} = \mathbb{1}_{-A}(t-s, x-y)e^{-\lambda(t-s)}$$

to finish the proof. Recall that $\rho = (\lambda e^{-\lambda t}\mathrm{d}t) \otimes \delta_{0,\mathbb{R}^d}$ by Theorem 3.7. We compute

$$g * \delta_0(t-s, x-y) = g(t-s, x-y) = \mathbb{1}_{-A}(t-s, x-y)e^{\frac{-\lambda\|x-y\|}{c}}$$

and

$$\begin{aligned} g * \rho(t-s, x-y) &= \int_0^{t-s}\int_{\mathbb{R}^d} g(t-s-\alpha, x-y-\beta)\rho(\mathrm{d}\alpha, \mathrm{d}\beta) \\ &= \int_0^{t-s}\int_{\mathbb{R}^d} \mathbb{1}_{-A}(t-s-\alpha, x-y-\beta)e^{-\frac{\lambda}{c}\|x-y-\beta\|}\rho(\mathrm{d}\alpha, \mathrm{d}\beta) \\ &= \int_0^{t-s}\int_{\mathbb{R}^d} \mathbb{1}_{A(t-s,x-y)}(\alpha,\beta)e^{-\frac{\lambda}{c}\|x-y-\beta\|}\rho(\mathrm{d}\alpha, \mathrm{d}\beta) = (*). \end{aligned}$$

Since ρ is supported on the time axis $\mathbb{R}^+ \times \{0\}$, we determine the intersection of $A(t-s, x-y)$ with the time axis. To do this define $\gamma \in \mathbb{R}$ as the time component of the intersection of $A(t-s, x-y) = A + (t-s, x-y)$ with the whole axis $\mathbb{R} \times \{0\}$, see Figure 3.

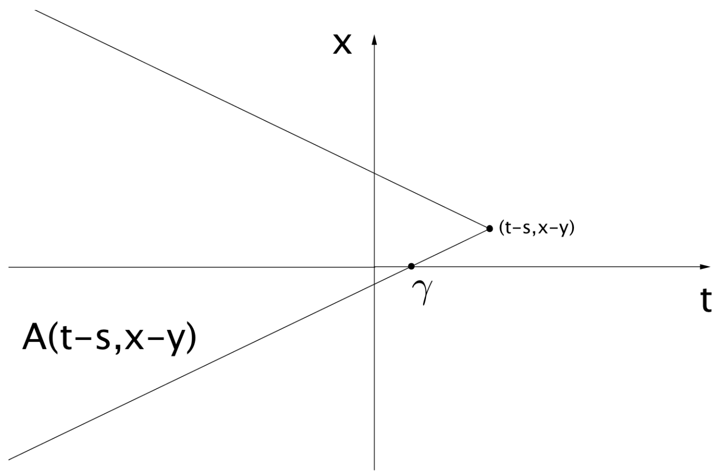

Figure 3: Intersection γ of $A(t-s, x-y)$ with $\mathbb{R} \times \{0\}$.

By definition of A, γ has to satisfy the equation $c((t-s)-\gamma) = \|x-y\|$, that is equivalent to
$$\gamma = t - s - \frac{\|x-y\|}{c}.$$

If $\gamma \geq 0$, then γ is the intersection of $A(t-s, x-y)$ with the positive time axis. Otherwise there is no intersection and the integral in $(*)$ becomes zero. Furthermore,

$$\gamma \geq 0 \iff t - s - \frac{\|x-y\|}{c} \geq 0 \iff c(t-s) \geq \|x-y\| \iff (t-s, x-y) \in -A$$

because $t - s \geq 0$. Consequently,

$$\begin{aligned}
(*) &= \mathbb{1}_{-A}(t-s, x-y) \int_0^\gamma e^{-\frac{\lambda}{c}\|x-y\|} \lambda e^{-\lambda s} \, \mathrm{d}s \\
&= \mathbb{1}_{-A}(t-s, x-y) e^{-\frac{\lambda}{c}\|x-y\|} [-e^{-\lambda s}]_0^\gamma \\
&= \mathbb{1}_{-A}(t-s, x-y) e^{-\frac{\lambda}{c}\|x-y\|} (1 - e^{-\lambda\gamma}) \\
&= \mathbb{1}_{-A}(t-s, x-y) (e^{-\frac{\lambda}{c}\|x-y\|} - e^{-\lambda(t-s)})
\end{aligned}$$

Thus,

$$\begin{aligned}
(g * (\delta_0 - \rho))(t-s, x-y) &= g * \delta_0(t-s, x-y) - g * \rho(t-s, x-y) \\
&= \mathbb{1}_{-A}(t-s, x-y) e^{\frac{-\lambda\|x-y\|}{c}} - \mathbb{1}_{-A}(t-s, x-y)(e^{-\frac{\lambda}{c}\|x-y\|} - e^{-\lambda(t-s)}) \\
&= \mathbb{1}_{-A}(t-s, x-y) e^{-\lambda(t-s)}.
\end{aligned}$$

□

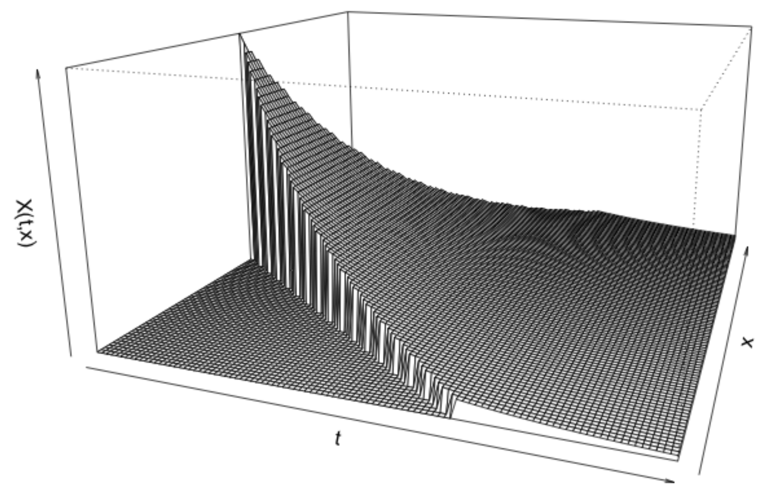

Figure 4: A sample of the tempo-spatial evolution of a single jump innovation in the second model. The peak belongs to the point of occurrence of the innovation in space-time. An exponential decay in time and an uniform propagation in space are observable. For pure-jump Lévy bases the second model can be understood as the superposition of a large number of these jump effects.

In a final remark on the second model the question of why the exponential function in equation (3.9) is only dependent on time is investigated. This is not a priori obvious in regard to the solution in equation (3.7). The second integral in equation (3.8) tells us that every innovation is damped with a rate of $\frac{\lambda}{c}$ in terms of space units. Further the innovations propagate with speed c, resulting in a damping at rate $c\frac{\lambda}{c} = \lambda$ in terms of time units. As a result, the damping rate is λ in every time interval leading to the solution formula in (3.9). This is why the reason for the simple form of this solution is the right choice of the damping rates in (3.8).

3.2.3 The third model

Using the same notation as in the second model, we discuss

$$X(t,x) = \int_0^t -\lambda X(s,x)\ \mathrm{d}s + \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y)\Lambda(\mathrm{d}s,\mathrm{d}y) \tag{3.10}$$

as our third model. The only change lies in omitting the exponential function in the second integral. Hence the dynamics of these two models are quite similar. The major difference is the effect that innovations do not lose their original magnitude when traveling from site to site. Once arrived the magnitude declines exponentially in time as usual due to the mean-reverting integral.

Theorem 3.13. *Let Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments. Then the unique solution to equation* (3.10) *is*

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y)e^{-\lambda(t-s-\frac{\|x-y\|}{c})}\Lambda(\mathrm{d}s,\mathrm{d}y). \tag{3.11}$$

Proof. The proof is completely analogous to the proof of Theorem 3.12. In this case we have

$$g * \delta_0(t-s,x-y) = \mathbb{1}_{-A}(t-s,x-y)$$

and

$$g * \rho(t-s,x-y) = \mathbb{1}_{-A}(t-s,x-y)(1-e^{-\lambda\gamma}),$$

where γ is again the intersection of $A(t-s,x-y)$ with $\mathbb{R}\times\{0\}$. $\square$

In comparison to the second model, the absence of the exponential function in the second integral of equation (3.10) leads to the additional factor $e^{\frac{\lambda\|x-y\|}{c}}$ in the solution formula.

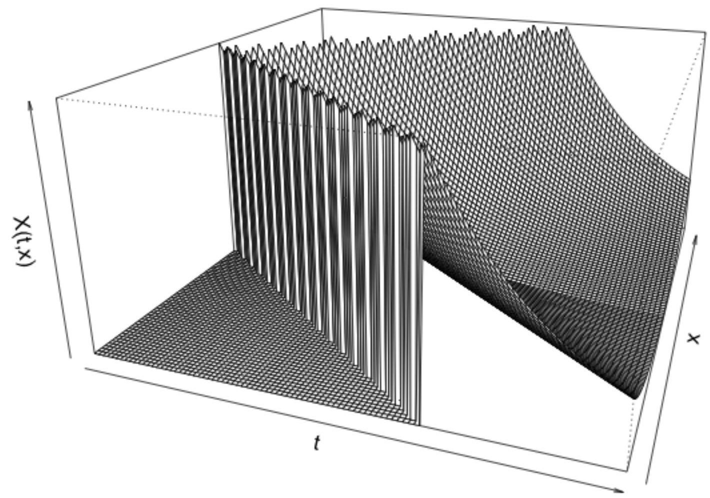

Figure 5: A sample of the tempo-spatial evolution of a single jump innovation in the third model. Two edges at the height of the jump shape the form of the ambit set A. Innovations do not lose their original magnitude when propagating through space. An uniform propagation in space and an exponential decay in time upon arrival are observable. For pure-jump Lévy bases the third model can be understood as the superposition of a large number of these jump effects.

4 Stationarity

In this section we study in which cases the solution to the stochastic convolution Volterra integral equation is stationary. More precisely:

Definition 4.1 A stochastic process X on $\mathbb{R}^+ \times \mathbb{R}^d$ is called *strictly stationary* if for every $n \in \mathbb{N}$, $\tilde{t}, t_1, ..., t_n \in \mathbb{R}^+$ and $\tilde{x}, x_1, ..., x_n \in \mathbb{R}^d$ the distribution of $(X(t_1, x_1), ..., X(t_n, x_n))$ is equal in law to the distribution of $(X(t_1 + \tilde{t}, x_1 + \tilde{x}), ..., X(t_n + \tilde{t}, x_n + \tilde{x}))$. $\square$

From the explicit solution formula in Theorem 3.6 we see that our processes are not strictly stationary (the process X is deterministic at time point $t = 0$, namely zero, but it is not deterministic in general for other time points). To circumvent this obstacle let us modify the integral equation to

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y)\mu(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y) + V(t,x), \quad (4.1)$$

where $V(t,x)$ is a stochastic process on $\mathbb{R}^+ \times \mathbb{R}^d$. Under mild conditions on V a slightly altered solution formula is obtained. This is then used to construct a stationary solution by choosing $\tilde{X}$ of a specific form.

Lemma 4.2. *Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\mu(\{0\} \times \mathbb{R}^d) = 0$, ρ be the resolvent of μ, $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded, Λ be a homogeneous Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with finite second moments and V be a stochastic process on $\mathbb{R}^+ \times \mathbb{R}^d$. Furthermore let f be an almost surely measurable version in equation* (3.1) *via Corollary 3.2 and Lemma 3.3. Under the assumptions*

- *V is almost surely measurable and*
- $\sup\limits_{[0,T] \times \mathbb{R}^d} \mathbb{E}(|V(t,x)|) < \infty$ *for every positive T,*

there is a unique (up to versions) solution of (4.1)*. This solution is given by*

$$\begin{aligned} X(t,x) = &\int_0^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s, x-y)\Lambda(\mathrm{d}s, \mathrm{d}y) + V(t,x) \\ &- \int_0^t \int_{\mathbb{R}^d} V(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \end{aligned} \quad (4.2)$$

or in short $X = (g * (\delta_0 - \rho)) * \Lambda + V - V * \rho$.

Proof. Defining $\tilde{f}(t,x) = f(t,x) + V(t,x)$ as the forcing function we deduce analogously to the proof of Theorem 3.4 that the unique solution of (4.1) is given by

$$\begin{aligned} X(t,x) &= \tilde{f}(t,x) - \int_0^t \int_{\mathbb{R}^d} \tilde{f}(t-s, x-y)\rho(\mathrm{d}s, \mathrm{d}y) \\ &= f(t,x) - \int_0^t \int_{\mathbb{R}^d} f(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y) + V(t,x) - \int_0^t \int_{\mathbb{R}^d} V(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y). \end{aligned}$$

Additionally the stochastic Fubini theorem is applicable as in Theorem 3.6 and the solution can be rewritten in the form

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) + V(t,x) - \int_0^t \int_{\mathbb{R}^d} V(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y).$$

□

Theorem 4.3. *Let $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\mu(\{0\} \times \mathbb{R}^d) = 0$, ρ be the resolvent of μ, $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ be bounded and Λ be a homogeneous Lévy basis on $\mathbb{R} \times \mathbb{R}^d$ with finite second moments. If $g * (\delta_0 - \rho) \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ and is bounded, then there exists a stochastic process V on $\mathbb{R}^+ \times \mathbb{R}^d$ such that equation (4.1) has a unique (up to versions) strictly stationary solution, namely*

$$X(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{4.3}$$

Proof. Set $\tilde{g} := g * (\delta_0 - \rho)$ for abbreviation. Then $\tilde{g} \in \mathrm{L}^1(\mathbb{R}^+ \times \mathbb{R}^d)$ and is bounded implies $\tilde{g} \in \mathrm{L}^2(\mathbb{R}^+ \times \mathbb{R}^d)$. Moreover $(t-s, x-y) \mapsto \tilde{g}(t-s,x-y)$ is bounded and integrable over $\mathbb{R}^- \times \mathbb{R}^d$ for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. By Lemma 3.1, we can define

$$\tilde{X}(t,x) := \int_{-\infty}^0 \int_{\mathbb{R}^d} \tilde{g}(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y),$$

and choose $\tilde{X}$ to be measurable due to Lemma 3.3. For any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ it holds

$$\begin{aligned} \mathbb{E}(|\tilde{X}(t,x)|) &\leq \sqrt{\mathbb{E}(|\tilde{X}(t,x)|^2)} = \sqrt{\mathrm{Var}(\tilde{X}(t,x)) + \mathbb{E}(\tilde{X}(t,x))^2} \\ &= \sqrt{\kappa_2 \int_t^\infty \int_{\mathbb{R}^d} \tilde{g}^2(s,y)\ \mathrm{d}s\mathrm{d}y + \kappa_1^2 \left(\int_t^\infty \int_{\mathbb{R}^d} \tilde{g}(s,y)\ \mathrm{d}s\mathrm{d}y \right)^2} \\ &\leq \sqrt{\kappa_2 \|\tilde{g}\|^2_{\mathrm{L}^2(\mathbb{R}^+\times\mathbb{R}^d)} + \kappa_1^2 \|\tilde{g}\|^2_{\mathrm{L}^1(\mathbb{R}^+\times\mathbb{R}^d)}} =: K < \infty, \end{aligned}$$

with the notation of Theorem 5.2. Set $\tilde{X} * \mu(t,x) := \int_0^t \int_{\mathbb{R}^d} \tilde{X}(t-s,x-y)\mu(\mathrm{d}s,\mathrm{d}y)$. Then

$$\mathbb{E}(|\tilde{X} * \mu(t,x)|) \leq K|\mu|([0,t] \times \mathbb{R}^d),$$

hence $\tilde{X} * \mu(t,x)$ is almost surely finite for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Further $\tilde{X} * \mu$ is almost surely measurable as a convolution of a measurable $\tilde{X}$ with a measure $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and

$$\sup_{[0,T]\times\mathbb{R}^d} \mathbb{E}(|\tilde{X} * \mu(t,x)|) \leq K|\mu|([0,T] \times \mathbb{R}^d) < \infty$$

for every positive T. Now choose $V(t,x) := \tilde{X}(t,x) - \tilde{X} * \mu(t,x)$. The conditions in Lemma 4.2 are satisfied, thus the unique solution can be written as

$$X(t,x) = \int_0^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) + V(t,x) - \int_0^t \int_{\mathbb{R}^d} V(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y)$$

Deploying the definition of V yields

$$\begin{aligned}
&V(t,x) - \int_0^t \int_{\mathbb{R}^d} V(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y) \\
=&\tilde{X}(t,x) - \int_0^t \int_{\mathbb{R}^d} \tilde{X}(t-s,x-y)\mu(\mathrm{d}s,\mathrm{d}y) - \int_0^t \int_{\mathbb{R}^d} \tilde{X}(t-s,x-y)\rho(\mathrm{d}s,\mathrm{d}y) \\
&+ \int_0^t \int_{\mathbb{R}^d} \tilde{X}(t-s,x-y)(\mu * \rho)(\mathrm{d}s,\mathrm{d}y) \\
=&\tilde{X}(t,x).
\end{aligned}$$

The first equation holds due to Proposition 2.6 and the second due to the definition of the resolvent. As a result we have

$$\begin{aligned}
X(t,x) &= \int_0^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) + \tilde{X}(t,x) \\
&= \int_0^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) + \int_{-\infty}^0 \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) \\
&= \int_{-\infty}^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d.
\end{aligned}$$

For strict stationarity consider

$$\begin{aligned}
X(t+\tilde{t},x+\tilde{x}) &= \int_{-\infty}^{t+\tilde{t}} \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t+\underbrace{\tilde{t}-s}_{-\alpha},x+\underbrace{\tilde{x}-y}_{-\beta})\Lambda(\mathrm{d}s,\mathrm{d}y) \\
&= \int_{-\infty}^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-\alpha,x-\beta)\Lambda(\tilde{t}+\mathrm{d}\alpha,\tilde{x}+\mathrm{d}\beta) \\
&\overset{\mathrm{d}}{=} \int_{-\infty}^t \int_{\mathbb{R}^d} (g*(\delta_0-\rho))(t-\alpha,x-\beta)\Lambda(\mathrm{d}\alpha,\mathrm{d}\beta) \\
&= X(t,x),
\end{aligned}$$

for every $t,\tilde{t} \in \mathbb{R}^+$ and $x,\tilde{x} \in \mathbb{R}^d$ under the homogeneity of Λ. The general case $n \in \mathbb{N}$ in Definition 4.1 is treated similarly. □

Remark 4.4 The conditions $g \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ and is bounded and $\rho \in M(\mathbb{R}^+ \times \mathbb{R}^d)$ are sufficient for the condition $g*(\delta_0-\rho) \in L^1(\mathbb{R}^+ \times \mathbb{R}^d)$ and is bounded due to Proposition 2.6. □

Let us apply this theorem to our examples. In the first model the measure $\mu = -\lambda \mathrm{Leb}_{\mathbb{R}^+} \otimes \delta_{0,\mathbb{R}^d}$ satisfies the conditions $\mu \in M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ and $\mu(\{0\} \times \mathbb{R}^d) = 0$. Recalling $g(s,y) = e^{-\lambda' \|y\|}$ and $(g*(\delta_0 - \rho))(s,y) = e^{-\lambda s - \lambda' \|y\|}$ we compute $\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} e^{-\lambda s - \lambda' \|y\|} \,\mathrm{d}s\mathrm{d}y = \int_{\mathbb{R}^+} e^{-\lambda s} \,\mathrm{d}s \int_{\mathbb{R}^d} e^{-\lambda' \|y\|} \,\mathrm{d}y = \frac{2\pi^{d\backslash 2}\Gamma(d)}{\lambda \lambda'^d \Gamma(\frac{d}{2})} < \infty$, where $\Gamma(\cdot)$ is the gamma function. Consequently $g * (\delta_0 - \rho)$ is integrable over $\mathbb{R}^+ \times \mathbb{R}^d$. Since $g * (\delta_0 - \rho)$ is also bounded by 1, Theorem 4.3 is applicable. The strictly stationary solution in the first model can then be represented as

$$X(t,x) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} e^{-\lambda(t-s) - \lambda' \|x-y\|} \Lambda(\mathrm{d}s, \mathrm{d}y).$$

The same argument holds for the second model, where $g(s,y) = \mathbb{1}_{-A}(s,y) e^{\frac{-\lambda \|y\|}{c}}$. Here we have $(g * (\delta_0 - \rho))(s,y) = \mathbb{1}_{-A}(s,y) e^{-\lambda s}$ and

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \mathbb{1}_{-A}(s,y) e^{-\lambda s} \,\mathrm{d}y\mathrm{d}s &= \int_{\mathbb{R}^+} e^{-\lambda s} (cs)^d \mathrm{Vol}_d(B_1(0)) \,\mathrm{d}s \\ &= \int_{\mathbb{R}^+} e^{-\lambda s} (cs)^d \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \,\mathrm{d}s \\ &= \frac{\pi^{d\backslash 2} c^d}{\Gamma(\frac{d}{2}+1)} \int_{\mathbb{R}^+} e^{-\lambda s} s^d \,\mathrm{d}s \\ &= \frac{\pi^{d\backslash 2} c^d \Gamma(d+1)}{\Gamma(\frac{d}{2}+1) \lambda^{d+1}} \\ &< \infty, \end{aligned} \tag{4.4}$$

where $\mathrm{Vol}_d(B_1(0))$ denotes the d-dimensional volume of the Euclidean unit ball in $\mathbb{R}^d$. The strictly stationary solution in the second model is then given by

$$X(t,x) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} \mathbb{1}_{A(t,x)}(s,y) e^{-\lambda(t-s)} \Lambda(\mathrm{d}s, \mathrm{d}y) = \int_{A(t,x)} e^{-\lambda(t-s)} \Lambda(\mathrm{d}s, \mathrm{d}y),$$

which is exactly the canonical $\mathrm{OU}_\wedge$ process introduced in Barndorff-Nielsen and Schmiegel [2]. Finally, $g(s,y) = \mathbb{1}_{-A}(s,y)$ and $(g * (\delta_0 - \rho))(s,y) = \mathbb{1}_{-A}(s,y) e^{-\lambda s + \lambda \frac{\|y\|}{c}}$ in the third model. However, $\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \mathbb{1}_{-A}(s,y) e^{-\lambda s + \lambda \frac{\|y\|}{c}} \,\mathrm{d}y\mathrm{d}s = \int_{\mathbb{R}^+} e^{-\lambda s} \int_{\|x\| \le cs} e^{\lambda \frac{\|y\|}{c}} \,\mathrm{d}y\mathrm{d}s = \int_{\mathbb{R}^+} e^{-\lambda s} \frac{2\pi^{d\backslash 2}}{\Gamma(d\backslash 2)} \int_0^{cs} r^{d-1} e^{\lambda \frac{r}{c}} \,\mathrm{d}r\mathrm{d}s = \infty$, i.e. $g * (\delta_0 - \rho)$ is not integrable over $\mathbb{R}^+ \times \mathbb{R}^d$. Thus we do not obtain a stationary solution in the sense of Theorem 4.3 in this case.

5 Second order structure

Definition 5.1 Let X be a stationary stochastic process on $\mathbb{R}^+ \times \mathbb{R}^d$. The function

$$\operatorname{acf}(\tilde{t}, \tilde{x}) = \operatorname{Cov}(X(0,0), X(\tilde{t}, \tilde{x})), \quad (\tilde{t}, \tilde{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$$

is called *the autocovariance function of* X. The function

$$\operatorname{acorrf}(\tilde{t}, \tilde{x}) = \operatorname{corr}(X(0,0), X(\tilde{t}, \tilde{x})), \quad (\tilde{t}, \tilde{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$$

is called *the autocorrelation function of* X. □

The autocovariance function, is an important tool for stochastic modeling and statistical inference. Note that for stationary X we have $\mathbb{E}(X(t,x)) = \mathbb{E}(X(0,0))$ and $\operatorname{Cov}(X(t,x), X(t+\tilde{t}, x+\tilde{x})) = \operatorname{Cov}(X(0,0), X(\tilde{t}, \tilde{x}))$ for every $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. The following theorem is useful for determining the second order structure.

Theorem 5.2. *Let $S \subseteq \mathbb{R}^d$, $h_1, h_2 \colon S \to \mathbb{R}$ be bounded and integrable functions and Λ be a homogeneous Lévy basis on S with finite second moments and characteristics (b, C, ν). Then it holds*

$$\mathbb{E}\left(\int_S h_1(x)\Lambda(\mathrm{d}x)\right) = \kappa_1 \int_S h_1(x) \, \mathrm{d}x$$

and

$$\operatorname{Cov}\left(\int_S h_1(x)\Lambda(\mathrm{d}x), \int_S h_2(x)\Lambda(\mathrm{d}x)\right) = \kappa_2 \int_S h_1(x)h_2(x) \, \mathrm{d}x,$$

where $\kappa_1 = b + \int_{\mathbb{R}\setminus(-1,1)} x \, \nu(\mathrm{d}x) \in \mathbb{R}$ and $\kappa_2 = C + \int_{\mathbb{R}} x^2 \, \nu(\mathrm{d}x) \in \mathbb{R}^+$ are the expectation and the variance of an infinitely divisible distribution with a finite second moment and characteristics (b, C, ν).

Proof. Define $H_1 := \int_S h_1(x)\Lambda(\mathrm{d}x)$ and $H_2 := \int_S h_2(x)\Lambda(\mathrm{d}x)$ using Theorem 3.1. The proof consists of two parts. In the first part we assume h_1 and h_2 to be simple functions whereas in the second part we deal with the general case.
Part 1. Assume $h_1 = \sum_{i=1}^n x_i \mathbb{1}_{A_i}$ and $h_2 = \sum_{j=1}^m x_j \mathbb{1}_{B_j}$ with $x_i, y_j \in \mathbb{R}$ and $A_i, B_j \in \mathcal{B}_\mathrm{b}(S)$. Then H_1 and H_2 have finite second moments for Λ has finite second moments. Employing that Λ is a homogeneous Lévy basis, we compute

$$\begin{aligned}
\mathbb{E}(H_1) = \mathbb{E}\Big(\sum_{i=1}^n x_i \Lambda(A_i)\Big) &= \sum_{i=1}^n x_i \mathbb{E}(\Lambda(A_i)) \\
&= \sum_{i=1}^n x_i \kappa_1 \operatorname{Leb}(A_i) = \kappa_1 \sum_{i=1}^n x_i \operatorname{Leb}(A_i) \\
&= \kappa_1 \int_S h_1(x) \, \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}(H_1, H_2) &= \mathrm{Cov}\Big(\sum_{i=1}^n x_i \Lambda(A_i), \sum_{j=1}^m x_j \Lambda(B_j)\Big) &&= \sum_{i=1}^n \sum_{j=1}^m x_i x_j \mathrm{Cov}(\Lambda(A_i), \Lambda(B_j)) \\
&= \sum_{i=1}^n \sum_{j=1}^m x_i x_j \mathrm{Var}(\Lambda(A_i \cap B_j)) &&= \sum_{i=1}^n \sum_{j=1}^m x_i x_j \kappa_2 \mathrm{Leb}(A_i \cap B_j) \\
&= \kappa_2 \sum_{i=1}^n \sum_{j=1}^m x_i x_j \mathrm{Leb}(A_i \cap B_j) &&= \kappa_2 \int_S h_1(x) h_2(x) \,\mathrm{d}x.
\end{aligned}$$

Part 2. Since h_1 and h_2 are bounded and integrable, they are also square-integrable, which further implies that $h_1 h_2$ is integrable by the Cauchy-Schwarz inequality. Take without loss of generality two approximating sequences $(h_{1,n})$ and $(h_{2,n})$ of simple functions such that

- $h_{1,n} \to h_1$ almost surely, in $\mathrm{L}^1(S)$ and in $\mathrm{L}^2(S)$,
- $h_{2,n} \to h_2$ almost surely, in $\mathrm{L}^1(S)$ and in $\mathrm{L}^2(S)$ and
- $h_{1,n} h_{2,n} \to h_1 h_2$ in $\mathrm{L}^1(S)$.

Set $H_{1,n} := \int_S h_{1,n}(x)\Lambda(\mathrm{d}x)$ and $H_{2,n} := \int_S h_{2,n}(x)\Lambda(\mathrm{d}x)$. Applying part 1, it holds

$$\begin{aligned}
\|H_{1,n} - H_{1,m}\|^2_{\mathrm{L}^2(\Omega)} &= \mathbb{E}((H_{1,n} - H_{1,m})^2) \\
&= \mathrm{Var}(H_{1,n} - H_{1,m}) + \mathbb{E}(H_{1,n} - H_{1,m})^2 \\
&= \mathrm{Var}\left(\int_S (h_{1,n} - h_{1,m})(x)\Lambda(\mathrm{d}x)\right) + \mathbb{E}\left(\int_S (h_{1,n} - h_{1,m})(x)\Lambda(\mathrm{d}x)\right)^2 \\
&= \kappa_2 \int_S (h_{1,n} - h_{1,m})^2(x) \,\mathrm{d}x + \left(\kappa_1 \int_S (h_{1,n} - h_{1,m})(x) \,\mathrm{d}x\right)^2 \\
&\le \kappa_2 \|h_{1,n} - h_{1,m}\|^2_{\mathrm{L}^2(S)} + \kappa_1^2 \|h_{1,n} - h_{1,m}\|^2_{\mathrm{L}^1(S)}
\end{aligned}$$

Since $h_{1,n}$ converges in $\mathrm{L}^1(S)$ and in $\mathrm{L}^2(S)$, $H_{1,n}$ is a Cauchy sequence in $\mathrm{L}^2(\Omega)$. It follows from the completeness of $\mathrm{L}^2(\Omega)$ that $H_{1,n}$ converges in $\mathrm{L}^2(\Omega)$ to a limit in $\mathrm{L}^2(\Omega)$. In light of Remark 2.12 this limit has to be H_1. Moreover $H_{1,n} \to H_1$ in $\mathrm{L}^1(\Omega)$, because convergence in $\mathrm{L}^2(\Omega)$ implies convergence in $\mathrm{L}^1(\Omega)$. Thus

$$\kappa_2 \int_S h_{1,n}^2(x) \,\mathrm{d}x = \mathrm{Var}(H_{1,n}) \to \mathrm{Var}(H_1)$$

and

$$\kappa_1 \int_S h_{1,n}(x) \,\mathrm{d}x = \mathbb{E}(H_{1,n}) \to \mathbb{E}(H_1).$$

Recalling $h_{1,n} \to h_1$ in $\mathrm{L}^1(S)$ and in $\mathrm{L}^2(S)$ we get

$$\mathbb{E}\left(\int_S h_1(x)\Lambda(\mathrm{d}x)\right) = \kappa_1 \int_S h_1(x)\ \mathrm{d}x$$

and

$$\mathrm{Var}\left(\int_S h_1(x)\Lambda(\mathrm{d}x)\right) = \kappa_2 \int_S h_1(x)^2\ \mathrm{d}x.$$

Analogously it can be shown that $H_{1,n}H_{2,n} \to H_1H_2$ in $\mathrm{L}^1(\Omega)$ and

$$\begin{aligned}\kappa_2 \int_S h_{1,n}(x)h_{2,n}(x)\ \mathrm{d}x &= \mathrm{Cov}(H_{1,n}, H_{2,n}) \\ &= \mathbb{E}(H_{1,n}H_{2,n}) - \mathbb{E}(H_{1,n})\mathbb{E}(H_{2,n}) \to \mathbb{E}(H_1H_2) - \mathbb{E}(H_1)\mathbb{E}(H_2) \\ &= \mathrm{Cov}(H_1, H_2).\end{aligned}$$

Finally

$$\mathrm{Cov}\left(\int_S h_1(x)\Lambda(\mathrm{d}x), \int_S h_2(x)\Lambda(\mathrm{d}x)\right) = \kappa_2 \int_S h_1(x)h_2(x)\ \mathrm{d}x$$

since $h_{1,n}h_{2,n} \to h_1h_2$ in $\mathrm{L}^1(S)$. □

For simplicity the second order structure is only stated in the stationary case in the next result although it is also possible to state it in the general case.

Corollary 5.3. *Under the conditions of Theorem 4.3 let X be the strictly stationary solution of equation* (4.1)*. Then the second order structure of X is given by*

$$\mathbb{E}(X(0,0)) = \kappa_1 \int_{\mathbb{R}^+}\int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(s,y)\ \mathrm{d}s\mathrm{d}y$$

$$\mathrm{acf}(\tilde{t},\tilde{x}) = \mathrm{Cov}(X(t,x), X(t+\tilde{t}, x+\tilde{x})) = \kappa_2 \int_{\mathbb{R}^+}\int_{\mathbb{R}^d} (g*(\delta_0-\rho))(s,y)(g*(\delta_0-\rho))(s+\tilde{t}, y+\tilde{x})\ \mathrm{d}s\mathrm{d}y,$$

for all $t, \tilde{t} \in \mathbb{R}^+$, $x, \tilde{x} \in \mathbb{R}^d$ and κ_1, κ_2 as in Theorem 5.2.

Proof. Recall

$$X(t,x) = \int_{-\infty}^t \int_{\mathbb{R}^d} (g * (\delta_0 - \rho))(t-s, x-y)\Lambda(\mathrm{d}s,\mathrm{d}y) = \int_{-\infty}^t \int_{\mathbb{R}^d} \tilde{g}(t-s,x-y)\Lambda(\mathrm{d}s,\mathrm{d}y)$$

with $\tilde{g} := g * (\delta_0 - \rho)$ for abbreviation. Invoking Theorem 5.2 allows us to compute

$$\mathbb{E}(X(0,0)) = \kappa_1 \int_{-\infty}^0 \int_{\mathbb{R}^d} \tilde{g}(-s, x-y)\ \mathrm{d}s\mathrm{d}y = \kappa_1 \int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \tilde{g}(s,y)\ \mathrm{d}s\mathrm{d}y$$

and

$$
\begin{aligned}
&\operatorname{acf}(\tilde{t},\tilde{x})\\
=&\operatorname{Cov}(X(0,0),X(\tilde{t},\tilde{x}))\\
=&\operatorname{Cov}\left(\int_{-\infty}^{0}\int_{\mathbb{R}^d}\tilde{g}(-s,-y)\Lambda(\mathrm{d}s,\mathrm{d}y),\int_{-\infty}^{\tilde{t}}\int_{\mathbb{R}^d}\tilde{g}(\tilde{t}-s,\tilde{x}-y)\Lambda(\mathrm{d}s,\mathrm{d}y)\right)\\
=&\operatorname{Cov}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{1}_{(-\infty,0]\times\mathbb{R}^d}\tilde{g}(-s,-y)\Lambda(\mathrm{d}s,\mathrm{d}y),\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{1}_{(-\infty,\tilde{t}]\times\mathbb{R}^d}\tilde{g}(\tilde{t}-s,\tilde{x}-y)\Lambda(\mathrm{d}s,\mathrm{d}y)\right)\\
=&\kappa_2\int_{\mathbb{R}}\int_{\mathbb{R}^d}\mathbb{1}_{(-\infty,0]\times\mathbb{R}^d}\tilde{g}(-s,-y)\mathbb{1}_{(-\infty,\tilde{t}]\times\mathbb{R}^d}\tilde{g}(\tilde{t}-s,\tilde{x}-y)\ \mathrm{d}s\mathrm{d}y\\
=&\kappa_2\int_{-\infty}^{0}\int_{\mathbb{R}^d}\tilde{g}(-s,-y)\tilde{g}(\tilde{t}-s,\tilde{x}-y)\ \mathrm{d}s\mathrm{d}y\\
=&\kappa_2\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\tilde{g}(s,y)\tilde{g}(s+\tilde{t},y+\tilde{x})\ \mathrm{d}s\mathrm{d}y.
\end{aligned}
$$

□

Example 5.4 An application of this corollary to the first model yields the following second order structure.

$$
\mathbb{E}(X(0,0))=\kappa_1\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}e^{-\lambda s-\lambda'\|y\|}\ \mathrm{d}s\mathrm{d}y=\kappa_1\int_{\mathbb{R}^+}e^{-\lambda s}\ \mathrm{d}s\int_{\mathbb{R}^d}e^{-\lambda'\|y\|}\ \mathrm{d}y=\frac{2\kappa_1\pi^{d\backslash 2}\Gamma(d)}{\lambda\lambda'^d\Gamma(\frac{d}{2})}
$$

$$
\begin{aligned}
\operatorname{acf}(\tilde{t},\tilde{x})&=\kappa_2\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}e^{-\lambda s-\lambda'\|y\|}e^{-\lambda(s+\tilde{t})-\lambda'\|y+\tilde{x}\|}\ \mathrm{d}s\mathrm{d}y\\
&=\kappa_2e^{-\lambda\tilde{t}}\int_{\mathbb{R}^+}e^{-2\lambda s}\ \mathrm{d}s\int_{\mathbb{R}^d}e^{-\lambda'\|y\|-\lambda'\|y+\tilde{x}\|}\ \mathrm{d}y\\
&=\frac{\kappa_2e^{-\lambda\tilde{t}}}{2\lambda}\int_{\mathbb{R}^d}e^{-\lambda'\|y\|-\lambda'\|y+\tilde{x}\|}\ \mathrm{d}y
\end{aligned}
$$

If $d=1$, i.e. in the case where the space domain is one-dimensional, the explicit formula of the above integral is given by

$$
\int_{\mathbb{R}}e^{-\lambda'\|y\|-\lambda'\|y+\tilde{x}\|}\ \mathrm{d}y=\frac{\lambda'\|\tilde{x}\|+1}{\lambda'}e^{-\lambda'\|\tilde{x}\|}.
$$

This implies an autocorrelation of the form

$$
\operatorname{acorrf}(\tilde{t},\tilde{x})=e^{-\lambda\tilde{t}-\lambda'\|\tilde{x}\|}(\lambda'\|\tilde{x}\|+1).
$$

□

Example 5.5 In the second model it holds similarly

$$
\mathbb{E}(X(0,0))=\kappa_1\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\mathbb{1}_{-A}(s,y)e^{-\lambda s}\ \mathrm{d}s\mathrm{d}y=\frac{\kappa_1\pi^{d\backslash 2}c^d\Gamma(d+1)}{\Gamma(\frac{d}{2}+1)\lambda^{d+1}}
$$

by equation (4.4) and

$$\operatorname{acf}(\tilde{t}, \tilde{x}) = \kappa_2 \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \mathbb{1}_{-A}(s, y) e^{-\lambda s} \mathbb{1}_{-A}(s + \tilde{t}, y + \tilde{x}) e^{-\lambda(s+\tilde{t})} \, \mathrm{d}s \mathrm{d}y.$$

The explicit form of the autocorrelation function in the second model is

$$\operatorname{acorrf}(\tilde{t}, \tilde{x}) = \min(\exp(-\lambda \tilde{t}), \exp(-\frac{\lambda|\tilde{x}|}{c})),$$

which is non-separable in t and x (see Nguyen and Veraart [6, Example 2]). $\square$

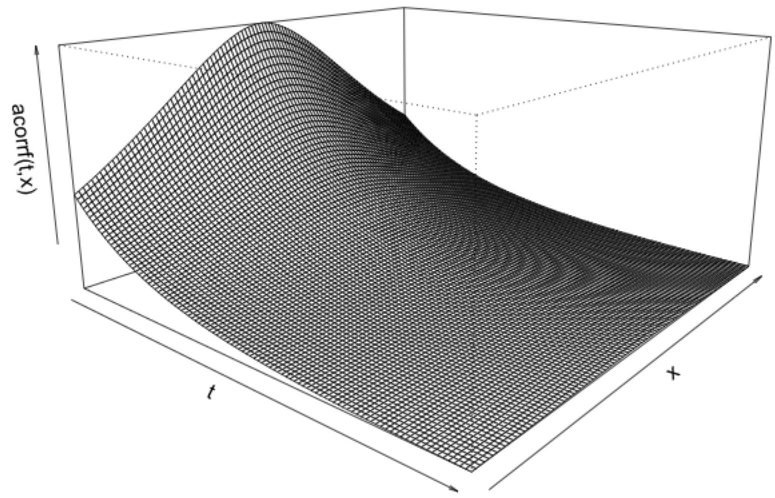

Figure 6: The autocorrelation function in the first model $\mathrm{acorrf}(t,x) = e^{-\lambda t-\lambda'\|x\|}(\lambda'\|x\|+1)$.

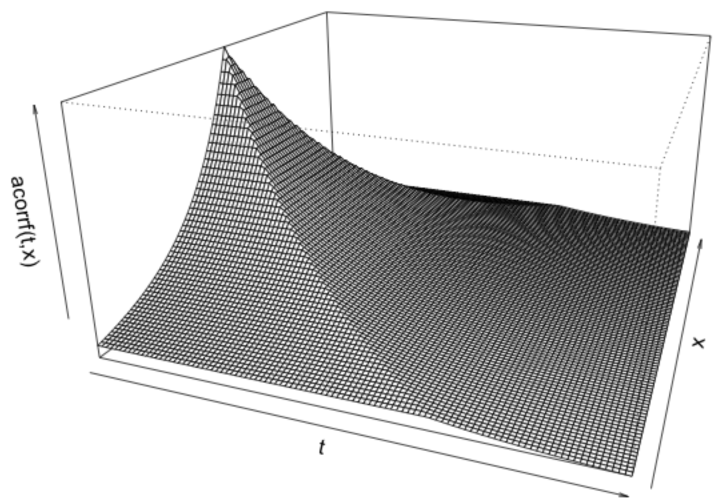

Figure 7: The autocorrelation function in the second model $\mathrm{acorrf}(t,x) = \min(\exp(-\lambda t), \exp(-\frac{\lambda|x|}{c}))$.

6 Index of Notation

$\lvert \cdot \rvert$	the absolute value in $\mathbb{R}$ or the total variation measure
$\lVert \cdot \rVert$	the Euclidean norm in $\mathbb{R}^d$ or the total variation norm of a measure
$\stackrel{\mathrm{d}}{=}$	equality in distribution
$\mathbb{1}_B(\cdot)$	the indicator function
$A(t,x)$	an ambit set
$\mathrm{acf}(\cdot,\cdot)$	the autocovariance function of a stationary process on $\mathbb{R}^+ \times \mathbb{R}^d$
$\mathrm{acorrf}(\cdot,\cdot)$	the autocorrelation function of a stationary process on $\mathbb{R}^+ \times \mathbb{R}^d$
$\mathcal{B}_{\mathrm{b}}(S)$	the collection of all bounded Borel sets in $S \subseteq \mathbb{R}^d$
(b, C, ν)	the characteristic triplet of a homogeneous Lévy basis or an infinitely divisible distribution
$\mu * \eta$, $h * \mu$	the convolution product of two measures or a function and a measure
δ_0	the Dirac measure in the origin
$\mathbb{E}(X)$	the expectation of a random variable
$\Gamma(\cdot)$	the gamma function
Λ	a Lévy basis
$\mathrm{Leb}(\cdot)$	the Lebesgue measure
$\mathrm{L}^p(\cdot)$	the L^p-spaces
$L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$	the set of real functions on $\mathbb{R}^+ \times \mathbb{R}^d$ which are Lebesgue integrable over $[0,T] \times \mathbb{R}^d$ when restricted to $[0,T] \times \mathbb{R}^d$ for all positive T
$M(S)$	the space of all signed complete Borel measures on S with finite total variation
$M_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$	the set of signed measures on $\mathbb{R}^+ \times \mathbb{R}^d$ which lie in $M([0,T] \times \mathbb{R}^d)$ when restricted to $[0,T] \times \mathbb{R}^d$ for all positive T
$\mathbb{N}$	the set $\{1, 2, \ldots\}$ of natural numbers
$\mathrm{P}\text{-}\lim\limits_{n\to\infty}$	limit in probability
$\mathbb{R}^+$	$[0, \infty)$
$\mathbb{R}^-$	$(-\infty, 0]$
$\tau(z)$	the truncation function $\tau(z) = z\mathbb{1}_{(-1,1)}(z)$
$\mathrm{Var}(X)$	the variance of a random variable
$\mathrm{Vol}_d(B_1(0))$	the d-dimensional volume of the Euclidean unit ball in $\mathbb{R}^d$

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