Stability of Gaussian Process Dynamical Systems


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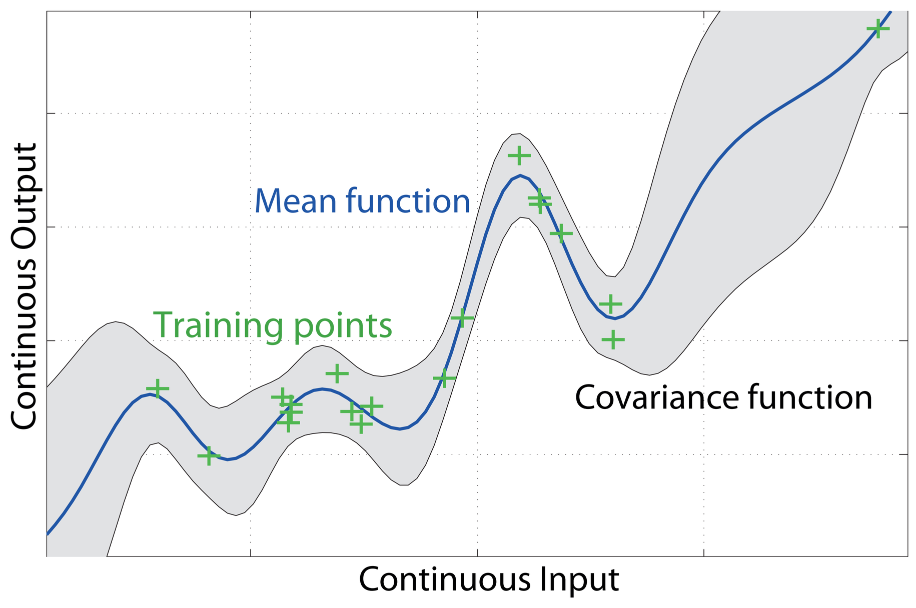
Motivation

Gaussian Process

+ Flexible nonlinear, nonparametric regression
+ Based on Bayesian probability mathematics
+ Knowledge about uncertainty of the estimation



Application

Usage of Gaussian Processes to identify nonlinear system dynamics in model-based control techniques e.g. control of chemical processes or prediction of human behavior. [Turner at al.]

Stability Conditions

Is the learned system $x_{k+1} = Ax_k + f_{err}(x_k)$ stable?

Mean square boundedness

The system must fulfill the following condition

$$\sup_{k \in \mathbb{N}_0} \mathrm{E}\left[\|x_k\|^2\right] < \infty$$

If the linear system part is stable it is possible to find an appropriate Lyapunov function $V(x) = x^\top P x$.

$$\mathrm{E}\left[V(x_{k+1})\right] - V(x_k) = x_k^\top (A^\top P A - P) x_k + \mathrm{E}\left[f^\top(x_k) P f(x_k)\right]$$

The expected value must be bounded to ensure stochastic stability.

$$\mathrm{E}\left[f^\top(x_k) P f(x_k)\right] = \sum_{i=1}^{n} \left(\mu_i^2(x_k) + \sigma_i^2(x_k)\right) P_{ii} + \sum_{j \neq i}^{n} \mu_i \mu_j P_{ij}$$

The a-posteriori mean μ and variance σ^2 of the Gaussian Process with squared exponential kernel is bounded.

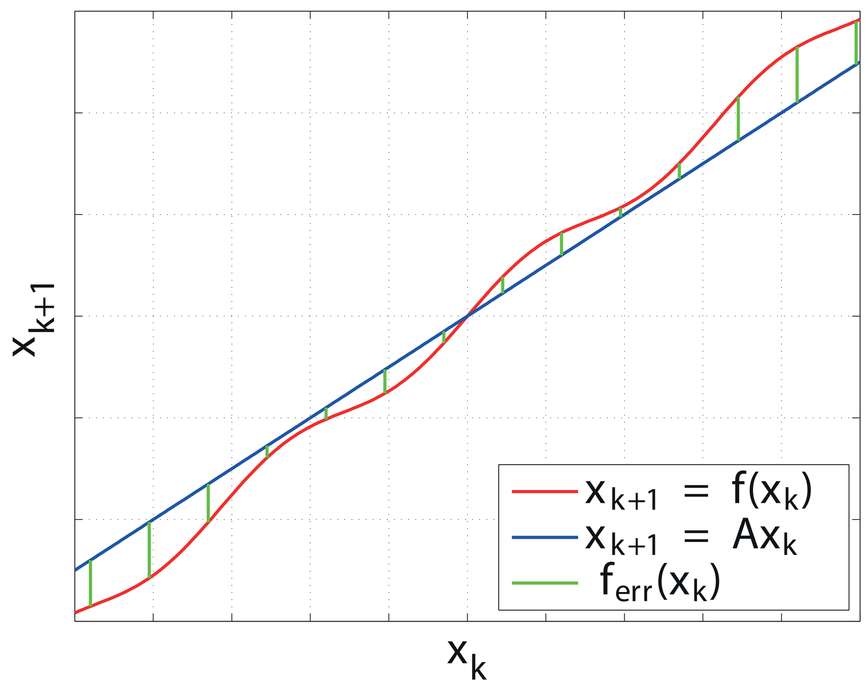
Conjecture

Stochastic asymptotic stability if the equilibrium point has no variance.

Gaussian Process Dynamical Systems

Assumption

- Nonlinear System $x_{k+1} = f(x_k)$
- Identification approach:
  Linear system with unknown nonlinear dynamics

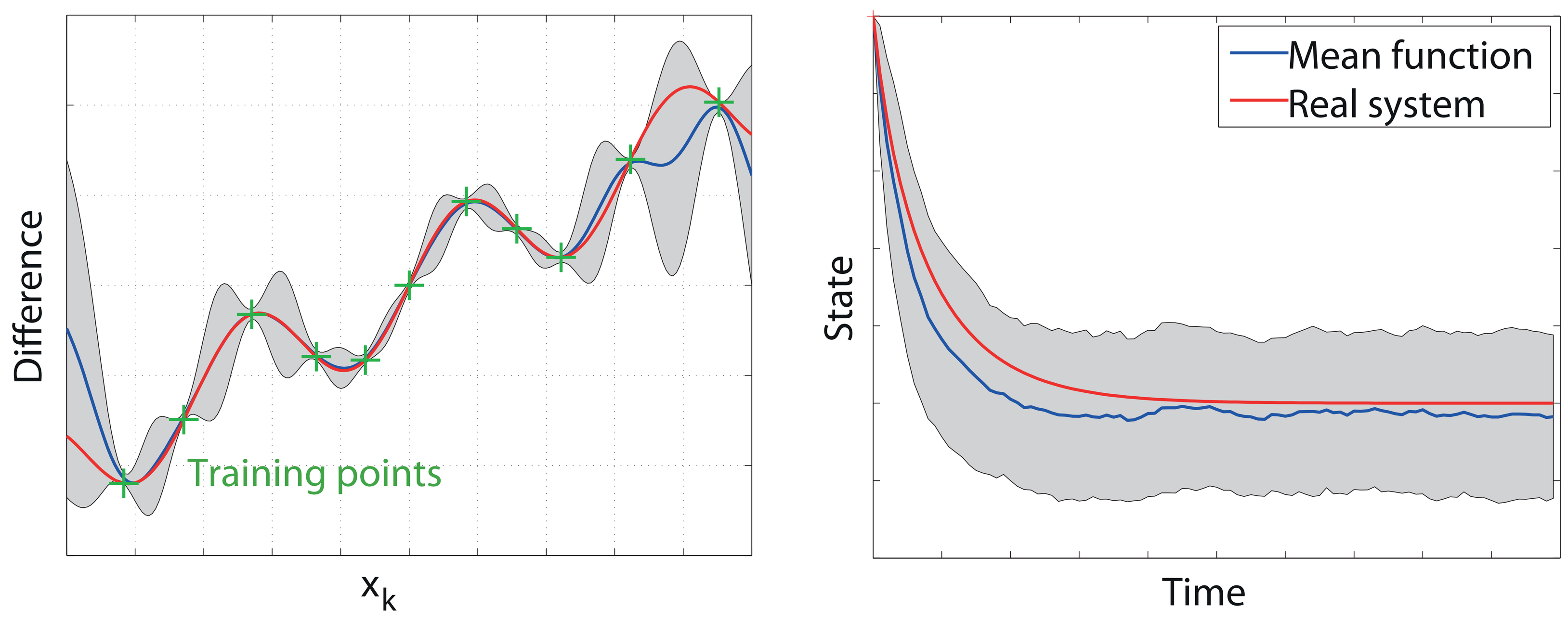


$$x_{k+1} = Ax_k + f_{err}(x_k)$$
$$f_{err} \sim \mathcal{GP}(m(x_k), k(x_k, x_k'))$$
$$f_{err}(x_k) \sim \mathcal{N}(\mu(x_k), \sigma^2(x_k))$$

- Nonlinear dynamics are modelled by Gaussian Processes
- State dependent Gaussian distributed probability variable $f(x_k)$ is defined by mean and variance
- Also known as discrete-time, continuous space Markov chain

Simulation

Numerical validation with example system



Real system: $x_{k+1} = 0.6x_k + 0.3\sin(x_k)$
Assumption: $x_{k+1} = 0.5x_k$

First and second moment of the Gaussian Process Dynamical System are bounded.

A linear stable system with additive uncertainties is mean square bounded if the unknown dynamics are modelled by a Gaussian Process with squared exponential kernel.

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References

1. Turner et al., "System identification in Gaussian process dynamical systems". In D. Görür, editor, NIPS Workshop on Nonparametric Bayes, 2009

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