Discrete-Time Linear Systems Primer

Wolfgang Polifke
Lehrstuhl für Thermodynamik, TU München

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This short write-up introduces notation and summarizes a few basic results about discrete-time linear systems. It is meant to be read as an appendix to [1], where this material could not be included due to page limitations. For further reference, the reader is referred to [2, 3].

The theory of discrete-time linear systems has been developed in the context of digital filter construction in signal processing (applications are typically communications, speech processing, speech or music synthesis, etc.), but is also related to control theory. The connection to continuous (real-world) processes is obvious – sampling of continuous (experimental) data produces a sequence of discrete (equally-spaced in time) data values. Transient numerical simulations, preferably with fixed time step dt, also produce such sequences.

Basic definitions

A *discrete-time* system may be defined as any algorithm which takes an input sequence $s(n)$ (the *signal*) and converts it into an output sequence $r(n)$ (the *response*),

$$s(n) \to r(n).$$

A *linear, time-invariant* system has the following property: if $s_1(n) \to r_1(n)$ and $s_2(n) \to r_2(n)$ for two arbitray input sequences s_1, s_2, then for arbitrary constants a, b and for all n_0

$$\begin{aligned} as_1 + bs_2 &\to ar_1 + br_2, \\ s_1(n - n_0) &\to r_1(n - n_0). \end{aligned}$$

Impulse Sequence & Impulse Response

The *impulse sequence* (or *unit sample*) is defined as follows:

$$u_0(n) = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases} \tag{1}$$

The response to this particular signal is known as the *impulse response* (or *unit sample response*) and the following important relation between signal $s(n)$, response $r(n)$ and impulse response $h(n)$ holds for linear, time-invariant (LTI) systems:

$$r(n) = \sum_{m=-\infty}^{+\infty} h(m)s(n - m). \tag{2}$$

which implies that the sequence $h(n)$ completely characterizes an LTI system. A formal proof of this result is given in [2]. Note the obvious analogy to continuous-time systems, where Green's function is the response to the Dirac-δ signal.

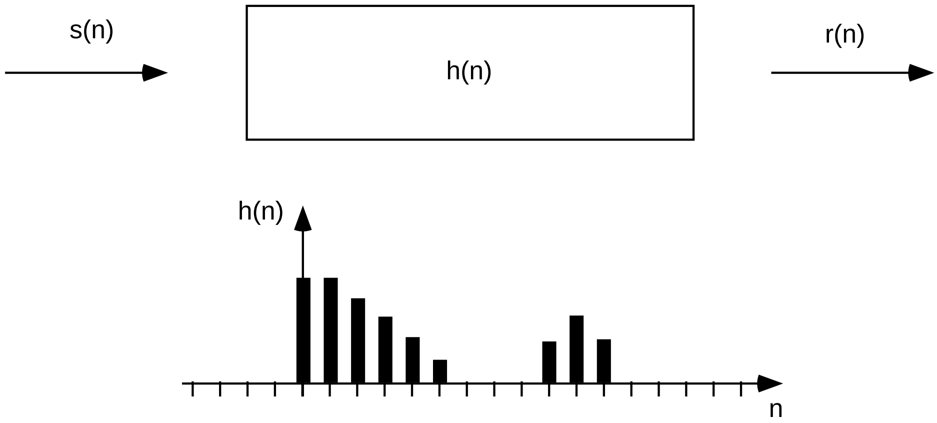

Figure 1: Representation of an LTI system.

Causality / Realizability

An LTI system is *causal* or *realizable* if the output at $n = n_0$ is dependent only on values of the input for $n \leq n_0$, i.e. the present state of the output depends only on present and previous states of the input. This implies that $h(n) = 0$ for $n < 0$.

Frequency Response of LTI systems

Consider now a special class of inputs, $s(n) = e^{i\omega\Delta t\, n}$, in order to investigate the frequency domain representation of LTI systems. Such a signal may be regarded as a sinusoid of frequency ω sampled with a time step Δt. When applied to an LTI system, the resulting output is

$$r(n) = \sum_{m=-\infty}^{+\infty} h(m) e^{i\omega\Delta t\,(n-m)} = e^{i\omega\Delta t\, n} \sum_{m=-\infty}^{+\infty} h(m) e^{-i\omega\Delta t\, m}.$$

With

$$F(\omega) \equiv \sum_{m=-\infty}^{+\infty} h(m) e^{-i\omega\Delta t\, m}, \tag{3}$$

we see that indeed for this particular type of input signal, the response is equal to the signal up to a complex-valued multiplicative factor $F(\omega)$, which depends solely on ω, i.e.

$$r(n) = F(\omega)\, s(n). \tag{4}$$

This multiplicative factor $F(\omega)$ is called the *frequency response* of the system and has the following properties

- The frequency $\omega\Delta t$ appears in the term $\exp(-i\omega\Delta t\, m)$ on the right hand side of (3), therefore the frequency response F is periodic in ω with period $2\pi/\Delta t$.
- if the signal $s(n)$ is purely real, then the absolute value of the frequency response F is symmetric about the points $\omega = 0$ and $\omega = \pi/\Delta t$, while the phase of the frequency response is antisymmetric.

For real-valued signals, it suffices therefore to specify the frequency response in the interval $\omega\Delta t = 0 \rightarrow \pi$ (normalized angular frequency) or $f = 0 \rightarrow 1/2\Delta t$ (frequency). The upper limit of this frequency range is the well-known *Nyquist frequency.*

z-Transform

The z-*Transform* of a sequence $s(n)$ is defined for a complex number z as

$$\hat{S}(z) = \sum_{m=-\infty}^{+\infty} s(m) z^{-m}. \tag{5}$$

From (3) we see that the frequency response F of an LTI system is equal to the z-transform of its impulse response $h(n)$ with argument $z = e^{i\omega\Delta t}$:

$$F(\omega) = \hat{H}(e^{i\omega\Delta t}) = \sum_{m=-\infty}^{+\infty} h(m) e^{-i\omega\Delta t\, m}. \tag{6}$$

In principle, an infinite number of coefficients $h(m),\ -\infty < m < +\infty$ is required to fully specifiy the frequency response. Of course, in practice a good estimate of the frequency response can often be obtained with a rather small number of coefficients ("low order filter"). For a causal system, only coefficients $h(m), m \geq 0$ are required.

Inverse z-Transform

Since the frequency response is periodic in $[-\pi/\Delta t, \pi/\Delta t]$, Eqn. (3) suggests that the coefficients $h(n)$ of the impulse response may be interpreted as the series coefficients of the Fourier series representation of the frequency response $F(\omega)$. This implies the following definition for an *inverse z-transform*

$$h(n) = \frac{\Delta t}{2\pi} \int_{-\pi/\Delta t}^{+\pi/\Delta t} F(\omega)\, e^{i\omega\Delta t\, n}\, d\omega, \tag{7}$$

(actually just the inverse Fourier transform of the frequency response).

One concludes that both the unit impulse response $h(n)$ and the frequency response $F(\omega)$ characterize completely a discrete time linear system and can be converted into each other with the (inverse) z-transform.

References

[1] W. Polifke and A. M. G. Gentemann. Order and realizability of impulse response filters for accurate identification of acoustic multi-ports from transient CFD. In *10th Int. Conf. on Sound and Vibration*, Stockholm, Sweden, July 2003. IIAV.

[2] L. R. Rabiner and B. Gold. *Theory and Application of Digital Signal Processing*. Prentice Hall, 1975.

[3] M. Bellanger. *Digital Processing of Signals*. Wiley Interscience, 3rd edition, 2000.