Spatial Risk Measures:

Local Specification and Boundary Risk

by

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0. Introduction

In a large network of financial institutions, the risk at a given node of the network is usually assessed in terms of some monetary risk measure that involves the marginal distribution at that node. But such an approach neglects the interactive effects that are not captured by the family of marginal distributions. This suggests to take a conditional approach, where the risk measure applied at a given node takes into account the situation at the other nodes of the network; see, for example, [AB]. The question is whether these conditional risk measures can be aggregated in a consistent manner to a global risk measure, and whether the global risk measure is uniquely determined by the local specification.

With this motivation in mind, we are going to focus on some of the purely mathematical problems which arise in such a spatial setting, and which can be viewed as non-linear analogues to some classical problems in the theory of Gibbs measures. In Dobrushin's probabilistic approach to the analysis of phase transitions in Statistical Mechanics, Gibbs measures are specified by a consistent family of local conditional probability distributions; cf. [Do] or [G]. In an infinite spatial network, the global Gibbs measure may not be uniquely determined by the local specification, and this is interpreted as a phase transition. In that case, Gibbs measures can be describes as mixtures of extreme points, and this can be done by using Dynkin's method of constructing the entrance boundary of a Markov process; cf. [Dy1] and [F1].

In analogy to Dobrushin's approach, we start with a given family $(\rho_V)_{V\in\mathcal{V}}$ of local conditional risk measures indexed by the class $\mathcal{V}$ of finite subsets of some infinite set of nodes. These conditional risk measures are convex, and they are assumed to be consistent in the usual sense, that is, $\rho_W(-\rho_V) = \rho_W$ if $V \subseteq W$. Our aim is to clarify the structure of the set $\mathcal{R}$ of global convex risk measures which are consistent with this local specification.

To this end, we assume that the local conditional risk measures ρ_V are absolutely continuous with respect to the local conditional probabilities π_V in the local specification of a Gibbs measure. In the locally law invariant case, the conditional risk of a financial position X would only depend on the distribution of X under the conditional probability measure π_V. In this special case, the local risk measures must be entropic, and the representation of global risk measures can be described in a rather explicit manner; see [F2].

In this paper we go beyond the case of local law invariance. But then the main difficulty consists in extending the local specification $(\rho_V)_{V\in\mathcal{V}}$ to a sufficiently regular conditional risk measure with respect to the tail field. We solve this problem by combining two methods. On the one hand, we use the supermartingale properties implied by local consistency, and in particular the non-linear extension of backwards martingale convergence in [FP2]. On the other hand, we use Dynkin's construction of a "boundary" which describes the extreme points of the convex set of Gibbs measures, and which corresponds to a sub-σ-field $\hat{\mathcal{F}}$ of the tail field. As our main result, we show that a sufficiently regular global risk measure ρ in $\mathcal{R}$ is uniquely determined by its behavior on the boundary field $\hat{\mathcal{F}}$. In particular, we show that we have non-uniqueness of the global risk measure if the underlying probabilistic structure admits a phase transition. From a financial point of view, this can be viewed as one mathematical aspect of the much broader issue of "systemic risk".

The paper is organized as follows. In section 2 we recall some basic facts from the theory of convex risk measures, and in particular the notion of a convex risk kernel introduced in [F2]. In section 3 we describe our spatial setting and the local specification of convex risk measures in terms of local risk kernels. The extension of this local specification to a sufficiently regular convex risk kernel with respect to the tail field is done in two steps. In Section 4 we use a straightforward definition of a limiting kernel ρ_∞ and show that it has good properties with respect to any given Gibbs measure P. But this kernel does not behave well enough simultaneously for all such Gibbs measures. To overcome this difficulty, we introduce an additional regularization that involves Dynkin's boundary construction. This second step is carried out in Section 5, and the resulting risk kernel $\hat{\rho}_\infty$ is shown to be the key to the structure of global risk measures in the class $\mathcal{R}$.

2. Preliminaries on convex risk kernels

In this section we recall some basic definitions and facts from the theory of convex risk measures initiated in [ADEH], [FRG], and [FS1], and also the notion of a convex risk kernel introduced in [F2]. For more details see, for example, [FS2] and [FK].

Let $(\Omega, \mathcal{F})$ be a measurable space, and denote by $M := M_b(\Omega, \mathcal{F})$ the space of all bounded measurable functions on $(\Omega, \mathcal{F})$. A real-valued functional ρ on M is called a *monetary risk measure* if it is *monotone*, i.e., $\rho(X) \geq \rho(Y)$ whenever $X \leq Y$, *cash-invariant*, i.e., $\rho(X+m) = \rho(X) - m$ for constants m, and *normalized*, i.e., $\rho(0) = 0$. If a monetary risk measure ρ is also convex on M, then ρ will be called a *convex risk measure.* A convex risk measure is called *coherent* if it is also positively homogeneous, that is, $\rho(\lambda X) = \lambda\rho(X)$ for any positive constant λ. We denote by $\mathcal{A} := \{X \in M \,|\, \rho(X) \leq 0\}$ the *acceptance set* of ρ; in the convex case the acceptance set is convex, in the coherent case a convex cone.

Now let P be a probability measure on $(\Omega, \mathcal{F})$. If ρ is a monetary risk measure on M such that $\rho(X) = \rho(Y)$ whenever $X = Y$ P-almost surely, then we say that ρ is *absolutely continuous with respect to* P, and we write $\rho \ll P$. In this case, ρ can also be considered as a monetary risk measure on the Banach space $L^\infty(\Omega, \mathcal{F}, P)$. Such a risk measure is called *law-invariant with respect to* P if $\rho(X) = \rho(Y)$ whenever X and Y have the same distribution under P.

Typically, a convex risk measure has a *dual representation*

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \big(E_Q[-X] - \alpha(Q)\big), \tag{2.1}$$

in terms of some set $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$ and some penalty function $\alpha : \mathcal{Q} \to [0, \infty]$. In this case, the representation also holds if we choose

$$\alpha(Q) = \sup_{X \in \mathcal{A}} E_Q[-X], \tag{2.2}$$

and this is the *minimal penalty function* such that (2.1) holds.

A necessary condition for (2.1) is the *Fatou property* of ρ, that is,

$$\lim_k X_k = X \quad \text{pointwise} \implies \rho(X) \leq \liminf_k \rho(X_k) \tag{2.3}$$

for any uniformly bounded sequence (X_k) in M. We say that ρ has the *Fatou property* if (2.3) is replaced by the stronger condition

$$\lim_k X_k = X \quad \text{pointwise} \implies \rho(X) = \lim_k \rho(X_k), \tag{2.4.}$$

Now suppose that $\rho \ll P$. Then the Fatou property is both necessary and sufficient for the dual representation (2.1) of ρ on $L^\infty(\Omega, \mathcal{F}, P)$. In this case we have $Q \ll P$ for any Q such that $\alpha(Q) < \infty$, and so we can restrict $\mathcal{Q}$ to probability measures which are absolutely continuous with respect to P; see Theorem 4.33 in [FS2]. If ρ satisfies the stronger Lebesgue property, then the supremum in (2.1) is actually attained by some Q depending on X; see Corollary 4.35 in in [FS2], and also [D] for a converse result.

Example 2.1. *Let P be a probability measure P on $(\Omega, \mathcal{F})$, and consider the entropic risk measure e_β with parameter $\beta \in [0, \infty)$, defined by*

$$e_\beta(X) = \frac{1}{\beta} \log E_P[e^{-\beta X}]; \tag{2.5}$$

for $\beta = 0$, this will be interpreted as the limiting linear case

$$e_0(\beta) := \lim_{\beta \downarrow 0} e_\beta(X) = E_P[-X]. \tag{2.6}$$

An entropic risk measure is clearly convex and law-invariant. It has the Lebesgue property, and the minimal penalty function in its dual representation (2.1) is given by $\alpha(Q) = \frac{1}{\beta} H(Q|P)$, where $H(Q|P)$ denotes the relative entropy of Q with respect to P; for $\beta = 0$ this is to be read as 0 if $Q = P$ and as $+\infty$ if not.

Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a sub-σ-field of $\mathcal{F}$, and denote by M_0 the space of bounded measurable functions on $(\Omega, \mathcal{F}_0)$. Let us first recall the definition of a *stochastic kernel* $\pi(\omega, d\eta)$ from a $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F})$: For any $\omega \in \Omega$, $\pi(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$, and for any $A \in \mathcal{F}$, the function $\pi(\cdot, A)$ on Ω is $\mathcal{F}_0$-measurable. For a probability measure P on $(\Omega, \mathcal{F}_0)$ we denote by $P\pi$ the probability measure on $(\Omega, \mathcal{F})$ defined by $P\pi[A] = \int \pi(\omega, A) P(d\omega)$. The stochastic kernel will be called *regular* if $\pi(\omega, \cdot) = \delta_\omega$ on $\mathcal{F}_0$. For two such kernels $\pi_i \quad (i = 0, 1)$, their composition $\pi_0\pi_1$ is defined as the stochastic kernel given by $\pi_0\pi_1(\omega, A) = \int \pi_1(\eta, A) d\pi_0(\omega, d\eta)$.

Let us now extend the classical definition of a stochastic kernel in the following manner.

Definition 2.2. *A* monetary risk kernel from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F})$ *is a real-valued function ρ on $\Omega \otimes M$ such that*

i) for each $\omega \in \Omega$, the functional $\rho(\omega_0, \cdot)$ is a monetary risk measure on M,

ii) for each $X \in M$, the function $\rho(\cdot, X)$ belongs to M_0.

Such a monetary risk kernel ρ_0 will be called convex *if all risk measures $\rho_0(\omega, \cdot)$ are convex. It will be called* regular *if*

$$\rho_0(\omega, f(X_0, X)) = \rho_0(\omega, f(X_0(\omega), X)) \tag{2.7}$$

for $\omega \in \Omega$, $X_0 \in M_0$, $X \in M$, and for any bounded measurable function f on R^2. We will say that the risk kernel ρ_0 has the Fatou property*, or the* Lebesgue property*, if condition (2.3) or condition (2.4) holds for each risk measure $\rho_0(\omega, \cdot)$.*

Note that regularity of a monetary risk kernel ρ_0 from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F})$ implies the following *local property*:

$$\rho_0(\omega, I_{A_0} X + I_{A_0^c} Y) = I_{A_0}(\omega)\rho_0(\omega, X) + I_{A_0^c}(\omega)\rho_0(\omega, Y) \tag{2.8}$$

for $\omega \in \Omega$, $X, Y \in M$, and any $A_0 \in \mathcal{F}_0$.

The composition $\rho_0(-\rho_1)$ of two monetary risk kernels ρ_0 and ρ_1 is defined as the monetary risk kernel given by

$$(\rho_0(-\rho_1))(\omega, X) := \rho_0(\omega, -\rho_1(\cdot, X)).$$

If ρ_0 and ρ_1 are both convex, then their composition $\rho_0(-\rho_1)$ is again convex.

If ρ_0 is a regular convex risk kernel from $(\Omega, \mathcal{F}_0)$ to $(\Omega, \mathcal{F})$ such that the risk measures $\rho_0(\omega, \cdot)$ satisfy the condition

$$\rho_0(\omega, \cdot) \ll P \quad P-\text{a.s.}, \tag{2.9}$$

then it is easy to check that ρ_0 can be regarded as a conditional convex risk measure in the usual sense:

Definition 2.7 *A map ρ_0 from $L^\infty(\Omega, \mathcal{F}, P)$ to $L^\infty(\Omega, \mathcal{F}_{\prime}, P)$ is called a* conditional monetary risk measure with respect to $\mathcal{F}_0$ and P *if it satisfies the following three properties for any $X, Y \in L^\infty(\Omega, \mathcal{F}, P)$:*
i) Monotonicity: $\rho_0(X) \geq \rho_0(Y)$ P-a.s. whenever $X \leq Y$ P-a.s.
ii) Conditional cash invariance: for all $m \in L_0^\infty$, $\rho_0(X + m) = \rho_0(X) - m$ P-a.s.
iii) Normalization: $\rho_0(0) = 0$ P-a.s.
Such a conditional risk measure ρ_0 is called convex *if*

$$\rho_0(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_0(X) + (1 - \lambda)\rho_0(Y) \quad P-\text{a.s.}$$

for any $\mathcal{F}_0$-measurable function λ such that $0 \leq \lambda \leq 1$ P-a.s.. It is said to have the Fatou property *if*

$$\lim_k X_k = X \;\; P-\text{a.s.} \implies \rho(X) \leq \liminf_k \rho(X_k) \;\; P-\text{a.s.}$$

for any uniformly bounded sequence (X_k) in $L^\infty(\Omega, \mathcal{F}, P)$, and the Lebesgue property *is defined in the same manner.*

Note that the Fatou or the Lebesgue property of the risk measures $\rho_0(\omega, \cdot)$ in (2.9) implies the corresponding property of ρ_0 regarded as a conditional risk measure with respect to P.

If a convex conditional risk measure ρ_0 with respect with respect to $\mathcal{F}_0$ and P has the Fatou property then it admits a conditional version of the dual representation (2.1). Denoting by

$$\mathcal{A}_0 = \{X \in L^\infty(\Omega, \mathcal{F}, P) \,|\, \rho_0(X) \leq 0 \;\; P-\text{a.s.}\}$$

the *acceptance set* of ρ_0, the dual representation takes the form

$$\rho_0(X) = \operatorname{ess\,sup} \big(E_Q[-X \,|\, \mathcal{F}_0] - \alpha_0(Q)\big), \tag{2.10}$$

where the essential supremum is taken with respect to P and over all probability measures $\mathcal{Q} \ll \mathcal{P}$ such that $Q \approx P$ on the σ-field $\mathcal{F}_0$, and where the minimal penalty function is given by

$$\alpha_0(Q) = \operatorname{ess\,sup}_{X \in \mathcal{A}_0} E_Q[-X \,|\, \mathcal{F}_0], \tag{2.11}$$

see [DS] or Theorem 11.2 in [FS2]. For a general $Q \ll P$, (2.11) is defined as an essential supremum under Q. But if Q satisfies the additional condition $Q \approx P$ on $\mathcal{F}_0$ as in (2.10), then it can as well be read as an essential supremum under P.

3. Local specification of spatial risk measures

Let I be a countable set of *sites*, and let S be some polish state space with Borel σ–field $\mathcal{S}$. We assume that each site $i \in I$ can be in some state $s \in S$, and we denote by $\Omega = S^I$ the set of possible *configurations* $\omega : I \to S$. For any subset $J \subseteq I$, we denote by ω_J the restriction of ω to $J \subseteq I$, by $\mathcal{F}_J$ the σ-field on Ω generated by the projection maps $\omega \to \omega(i)$ for any $i \in J$, and we write $\mathcal{F} = \mathcal{F}_I$. A probability measure P on $(\Omega, \mathcal{F})$ is also called a *random field.*

Let $\mathcal{V}$ denote the class of non-empty finite subsets $V \subseteq I$. For a given set $V \in \mathcal{V}$, the σ-field $\mathcal{F}_V$ describes what is observable on V, while $\mathcal{F}_{V^c}$ describes the situation on $V^c := I - V$, also called the *environment* of V.

Definition 3.1. *A collection $(\rho_V)_{V\in\mathcal{V}}$ of regular convex risk kernels ρ_V from $(\Omega, \mathcal{F}_{V^c})$ to $(\Omega, \mathcal{F})$ is called a* local specification of a convex risk measure *if it satisfies the consistency condition*

$$\rho_W(-\rho_V) = \rho_W \tag{3.1}$$

for any $V, W \in \mathcal{V}$ such that $V \subseteq W$, and if each kernel is regular in the sense of (1.7) and has the Fatou property.

From now on we fix a local specification $(\rho_V)_{V\in\mathcal{V}}$ of a convex risk measure.

Definition 3.2. *Let $\mathcal{R}$ denote the set of all convex risk measures ρ on M which are consistent with the local specification $(\rho_V)_{V\in\mathcal{V}}$, that is,*

$$\rho(-\rho_V) = \rho \quad \text{for any } V \in \mathcal{V}. \tag{3.2}$$

Our aim is to clarify the structure of the global risk measures in $\mathcal{R}$. At the general level of Definition 3.1 there is not much to be said. The situation becomes clearer if we introduce an underlying probabilistic structure, described by the local specification of a random field; cf. [Do] and [G].

Definition 3.3. *A collection $(\pi_V)_{V\in\mathcal{V}}$ of regular stochastic kernels π_V from $(\Omega, \mathcal{F}_{V^c})$ to $(\Omega, \mathcal{F})$ is called a* local specification of a random field *if it satisfies the consistency condition*

$$\pi_W \pi_V = \pi_W \tag{3.3}$$

for any $V, W \in \mathcal{V}$ such that $V \subseteq W$.

Definition 3.4. *We denote by $\mathcal{P}$ the convex set of all random fields P which are consistent with this local specification in the sense that*

$$P\pi_V = P \quad \text{for any } V \in \mathcal{V}. \tag{3.4}$$

A random field $P \in \mathcal{P}$ is also called a Gibbs measure. *The case $|\mathcal{P}| > 1$, where the global random field is not uniquely determined by the local specification $(\pi_V)_{V\in\mathcal{V}}$, is often referred to as a* phase transition.

For any $V \in \mathcal{V}$, the stochastic kernel π_V serves as a conditional probability distribution with respect to $\mathcal{F}_{V^c}$ which is common to all probability measures $\mathcal{P}$, and so we can write

$$E_P[\, f \,|\, \mathcal{F}_{V^c}](\omega) = \int f(\eta)\pi_V(\omega, d\eta) \tag{3.5}$$

for any $P \in \mathcal{P}$ and any measurable function $f \geq 0$ on $(\Omega, \mathcal{F})$.

Let us now fix a local specification $(\pi_V)_{V\in\mathcal{V}}$ of a random field such that

$$\mathcal{P} \neq \emptyset. \tag{3.6}$$

We connect our local specification (ρ_V) of a convex risk measure with the local specification (π_V) by the following assumption:

Assumption 3.5. *For any $\omega \in \Omega$ and any $V \in \mathcal{V}$, the convex risk measure $\rho_V(\omega, \cdot)$ has the following two properties:*

i) $\rho_V(\omega,\cdot) \ll \pi_V(\omega,\cdot)$

ii) If X *is acceptable for* $\rho_V(\omega,\cdot)$ *then the expected loss under the measure* $\pi_V(\omega,\cdot)$ *is uniformly bounded from below, i.e., there is a constant* $c \geq 0$ *such that*

$$\rho_V(\omega, X) \leq 0 \Longrightarrow \int (-X)(\eta) d\pi_V(\omega, d\eta) \leq c. \tag{3.7}$$

Remark 3.6. *The local specification* $(\rho_V)_{V\in\mathcal{V}}$ *is called* law-invariant *if condition i) is replaced by the much stronger assumption that each convex risk measure* $\rho_V(\omega,\cdot)$ *is not only absolutely continuous but even law-invariant with respect to the probability measure* $\pi_V(\omega,\cdot)$. *This implies*

$$\rho_V(\omega, X) \leq \int (-X(\eta)\pi_V(\omega, d\eta)$$

for any $X \in M$, *and so condition (3.7) holds with* $c = 0$; *see Corollary 4.65 in [FS2]. Actually much more is true: Local law invariance together with consistency of* (ρ_V) *implies that the risk measures* $\rho_V(\omega,\cdot)$ *must be* entropic; *see [F] and also [KS]. More precisely, the risk kernel* ρ_V *takes the form*

$$\rho_V(\omega, X) = \frac{1}{\beta_\infty(\omega)} \log \int e^{-\beta_\infty(\omega)X(\eta)} \pi_V(\omega, d\eta) \tag{3.8}$$

with $\beta_\infty(\omega) \in [0,\infty)$ *as in Example 2.1. The parameter* $\beta_\infty(\omega)$ *does not depend on* V, *and this implies that the function* $\beta_\infty(\cdot)$ *is measurable with respect to the tail field* $\mathcal{F}_\infty$ *introduced in Section 4 below.*

Lemma 3.7. *For any* $P \in \mathcal{P}$, *the risk kernel* ρ_V *can be regarded as a conditional risk measure*

$$\rho_V : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_{V^c}, P),$$

and this conditional risk measure has the Fatou property with respect to P.

Proof. Take X and Y in M such that $X = Y$ P-a.s.. We have to show that $\rho_V(\cdot, X) = \rho_V(\cdot, Y)$ P-a.s.. Indeed, the consistency condition $P = P\pi_V$ implies $\pi_V(\cdot, X) = \pi_V(\cdot Y)$ P-a.s., hence $\rho_V(\cdot, X) = \rho_V(\cdot, Y)$ P-a.s. due to part i) of our Assumption 3.5. The Fatou property of the conditional risk measure with respect to P follows from the Fatou property of the risk kernel ρ_V.

We now take a closer look at our consistency condition (3.1). For a given probability measure $P \in \mathcal{P}$, this can be read as a consistency condition for two conditional risk measures with respect to P, as shown in Lemma 3.6. As such, it can be characterized at the level of the corresponding acceptance sets and also at the level of penalty functions; see, for example, [BN] and [FP1]. For our purposes, however, we will need the following property; see [FP1] and Theorem 2 in [AP].

Proposition 3.8. *For any* $P \in \mathcal{P}$ *and any* $V, W \in \mathcal{V}$ *such that* $V \subseteq W$, *the consistency condition* $\rho_W(-\rho_V) = \rho_W$ *yields the supermartingale inequality*

$$\rho_W(X) + \alpha_W(Q) \geq E_Q[\rho_V(X) + \alpha_V(Q) \,|\, \mathcal{F}_{W^c}] \quad P-\text{a.s.} \tag{3.9}$$

for any $X \in L^\infty(\Omega, \mathcal{F}, P)$ *and any probability measure* $Q \ll P$.

4. Passing to the tail field

Our aim is to clarify the structure of the class $\mathcal{R}$ of global convex risk measures which are consistent with our local specification $(\rho_V)_{V\in\mathcal{V}}$, in analogy to the classical analysis of the class $\mathcal{P}$ of global random fields which are consistent with the local specification $(\pi_V)_{V\in\mathcal{V}}$.

This problem is trivial if I is finite: In this case we have $I \in \mathcal{V}$ and $\mathcal{F}_{I^c} = \{\emptyset, \Omega\}$, and so $\rho_I(\omega, \cdot)$ does not depend on ω. Thus there is exactly one risk measure $\rho \in \mathcal{R}$, namely $\rho = \rho_I$.

From now we assume $|I| = \infty$, and so $(\Omega, \mathcal{F})$ is an infinite product space. Here we will proceed in two steps. In this section we are going to extend the local specification $(\rho_V)_{V\in\mathcal{V}}$ in a consistent manner to a risk kernel ρ_∞ with respect to the *tail field*

$$\mathcal{F}_\infty := \bigcap_{V\in\mathcal{V}} \mathcal{F}_{V^c},$$

and we shall describe the properties of ρ_∞ as a conditional risk measure with respect to any given measure $P \in \mathcal{P}$. The second step will be done in the next section. It involves a regularization of the initial kernel ρ_∞, and this will be the key to the structure of global risk measures.

Let us fix a sequence $(V_n) \subseteq \mathcal{V}$ increasing to I, and let us use the notation

$$\rho_n := \rho_{V_n}, \quad n = 1, 2, \dots$$

for the corresponding sequence of risk kernels. Now consider the risk kernel ρ_∞ defined by

$$\rho_\infty(\omega, X) := \limsup_{n\to\infty} \rho_n(\omega, X). \tag{4.1}$$

for any $X \in M$ and any $\omega \in \Omega$. We denote by

$$M_\infty := M_b(\Omega, \mathcal{F}_\infty, P)$$

the space of all bounded measurable functions on $(\Omega, \mathcal{F}_\infty)$. For any $X \in M$, the function $\rho_\infty(\cdot, X)$ belongs to M_∞, since it is bounded by $||X||$ and clearly measurable with respect to the tail field $\mathcal{F}_\infty$.

Lemma 4.1. *The functional $\rho_\infty : M \to M_\infty$ defined by (4.1) is a regular convex risk kernel from $(\Omega, \mathcal{F}_\infty)$ to $(\Omega, \mathcal{F})$, and it satisfies the consistency condition*

$$\rho_\infty(-\rho_V) = \rho_\infty \tag{4.2}$$

for any $V \in \mathcal{V}$.

Proof. For any $\omega \in \Omega$, the functional $\rho_\infty(\omega, \cdot)$ on M inherits from the sequence (ρ_n) the properties of a convex risk measure and also the regularity property (2.7). Moreover we have

$$\rho_\infty(-\rho_V(X)) = \limsup_n \rho_n(-\rho_V(X)) = \limsup_n \rho_n(X) = \rho_\infty(X)$$

for any $V \in \mathcal{V}$, since $\rho_n(-\rho_V(X)) = \rho_n(X)$ as soon as $V \subset V_n$, due to the consistency condition (3.1).

For the rest of this section we fix a probability measure $P \in \mathcal{P}$. We are going to show that the limit superior in (4.1) is P-almost surely a limit, and that ρ_∞ has good properties as a conditional risk measure with respect to P.

Lemma (3.7) shows that the risk kernel ρ_n can be regarded as a conditional risk measure under P with respect to $\mathcal{F}_{V_n^c}$, and that it has the Fatou property with respect to P. We denote by $\mathcal{A}_n(P)$ its acceptance set and by

$$\alpha_n(Q) = \operatorname{ess\,sup}_{X\in\mathcal{A}_n(P)} E_Q[-X|\mathcal{F}_{V_n^c}].$$

its penalty function. It follows that ρ_n admits the dual representation

$$\rho_n(X) = \operatorname{ess\,sup}\big(E_Q[-X|\mathcal{F}_{V_n^c}] - \alpha_n(Q)\big), \tag{4.3}$$

where the essential supremum is taken over all $Q \ll P$ such that $Q \approx P$ on $\mathcal{F}_{V_n^c}$.

Let us also introduce the set

$$\mathcal{Q}(P) := \{Q \in \mathcal{M}_1(P) | Q = P \text{ on } \mathcal{F}_\infty, \ \sup_{\mathrm{n}} \mathrm{E_Q}[\alpha_{\mathrm{n}}(\mathrm{Q})] < \infty\}.$$

As we shall see in the proof of the following Theorem, we have $P \in \mathcal{Q}(P)$, hence $\mathcal{Q}(P) \neq \emptyset$.

Lemma 4.2. *For any $Q \in \mathcal{Q}(P)$, the limit*

$$\alpha_\infty(Q) = \lim_{n\to\infty} \alpha_n(Q) \tag{4.4}$$

exists P-a.s. and satisfies

$$E_P[\alpha_\infty(Q)] < \infty. \tag{4.5}$$

Proof. Take $Q \in \mathcal{Q}(P)$. Applying Proposition 3.8 for $X = 0$, we see that the consistency condition $\rho_{n+1} = \rho_{n+1}(-\rho_n)$ implies the backwards supermartingale inequality

$$\alpha_{n+1}(Q) \geq E_Q[\alpha_n(Q) | \mathcal{F}_{V_n^c}], \qquad n = 1, 2, \ldots$$

It follows that $(\alpha_n(Q))_{n=1,2...}$ is a non-negative backwards supermartingale under Q which is bounded in $L^1(Q)$. It is thus convergent, Q-a.s. and in $L^1(Q)$, to a finite limit $\alpha_\infty(Q)$ such that

$$E_Q[\alpha_\infty(Q)] = \lim_n E_Q[\alpha_n(Q)] < \infty.$$

This implies (4.5) and also the P-almost sure convergence in (4.4), since $Q = P$ on $\mathcal{F}_\infty$.

Combining Lemma 4.2 with the supermartingale inequality (3.9), we obtain the first part of the following Proposition. The second part will follow by applying the results in [FP2] on the behavior of consistent conditional risk measures along decreasing σ-fields.

Proposition 4.3. *We have*

$$\rho_\infty(\cdot, X) = \lim_{n\to\infty} \rho_n(\cdot, X) \quad P-\text{a.s.} \tag{4.6}$$

for any $X \in M$, and the kernel ρ_∞ defines a conditional convex risk measure

$$\rho_\infty : L^\infty(\Omega, \mathcal{F}, P) \to L^\infty(\Omega, \mathcal{F}_\infty, P) \tag{4.7}$$

under P with respect to the tail-field $\mathcal{F}_\infty$. This conditional risk measure has the Fatou property, and its dual representation is given by

$$\rho_\infty(X) = \operatorname{ess\,sup}_{Q\in\mathcal{Q}_P}(E_Q[-X|\mathcal{F}_{-\infty}] - \alpha_\infty(Q)), \quad X \in M, \tag{4.8}$$

where $\alpha_\infty(Q)$ is given by (4.4). Moreover, α_∞ coincides with the minimal penalty function of ρ_∞, i.e.,

$$\alpha_\infty(Q) = \operatorname{ess\,sup}_{X\in\mathcal{A}_\infty(P)} E_Q[-X|\mathcal{F}_\infty] \tag{4.9}$$

for any $Q \in \mathcal{Q}(P)$.

Proof. 1) Take any $X \in M$ and consider the process

$$V_n(P, X) = \rho_n(X) + \alpha_n(P), \qquad n = 1, 2, \ldots.$$

This process is bounded from below by $-||X||$, and the consistency condition $\rho_{n+1} = \rho_{n+1}(-\rho_n)$ implies the backward supermartingale inequality

$$V_{n+1}(P,X) \geq E_P[V_n(P,X))|\mathcal{F}_{V_n^c}];$$

see Proposition 3.8 for $Q = P$.
2) Take any $X \in \mathcal{A}_n(P)$. Since $\rho_n(\cdot, X) \leq 0$ P-a.s., we have

$$\rho_n(\cdot, X) \leq 0 \qquad \pi_n(\omega, \cdot) - \text{a.s.}$$

for P-almost all ω. Using (3.5) and our assumption (3.7), this implies

$$E_P[-X|\mathcal{F}_{V_n^c}](\omega) = \int (-X)(\eta)\pi_n(\omega, d\eta) \leq c$$

for P-almost all ω. In view of (2.11), this yields the estimate

$$\alpha_n(P) \leq c \qquad P - \text{a.s.}.$$

This bound is valid for any n, and so we have $P \in \mathcal{Q}(P)$.
3) Since $P \in \mathcal{Q}(P)$, the process $(V_n(P,X))_{n=1,2,...}$ is a backwards supermartingale with respect to P and bounded in $L^1(P)$, hence convergent P-a.s. to some finite limit $V_\infty(P,X)$. Combined with Lemma 4.2, this yields P-almost sure convergence of the sequence

$$\rho_n(X) = V_n(P,X) + \alpha_n(P), \qquad n = 1, 2, \ldots$$

to $\rho_\infty(X)$ and the equality

$$\rho_\infty(X) = V_\infty(P,X) + \alpha_\infty(P) \quad P - \text{a.s.}.$$

4) Since the backwards supermartingale $(\alpha_n(P))_{n=1,2,...}$ is bounded in $L^1(P)$, we can now apply the results of [FP2] on the limiting behavior of consistent conditional risk measures along decreasing σ-fields under a fixed reference measure P. Lemma 2 in [FP2] shows that ρ_∞ has the Fatou property under P, and Theorem 4 in [FP2] yields the dual representation (4.8) and the identification of α_∞ as the minimal penalty function of ρ_∞.

5. Dynkin boundary and boundary risk

In this section we are going to modify the risk kernel ρ_∞ in such a way, that the resulting kernel $\hat{\rho}_\infty$ has good properties in terms of the class $\mathcal{P}$ of Gibbs measures. To this end, we use a method developed by E.B. Dynkin [Dy1] for the construction of the entrance boundary of a Markov process, as it was applied in [F1] to the integral representation of the class $\mathcal{P}$. This involves an extension of the local specification $(\pi_V)_{V\in\mathcal{V}}$ to a conditional probability distribution π_∞ with respect to the tail field $\mathcal{F}_\infty$ which is common to all probability measures $P \in \mathcal{P}$. More precisely, there exists a stochastic kernel π_∞ from $(\Omega, \mathcal{F}_\infty)$ to $(\Omega, \mathcal{F})$ with the following properties:

i) For any $\omega \in \Omega$, the random field $\pi_\infty(\omega, \cdot)$ belongs to $\mathcal{P}$ and is actually an extreme point of the convex set $\mathcal{P}$. In particular we have

(5.1) $$\pi_\infty \pi_V = \pi_\infty \quad \text{for any } V \in \mathcal{V}.$$

ii) For any $\omega \in \Omega$, the probability measure $\pi_\infty(\omega, \cdot)$ is ergodic on the tail field, that is, $\pi_\infty(\omega, A) \in \{0, 1\}$ for any $A \in \mathcal{F}_\infty$, and this implies

(5.2) $$\pi_\infty(\eta, \cdot) = \pi_\infty(\omega, \cdot) \quad \pi_\infty(\omega, \cdot) - \text{a.s.};$$

see [Dy1], [Dy2], and [F1].

Due to (5.1), the kernel π_∞ serves, simultaneously for any $P \in \mathcal{P}$, as a conditional distribution with respect to the tail field $\mathcal{F}_\infty$, that is,

$$E_P[\,f\,|\,\mathcal{F}_\infty\,](\omega) = \int f(\eta)\pi_\infty(\omega, d\eta) \tag{5.3}$$

P-a.s. for any $P \in \mathcal{P}$ and for any measurable function $f \geq 0$ on $(\Omega, \mathcal{F})$.

We endow the set $\mathcal{P}$ with the canonical σ-field $\mathcal{B}$ generated by the maps $P \to P[A] \quad (A \in \mathcal{F})$. Then the kernel π_∞ can be viewed as a measurable map from $(\Omega, \mathcal{F}_\infty)$ to $(\mathcal{P}, \mathcal{B})$. We denote by

$$\hat{\mathcal{F}} := \sigma(\pi_\infty) \subseteq \mathcal{F}_\infty$$

the σ-field on Ω generated by this map, and by

$$\hat{M} := M_b(\Omega, \hat{\mathcal{F}}) \subseteq M_\infty$$

the corresponding space of bounded measurable functions. We will call $(\Omega, \hat{\mathcal{F}})$ the *Dynkin boundary* of the local specification $(\pi_V)_{V \in \mathcal{V}}$, and $\hat{\mathcal{F}}$ will be called the *boundary field.* Thus, any random field $P \in \mathcal{P}$ admits a representation by a probability measure on the Dynkin boundary, namely

$$P = \hat{P}\pi_\infty := \int \pi_\infty(\omega, \cdot)\,\hat{P}\,(d\omega), \tag{5.4}$$

where $\hat{P}$ denotes the restriction of P to the σ-field $\hat{\mathcal{F}}$. Conversely, any probability measure $\hat{P}$ on $(\Omega, \hat{\mathcal{F}})$ defines via (5.4) a random field $P \in \mathcal{P}$, due to (5.1). In this way, we obtain an *integral representation* of the convex set $\mathcal{P}$ that is coupled to the tail field by the kernel π_∞:

$$\mathcal{P} = \{\hat{P}\pi_\infty \,|\, \hat{P} \ \text{is a probability measure on}\ (\Omega, \hat{\mathcal{F}})\}. \tag{5.5}$$

In particular, a phase transition $|\mathcal{P}| > 1$ occurs if and only if the Dynkin boundary is non-trivial in the sense that the kernel π_∞ really depends on the tail field, that is, not all measures $\pi_\infty(\omega, \cdot)$ coincide, and so $\hat{\mathcal{F}}$ does not reduce to the trivial σ-field $\{\emptyset, \Omega\}$.

Remark 5.1. *The integral representation (5.5) shows that the set of extreme points of the convex set $\mathcal{P}$ is given by*

$$\mathcal{P}_e := \{\pi_\infty(\omega, \cdot) | \omega \in \Omega\}.$$

In particular, $\mathcal{P}_e$ is a measurable subset of $\mathcal{P}$. Denoting by μ_P the image of P under the map $\pi_\infty : \Omega \to \mathcal{P}_e$, the representation (5.4) takes the form

$$P = \int_{\mathcal{P}_e} Q\,\mu_P(dQ). \tag{5.6}$$

Conversely, any probability measure μ on $\mathcal{P}_e$ defines via (5.6) a random field $P \in \mathcal{P}$, and we have $\mu = \mu_P$. Thus we obtain a Choquet type integral representation of the convex set $\mathcal{P}$, that is, any $P \in \mathcal{P}$ is barycenter of a unique probability measure μ_P on the set $\mathcal{P}_e$ of extreme points; see [Dy1], [Dy2], and [F1].

Let us now regularize the kernel ρ_∞ by introducing the risk kernel $\hat{\rho}_\infty = \pi_\infty \rho_\infty$ defined by

$$\hat{\rho}_\infty(\omega, X) = \int \rho_\infty(\eta, X)\pi_\infty(\omega, d\eta) \tag{5.7}$$

for $\omega \in \Omega$ and $X \in M$. In order to describe its properties, we first take a closer look at the functions in the space $\hat{M}$.

Lemma 5.2. *For any function* $\hat{X} \in \hat{M}$ *and any* $\omega \in \Omega$*, we have*

$$\hat{X}(\omega) = \int \hat{X}(\eta)\pi_\infty(\omega, d\eta) \tag{5.8}$$

and

$$\hat{X}(\cdot) = \hat{X}(\omega) \quad \pi_\infty(\omega, \cdot) - \text{a.s.} \tag{5.9}$$

Proof. Since $\hat{\mathcal{F}}$ is generated by the map $\pi_\infty : \Omega \to \mathcal{P}$, there is a measurable function f on $\mathcal{P}$ such that $\hat{X}(\omega) = f(\pi_\infty(\omega, \cdot))$ for all $\omega \in \Omega$. Due to (5.3), we have $\pi_\infty(\eta, \cdot) = P$ for P-almost all η. But for any such η we obtain

$$\int \hat{X}(\eta)\pi_\infty(\omega, d\eta) = \int f(\pi_\infty(\eta, \cdot))P(d\eta) = f(\pi_\infty(\omega, \cdot)) = \hat{X}(\omega),$$

and also

$$\hat{X}(\eta) = \int \hat{X}(\zeta)\pi_\infty(\eta, d\zeta) = \int \hat{X}(\zeta)\pi_\infty(\omega, d\zeta) = \hat{X}(\omega).$$

Proposition 5.3. $\hat{\rho}_\infty$ *is a regular convex risk kernel from* $(\Omega, \hat{\mathcal{F}})$ *to* $(\Omega, \mathcal{F})$*, and it satisfies the consistency condition*

$$\hat{\rho}_\infty(-\rho_V) = \hat{\rho}_\infty \tag{5.10}$$

for any $V \in \mathcal{V}$*. For fixed* $\omega \in \Omega$*, we have*

$$\hat{\rho}_\infty(\omega, \cdot) \ll \pi_\infty(\omega, \cdot), \tag{5.11}$$

and the convex risk measure $\hat{\rho}_\infty(\omega, \cdot)$ *has the Fatou property with respect to the probability measure* $\pi_\infty(\omega, \cdot)$*.*

Proof. For any $X \in M$, the function $\hat{\rho}_\infty(\cdot, X)$ is clearly $\hat{\mathcal{F}}$-measurable. For fixed $\omega \in \Omega$, the functional $\hat{\rho}_\infty(\omega, \cdot)$ on M inherits from ρ_∞ the properties of a convex risk measure and also the consistency condition:

$$\begin{aligned}\hat{\rho}_\infty(\omega, -\rho_V(X)) &= \int \rho_\infty(\eta, -\rho_V(X))\pi_\infty(\omega, d\eta) \\ &= \int \rho_\infty(\eta, X)\pi_\infty(\omega, d\eta) \\ &= \hat{\rho}_\infty(\omega, X).\end{aligned}$$

Thus $\hat{\rho}_\infty$ is a convex kernel from $(\Omega, \hat{\mathcal{F}})$ to $(\Omega, \mathcal{F})$ such that $\hat{\rho}_\infty(\omega, \cdot) \in \mathcal{R}$ for any $\omega \in \Omega$. To check its regularity, take $\hat{X} \in \hat{M}$, $X \in M$, and any bounded measurable function f on R^2. Since ρ_∞ is regular by Lemma 4.1, and since $\hat{X}(\eta) = \hat{X}(\omega)$ for $\pi_\infty(\omega, \cdot)$-almost all η by (5.9), we obtain

$$\begin{aligned}\hat{\rho}_\infty(\omega, f(\hat{X}, X)) &= \int \rho_\infty(\eta, f(\hat{X}, X))\pi_\infty(\omega.d\eta) \\ &= \int \rho_\infty(\eta, f(\hat{X}(\eta), X))\pi_\infty(\omega.d\eta) \\ &= \int \rho_\infty(\eta, f(\hat{X}(\omega), X))\pi_\infty(\omega.d\eta) \\ &= \hat{\rho}_\infty(\omega, f(\hat{X}(\omega), X)).\end{aligned}$$

It remains to verify the Fatou property of $\hat{\rho}_\infty(\omega, \cdot)$ with respect to the measure $P := \pi_\infty(\omega, \cdot)$. Take any uniformly bounded sequence (X_k) in M such that X_k converges P-a.s. to some $X \in M$. Since $P \in \mathcal{P}$, Proposition (4.3) implies

$$\rho_\infty(\cdot, X) \leq \liminf_k \rho_\infty(\cdot, X_k) \quad P - \text{a.s.}.$$

Applying Fatou's lemma, we obtain

$$\begin{aligned}\hat{\rho}_\infty(\omega, X) &= E_P[\rho_\infty(\cdot, X)] \\ &\le E_P[\liminf_k \rho_\infty(\cdot, X_k)] \\ &\le \liminf_k E_P[\rho_\infty(\cdot, X_k)] \\ &= \liminf_k \hat{\rho}_\infty(\omega, X_k).\end{aligned}$$

In the special case $X_k \equiv Y$ we see that $\hat{\rho}_\infty(\omega, X) \le \hat{\rho}_\infty(\omega, Y)$ whenever $X \le Y$ $\pi_\infty(\omega, \cdot)$-a.s., and this implies $\hat{\rho}_\infty(\omega, \cdot) \ll \pi_\infty(\omega, \cdot)$.

Definition 5.4. *Let us say that a monetary risk measure ρ on M has the* Lebesgue property with respect to the class $\mathcal{P}$ *if* $\lim_k \rho(X_k) = \rho(X)$ *whenever (X_k) is a uniformly bounded sequence in M such that*

$$\lim_k X_k = X \quad \mathcal{P}-\text{almost surely},$$

that is, the convergence takes place P-a.s. for any $P \in \mathcal{P}$. We denote by $\mathcal{R}_L$ the class of all risk measures $\rho \in \mathcal{R}$ which have the Lebesgue property with respect to $\mathcal{P}$.

Remark 5.5. *For a monetary risk measure $\hat{\rho}$ on $\hat{M}$, the Lebesgue property with respect to $\mathcal{P}$ is equivalent to the Lebesgue property with respect to pointwise convergence, that is, $\lim_k \hat{\rho}(\hat{X}_k) = \hat{\rho}(\hat{X})$ whenever $(\hat{X}_k)$ is a uniformly bounded sequence in $\hat{M}$ such that $\lim_k \hat{X}_k(\omega) = \hat{X}(\omega)$ for any $\omega \in \Omega$. Indeed, if $\lim_n \hat{X}_n = \hat{X}$ $\mathcal{P}$-a.s. then the sequence converges $\pi_\infty(\omega, \cdot)$-a.s. for each $\omega \in \Omega$, and this amounts to pointwise convergence on Ω, due to Lemma 5.2.*

The following theorem shows that any risk measure $\rho \in \mathcal{R}_L$ is uniquely determined by its behavior on the Dynkin boundary, that is, by its restriction $\hat{\rho}$ to the space $\hat{M}$.

Theorem 5.6. *Any risk measure $\rho \in \mathcal{R}_L$ has the form*

$$\rho = \hat{\rho}(-\hat{\rho}_\infty), \tag{5.12}$$

where $\hat{\rho}$ denotes the restriction of ρ to $\hat{M}$.

Proof. Take $\rho \in \mathcal{R}_L$ and any $X \in M$. Since $\rho \in \mathcal{R}$, we have

$$\rho(-\rho_n(X)) = \rho(X)$$

for any $n \ge 1$. The sequence sequence $(\rho_n(X))_{n=1,2,\dots}$ is uniformly bounded by $||X||$, and Proposition 4.3 shows that

$$\lim_n \rho_n(\cdot, X) = \rho_\infty(\cdot, X) \quad \mathcal{P}-\text{almost surely}.$$

Now note that, for any $\omega \in \Omega$, the equality

$$\rho_\infty(\cdot, X) = \int \rho_\infty(\cdot, X)\pi_\infty(\omega, d\eta) = \hat{\rho}_\infty(\omega, X) = \hat{\rho}_\infty(\cdot, X)$$

holds $\pi_\infty(\omega, \cdot)$-almost surely, using first the ergodicity of $\pi_\infty(\omega, \cdot)$ and then (5.9). In view of the integral representation (5.4), this implies $\rho_\infty(\cdot, X) = \hat{\rho}_\infty(\cdot, X)$ P-a.s. for any $P \in \mathcal{P}$, and so we get

$$\lim_n \rho_n(X) = \hat{\rho}_\infty(\cdot, X) \quad \mathcal{P}-\text{almost surely}.$$

Applying the Lebesgue property of ρ with respect to $\mathcal{P}$, we obtain

$$\rho(X) = \lim_n \rho(-\rho_n(X)) = \rho(-\hat{\rho}_\infty(X)) = \hat{\rho}(-\hat{\rho}_\infty(X)),$$

and this proves the representation (5.12).

Remark 5.7. *If a risk measure $\rho \in \mathcal{R}$ has the Fatou property with respect to $\mathcal{P}$ but not the Lebesgue property, then the preceding proof yields the inequality $\rho \leq \hat{\rho}(-\hat{\rho}_\infty)$.*

Now suppose that, as in the special entropic case (5.16) below, the risk kernel $\hat{\rho}_\infty$ is such that each risk measure $\hat{\rho}_\infty(\omega, \cdot)$ has not only the Fatou property but also the Lebesgue property with respect to the measure $\pi_\infty(\omega, \cdot)$. In such a situation, we have $\mathcal{R}_L \neq \emptyset$, and there is a one-to-one correspondance between the class $\mathcal{R}_L$ and the class $\hat{\mathcal{R}}_L$ of all convex risk measures $\hat{\rho}$ on $\hat{M}$ that have the Lebesgue property with respect to pointwise convergence:

Corollary 5.8. *If each risk measure $\hat{\rho}_\infty(\omega, \cdot)$ has the Lebesgue property with respect to the measure $\pi_\infty(\omega, \cdot)$, then we have*

$$\mathcal{R}_L = \{\rho(-\hat{\rho}_\infty) | \hat{\rho} \in \hat{\mathcal{R}}_L\}, \tag{5.13}$$

and in particular $\mathcal{R}_L \neq \emptyset$.

Proof. The inclusion "$\subseteq$" follows from the preceding theorem. Conversely, if $\hat{\rho} \in \hat{\mathcal{R}}_L$ then $\rho := \hat{\rho}(-\hat{\rho}_\infty)$ clearly defines a convex risk measure on M which belongs to the class $\mathcal{R}$. To see that ρ has the Lebesgue property with respect to $\mathcal{P}$ and thus belongs to $\mathcal{R}_L$, take a uniformly bounded sequence (X_n) in M such that $X_n \to X$ $\mathcal{P}$-a.s.. In particular, the convergence holds $\pi_\infty(\omega, \cdot)$-a.s. for any $\omega \in \Omega$, and this implies $\lim_n \hat{\rho}_\infty(\omega, X_n) = \hat{\rho}_\infty(\omega, X)$. Thus we have pointwise convergence of the uniformly bounded sequence $(\hat{\rho}_\infty(\cdot, X_n))_{n=1,2,\ldots}$ in $\hat{M}$. Since $\hat{\rho}$ belongs to $\hat{\mathcal{R}}_L$, we get

$$\lim_n \rho(X_n) = \lim_n \hat{\rho}(-\hat{\rho}_\infty(\cdot, X_n)) = \hat{\rho}(-\hat{\rho}_\infty(\cdot, X)) = \rho(X).$$

This proves the converse inclusion "$\supseteq$". In particular we have $\mathcal{R}_L \neq \emptyset$, since $\hat{\mathcal{R}}_L \neq \emptyset$. Indeed, any probability measure $\hat{P}$ on the Dynkin boundary induces via

$$\hat{\rho}(X) = \int (-X) d\hat{P} \tag{5.14}$$

a convex risk measures $\hat{\rho} \in \hat{\mathcal{R}}_L$.

Corollary 5.9. *A risk measure $\rho \in \mathcal{R}_L$ is uniquely determined by the local specification $(\rho_V)_{V \in \mathcal{V}}$ if and only if the local specification $(\pi_V)_{V \in \mathcal{V}}$ admits no phase transition, i.e.,*

$$|\mathcal{R}_L| = 1 \iff |\mathcal{P}| = 1. \tag{5.15}$$

Proof. If $|\mathcal{P}| = 1$ then $\hat{\mathcal{F}}$ is trivial, $\hat{M}$ can be identified with R^1, and there is only one monetary risk measure on $\hat{M}$ given by $\hat{\rho}(m) = -m$. Thus (5.13) implies $|\mathcal{R}_L| = 1$. Conversely, if $|\mathcal{P}| > 1$ then we can choose $\omega_1, \omega_2 \in \Omega$ such that $\pi_\infty(\omega_1, \cdot) \neq \pi_\infty(\omega_2, \cdot)$. Taking

$$A = \{\omega \in \Omega \,|\, \pi_\infty(\omega, \cdot) = \pi_\infty(\omega_1, \cdot)\} \in \hat{\mathcal{F}},$$

we obtain $\pi_\infty(\omega_1, A) = 1$ und $\pi_\infty(\omega_2, A) = 0$ due to (5.2). But $\hat{\rho}_\infty(\omega_i, \cdot) \ll \pi_\infty(\omega_i, \cdot)$ for $i = 1, 2$ by Proposition 5.3, and so we get $\hat{\rho}_\infty(\omega_1, -I_A) = 1$ and $\hat{\rho}_\infty(\omega_2, -I_A) = 0$. This shows that the two risk measures $\hat{\rho}_i := \hat{\rho}_i(\omega, \cdot) \in \mathcal{R}_L$ do not coincide, and so we have $|\mathcal{R}_L| > 1$.

The absence of a phase transition at the underlying probabilistic level implies $|\mathcal{R}_L| = 1$, but not $|\mathcal{R}| = 1$, as illustrated by the following remark on the entropic case.

Remark 5.10. *Let us return to the special case of local law invariance in Remark 3.6, where the local risk measures $\rho_V(\omega, \cdot)$ are of the entropic form (3.8) with some parameter $\beta_\infty(\omega)$ which depends on the tail field $\mathcal{F}_\infty$. For fixed $\omega \in \Omega$, the measure $\pi_\infty(\omega, \cdot)$ is ergodic on $\mathcal{F}_\infty$, and so we have*

$$\beta_\infty(\eta) = \hat{\beta}(\omega) := \int \beta_\infty(\zeta) \pi_\infty(\omega, d\zeta)$$

for $\pi_\infty(\omega,\cdot)$-almost all $\eta \in \Omega$. Thus the risk kernel $\hat{\rho}_\infty = \pi_\infty \rho_\infty$ in (5.7) takes the form

$$\hat{\rho}_\infty(\omega, X) = \frac{1}{\hat{\beta}(\omega)} \log \int e^{-\hat{\beta}(\omega)X(\eta)} \pi_\infty(\omega, d\eta). \tag{5.16}$$

Clearly, the convex risk measure $\hat{\rho}_\infty(\omega,\cdot)$ has not only the Fatou property but also the Lebesgue property with respect to the probability measure $\pi_\infty(\omega,\cdot)$. Thus we can apply Corollary 5.8 and Corollory 5.9.

In the absence of a phase transition we have $\mathcal{P} = \{P\}$ for a single random field P, the $\hat{\mathcal{F}}$-measurable function $\hat{\beta}$ reduces to the constant

$$\beta := \int \beta_\infty(\omega)P(d\omega) \in [0,\infty),$$

and the unique risk measure ρ in $\mathcal{R}_L$ is given by the entropic risk measure (2.5) with respect to P and β. In particular we obtain $\rho(X) = E_P[-X]$ for any function $X \in M_\infty$, since $X(\cdot) = E_P[X]$ P-almost surely, due to the ergodicity of P on $\mathcal{F}_\infty$. On the other hand, the convex risk measures $\rho_\infty(\omega,\cdot)$ in (4.1) all belong to $\mathcal{R}$ due to (4.2), and they are different from ρ since regularity of the kernel ρ_∞ implies $\rho_\infty(\omega, X) = -X(\omega)$ for any $X \in M_\infty$.

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